Analyse et contrôle de modèles d'écoulements fluides

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Analyse et contrôle de modèles d'écoulements fluides

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À ma famille,

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Introduction

Ma thèse s'inscrit dans le domaine du contrôle des équations aux dérivées partielles. Je m'intéresse plus particulièrement aux équations de Navier-Stokes qui fournissent des modèles physiques d'écoulements fluides. Ces équations sont au coeur de nombreuses disciplines actuelles. Les avancées de la recherche sur ces modèles jouent un rôle crucial dans le domaine aéronautique et spatial. Au cours de ma thèse, je me suis intéressé à trois problèmes bien distincts.

Tout d'abord, je me suis intéressé au contrôle local au trajectoire des équations de Navier-Stokes compressibles 1D. Conjointement avec Sylvain Ervedoza, nous avons prouvé un résultat de contrôlabilité autour de trajectoires régulières sous une hypothèse géométrique s'apparentant à un 'vidage' du domaine de son fluide initial lorsque celui-ci suit son flot. Ensuite, je me suis intéressé à l'existence de solutions fortes d'un système de Navier-Stokes incompressible 2D et 3D dans un domaine borné incluant une frontière mouvante dont la dynamique est donnée par le flot et qui génère une force sur le fluide. Le but est de faire un premier pas vers l'étude du système de la méthode des frontières immergées. Je prouve un résultat d'existence de solutions fortes local en temps et un résultat d'existence de solutions fortes en tout temps pour des données petites. Enfin, je m'intéresse à la stabilisation exponentielle à tout taux de décroissance de l'interface entre deux couches de fluides non miscibles, visqueux, incompressibles soumis à l'effet de tension de surface par un contrôle frontière dans un domaine horizontalement périodique. Je prouve un résultat de stabilisation autour de la configuration plate avec fluides au repos. Ce travail est un premier pas vers l'étude de la stabilisation des instabilités de Rayleigh Taylor.

1 Contrôle local au trajectoire des équations de Navier-Stokes compressibles 1D

Ce chapitre est un travail réalisé en collaboration avec Sylvain Ervedoza.

1 .1 Présentation

On s'intéresse dans un premier temps aux équations de Navier-Stokes compressibles 1D suivantes :

$$\left\{ \begin{array}{rcl} \partial_t \rho_s + \partial_x(\rho_s u_s) & = & 0 \quad \text{dans } (0,T)\times(0,L), \\ \rho_s(\partial_t u_s + u_s\partial_x u_s) - \nu\partial_{xx}u_s + \partial_x p(\rho_s) & = & 0 \quad \text{dans } (0,T)\times(0,L). \end{array} \right. \qquad (1\ .1)$$

Les inconnues sont la densité du fluide ρ_s et la vitesse du fluide u_s. On suppose que le terme de pression $p = p(\rho_s)$ ne dépend que de la densité. La constante $\nu > 0$ représente la viscosité du fluide. Pour être bien posées, ces équations doivent être complétées par des conditions de bords et des données initiales que l'on ne fait pas apparaître intentionnellement.

La question que l'on se pose est celle de la contrôlabilité locale aux trajectoires en temps fini par un contrôle frontière.

1 .2 Résultats antérieurs

Les résultats sur la contrôlabilité locale exacte des équations de Navier-Stokes compressibles sont très récents. Le premier est du à Amosova : dans [2], elle prouve un résultat de contrôlabilité locale exacte pour des équations de Navier-Stokes compressibles 1D écrites sous forme Lagrangienne. Une année plus tard, Ervedoza, Glass, Guerrero et Puel [28] prouvent un résultat de contrôlabilité locale exacte pour le système (1 .1) autour de trajectoires constantes $(\bar{\rho},\bar{u})$ sous l'hypothèse géométrique $T|\bar{u}|>L$. Ce résultat est élargi dans [27] au cas 2D et 3D en domaine borné sous l'hypothèse semblable $T|\bar{u}|>L_0$ où $L_0>0$ est supérieur à l'épaisseur du domaine dans la direction $\bar{u}$. On peut aussi trouver des résultats de contrôlabilité sur les équations 1D linéarisées autour d'un état constant $(\bar{\rho},\bar{u})$, $\bar{u}\neq 0$, avec un contrôle n'agissant que sur la vitesse [16].

Il existe quelques résultats de contrôlabilité des équations 1D de fluide compressible parfait. Dans le cas de solutions régulières, on peut citer [46]. Des résultats de contrôlabilité locale exacte ont été obtenus dans le cas de solutions isentropiques [37], généralisés un peu plus tard dans le cas des équations d'Euler non-isentropiques [38].

Le contrôle des fluides incompressibles a été beaucoup plus étudié depuis le premier travail de Fursikov-Imanuvilov [35]. Les articles [6, 17, 18, 31, 41] constituent une liste non exhaustive de travaux récents sur les équations de Navier-Stokes incompressibles. Concernant les équations d'Euler incompressibles, citons les travaux [17, 36] qui se basent sur la méthode de retour.

1 .3 Résultat obtenu

Soit $(\overline{\rho},\overline{u})$ une solution de (1 .1) telle que

$$(\overline{\rho},\overline{u})\in C^2([0,T]\times[0,L])\times C^2([0,T]\times[0,L]),\qquad \text{avec}\ \inf_{[0,T]\times[0,L]}\overline{\rho}(t,x)>0. \tag{1 .2}$$

On étend $(\overline{\rho},\overline{u})$ sur $[0,T]\times\mathbb{R}$ de telle manière à ce que l'hypothèse (1 .2) soit vérifiée sur $[0,T]\times\mathbb{R}$:

$$(\overline{\rho},\overline{u})\in C^2([0,T]\times\mathbb{R})\times C^2([0,T]\times\mathbb{R}),\qquad \text{avec}\ \inf_{[0,T]\times\mathbb{R}}\overline{\rho}(t,x)>0. \tag{1 .3}$$

Théorème 1 .1. *Soit $T>0$, $(\overline{\rho},\overline{u})\in(C^2([0,T]\times[0,L]))^2$ une solution de (1 .1) satisfaisant (1 .2) étendue sur $[0,T]\times\mathbb{R}$ de manière à vérifier (1 .3), $\overline{X}$ le flot associé à $\overline{u}$.*

On suppose la condition géométrique suivante

$$\forall x\in[0,L],\ \exists t_x\in(0,T),\ \overline{X}(t_x,0,x)\notin[0,L]. \tag{1 .4}$$

Alors il existe $\varepsilon>0$ tel que pour tout $(\rho_0,u_0)\in H^1(0,L)\times H^1(0,L)$ vérifiant

$$\|(\rho_0,u_0)\|_{H^1(0,L)\times H^1(0,L)}\leq\varepsilon, \tag{1 .5}$$

il existe une trajectoire contrôlée (ρ_s,u_s) de (1 .1) qui satisfait la condition initiale

$$(\rho_s(0,\cdot),u_s(0,\cdot))=(\overline{\rho}(0,\cdot),\overline{u}(0,\cdot))+(\rho_0,u_0)\qquad \textit{dans}\ (0,L), \tag{1 .6}$$

et la condition finale

$$(\rho_s(T,\cdot),u_s(T,\cdot))=(\overline{\rho}(T,\cdot),\overline{u}(T,\cdot))\qquad \textit{dans}\ (0,L).$$

De plus, on a les régularités suivantes :

$$\rho_s\in C^0([0,T];H^1(0,L))\cap C^1([0,T];L^2(0,L))\ \textit{et}\ u_s\in H^1(0,T;L^2(0,L))\cap L^2(0,T;H^2(0;L)).$$

Les contrôles n'apparaissent pas explicitement dans le Théorème 1 .1. Comme mentionné plus haut, ils sont appliqués sur le bord du domaine, c'est-à-dire sur $(0,T)\times\{0,L\}$. Le système (1 .1)–(1 .6) peut ainsi être complété par les conditions aux bords suivantes :

$$\begin{cases} \rho_s(t,0)=\overline{\rho}(t,0)+v_{0,\rho}(t) & \text{pour } t\in(0,T) \text{ tel que } u_s(t,0)>0,\\ \rho_s(t,L)=\overline{\rho}(t,L)+v_{L,\rho}(t) & \text{pour } t\in(0,T) \text{ tel que } u_s(t,L)<0,\\ u_s(t,0)=\overline{u}(t,0)+v_{0,u}(t) & \text{pour } t\in(0,T),\\ u_s(t,L)=\overline{u}(t,L)+v_{L,u}(t) & \text{pour } t\in(0,T), \end{cases}$$

où $v_{0,\rho}, v_{L,\rho}, v_{0,u}, v_{L,u}$ sont les contrôles. Notons toutefois que la preuve ne fait pas apparaître explicitement ces contrôles.

L'hypothèse principale du Théorème 1 .1 est l'hypothèse (1 .4). Elle est de nature géométrique et elle ne dépend pas du choix de l'extension de $(\overline{\rho},\overline{u})$ sur $[0,T]\times\mathbb{R}$, mais bien uniquement de la trajectoire cible $(\overline{\rho},\overline{u})$ définie sur $[0,T]\times[0,L]$. Elle provient du contrôle de la densité et est indispensable pour le contrôle des équations linéarisées, comme l'a montré [48]. Dans le cas de trajectoires constantes, elle se réduit à $T>L/|\overline{u}|$ qui correspond bien à l'hypothèse de [28]. Elle apparaît aussi pour obtenir un résultat de contrôlabilité locale exacte pour les équations de Navier-Stokes incompressibles [6] et compressibles [27] en 2D et 3D.

Le Théorème 1 .1 généralise le résultat de [28] qui travaille autour de trajectoire constante $(\overline{\rho},\overline{u})$ avec $\overline{u}\neq 0$. Remarquez d'ailleurs que même autour de trajectoire constante, ce résultat est plus précis que [28] car l'hypothèse de petitesse (1 .5) est en norme $H^1(0,L)\times H^1(0,L)$ alors qu'elle est en norme $H^3(0,L)\times H^3(0,L)$ dans [28].

1 .4 Schéma de preuve

La stratégie de démonstration est la même que [28]. Elle repose sur le contrôle d'un système principalement linéarisé, les termes non linéaires étant traités par un argument de point fixe. La principale nouveauté de notre approche est dans le contrôle de la densité. Contrairement à [28] où la vitesse cible est constante non nulle, notre vitesse cible $\overline{u}$ peut s'annuler au bord du domaine. La difficulté sous jacente est que les caractéristiques sont alors tangentes au domaine. Nous traitons cette difficulté à travers des considérations géométriques basées sur l'hypothèse (1 .4), qui aboutissent sur un élargissement de domaine et sur la transformation du contrôle frontière en un contrôle intérieur. Cette modification de domaine et de contrôle a des répercussions importantes sur la construction de la trajectoire contrôlée de la densité et sur son estimation. La seconde nouveauté de ce travail est l'utilisation d'un seul paramètre de Carleman dans l'argument de point fixe. Cela simplifie la preuve de [28] qui reposait sur l'usage de deux paramètres issus des inégalités de Carleman.

1 .4.1 Considérations géométriques

Comme mentionné plus haut, cette partie est le coeur de la preuve. Il s'agit d'élargir le domaine et de passer d'un contrôle frontière à un contrôle intérieur afin d'éviter le cas délicat où les caractéristiques sont tangentes au domaine et ne permettent pas le transfert d'information du bord du domaine à l'intérieur du domaine.

L'idée est de considérer le point $x_0\in[0,L]$ défini par

$$x_0=\sup\left\{x\in[0,L]\mid \exists t_x\in[0,T),\ \overline{X}(t_x,0,x)=0 \text{ et } \forall t\in(0,t_x),\ \overline{X}(t,0,x)\in(0,L)\right\}.$$

Ce point a la particularité suivante : tous les points de $[0,x_0)$ sortent du domaine par le bord $x=0$ et tous les points de $(x_0,L]$ sortent du domaine par le bord $x=L$. On se retrouve donc dans l'un des cas de la Figure 1.

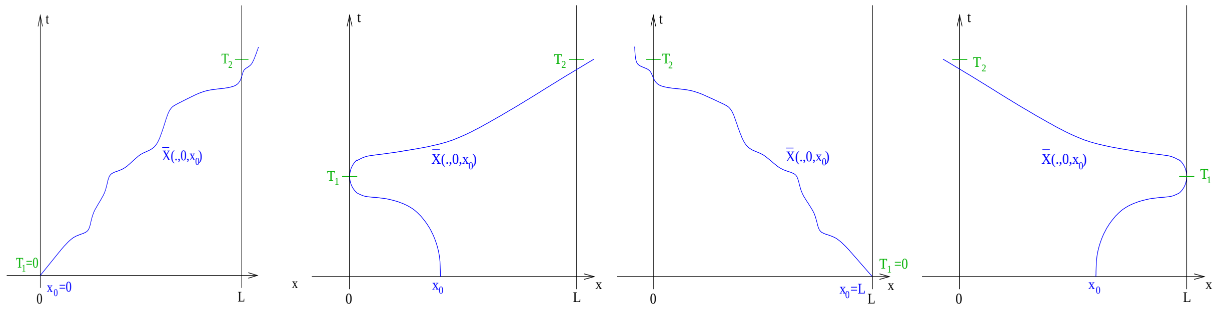

FIGURE 1 : Trajectoires possibles pour $t \mapsto \overline{X}(.,0,x_0)$.

Remarquons que quitte à remplacer x par $L - x$, on peut se placer dans l'un des deux premiers cas sans perte de généralité . D'après le Théorème de Cauchy-Lipschitz, les caractéristiques ne se croisent pas. On est donc en mesure de créer un point particulier $x_1 < x_0$ tel qu'il existe deux temps $0 < 2T_0 < T_L < T$ vérifiant $\overline{X}(2T_0, 0, x_1) < 0$ et $\overline{X}(T_L, 0, x_1) > L$. Il en découle l'existence d'un domaine (a, b) contenant $[0, L]$ et vérifiant

$$\forall x \in [a,b], \quad \exists t_x \in (0, T_L), \quad \overline{X}(t_x, 0, x) \notin [a,b].$$

Ceci est résumé dans la Figure 2.

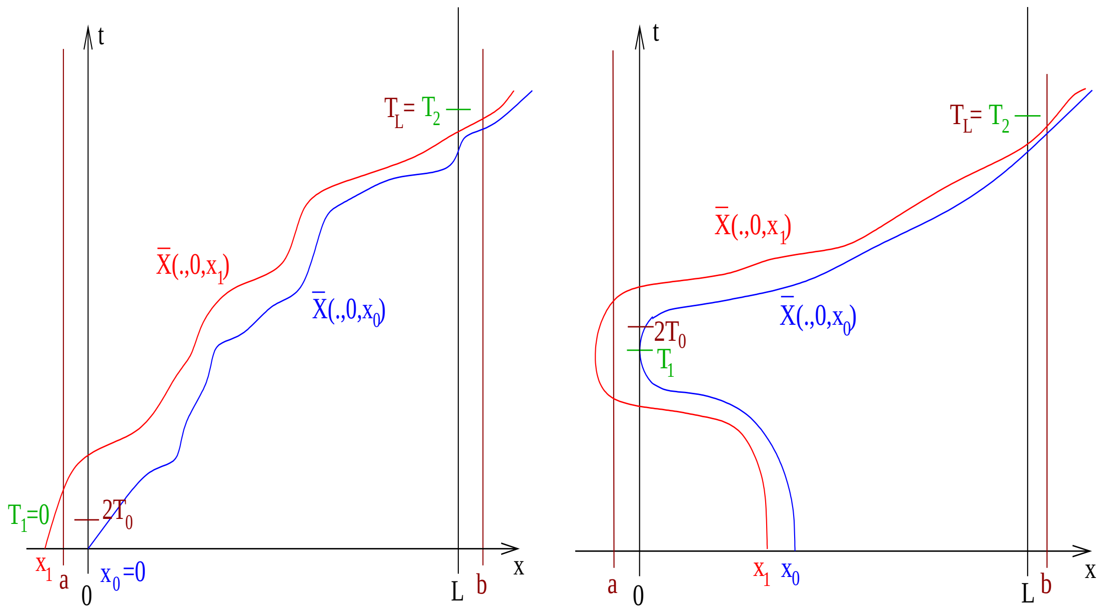

FIGURE 2 : Construction du domaine (a, b) : à gauche, $x_0 = 0$, à droite, $x_0 \in (0, L)$.

1 .4.2 Le système quasi-linéaire

On s'intéresse alors au contrôle à 0 du système suivant avec un contrôle distribué sur la densité :

$$\left\{ \begin{array}{rcll} \partial_t \rho + (\overline{u} + u)\partial_x \rho + \overline{\rho}\partial_x u + \left(\dfrac{\overline{\rho}}{\nu} p'(\overline{\rho}) + \partial_x \overline{u} \right) \rho & = & f + v_\rho \chi & \text{dans } (0,T) \times (a,b), \\ \overline{\rho}(\partial_t u + \overline{u}\partial_x u) - \nu \partial_{xx} u & = & g & \text{dans } (0,T) \times (a,b), \end{array} \right. \qquad (1\ .7)$$

où χ est la fonction indicatrice de $(0,a) \cap (L,b)$ et v_ρ le contrôle sur la densité. Les conditions aux bords associés au système (1 .7) sont les suivantes

$$\begin{cases} \rho(t,a) = 0 & \text{pour } t \in (0,T) \text{ tel que } (\overline{u}+u)(t,a) > 0, \\ \rho(t,b) = 0 & \text{pour } t \in (0,T) \text{ tel que } (\overline{u}+u)(t,b) < 0, \\ u(t,a) = v_u(t) & \text{pour } t \in (0,T), \\ u(t,b) = 0 & \text{pour } t \in (0,T), \end{cases} \tag{1 .8}$$

où v_u est le contrôle sur la vitesse. Chaque équation du système (1 .7) considérée seule est linéaire, mais le système (1 .7) n'est pas à proprement parler linéaire à cause du terme de transport $u\partial_x\rho$. Nous l'incluons dans la partie linéaire car les normes introduites ne permettent pas de le traiter en terme perturbatif.

Pour le contrôle de (1 .7), la stratégie consiste à commencer par contrôler l'équation de la vitesse $(1 .7)_2$, et une fois la trajectoire u obtenue, à construire une trajectoire contrôlée ρ de $(1 .7)_1$. Le contrôle de l'équation de la vitesse est classique et ne diffère que très peu de ce qui peut être trouvé dans [28] (voir aussi les travaux de Fursikov-Imanuvilov [35]). On introduit des poids Carleman qui nous permettent d'avoir l'inégalité de Carleman [6, Theorem 2.5] sur le problème adjoint à $(1 .7)_2$. L'existence d'une trajectoire contrôlée u de $(1 .7)_2$ découle de cette inégalité et les estimations sur la trajectoire contrôlée sont obtenues par des méthodes d'énergie. Nous ne nous étendons pas sur le contrôle de la vitesse et nous portons maintenant notre attention sur le contrôle de la densité.

1 .4.3 Construction de la densité

L'objectif est de contrôler à 0 le système suivant

$$\begin{cases} \partial_t\rho + (\overline{u}+u)\partial_x\rho + \overline{\rho}\partial_x u + \left(\dfrac{\overline{\rho}}{\nu}p'(\overline{\rho}) + \partial_x\overline{u}\right)\rho & = f + v_\rho\chi \quad \text{dans } (0,T)\times(a,b), \\ \rho(t,a) = 0 & \text{pour } t \in (0,T) \text{ t.q. } (\overline{u}+u)(t,a) > 0, \\ \rho(t,b) = 0 & \text{pour } t \in (0,T) \text{ t.q. } (\overline{u}+u)(t,b) < 0, \\ \rho(0,\cdot) = \rho_0 & \text{dans } (a,b). \end{cases} \tag{1 .9}$$

L'idée est de raccorder une solution *forward* en temps de condition initiale ρ_0 et une solution *backward* en temps de condition finale 0. On définit ainsi ρ_f et ρ_b les solutions de

$$\begin{cases} \partial_t\rho_f + \eta(\overline{u}+u)\partial_x\rho_f + \left(\dfrac{\overline{\rho}}{\nu}p'(\overline{\rho}) + \partial_x\overline{u}\right)\rho_f & = & \eta(f - \overline{\rho}\partial_x u) & \text{dans } (0,T)\times(a,b), \\ \rho_f(0,\cdot) & = & \rho_0 & \text{dans } (a,b), \end{cases} \tag{1 .10}$$

et

$$\begin{cases} \partial_t\rho_b + \eta(\overline{u}+u)\partial_x\rho_b + \left(\dfrac{\overline{\rho}}{\nu}p'(\overline{\rho}) + \partial_x\overline{u}\right)\rho_b & = & \eta(f - \overline{\rho}\partial_x u) & \text{dans } (0,T)\times(a,b), \\ \rho_b(T,\cdot) & = & 0 & \text{dans } (a,b), \end{cases} \tag{1 .11}$$

où η est la function cut off définie par

$$\forall x \in \mathbb{R}, \quad \eta(x) = \begin{cases} 1 & \text{pour } x \in [0,L], \\ 0 & \text{pour } x \in \mathbb{R} \setminus \left(\dfrac{a}{2}, \dfrac{L+b}{2}\right). \end{cases} \tag{1 .12}$$

Puisque $(\eta(\overline{u}+u))(t,a) = (\eta(\overline{u}+u))(t,b) = 0$, il n'est pas nécessaire de spécifier de condition de bord pour les systèmes (1 .10)–(1 .11). De plus, puisque $\rho_0 \in H_0^1(a,b)$, on a $\rho_f(\cdot,a) = \rho_f(\cdot,b) = \rho_b(\cdot,a) = \rho_b(\cdot,b) = 0$ sur $(0,T)$.

Pour créer une solution de (1 .7)$_1$, on raccorde ρ_f et ρ_b grâce à une fonction $\tilde{\eta}$ vérifiant l'équation

$$\partial_t \tilde{\eta} + (\overline{u} + u)\partial_x \tilde{\eta} = w\chi \text{ dans } (0,T) \times (a,b),$$

pour une certaine fonction w, (on rappelle que χ est la fonction indicatrice de $(a,0) \cup (L,b)$), et tel que

$$\begin{aligned} \tilde{\eta}(t,x) &= 1 \quad \text{pour tout } (t,x) \in [0,T_0] \times [a,b], \\ \tilde{\eta}(t,x) &= 0 \quad \text{pour tout } (t,x) \in [T_L,T] \times [a,b]. \end{aligned} \tag{1 .13}$$

On définit alors

$$\rho(t,x) = \tilde{\eta}(t,x)\rho_f(t,x) + (1-\tilde{\eta}(t,x))\rho_b(t,x) \text{ pour tout } (t,x) \in [0,T] \times [a,b].$$

Ainsi défini, ρ est solution de (1 .9) pour un certain v_ρ et vérifie la condition de contrôle

$$\rho(T,\cdot) = 0 \quad \text{dans } (a,b).$$

1 .4.4 Poids Carleman

Les inégalités de Carleman sont des inégalités d'observabilité avec des poids de la forme $\mathrm{e}^{-s\varphi}$, le paramètre $s \geq 1$ étant choisi suffisamment grand et le poids $\varphi = \varphi(t,x)$ étant strictement pseudo-convexe (voir [45]). Elles sont utilisées pour contrôler l'équation de la vitesse (1 .7)$_2$. Deux aspects sont pris en compte pour la construction du poids φ.

Premièrement, il rentre dans le cadre d'application de l'inégalité de Carleman [6, Theorem 2.5] qui permet de prouver l'existence d'une trajectoire contrôlée u et d'obtenir une estimée $\mathrm{e}^{s\varphi}\, u \in L^2(H^2)$ sous l'hypothèse $\mathrm{e}^{s\varphi}\, g \in L^2(L^2)$.

Deuxièmement, les poids Carleman interviennent dans l'espace fonctionnel permettant l'utilisation d'un théorème de point fixe. Notamment, nous devons être capable d'estimer la densité ρ construite en Section 1 .4.3 dans des espaces de Sobolev à poids Carleman appropriés. Il est donc naturel que notre poids φ ait des propriétés particulières le long des caractéristiques associées à $\overline{u}$. Nous le construisons de manière à respecter les inégalités suivantes

$$\begin{aligned} \partial_t \varphi + \overline{u}\partial_x \varphi \leq 0 \quad \text{dans } (0,T_L) \times (a,b), \\ \partial_t \varphi + \overline{u}\partial_x \varphi \geq 0 \quad \text{dans } (T_0,T) \times (a,b). \end{aligned}$$

Ces inégalités permettent d'avoir des estimées sur $\mathrm{e}^{s\varphi}\, \rho_f$ dans $(0,T_L)$ et sur $\mathrm{e}^{s\varphi}\, \rho_b$ dans (T_0,T). Notons ici l'importance de la propriété (1 .13) de la fonction $\tilde{\eta}$ qui permet d'avoir

$$\begin{aligned} \rho(t,x) &= \rho_f(t,x) \quad \text{pour tout } (t,x) \in [0,T_0] \times [a,b], \\ \rho(t,x) &= \rho_b(t,x) \quad \text{pour tout } (t,x) \in [T_L,T] \times [a,b]. \end{aligned}$$

1 .4.5 Estimation de la densité

Les estimées sur la densité ρ sont obtenues sur les solutions *forward* ρ_f et *backward* ρ_b. L'estimée délicate concerne le terme $\mathrm{e}^{s\varphi}\, \partial_x \rho$. En dérivant (1 .9) par rapport à la variable d'espace x, il semble en effet que $\partial_x \rho$ peut être estimé dans $L^\infty(L^2)$ par l'estimée $L^1(L^2)$ de $\partial_{xx} u$. Cependant, cette estimée semble insuffisante pour entrer dans le cadre d'application d'un théorème de point fixe (voir la définition des espaces fonctionnels (1.2 .31)–(1.2 .32)). L'idée est de changer d'inconnue en combinant les deux équations (1 .7) pour affaiblir les effets de couplage. Nous introduisons alors

$$\mu_f = \eta u + \frac{\nu}{\overline{\rho}^2}\partial_x \rho_f, \qquad \text{et} \qquad \mu_b = \eta u + \frac{\nu}{\overline{\rho}^2}\partial_x \rho_b,$$

où η est défini en (1 .12). Cette quantité s'appelle la vitesse effective et a déjà été introduite dans la littérature pour l'étude du caractère bien posé de quelques modèles de fluides visqueux compressibles [14, 15]. Les vitesses effectives μ_f et μ_b sont solutions de

$$\begin{cases} \partial_t \mu_f + \eta(\overline{u}+u)\partial_x \mu_f + k\mu_f &= h - \dfrac{\nu}{\overline{\rho}^2}\partial_x\left(\dfrac{\overline{\rho}}{\nu}p'(\overline{\rho}) + \partial_x \overline{u}\right)\rho_f \quad \text{dans } (0,T)\times(a,b), \\ \mu_f(0,\cdot) &= \eta u_0 + \dfrac{\overline{\rho}^2}{\nu}\partial_x \rho_0 \quad \text{dans } (a,b), \end{cases}$$

et

$$\begin{cases} \partial_t \mu_b + \eta(\overline{u}+u)\partial_x \mu_b + k\mu_b &= h - \dfrac{\nu}{\overline{\rho}^2}\partial_x\left(\dfrac{\overline{\rho}}{\nu}p'(\overline{\rho}) + \partial_x \overline{u}\right)\rho_b \quad \text{dans } (0,T)\times(a,b), \\ \mu_b(T,\cdot) &= 0 \quad \text{dans } (a,b), \end{cases} \tag{1 .14}$$

où les formulations précises de k et h sont données dans (1.3 .25)–(1.3 .26). Le point important est de remarquer que le terme $\partial_{xx}u$ n'apparaît pas dans h et que désormais $\mu_{f,b}$ peut être estimé dans $L^\infty(L^2)$ par la norme $L^1(L^2)$ de $\partial_x u$. Par conséquent, le terme $\partial_x \rho_{f,b}$ peut être estimé dans $L^\infty(L^2)$ par la norme $L^1(L^2)$ de $\partial_x u$ et $L^\infty(L^2)$ de u. Les estimées sur μ_f, μ_b, ρ_f, ρ_b sont alors obtenues par des méthodes d'énergie à poids.

1 .5 Remarques complémentaires et perspectives

1 .5.1 Remarques complémentaires

Si l'on suppose $\overline{u}(t,L) > 0$, seul un contrôle en $x = 0$ est nécessaire. En effet, dans la discussion géométrique de la Section 1 .4.1, il suffirait d'étendre le domaine en $x = 0$. Au regard des conditions de bord (1 .8), seul un contrôle en $x = a$ est nécessaire pour contrôler la vitesse u. Pour le contrôle de la densité ρ, il faudrait alors compléter le système de la solution *backward* ρ_b (1 .11) par la condition de bord $\rho_b(\cdot, L) = 0$. Nous aurions alors une condition de bord pour compléter le système de μ_b (1 .14) que l'on n'explicite pas mais qui ne présenterait aucune singularité par hypothèse $\overline{u}(t,L) > 0$ (voir [28]). Cependant, l'introduction d'un deuxième paramètre de Carleman serait nécessaire pour estimer ce terme de bord $\mu_b(\cdot, L)$ et appliquer le théorème de point fixe (voir [28]).

1 .5.2 Perspectives

On peut se demander si l'on peut prouver un résultat de même nature en dimension supérieure sous des hypothèses similaires. Ervedoza, Glass et Guerrero [27] ont prouvé un résultat de contrôlabilité locale exacte des équations de Navier-Stokes compressibles 2D et 3D autour de trajectoires constantes sous une hypothèse géométrique similaire à (1 .4) en contrôlant sur toute la frontière. La difficulté majeure de la contrôlabilité autour de trajectoires non constantes se trouve dans l'adaptation à la dimension supérieure de la discussion géométrique de la Section 1 .4.1.

2 Solutions fortes pour les équations de Navier-Stokes incompressibles 2D et 3D avec une frontière immergée

2 .1 Présentation

On travaille dans un domaine borné Ω de $\mathbb{R}^d$, $d \in \{2,3\}$. Soit Ω^- un domaine de Ω tel que $\overline{\Omega^-} \subset \Omega$ et $\Omega^+ = \Omega \setminus \overline{\Omega^-}$. On note $\partial\Omega$ la frontière de Ω et Γ la frontière Ω^-. Cette configuration est dessinée en Figure 3 dans le cas de la dimension $d = 2$.

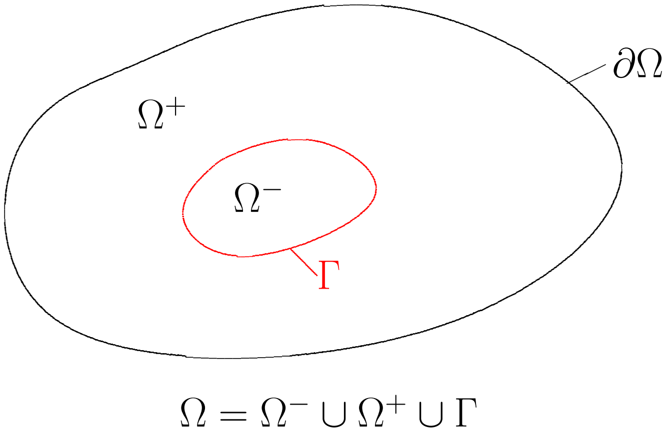

FIGURE 3 : Configuration initiale pour $d = 2$.

On s'intéresse au système suivant :

$$\left\{\begin{array}{rcll} \partial_t u_s + (u_s \cdot \nabla) u_s - \mu \Delta u_s + \nabla p_s & = & (f \circ Y)\delta_{\Gamma(t)} & \text{dans } (0,T) \times \Omega, \\ \nabla \cdot u_s & = & 0 & \text{dans } (0,T) \times \Omega, \\ u_s & = & 0 & \text{sur } (0,T) \times \partial\Omega, \\ u_s|_{t=0} & = & u_0 & \text{dans } \Omega, \\ \Gamma(t) & = & X(t,\Gamma) & \text{pour } t \in (0,T), \end{array}\right. \tag{2 .1}$$

où on définit le flot $X(\cdot, y)$ pour $y \in \Omega$ par

$$\left\{\begin{array}{rcll} \dfrac{dX}{dt}(t,y) & = & u_s(t, X(t,y)) & \text{sur } (0,T), \\ X(0,y) & = & y, & \end{array}\right. \tag{2 .2}$$

et où pour tout $t \in (0,T)$, $Y(t,\cdot) = X(t,\cdot)^{-1}$ est la transformation inverse de celle induite par le flot.

Il s'agit d'un système de Navier-Stokes incompressible. Les inconnues sont la vitesse du fluide u_s, la pression du fluide p_s et le flot X. La constante $\mu > 0$ représente la densité du fluide. Enfin le terme de force $(f \circ Y)\delta_{\Gamma(t)}$ est défini pour $t \in (0,T)$ par

$$\forall \varphi \in \mathscr{D}(\Omega), \quad \langle (f \circ Y)(t)\delta_{\Gamma(t)}, \varphi \rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}(t) = \langle f \circ Y(t), \varphi|_{\Gamma(t)} \rangle_{\mathscr{D}'(\Gamma(t)),\mathscr{D}(\Gamma(t))}(t), \quad \text{dans } \mathscr{D}'(0,T).$$

Ce terme de force $(f \circ Y)\delta_{\Gamma(t)}$ peut aussi être compris comme une condition de transfert à travers $\Gamma(t)$. Cette condition s'exprime sous la forme d'une continuité des vitesses et d'un saut du tenseur des contraintes :

$$[u] = 0 \quad \text{et} \quad [\sigma(u,p) n_{\Gamma(t)}] = f \quad \text{sur } \Gamma(t), \text{ pour } t \in (0,T), \tag{2 .3}$$

où $[w]$ désigne le saut de la quantité w à travers $\Gamma(t)$ et le tenseur des contraintes $\sigma(u,p)$ est défini par

$$\sigma(u,p) = \mu(\nabla u + (\nabla u)^T) - pI.$$

La question que l'on se pose est celle de l'existence de solutions fortes pour le système (2 .1)–(2 .2) en dimension $d \in \{2, 3\}$.

2 .2 Motivation

La motivation principale de l'étude du système (2 .1)–(2 .2) est de fournir un cadre mathématique à la méthode des frontières immergées.

La méthode des frontières immergées est une méthode d'origine numérique introduite par Peskin en 1972 [52] pour simuler l'écoulement du sang autour des valves du coeur. Le principe consiste à modéliser les valves par une frontière immergée $\Gamma(t)$ dans un fluide visqueux et générant une force $f\delta_{\Gamma(t)}$. L'intérêt numérique du système (2 .1) est qu'il peut être discrétisé sur un maillage fixe. La frontière $\Gamma(t)$ est représentée par un ensemble de points lagrangiens évoluant à chaque pas de temps. Travailler sur un maillage fixe permet de réduire considérablement les temps de calcul. Depuis son introduction en 1972, la méthode des frontières immergées a été utilisée à de maintes reprises pour simuler de nombreux modèles d'interaction fluide-structure. Fogelson l'utilisa pour modéliser l'agrégation des plaquettes sanguines lors de la coagulation du sang [32]. McQueen et Peskin ont simulé un modèle d'écoulement du sang dans le coeur en 3D [54]. Fauci et Peskin ont utilisé la méthode des frontières immergées pour modéliser le mouvement d'animaux aquatiques [30]. On peut encore citer les travaux [25, 26, 74]. Il ne s'agit là que d'un choix subjectif tant la liste est longue et s'allonge encore. Pour des références plus complètes sur la méthode des frontières immergées et ses applications, on pourra se reporter à l'article de Peskin [53] ou à la conférence plus récente de Stockie [65].

Dans la méthode des frontières immergées, la densité de force f générée par la frontière dépend des propriétés élastiques de la structure et de la forme de $\Gamma(t)$ à l'instant t. Dans les cas les plus classiques, on retrouve une force de compression/étirement avec une formulation 2D :

$$f_c = \frac{\partial}{\partial s}\left[\left(\left|\frac{\partial X}{\partial s}\right| - 1\right)\tau\right], \tag{2 .4}$$

où s représente l'abscisse curviligne et τ le vecteur tangent à $\Gamma(t)$.

Dans notre étude, nous supposons que f est une donnée pour le système (2 .1)–(2 .2) et nous ne traitons pas le couplage qu'elle induit. Nous n'entrons donc pas dans le cadre de la méthode des frontières immergées. Il s'agit d'un travail préliminaire voué à être élargi à la version complète de la méthode des frontières immergées (voir Section 2 .6).

2 .3 Résultats antérieurs

Les résultats théoriques sur le système (2 .1)–(2 .2) sont peu développés. Ton a montré un résultat d'existence de solution faible en 2D en régularisant l'équation du flot X (2 .2) dans [71]. Il s'agit du seul résultat connu par l'auteur sur le système (2 .1)–(2 .2).

Négligeant les effets inertiels et prenant une loi pour f de type (2 .4), Lin et Tong [47] et Mori et al. [49] ont récemment montré des résultats d'existence en temps petit et en temps long pour l'écoulement de Stokes 2D. Lin et Tong [47] travaillent dans les espaces de Sobolev et Mori et al. [49] dans les espaces de Hölder.

Le système (2 .1)–(2 .2) est à rapprocher des modèles à deux fluides visqueux non miscibles. La littérature sur les systèmes bifluides est abondante. Le système bifluide sans tension de surface a de multiples similarités avec le système (2 .1)–(2 .2), les différences étant que la densité de force $f = 0$ mais que les densités et des viscosités sont différentes sur chaque fluide. Les résultats d'existence et d'unicité de solutions faibles et fortes sont nombreux [1, 21, 22, 66]. Les systèmes avec tension de surface s'éloignent sans doute un peu plus du système (2 .1)–(2 .2), puisque la densité de force de tension de surface f dépend de la géométrie de l'interface $\Gamma(t)$ entre les deux fluides (voir Chapitre 3). On peut citer les travaux de Denisova et Solonikov qui montrent l'existence et l'unicité de solutions fortes dans les espaces de Sobolev et de Hölder pour différentes géométries 3D [19, 20, 23, 24]. D'autres résultats peuvent être trouvés dans [43, 57]. Au vu de l'abondance de la littérature sur les systèmes bifluides, ces listes sont subjectives.

Enfin, le système (2 .1)–(2 .2) est aussi à comparer aux modèles de frontière libre. Sur ce sujet aussi, la littérature est prolifique. Sans être exhaustif, on pourra se reporter aux travaux [64, 10, 11, 7, 40, 50, 68] et à leurs références.

2 .4 Résultats obtenus

Les résultats obtenus sont exprimés en coordonnées lagrangiennes. Le système (2 .1)–(2 .2) en coordonnées lagrangiennes s'écrit

$$\left\{\begin{array}{rcll} \partial_t u - \nabla\cdot\sigma(u,p) &=& (f+H(u,p))\delta_\Gamma + F(u,p) & \text{dans } (0,T)\times\Omega, \\ \nabla\cdot u &=& \nabla\cdot G(u) & \text{dans } (0,T)\times\Omega, \\ u &=& 0 & \text{sur } (0,T)\times\partial\Omega, \\ u|_{t=0} &=& u_0 & \text{dans } \Omega. \end{array}\right. \tag{2 .5}$$

et pour $y\in\Omega$,

$$\left\{\begin{array}{rcll} \dfrac{dX^\pm}{dt}(t,y) &=& u^\pm(t,y) & \text{sur } (0,T), \\ X^\pm(0,y) &=& y\in\Omega. & \end{array}\right. \tag{2 .6}$$

Les formulations précises des termes non linéaires F, G, H sont données dans (2.2 .2)–(2.2 .7).

Les résultats obtenus sur (2 .5)–(2 .6) sont des résultats d'existence de solutions fortes de deux types : des résultats locaux en temps pour toute donnée et des résultats pour tout temps à petite donnée. Ces résultats ne font pas mention d'unicité.

On introduit tout d'abord les notations suivantes pour $r, s, T > 0$,

$$\begin{array}{rcl} V(\Omega) &=& \left\{u\in H_0^1(\Omega)\mid \nabla\cdot u = 0 \text{ dans } \Omega\right\}, \\ H^{r,s}((0,T)\times\Omega^\pm) &=& L^2(0,T;H^r(\Omega^\pm))\cap H^s(0,T;L^2(\Omega^\pm)), \\ H^{r,s}((0,T)\times\Gamma) &=& L^2(0,T;H^r(\Gamma))\cap H^s(0,T;L^2(\Gamma)). \end{array}$$

2 .4.1 En dimension 2

Théorème 2 .1. *Soit*

$$\begin{gathered} d=2, \quad \ell\in(0,1/2), \quad \Gamma,\ \partial\Omega\in C^4, \\ f\in L^2_{loc}(0,+\infty;H^{1/2+\ell}(\Gamma))\cap H^{1/4+\ell/2}_{loc}(0,+\infty;L^2(\Gamma)), \\ u_0\in V(\Omega), \quad u_0^\pm\in H^{1+\ell}(\Omega^\pm). \end{gathered}$$

Il existe un temps $T>0$ tel que le système (2 .5)–(2 .6) admet une solution (u,p,X) vérifiant

$$\begin{gathered} u\in L^2(0,T;H^1(\Omega)), \quad u^\pm\in H^{2+\ell,1+\ell/2}((0,T)\times\Omega^\pm), \\ p\in L^2((0,T)\times\Omega), \quad \nabla p^\pm\in H^{\ell,\ell/2}((0,T)\times\Omega^\pm), \quad [p]\in H^{1/2+\ell,1/4+\ell/2}((0,T)\times\Gamma), \\ X\in L^2((0,T)\times\Omega), \quad X^\pm\in H^1(0,T;H^{2+\ell}(\Omega^\pm))\cap H^{2+\ell/2}(0,T;L^2(\Omega^\pm)). \end{gathered}$$

De plus, pour tout temps $t\in(0,T)$, $X(t,\cdot)$ est inversible et les inconnues $u_s(t,x)=u(t,Y(t,x))$ et $p_s(t,x)=p(t,Y(t,x))$, où $Y(t,.)$ est la transformation inverse de $X(t,\cdot)$, sont tels que (u_s,p_s,X) est une solution forte de (2 .1)-(2 .2).

Théorème 2 .2. *Soit*

$$d=2, \quad \ell\in(0,1/2), \quad \Gamma,\ \partial\Omega\in C^4.$$

Pour tout temps $T > 0$, il existe $r > 0$ tel que pour tout

$$f \in H^{1/2+\ell,1/4+\ell/2}((0,T)\times\Gamma),$$
$$u_0 \in V(\Omega), \quad u_0^\pm \in H^{1+\ell}(\Omega^\pm).$$

tel que

$$\|f\|_{H^{1/2+\ell,1/4+\ell/2}((0,T)\times\Gamma)} + \sum_\pm \|u_0^\pm\|_{H^{1+\ell}(\Omega^\pm)} \leq r,$$

le système (2 .5)–(2 .6) *admet une solution* (u,p,X) *vérifiant*

$$\begin{gathered} u \in L^2(0,T;H^1(\Omega)), \quad u^\pm \in H^{2+\ell,1+\ell/2}((0,T)\times\Omega^\pm), \\ p \in L^2((0,T)\times\Omega), \quad \nabla p^\pm \in H^{\ell,\ell/2}((0,T)\times\Omega^\pm), \quad [p] \in H^{1/2+\ell,1/4+\ell/2}((0,T)\times\Gamma), \\ X \in L^2((0,T)\times\Omega), \quad X^\pm \in H^1(0,T;H^{2+\ell}(\Omega^\pm)) \cap H^{2+\ell/2}(0,T;L^2(\Omega^\pm)). \end{gathered}$$

De plus, pour tout temps $t \in (0,T)$, $X(t,\cdot)$ est inversible et les inconnues $u_s(t,x) = u(t,Y(t,x))$ et $p_s(t,x) = p(t,Y(t,x))$, où $Y(t,.)$ est la transformation inverse de $X(t,\cdot)$, sont tels que (u_s,p_s,X) est une solution forte de (2 .1)-(2 .2).

2 .4.2 En dimension 3

Théorème 2 .3. *Soit*

$$\begin{gathered} d = 3, \quad \ell \in (1/2,1), \quad \Gamma,\ \partial\Omega \in C^4, \\ f \in L^2_{loc}(0,+\infty;H^{1/2+\ell}(\Gamma)) \cap H^{1/4+\ell/2}_{loc}(0,+\infty;L^2(\Gamma)), \\ u_0 \in V(\Omega), \quad u_0^\pm \in H^{1+\ell}(\Omega^\pm). \end{gathered}$$

On suppose que nos données remplissent la condition de compatibilité suivante

$$2\mu([\nabla u_0 + (\nabla u_0)^T]n_\Gamma)_\tau = f_\tau|_{t=0} \quad \textit{sur } \Gamma, \tag{2 .7}$$

où l'indice τ désigne la composante suivant l'espace tangent à Γ. Alors il existe un temps $T > 0$ tel que le système (2 .5)–(2 .6) *admet une solution* (u,p,X) *vérifiant*

$$\begin{gathered} u \in L^2(0,T;H^1(\Omega)), \quad u^\pm \in H^{2+\ell,1+\ell/2}((0,T)\times\Omega^\pm), \\ p \in L^2((0,T)\times\Omega), \quad \nabla p^\pm \in H^{\ell,\ell/2}((0,T)\times\Omega^\pm), \quad [p] \in H^{1/2+\ell,1/4+\ell/2}((0,T)\times\Gamma), \\ X \in L^2((0,T)\times\Omega), \quad X^\pm \in H^1(0,T;H^{2+\ell}(\Omega^\pm)) \cap H^{2+\ell/2}(0,T;L^2(\Omega^\pm)). \end{gathered}$$

De plus, pour tout temps $t \in (0,T)$, $X(t,\cdot)$ est inversible et les inconnues $u_s(t,x) = u(t,Y(t,x))$ et $p_s(t,x) = p(t,Y(t,x))$, où $Y(t,.)$ est la transformation inverse de $X(t,\cdot)$, sont tels que (u_s,p_s,X) est une solution forte de (2 .1)-(2 .2).

Théorème 2 .4. *Soit*

$$d = 3, \quad \ell \in (1/2,1), \quad \Gamma,\ \partial\Omega \in C^4.$$

Pour tout temps $T > 0$, il existe $r > 0$ tel que pour tout

$$f \in H^{1/2+\ell,1/4+\ell/2}((0,T)\times\Gamma),$$
$$u_0 \in V(\Omega), \quad u_0^\pm \in H^{1+\ell}(\Omega^\pm).$$

vérifiant la condition de compatibilité

$$2\mu([\nabla u_0 + (\nabla u_0)^T]n_\Gamma)_\tau = f_\tau|_{t=0} \quad \textit{sur } \Gamma,$$

où l'indice τ représente la composante suivant l'espace tangent à Γ, et la condition de petitesse

$$\|f\|_{H^{1/2+\ell,1/4+\ell/2}((0,T)\times\Gamma)} + \sum_{\pm}\|u_0^{\pm}\|_{H^{1+\ell}(\Omega^{\pm})} \leq r,$$

le système (2 .5)–(2 .6) *admet une solution* (u, p, X) *vérifiant*

$$\begin{gathered} u \in L^2(0,T;H^1(\Omega)), \quad u^{\pm} \in H^{2+\ell,1+\ell/2}((0,T)\times\Omega^{\pm}), \\ p \in L^2((0,T)\times\Omega), \quad \nabla p^{\pm} \in H^{\ell,\ell/2}((0,T)\times\Omega^{\pm}), \quad [p] \in H^{1/2+\ell,1/4+\ell/2}((0,T)\times\Gamma), \\ X \in L^2((0,T)\times\Omega), \quad X^{\pm} \in H^1(0,T;H^{2+\ell}(\Omega^{\pm})) \cap H^{2+\ell/2}(0,T;L^2(\Omega^{\pm})). \end{gathered}$$

De plus, pour tout temps $t \in (0,T)$, $X(t,\cdot)$ est inversible et les inconnues $u_s(t,x) = u(t,Y(t,x))$ et $p_s(t,x) = p(t,Y(t,x))$, où $Y(t,.)$ est la transformation inverse de $X(t,\cdot)$, sont tels que (u_s, p_s, X) est une solution forte de (2 .1)-(2 .2).

2 .4.3 Remarques sur les résultats obtenus

Notons tout d'abord que nous travaillons avec une régularité supérieure à celle qui permet de prouver l'existence de solutions fortes pour le système classique de Navier-Stokes dans un domaine borné (voir par exemple [13, 70]). De plus, cette régularité dépend de la dimension dans laquelle on travaille. Ce sont évidemment les non linéarités issues du passage en coordonnées Lagrangiennes qui sont à l'origine de ce besoin de régularité (voir Section 2 .5.2).

Concernant les conditions de compatibilité : tout d'abord, remarquons que la transformation Lagrangienne X (2 .2) est l'identité en $t = 0$. De fait, nous avons $F|_{t=0} = G|_{t=0} = H|_{t=0} = 0$. De plus, les équations $(2\ .1)_{2,3}$ imposent $u_0 \in V(\Omega)$. En dimension 3, la deuxième condition de transmission (2 .3) a une trace en $t = 0$. C'est la partie tangentielle de cette trace qui donne la condition de compatibilité (2 .7). La partie normale donne quant à elle une condition sur $[p]|_{t=0}$:

$$[p]|_{t=0} = (-f|_{t=0} + 2\mu([Du_0]n_\Gamma)) \cdot n_\Gamma \quad \text{sur } \Gamma.$$

Cependant cette condition n'est pas une condition de compatibilité sur les données. Il s'agirait plutôt d'une condition de transmission dans le problème de Cauchy qui permettrait de définir $p|_{t=0}$ si l'on montait en régularité.

2 .5 Schéma de preuve

La démonstration s'articule en deux parties : l'étude d'un système linéaire de Stokes avec second membre mesure associé à (2 .5)–(2 .6), et l'estimation des termes non linéaires afin d'utiliser un point fixe de Picard.

2 .5.1 Système de Stokes avec un second membre mesure

Le système linéaire associé à (2 .5)–(2 .6) est le suivant

$$\left\{\begin{array}{rcll} \partial_t u - \nabla\cdot\sigma(u,p) &=& f\delta_\Gamma + F & \text{dans } (0,T)\times\Omega, \\ \nabla\cdot u &=& \nabla\cdot G & \text{dans } (0,T)\times\Omega, \\ u &=& 0 & \text{sur } (0,T)\times\partial\Omega, \\ u|_{t=0} &=& u_0 & \text{dans } \Omega. \end{array}\right. \tag{2 .8}$$

avec les conditions de compatibilité : $G = u_0 = 0$ sur $\partial\Omega$, $\nabla\cdot G(0) = \nabla\cdot u_0 = 0$ et enfin (2 .7) pour $d = 3$.

Pour ce système, nous avons un résultat d'existence et d'unicité de solutions dans les espaces des Théorèmes 2 .1–2 .4. La preuve de ce résultat est basée sur un relèvement des singularités engendrées par Γ. Le terme f n'est pas le seul mis en jeu : les termes non homogènes F et G (voir leur expression dans (2.2 .2)–(2.2 .7)) et la condition initiale u_0 ne sont réguliers que de chaque côté de Γ. Remarquons tout de même qu'une fonction $L^2(\Omega)$ qui est $H^\ell(\Omega^\pm)$ avec $\ell \in (0,1/2)$ est en fait $H^\ell(\Omega)$. Ainsi, en dimension $d=2$, travaillant avec $\ell \in (0,1/2)$, la seule singularité à relever est f. En dimension $d=3$ en revanche, $\ell \in (1/2,1)$, et nous relevons alors le saut de trace à travers Γ du terme non homogène F.

Pour être plus précis, en dimension $d=2$, le problème de relèvement consiste à construire une solution (v,q) de

$$\left\{\begin{array}{rcll} \nabla\cdot v^\pm &=& \nabla\cdot G^\pm & \text{dans } (0,T)\times\Omega^\pm, \\ v^- &=& v^+ & \text{sur } (0,T)\times\Gamma, \\ [\sigma(v,q)n_\Gamma] &=& f & \text{sur } (0,T)\times\Gamma, \\ v &=& 0 & \text{sur } (0,T)\times\partial\Omega, \\ v|_{t=0} &=& u_0 & \text{dans } \Omega. \end{array}\right.$$

En dimension 3, s'ajoute à ce système l'équation suivante

$$[\nabla\cdot\sigma(v,q)] = -[F] \quad \text{sur } (0,T)\times\Gamma. \tag{2 .9}$$

Les lemmes de relèvement associés (voir Lemma 2.3 .3 en dimension $d=2$ et Lemma 2.3 .5 en dimension $d=3$) sont la clef pour traiter le système linéaire (2 .8). Ils nous permettent de nous ramener à une équation de Stokes classique sans interface dans un domaine borné et avec des données dont la régularité est donnée sur tout le domaine.

Il est à noter que ce raisonnement n'est pas avantageusement adaptable au modèle multi fluide : dans le cas de viscosités et/ou densités différentes pour chaque fluide, le relèvement de l'ensemble des singularités induites par l'interface ne semble pas une stratégie optimale, car l'interface sera toujours présente dans le système à travers les viscosités et/ou densités. Cela constitue la principale limite de ce raisonnement mais c'est aussi ce qui en fait son originalité.

2 .5.2 Les estimées des termes non linéaires

Cette partie est plutôt classique (voir par exemple [63] pour des estimées similaires), le point délicat étant de faire apparaître explicitement le temps dans les constantes des estimées.

Notons tout de même que contrairement à [63], les termes non linéaires ne font apparaître que la transformation directe X (voir leur expression précise dans (2.2 .2)–(2.2 .7)). En effet, lors du passage en coordonnées Lagrangiennes, nous avons utilisé le fait que $\nabla Y(X) = \text{cof}(\nabla X)^T$ par incompressibilité. Nous avons donc pu éliminer la transformation indirecte Y de l'expression des termes non linéaires, ce qui simplifie grandement les estimées.

Par ailleurs, puisque $u^\pm \in H^{2+\ell,1+\ell/2}((0,T)\times\Omega^\pm)$, nous avons $\nabla X^\pm \in H^1(0,T;H^{1+\ell}(\Omega^\pm))\cap H^{3/2+\ell/2}(0,T;L^2(\Omega))$. Du fait que $\ell\in(0,1/2)$ pour $d=2$ et $\ell\in(1/2,1)$ pour $d=3$, ce premier espace $H^1(0,T;H^{1+\ell}(\Omega^\pm))$ a une structure d'algèbre normée. Il constitue la pierre angulaire des estimées des termes non linéaires.

2 .6 Remarques complémentaires et perspectives

Je vois trois perspectives principales à ce travail :

2 .6.1 Le problème de l'unicité

La question de l'unicité des solutions des Théorèmes 2 .1 et 2 .3 est la suivante : soit $T>0$ le temps d'existence et (u,p,X) la solution de (2 .5)–(2 .6) sur $(0,T)$ donnée par ce théorème, si

$(\tilde{u}, \tilde{p}, \tilde{X})$ est solution de (2 .5)–(2 .6) sur $(0, T)$ dans les espaces des Théorèmes 2 .1 et 2 .3, a-t-on $(\tilde{u}, \tilde{p}, \tilde{X}) = (u, p, X)$?

Nous aimerions appliquer le raisonnement classique suivant : par unicité du point fixe de Picard, les deux solutions coïncident sur un temps $0 < T' \leq T$. Notons T_{max} le temps maximum où ces deux solutions coïncident et supposons $T_{max} < T$. On peut alors utiliser le théorème d'existence locale en partant de T_{max} et en utilisant de nouveau un argument d'unicité du point fixe de Picard, on montre que les deux solutions coïncident sur un temps $T'' > T_{max}$. Cela contredit la maximalité de T_{max}.

Dans notre cas, ce raisonnement ne peut pas être mis en place à cause d'une perte de régularité sur $\Gamma(t)$ qui ne nous permet pas d'appliquer nos résultats d'existence à partir du temps T_{max}. Remarquons que l'on a en effet $\Gamma(t)$ de classe $H^{3/2+\ell}$ alors que $\Gamma(0) = \Gamma$ est de classe C^4. C'est dans les lemmes de relèvement (voir les Lemmes 2.3 .3 en dimension $d = 2$ et 2.3 .5 en dimension $d = 3$) que ce besoin de régularité sur Γ apparaît. Pour abaisser cette régularité, il faudrait employer une autre technique de relèvement pour le terme de divergence puis pratiquer une décomposition de Helmholtz en aval du redressement local de Γ.

2 .6.2 Une frontière ouverte

Une autre piste de recherche intéressante est de considérer une frontière ouverte. La difficulté est complètement différente du problème actuel. On peut raisonnablement attendre des pertes de régularité autour des bords de la frontière. Je ne me suis pas encore consacré à cet aspect. Cependant, une première recherche bibliographique laisse à penser que ce sujet est relativement ouvert.

2 .6.3 Des lois pour f

Une dernière piste de recherche intéressante est de se donner des lois pour f qui vont dépendre de l'état de la structure, comme par exemple la loi (2 .4). Remarquons que cette loi ne peut pas se traiter en termes perturbatifs par manque de régularité sur le flot X. Elles induisent donc un couplage fort qu'il faut intégrer dans le système linéaire.

Dans le cas d'une interface entre deux fluides, on considère souvent l'effet de tension de surface f sous la forme

$$f = \sigma \kappa n_{\Gamma(t)},$$

où $\sigma > 0$ est le coefficient de tension de surface, κ la courbure de $\Gamma(t)$ et $n_{\Gamma(t)}$ sa normale. La littérature sur ce système est abondante comme mentionné plus haut.

En revanche, la littérature sur des lois de structure élastique, comme des densités de force de compression/étirement (2 .4), me semble peu développée. Je ne connais aujourd'hui que les travaux récents de Lin et Tong [47] et de Mori et al. [49] qui traitent le cas d'une densité de force de compression/étirement (2 .4) dans le cadre de l'écoulement de Stokes en dimension $d = 2$.

3 Stabilisation d'un problème bifluide avec tension de surface

3 .1 Présentation

On travaille dans $\Omega = \mathbb{T}_d \times (-1, 1)$ où $\mathbb{T}_d = \mathbb{R}^{d-1}/2\pi\mathbb{Z}^{d-1}$ désigne le tore et $d \in \{2, 3\}$. On se donne $k_0 : \mathbb{T}_d \to (-1, 1)$, $\Gamma_0 = \{(x', k_0(x')) \in \overline{\Omega} \mid x' \in \mathbb{T}_d\}$, $\Omega_0^+ = \{(x', x_d) \in \Omega \mid x_d > k_0(x')\}$, $\Omega_0^- = \{(x', x_d) \in \Omega \mid x_d < k_0(x')\}$, $\Gamma_d^\pm = \mathbb{T} \times \{\pm 1\}$, $\Gamma_d = \Gamma_d^+ \cup \Gamma_d^-$, et enfin $\Gamma_c \subset\subset \Gamma_d^+$ ou $\Gamma_c \subset\subset \Gamma_d^-$ la zone de contrôle localisée sur l'une des deux frontières Γ_d^+ ou Γ_d^-. La configuration est résumée dans la Figure 4.

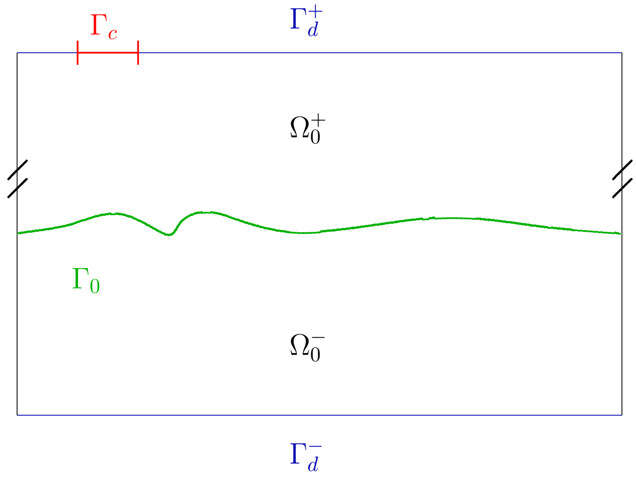

FIGURE 4 : Configuration initiale pour $d = 2$ avec $\Gamma_c \subset \Gamma_d^+$.

On s'intéresse à la stabilisation du système suivant :

$$\left\{\begin{array}{rcll}\rho^\pm\partial_t v^\pm + \rho^\pm(v^\pm\cdot\nabla)v^\pm &=& \mu^\pm\Delta v^\pm - \nabla q^\pm & \text{dans } \Omega^\pm(t),\ t>0,\\ \nabla\cdot v^\pm &=& 0 & \text{dans } \Omega^\pm(t),\ t>0,\\ v &=& g & \text{sur } (0,+\infty)\times\Gamma_d,\\ v^- &=& v^+ & \text{sur } \Gamma(t),\ t>0,\\ [\mathbb{S}(v,q)n_{\Gamma(t)}] &=& -\sigma\kappa n_{\Gamma(t)} & \text{sur } \Gamma(t),\ t>0,\\ V(t,x') &=& v(t,x',k(t,x'))\cdot n_{\Gamma(t)} & \text{pour } (t,x')\in(0,+\infty)\times\mathbb{T}_d,\\ v(0) &=& v_0 & \text{dans } \Omega,\\ k(0) &=& k_0 & \text{sur } \mathbb{T}_d,\end{array}\right. \tag{3 .1}$$

où le tenseur des contraintes $\mathbb{S}^\pm(v^\pm, q^\pm)$ est défini par

$$\mathbb{S}^\pm(v^\pm,q^\pm) = \mu^\pm(\nabla v^\pm + (\nabla v^\pm)^T) - q^\pm I,$$

la notation $[w]$ désigne la saut de w à travers $\Gamma(t)$:

$$[w] = (w|_{\Omega^+(t)} - w|_{\Omega^-(t)})|_{\Gamma(t)},$$

et pour $t \in \mathbb{R}_+^*$,

$$\begin{array}{rcl}\Gamma(t) &=& \{(x',k(t,x'))\mid x'\in\mathbb{T}_d\},\\ \Omega^+(t) &=& \{(x',x_d)\in\Omega\mid x_d>k(t,x')\},\\ \Omega^-(t) &=& \{(x',x_d)\in\Omega\mid x_d<k(t,x')\}.\end{array}$$

Les inconnues sont la vitesse du fluide $v^\pm$, la pression du fluide $q^\pm$ et la hauteur de l'interface k. Les densités $\rho^\pm > 0$ et les viscosités $\mu^\pm > 0$ sont supposées constantes. Le terme non homogène du saut du tenseur des contraintes $\sigma\kappa n_{\Gamma(t)}$ représente l'effet de tension de surface à l'interface $\Gamma(t)$: la constante σ est le coefficient de surface tension, $\kappa(t,\cdot)$ représente le rayon de courbure de $\Gamma(t)$ orienté dans le sens de la normale $n_{\Gamma(t)}$ de $\Gamma(t)$ qui pointe vers l'extérieur de $\Omega^-(t)$. Les formulations précises sont les suivantes

$$n_{\Gamma(t)} = \frac{1}{\sqrt{1+|\nabla_{x'}k|^2}}\begin{pmatrix}-\nabla_{x'}k\\ 1\end{pmatrix},$$

et

$$\kappa = \nabla\cdot\left(\frac{\nabla_{x'}k}{\sqrt{1+|\nabla_{x'}k|^2}}\right).$$

Le terme $V(t,x')$ pour $(t,x') \in (0,+\infty) \times \mathbb{T}_d$ représente la vitesse normale de $\Gamma(t)$ et est donné par

$$V(t,x') = \frac{\partial_t k(t,x')}{\sqrt{1+|\nabla_{x'} k(t,x')|^2}}.$$

Enfin, g représente notre contrôle frontière. On suppose qu'il est de dimension finie et qu'il n'agit que sur Γ_c :

$$\forall (t,x) \in (0,+\infty) \times \Gamma_d, \quad g(t,x) = \sum_{i=1}^{n_c} g_i(t) w_i(x) \quad \text{avec } \mathbf{g} = (g_i)_{i \in [\![1,n_c]\!]}, \tag{3 .2}$$

où les fonctions de contrôle $(w_i)_{i\in[\![1,n_c]\!]}$ sont définies dans la preuve et satisfont

$$\forall i \in [\![1,n_c]\!], \quad \mathrm{Supp}(w_i) \subset \Gamma_c \quad \text{et} \quad \int_{\Gamma_c} w_i \cdot n = 0.$$

La question que l'on se pose est celle de la stabilisation locale du système (3 .1) en dimension $d \in \{2,3\}$ par un contrôle frontière g sous la forme (3 .2). En d'autres termes, on aimerait montrer que pour tout taux de décroissance $\omega > 0$, il existe un contrôle frontière g sous la forme (3 .2) tel que la solution (v,q,k) du système (3 .1) satisfait l'estimée

$$\exists C > 0, \quad \forall t \in [0,+\infty), \quad \|(v(t),k(t))\| \leq C\,\mathrm{e}^{-\omega t}\,\|(v_0,k_0)\|,$$

pour une norme $\|\cdot\|$ à définir.

3 .2 Motivation

C'est d'avantage dans des modèles physiques à petite échelle que la tension de surface et la viscosité affectent le comportement du fluide. On peut trouver de nombreux exemples d'applications de contrôle d'interface fluide dans [3, 55]. Par exemple, certaines technologies microfluidiques utilisent des gouttelettes immergées dans un fluide comme contenant et moyen de transport ou comme catalyseur pour des réactions chimiques. Le contrôle de l'interface entre les gouttelettes et le fluide peut permettre d'éviter les pertes dans le transport, d'en optimiser sa vitesse, d'observer les réactions chimiques. Un autre exemple plus proche de notre géométrie, fourni par [8], concerne la production d'aluminium par électrolyse. Dans un tel procédé, le liquide de la couche basse est de l'aluminium liquide et celui de la couche haute de l'alumine liquide. Les électrodes sont plongées dans l'alumine et le passage des ions transforme l'alumine liquide en aluminium liquide qui rejoint la couche basse. L'aluminium liquide ne doit pas entrer en contact avec les électrodes durant l'électrolyse. Il faut donc éviter des effets oscillants de l'interface entre l'aluminium et l'alumine.

Une autre application est celle de la stabilisation des instabilités de Rayleigh-Taylor. Ces instabilités se produisent lorsqu'on superpose deux fluides non miscibles sur des couches parallèles, le plus lourd surplombant le plus léger. Cette configuration est un état d'équilibre, fortement instable sous l'effet de l'attraction terrestre. Cette instabilité a été découverte par Rayleigh à la fin du dix-neuvième siècle [59]. L'apport de Taylor au milieu du vingtième siècle a été de montrer qu'il s'agit exactement de la même instabilité que lorsque l'on accélère le fluide léger dans la direction du fluide lourd [69]. Pour plus d'informations sur les instabilités de Rayleigh-Taylor, on peut se référer à [44].

3 .3 Résultats antérieurs

Les premiers résultats sur les systèmes à deux fluides non miscibles sont très récents. Dans le cas où l'effet de tension de surface est prise en compte, nous avons des résultats d'existence locale en temps de solutions fortes dans des espaces de Sobolev et de Hölder en dimension 3 [19, 20, 23, 24] et des résultats d'existence globale en temps à données petites : [67] en dimension 3, [43, 57] en dimension d et dans les espaces L^p vérifiant $p > d + 2$ pour [43] et $p > d + 3$ pour [57]. Dans le cas d'une interface quasi plate, Prüss et Simonnett [57] montrent que les équations sont vérifiées au sens classique et que les solutions deviennent instantanément analytiques. Dans le cas où l'on ne prend pas en compte l'effet de tension de surface, d'autres résultats d'existence existent : existence de solutions faibles [1, 51, 66], de solutions fortes [22], de solutions classiques [21].

Les systèmes à deux fluides non miscibles sont aussi très fortement reliés aux problèmes de frontière libre. Dans son travail pionnier sur les ondes de surface, Beale [11] prouve un résultat d'existence et unicité globale en temps à données petites. Sur le même système dans une géométrie périodique, Nishida et al. [50] montrent que la solution décroit exponentiellement vers la configuration plate à vitesse et pression nulles. Si l'on ne considère pas l'effet de tension de surface, d'autres résultats d'existence et de décroissance existent [64, 10, 40, 39].

A ma connaissance, la littérature sur la stabilisation et le contrôle d'interfaces entre deux fluides visqueux est peu développée. Cependant, la stabilisation du système de Navier-Stokes sans interface par contrôle frontière a connu un rapide essor ces 20 dernières années. On peut citer les travaux pionniers de Fursikov [33] en dimension 2 et [34] en dimension 3, qui montrent l'existence d'un contrôle sous forme d'un feedback dépendant du temps. Ces travaux ont été améliorées en dimension 2 par Raymond [60] par résolution d'une équation de Riccati algébrique et obtention d'un feedback indépendant du temps. En dimension 3, des conditions de compatibilité sont obligatoires pour un contrôle sous forme de feedback. Raymond [61] propose donc l'utilisation d'un feedback qui dépend du temps seulement sur un voisinage de $t = 0$. D'autres résultats de stabilization existent en dimension 3 : par exemple, avec des contrôles frontières tangentielles [9], avec un contrôle sous forme d'intégrateur [4], avec un feedback mais un ensemble restreint de conditions initiales [5], avec un feedback de dimension finie [62].

Enfin, concernant les instabilités de Rayleigh Taylor, Prüss et Simonett [56] ont montré que dans $\mathbb{R}^d$, l'état d'équilibre plat pour $[\rho] > 0$ était inconditionnellement instable. Tice et Wang [72] ont montré que dans la géométrie du tore dans laquelle nous travaillons, il existe une surface de tension critique $\sigma_c > 0$ tel que l'état d'équilibre plat pour $[\rho] > 0$ est stable pour $\sigma > \sigma_c$ et instable pour $\sigma \in (0, \sigma_c)$.

3 .4 Résultat obtenu

De la même manière que pour le précédent chapitre, nous procédons à un changement d'inconnue afin d'énoncer précisément le résultat obtenu. On définit

$$X(t,\cdot): \begin{array}{ccc} \Omega & \to & \Omega \\ (y', y_d) & \mapsto & (y', k_e(t, y', y_d)(1 - y_d^2) + y_d) \end{array}. \tag{3 .3}$$

où k_e est un relèvement de k dans le domaine Ω choisie de manière à ce que la transformation $X(t,\cdot)$ soit inversible en tout temps $t > 0$. Appelons $Y(t,\cdot)$ sa transformation inverse. Le domaine $Y(t,\Omega)$ est fixe au cours du temps avec une interface plate comme dessiné dans la figure Figure 5 avec

$$\begin{array}{rcccl} \Gamma & = & \mathbb{T}_d \times \{0\} & = & Y(t, \Gamma(t)), \\ \Omega^+ & = & \mathbb{T}_d \times (-1, 0) & = & Y(t, \Omega^+(t)), \\ \Omega^- & = & \mathbb{T}_d \times (0, 1) & = & Y(t, \Omega^-(t)), \end{array}$$

pour tout $t \in (0, +\infty)$.

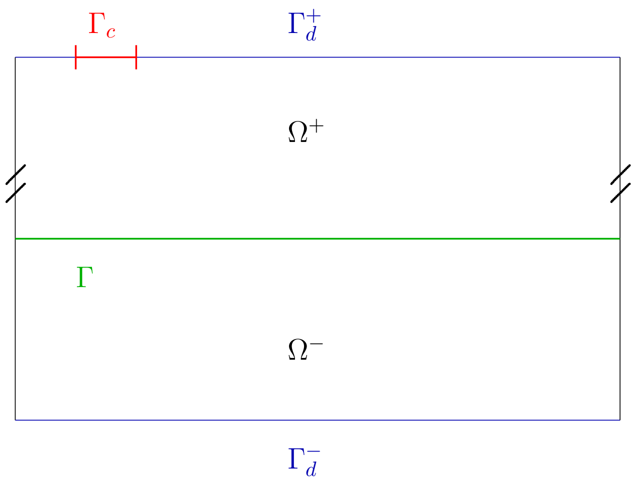

FIGURE 5 : Configuration fixe en dimension $d = 2$ avec $\Gamma_c \subset \Gamma_d^+$.

Pour un taux de décroissance $\omega > 0$ donné, on considère les inconnues

$$u^\pm = \mathrm{e}^{\omega t}\,\mathrm{cof}(\nabla X^\pm)^T v^\pm(X^\pm), \qquad p^\pm = \mathrm{e}^{\omega t}\,q(X^\pm), \qquad h = \mathrm{e}^{\omega t}\,k.$$

On s'intéresse au système vérifié par ces nouvelles inconnues :

$$\left\{\begin{array}{rcll} \partial_t u^\pm - \dfrac{1}{\rho^\pm}\nabla\cdot\mathbb{S}^\pm(u^\pm,p^\pm) - \omega u^\pm & = & F^\pm(u,p,h) & \text{dans } (0,+\infty)\times\Omega^\pm, \\ \nabla\cdot u^\pm & = & 0 & \text{dans } (0,+\infty)\times\Omega^\pm, \\ u & = & \mathrm{e}^{\omega t}\,g & \text{sur } (0,+\infty)\times\Gamma_d, \\ u^- & = & u^+ & \text{sur } (0,+\infty)\times\Gamma, \\ [\mathbb{S}(u,p)n_\Gamma] + (\sigma\Delta_{x'}h)n_\Gamma & = & H(u,p,h) & \text{sur } (0,+\infty)\times\Gamma, \\ \partial_t h - u_d - \omega h & = & 0 & \text{sur } (0,+\infty)\times\Gamma, \\ u(0) & = & u_0 & \text{dans } \Omega, \\ h(0) & = & h_0 & \text{sur } \Gamma, \end{array}\right. \tag{3 .4}$$

où l'expression précise des termes non linéaires F et H sont donnés dans (3.2 .4)–(3.2 .5), $u_0 = \mathrm{cof}(X_0)^T v_0(X_0)$, $h_0 = k_0$ et X_0 est défini par (3 .3) pour $k(0) = h_0$.
On introduit tout d'abord les notations suivantes pour $\gamma, r, s > 0$,

$$\begin{array}{rcl} V(\Omega) & = & \left\{u \in H_0^1(\Omega) \mid \nabla\cdot u = 0 \text{ dans } \Omega\right\}, \\ H_m^\gamma(\Gamma) & = & \left\{h \in H^\gamma(\Gamma) \mid \displaystyle\int_\Gamma h = 0\right\}, \\ H^{r,s}((0,T)\times\Omega^\pm) & = & L^2(0,T;H^r(\Omega^\pm)) \cap H^s(0,T;L^2(\Omega^\pm)). \end{array}$$

Dans la suite, on omettra l'indice x' dans $\Delta_{x'}h$.

Théorème 3 .1. *Soit*

$$\ell \in \left\{\begin{array}{ll} (0,1/2) & \textit{pour } d = 2, \\ (1/2,1) & \textit{pour } d = 3. \end{array}\right.$$

Pour tout taux de décroissance $\omega > 0$, il existe $r > 0$ tel que pour tout $(u_0, h_0) \in V(\Omega) \times H_m^{2+\ell}(\Gamma)$ vérifiant $u_0^\pm \in H^{1+\ell}(\Omega^\pm)$, la condition de petitesse

$$\|(u_0^\pm, h_0)\|_{H^{1+\ell}(\Omega^\pm)\times H^{2+\ell}(\Gamma)} \le r,$$

et la condition de compatibilité suivante en dimension 3 *:*

$$\left[\mu\left(\nabla\left(\mathrm{cof}(X_0^{-1})^T u_0(X_0^{-1})\right)+\left(\nabla\left(\mathrm{cof}(X_0^{-1})^T u_0(X_0^{-1})\right)\right)^T\right)n_{\Gamma_0}\right]_\tau=0 \quad \text{on } \Gamma_0, \tag{3 .5}$$

où X_0 est défini en (3 .3) pour h_0 et où l'indice τ désigne la composante suivant l'espace tangent à Γ_0, il existe un contrôle de dimension finie $g\in H^{1+\ell}(0,+\infty;C^\infty(\Gamma_d))$ de la forme (3 .2) tel que le système (3 .4) admet une solution (u,p,h) et qu'elle vérifie

$$\begin{aligned}&\|u^\pm\|_{H^{2+\ell,1+\ell/2}((0,+\infty)\times\Omega^\pm)}+\|h\|_{H^{7/4+\ell/2}(0,+\infty;L^2_m(\Gamma))\cap H^1(0,+\infty;H_m^{3/2+\ell}(\Gamma))\cap L^2(0,+\infty;H_m^{5/2+\ell}(\Gamma))}\\&\quad+\|\nabla p^\pm\|_{H^{\ell,\ell/2}((0,+\infty)\times\Omega^\pm)}+\|[p]\|_{H^{1/2+\ell,1/4+\ell/2}((0,+\infty)\times\Gamma)}\le C\|(u_0^\pm,h_0)\|_{H^{1+\ell}(\Omega^\pm)\times H_m^{2+\ell}(\Gamma)}.\end{aligned}$$

De plus, pour tout temps $t>0$, la transformation $X(t,\cdot)$ défini en (3 .3) pour $k(t,\cdot)=\mathrm{e}^{-\omega t}\,h(t,\cdot)$, est inversible et le triplet $(v,q,k)=\mathrm{e}^{-\omega t}(\mathrm{cof}(\nabla Y)^T u(Y),p(Y),h)$, où $Y(t,.)$ est la transformation inverse de $X(t,\cdot)$, est solution de (3 .1) et satisfait pour tout $t>0$,

$$\|(v^\pm(t),k(t))\|_{H^{1+\ell}(\Omega^\pm(t))\times H_m^{2+\ell}(\Gamma)}\le C\,\mathrm{e}^{-\omega t}\,\|(v_0^\pm,k_0)\|_{H^{1+\ell}(\Omega_0^\pm)\times H_m^{2+\ell}(\Gamma)}.$$

La régularité de $\Gamma(t)$ est donnée par la régularité de h. Remarquons qu'avec l'injection $H^1(0,+\infty;H^{3/2+\ell}(\Gamma))\cap L^2(0,+\infty;H^{5/2+\ell}(\Gamma))\hookrightarrow C([0,+\infty],H^{2+\ell}(\Gamma))$, nous avons $\Gamma(t)$ de classe $H^{2+\ell}$ qui correspond à la régularité initiale de Γ_0. Nous n'atteignions pas cette régularité dans le chapitre précédent et en ce sens, l'effet de tension de surface vient régulariser l'interface.

Une fois de plus, le fait de travailler dans des régularités supérieures à $H^{2,1}$ pour la vitesse provient du traitement des non linéarités que l'on ne sait pas traiter pour des régularités inférieures.

La condition de compatibilité (3 .5) est issue de la composante tangentielle de $(3\ .1)_4$.

Le contrôle $\mathbf{g}$ défini en (3 .2) est trouvé sous la forme d'un intégrateur

$$\mathbf{g}'+\Lambda\mathbf{g}=\mathbf{f},\quad \mathbf{g}(0)=0,$$

où $\mathbf{f}$ est lui même trouvé sous la forme d'un feedback

$$\mathbf{f}(t)=K(u,p,h,\mathbf{g})(t).$$

Ce feedback K, linéaire en $(u,p,h,\mathbf{g})$, est non linéaire en les inconnues (v,g,k) du système (3 .1) à cause de la multiplication de la vitesse u par $\mathrm{cof}(\nabla Y)^T$.

3 .5 Schéma de preuve

La démonstration est basée sur la stabilisation d'un système linéaire associé au système (3 .4) :

$$\left\{\begin{array}{rcll}\partial_t u^\pm-\dfrac{1}{\rho^\pm}\nabla\cdot\mathbb{S}^\pm(u^\pm,p^\pm)-\omega u^\pm&=&f^\pm&\text{dans } (0,+\infty)\times\Omega^\pm,\\ \nabla\cdot u^\pm&=&0&\text{dans } (0,+\infty)\times\Omega^\pm,\\ u&=&\displaystyle\sum_{i=1}^p g_i w_i&\text{sur } (0,+\infty)\times\Gamma_d,\\ u^-&=&u^+&\text{sur } (0,+\infty)\times\Gamma,\\ [\mathbb{S}(u,p)n_\Gamma]+(\sigma\Delta h)n_\Gamma&=&f_{jump}&\text{sur } (0,+\infty)\times\Gamma,\\ \partial_t h-u_d-\omega h&=&0&\text{sur } (0,+\infty)\times\Gamma,\\ \mathbf{g}'+\Lambda\mathbf{g}&=&\mathbf{f},&\\ u(0)&=&u_0&\text{dans } \Omega,\\ h(0)&=&h_0&\text{sur } \Gamma,\\ \mathbf{g}(0)&=&0.&\end{array}\right. \tag{3 .6}$$

Nous déroulons une stratégie classique de stabilisation autour d'une trajectoire stationnaire en s'appuyant sur la théorie des semi-groupes. On écrit (3 .6) sous forme d'opérateur

$$\left\{\begin{array}{rcl} z' & = & (A_e+\omega I)z + B\mathbf{f} + F, \\ z(0) & = & z_0, \end{array}\right. \tag{3 .7}$$

notre espace de travail étant

$$Z_e = PL^2(\Omega)\times H^{3/2}_m(\Gamma)\times \mathbb{R}^p,$$

où P est un projecteur légèrement différent d'un projecteur de Leray classique (voir Proposition 3.3 .9 pour sa définition précise), $z=(Pu,h,\mathbf{g})$, et $(A_e, D(A_e;Z_e))$ est défini en (3.4 .3)–(3.4 .4) et B est défini dans la Proposition 3.5 .1. La démonstration de l'analyticité du semigroupe d'opérateur $(A_e, D(A_e;Z_e))$ est un point délicat et propre à ce travail (Section 3 .5.1 ci-dessous). Nous montrons ensuite par un critère d'Hautus que la paire $(A_e+\omega I, B)$ est stabilisable et utilisons des équations de Riccati pour trouver une loi de feedback. Les estimées linéaires en temps infini relatives à Z_e sont obtenues avec [12, Part II, Chapter 1, Theorem 3.1, page 43]. Enfin, afin de pouvoir traiter le système non linéaire (3 .4), nous obtenons des estimées pour les solutions du système (3 .6) avec feedback dans le cadre des régularités du Théorème 3 .1. Cette dernière étape est un des points délicats et techniques de la preuve.

La difficulté intrinsèque au système de tension de surface (3 .6) est mise en évidence à travers son énergie. Dans le cas simplifié où $f=f_{jump}=g=0$ et $\omega=0$, la conservation de l'énergie s'écrit

$$\frac{d}{dt}\left(\int_\Omega \rho|u|^2+\sigma\int_\Gamma |\nabla h|^2\right)(t)+\int_\Omega \mu|\nabla u+(\nabla u)^T|^2(t)=0$$

Dans cette égalité d'énergie, il n'y a pas de terme dissipatif en h. Pourtant, le système (3 .6) est régularisant en h. L'espace induit par l'énergie est de fait incomplet car il ne fait pas apparaître l'espace de régularité optimale pour h.

3 .5.1 Le problème stationnaire et l'estimée d'analyticité

Nous nous intéressons au problème stationnaire suivant

$$\left\{\begin{array}{rcll} \lambda u^\pm - \dfrac{1}{\rho^\pm}\nabla\cdot\mathbb{S}^\pm(u^\pm,p^\pm) & = & f^\pm & \text{dans } \Omega^\pm, \\ \nabla\cdot u^\pm & = & 0 & \text{dans } \Omega^\pm, \\ u & = & 0 & \text{sur } \Gamma_d^\pm, \\ u^+ & = & u^- & \text{sur } \Gamma, \\ [\mathbb{S}(u,p)n_\Gamma]+(\sigma\Delta h)n_\Gamma & = & 0 & \text{sur } \Gamma, \\ \lambda h - u_d & = & f_h & \text{sur } \Gamma, \end{array}\right. \tag{3 .8}$$

d'énergie

$$\lambda\int_\Omega \rho|u|^2+\sigma\overline{\lambda}\int_\Gamma |\nabla h|^2+\frac{1}{2}\int_\Omega \mu|\nabla u+(\nabla u)^T|^2=\int_\Omega \rho f u+\sigma\int_\Gamma \nabla f_h\cdot\nabla h. \tag{3 .9}$$

Le problème stationnaire présente la difficulté suivante : la régularité de h est donnée par le couplage $(3\ .8)_5$ et non par l'équation d'évolution de h $(3\ .8)_6$. Afin d'être précis sur les régularités des termes sources, nous proposons une définition de *solution faible* basée sur relèvement de f_h par un terme de vitesse. Introduisons les espaces suivants

$$\begin{array}{rcl} \mathscr{V}_\lambda(\Omega) & = & \{(\varphi,\psi)\in V(\Omega)\times H^1_m(\Gamma) \mid \varphi_d=\lambda\psi \text{ sur } \Gamma\}, \\ H^{-1}_\rho(\Omega) & = & \{f\in\mathscr{D}'(\Omega)\mid \rho f\in H^{-1}(\Omega)\}. \end{array}$$

Pour $(f, f_h) \in H_\rho^{-1}(\Omega) \times H_m^{1/2}(\Gamma)$, nous définissons une *solution faible* de (1 .7) au sens suivant : $(u, h) \in V(\Omega) \times H_m^1(\Gamma)$ est une *solution faible* du système (3 .6) si $u = v + w$ avec $v \in V(\Omega)$ vérifiant $v_d = -f_h$ sur Γ, et $(w, h) \in \mathscr{V}_\lambda(\Omega)$ tel que pour tout $(\varphi, \psi) \in \mathscr{V}_\lambda(\Omega)$,

$$\lambda(\rho w, \varphi) + \frac{1}{2}(\mu D(w), D(\varphi)) + \sigma\bar{\lambda}(\partial_x h, \partial_x \psi)_\Gamma = \langle \rho f, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \lambda(\rho v, \varphi) - \frac{1}{2}(\mu D(v), D(\varphi)), \quad (3\ .10)$$

où $D(w) = \nabla w + (\nabla w)^T$ est le gradient symétrisé. Dans cette formulation, remarquons que la dissipation en h n'est plus nécessaire pour traiter les seconds membres. Notamment, le terme f_h est considéré en norme $H^{1/2}$, alors qu'il apparaît en norme H^1 dans le bilan d'énergie (3 .9).

Remarquez que de telles régularités ne permettent pas de remonter de la définition de *solution faible* au système (3 .8). Cependant, en prenant $f \in L^2(\Omega)$, la formulations variationnelle (3 .10) et le système (3 .8) sont équivalentes et en particulier $h \in H_m^{3/2}(\Gamma)$.

Des méthodes classiques (voir [13, Chapter IV, Proposition IV.6.2, page 94]) nous permettent alors de monter en régularité : pour des données $(f, f_h) \in L^2(\Omega) \times H_m^{3/2}(\Gamma)$, notre solution (u, h) satisfait $u^\pm \in H^2(\Omega^\pm)$ et $h \in H_m^{5/2}(\Gamma)$. Cependant, l'analyticité ne peut être obtenue à partir de la formulation variationnelle (3 .10). En effet, la formulation variationnelle est écrite de manière à optimiser la régularité des termes sources en se basant sur l'équation de couplage $(3\ .6)_5$. Cependant, c'est l'équation $(3\ .6)_6$ qui permet de gagner de l'analyticité pour h. C'est l'égalité d'énergie (3 .9) qui nous permet d'obtenir l'estimée de résolvante suivante

$$\|u^\pm\|_{H^2(\Omega^\pm)} + \|h\|_{H_m^{5/2}(\Gamma)} + |\lambda|(\|u\|_{L^2(\Omega)} + \|h\|_{H_m^{3/2}(\Gamma)}) \le C(\|f\|_{L^2(\Omega)} + \|f_h\|_{H_m^{3/2}(\Gamma)}), \quad (3\ .11)$$

où C est une constante indépendante de λ. A la connaissance de l'auteur, c'est la première fois que l'on obtient ce genre d'estimée dans ce cadre fonctionnel pour le modèle bifluide avec tension de surface. Dans le cas des frontières libres et sans terme source en h ($f_h = 0$), Beale [11] utilise une méthode similaire et obtient une estimée de même nature dans un cadre fonctionnel plus régulier. Toujours dans le cas des frontières libres, Nishida et al. [50] obtiennent exactement l'estimée (3 .11). Toutefois, leur méthode repose sur une décomposition de Fourier et sur des estimées plus complexes. La méthode présentée ici permet d'éviter ces calculs et d'obtenir plus simplement (3 .11).

3 .5.2 Stabilisation du problème linéaire

La stabilisabilité du système linéaire (3 .6) est montré par un critère d'Hautus (voir [12, Part V, Chapter 1, Proposition 3.3, page 492]) qui consiste à travailler sur les valeurs propres du système stationnaire adjoint à (3 .6) :

$$\left\{\begin{array}{rcll}
\lambda v^\pm - \dfrac{1}{\rho^\pm}\nabla \cdot \mathbb{S}^\pm(v^\pm, q^\pm) & = & \varphi & \text{dans } \Omega^\pm, \\
\nabla \cdot v^\pm & = & 0 & \text{dans } \Omega^\pm, \\
v & = & 0 & \text{sur } \Gamma_d, \\
v^+ & = & v^- & \text{sur } \Gamma, \\
[\mathbb{S}(v, q)n_\Gamma] - \sigma(\Delta k)n_\Gamma & = & 0 & \text{sur } \Gamma, \\
\lambda k + v_3 & = & \varphi_k & \text{sur } \Gamma, \\
\lambda\tilde{\mathbf{g}} + \Lambda\tilde{\mathbf{g}} + \left(\langle \mathbb{S}(v, q)n, w_i\rangle_{H^{-1/2}(\Gamma_d), H^{1/2}(\Gamma_d)}\right)_{i\in[\![1,p]\!]} & = & \varphi_{\tilde{g}}. &
\end{array}\right. \quad (3\ .12)$$

Une des subtilités du calcul de l'adjoint est que l'espace pivot choisi n'est pas l'espace de base $Z_e = PL^2(\Omega) \times H_m^{3/2}(\Gamma) \times \mathbb{R}^d$ du système directe (3 .6), mais l'espace naturel d'énergie $H_e = PL^2(\Omega) \times H_m^1(\Gamma) \times \mathbb{R}^d$. L'espace de base du sytème adjoint est donc $Z_e' = PL^2(\Omega) \times H_m^{1/2}(\Gamma) \times \mathbb{R}^d$.

Remarquons que grâce à la définition de *solution faible* établie en (3 .10), on sait prouver existence et unicité d'une solution $(v, q, k, \tilde{g}) \in V(\Omega) \times L^2(\Omega) \times H_m^{3/2}(\Gamma) \times \mathbb{R}^p$ du système (3 .12) pour des données $(\varphi, \varphi_k, \varphi_{\tilde{g}}) \in Z_e'$. De nouveau, on peut remarquer le gain de la formulation variationnelle (3 .10) qui nous permet de considérer f_h et φ_k dans $H_m^{1/2}(\Gamma)$ par rapport à une formulation variationnelle basée sur l'énergie (3 .9) qui ne nous permettrait que de considérer f_h et φ_k dans $H_m^1(\Gamma)$.

L'argument de prolongement unique permettant d'appliquer le critère d'Hautus est une conséquence du travail de Fabre-Lebeau [29]. La résolution des équations de Riccati (voir [42]) donnent alors l'existence d'une loi de feedback K qui stabilise (3 .6) dans Z_e.

3 .5.3 Gain de régularité sur le système linéaire

Le résultat de régularité sur les semi groupes analytiques exponentiellement stables [12, Part II, Chapter 1, Theorem 3.1, page 43] nous donne $z \in H^1(0, +\infty; Z_e) \cap L^2(0, +\infty; D(A_e; Z_e))$. Cependant cette régularité n'est pas suffisante pour estimer convenablement les termes non linéaires.

Pour augmenter la régularité, la stratégie classique consiste à dériver l'équation en z (3 .7) et obtenir $z' \in H^1(0, +\infty; Z_e) \cap L^2(0, +\infty; D(A_e; Z_e))$, puis à écrire $(3\ .7)_1$ sous la forme $A_e z(t) = z'(t) - F(t) - B\mathbf{f}(t) - \omega z(t)$ pour tout temps $t > 0$ et à utiliser des résultats de régularité obtenues sur le problème stationnaire. Cependant, dans notre cas, les régularités stationnaires pour un second membre $z' \in L^2(0, +\infty; D(A_e; Z_e))$ ne sont pas satisfaisantes car $D(A_e; Z_e)$ n'est pas un niveau de régularité stationnaire optimale pour des seconds membres. Cela vient essentiellement du fait que le gain de régularité stationnaire sur la vitesse est de 2 alors que le gain de régularité sur la hauteur est de 1 (voir l'estimée (3 .11)). Il manque donc une régularité de 1 en espace sur $\partial_t h$. Pour atteindre cette régularité, notre idée est de dériver deux fois le système (3 .6) dans la direction tangentielle de Γ puis de réutiliser [12, Part II, Chapter 1, Theorem 3.1, page 43] afin d'en déduire une estimée maximale en espace sur $\partial_t h$. La régularité sur z' est alors optimale pour utiliser les résultats de régularité sur le problème stationnaire.

Les estimées dans des espaces de Sobolev intermédiaires sont obtenus par interpolation.

3 .5.4 Stabilisation du système non linéaire

On se place dans le cadre du théorème de point fixe de Picard. Les estimées sont classiques et les méthodes similaires à celles du Chapitre 2. Remarquons que la multiplication de la vitesse v du système (3 .1) par $\mathrm{cof}(\nabla X)^T$ transforme l'équation d'évolution de k $(3\ .1)_6$ en $\partial_t(e^{-\omega t} h) = \mathrm{e}^{-\omega t}\, u_d$ et permet de conserver la condition d'incompressibilité. Cette multiplication par $\mathrm{cof}(\nabla X)^T$ a pour inconvénient de demander une forte régularité en espace sur h. Nous l'obtenons grâce à l'effet régularisant de la tension de surface à travers le couplage $(3\ .6)_5$, puis par un argument de relèvement de h sur tout le domaine Ω.

3 .6 Remarques additionnelles et perspectives

Ce résultat ainsi que sa démonstration ouvrent plusieurs perspectives :

3 .6.1 L'unicité

De nouveau, l'unicité de la solution construite dans le Théorème 3 .1 n'a pas été prouvée. Cependant, contrairement au Chapitre 2, rien ne semble s'y opposer, car les solutions sont dans des espaces de fonctions continues en temps à valeurs dans l'espace des conditions initiales : $u^\pm \in C([0, +\infty); H^{1+\ell}(\Omega^\pm))$ et $h \in C([0, +\infty); H_m^{2+\ell}(\Gamma))$.

3 .6.2 Etat stationnaire excité

On peut se demander si on est capable de stabiliser notre système autour d'un état stationnaire excité. Cette étude nous permettrait par exemple de traiter les instabilités de type Rayleigh-Taylor. Un premier pas vers cette démarche consisterait à traiter les équations d'Oseen dans l'étude du linéaire. Cependant il est à noter que le système linéarisé autour d'une solution stationnaire excitée ne se limite pas à un changement de l'équation de Stokes par l'équation d'Oseen dans (3 .6) (voir le système linéarisé dans l'étude des instabilités de Rayleigh-Taylor [56, 72, 73]).

En ce qui concerne l'étude des équations d'Oseen, la preuve actuelle semble tout à fait adaptable. Cependant, je pense que le contrôle devra se faire de chaque côté de la frontière. En effet, pour prouver la propriété de prolongement unique, nous utilisons le fait que 0 ne peut pas être valeur propre pour le système adjoint (3 .12). Cela nous permet de déduire $k = 0$ par l'équation $(3\ .12)_6$ et de transmettre le tenseur des contraintes à travers Γ par $(3\ .12)_5$. Dans le cas des équations d'Oseen, il n'y a à priori pas de raison pour que 0 ne soit pas valeur propre du système adjoint. Lorsque $\lambda = 0$, l'équation $(3\ .12)_6$ ne fait plus intervenir k. La transmission des informations à travers Γ est donc moins claire, ce qui laisse penser qu'un contrôle sur le second fluide est nécessaire.

3 .6.3 Des géométries différentes

Une autre possibilité d'ouverture est de traiter des géométries différentes. Par exemple, je pense que l'actuelle démonstration est tout à fait adaptable au cas de la géométrie du Chapitre 2. On peut aussi penser à garder une géométrie similaire non torique. Il faudrait alors définir convenablement des conditions sur les bords introduits. La stabilisation d'interface sur de telles géométries est une perspective très intéressante car elle connaîtrait des répercussions directes en industrie (voir Section 3 .2).

3 .6.4 Observabilité partielle

Toujours dans l'optique de se rapprocher des applications, on pourrait se demander si on arriverait à stabiliser l'interface en n'observant qu'une seule partie de l'état. Dans la preuve actuelle, nous observons tout l'état, mais la pratique ne permet généralement que d'observer la pression à travers des capteurs ponctuels.

3 .6.5 La simulation numérique

Enfin, la dernière mais non moins intéressante perspective est celle du développement d'une partie numérique. La simulation pourrait en effet mettre en évidence le coût du contrôle ou sa robustesse par rapport aux différents paramètres de (3 .1) (voir par exemple [58] sur l'analyse de robustesse d'un contrôle d'un système fluide-structure). Elle pourrait aussi constituer une base pour des géométries plus physiques comme celles mentionnées plus haut.

Bibliographie

[1] H. Abels. On generalized solutions of two-phase flows for viscous incompressible fluids. *Interfaces Free Bound.*, 9(1) :31–65, 2007.

[2] E. V. Amosova. Exact local controllability for equations of the dynamics of a viscous gas. *Differ. Uravn.*, 47(12) :1754–1772, 2011.

[3] B. Atencia and J. Beebe. Controlled microfluidic interfaces. *Nature*, 437 :648–655, 2005.

[4] M. Badra. Feedback stabilization of the 2-D and 3-D Navier-Stokes equations based on an extended system. *ESAIM Control Optim. Calc. Var.*, 15(4) :934–968, 2009.

[5] M. Badra. Lyapunov function and local feedback boundary stabilization of the Navier-Stokes equations. *SIAM J. Control Optim.*, 48(3) :1797–1830, 2009.

[6] M. Badra, S. Ervedoza, and S. Guerrero. Local controllability to trajectories for non-homogeneous incompressible Navier-Stokes equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 33(2) :529–574, 2016.

[7] H. Bae. Solvability of the free boundary value problem of the Navier-Stokes equations. *Discrete Contin. Dyn. Syst.*, 29(3) :769–801, 2011.

[8] L. Bañas, M. Klein, and A. Prohl. Control of interface evolution in multiphase fluid flows. *SIAM J. Control Optim.*, 52(4) :2284–2318, 2014.

[9] V. Barbu, I. Lasiecka, and R. Triggiani. Tangential boundary stabilization of Navier-Stokes equations. *Mem. Amer. Math. Soc.*, 181(852) :x+128, 2006.

[10] J. T. Beale. The initial value problem for the Navier-Stokes equations with a free surface. *Comm. Pure Appl. Math.*, 34(3) :359–392, 1981.

[11] J. T. Beale. Large-time regularity of viscous surface waves. *Arch. Rational Mech. Anal.*, 84(4) :307–352, 1983/84.

[12] A. Bensoussan, G. Da Prato, M. C. Delfour, and S. K. Mitter. *Representation and control of infinite dimensional systems*. Systems & Control : Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA, second edition, 2007.

[13] F. Boyer and P. Fabrie. *Mathematical tools for the study of the incompressible Navier-Stokes equations and related models*, volume 183 of *Applied Mathematical Sciences*. Springer, New York, 2013.

[14] D. Bresch and B. Desjardins. Existence of global weak solutions for a 2D viscous shallow water equations and convergence to the quasi-geostrophic model. *Comm. Math. Phys.*, 238(1-2) :211–223, 2003.

[15] D. Bresch, B. Desjardins, and C.-K. Lin. On some compressible fluid models : Korteweg, lubrication, and shallow water systems. *Comm. Partial Differential Equations*, 28(3-4) :843–868, 2003.

[16] S. Chowdhury, D. Mitra, M. Ramaswamy, and M. Renardy. Null controllability of the linearized compressible Navier Stokes system in one dimension. *J. Differential Equations*, 257(10) :3813–3849, 2014.

[17] J.-M. Coron. On the controllability of 2-D incompressible perfect fluids. *J. Math. Pures Appl. (9)*, 75(2) :155–188, 1996.

[18] J.-M. Coron and A. V. Fursikov. Global exact controllability of the 2D Navier-Stokes equations on a manifold without boundary. *Russian J. Math. Phys.*, 4(4) :429–448, 1996.

[19] I. V. Denisova. A priori estimates for the solution of the linear nonstationary problem connected with the motion of a drop in a liquid medium. *Trudy Mat. Inst. Steklov.*, 188 :3–21, 191, 1990. Translated in Proc. Steklov Inst. Math. **1**991, no. 3, 1–24, Boundary value problems of mathematical physics, 14 (Russian).

[20] I. V. Denisova. Problem of the motion of two viscous incompressible fluids separated by a closed free interface. *Acta Appl. Math.*, 37(1-2) :31–40, 1994. Mathematical problems for Navier-Stokes equations (Centro, 1993).

[21] I. V. Denisova. Global solvability of a problem on two fluid motion without surface tension. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 348(Kraevye Zadachi Matematicheskoi Fiziki i Smezhnye Voprosy Teorii Funktsii. 38) :19–39, 303, 2007.

[22] I. V. Denisova. Global L_2-solvability of a problem governing two-phase fluid motion without surface tension. *Port. Math.*, 71(1) :1–24, 2014.

[23] I. V. Denisova and V. A. Solonnikov. Solvability of a linearized problem on the motion of a drop in a fluid flow. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 171(Kraev. Zadachi Mat. Fiz. i Smezh. Voprosy Teor. Funktsii. 20) :53–65, 184, 1989.

[24] I. V. Denisova and V. A. Solonnikov. Solvability in Hölder spaces of a model initial-boundary value problem generated by a problem on the motion of two fluids. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 188(Kraev. Zadachi Mat. Fiz. i Smezh. Voprosy Teor. Funktsiĭ. 22) :5–44, 186, 1991.

[25] R. H. Dillon and L. J. Fauci. An integrative model of internal axoneme mechanics and external fluid dynamics in ciliary beating. *Journal of Theoretical Biology*, 207(3) :415 – 430, 2000.

[26] R. H. Dillon, L. J. Fauci, and C. Omoto. Mathematical modeling of axoneme mechanics and fluid dynamics in ciliary and sperm motility. *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, 10(5) :745–757, 2003. Progress in partial differential equations (Pullman, WA, 2002).

[27] S. Ervedoza, O. Glass, and S. Guerrero. Local exact controllability for the two- and three-dimensional compressible Navier-Stokes equations. *Comm. Partial Differential Equations*, 41(11) :1660–1691, 2016.

[28] S. Ervedoza, O. Glass, S. Guerrero, and J.-P. Puel. Local exact controllability for the one-dimensional compressible Navier-Stokes equation. *Arch. Ration. Mech. Anal.*, 206(1) :189–238, 2012.

[29] C. Fabre and G. Lebeau. Prolongement unique des solutions de l'equation de Stokes. *Comm. Partial Differential Equations*, 21(3-4) :573–596, 1996.

[30] L. J. Fauci and C. S. Peskin. A computational model of aquatic animal locomotion. *J. Comput. Phys.*, 77(1) :85–108, 1988.

[31] E. Fernández-Cara, S. Guerrero, O. Y. Imanuvilov, and J.-P. Puel. Local exact controllability of the Navier-Stokes system. *J. Math. Pures Appl. (9)*, 83(12) :1501–1542, 2004.

[32] A. L. Fogelson. A mathematical model and numerical method for studying platelet adhesion and aggregation during blood clotting. *J. Comput. Phys.*, 56(1) :111–134, 1984.

[33] A. V. Fursikov. Stabilizability of two-dimensional Navier-Stokes equations with help of a boundary feedback control. *J. Math. Fluid Mech.*, 3(3) :259–301, 2001.

[34] A. V. Fursikov. Stabilization for the 3D Navier-Stokes system by feedback boundary control. *Discrete Contin. Dyn. Syst.*, 10(1-2) :289–314, 2004. Partial differential equations and applications.

[35] A. V. Fursikov and O. Y. Imanuvilov. Local exact controllability of the Navier-Stokes equations. *C. R. Acad. Sci. Paris Sér. I Math.*, 323(3) :275–280, 1996.

[36] O. Glass. Exact boundary controllability of 3-D Euler equation. *ESAIM Control Optim. Calc. Var.*, 5 :1–44, 2000.

[37] O. Glass. On the controllability of the 1-D isentropic Euler equation. *J. Eur. Math. Soc. (JEMS)*, 9(3) :427–486, 2007.

[38] O. Glass. On the controllability of the non-isentropic 1-D Euler equation. *J. Differential Equations*, 257(3) :638–719, 2014.

[39] Y. Guo and I. Tice. Decay of viscous surface waves without surface tension in horizontally infinite domains. *Anal. PDE*, 6(6) :1429–1533, 2013.

[40] Y. Guo and I. Tice. Local well-posedness of the viscous surface wave problem without surface tension. *Anal. PDE*, 6(2) :287–369, 2013.

[41] O. Y. Imanuvilov. Remarks on exact controllability for the Navier-Stokes equations. *ESAIM Control Optim. Calc. Var.*, 6 :39–72, 2001.

[42] S. Kesavan and J.-P. Raymond. On a degenerate Riccati equation. *Control Cybernet.*, 38(4B) :1393–1410, 2009.

[43] M. Köhne, J. Prüss, and M. Wilke. On quasilinear parabolic evolution equations in weighted L_p-spaces. *J. Evol. Equ.*, 10(2) :443–463, 2010.

[44] H.-J. Kull. Theory of the rayleigh-taylor instability. *Physics Reports*, 206 :197–325, 1991.

[45] J. Le Rousseau and G. Lebeau. On Carleman estimates for elliptic and parabolic operators. Applications to unique continuation and control of parabolic equations. *ESAIM Control Optim. Calc. Var.*, 18(3) :712–747, 2012.

[46] T.-T. Li and B.-P. Rao. Exact boundary controllability for quasi-linear hyperbolic systems. *SIAM J. Control Optim.*, 41(6) :1748–1755, 2003.

[47] F.-H. Lin and J. Tong. Solvability of the Stokes Immersed Boundary Problem in Two Dimension. *ArXiv e-prints*, March 2017.

[48] D. Maity. Some controllability results for linearized compressible Navier-Stokes system. *ESAIM Control Optim. Calc. Var.*, 21(4) :1002–1028, 2015.

[49] Y. Mori, A. Rodenberg, and D. Spirn. Well-posedness and global behavior of the Peskin problem of an immersed elastic filament in Stokes flow. *ArXiv e-prints*, April 2017.

[50] T. Nishida, Y. Teramoto, and H. Yoshihara. Global in time behavior of viscous surface waves : horizontally periodic motion. *J. Math. Kyoto Univ.*, 44(2) :271–323, 2004.

[51] A. Nouri and F. Poupaud. An existence theorem for the multifluid Navier-Stokes problem. *J. Differential Equations*, 122(1) :71–88, 1995.

[52] C. S. Peskin. Flow patterns around heart valves : A numerical method. *Journal of Computational Physics*, 10(2) :252 – 271, 1972.

[53] C. S. Peskin. The immersed boundary method. *Acta Numer.*, 11 :479–517, 2002.

[54] C. S. Peskin and D. M. McQueen. A three-dimensional computational method for blood flow in the heart. I. Immersed elastic fibers in a viscous incompressible fluid. *J. Comput. Phys.*, 81(2) :372–405, 1989.

[55] N. Petit. Control problems for one-dimensional fluids and reactive fluids with moving interfaces. In *Advances in the theory of control, signals and systems with physical modeling*, volume 407 of *Lect. Notes Control Inf. Sci.*, pages 323–337. Springer, Berlin, 2010.

[56] J. Prüss and G. Simonett. On the Rayleigh-Taylor instability for the two-phase Navier-Stokes equations. *Indiana Univ. Math. J.*, 59(6) :1853–1871, 2010.

[57] J. Prüss and G. Simonett. On the two-phase Navier-Stokes equations with surface tension. *Interfaces Free Bound.*, 12(3) :311–345, 2010.

[58] L. Ravanbod, D. Noll, J.-P. Raymond, and J.-M. Buchot. Robustified H_2-control of a system with large state dimension. *Eur. J. Control*, 31 :59–71, 2016.

[59] L. Rayleigh. Analytic solutions of the rayleigh equation for linear density profiles. *Proc. London Math. Soc.*, 14 :170–177, 1883.

[60] J.-P. Raymond. Feedback boundary stabilization of the two-dimensional Navier-Stokes equations. *SIAM J. Control Optim.*, 45(3) :790–828, 2006.

[61] J.-P. Raymond. Feedback boundary stabilization of the three-dimensional incompressible Navier-Stokes equations. *J. Math. Pures Appl. (9)*, 87(6) :627–669, 2007.

[62] J.-P. Raymond and L. Thevenet. Boundary feedback stabilization of the two dimensional Navier-Stokes equations with finite dimensional controllers. *Discrete Contin. Dyn. Syst.*, 27(3) :1159–1187, 2010.

[63] J.-P. Raymond and M. Vanninathan. A fluid-structure model coupling the Navier-Stokes equations and the Lamé system. *J. Math. Pures Appl. (9)*, 102(3) :546–596, 2014.

[64] V. A. Solonnikov. Solvability of the problem of the motion of a viscous incompressible fluid that is bounded by a free surface. *Izv. Akad. Nauk SSSR Ser. Mat.*, 41(6) :1388–1424, 1448, 1977.

[65] J. Stockie. Immersed boundary method : Recent developments in analysis, algorithms and applications, August 2015. Beijing, ICIAM.

[66] S. Takahashi. On global weak solutions of the nonstationary two-phase Navier-Stokes flow. *Adv. Math. Sci. Appl.*, 5(1) :321–342, 1995.

[67] N. Tanaka. Global existence of two phase nonhomogeneous viscous incompressible fluid flow. *Comm. Partial Differential Equations*, 18(1-2) :41–81, 1993.

[68] A. Tani and N. Tanaka. Large-time existence of surface waves in incompressible viscous fluids with or without surface tension. *Arch. Rational Mech. Anal.*, 130(4) :303–314, 1995.

[69] G. Taylor. The instability of liquid surfaces when accelerated in a direction perpendicular to their planes. I. *Proc. Roy. Soc. London. Ser. A.*, 201 :192–196, 1950.

[70] R. Temam. *Navier-Stokes equations. Theory and numerical analysis.* North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977. Studies in Mathematics and its Applications, Vol. 2.

[71] B. A. Ton. A free boundary problem for the Navier-Stokes equations with measure data. *Nonlinear Anal.*, 64(1) :1–21, 2006.

[72] Y. Wang and I. Tice. The viscous surface-internal wave problem : nonlinear Rayleigh-Taylor instability. *Comm. Partial Differential Equations*, 37(11) :1967–2028, 2012.

[73] M. Wilke. Rayleigh-Taylor instability for the two-phase Navier-Stokes equations with surface tension in cylindrical domains. *ArXiv e-prints*, March 2017.

[74] L. Zhu and C. S. Peskin. Simulation of a flapping flexible filament in a flowing soap film by the immersed boundary method. *J. Comput. Phys.*, 179(2) :452–468, 2002.

Chapter 1

Local boundary controllability to trajectories for the 1d compressible Navier Stokes equations

In this article, we show a local exact boundary controllability result for the 1d isentropic compressible Navier Stokes equations around a smooth target trajectory. Our controllability result requires a geometric condition on the flow of the target trajectory, which comes naturally when dealing with the linearized equations. The proof of our result is based on a fixed point argument in weighted spaces and follows the strategy already developed in [13] in the case of a non-zero constant velocity field. The main novelty of this article is in the construction of the controlled density in the case of possible oscillations of the characteristics of the target flow on the boundary.

1.1 Introduction

Let us consider the 1d compressible Navier-Stokes equations stated in a bounded domain $(0, L)$ and in finite time horizon $T > 0$:

$$\begin{cases} \partial_t \rho_s + \partial_x(\rho_s u_s) = 0 & \text{in } (0,T)\times(0,L), \\ \rho_s(\partial_t u_s + u_s\partial_x u_s) - \nu\partial_{xx}u_s + \partial_x p(\rho_s) = 0 & \text{in } (0,T)\times(0,L). \end{cases} \tag{1.1 .1}$$

Here, ρ_s is the density of the fluid, and u_s represents its velocity. The pressure term $p = p(\rho_s)$ is assumed to depend on the density ρ_s, according to a law which can take different forms. In this article, we will only require $p \in C^2(\mathbb{R}_+^*; \mathbb{R})$, which encompasses the cases of pressure of the form

$$p(\rho_s) = A\rho_s^\gamma, \text{ with } A > 0 \text{ and } \gamma \geq 1, \tag{1.1 .2}$$

corresponding to the isentropic law for perfect gases. The constant $\nu > 0$ stands for the viscosity of the fluid.

To be well posed, system (1.1 .1) should be completed with initial data at $t = 0$ and boundary conditions, but we will not make them precise yet.

Instead, we assume that we have a solution $(\overline{\rho}, \overline{u})$ of (1.1 .1) in $(0,T)\times(0,L)$, which enjoys the following regularity:

$$(\overline{\rho}, \overline{u}) \in C^2([0,T]\times[0,L]) \times C^2([0,T]\times[0,L]), \quad \text{with} \inf_{[0,T]\times[0,L]} \overline{\rho}(t,x) > 0. \tag{1.1 .3}$$

The question of local exact boundary controllability we aim at addressing is the following: if we consider a solution (ρ_s, u_s) of (1.1 .1) starting at $t = 0$ from an initial data close to

$(\overline{\rho}(0,\cdot),\overline{u}(0,\cdot))$, can we find boundary controls such that the corresponding solution of (1.1 .1) reaches exactly $(\overline{\rho}(T,\cdot),\overline{u}(T,\cdot))$ at time T?

In order to state our result precisely, we introduce an extension of $(\overline{\rho},\overline{u})$ on $[0,T]\times\mathbb{R}$, still denoted in the same way for sake of simplicity, enjoying the same conditions as in (1.1 .3):

$$(\overline{\rho},\overline{u}) \in C^2([0,T]\times\mathbb{R})\times C^2([0,T]\times\mathbb{R}), \quad \text{with } \inf_{[0,T]\times\mathbb{R}} \overline{\rho}(t,x) > 0. \tag{1.1 .4}$$

These regularity assumptions allow to define the flow $\overline{X}=\overline{X}(t,\tau,x)$ corresponding to $\overline{u}$ defined for $(t,\tau,x)\in[0,T]\times[0,T]\times\mathbb{R}$ as follows:

$$\begin{cases} \dfrac{d\overline{X}}{dt}(t,\tau,x) &= \overline{u}(t,\overline{X}(t,\tau,x)) \quad \text{in } (0,T), \\ \overline{X}(\tau,\tau,x) &= x. \end{cases} \tag{1.1 .5}$$

Our main assumption then is the following one:

$$(\overline{X},T) \text{ is such that } \forall x\in[0,L],\ \exists t_x\in(0,T) \text{ such that } \overline{X}(t_x,0,x)\notin[0,L]. \tag{1.1 .6}$$

Before going further, let us emphasize that Condition (1.1 .6) is of geometric nature and does not depend on the choice of the extension of $(\overline{\rho},\overline{u})$ on $\mathbb{R}$, but only depends on the original target trajectory $(\overline{\rho},\overline{u})$ defined only on $[0,T]\times[0,L]$.

The main result of this article then states as follows:

Theorem 1.1 .1. *Let $T>0$, and $(\overline{\rho},\overline{u})\in(C^2([0,T]\times[0,L]))^2$ as in (1.1 .3) be a solution of (1.1 .1) in $(0,T)\times(0,L)$, and such that there exists an extension (still denoted the same) satisfying the regularity conditions (1.1 .4). Assume that the flow $\overline{X}$ in (1.1 .5) satisfies (1.1 .6).*

Then there exists $\varepsilon>0$ such that for all $(\rho_0,u_0)\in H^1(0,L)\times H^1(0,L)$ satisfying

$$\|(\rho_0,u_0)\|_{H^1(0,L)\times H^1(0,L)} \le \varepsilon, \tag{1.1 .7}$$

there exists a controlled trajectory (ρ_s,u_s) of (1.1 .1) satisfying

$$(\rho_s(0,\cdot),u_s(0,\cdot)) = (\overline{\rho}(0,\cdot),\overline{u}(0,\cdot)) + (\rho_0,u_0) \quad \text{in } (0,L), \tag{1.1 .8}$$

and the control requirement

$$(\rho_s(T,\cdot),u_s(T,\cdot)) = (\overline{\rho}(T,\cdot),\overline{u}(T,\cdot)) \quad \text{in } (0,L). \tag{1.1 .9}$$

Furthermore, (ρ_s,u_s) enjoys the following regularity:

$$\rho_s\in C^0([0,T];H^1(0,L))\cap C^1([0,T];L^2(0,L)) \text{ and } u_s\in H^1(0,T;L^2(0,L))\cap L^2(0,T;H^2(0;L)). \tag{1.1 .10}$$

Note that the control functions do not appear explicitly in Theorem 1.1 .1. As said earlier, the controls are applied on the boundary of the domain, i.e. on $(0,T)\times\{0,L\}$. If one wants to make them appear explicitly, the equation (1.1 .1)–(1.1 .8) should be completed with

$$\begin{cases} \rho_s(t,0)=\overline{\rho}(t,0)+v_{0,\rho}(t) & \text{for } t\in(0,T) \text{ with } u_s(t,0)>0, \\ \rho_s(t,L)=\overline{\rho}(t,L)+v_{L,\rho}(t) & \text{for } t\in(0,T) \text{ with } u_s(t,L)<0, \\ u_s(t,0)=\overline{u}(t,0)+v_{0,u}(t) & \text{for } t\in(0,T), \\ u_s(t,L)=\overline{u}(t,L)+v_{L,u}(t) & \text{for } t\in(0,T), \end{cases}$$

where $v_{0,\rho}, v_{L,\rho}, v_{0,u}, v_{L,u}$ are the control functions. Our strategy will not make them appear explicitly, but we emphasize that these are the control functions used to control the fluid in Theorem 1.1 .1.

Theorem 1.1 .1 generalizes [13] where the same local boundary controllability result is proved when $(\overline{\rho}, \overline{u})$ is a constant trajectory with $\overline{u} \neq 0$ and $\|(\rho_0, u_0)\|_{H^3(0,L)\times H^3(0,L)} \leq \varepsilon$ instead of (1.1 .7). In this case, the geometric condition (1.1 .6) reduces to $T > L/|\overline{u}|$ and coincides with the geometric constraint in [13]. Note that even when considering the case of constant reference trajectories $(\overline{\rho}, \overline{u})$ with $\overline{u} \neq 0$, Theorem 1.1 .1 is more precise than the result in [13] as the smallness of (ρ_0, u_0) is required in the $H^1(0, L) \times H^1(0, L)$ norm in (1.1 .7), while it was required in the $H^3(0, L) \times H^3(0, L)$ norm in [13] due to the use of the work [24]. This improvement is obtained by using the Carleman estimates recently derived in [3] instead of the ones in [16].

The proof of Theorem 1.1 .1 follows the one in [13] in the case of constant trajectory and is based on a fixed point argument on a kind of linearized system. The main novelty in our approach is on the control of the density, which is more delicate than in [13] due to possible cancellations on the target velocity $\overline{u}$ at $x = 0$ and $x = L$. The resulting difficulty is that the characteristics may be tangent to the boundary at some times. To overcome this difficulty which arises on the boundary, we transform our boundary control into a distributed control problem on an extended domain. This enlargement of the domain is carried out through a careful geometric discussion (Section 1.2 .2) which constitutes the main difference with the approach in [13] and the main novelty of our work. This geometric consideration has important consequences on the construction of the controlled density and its estimates (Section 1.3). The second main interest of this construction is that it simplifies the proof in [13] in the sense that we now use only one parameter in the Carleman estimates we shall use in our fixed point argument, while [13] requires the use of two parameters in the Carleman estimates. Of course, the geometric condition (1.1 .6) originates from the control of the density and it cannot be avoided when working on the linearized equations, as pointed out for instance in the recent work [22]. Such geometric conditions also appeared in [3] in order to obtain a local exact controllability result for the incompressible Navier-Stokes equations in 2 and 3d, but in that context, the authors did not require regularity conditions on the controlled density, while we need them in our problem.

Results on the local exact controllability of viscous compressible flows are very recent. The first one is due to [2] in the one-dimensional case when the equations are stated in Lagrangian forms and the density coincides with the target density at the time $t = 0$. The work [13] then obtained a similar result as Theorem 1.1 .1 but in the context of constant trajectories $(\overline{\rho}, \overline{u})$ with $\overline{u} \neq 0$ and $T > L/|\overline{u}|$. This geometric condition appears naturally when dealing with the linearized equations and can already be found in [25] for a structurally damped wave equation, which is of similar nature as the 1d linearized compressible Navier-Stokes equations around the constant state $(\overline{\rho}, 0)$. This has later been thoroughly discussed in the work [8] on the stabilization of the linearized compressible Navier-Stokes equation when the actuator is located in the velocity equation. When the equations are linearized around constant trajectories $(\overline{\rho}, \overline{u})$ with $\overline{u} \neq 0$, some controllability results were obtained in the case of controls acting only on the velocity field [7]. These results are also deeply related to the ones obtained on the structurally damped wave equation with moving controls [23, 6].

The fact that we are dealing with the non-linear problem induces the use of a flexible tool to control the linearized equations. Therefore, we shall use Carleman estimates in the spirit of [16]. But in our context, the linearized equations encompass both parabolic and hyperbolic behaviors. We are therefore led to using Carleman estimates with a weight function traveling along the characteristics of the reference velocity, following an idea already used in several previous works, e.g. [1] for linear thermoelasticity, [13, 3] for non-homogeneous viscous fluids, or [6] for linear viscoelasticity with moving controls.

Let us also mention that other controllability results were derived in the case of compressible perfect fluids corresponding to $\nu = 0$. In that context, several results were obtained depending if one considers regular solutions [21] or BV solutions [18, 19]. But the methods developed in this context are very different from the ones used for viscous flows.

The literature is also very rich for what concerns incompressible fluids (with homogeneous density) in the two and three dimensional setting. With no aim at exhaustivity, we refer for instance to [10, 17] for the controllability of the incompressible Euler equations, and to [9, 12, 15, 20, 14] for the controllability of the incompressible Navier-Stokes equations.

Let us also mention that Theorem 1.1 .1 states a result of local exact controllability to smooth trajectories under the geometric condition (1.1 .6). But this geometric condition may not be necessary in order to get a local controllability result for the non-linear equations (1.1 .1) based on the use of the non-linearity, for instance in the spirit of the return method, see e.g. [10, 17] where this idea has been developed in the context of Euler equations, and [11] for several examples in which the non-linear effects help in controlling the equations at hand. To our knowledge, this idea has not been developed yet in the context of compressible Navier-Stokes equations.

The article is organized as follows. Section 1.2 presents the global strategy of the proof and introduces the mathematical framework. We will in particular present some geometrical considerations and we will define a fixed point map subsequently. Section 1.3 deals with the control of the density equation, which is the main contribution of this article. Section 1.4 is dedicated to the control of the velocity. Finally, Section 1.5 puts together the ingredients developed in the preceding sections and concludes the proof of Theorem 1.1 .1. The appendix recalls a weighted Poincaré's inequality proved in [13, Lemma 4.9].

1.2 Main steps of the proof of Theorem 1.1 .1

The aim of this section is to give the structure of the proof of Theorem 1.1 .1 and its main steps.

1.2 .1 Reformulation of the problem

We start by introducing the variation (ρ, u) of (ρ_s, u_s) around the trajectory $(\overline{\rho}, \overline{u})$, defined by

$$(\rho, u) = (\rho_s, u_s) - (\overline{\rho}, \overline{u}) \text{ in } (0,T) \times (0,L). \tag{1.2 .1}$$

The new unknowns (ρ, u) then satisfy

$$\begin{cases} \partial_t \rho + (\overline{u} + u)\partial_x \rho + \overline{\rho}\partial_x u + \left(\dfrac{\overline{\rho}}{\nu} p'(\overline{\rho}) + \partial_x \overline{u}\right)\rho = f(\rho, u) & \text{in } (0,T)\times(0,L), \\ \overline{\rho}(\partial_t u + \overline{u}\partial_x u) - \nu \partial_{xx} u = g(\rho, u) & \text{in } (0,T)\times(0,L), \end{cases} \tag{1.2 .2}$$

where

$$\begin{cases} f(\rho, u) = -\rho \partial_x u + \dfrac{\overline{\rho}}{\nu} p'(\overline{\rho})\rho - \partial_x \overline{\rho} u, \\ g(\rho, u) = -\rho(\partial_t(\overline{u} + u) + (\overline{u} + u)\partial_x(\overline{u} + u)) - \overline{\rho} u \partial_x(\overline{u} + u) \\ \qquad\qquad\qquad -p'(\overline{\rho} + \rho)\partial_x(\overline{\rho} + \rho) + p'(\overline{\rho})\partial_x \overline{\rho}, \end{cases} \tag{1.2 .3}$$

with initial data

$$(\rho(0,\cdot), u(0,\cdot)) = (\rho_0, u_0) \qquad \text{in } (0,L), \tag{1.2 .4}$$

and controls acting on the boundary conditions, that we do not specify here. The control requirement (1.1 .9) then reads as follows:

$$(\rho(T,\cdot), u(T,\cdot)) = (0,0) \qquad \text{in } (0,L). \tag{1.2 .5}$$

The strategy will then consist in finding a trajectory (ρ, u) solving (1.2 .2)–(1.2 .4) and satisfying the control requirement (1.2 .5). This will be achieved by a fixed point argument that we will present in Section 1.2 .4 and that will be detailed in Section 1.5 .

1.2 .2 Geometric considerations

To deal with system (1.2 .2), we introduce a suitable extension of the spatial domain $(0, L)$ designed from the extension $\overline{u}$ on $\mathbb{R}$. This will allow us to pass from boundary controls to distributed controls which is easier to deal with from a theoretical point of view.

We first introduce the point $x_0 \in [0, L]$ defined by

$$x_0 = \sup \left\{ x \in [0, L] \;\middle|\; \exists t_x \in [0, T), \quad \begin{array}{l} \overline{X}(t_x, 0, x) = 0, \\ \forall t \in (0, t_x), \ \overline{X}(t, 0, x) \in (0, L) \end{array} \right\}, \tag{1.2 .6}$$

where $\overline{X}$ is the flow corresponding to $\overline{u}$ (recall (1.1 .5)). This point x_0 has interesting properties due to the uniqueness of Cauchy-Lipschitz's theorem forbidding the crossing of characteristics (recall that $\overline{u}$ belongs to $C^2([0, T] \times \mathbb{R})$ by Assumption (1.1 .4)):

- All trajectories $t \mapsto \overline{X}(t, 0, x)$ starting from $x \in [0, x_0)$ first exit the domain $[0, L]$ through $x = 0$;
- According to the geometric assumption (1.1 .6), all trajectories $t \mapsto \overline{X}(t, 0, x)$ starting from $x \in (x_0, L]$ necessary exit the domain $[0, L]$ through $x = L$, and thus x_0 in (1.2 .6) can also be defined as

$$x_0 = \inf \left\{ x \in [0, L] \;\middle|\; \exists t_x \in [0, T), \quad \begin{array}{l} \overline{X}(t_x, 0, x) = L, \\ \forall t \in (0, t_x), \ \overline{X}(t, 0, x) \in (0, L) \end{array} \right\}. \tag{1.2 .7}$$

Besides, using again (1.1 .6), there exists $T_2 < T$ such that $\overline{X}(T_2, 0, x_0) \notin [0, L]$.
If $\overline{X}(T_2, 0, x_0) > L$, by definition of x_0, there exists $T_1 \in [0, T_2)$ such that $\overline{X}(T_1, 0, x_0) = 0$.
If $\overline{X}(T_2, 0, x_0) < 0$, from (1.2 .7), there exists $T_1 \in [0, T_2)$ such that $\overline{X}(T_1, 0, x_0) = L$.

We are therefore in one of the configurations displayed in Figure 1.1. Using the change of

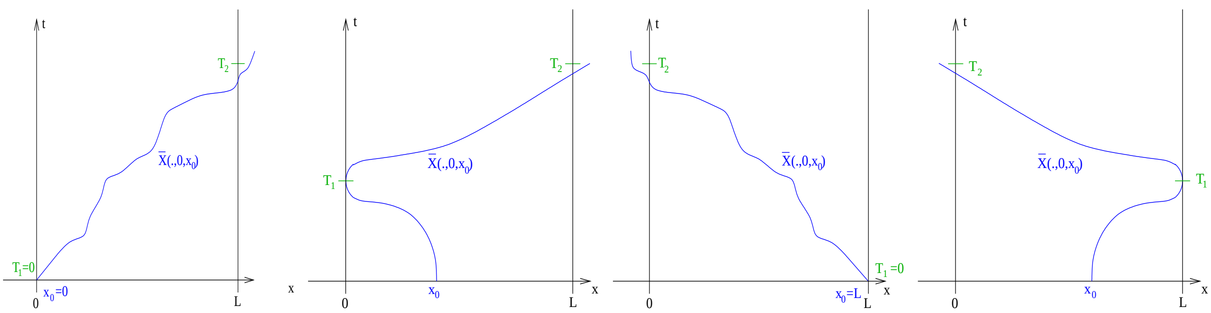

Figure 1.1: Some possible trajectories $t \mapsto \overline{X}(., 0, x_0)$.

variables $x \mapsto L - x$ if needed, we now assume without loss of generality that

$$\overline{X}(T_2, 0, x_0) > L \text{ and } T_1 \in [0, T_2) \text{ with } \overline{X}(T_1, 0, x_0) = 0.$$

Using the continuity of $\overline{X}(T_2, 0, \cdot)$, there exists $x_1 < x_0$ (x_1 may not belong to $[0, L]$ if $x_0 = 0$) such that $\overline{X}(T_2, 0, x_1) > L$, while due to Cauchy Lipschitz's theorem, $\overline{X}(T_1, 0, x_1) < 0$. By continuity of $\overline{X}(\cdot, 0, x_1)$, we also have the existence of $T_1' \in (0, T_2)$ such that $\overline{X}(T_1', 0, x_1) = \overline{X}(T_1, 0, x_1)/2$. We then define

$$a = \frac{\overline{X}(T_1', 0, x_1)}{2} \quad (< 0), \qquad b = \frac{\overline{X}(T_2, 0, x_1) + L}{2} \quad (> L), \tag{1.2 .8}$$

and we set

$$T_0 = \frac{T_1'}{2} \quad (>0), \qquad T_L = T_2 \quad (\in (T_1', T)), \tag{1.2 .9}$$

so that

$$\overline{X}(2T_0, 0, x_1) < a \quad \text{and} \quad \overline{X}(T_L, 0, x_1) > b. \tag{1.2 .10}$$

Lastly, remark that due to the fact that two different characteristics do not cross each other due to the regularity of $\overline{u}$ in (1.1 .4), we have the following counterpart of (1.1 .6) on the domain (a, b):

$$\forall x \in [a, b], \quad \exists t_x \in (0, T_L), \quad \overline{X}(t_x, 0, x) \notin [a, b]. \tag{1.2 .11}$$

We refer to Figure 1.2 for a summary of the notations introduced above.

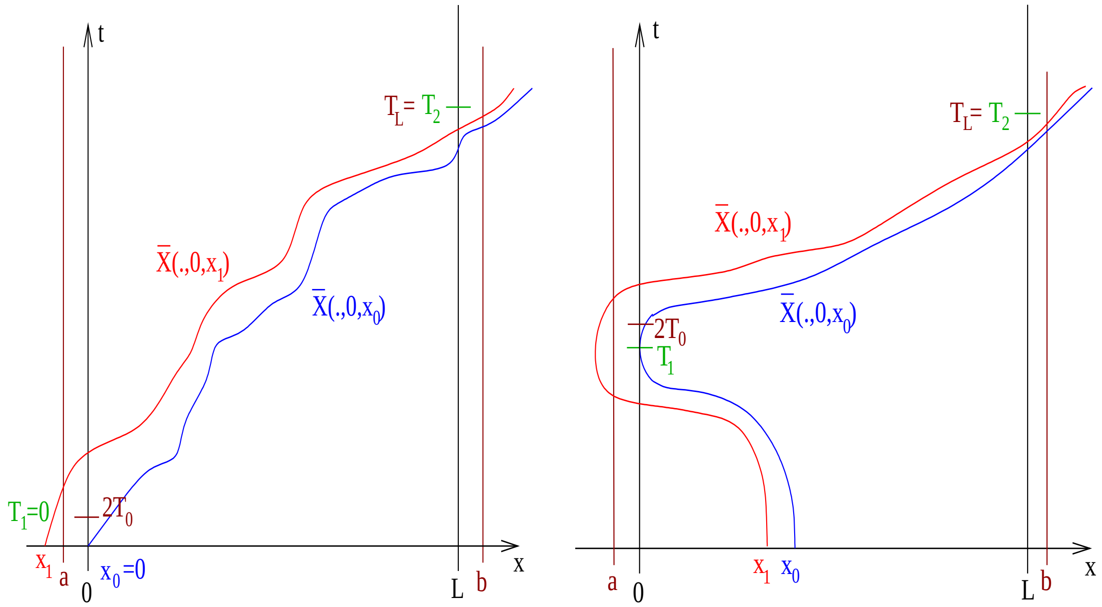

Figure 1.2: Choice of x_1 and construction of the extended domain (a, b) and the times (T_0, T_L) in two different configurations: Left, $x_0 = 0$, right, $x_0 \in (0, L)$.

For convenience, we then redefine the time horizon T by reducing it if necessary by setting

$$T := \min\{T, T_L + 1\}. \tag{1.2 .12}$$

This can be done without loss of generality as a local exact controllability result on a given trajectory in some time obviously implies a local exact controllability result on the target trajectory in all larger times.

1.2 .3 Carleman weights

Our fixed point argument will be done in suitable weighted spaces coming from the use of Carleman estimates, namely the ones in [3, Theorem 2.5]. In this section, we construct once for all the Carleman weights that will be used along the article.

Let us introduce a cut-off function $\eta \in C^\infty(\mathbb{R}; [0, 1])$ such that

$$\forall x \in \mathbb{R}, \quad \eta(x) = \begin{cases} 1 & \text{for } x \in [0, L], \\ 0 & \text{for } x \in \mathbb{R} \setminus \left(\frac{a}{2}, \frac{L+b}{2}\right). \end{cases} \tag{1.2 .13}$$

We then construct a weight function ψ as follows:

Lemma 1.2 .1. *Let η as in (1.2 .13). There exists $\psi \in C^2([0,T]\times\mathbb{R})$ such that:*

$$\forall (t,x) \in [0,T]\times\mathbb{R}, \quad \partial_t \psi + \eta \overline{u} \partial_x \psi = 0, \tag{1.2 .14}$$
$$\forall (t,x) \in [0,T]\times\mathbb{R}, \quad \psi(t,x) \in [0,1], \tag{1.2 .15}$$
$$\forall t \in (0,T), \quad \partial_x \psi(t,2a) \geq 0, \tag{1.2 .16}$$
$$\sup_{[0,T]\times[a,b]} \partial_x \psi < 0. \tag{1.2 .17}$$

Proof of Lemma 1.2 .1. Let $\psi_0 \in C^2(\mathbb{R};[0,1])$ such that $\partial_x \psi_0(2a) \geq 0$ and

$$\sup_{[a,b]} \partial_x \psi_0 < 0. \tag{1.2 .18}$$

We then define ψ as the solution of the equation

$$\begin{cases} \partial_t \psi + \eta \overline{u} \partial_x \psi = 0 & \text{in } (0,T)\times\mathbb{R}, \\ \psi(0,\cdot) = \psi_0 & \text{in } \mathbb{R}. \end{cases} \tag{1.2 .19}$$

It is easy to check that $\psi \in C^2([0,T]\times\mathbb{R})$ as $\overline{u} \in C^2([0,T]\times\mathbb{R})$ (recall (1.1 .4)) and $\psi_0 \in C^2(\mathbb{R})$. The function ψ takes value in $[0,1]$ as $\psi_0 \in C^2(\mathbb{R};[0,1])$. In order to check conditions (1.2 .16)–(1.2 .17), we simply look at the equation satisfied by $\partial_x \psi$:

$$\begin{cases} \partial_t \partial_x \psi + \eta \overline{u} \partial_x(\partial_x \psi) + \partial_x(\eta \overline{u}) \partial_x \psi = 0 & \text{in } (0,T)\times\mathbb{R}, \\ \partial_x \psi(0,\cdot) = \partial_x \psi_0 & \text{in } \mathbb{R}. \end{cases} \tag{1.2 .20}$$

It follows that $\partial_x \psi(t,2a) = \partial_x \psi_0(2a)$ for all $t \in [0,T]$, so that (1.2 .16) holds true, and that the critical points of $\psi(t,\cdot)$ are transported along the characteristics of $\eta\overline{u}$. As the velocity field $\eta\overline{u}$ vanishes outside $(a/2,(L+b/2))$, the fact that ψ_0 does not have any critical point in $[a,b]$ (recall (1.2 .18)) implies that $\psi(t,\cdot)$ cannot have a critical point in $[a,b]$. Thus, $\partial_x \psi$ cannot vanish in $[0,T]\times[a,b]$, and the sign condition in (1.2 .18) implies (1.2 .17), therefore concluding the proof of Lemma 1.2 .1. □

For $s \geq 1$ and $\lambda \geq 1$, we set $\mu = s\lambda^2 e^{2\lambda}$ and we define $\theta \in C^2([0,T);\mathbb{R})$ as follows:

$$\forall t \in [0,T), \quad \theta(t) = \begin{cases} 1 + \left(1 - \dfrac{t}{T_0}\right)^{\mu} & \text{for } t \in [0,T_0], \\ 1 & \text{for } t \in [T_0,T_L], \\ \text{nondecreasing} & \text{for } t \in \left[T_L, \dfrac{T+T_L}{2}\right], \\ \dfrac{1}{T-t} & \text{for } t \in \left[\dfrac{T+T_L}{2}, T\right), \end{cases} \tag{1.2 .21}$$

where T_0 and T_L are defined in (1.2 .9). Note that such construction is possible according to (1.2 .12). We also define $\varphi = \varphi(t,x)$ by

$$\varphi(t,x) = \theta(t)(\lambda e^{12\lambda} - e^{\lambda(\psi(t,x)+6)}) \quad \text{for all } (t,x) \in [0,T]\times\mathbb{R}. \tag{1.2 .22}$$

These functions θ and φ depend on the parameters $s \geq 1$ and $\lambda \geq 1$, but we shall omit these dependencies for simplicity of notations. Actually, the parameter $\lambda \geq 1$ in the definitions (1.2 .21)-(1.2 .22) of θ and φ is chosen sufficiently large for Theorem 2.5 in [3], recalled in the present paper in Theorem 1.4 .2, to be true and is fixed in all the article. Yet, the Carleman parameter $s \geq 1$ is not fixed and will be chosen at the end of the proof. Consequently, in all the article, constants C never depend on the parameter s, except if specifically said.

These functions will appear naturally as weights when using Carleman estimates, see Section 1.4 . Our approach here is based on the Carleman estimate developed in [3, Theorem 2.5], but it shares very close features to the ones developed in [16].

Here, we will use the Carleman estimates developed in [3, Theorem 2.5] as they allow to avoid doing a lifting of the initial data and the use of technical results concerning the Cauchy problem (for instance [24] which was used in [13]). The Carleman estimate in [3, Theorem 2.5] has also been developed for weight functions $\psi = \psi(t,x)$ depending on the time and space variables, which will be needed in our analysis to get estimates on the controlled density, see Section 1.3 .2.3. In this step indeed, we shall strongly use that ψ solves the transport equation (1.2 .14).

1.2 .4 Definition of the fixed point map

We now make precise the fixed point map and the intermediate results which will be needed to conclude Theorem 1.1 .1.

The controllability of (1.2 .2)–(1.2 .4) will be studied in the extended domain (a,b), where a and b are defined in (1.2 .8). Let $(\rho_0,u_0) \in H^1(0,L)\times H^1(0,L)$ satisfying (1.1 .7) with $\varepsilon > 0$. This parameter ε will be chosen small enough within our proof of Theorem 1.1 .1, see Section 1.5 and more precisely Lemma 1.5 .1. We thus extend (ρ_0,u_0) such that $(\rho_0,u_0) \in H^1_0(a,b)\times H^1_0(a,b)$ and

$$\|(\rho_0,u_0)\|_{H^1_0(a,b)\times H^1_0(a,b)} \leq C\varepsilon, \tag{1.2 .23}$$

where C is independent of ε. We also introduce the characteristic function of $(a,b)\setminus[0,L]$

$$\chi(x) = \begin{cases} 1 & \text{for } x \in (a,0)\cup(L,b), \\ 0 & \text{for } x\in[0,L]. \end{cases} \tag{1.2 .24}$$

Instead of considering the equations (1.2 .2)–(1.2 .4), we now consider the following system of equations:

$$\begin{cases} \partial_t\rho + (\overline{u}+u)\partial_x\rho + \overline{\rho}\partial_x u + \left(\dfrac{\overline{\rho}}{\nu}p'(\overline{\rho}) + \partial_x\overline{u}\right)\rho = f(\rho,u) + v_\rho\chi & \text{in } (0,T)\times(a,b), \\ \overline{\rho}(\partial_t u + \overline{u}\partial_x u) - \nu\partial_{xx}u = g(\rho,u) & \text{in } (0,T)\times(a,b), \end{cases} \tag{1.2 .25}$$

with initial data

$$(\rho(0,\cdot),u(0,\cdot)) = (\rho_0,u_0) \quad \text{in } (a,b), \tag{1.2 .26}$$

source terms $f(\rho,u)$, $g(\rho,u)$ as in (1.2 .3), and with boundary conditions

$$\begin{cases} \rho(t,a) = 0 & \text{for } t\in(0,T) \text{ with } (\overline{u}+u)(t,a) > 0, \\ \rho(t,b) = 0 & \text{for } t\in(0,T) \text{ with } (\overline{u}+u)(t,b) < 0, \\ u(t,a) = v_u(t) & \text{for } t\in(0,T), \\ u(t,b) = 0 & \text{for } t\in(0,T). \end{cases} \tag{1.2 .27}$$

In (1.2 .25) and (1.2 .27), the functions $v_\rho \in L^2((0,T)\times((a,0)\cup(L,b)))$ and $v_u \in L^2(0,T)$ are control functions and will be chosen such that the solution (ρ,u) of (1.2 .25)–(1.2 .27) satisfies the control requirement

$$(\rho(T,\cdot),u(T,\cdot)) = 0 \qquad \text{in } (a,b). \tag{1.2 .28}$$

The existence of a controlled trajectory (ρ,u) satisfying (1.2 .25)–(1.2 .28) will be obtained through a fixed point argument that we now introduce.

Let us first describe the functional space in which we will define our fixed point map. For $s \geq 1$, we introduce the norms $\|.\|_{\mathscr{X}_s}$ and $\|.\|_{\mathscr{Y}_s}$ defined by

$$\|\rho\|_{\mathscr{X}_s} = s\|e^{s\varphi}\rho\|_{L^\infty(0,T;L^2(a,b))} + \|\theta^{-1}e^{s\varphi}\partial_x\rho\|_{L^\infty(0,T;L^2(a,b))} + \|e^{s\varphi/2}\rho\|_{L^\infty(0,T;L^\infty(a,b))} + \|\partial_t\rho\|_{L^2(0,T;L^2(a,b))}, \quad (1.2\ .29)$$

and

$$\|u\|_{\mathscr{Y}_s} = s^{3/2}\|e^{s\varphi}u\|_{L^2(0,T;L^2(a,b))} + s^{1/2}\|\theta^{-1}e^{s\varphi}\partial_x u\|_{L^2(0,T;L^2(a,b))} + s^{-1/2}\|\theta^{-2}e^{s\varphi}\partial_{xx}u\|_{L^2(0,T;L^2(a,b))} + s^{-1/2}\|\theta^{-2}e^{s\varphi}\partial_t u\|_{L^2(0,T;L^2(a,b))} + s^{1/2}\|\theta^{-1}e^{s\varphi}u\|_{L^\infty(0,T;L^2(a,b))}, \quad (1.2\ .30)$$

and the corresponding spaces

$$\mathscr{X}_s = \left\{\rho \in L^2(0,T;H_0^1(a,b)) \text{ with } \|\rho\|_{\mathscr{X}_s} < \infty\right\}, \quad (1.2\ .31)$$

$$\mathscr{Y}_s = \left\{u \in L^2(0,T;L^2(a,b)) \text{ with } \|u\|_{\mathscr{Y}_s} < \infty\right\}. \quad (1.2\ .32)$$

For $s \geq 1$ and $R_\rho, R_u > 0$, we also introduce the corresponding balls

$$\mathscr{X}_{s,R_\rho} = \{\rho \in \mathscr{X}_s \text{ with } \|\rho\|_{\mathscr{X}_s} \leq R_\rho\}, \qquad \mathscr{Y}_{s,R_u} = \{u \in \mathscr{Y}_s \text{ with } \|u\|_{\mathscr{Y}_s} \leq R_u\}. \quad (1.2\ .33)$$

The fixed point map is then constructed as follows: For $(\widehat{\rho}, \widehat{u}) \in \mathscr{X}_{s,R_\rho} \times \mathscr{Y}_{s,R_u}$, find control functions (v_ρ, v_u) such that the controlled trajectories (ρ, u) solving

$$\begin{cases} \partial_t\rho + (\overline{u}+u)\partial_x\rho + \overline{\rho}\partial_x u + \left(\dfrac{\overline{\rho}}{\nu}p'(\overline{\rho}) + \partial_x\overline{u}\right)\rho = f(\widehat{\rho},\widehat{u}) + v_\rho\chi & \text{in } (0,T)\times(a,b), \\ \overline{\rho}(\partial_t u + \overline{u}\partial_x u) - \nu\partial_{xx}u = g(\widehat{\rho},\widehat{u}) & \text{in } (0,T)\times(a,b), \end{cases} \quad (1.2\ .34)$$

with initial data (1.2 .26), source terms

$$f(\widehat{\rho},\widehat{u}) = -\widehat{\rho}\partial_x\widehat{u} + \frac{\bar{\rho}}{\nu}p'(\bar{\rho})\widehat{\rho} - \partial_x\bar{\rho}\widehat{u}, \quad (1.2\ .35)$$

$$g(\widehat{\rho},\widehat{u}) = -\widehat{\rho}(\partial_t(\bar{u}+\widehat{u}) + (\bar{u}+\widehat{u})\partial_x(\bar{u}+\widehat{u})) - \bar{\rho}\widehat{u}\partial_x(\bar{u}+\widehat{u}) - p'(\bar{\rho}+\widehat{\rho})\partial_x(\bar{\rho}+\widehat{\rho}) + p'(\bar{\rho})\partial_x\bar{\rho}, \quad (1.2\ .36)$$

boundary conditions (1.2 .27) (involving the control v_u), satisfies the controllability requirement(1.2 .28).

Our main task is then to show that the above construction is well-defined for $(\widehat{\rho},\widehat{u}) \in \mathscr{X}_{s,R_\rho} \times \mathscr{Y}_{s,R_u}$ for a suitable choice of parameters $s \geq 1$, $R_\rho > 0$ and $R_u > 0$, that the corresponding controlled trajectory (ρ, u) can be constructed such that $(\rho,u) \in \mathscr{X}_{s,R_\rho} \times \mathscr{Y}_{s,R_u}$ and then to apply Schauder's fixed point theorem.

System (1.2 .34) is not properly speaking the linearized system of (1.2 .25) due to the term $u\partial_x\rho$ in the equation $(1.2\ .34)_{(1)}$, which is quadratic. However, as in [13], this term cannot be handled as a source term due to regularity issues. But still, the controllability of (1.2 .34) can be solved using subsequently two controllability results for linear equations. Indeed, one can first control the equation $(1.2\ .34)_{(2)}$ of the velocity and obtain u and v_u from $(\widehat{\rho},\widehat{u})$, and once u is constructed, the equation $(1.2\ .34)_{(1)}$ of the density is a linear transport equation which can be controlled independently. Our approach will then follows this 2-step construction.

Let us start with a controllability result for the equation of the velocity:

Theorem 1.2 .2. *There exist $C_u > 0$, $s_0 \geq 1$, $\lambda \geq 1$, such that for all $s \geq s_0$, for all $R_u \in (0,1)$, for all $R_\rho \in (0, \min\{1, \min_{[0,T]\times\mathbb{R}} \overline{\rho}/2\})$, for all $u_0 \in H_0^1(a,b)$ and for all $(\widehat{\rho}, \widehat{u}) \in \mathscr{X}_{s,R_\rho} \times \mathscr{Y}_{s,R_u}$, there exists $v_u \in L^2(0,T)$ such that the solution u of*

$$\begin{cases} \overline{\rho}(\partial_t u + \overline{u}\partial_x u) - \nu \partial_{xx} u = g(\widehat{\rho}, \widehat{u}) & \text{in } (0,T)\times(a,b), \\ u(t,a) = v_u(t) & \text{in } (0,T), \\ u(t,b) = 0 & \text{in } (0,T), \\ u(0,\cdot) = u_0 & \text{in } (a,b), \end{cases} \tag{1.2 .37}$$

where $g(\widehat{\rho}, \widehat{u})$ is defined as in (1.2 .36), satisfies

$$u(T,\cdot) = 0 \text{ in } (a,b). \tag{1.2 .38}$$

Besides, $u \in \mathscr{Y}_s$ and satisfies the estimate

$$\|u\|_{\mathscr{Y}_s} \leq C_u'(s)\|u_0\|_{H^1(a,b)} + C_u R_\rho + \frac{C_u}{s} R_u + C_u R_u^2, \tag{1.2 .39}$$

where $C_u'(s)$ depends on the parameter s.

Theorem 1.2 .2 does not present any significant new difficulty compared to [13, 3]. For sake of completeness, we shall nonetheless provide some details in Section 1.4 .

In a second step, we analyze the controllability properties of the transport equation $(1.2\ .34)_{(1)}$:

Theorem 1.2 .3. *Let λ and s_0 as in Theorem 1.2 .2. There exist $C_\rho > 0$, $s_1 \geq s_0$ such that for all $s \geq s_1$, there exists $\varepsilon(s) > 0$ such that for all $\varepsilon \in (0, \varepsilon(s)]$, for all $(\rho_0, u_0) \in H_0^1(a,b) \times H_0^1(a,b)$ satisfying (1.2 .23), for all $R_\rho \in (0, \min\{1, \min_{[0,T]\times\mathbb{R}} \overline{\rho}/2\})$, for all $R_u \in (0,1)$, for all $(\widehat{\rho}, \widehat{u}) \in \mathscr{X}_{s,R_\rho} \times \mathscr{Y}_{s,R_u}$ and for u constructed in Theorem 1.2 .2, there exists $v_\rho \in L^2(0,T; L^2((a,0)\cup(L,b)))$ such that the solution ρ of*

$$\begin{cases} \partial_t \rho + (\overline{u}+u)\partial_x \rho + \left(\dfrac{\overline{\rho}}{\nu} p'(\overline{\rho}) + \partial_x \overline{u}\right)\rho & = f(\widehat{\rho}, \widehat{u}) - \overline{\rho}\partial_x u + v_\rho \chi \text{ in } (0,T)\times(a,b), \\ \rho(t,a) = 0 & \text{for } t \in (0,T) \text{ with } (\overline{u}+u)(t,a) > 0, \\ \rho(t,b) = 0 & \text{for } t \in (0,T) \text{ with } (\overline{u}+u)(t,b) < 0, \\ \rho(0,\cdot) = \rho_0 & \text{in } (a,b), \end{cases} \tag{1.2 .40}$$

where $f(\widehat{\rho}, \widehat{u})$ is defined as in (1.2 .35), satisfies

$$\rho(T,\cdot) = 0 \text{ in } (a,b). \tag{1.2 .41}$$

Besides, $\rho \in \mathscr{X}_s$ and satisfies the estimate

$$\|\rho\|_{\mathscr{X}_s} \leq C_\rho'(s)\varepsilon + \frac{C_\rho}{\sqrt{s}}(R_\rho + R_u) + C_\rho(R_\rho^2 + R_u^2). \tag{1.2 .42}$$

where $C_\rho'(s)$ depends on the parameter s.

The proof of Theorem 1.2 .3 is developed in Section 1.3 and is the main contribution of our work.

The end of the proof of Theorem 1.1 .1 then consists in putting together the aforementioned steps and show that Schauder's fixed point theorem applies. This last point will be explained in Section 1.5 .

Remark 1.2 .4. *Theorems 1.2 .2 and 1.2 .3 are designed to be used later in the proof of Theorem 1.1 .1 in Section 1.5 . Therefore, we only focused on giving precise estimates of the controlled trajectories (see* (1.2 .39) *and* (1.2 .42)*). In particular, we did not provide precise regularity results on the controls v_u in Theorem 1.2 .2 nor v_ρ in Theorem 1.2 .3. One can check that the proofs of Theorems 1.2 .2 and 1.2 .3 provide controls which are more regular than simply L^2:*

- *The control v_u in Theorem 1.2 .2 is in $H^{3/4}(0,T)$, the space of trace of $L^2(0,T;H^2(a,b)) \cap H^1(0,T;L^2(a,b))$.*
- *The control v_ρ in Theorem 1.2 .3 belongs to $L^\infty(0,T;L^2((a,0)\cup(L,b))$. Note that this regularity result on v_ρ is not sufficient to guarantee that the solution ρ of* (1.2 .40) *belongs to $L^\infty(0,T;H^1(a,b))$. In fact, this regularity will be deduced during the construction of the controlled trajectory ρ.*

Notations. For simplicity of notations, we shall often use the notations

$$\widehat{f} := f(\widehat{\rho},\widehat{u}), \quad \widehat{g} := g(\widehat{\rho},\widehat{u}), \tag{1.2 .43}$$

where $f(\widehat{\rho},\widehat{u})$, $g(\widehat{\rho},\widehat{u})$ are respectively defined in (1.2 .35) and (1.2 .36).

1.3 Control of the density

The proof of Theorem 1.2 .3 is divided into two steps: the first step presents the construction of the controlled trajectory and the second step is devoted to get estimates on it.

1.3 .1 Construction of a controlled trajectory ρ

For the time being, let us fix $f \in L^2(0,T;L^2(a,b))$ and assume that u satisfies

$$\|u\|_{L^1(0,T;W^{1,\infty}(a,b))\cap L^\infty(0,T;L^\infty(a,b))} \le \varepsilon_0, \tag{1.3 .1}$$

for some $\varepsilon_0 > 0$ small enough that will be defined later as the one given by Lemma 1.3 .1.

We then focus on the following controllability problem: find a control $v_\rho \in L^2(0,T;L^2((a,0)\cup(L,b))$ such that the solution ρ of

$$\begin{cases} \partial_t\rho + (\overline{u}+u)\partial_x\rho + \left(\dfrac{\overline{\rho}}{\nu}p'(\overline{\rho}) + \partial_x\overline{u}\right)\rho & = f - \overline{\rho}\partial_x u + v_\rho\chi \text{ in } (0,T)\times(a,b), \\ \rho(t,a) = 0 & \text{for } t\in(0,T) \text{ with } (\overline{u}+u)(t,a) > 0, \\ \rho(t,b) = 0 & \text{for } t\in(0,T) \text{ with } (\overline{u}+u)(t,b) < 0, \\ \rho(0,\cdot) = \rho_0 & \text{in } (a,b), \end{cases} \tag{1.3 .2}$$

satisfies (1.2 .41), where χ is as in (1.2 .24).

The construction of such solution is done following the spirit of the construction in [13] by gluing forward and backward solutions of the transport equation. More precisely, using the function η in (1.2 .13), we define ρ_f as the solution of

$$\begin{cases} \partial_t\rho_f + \eta(\overline{u}+u)\partial_x\rho_f + \left(\dfrac{\overline{\rho}}{\nu}p'(\overline{\rho}) + \partial_x\overline{u}\right)\rho_f = \eta(f - \overline{\rho}\partial_x u) & \text{in } (0,T)\times(a,b), \\ \rho_f(0,\cdot) = \rho_0 & \text{in } (a,b), \end{cases} \tag{1.3 .3}$$

and ρ_b as the solution of

$$\begin{cases} \partial_t\rho_b + \eta(\overline{u}+u)\partial_x\rho_b + \left(\dfrac{\overline{\rho}}{\nu}p'(\overline{\rho}) + \partial_x\overline{u}\right)\rho_b = \eta(f - \overline{\rho}\partial_x u) & \text{in } (0,T)\times(a,b), \\ \rho_b(T,\cdot) = 0 & \text{in } (a,b). \end{cases} \tag{1.3 .4}$$

Remark that in (1.3 .3) and (1.3 .4), we do not need to specify any boundary condition as $(\eta(\overline{u}+u))(t,a) = (\eta(\overline{u}+u))(t,b) = 0$. Besides, since $\rho_0 \in H_0^1(a,b)$,

$$\rho_f(t,a) = \rho_f(t,b) = 0 \text{ for all } t \in (0,T). \tag{1.3 .5}$$

Similarly, we also have

$$\rho_b(t,a) = \rho_b(t,b) = 0 \text{ for all } t \in (0,T). \tag{1.3 .6}$$

We now construct a suitable cut-off function $\tilde{\eta}$ traveling along the characteristics:

Lemma 1.3 .1. *Let χ as in (1.2 .24). There exist positive constants $C > 0$ and $\varepsilon_0 \in (0,1)$ such that for all u belonging to $L^1(0,T;W^{1,\infty}(a,b)) \cap L^\infty(0,T;L^\infty(a,b))$ and satisfying (1.3 .1), there exists $\tilde{\eta} = \tilde{\eta}(t,x)$ satisfying the equation*

$$\partial_t \tilde{\eta} + (\overline{u}+u)\partial_x \tilde{\eta} = w\chi \text{ in } (0,T) \times (a,b), \tag{1.3 .7}$$

for some $w \in L^2(0,T;L^2((a,0) \cup (L,b))$, (recall χ in (1.2 .24) is the indicator function of $(a,0) \cup (0,b)$), the conditions

$$\begin{aligned} \tilde{\eta}(t,x) &= 1 \text{ for all } (t,x) \in [0,T_0] \times [a,b], & (1.3\ .8)\\ \tilde{\eta}(t,x) &= 0 \text{ for all } (t,x) \in [T_L,T] \times [a,b], & (1.3\ .9) \end{aligned}$$

and the bound

$$\|\tilde{\eta}\|_{W^{1,\infty}((0,T)\times(a,b))} \le C, \tag{1.3 .10}$$

for some constant C independent of ε_0.

Before going into the proof of Lemma 1.3 .1, let us briefly explain how we can conclude the construction of a controlled trajectory ρ satisfying (1.3 .2) when $u \in L^1(0,T;W^{1,\infty}(a,b)) \cap L^\infty(0,T;L^\infty(a,b))$ satisfies (1.3 .1) for $\varepsilon_0 \in (0,1)$ as in Lemma 1.3 .1.

It simply consists in setting

$$\rho(t,x) = \tilde{\eta}(t,x)\rho_f(t,x) + (1-\tilde{\eta}(t,x))\rho_b(t,x) \text{ for all } (t,x) \in [0,T] \times [a,b], \tag{1.3 .11}$$

where $\tilde{\eta}$ is the cut-off function constructed in Lemma 1.3 .1. Indeed, one easily checks that ρ defined in that way solves (1.3 .2) with control function

$$v_\rho = (1-\eta)\left[(\overline{u}+u)(\tilde{\eta}\partial_x\rho_f + (1-\tilde{\eta})\partial_x\rho_b) - f + \overline{\rho}\partial_x u\right] + w(\rho_f - \rho_b)\chi, \tag{1.3 .12}$$

and satisfies

$$\rho(t,a) = \rho(t,b) = 0, \qquad \text{for all } t \in (0,T). \tag{1.3 .13}$$

Furthermore, using (1.3 .8)–(1.3 .9), we immediately get the two following identities:

$$\begin{aligned} \rho(t,x) &= \rho_f(t,x) \text{ for all } (t,x) \in [0,T_0] \times [a,b], & (1.3\ .14)\\ \rho(t,x) &= \rho_b(t,x) \text{ for all } (t,x) \in [T_L,T] \times [a,b]. & (1.3\ .15) \end{aligned}$$

We now prove Lemma 1.3 .1.

Proof of Lemma 1.3 .1. We first extend u to $(0,T) \times \mathbb{R}$ such that $u \in L^1(0,T;W^{1,\infty}(\mathbb{R})) \cap L^\infty(0,T;L^\infty(\mathbb{R}))$ satisfies

$$\|u\|_{L^1(0,T;W^{1,\infty}(\mathbb{R}))\cap L^\infty(0,T;L^\infty(\mathbb{R}))} \le C\varepsilon_0 \quad (\le C \text{ as } \varepsilon_0 \le 1). \tag{1.3 .16}$$

For $(t, \tau, x) \in [0, T] \times [0, T] \times \mathbb{R}$, we define the flow X associated to $\overline{u} + u$ as the solution of:

$$\begin{cases} \dfrac{dX}{dt}(t, \tau, x) = (\overline{u} + u)(t, X(t, \tau, x)) \quad \text{in } (0, T), \\ X(\tau, \tau, x) = x. \end{cases} \tag{1.3 .17}$$

Writing

$$\begin{cases} \dfrac{d}{dt}(X - \overline{X})(t, \tau, x) = \overline{u}(t, X(t, \tau, x)) - \overline{u}(t, \overline{X}(t, \tau, x)) + u(t, X(t, \tau, x)) \quad \text{in } (0, T), \\ (X - \overline{X})(\tau, \tau, x) = 0, \end{cases}$$

using (1.1 .4) and (1.3 .16), Gronwall's lemma yields

$$\sup_{(t,\tau,x) \in [0,T] \times [0,T] \times \mathbb{R}} \left\{ |X(t, \tau, x) - \overline{X}(t, \tau, x)| \right\} \leq C \varepsilon_0.$$

Therefore, using (1.2 .10), for $\varepsilon_0 > 0$ small enough,

$$X(2T_0, 0, x_1) < a \text{ and } X(T_L, 0, x_1) > b. \tag{1.3 .18}$$

We then perform the construction of $\tilde{\eta}$ in three steps, defining it separately on each time interval $(0, 2T_0)$, $(2T_0, T_L)$ and (T_L, T).

On $[0, 2T_0]$, we simply set

$$\tilde{\eta}(t, x) = 1 - \eta_1(t)\eta_2(x), \text{ for all } (t, x) \in [0, 2T_0] \times \mathbb{R}, \tag{1.3 .19}$$

where $\eta_1 = \eta_1(t) \in C^\infty([0, 2T_0])$ such that $\eta_1(t) = 0$ for all $t \in [0, T_0]$ and $\eta_1(2T_0) = 1$, and $\eta_2 = \eta_2(x) \in C^\infty(\mathbb{R})$ such that $\eta_2(x) = 0$ in $[0, L]$ and $\eta_2(x) = 1$ in $(-\infty, a] \cup [b, \infty)$. On $(2T_0, T_L)$, we solve the equation

$$\begin{cases} \partial_t \tilde{\eta} + (\overline{u} + u)\partial_x \tilde{\eta} = 0 & \text{in } (2T_0, T_L) \times \mathbb{R}, \\ \tilde{\eta}(2T_0, \cdot) = 1 - \eta_1(2T_0)\eta_2(\cdot) & \text{in } \mathbb{R}. \end{cases} \tag{1.3 .20}$$

On the time interval $[T_L, T]$, we set

$$\tilde{\eta}(t, x) = 0, \text{ for all } (t, x) \in [T_L, T] \times \mathbb{R}. \tag{1.3 .21}$$

The above piecewise construction (1.3 .19)–(1.3 .20)–(1.3 .21) defines $\tilde{\eta}$ on the whole time interval $[0, T]$. One easily checks that this $\tilde{\eta}$ solves (1.3 .7) on $(0, T) \times (a, b)$ (with $w(t, x) = -1_{[0,2T_0]}(\partial_t \eta_1(t)\eta_2(x) + \eta_1(t)(\overline{u} + u)\partial_x \eta_2(x)))$ as $\tilde{\eta}(2T_0^-, x) = \tilde{\eta}(2T_0^+, x)$ for all $x \in (a, b)$ and $\tilde{\eta}(T_L^-, x) = 0$ for all $x \in (a, b)$ according to (1.3 .18) and the fact that $\tilde{\eta}(2T_0, x) = 0$ for all $x \leq a$. Besides, $\tilde{\eta}$ obviously satisfies (1.3 .8)–(1.3 .9).

We then check that $\tilde{\eta}$ belongs to $W^{1,\infty}((0, T) \times (a, b))$. Of course, the only difficulty is on the time interval $(2T_0, T_L)$. But we can then look at the equation satisfied by $\partial_x \tilde{\eta}$, i.e.

$$\begin{cases} \partial_t(\partial_x \tilde{\eta}) + (\overline{u} + u)\partial_x(\partial_x \tilde{\eta}) + \partial_x(\overline{u} + u)\partial_x \tilde{\eta} = 0 & \text{in } (2T_0, T_L) \times \mathbb{R}, \\ \partial_x \tilde{\eta}(2T_0, \cdot) = -\partial_x \eta_2(\cdot) & \text{in } \mathbb{R}, \end{cases}$$

and solve it using the flow X in (1.3 .17). The regularity (1.1 .4) and the bound (1.3 .16) then yields $\partial_x \tilde{\eta} \in L^\infty((0, T) \times (a, b))$ with an explicit bound independent of $\varepsilon_0 \in (0, 1)$. We then deduce that $\partial_t \tilde{\eta} \in L^\infty((0, T) \times (a, b))$ from the equation (1.3 .7) and the fact that $w(t, x) = -1_{[0,2T_0]}(\partial_t \eta_1(t)\eta_2(x) + \eta_1(t)(\overline{u} + u)\partial_x \eta_2(x)) \in L^\infty((0, T) \times ((a, 0) \cup (0, b)))$. This concludes the proof of (1.3 .10). $\square$

1.3 .2 Estimates on ρ

The purpose of this section is to prove that the controlled trajectory ρ constructed in (1.3 .11) satisfies the estimate (1.2 .42) claimed in Theorem 1.2 .3.

We shall then put ourselves in the setting of Theorem 1.2 .3. In particular, in the whole section we assume the following:

Assumption and Setting 1. *Let* $(\rho_0, u_0) \in H_0^1(a,b) \times H_0^1(a,b)$ *satisfying* (1.2 .23) *for* $\varepsilon > 0$, $(\widehat{\rho}, \widehat{u}) \in \mathscr{X}_{s,R_\rho} \times \mathscr{Y}_{s,R_u}$ *for some* $R_\rho \in (0, \min\{1, \min_{[0,T]\times\mathbb{R}} \overline{\rho}/2\})$, $R_u \in (0,1)$, *and* u *the controlled trajectory given by Theorem 1.2 .2.*

Further assume the condition (1.3 .1) *for* $\varepsilon_0 > 0$ *small enough so that the construction in Section 1.3 .1 can be done. The controlled trajectory* ρ *in* (1.3 .11) *is constructed for* $f = \widehat{f} = f(\widehat{\rho}, \widehat{u})$ *defined in* (1.2 .35).

Note that due to the continuity of the embedding of $L^2(0,T;H^2(a,b)) \cap H^1(0,T;L^2(a,b))$ into $L^1(0,T;W^{1,\infty}(a,b)) \cap L^\infty(0,T;L^\infty(a,b))$ and $\inf\{s^{3/2}e^{s\varphi}, s^{1/2}\theta^{-1}e^{s\varphi}, s^{-1/2}\theta^{-2}e^{s\varphi}\} \geq Ce^s$, we have

$$\|u\|_{L^1(W^{1,\infty})\cap L^\infty(L^\infty)} \leq Ce^{-s}\|u\|_{\mathscr{Y}_s} \leq Ce^{-s}(C_u'(s)\varepsilon + C_u + \frac{C_u}{s} + C_u),$$

with (1.2 .39), (1.2 .23) and $R_\rho, R_u \leq 1$. Thus, the condition (1.3 .1) for $\varepsilon_0 > 0$ given in Lemma 1.3 .1 can be imposed by choosing $s \geq s_1'$ large enough and $\varepsilon \leq 1/C_u'(s)$.

1.3 .2.1 The effective velocity

In order to derive estimates on the controlled trajectory ρ constructed in (1.3 .11), similarly as in [13], we introduce the following quantities, defined for $(t,x) \in [0,T] \times [a,b]$:

$$\mu_f = \eta u + \frac{\nu}{\overline{\rho}^2}\partial_x\rho_f, \qquad \text{and} \qquad \mu_b = \eta u + \frac{\nu}{\overline{\rho}^2}\partial_x\rho_b, \tag{1.3 .22}$$

where η denotes the cut-off function in (1.2 .13).

Tedious computations show that μ_f satisfies

$$\begin{cases} \partial_t\mu_f + \eta(\overline{u}+u)\partial_x\mu_f + k\mu_f = h - \frac{\nu}{\overline{\rho}^2}\partial_x\left(\frac{\overline{\rho}}{\nu}p'(\overline{\rho}) + \partial_x\overline{u}\right)\rho_f & \text{in } (0,T)\times(a,b), \\ \mu_f(0,\cdot) = \eta u_0 + \frac{\overline{\rho}^2}{\nu}\partial_x\rho_0 & \text{in } (a,b), \end{cases} \tag{1.3 .23}$$

and μ_b satisfies

$$\begin{cases} \partial_t\mu_b + \eta(\overline{u}+u)\partial_x\mu_b + k\mu_b = h - \frac{\nu}{\overline{\rho}^2}\partial_x\left(\frac{\overline{\rho}}{\nu}p'(\overline{\rho}) + \partial_x\overline{u}\right)\rho_b & \text{in } (0,T)\times(a,b), \\ \mu_b(T,\cdot) = 0 & \text{in } (a,b), \end{cases} \tag{1.3 .24}$$

where

$$k = \frac{\overline{\rho}}{\nu}p'(\overline{\rho}) + \partial_x\overline{u} + \partial_x(\eta(\overline{u}+u)) + 2\frac{\partial_t\overline{\rho}}{\overline{\rho}} + 2\eta(\overline{u}+u)\frac{\partial_x\overline{\rho}}{\overline{\rho}}, \tag{1.3 .25}$$

and

$$\begin{aligned} h = {}& \frac{\eta}{\overline{\rho}}(\frac{\nu}{\overline{\rho}}\partial_x\widehat{f} + \widehat{g}) + \frac{\nu}{\overline{\rho}^2}\partial_x\eta\widehat{f} + \left[\eta(\eta(\overline{u}+u) - \overline{u} - \frac{\nu}{\overline{\rho}^2}\partial_x\overline{\rho}) - \frac{\nu}{\overline{\rho}}\partial_x\eta\right]\partial_x u \\ & + \left[\frac{\overline{\rho}}{\nu}p'(\overline{\rho}) + \partial_x\overline{u} + \partial_x(\eta(\overline{u}+u)) + 2\frac{\partial_t\overline{\rho}}{\overline{\rho}} + 2\eta(\overline{u}+u)\frac{\partial_x\overline{\rho}}{\overline{\rho}} + \partial_x\eta(\overline{u}+u)\right]\eta u, \end{aligned} \tag{1.3 .26}$$

in which $\widehat{f} = f(\widehat{\rho}, \widehat{u})$ and $\widehat{g} = g(\widehat{\rho}, \widehat{u})$ have been defined in (1.2 .35)–(1.2 .36).

The quantities μ_f and μ_b in (1.3 .22) correspond to what is known as the effective velocity, and which has been used for instance in [4, 5] in the study of the well-posedness of some models of viscous compressible fluids. These quantities have also been used in [13] in order to get good estimates on the controlled density. However, the approach in [13] looks for a controlled density on the space interval $(0, L)$ and thus requires estimates on the trace of μ at $x = 0$ and $x = L$, which cannot be achieved in our case as $t \mapsto \overline{u}(t, a)$ and $t \mapsto \overline{u}(t, b)$ may vanish for some values of time. Our approach avoids this difficulty by a careful discussion of the geometry of the problem. In particular, this allows us to avoid the use of the second parameter λ in the weight function φ in (1.2 .22), contrarily to what is done in [13].

The interest of using the quantities μ_f and μ_b can be understood in terms of regularity. Indeed, when looking at the linearized version of (1.3 .3), it seems that $\partial_x \rho_f$ in $L^\infty(L^2)$ can be estimated in terms of $\partial_{xx} u$ in $L^1(L^2)$. But considering the linearized version of (1.3 .23) instead, it rather seems that μ_f in $L^\infty(L^2)$ can be estimated in terms of $\partial_x u$ in $L^1(L^2)$, and as $\partial_x \rho_f$ in $L^\infty(L^2)$ can be estimated immediately from μ_f and u in $L^\infty(L^2)$, this latter estimate seems better. The goal of the next sections is to make this argument completely rigorous.

1.3 .2.2 Estimates on the coefficients

We start by estimating the coefficients appearing in (1.3 .23)–(1.3 .24):

Lemma 1.3 .2. *There exists $C > 0$ independent of $s \geq 1$ and $R_\rho, R_u \in (0,1)$ such that the following estimates hold true for all $s \geq 1$, $R_\rho \in (0, \min\{1, \min_{[0,T]\times\mathbb{R}} \bar{\rho}/2\})$, $R_u \in (0,1)$, $(\widehat{\rho}, \widehat{u}) \in \mathscr{X}_{s,R_\rho} \times \mathscr{Y}_{s,R_\rho}$:*

$$\|\theta^{-1} e^{s\varphi} \widehat{f}\|_{L^2(0,T;L^2(a,b))} \leq \frac{C}{s}(R_\rho + R_u) + C(R_\rho^2 + R_u^2), \tag{1.3 .27}$$

$$\|\theta^{-1} e^{s\varphi} h\|_{L^2(0,T;L^2(a,b))} \leq C'(s)\|u_0\|_{H^1(a,b)} + \frac{C}{\sqrt{s}}(R_\rho + R_u) + C(R_\rho^2 + R_u^2), \tag{1.3 .28}$$

$$\|k\|_{L^1(0,T;L^\infty(a,b))} \leq C_k'(s)\|u_0\|_{H^1(a,b)} + C, \tag{1.3 .29}$$

where $\widehat{f} = f(\widehat{\rho}, \widehat{u})$ is defined in (1.2 .35), h in (1.3 .26), k in (1.3 .25) and $C'(s)$ and $C_k'(s)$ are constants depending on the parameter s.

Proof. In the proof below, we will often denote the norms by omitting the mention of the time and space intervals, e.g. $\|\cdot\|_{L^2(L^2)}$ for denoting $\|\cdot\|_{L^2(0,T;L^2(a,b))}$.
We deal with each estimate separately.

• *Proof of estimate* (1.3 .27). We derive the estimate on $\widehat{f} = f(\widehat{\rho}, \widehat{u})$ in (1.2 .35) term by term:

$$\|\theta^{-1} e^{s\varphi} \widehat{\rho} \partial_x \widehat{u}\|_{L^2(L^2)} \leq \|\widehat{\rho}\|_{L^\infty(L^\infty)} \|\theta^{-1} e^{s\varphi} \partial_x \widehat{u}\|_{L^2(L^2)} \leq C R_\rho R_u,$$
$$\|\theta^{-1} e^{s\varphi} \frac{\overline{\rho}}{\nu} p'(\overline{\rho}) \widehat{\rho}\|_{L^2(L^2)} \leq C \|e^{s\varphi} \widehat{\rho}\|_{L^2(L^2)} \leq \frac{C}{s} R_\rho,$$
$$\|\theta^{-1} e^{s\varphi} \partial_x \overline{\rho} \widehat{u}\|_{L^2(L^2)} \leq C \|\theta^{-1} e^{s\varphi} \widehat{u}\|_{L^2(L^2)} \leq \frac{C}{s^{3/2}} R_u.$$

Estimate (1.3 .27) immediately follows.

• *Proof of estimate* (1.3 .28). Taking the definition of h in (1.3 .26) and using the bound (1.3 .10), we remark that for all $(t,x) \in (0,T) \times (a,b)$,

$$|h| \leq C\Big[|\frac{\nu}{\overline{\rho}} \partial_x \widehat{f} + \widehat{g}| + |\widehat{f}| + |u| + |\partial_x u| + |u|^2 + |u \partial_x u|\Big]. \tag{1.3 .30}$$

The estimation for the term $\widehat{f}$ is already done, see (1.3 .27). The terms involving u can be estimated as follows:

$$\begin{aligned}
&\|\theta^{-1}e^{s\varphi}(|u|+|\partial_x u|+|u|^2+|u\partial_x u|)\|_{L^2(L^2)}\\
&\le \|\theta^{-1}e^{s\varphi}u\|_{L^2(L^2)}+\|\theta^{-1}e^{s\varphi}\partial_x u\|_{L^2(L^2)}+\|\theta^{-1}e^{s\varphi}u^2\|_{L^2(L^2)}+\|\theta^{-1}e^{s\varphi}u\partial_x u\|_{L^2(L^2)}\\
&\le \frac{C}{s^{3/2}}\|u\|_{\mathscr{Y}_s}+\frac{C}{s^{1/2}}\|u\|_{\mathscr{Y}_s}+\|u\|_{L^\infty(L^\infty)}\|\theta^{-1}e^{s\varphi}u\|_{L^2(L^2)}+\|u\|_{L^\infty(L^\infty)}\|\theta^{-1}e^{s\varphi}\partial_x u\|_{L^2(L^2)}\\
&\le \frac{C}{s^{1/2}}\|u\|_{\mathscr{Y}_s}+C\|u\|^2_{\mathscr{Y}_s},
\end{aligned}$$

where we used

$$\|u\|_{L^\infty(L^\infty)}\le\|u\|_{L^\infty(H^1)}\le C\|u\|^{1/2}_{H^1(L^2)}\|u\|^{1/2}_{L^2(H^2)}\le C\|u\|_{\mathscr{Y}_s}. \tag{1.3 .31}$$

Consequently, the proof of (1.3 .28) will follow first from the above estimates and (1.2 .39) from one side, and second from an estimate on $\theta^{-1}e^{s\varphi}(\nu\partial_x\widehat{f}/\overline{\rho}+\widehat{g})$, on which we will focus from now. Recalling (1.2 .35)–(1.2 .36), we obtain

$$\begin{aligned}
\frac{\nu}{\overline{\rho}}\partial_x\widehat{f}+\widehat{g}=&-\frac{\nu}{\overline{\rho}}\partial_x\widehat{\rho}\partial_x\widehat{u}-\frac{\nu}{\overline{\rho}}\widehat{\rho}\partial_{xx}\widehat{u}+\frac{\nu}{\overline{\rho}}\partial_x\Big(\frac{\overline{\rho}}{\nu}p'(\overline{\rho})\Big)\widehat{\rho}+p'(\overline{\rho})\partial_x\widehat{\rho}-\frac{\nu}{\overline{\rho}}\partial_{xx}\overline{\rho}\widehat{u}-\frac{\nu}{\overline{\rho}}\partial_x\overline{\rho}\partial_x\widehat{u}\\
&-\widehat{\rho}(\partial_t(\overline{u}+\widehat{u})+(\overline{u}+\widehat{u})\partial_x(\overline{u}+\widehat{u}))-\overline{\rho}\widehat{u}\partial_x(\overline{u}+\widehat{u})-p'(\overline{\rho}+\widehat{\rho})\partial_x\overline{\rho}+p'(\overline{\rho})\partial_x\overline{\rho}-p'(\overline{\rho}+\widehat{\rho})\partial_x\widehat{\rho}.
\end{aligned} \tag{1.3 .32}$$

We then use the following estimates:

$$\begin{aligned}
&\|\theta^{-1}e^{s\varphi}\frac{\nu}{\overline{\rho}}\partial_x\widehat{\rho}\partial_x\widehat{u}\|_{L^2(L^2)}\le C\|\theta^{-1}e^{s\varphi}\partial_x\widehat{\rho}\|_{L^\infty(L^2)}\|\partial_x\widehat{u}\|_{L^2(L^\infty)}\le CR_\rho\|\partial_{xx}u\|_{L^2(L^2)}\le CR_\rho R_u,\\
&\|\theta^{-1}e^{s\varphi}\frac{\nu}{\overline{\rho}}\widehat{\rho}\partial_{xx}\widehat{u}\|_{L^2(L^2)}\le C\|\widehat{\rho}\|_{L^\infty(L^\infty)}\|\theta^{-1}e^{s\varphi}\partial_{xx}\widehat{u}\|_{L^2(L^2)}\le CR_\rho R_u,\\
&\|\theta^{-1}e^{s\varphi}\frac{\nu}{\overline{\rho}}\partial_x\Big(\frac{\overline{\rho}}{\nu}p'(\overline{\rho})\Big)\widehat{\rho}\|_{L^2(L^2)}\le C\|e^{s\varphi}\widehat{\rho}\|_{L^2(L^2)}\le\frac{C}{s}R_\rho,\\
&\|\theta^{-1}e^{s\varphi}(p'(\overline{\rho})\partial_x\widehat{\rho}-p'(\overline{\rho}+\widehat{\rho})\partial_x\widehat{\rho})\|_{L^2(L^2)}\le C\|\widehat{\rho}\|_{L^\infty(L^\infty)}\|\theta^{-1}e^{s\varphi}\partial_x\widehat{\rho}\|_{L^2(L^2)}\le CR_\rho^2, \qquad (1.3\ .33)\\
&\|\theta^{-1}e^{s\varphi}\frac{\nu}{\overline{\rho}}\partial_{xx}\overline{\rho}\widehat{u}\|_{L^2(L^2)}\le C\|\theta^{-1}e^{s\varphi}\widehat{u}\|_{L^2(L^2)}\le\frac{C}{s^{3/2}}R_u,\\
&\|\theta^{-1}e^{s\varphi}\frac{\nu}{\overline{\rho}}\partial_x\overline{\rho}\partial_x\widehat{u}\|_{L^2(L^2)}\le C\|\theta^{-1}e^{s\varphi}\partial_x\widehat{u}\|_{L^2(L^2)}\le\frac{C}{s^{1/2}}R_u,\\
&\|\theta^{-1}e^{s\varphi}(-p'(\overline{\rho}+\widehat{\rho})\partial_x\overline{\rho}+p'(\overline{\rho})\partial_x\overline{\rho})\|_{L^2(L^2)}\le C\|\theta^{-1}e^{s\varphi}\widehat{\rho}\|_{L^2(L^2)}\le\frac{C}{s}R_\rho.
\end{aligned}$$

For this last estimate, we have used $p'\in C^1(\mathbb{R}_+^*;\mathbb{R})$ and $\|\rho\|_{L^\infty(L^\infty)}\le R_\rho\le\min_{[0,T]\times\mathbb{R}}\bar{\rho}/2$. Using (1.3 .31) for $\widehat{u}$, we also get

$$\begin{aligned}
&\|\theta^{-1}e^{s\varphi}\widehat{\rho}(\partial_t(\overline{u}+\widehat{u})+(\overline{u}+\widehat{u})\partial_x(\overline{u}+\widehat{u}))\|_{L^2(L^2)}\\
&\le\frac{C}{s}R_\rho+C\|s^{1/2}\theta\widehat{\rho}\|_{L^\infty(L^\infty)}\|s^{-1/2}\theta^{-2}e^{s\varphi}\partial_t\widehat{u}\|_{L^2(L^2)}\\
&\qquad+C\|\widehat{\rho}\|_{L^\infty(L^\infty)}\left(\|\theta^{-1}e^{s\varphi}(|\widehat{u}|+|\partial_x\widehat{u}|)\|_{L^2(L^2)}+\|\widehat{u}\|_{L^\infty(L^\infty)}\|\theta^{-1}e^{s\varphi}\partial_x\widehat{u}\|_{L^2(L^2)}\right)\\
&\le\frac{C}{s}R_\rho+CR_\rho R_u+CR_\rho R_u^2,
\end{aligned} \tag{1.3 .34}$$

and

$$\begin{aligned}
\|\theta^{-1}e^{s\varphi}\overline{\rho}\widehat{u}\partial_x(\overline{u}+\widehat{u})\|_{L^2(L^2)}&\le C\|\theta^{-1}e^{s\varphi}\widehat{u}\partial_x\overline{u}\|_{L^2(L^2)}+C\|\theta^{-1}e^{s\varphi}\widehat{u}\partial_x\widehat{u}\|_{L^2(L^2)}\\
&\le C\|\theta^{-1}e^{s\varphi}\widehat{u}\|_{L^2(L^2)}+C\|\widehat{u}\|_{L^\infty(L^\infty)}\|\theta^{-1}e^{s\varphi}\partial_x\widehat{u}\|_{L^2(L^2)}\le\frac{C}{s^{3/2}}R_u+CR_u^2.
\end{aligned} \tag{1.3 .35}$$

Combining the above estimates yields (1.3 .28).

• *Proof of estimate* (1.3 .29). From the definition of k in (1.3 .25), we have

$$|k| \leq C(1 + |u| + |\partial_x u|).$$

Therefore,

$$\|k\|_{L^1(L^\infty)} \leq C + \|u\|_{L^1(L^\infty)} + \|\partial_x u\|_{L^1(L^\infty)} \leq C + \|u\|_{L^2(H^1)} + \|u\|_{L^2(H^2)} \leq C + C\|u\|_{\mathscr{Y}_s}.$$

Using Theorem 1.2 .2 and $R_\rho, R_u \leq 1$, we deduce (1.3 .29). □

Remark 1.3 .3. *Let us point out that the estimate on h in (1.3 .28) is based on the fact that the combination of the terms $p'(\overline{\rho})\partial_x\widehat{\rho}$ and $p'(\overline{\rho}+\widehat{\rho})\partial_x\widehat{\rho}$ coming from the pressure in $\nu\partial_x\widehat{f}/\overline{\rho}+\widehat{g}$ cancels out at first order in $\widehat{\rho}$, see (1.3 .33). This cancellation motivates the introduction of the term $\overline{\rho}p'(\overline{\rho})\rho/\nu$ in the left hand side of (1.2 .34)$_{(1)}$ and $\overline{\rho}p'(\overline{\rho})\widehat{\rho}/\nu$ in the source term $f(\widehat{\rho},\widehat{u})$ in (1.2 .35).*

1.3 .2.3 Energy Lemma

In order to get estimates on μ_f solving (1.3 .23) and μ_b solving (1.3 .24), we remark that both quantities μ_f and μ_b satisfy transport-type equation. Therefore, in this section we explain how to derive weighted estimates on μ_f and μ_b using weighted energy methods. It turns out that we will only be able to get good $L^\infty(L^2)$ estimates on $\theta^{-1}e^{s\varphi}\mu_f$ on the time interval $(0, T_L)$ and on $\theta^{-1}e^{s\varphi}\mu_b$ on the time interval (T_0, T) as $\partial_t\theta$ has constant sign on each of these intervals (recall (1.2 .21)).

We start with the estimates on the time interval $(0, T_L)$.

Lemma 1.3 .4. *There exists $C > 0$ such that for all $s \geq 1$, for all u such that $\theta u \in L^1(0, T_L; W^{1,\infty}(a,b))$, for all $K \in L^1(0, T_L; L^\infty(a,b))$, for all $H \in L^2(0, T_L; L^2(a,b))$ with*

$$\|\theta^{-1}e^{s\varphi}H\|_{L^2(0,T_L;L^2(a,b))} < +\infty, \tag{1.3 .36}$$

and for all $c_0 \in L^2(a,b)$, the solution c of

$$\begin{cases} \partial_t c + \eta(\overline{u}+u)\partial_x c + Kc = H & \text{in } (0,T_L)\times(a,b), \\ c(0,\cdot) = c_0 & \text{in } (a,b), \end{cases} \tag{1.3 .37}$$

satisfies

$$\begin{aligned} &\|\theta^{-1}e^{s\varphi}c\|_{L^\infty(0,T_L;L^2(a,b))} \\ &\leq Ce^{C(1+\|\theta u\|_{L^1(0,T_L;W^{1,\infty}(a,b))}+\|K\|_{L^1(0,T_L;L^\infty(a,b))})}\left(\|\theta^{-1}e^{s\varphi}H\|_{L^2(0,T_L;L^2(a,b))} + C'(s)\|c_0\|_{L^2(a,b)}\right), \end{aligned} \tag{1.3 .38}$$

where $C'(s)$ depends on the parameter s.

Proof. Multiplying the equation (1.3 .37) by $\theta^{-2}e^{2s\varphi}c$ and integrating in space, we get:

$$\begin{aligned} \frac{d}{dt}\left(\int_a^b \theta^{-2}e^{2s\varphi}|c|^2dx\right) &= \int_a^b \left(\partial_t(\theta^{-2}e^{2s\varphi}) + \partial_x(\theta^{-2}e^{2s\varphi}\eta(\overline{u}+u))\right)|c|^2dx \\ &\quad - 2\int_a^b \theta^{-2}e^{2s\varphi}K|c|^2dx + 2\int_a^b \theta^{-2}e^{2s\varphi}Hc\,dx. \end{aligned} \tag{1.3 .39}$$

Using then the choice of the weight function ψ in (1.2 .1), which satisfies the transport equation (1.2 .14), θ in (1.2 .21) and φ in (1.2 .22),

$$\begin{aligned}
&\partial_t(\theta^{-2}e^{2s\varphi}) + \partial_x(\theta^{-2}e^{2s\varphi}\eta(\overline{u}+u)) = \partial_t(\theta^{-2}e^{2s\varphi}) + \eta(\overline{u}+u)\partial_x(\theta^{-2}e^{2s\varphi}) + \partial_x(\eta(\overline{u}+u))\theta^{-2}e^{2s\varphi}\\
&= 2\partial_t\theta\theta^{-3}e^{2s\varphi}(-1+s\varphi) - 2s\lambda e^{\lambda(\psi+6)}u\partial_x\psi\theta^{-1}e^{2s\varphi} + \partial_x(\eta(\overline{u}+u))\theta^{-2}e^{2s\varphi}\\
&\leq C(1+\|\theta u(t)\|_{W^{1,\infty}(a,b)})\theta^{-2}e^{2s\varphi},
\end{aligned}$$

as $\partial_t\theta(t) \leq 0$ for $t \in (0,T_L)$ and $s\varphi \geq 2$ for $s \geq 1$.

Therefore, (1.3 .39) yields:

$$\frac{d}{dt}\left(\int_a^b \theta^{-2}e^{2s\varphi}|c|^2dx\right) \leq C\left(1+\|\theta u(t)\|_{W^{1,\infty}(a,b)} + \|K(t)\|_{L^\infty(a,b)}\right)\int_a^b \theta^{-2}e^{2s\varphi}|c|^2dx + \int_a^b \theta^{-2}e^{2s\varphi}H^2dx. \quad (1.3\ .40)$$

The estimate (1.3 .38) easily follows from Gronwall's Lemma. □

Using now that $\partial_t\theta(t) \geq 0$ for all $t \in (T_0,T)$ by construction, recall (1.2 .21), we get the following counterpart of Lemma 1.3 .4:

Lemma 1.3 .5. *There exists $C > 0$ such that for all $s \geq 1$, for all u such that $\theta u \in L^1(T_0,T;W^{1,\infty}(a,b))$, for all $K \in L^1(T_0,T;L^\infty(a,b))$, for all $H \in L^2(T_0,T;L^2(a,b))$ with*

$$\|\theta^{-1}e^{s\varphi}H\|_{L^2(T_0,T;L^2(a,b))} < +\infty, \quad (1.3\ .41)$$

the solution c of

$$\begin{cases} \partial_t c + \eta(\overline{u}+u)\partial_x c + Kc = H & \text{in } (T_0,T)\times(a,b),\\ c(T,\cdot) = 0 & \text{in } (a,b), \end{cases} \quad (1.3\ .42)$$

satisfies

$$\|\theta^{-1}e^{s\varphi}c\|_{L^\infty(T_0,T;L^2(a,b))} \leq Ce^{C(1+\|\theta u\|_{L^1(T_0,T;W^{1,\infty}(a,b))}+\|K\|_{L^1(T_0,T;L^\infty(a,b))})}\|\theta^{-1}e^{s\varphi}H\|_{L^2(T_0,T;L^2(a,b))}. \quad (1.3\ .43)$$

Proof. The proof is exactly the same as Lemma 1.3 .4. The only minor difference is that φ is singular at the time $t = T$. In order to avoid this difficulty, we introduce $\theta_\delta(t) = \theta(t-\delta)$ for $t \in [T_0+\delta, T]$ and $\theta_\delta(t) = 1$ for $t \in [T_0, T_0+\delta]$. We thus prove the estimate (1.3 .43) with φ replaced by $\varphi_\delta(t,x) = \theta_\delta(t)(\lambda e^{12\lambda} - e^{\lambda(\psi(t,x)+6)})$, uniformly with respect to the parameter $\delta > 0$, and we pass to the limit $\delta \to 0$. □

1.3 .2.4 Proof of Theorem 1.2 .3

We are then in position to prove Theorem 1.2 .3.

We start by choosing $\varepsilon \in (0,\varepsilon(s))$ with

$$\varepsilon(s) = \min\{1/C'_u(s), 1/C'_k(s)\},$$

where $C'_u(s)$ and $C'_k(s)$ are the constants in (1.2 .39) and (1.3 .29), so that $\|u\|_{\mathscr{Y}_s} \leq C$ and $\|k\|_{L^1(0,T;L^\infty(a,b))} \leq C$ where k is defined in (1.3 .25). Remark that the continuity of the embedding $H^1(0,T;L^2(a,b)) \cap L^2(0,T;H^2(a,b))$ into $L^1(0,T;W^{1,\infty}(a,b)) \cap L^\infty(0,T;L^\infty(a,b))$ implies

$$\|\theta u\|_{L^1(W^{1,\infty})\cap L^\infty(L^\infty)} \leq C. \quad (1.3\ .44)$$

Indeed, introduce

$$\forall t \in [0,T), \quad \theta_0(t) = \begin{cases} 1 & \text{for } t \in [0,T_0], \\ \theta(t) & \text{for } t \in [T_0,T]. \end{cases}$$

Remark that $\theta_0 \le \theta$ and $0 \le \partial_t \theta_0 \le C\theta_0^2$ where C is independent of s. We can then easily check that $\|e^{\theta_0} u\|_{L^2(H^2)\cap H^1(L^2)} \le C\|u\|_{\mathscr{Y}_s}$. Inequality (1.3 .44) follows from the continuity of the embedding $H^1(0,T;L^2(a,b)) \cap L^2(0,T;H^2(a,b))$ into $L^1(0,T;W^{1,\infty}(a,b)) \cap L^\infty(0,T;L^\infty(a,b))$ and $e^{\theta_0} \ge \theta$.

Recall then that ρ_f and ρ_b have been constructed as solutions of (1.3 .3) and (1.3 .4) respectively, with source term $f = \widehat{f}$. Using then Lemma 1.3 .2 and Lemma 1.3 .4 for ρ_f, or Lemma 1.3 .5 for ρ_b, we get that $\theta^{-1} e^{s\varphi} \rho_f \in L^\infty(0,T_L;L^2(a,b))$ and $\theta^{-1} e^{s\varphi}\rho_b \in L^\infty(T_0,T;L^2(a,b))$.

Using Lemma 1.3 .2, we can apply Lemma 1.3 .4 to the solution μ_f of (1.3 .23) with $K = k$ in (1.3 .25),

$$H = h - \frac{\nu}{\overline{\rho}^2}\partial_x \left(\frac{\overline{\rho}}{\nu} p'(\overline{\rho}) + \partial_x \overline{u}\right)\rho_f,$$

where h is defined in (1.3 .26) and $c = \mu_f$ in (1.3 .23):

$$\begin{aligned}
&\|\theta^{-1} e^{s\varphi} \mu_f\|_{L^\infty(0,T_L;L^2(a,b))} \\
&\le C\left(\|\theta^{-1} e^{s\varphi} h\|_{L^2(0,T_L;L^2(a,b))} + \|\theta^{-1} e^{s\varphi}\rho_f\|_{L^2(0,T_L;L^2(a,b))} + C'(s)\|\mu_f(0,\cdot)\|_{L^2(a,b)}\right) \\
&\le C'(s)\varepsilon + \frac{C}{\sqrt{s}}(R_\rho + R_u) + C(R_\rho^2 + R_u^2) + \|\theta^{-1} e^{s\varphi}\rho_f\|_{L^2(0,T_L;L^2(a,b))}.
\end{aligned}$$

Using the definition of μ_f in (1.3 .22), we deduce

$$\begin{aligned}
&\|\theta^{-1} e^{s\varphi}\partial_x \rho_f\|_{L^\infty(0,T_L;L^2(a,b))} \\
&\qquad \le \frac{C}{\sqrt{s}}\|u\|_{\mathscr{Y}_s} + C'(s)\varepsilon + \frac{C}{\sqrt{s}}(R_\rho + R_u) + C(R_\rho^2 + R_u^2) + \|\theta^{-1} e^{s\varphi}\rho_f\|_{L^2(0,T_L;L^2(a,b))}.
\end{aligned}$$

We then use the weighted Poincaré inequality in Lemma 1.A.1 for ρ_f (recall that ρ_f vanishes at $x = a$ and $x = b$, see (1.3 .5)): for $s \ge s_1$ large enough,

$$\begin{aligned}
&s\|e^{s\varphi}\rho_f\|_{L^\infty(0,T_L;L^2(a,b))} + \|\theta^{-1} e^{s\varphi}\partial_x\rho_f\|_{L^\infty(0,T_L;L^2(a,b))} \\
&\le \frac{C}{\sqrt{s}}\|u\|_{\mathscr{Y}_s} + C'(s)\varepsilon + \frac{C}{\sqrt{s}}(R_\rho + R_u) + C(R_\rho^2 + R_u^2) \le C'(s)\varepsilon + \frac{C}{\sqrt{s}}(R_\rho + R_u) + C(R_\rho^2 + R_u^2),
\end{aligned}$$

where the last estimate comes from (1.2 .39).

Using then the equation (1.3 .3) satisfied by ρ_f and the estimates (1.3 .27) on $f = \widehat{f}$ and (1.2 .39) on $\partial_x u$, we deduce

$$\begin{aligned}
&s\|e^{s\varphi}\rho_f\|_{L^\infty(0,T_L;L^2(a,b))} + \|\theta^{-1} e^{s\varphi}\partial_x\rho_f\|_{L^\infty(0,T_L;L^2(a,b))} + \|\partial_t \rho_f\|_{L^2(0,T_L;L^2(a,b))} \\
&\le \frac{C}{\sqrt{s}}\|u\|_{\mathscr{Y}_s} + C'(s)\varepsilon + \frac{C}{\sqrt{s}}(R_\rho + R_u) + C(R_\rho^2 + R_u^2) \le C'(s)\varepsilon + \frac{C}{\sqrt{s}}(R_\rho + R_u) + C(R_\rho^2 + R_u^2),
\end{aligned}$$

We also have

$$\begin{aligned}
\|e^{s\varphi/2}\rho_f\|_{L^\infty(0,T_L;L^\infty(a,b))} &\le C\|e^{s\varphi/2}\rho_f\|_{L^\infty(0,T_L;H^1(a,b))} \\
&\le Cs\|\theta e^{s\varphi/2}\rho_f\|_{L^\infty(0,T_L;L^2(a,b))} + C\|e^{s\varphi/2}\partial_x\rho_f\|_{L^\infty(0,T_L;L^2(a,b))} \\
&\le Cs\|e^{s\varphi}\rho_f\|_{L^\infty(0,T_L;L^2(a,b))} + C\|\theta^{-1} e^{s\varphi}\partial_x\rho_f\|_{L^\infty(0,T_L;L^2(a,b))}.
\end{aligned}$$

Combining the above estimates on ρ_f we get

$$s\|e^{s\varphi}\rho_f\|_{L^\infty(0,T_L;L^2(a,b))} + \|\theta^{-1}e^{s\varphi}\partial_x\rho_f\|_{L^\infty(0,T_L;L^2(a,b))} + \|e^{s\varphi/2}\rho_f\|_{L^\infty(0,T_L;L^\infty(a,b))} + \|\partial_t\rho_f\|_{L^2(0,T_L;L^2(a,b))} \leq C'(s)\varepsilon + \frac{C}{\sqrt{s}}(R_\rho + R_u) + C(R_\rho^2 + R_u^2). \quad (1.3\ .45)$$

Similar computations can be done for μ_b based on Lemma 1.3 .2, Lemma 1.3 .5, the boundary conditions (1.3 .6) satisfied by ρ_b and the weighted Poincaré inequality in Lemma 1.A.1. That way, we achieve:

$$s\|e^{s\varphi}\rho_b\|_{L^\infty(T_0,T;L^2(a,b))} + \|\theta^{-1}e^{s\varphi}\partial_x\rho_b\|_{L^\infty(T_0,T;L^2(a,b))} + \|e^{s\varphi/2}\rho_b\|_{L^\infty(T_0,T;L^\infty(a,b))} + \|\partial_t\rho_b\|_{L^2(T_0,T;L^2(a,b))} \leq C'(s)\varepsilon + \frac{C}{\sqrt{s}}(R_\rho + R_u) + C(R_\rho^2 + R_u^2). \quad (1.3\ .46)$$

We then recall that ρ is given by (1.3 .11) with $\tilde\eta$ constructed in Lemma 1.3 .1 and satisfying the bound (1.3 .10). Using then the above estimates (1.3 .45)–(1.3 .46) and (1.3 .14)–(1.3 .15), we get (1.2 .42).

1.4 Control of the velocity

The purpose of this section is to present the main ingredients of the proof of Theorem 1.2 .2, which mainly consists in a suitable adaptation of [3, Theorems 2.5 and 2.6].

1.4 .1 Construction of a controlled trajectory u

For the time being let us fix $g \in L^2(0,T;L^2(a,b))$ and $u_0 \in H_0^1(a,b)$, and consider the following controllability problem: Find a control function $v_u \in L^2(0,T)$ such that the solution u of

$$\begin{cases} \overline{\rho}\partial_t u + \overline{\rho u}\partial_x u - \nu\partial_{xx}u = g & \text{in } (0,T)\times(a,b), \\ u(t,a) = v_u & \text{in } (0,T), \\ u(t,b) = 0 & \text{in } (0,T), \\ u(0,\cdot) = u_0 & \text{in } (a,b), \end{cases} \quad (1.4\ .1)$$

satisfies the controllability requirement (1.2 .38).

We then claim the following result, strongly inspired in [3, Theorems 2.5 and 2.6]:

Theorem 1.4 .1. *There exist $C > 0$ and $s_0 \geq 1$ such that for all $s \geq s_0$, for all $g \in L^2(0,T;L^2(a,b))$ with*

$$\|\theta^{-3/2}e^{s\varphi}g\|_{L^2(0,T;L^2(a,b))} < +\infty, \quad (1.4\ .2)$$

for all $u_0 \in H_0^1(a,b)$, there exists a controlled trajectory u of (1.4 .1) satisfying (1.2 .38) with the estimate

$$\|u\|_{\mathscr{Y}_s} \leq C'(s)\|u_0\|_{H_0^1(a,b)} + C\|\theta^{-3/2}e^{s\varphi}g\|_{L^2(0,T;L^2(a,b))}. \quad (1.4\ .3)$$

where $C'(s)$ depends on the parameter s.

Sketch of the proof. As in [13, Section 3], we first extend the domain (a,b) into $(2a,b)$ (recall $a<0$), with g and u_0 both extended by 0 on $(2a,a)$, and instead of solving the controllability problem (1.4 .1), (1.2 .38) directly, we consider a distributed control v supported in space in $(2a,a)$. We therefore focus on the following system:

$$\begin{cases} \overline{\rho}\partial_t u + \overline{\rho u}\partial_x u - \nu\partial_{xx}u = g + v\chi_{(2a,a)} & \text{in } (0,T)\times(2a,b), \\ u(t,2a) = u(t,b) = 0 & \text{in } (0,T), \\ u(0,\cdot) = u_0 & \text{in } (2a,b). \end{cases} \quad (1.4\ .4)$$

Here, v is the control function and $\chi_{(2a,a)} = \chi_{(2a,a)}(x)$ is the indicator of the space interval $(2a, a)$: $\chi_{(2a,a)}(x) = 1$ for $x \in (2a, a)$ and $\chi_{(2a,a)}(x) = 0$ for $x \in (a, b)$.

Our purpose now is to find $v \in L^2(0, T; L^2(2a, a))$ such that the solution u of (1.4 .4) satisfies

$$u(T, \cdot) = 0 \quad \text{in } (2a, b). \tag{1.4 .5}$$

If we have such a controlled trajectory, then u restricted to (a, b) provides a solution of the controllability problem (1.4 .1), (1.2 .38).

We then use the following weighted observability result, obtained in [3, Theorem 2.5]:

Theorem 1.4 .2. *There exist $C > 0$, $s_0 \geq 1$ and $\lambda \geq 1$ large enough such that for all $z \in C^\infty([0, T] \times [2a, b])$ with $z(t, 2a) = z(t, b) = 0$ for all $t \in (0, T)$ and for all $s \geq s_0$, we have*

$$\begin{aligned} s^3 \int\int_{(0,T)\times(2a,b)} \theta^3 e^{-2s\varphi} |z|^2 \, dtdx + s^2 \int_{2a}^{b} e^{-2s\varphi(0,\cdot)} |z(0, \cdot)|^2 dx \\ + s \int\int_{(0,T)\times(2a,b)} \theta e^{-2s\varphi} |\partial_x z|^2 \, dtdx \leq C \int\int_{(0,T)\times(2a,b)} e^{-2s\varphi} |(-\overline{\rho}\partial_t - \nu \partial_{xx}) z|^2 \, dtdx \\ + Cs^3 \int\int_{(0,T)\times(2a,a)} \theta^3 e^{-2s\varphi} |z|^2 \, dtdx. \end{aligned} \tag{1.4 .6}$$

As it is done classically for control problems, see e.g. [16], we then use duality. From now, our approach follows very closely the one in [3, Theorem 2.6 and Appendix A.2]. Namely, for $z \in C^\infty([0, T] \times [2a, b])$ with $z(t, 2a) = z(t, b) = 0$ for all $t \in (0, T)$, we define

$$\begin{aligned} J(z) = \frac{1}{2} \int\int_{(0,T)\times(2a,b)} e^{-2s\varphi} |(-\partial_t(\overline{\rho} z) - \partial_x(\overline{\rho}\overline{u} z) - \nu \partial_{xx} z|^2 dtdx \\ + \frac{s^3}{2} \int\int_{(0,T)\times(2a,a)} \theta^3 e^{-2s\varphi} |z|^2 dtdx - \int_{2a}^{b} \overline{\rho} u_0 z(0) dx - \int\int_{(0,T)\times(2a,b)} gz dtdx. \end{aligned} \tag{1.4 .7}$$

According to Theorem 1.4 .2, the assumptions (1.1 .4) and condition (1.4 .2), for s large enough, the functional J can be extended as a continuous, strictly convex and coercive function on the set

$$Y_{obs} = \overline{\{z \in C^\infty([0, T] \times [2a, b]) \text{ with } z(t, 2a) = z(t, b) = 0 \text{ for all } t \in (0, T)\}}^{\|\cdot\|_{obs}}$$

where $\| \cdot \|_{obs}$ is given by

$$\|z\|_{obs}^2 = \int\int_{(0,T)\times(2a,b)} e^{-2s\varphi} | - \partial_t(\overline{\rho} z) - \partial_x(\overline{\rho}\overline{u} z) - \nu \partial_{xx} z|^2 + s^3 \int\int_{(0,T)\times(2a,a)} \theta^3 e^{-2s\varphi} |z|^2.$$

Consequently, J admits a minimum z_{min} on Y_{obs}. Using the Euler-Lagrange equation for J at z_{min}, (u, v) defined by

$$\begin{cases} u = e^{-2s\varphi}(-\partial_t(\overline{\rho} z_{min}) - \partial_x(\overline{\rho}\overline{u} z_{min}) - \nu \partial_{xx} z_{min}) & \text{in } (0, T) \times (2a, b), \\ v = -s^3 \theta^3 e^{-2s\varphi} z_{min} & \text{in } (0, T) \times (2a, a), \end{cases} \tag{1.4 .8}$$

solves (1.4 .4)–(1.4 .5), see [3, Theorem 2.6].

The coercivity of the functional J in (1.4 .7) immediately yields an estimate on the $L^2(L^2)$-norm of $e^{s\varphi} u$ and $\theta^{-3/2} e^{s\varphi} v$ in terms of the $L^2(L^2)$-norm of $\theta^{-3/2} e^{s\varphi} g$ and the L^2 norm of $e^{s\varphi(0,\cdot)} u_0$:

$$\begin{aligned} s^{3/2} \|e^{s\varphi} u\|_{L^2(0,T;L^2(2a,b))} + \|\theta^{-3/2} e^{s\varphi} v\|_{L^2(0,T;L^2(2a,a))} \\ \leq C \|\theta^{-3/2} e^{s\varphi} g\|_{L^2(0,T;L^2(2a,b))} + Cs^{1/2} \|e^{s\varphi(0,\cdot)} u_0\|_{L^2(2a,b)}. \end{aligned}$$

The computations to get the $L^2(L^2)$ estimates on $\theta^{-1}e^{s\varphi}\partial_x u$, $\theta^{-2}e^{s\varphi}\partial_{xx}u$ and $\theta^{-2}e^{s\varphi}\partial_t u$ closely follows the ones in [3, Appendix A.2]. The only difference is that [3] considers an initial datum $u_0 = 0$ while we are not. This introduces boundary terms in time $t = 0$ when doing the weighted energy estimates [3, Appendix A.2], which are all bounded by the $H^1_0(a,b)$-norm of $u_0 e^{s\varphi(0,\cdot)}$. To be more precise, following [3, Appendix A.2], we get

$$\begin{aligned} s^{3/2}\|e^{s\varphi}u\|_{L^2(0,T;L^2(2a,b))} &+ s^{1/2}\|\theta^{-1}e^{s\varphi}\partial_x u\|_{L^2(0,T;L^2(a,b))} \\ &+ \|\theta^{-3/2}e^{s\varphi}v\|_{L^2(0,T;L^2(2a,a))} + s^{-1/2}\|\theta^{-2}e^{s\varphi}(\partial_t u, \partial_{xx}u)\|_{L^2(0,T;L^2(a,b))} \\ &\leq C\|\theta^{-3/2}e^{s\varphi}g\|_{L^2(0,T;L^2(2a,b))} + Cs^{1/2}\|e^{s\varphi(0,\cdot)}u_0\|_{H^1_0(2a,b)}. \end{aligned} \tag{1.4 .9}$$

In order to conclude (1.4 .3), we shall also get an $L^\infty(0,T;L^2(a,b))$ norm of $\theta^{-1}e^{s\varphi}u$ which has not been derived in [3, Theorem 2.6], though this estimate can also be obtained by weighted energy estimates. Indeed, if we multiply (1.4 .4) by $s\theta^{-2}e^{2s\varphi}u$ and integrate on $(0,t)\times(2a,b)$, we easily obtain:

$$\begin{aligned} s\int_{2a}^b \overline{\rho}(t)\theta(t)^{-2}e^{2s\varphi(t)}|u(t)|^2dx &= s\int\int_{(0,t)\times(2a,b)} \partial_t(\overline{\rho}\theta^{-2}e^{2s\varphi})u^2 \\ &- 2s\int\int_{(0,t)\times(2a,b)} \overline{\rho u}\theta^{-2}e^{2s\varphi}u\partial_x u + 2s\nu\int\int_{(0,t)\times(2a,b)} \overline{\rho}\theta^{-2}e^{2s\varphi}u\partial_{xx}u \\ &+ 2s\int\int_{(0,t)\times(2a,b)} \theta^{-2}e^{2s\varphi}(g + v\chi_{(2a,a)})u + s\int_{2a}^b \overline{\rho}(0,\cdot)\theta(0)^{-2}e^{2s\varphi(0,\cdot)}|u_0|^2dx. \end{aligned} \tag{1.4 .10}$$

Using then (1.4 .9) and the fact that

$$\partial_t(\theta^{-2}e^{2s\varphi}) \leq Cse^{2s\varphi} \text{ on } (0,T),$$

we easily conclude (1.4 .3). □

1.4 .2 Estimates on $\widehat{g} = g(\widehat{\rho},\widehat{u})$ in (1.2 .36)

We wish to apply Theorem 1.4 .1 to $g = \widehat{g} = g(\widehat{\rho},\widehat{u})$ defined in (1.2 .36). We shall therefore show that for $(\widehat{\rho},\widehat{u}) \in \mathscr{X}_{s,R_\rho}\times\mathscr{Y}_{s,R_u}$, $\widehat{g}$ satisfies (1.4 .2):

Lemma 1.4 .3. *There exists $C > 0$ such that for all $s \geq 1$, $R_\rho \in (0, \min\{1, \min_{[0,T]\times\mathbb{R}}\overline{\rho}/2\})$, $R_u \in (0,1)$ and $(\widehat{\rho},\widehat{u}) \in \mathscr{X}_{s,R_\rho}\times\mathscr{Y}_{s,R_u}$, $\widehat{g} = g(\widehat{\rho},\widehat{u})$ in (1.2 .36) satisfies*

$$\|\theta^{-3/2}e^{s\varphi}\widehat{g}\|_{L^2(0,T;L^2(a,b))} \leq CR_\rho + \frac{C}{s}R_u + CR_u^2. \tag{1.4 .11}$$

Proof. Some terms in $\widehat{g}$ were already estimated in (1.3 .34)–(1.3 .35):

$$\begin{aligned} &\|\theta^{-3/2}e^{s\varphi}\widehat{\rho}(\partial_t(\overline{u}+\widehat{u}) + (\overline{u}+\widehat{u})\partial_x(\overline{u}+\widehat{u}))\|_{L^2(L^2)} + \|\theta^{-3/2}e^{s\varphi}\overline{\rho}\widehat{u}\partial_x(\overline{u}+\widehat{u})\|_{L^2(L^2)} \\ &\leq \|\theta^{-1}e^{s\varphi}\widehat{\rho}(\partial_t(\overline{u}+\widehat{u}) + (\overline{u}+\widehat{u})\partial_x(\overline{u}+\widehat{u}))\|_{L^2(L^2)} + \|\theta^{-1}e^{s\varphi}\overline{\rho}\widehat{u}\partial_x(\overline{u}+\widehat{u})\|_{L^2(L^2)} \\ &\leq \frac{C}{s}(R_\rho + R_u) + CR_u^2. \end{aligned}$$

We then only have to estimate the remaining terms in $\widehat{g}$. Since $p' \in C^1(\mathbb{R}^*_+;\mathbb{R})$ and $\|\widehat{\rho}\|_{L^\infty(L^\infty)} \leq R_\rho \leq \min_{[0,T]\times\mathbb{R}}\overline{\rho}/2$, one has

$$\begin{aligned} &\|\theta^{-3/2}e^{s\varphi}\left(-p'(\overline{\rho}+\widehat{\rho})\partial_x(\overline{\rho}+\widehat{\rho}) + p'(\overline{\rho})\partial_x\overline{\rho}\right)\|_{L^2(L^2)} \\ &\leq \|\theta^{-3/2}e^{s\varphi}p'(\overline{\rho}+\widehat{\rho})\partial_x\widehat{\rho}\|_{L^2(L^2)} + \|\theta^{-3/2}e^{s\varphi}(p'(\overline{\rho}) - p'(\overline{\rho}+\widehat{\rho}))\partial_x\overline{\rho})\|_{L^2(L^2)} \\ &\leq C(1 + \|\widehat{\rho}\|_{L^\infty(L^\infty)})\|\theta^{-3/2}e^{s\varphi}\partial_x\widehat{\rho}\|_{L^2(L^2)} + C\|\theta^{-3/2}e^{s\varphi}\widehat{\rho}\|_{L^2(L^2)} \\ &\leq CR_\rho + CR_\rho^2 \leq CR_\rho. \end{aligned}$$

Combining the above estimates yields (1.4 .11) and concludes the proof of Lemma 1.4 .3. $\square$

1.4 .3 End of the proof of Theorem 1.2 .2

For $s \geq 1$, $R_\rho \in (0, \min\{1, \min_{[0,T]\times\mathbb{R}} \bar{\rho}/2\})$, $R_u \in (0,1)$, $(\widehat{\rho}, \widehat{u}) \in \mathscr{X}_{s,R_\rho} \times \mathscr{Y}_{s,R_u}$, and $\widehat{g} = g(\widehat{\rho}, \widehat{u})$ in (1.2 .36), applying Lemma 1.4 .3, $\widehat{g}$ satisfies (1.4 .2). We can then apply Theorem 1.4 .1 to $g = \widehat{g}$ and concludes Theorem 1.2 .2 simply by putting together estimates (1.4 .3) and (1.4 .11).

1.5 The fixed point argument

Theorems 1.2 .2 and 1.2 .3 allow to define, for $s \geq s_1$ large enough, $R_\rho \in (0, \min\{1, \min_{[0,T]\times\mathbb{R}} \bar{\rho}/2\})$, $R_u \in (0,1)$ and $\varepsilon \in (0, \varepsilon(s))$ in (1.2 .23) small enough, a map $\mathscr{F} : (\widehat{\rho}, \widehat{u}) \in \mathscr{X}_{s,R_\rho} \times \mathscr{Y}_{s,R_u} \mapsto (\rho, u) \in \mathscr{X}_s \times \mathscr{Y}_s$, where

- u is the solution of the control problem (1.2 .37)–(1.2 .38) given by Theorem 1.2 .2 with $g(\widehat{\rho}, \widehat{u})$ defined in (1.2 .36),
- ρ is the solution of the control problem (1.2 .40)–(1.2 .41) given by Theorem 1.2 .3 with $f(\widehat{\rho}, \widehat{u})$ defined in (1.2 .35).

We then choose the parameters $s \geq s_1$, $R_\rho \in (0, \min\{1, \min_{[0,T]\times\mathbb{R}} \bar{\rho}/2\})$, $R_u \in (0,1)$ and $\varepsilon \in (0, \varepsilon(s)]$ in (1.2 .23) where $\varepsilon(s)$ is given in Theorem 1.2 .3, such that $\mathscr{F}$ maps $\mathscr{X}_{s,R_\rho} \times \mathscr{Y}_{s,R_u}$ into itself. This can be done according to the following lemma:

Lemma 1.5 .1. *Let $C_u, C_u'(s)$ and $C_\rho, C_\rho'(s), s_1, \varepsilon(s)$ be the constants in Theorem 1.2 .2 and 1.2 .3 respectively. There exist $s \geq s_1$, $R_u \in (0,1)$, $R_\rho \in (0, \min\{1, \min_{[0,T]\times\mathbb{R}} \bar{\rho}/2\})$, $\varepsilon \in (0, \varepsilon(s)]$ such that*

$$\begin{aligned} &C_u'(s)\varepsilon + C_u R_\rho + \tfrac{C_u}{s} R_u + C_u R_u^2 \leq R_u, \\ &C_\rho'(s)\varepsilon + \tfrac{C_\rho}{\sqrt{s}}(R_\rho + R_u) + C_\rho(R_\rho^2 + R_u^2) \leq R_\rho. \end{aligned} \tag{1.5 .1}$$

Proof. We set $C_0 = \max\{C_\rho, C_u, 1, (2 \min_{[0,T]\times\mathbb{R}} \bar{\rho})^{-1}\}$ and $C_0'(s) = \max\{C_\rho'(s), C_u'(s)\}$, and we look for parameters s, R_ρ, R_u and ε such that

$$C_0'(s)\varepsilon + C_0 R_\rho + \frac{C_0}{s} R_u + C_0 R_u^2 \leq R_u, \quad \text{and} \quad C_0'(s)\varepsilon + \frac{C_0}{\sqrt{s}}(R_\rho + R_u) + C_0(R_\rho^2 + R_u^2) \leq R_\rho.$$

We thus choose

$$R_u = \frac{1}{12C_0^2 + 3}, \quad R_\rho = \frac{1}{4C_0} R_u.$$

so that $R_u \in (0,1)$, $R_\rho \in (0, \min\{1, \min_{[0,T]\times\mathbb{R}} \bar{\rho}/2\})$ and satisfies:

$$C_0 R_u^2 \leq \frac{R_u}{4}, \qquad C_0 R_\rho \leq \frac{R_u}{4}, \qquad C_0(R_\rho^2 + R_u^2) \leq \frac{R_\rho}{3}.$$

We then choose

$$s = \max\{s_1, 4C_0, 9C_0^2(1 + 4C_0)^2\},$$

which guarantees

$$\frac{C_0}{s} R_u \leq \frac{R_u}{4}, \qquad \frac{C_0}{\sqrt{s}}(R_\rho + R_u) \leq \frac{R_\rho}{3}.$$

Lastly, we choose $\varepsilon > 0$ as follows:

$$\varepsilon = \min\left\{\varepsilon(s), \frac{R_\rho}{3C_0'(s)}, \frac{R_u}{4C_0'(s)}\right\}.$$

One then easily checks that the inequalities (1.5 .1) are satisfied with these choices of parameters, and this concludes the proof of Lemma 1.5 .1. □

We thus fix the parameters $s \geq s_1$, $R_\rho \in (0, \min\{1, \min_{[0,T]\times\mathbb{R}} \bar{\rho}/2\})$, $R_u \in (0,1)$ and $\varepsilon \in (0, \varepsilon(s)]$ such that the inequalities (1.5 .1) are satisfied. Using Theorems 1.2 .2 and 1.2 .3, we then have that $\mathscr{F}$ maps $\mathscr{X}_{s,R_\rho} \times \mathscr{Y}_{s,R_u}$ into itself.

We are then left to check that $\mathscr{F}$ satisfies the assumptions of Schauder's fixed point theorem. In order to do that, we endow the set $\mathscr{X}_{s,R_\rho} \times \mathscr{Y}_{s,R_u}$ with the $L^2(0,T;L^2(a,b))$-topology, which makes this set compact (for the $L^2(0,T;L^2(a,b))$-topology) by Aubin-Lions' theorem, see [26].

It thus remains to prove that the map $\mathscr{F}$ is continuous on $\mathscr{X}_{s,R_\rho}\times\mathscr{Y}_{s,R_u}$ for the $L^2(0,T;L^2(a,b))$ topology. This can be done as in [13, Section 5.3], but we recall the main ingredients for the convenience of the reader. Let us then consider a sequence $(\widehat{\rho}_n, \widehat{u}_n)$ in $\mathscr{X}_{s,R_\rho} \times \mathscr{Y}_{s,R_u}$ converging in $L^2(0,T;L^2(a,b))$ towards some $(\widehat{\rho}, \widehat{u})$, and set $(\rho_n, u_n) = \mathscr{F}(\widehat{\rho}_n, \widehat{u}_n)$, $(\rho, u) = \mathscr{F}(\widehat{\rho}, \widehat{u})$. As the sequence $(\widehat{\rho}_n, \widehat{u}_n)$ is bounded in $\mathscr{X}_s \times \mathscr{Y}_s$, we then also have the following weak convergences:

$$\widehat{\rho}_n \underset{n\to\infty}{\rightharpoonup} \widehat{\rho} \quad \text{weakly * in } L^\infty(0,T;H^1(a,b)) \cap H^1(0,T;L^2(a,b)), \tag{1.5 .2}$$

$$\widehat{u}_n \underset{n\to\infty}{\rightharpoonup} \widehat{u} \quad \text{weakly in } L^2(0,T;H^2(a,b)) \cap H^1(0,T;L^2(a,b)). \tag{1.5 .3}$$

Using interpolations of L^p and H^1 spaces (see [27, Section 4.3.1, Theorem 1]), the functional space $L^p(0,T;H^1(a,b)) \cap H^1(0,T;L^2(a,b))$ is continuously embedded into $W^{1/4,q}(0,T;H^{3/4}(a,b))$ (see [27, Section 2.3.1 & Section 4.3.1]) with q given by

$$\frac{1}{q} = \frac{3}{4}\cdot\frac{1}{p} + \frac{1}{4}\cdot\frac{1}{2} = \frac{3}{4p} + \frac{1}{8}.$$

But, for $p > 6$, we have $q > 4$ and thus $W^{1/4,q}(0,T)$ embeds into some Hölder spaces $C^{0,\alpha(q)}(0,T)$ with $\alpha(q) > 0$ (see [27, Section 4.6.1]). Therefore, using the compact embedding of the spaces of Hölder spaces into the space of continuous function (Ascoli's theorem), the space $L^\infty(0,T;H^1(a,b)) \cap H^1(0,T;L^2(a,b))$ is compactly embedded into the set of continuous functions $C^0([0,T]\times[a,b])$. Furthermore, using Aubin Lions' Lemma, we also have a compact embedding of $L^2(0,T;H^2(a,b)) \cap H^1(0,T;L^2(a,b))$ in $L^\infty(0,T;L^\infty(a,b))$. We finally obtain the following strong convergences:

$$\widehat{\rho}_n \underset{n\to\infty}{\rightarrow} \widehat{\rho} \quad \text{strongly in } L^\infty(0,T;L^\infty(a,b)), \tag{1.5 .4}$$

$$\widehat{u}_n \underset{n\to\infty}{\rightarrow} \widehat{u} \quad \text{strongly in } L^\infty(0,T;L^\infty(a,b)). \tag{1.5 .5}$$

We then easily show that

$$(f(\widehat{\rho}_n, \widehat{u}_n), g(\widehat{\rho}_n, \widehat{u}_n)) \underset{n\to\infty}{\rightharpoonup} (f(\widehat{\rho}, \widehat{u}), g(\widehat{\rho}, \widehat{u})) \quad \text{in } (\mathscr{D}'((0,T)\times(a,b)))^2. \tag{1.5 .6}$$

The control process in Theorem 1.4 .1 is linear in (u_0, g), and therefore, following the construction done in Section 1.4 , u_n weakly converges to u in $\mathscr{D}'((0,T)\times(a,b))$. As $\mathscr{F}$ maps $\mathscr{X}_{s,R_\rho} \times \mathscr{Y}_{s,R_u}$ into itself, u_n is bounded in $\mathscr{Y}_s$ and we can therefore also deduce the convergences

$$u_n \underset{n\to\infty}{\rightharpoonup} u \quad \text{weakly in } L^2(0,T;H^2(a,b)) \cap H^1(0,T;L^2(a,b)), \tag{1.5 .7}$$

$$u_n \underset{n\to\infty}{\rightarrow} u \quad \text{strongly in } L^\infty(0,T;L^\infty(a,b)). \tag{1.5 .8}$$

We then focus on the construction of ρ_n, ρ performed in Section 1.3 .1 and its continuity with respect to u_n and $f(\widehat{\rho}_n, \widehat{u}_n)$. We then introduce $\rho_{f,n}$ the solution of (1.3 .3) with u_n instead of u and $f = f(\widehat{\rho}_n, \widehat{u}_n)$. Due to (1.3 .45), $\rho_{f,n}$ is uniformly bounded in $H^1((0,T_L)\times(a,b))$ and therefore weakly converges to some ρ_f^* in $H^1((0,T_L)\times(a,b))$. Using the convergences (1.5 .8) and (1.5 .6), we can pass to the limit in the equation satisfied by $\rho_{f,n}$ and then obtain that ρ_f^* is the solution ρ_f of (1.3 .3) with u and $f = f(\widehat{\rho}, \widehat{u})$. Similarly, the solutions $\rho_{b,n}$ of (1.3 .4) with u_n instead of u and $f = f(\widehat{\rho}_n, \widehat{u}_n)$ weakly converge in $H^1((T_0,T)\times(a,b))$ to the solution ρ_b of (1.3 .4) with u and $f = f(\widehat{\rho}, \widehat{u})$. It is then easy to check that the construction of the cut-off function $\tilde{\eta}$ in Lemma 1.3 .1 is continuous with respect to u. Indeed, if we call $\tilde{\eta}_n$ the cut-off functions constructed in Lemma 1.3 .1 corresponding to u_n, as the sequence $\tilde{\eta}_n$ is uniformly bounded in $H^1((0,T)\times(a,b))$, recall (1.3 .10), we can pass to the limit in the construction and get that $\tilde{\eta}_n$ weakly converges in $H^1((0,T)\times(a,b))$ to $\tilde{\eta}$, the cut-off function constructed in Lemma 1.3 .1 corresponding to u, and thus strongly converges in $L^2(0,T;L^2(a,b))$ according to Aubin-Lions' Lemma. Therefore, the sequence $\rho_n = \tilde{\eta}_n\rho_{f,n} + (1-\tilde{\eta}_n)\rho_{b,n}$ weakly converges to $\rho = \tilde{\eta}\rho_f + (1-\tilde{\eta})\rho_b$ in $\mathscr{D}'((0,T)\times(0,L))$.

We have thus shown that $\mathscr{F}(\widehat{\rho}_n, \widehat{u}_n)$ weakly converges as $n\to\infty$ towards $\mathscr{F}(\widehat{\rho}, \widehat{u})$ in $(\mathscr{D}'((0,T)\times(a,b)))^2$. Moreover, the sequence $\mathscr{F}(\widehat{\rho}_n, \widehat{u}_n)$ is bounded in $\mathscr{X}_{s,R_\rho}\times\mathscr{Y}_{s,R_u}$. Recall then that this set is compact for the $(L^2(0,T;L^2(a,b)))^2$ topology, so that the sequence $\mathscr{F}(\widehat{\rho}_n, \widehat{u}_n)$ strongly converges in $(L^2(0,T;L^2(a,b)))^2$ to $\mathscr{F}(\widehat{\rho}, \widehat{u})$. This proves that $\mathscr{F}$ is continuous on $\mathscr{X}_{s,R_\rho}\times\mathscr{Y}_{s,R_u}$ endowed with the $(L^2(0,T;L^2(a,b)))^2$ topology.

We can then use Schauder's fixed point theorem for $\mathscr{F}$ defined on the set $\mathscr{X}_{s,R_\rho}\times\mathscr{Y}_{s,R_u}$ endowed with the $(L^2(0,T;L^2(a,b)))^2$ topology. Let (ρ,u) be a fixed point of $\mathscr{F}$. By construction, $(\rho,u)\in\mathscr{X}_{s,R_\rho}\times\mathscr{Y}_{s,R_u}$ and solves the controllability problem (1.2 .25)–(1.2 .26)–(1.2 .28) for some $v_\rho\in L^2(0,T;L^2((a,0)\cup(L,b))$ and $v_u\in L^2(0,T)$. The restriction of (ρ,u) on the space interval $(0,L)$ provides a solution of (1.2 .2)–(1.2 .4)–(1.2 .5). Therefore, (ρ_s,u_s) in (1.2 .1) is a controlled solution of (1.1 .1) satisfying the initial condition (1.1 .8) and the controllability requirement (1.1 .9) with the regularity stated in (1.1 .10). Besides, remark that the positivity of the density ρ_s is assured by Lemma 1.5 .1 since $R_\rho < \min_{[0,T]\times\mathbb{R}} \bar{\rho}/2$. This concludes the proof of Theorem 1.1 .1.

1.A A weighted Poincaré inequality

Here, we recall the following result, proved for instance in [13, Lemma 4.9]:

Lemma 1.A.1 (A weighted Poincaré inequality [13, Lemma 4.9])**.** *Let φ as in (1.2 .22) with θ and ψ as in (1.2 .21) and Lemma 1.2 .1. There exist constants $C>0$ and $s_2\geq 1$ such that for all $s\geq s_2$, for all $t\in[0,T)$ and for all $f\in H_0^1(a,b)$,*

$$s\|e^{s\varphi(t,\cdot)}f\|_{L^2(a,b)} \leq C\|\theta^{-1}(t)e^{s\varphi(t,\cdot)}\partial_x f\|_{L^2(a,b)}. \tag{1.A.1}$$

The proof of Lemma 1.A.1 is not difficult and simply requires that ψ in Lemma 1.2 .1 does not have any critical point in $[a,b]$, i.e. Assumption (1.2 .17). We refer the interested reader to [13, Lemma 4.9] for a detailed proof.

Bibliography

[1] P. Albano and D. Tataru. Carleman estimates and boundary observability for a coupled parabolic-hyperbolic system. *Electron. J. Differential Equations*, pages No. 22, 15 pp. (electronic), 2000.

[2] E.V. Amosova. Exact local controllability for the equations of viscous gas dynamics. *Differentsial'nye Uravneniya*, 47(12):1754–1772, 2011.

[3] M. Badra, S. Ervedoza, and S. Guerrero. Local controllability to trajectories for non-homogeneous incompressible Navier-Stokes equations. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, (0):–, 2014.

[4] D. Bresch and B. Desjardins. Existence of global weak solutions for a 2D viscous shallow water equations and convergence to the quasi-geostrophic model. *Comm. Math. Phys.*, 238(1-2):211–223, 2003.

[5] D. Bresch, B. Desjardins, and C.-K. Lin. On some compressible fluid models: Korteweg, lubrication, and shallow water systems. *Comm. Partial Differential Equations*, 28(3-4):843–868, 2003.

[6] F. W. Chaves-Silva, L. Rosier, and E. Zuazua. Null controllability of a system of viscoelasticity with a moving control. *J. Math. Pures Appl. (9)*, 101(2):198–222, 2014.

[7] S. Chowdhury, M. Debanjana, M. Ramaswamy, and M. Renardy. Null controllability of the linearized compressible Navier Stokes system in one dimension. *J. Differential Equations*, 257(10):3813–3849, 2014.

[8] S. Chowdhury, M. Ramaswamy, and J.-P. Raymond. Controllability and stabilizability of the linearized compressible Navier-Stokes system in one dimension. *SIAM J. Control Optim.*, 50(5):2959–2987, 2012.

[9] J.-M. Coron. On the controllability of the 2-D incompressible Navier-Stokes equations with the Navier slip boundary conditions. *ESAIM Control Optim. Calc. Var.*, 1:35–75 (electronic), 1995/96.

[10] J.-M. Coron. On the controllability of 2-D incompressible perfect fluids. *J. Math. Pures Appl. (9)*, 75(2):155–188, 1996.

[11] J.-M. Coron. *Control and nonlinearity*, volume 136 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.

[12] J.-M. Coron and A. V. Fursikov. Global exact controllability of the 2D Navier-Stokes equations on a manifold without boundary. *Russian J. Math. Phys.*, 4(4):429–448, 1996.

[13] S. Ervedoza, O. Glass, S. Guerrero, and J.-P. Puel. Local exact controllability for the one-dimensional compressible Navier-Stokes equation. *Arch. Ration. Mech. Anal.*, 206(1):189–238, 2012.

[14] E. Fernández-Cara, S. Guerrero, O. Y. Imanuvilov, and J.-P. Puel. Local exact controllability of the Navier-Stokes system. *J. Math. Pures Appl. (9)*, 83(12):1501–1542, 2004.

[15] A. V. Fursikov and O. Yu. Èmanuilov. Exact controllability of the Navier-Stokes and Boussinesq equations. *Uspekhi Mat. Nauk*, 54(3(327)):93–146, 1999.

[16] A. V. Fursikov and O. Y. Imanuvilov. *Controllability of evolution equations*, volume 34 of *Lecture Notes Series*. Seoul National University Research Institute of Mathematics Global Analysis Research Center, Seoul, 1996.

[17] O. Glass. Exact boundary controllability of 3-D Euler equation. *ESAIM Control Optim. Calc. Var.*, 5:1–44 (electronic), 2000.

[18] O. Glass. On the controllability of the 1-D isentropic Euler equation. *J. Eur. Math. Soc. (JEMS)*, 9(3):427–486, 2007.

[19] O. Glass. On the controllability of the non-isentropic 1-D Euler equation. *J. Differential Equations*, 257(3):638–719, 2014.

[20] O. Y. Imanuvilov. Remarks on exact controllability for the Navier-Stokes equations. *ESAIM Control Optim. Calc. Var.*, 6:39–72 (electronic), 2001.

[21] T.-T. Li and B.-P. Rao. Exact boundary controllability for quasi-linear hyperbolic systems. *SIAM J. Control Optim.*, 41(6):1748–1755 (electronic), 2003.

[22] D. Maity. Some controllability results for linearized compressible Navier-Stokes system. *ESAIM Control Optim. Calc. Var.*, 21(4):1002–1028 (electronic), 2015.

[23] P. Martin, L. Rosier, and P. Rouchon. Null controllability of the structurally damped wave equation with moving control. *SIAM J. Control Optim.*, 51(1):660–684, 2013.

[24] A. Matsumura and T. Nishida. The initial value problem for the equations of motion of viscous and heat-conductive gases. *J. Math. Kyoto Univ.*, 20(1):67–104, 1980.

[25] L. Rosier and P. Rouchon. On the controllability of a wave equation with structural damping. *Int. J. Tomogr. Stat.*, 5(W07):79–84, 2007.

[26] J. Simon. Compact sets in the space $L^p(0, T; B)$. *Ann. Mat. Pura Appl. (4)*, 146:65–96, 1987.

[27] H. Triebel Interpolation theory, function spaces, differential operators, *North-Holland Publishing Co.*, Amsterdam-New York, 1978.

Chapter 2

Strong solutions for incompressible Navier-Stokes equations with a moving boundary immersed in a bounded domain

We are interested in the incompressible Navier-Stokes equations in a bounded domain containing a boundary which follows the flow. The purpose is to take the first step toward the analysis of the Immersed Boundary Method (IBM). The IBM is a numerical method introduced by Peskin in 1972 to model the flow patterns around heart valves [15] and has been a very successful tool to simulate fluid structure interaction. Unlike the system of the IBM, we consider that the boundary force generated by the moving boundary is a given data and does not depend on the elastic structure nor the geometry of the boundary. We focus our attention on existence of solution. Rewriting the system in Lagrangian variables, we state an existence result of strong solutions locally in time and an existence result of strong solutions at all time for small data. The non linearity coming from the moving boundary is treated with a fixed point argument. The linearized system expressed in a fixed domain is a non homogeneous Stokes equation with a force supported along the fixed interface. We use a lifting of this force to obtain regularity of the solution on each side of the boundary.

2.1 Introduction

Let Ω be a bounded open subset of $\mathbb{R}^d$, $d = 2$ or 3. Let Ω^- be an open subset such that $\overline{\Omega^-} \subset \Omega$ and $\Omega^+ = \Omega \setminus \overline{\Omega^-}$. We denote by $\partial\Omega$ the boundary of Ω and by Γ the boundary of Ω^-. This configuration is drawn in Figure 2.1.

We are interested in the existence of solutions to the following system:

$$\left\{ \begin{array}{rcll} \partial_t u_s + (u_s \cdot \nabla)u_s - \mu\Delta u_s + \nabla p_s &=& (f \circ Y)\delta_{\Gamma(t)} & \text{in } (0,T) \times \Omega, \\ \nabla \cdot u_s &=& 0 & \text{in } (0,T) \times \Omega, \\ u_s &=& 0 & \text{on } (0,T) \times \partial\Omega, \\ u_s|_{t=0} &=& u_0 & \text{in } \Omega, \\ \Gamma(t) &=& X(t,\Gamma) & \text{for } t \in (0,T), \end{array} \right. \tag{2.1 .1}$$

where for $y \in \Omega$, we define the flow X:

$$\left\{ \begin{array}{rcll} \dfrac{dX}{dt}(t,y) &=& u_s(t, X(t,y)) & \text{on } (0,T), \\ X(0,y) &=& y, & \end{array} \right. \tag{2.1 .2}$$

and for $t \in (0,T)$, $Y(t,\cdot)$ is the inverse transformation induced by the flow $X(t,\cdot)$.

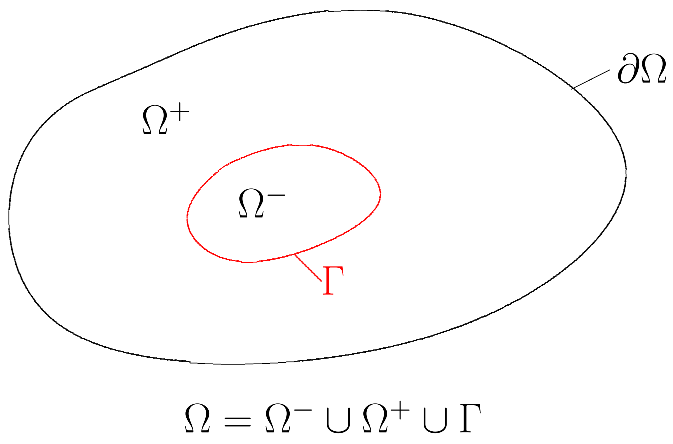

Figure 2.1: Initial configuration for $d = 2$.

This system is an homogeneous incompressible Navier-Stokes system. The unknowns are the velocity u_s, the pressure p_s and the flow X. The viscosity $\mu > 0$ is assumed to be constant. Lastly, the right hand side $(f \circ Y)\delta_{\Gamma(t)} = f(t, Y(t,x))\delta_{\Gamma(t)}$ for $t \in (0,T)$ and $x \in \Gamma(t)$, is defined as follow :

$$\forall \varphi \in \mathscr{D}(\Omega) \qquad \langle (f \circ Y)\delta_{\Gamma(t)}, \varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}(t) = \langle f \circ Y, \varphi|_{\Gamma(t)}\rangle_{\mathscr{D}'(\Gamma(t)),\mathscr{D}(\Gamma(t))}(t), \quad \text{in } \mathscr{D}'(0,T).$$

This force term $(f \circ Y)\delta_{\Gamma(t)}$ can also be read as a transmission through the boundary $\Gamma(t)$. This transmission is expressed with a nonslip condition and a jump of the stress tensor:

$$[u_s] = 0 \quad \text{and} \quad [\sigma(u_s, p_s)n_{\Gamma(t)}] = f \circ Y \quad \text{on } \Gamma(t), \text{ for } t \in (0,T),$$

where $[w]$ stands for the jump of w across $\Gamma(t)$ and $\sigma(u,p)$ is the stress tensor and writes

$$\sigma(u_s, p_s) = \mu D(u) - pI \quad \text{where} \quad D(u_s) = \nabla u_s + (\nabla u_s)^T.$$

Therefore, system (2.1 .1)–(2.1 .2) can be rewritten in the following form

$$\left\{\begin{array}{rcll} \partial_t u_s^\pm + (u_s^\pm \cdot \nabla)u_s^\pm - \mu\Delta u_s^\pm + \nabla p_s^\pm & = & 0 & \text{in } \Omega^\pm(t), \text{ for } t \in (0,T), \\ \nabla \cdot u_s^\pm & = & 0 & \text{in } \Omega^\pm(t), \text{ for } t \in (0,T), \\ u_s^+ & = & 0 & \text{on } (0,T) \times \partial\Omega, \\ u_s^- & = & u_s^+ & \text{on } \Gamma(t), \text{ for } t \in (0,T), \\ [\sigma(u_s,p_s)n_{\Gamma(t)}] & = & f \circ Y^- & \text{on } \Gamma(t), \text{ for } t \in (0,T), \\ u_s^\pm|_{t=0} & = & u_0^\pm & \text{in } \Omega^\pm, \\ \Gamma(t) & = & X^-(t,\Gamma) & \text{for } t \in (0,T), \\ \Omega^\pm(t) & = & X^\pm(t,\Omega^\pm) & \text{for } t \in (0,T). \end{array}\right. \tag{2.1 .3}$$

where for $y \in \Omega^\pm$, we define the flow $X^\pm$:

$$\left\{\begin{array}{rcll} \dfrac{dX^\pm}{dt}(t,y) & = & u_s(t, X^\pm(t,y)) & \text{on } (0,T), \\ X^\pm(0,y) & = & y. & \end{array}\right. \tag{2.1 .4}$$

Notice that with $(2.1\ .3)_4$, we have $X^+(\cdot,y) = X^-(\cdot,y)$ for $y \in \Gamma$ and therefore, $Y^+(\cdot,x) = Y^-(\cdot,x)$ for $x \in \Gamma(t)$. Furthermore, the form of system (2.1 .3)–(2.1 .4) shows that system

(2.1 .1)–(2.1 .2) cannot have a classical solution and that it is not possible to get regularity results on the whole domain.

The question we aim at addressing is the existence of strong solutions of system (2.1 .1)-(2.1 .2) or equivalently system (2.1 .3)-(2.1 .4) in dimension $d \in \{2, 3\}$.

Motivation: The main motivation is to provide a mathematical framework for the Immersed Boundary Method (IBM) system.

The IBM was originally introduced by Peskin in 1972 [15] to model the flow patterns around the heart valves by approximating the leaflet of the valve by a force-generating boundary immersed in a viscous incompressible fluid. The main interest was purely numerical, for equations of the IBM can be discretized with a fixed mesh, the boundary being represented by a discrete set of Lagrangian points. Working with a fixed mesh significantly reduces the time of calculation. Since then, the IBM has been very successful in simulation of fluid structure models and has been applied in many different fields. Fogelson used the IBM to model platelet adhesion and aggregation during blood clotting [7]. McQueen and Peskin derived a model for blood flow in the heart in 3D using the IBM [17]. Fauci and Peskin applied the IBM to aquatic animal locomotion [6]. Dillon, Fauci, Omoto developed a model of ciliary and sperm motility in 2D [4], [5]. Peskin and Zhu derived a model of flapping flag in the wind in 2D and showed that the system is bistable, the stable states being the rest state and the sustained flapping [28]. This list is far from being exhaustive, and for more references and applications, we refer the interested reader to Peskin's article [16] or to Stockie's lecture [22]. They both draw a clear overview on the IBM. In [16], Peskin derives the equations of the IBM and presents the applications and some directions of improvement. For more recent works and advances, [22] provides a very appropriate continuity to Peskin's article.

In the IBM, the force f actually depends on the elastic properties of the immersed structure and on the form of $\Gamma(t)$ at time $t \in (0, T)$. When the immersed structure is an elastic solid, f generally includes a stretching and compression force f_s which takes the form in dimension 2:

$$f_s = \frac{\partial}{\partial s}(T\tau), \tag{2.1 .5}$$

where

$$\begin{aligned} T &= K_s\Big(\Big|\frac{\partial X}{\partial s}\Big| - 1\Big), \\ \tau &= \frac{\partial X/\partial s}{|\partial X/\partial s|}. \end{aligned}$$

The constant K_s stands for the boundary stretching coefficient, s is the curvilinear abscissa of $\Gamma(t)$, τ is the unit tangent vector and T is the tension of the boundary given by Hook Law (see [1, 18, 28]). Besides, when we work with two fluids, f may represent the surface tension:

$$f_f = \sigma\kappa n_{\Gamma(t)}, \tag{2.1 .6}$$

where

$$\kappa(t, .) = -\nabla_{\Gamma(t)} \cdot n_{\Gamma(t)}. \tag{2.1 .7}$$

The constant σ stands for the surface tension and $\kappa(t, .)$ is the curvature of $\Gamma(t)$ with respect to $n_{\Gamma(t)}$ (see for example [19]). Furthermore, f may include many different forces that depend on the modeled mechanisms. Most of the boundary forces f appearing in the applications depend on the geometry of $\Gamma(t)$ and may directly depend on the flow X as shown in the examples above.

In our study, we assume that f is a given data for system (2.1 .1)-(2.1 .2) and we do not deal with the induced coupling of a boundary force like (2.1 .5)-(2.1 .7). Notice that this coupling is always very strong and cannot be seen as a perturbative term. It is thus important to note that we are not in the framework of the IBM. This article is a preliminary work intended to be enlarge to the case of the complete IBM.

Previous works: To the best of the author knowledge, system (2.1 .1)–(2.1 .2) has only been considered by Ton in 2006 in [26]. In his article, Ton proved a result of existence of weak solution for system (2.1 .1) in 2D but with a regularization of the equation of the flow (2.1 .2). In 2003, Ton also showed existence and uniqueness of weak solution for the Stokes equations in 3D with a Radon measure in [25].

Neglecting the inertial effects and considering a stretching/compression force f like in (2.1 .5), Lin and Tong [11] and Mori et al. [14] have recently proved existence results for the 2D Stokes flow. Lin and Tong [11] work in Sobolev spaces and show local existence and uniqueness and existence and uniqueness of global solutions for data close to an equilibrum with exponential convergence towards the equilibrum. Mori et al. [14] work in Hölder spaces : they show local well-posedness for low-regularity initial data, they also study equilibria and prove their stability with an exponential decay.

System (2.1 .3)–(2.1 .4) shares numerous similarities with free surface problem, especially two-phase free surface problem of viscous incompressible fluids. This type of model can also consider capillary effect. The main difference between system (2.1 .3)–(2.1 .4) and two-phase free surface model of viscous incompressible non-capillary fluids lies in the force term f and the difference of density/viscosity that we do not take into account. The two-phase free surface model of viscous incompressible capillary fluids considers surface tension effect, that is a force f like (2.1 .6), which induces a strong coupling. Results on the two phase problem are very recent but very rich. On a geometry like ours, we can quote [23, 24, 10] which take surface tension into account and [2, 3] which neglect the surface tension effect. On a geometry of two layers, one can refer to [9, 19, 27].

2.2 The main results

Our goal is to study the existence of strong solutions for system (2.1 .1)–(2.1 .2). It turns out that it will thoroughly depend on the dimension $d \in \{2,3\}$. The purpose of this first section is to state our two existence results for system (2.1 .1)–(2.1 .2).

In Section 2.2 .1, we express system (2.1 .1)–(2.1 .2) in Lagrangian coordinates. In Section 2.2 .2, we present our main results. Then, we present a quick overview of the proof in Section 2.2 .3. Laslty, in Section 2.2 .4, the notations and functional settings are introduced.

2.2 .1 Lagrangian coordinates

In this subsection, we formally pass from Eulerian coordinates to Lagrangian coordinates. These formal calculations will be justified later when the regularity result on (u, p, X) is proved. This change of coordinates is carried out on system (2.1 .3)–(2.1 .4).

By definition $Y^{\pm}$ is the inverse transformation of the flow $X^{\pm}$:

$$\begin{array}{rcl} X^{\pm}(t, Y^{\pm}(t,x)) &=& x, \quad \text{pour } x \in \Omega^{\pm}(t),\ t \in (0,T), \\ Y^{\pm}(t, X^{\pm}(t,y)) &=& y, \quad \text{pour } y \in \Omega^{\pm},\ t \in (0,T), \end{array}$$

Remark that thanks to $(2.1\ .3)_4$, for all $t \in (0,T)$,

$$\begin{array}{rcl} X^{-}(t,\cdot) &=& X^{+}(t,\cdot) \quad \text{on } \Gamma, \\ Y^{-}(t,\cdot) &=& Y^{+}(t,\cdot) \quad \text{on } \Gamma(t). \end{array}$$

We denote $J_{X^\pm}(t,y)$ the Jacobian of $X^\pm(t,\cdot)$ for $(t,y)\in(0,T)\times\Omega^\pm$ and $J_{Y^\pm}(t,x)$ the Jacobian of $Y^\pm(t,\cdot)$ for $t\in(0,T)$ and $x\in\Omega^\pm(t)$. Finally, we introduce the Lagrangian unknowns:

$$u^\pm(t,y)=u_s^\pm(t,X^\pm(t,y)) \quad \text{and} \quad p^\pm(t,y)=p_s^\pm(t,X^\pm(t,y)) \quad \text{for} \quad (t,y)\in(0,T)\times\Omega.$$

Straightforward computations show that

$$\begin{aligned}
\partial_t u_s^\pm+(u_s^\pm.\nabla)u_s^\pm &= \partial_t u^\pm\circ Y^\pm,\\
\Delta u_s^\pm &= \mu\sum_{k,l=1}^d(\partial_{x_l x_l}Y_k^\pm)\partial_{y_k}u^\pm\circ Y^\pm+\mu\sum_{k,l,j=1}^d(\partial_{x_j}Y_l^\pm)(\partial_{x_j}Y_k^\pm)\partial_{y_k y_l}u^\pm\circ Y^\pm,\\
\nabla p_s^\pm &= J_{Y^\pm}^T\nabla p^\pm\circ Y^\pm,\\
\nabla\cdot u_s^\pm &= \nabla u^\pm\circ Y^\pm : J_{Y^\pm}^T,\\
\sigma(u_s^\pm,p_s^\pm) &= \mu((\nabla u^\pm\circ Y^\pm)J_{Y^\pm}+J_{Y^\pm}^T(\nabla u^\pm\circ Y^\pm)^T)-(p^\pm\circ Y^\pm)I,\\
n_{\Gamma(t)} &= \frac{\mathrm{cof}(J_{X^\pm})n_\Gamma}{|\mathrm{cof}(J_{X^\pm})n_\Gamma|},
\end{aligned}$$

the vector n_Γ denoting the outward normal of Ω^-. Remark that

$$J_{Y^\pm}\circ X^\pm=\frac{1}{\det J_{X^\pm}}\mathrm{cof}(J_{X^\pm})^T=\mathrm{cof}(J_{X^\pm})^T,$$

since $\nabla\cdot u_s^\pm=0$ implies that $\det J_{X^\pm}=1$. This remark enables us to get rid off the inverse of the flow X and to express the non linear terms only with the flow X. It is very useful in the estimations of the non linear terms. The new unknown $(u^\pm,p^\pm)$ formally satisfies:

$$\left\{\begin{aligned}
\partial_t u^\pm-\nabla\cdot\sigma(u^\pm,p^\pm) &= F^\pm(u^\pm,p^\pm) && \text{in } (0,T)\times\Omega^\pm,\\
\nabla\cdot u^\pm &= \nabla\cdot G^\pm(u^\pm) && \text{in } (0,T)\times\Omega^\pm,\\
u^+ &= 0 && \text{on } (0,T)\times\partial\Omega,\\
u^- &= u^+ && \text{on } (0,T)\times\Gamma,\\
[\sigma(u,p)n_\Gamma] &= f+H(u,p) && \text{on } (0,T)\times\Gamma,\\
u^\pm|_{t=0} &= u_0^\pm && \text{in } \Omega^\pm,
\end{aligned}\right. \tag{2.2 .1}$$

where

$$F^\pm(u^\pm,p^\pm) = F_1^\pm(u^\pm)+F_2^\pm(u^\pm)+F_3^\pm(u^\pm,p^\pm)-\mu\nabla(\nabla\cdot G^\pm(u^\pm)), \tag{2.2 .2}$$

$$F_1^\pm(u^\pm) = \mu\sum_{i,j=1}^d\partial_{x_j}\mathrm{cof}_{j,i}(J_{X^\pm})\partial_{y_i}u^\pm, \tag{2.2 .3}$$

$$F_2^\pm(u^\pm) = \mu\sum_{i,j,k=1}^d\mathrm{cof}_{k,j}(J_{X^\pm})\mathrm{cof}_{k,i}(J_{X^\pm})\partial_{y_i y_j}u^\pm-\mu\Delta u, \tag{2.2 .4}$$

$$F_3^\pm(u^\pm,p^\pm) = (I-\mathrm{cof}(J_{X^\pm}))\nabla p^\pm, \tag{2.2 .5}$$

$$G^\pm(u^\pm) = (I-\mathrm{cof}(J_{X^\pm})^T)u^\pm, \tag{2.2 .6}$$

$$\begin{aligned}H(u,p)=-\Big[&\mu((\nabla u)\mathrm{cof}(J_X)^T+\mathrm{cof}(J_X)(\nabla u)^T)\frac{\mathrm{cof}(J_X)n_\Gamma}{|\mathrm{cof}(J_X)n_\Gamma|}\\
&-\mu(\nabla u+(\nabla u)^T)n_\Gamma-p(\frac{\mathrm{cof}(J_X)n_\Gamma}{|\mathrm{cof}(J_X)n_\Gamma|}-n_\Gamma)\Big],\end{aligned} \tag{2.2 .7}$$

and $|\mathrm{cof}(J_X)n_\Gamma|$ is the euclidean norm of $\mathbb{R}^d$ of $\mathrm{cof}(J_X)n_\Gamma$, and $[f]$ is the jump of f across Γ:

$$[f]=f^+-f^- \quad \text{on } \Gamma.$$

Remark that to establish system (2.2 .1), we have used the Piola identity to get

$$\nabla u^{\pm} : (I - \mathrm{cof}(J_{X^{\pm}})) = \nabla \cdot ((I - \mathrm{cof}(J_{X^{\pm}})^T)u^{\pm}).$$

With the new unknowns, the flow X is defined by: for all $y \in \Omega^{\pm}$,

$$\left\{\begin{array}{rcll} \dfrac{dX^{\pm}}{dt}(t,y) & = & u^{\pm}(t,y) & \text{on } (0,T), \\ X^{\pm}(0,y) & = & y. & \end{array}\right. \tag{2.2 .8}$$

Remark that

$$\nabla \cdot G(u) = \mathbb{1}_{\Omega^-} \nabla \cdot G^-(u^-) + \mathbb{1}_{\Omega^+} \nabla \cdot G^+(u^+),$$

because $(2.2\ .1)_4$ imposes $\mathrm{cof}(J_{X^-})n_\Gamma = \mathrm{cof}(J_{X^+})n_\Gamma$. System (2.2 .1)-(2.2 .8) can be rewritten in the form

$$\left\{\begin{array}{rcll} \partial_t u - \nabla \cdot \sigma(u,p) & = & (f + H(u,p))\delta_\Gamma + F(u,p) & \text{in } (0,T) \times \Omega, \\ \nabla \cdot u & = & \nabla \cdot G(u) & \text{in } (0,T) \times \Omega, \\ u & = & 0 & \text{on } (0,T) \times \partial\Omega, \\ u|_{t=0} & = & u_0 & \text{in } \Omega. \end{array}\right. \tag{2.2 .9}$$

where

$$\begin{array}{rcl} F(u,p) & = & \mathbb{1}_{\Omega^-} F^-(u^-,p^-) + \mathbb{1}_{\Omega^+} F^+(u^+,p^+), \\ G(u) & = & \mathbb{1}_{\Omega^-} G^-(u^-) + \mathbb{1}_{\Omega^+} G^+(u^+). \end{array} \tag{2.2 .10}$$

and

$$u = \mathbb{1}_{\Omega^+} u^+ + \mathbb{1}_{\Omega^-} u^- \qquad \text{and} \qquad p = \mathbb{1}_{\Omega^+} p^+ + \mathbb{1}_{\Omega^-} p^-,$$

the flow X satisfying: for all $y \in \Omega$,

$$\left\{\begin{array}{rcll} \dfrac{dX}{dt}(t,y) & = & u(t,y) & \text{on } (0,T), \\ X(0,y) & = & y. & \end{array}\right. \tag{2.2 .11}$$

Remark 2.2 .1. *We would like to point out that if (u,p,X) solves system (2.2 .9)–(2.2 .11) and $X(t,\cdot)$ is invertible for all $t \in (0,T)$, then $u_s = u \circ Y$ satisfies $\nabla \cdot u_s = 0$ on Ω and therefore* $\det J_{X^{\pm}} = 1$.

2.2 .2 Existence of strong solutions for (2.2 .9)–(2.2 .11)

We can now state the main results of this article. We recall that we work in $\mathbb{R}^d$, $d = 2$ or 3. We state the results separately in each dimension.

We first introduce the following spaces for $r, s, T > 0$:

$$\begin{array}{rcl} V(\Omega) & = & \{u \in H^1(\Omega) \mid \nabla \cdot u = 0 \text{ and } u = 0 \text{ on } \partial\Omega\}, \\ H^{r,s}((0,T) \times \Omega^{\pm}) & = & H^s(0,T; L^2(\Omega^{\pm})) \cap L^2(0,T; H^r(\Omega^{\pm})), \\ H^{r,s}((0,T) \times \Gamma) & = & H^s(0,T; L^2(\Gamma)) \cap L^2(0,T; H^r(\Gamma)). \end{array}$$

2.2 .2.1 Existence of strong solutions for (2.2 .9)–(2.2 .11) in dimension $d = 2$

The first Theorem is a result of local in time existence of strong solutions for given data.

Theorem 2.2 .2. *Let*

$$d = 2, \quad \ell \in (0, 1/2), \quad \Gamma,\ \partial\Omega \in C^4,$$
$$f \in L^2_{loc}(0, +\infty; H^{1/2+\ell}(\Gamma)) \cap H^{1/4+\ell/2}_{loc}(0, +\infty; L^2(\Gamma)),$$
$$u_0 \in V(\Omega), \quad u_0^\pm \in H^{1+\ell}(\Omega^\pm).$$

There exists a time $T > 0$ such that system (2.2 .9)–(2.2 .11) admits a solution (u, p, X) with

$$u \in L^2(0, T; H^1(\Omega)), \quad u^\pm \in H^{2+\ell,1+\ell/2}((0, T) \times \Omega^\pm),$$
$$p \in L^2((0, T) \times \Omega), \quad \nabla p^\pm \in H^{\ell,\ell/2}((0, T) \times \Omega^\pm), \quad [p] \in H^{1/2+\ell,1/4+\ell/2}((0, T) \times \Gamma),$$
$$X \in L^2((0, T) \times \Omega), \quad X^\pm \in H^1(0, T; H^{2+\ell}(\Omega^\pm)) \cap H^{2+\ell/2}(0, T; L^2(\Omega^\pm)).$$

Furthermore, for all time $t \in (0, T)$, $X(t, \cdot)$ is invertible and if we set $u_s(t, x) = u(t, Y(t, x))$ and $p_s(t, x) = p(t, Y(t, x))$, where $Y(t, \cdot)$ is the inverse transformation of $X(t, \cdot)$, then (u_s, p_s, X) is a strong solution of (2.1 .1)–(2.1 .2).

The second Theorem is a result of global in time existence of strong solutions for small data.

Theorem 2.2 .3. *Let*

$$d = 2, \quad \ell \in (0, 1/2), \quad \Gamma,\ \partial\Omega \in C^4.$$

For all time $T > 0$, there exists $r > 0$ such that, for all

$$f \in H^{1/2+\ell,1/4+\ell/2}((0, T) \times \Gamma),$$
$$u_0 \in V(\Omega), \quad u_0^\pm \in H^{1+\ell}(\Omega^\pm).$$

satisfying

$$\|f\|_{H^{1/2+\ell,1/4+\ell/2}((0,T)\times\Gamma)} + \sum_\pm \|u_0^\pm\|_{H^{1+\ell}(\Omega^\pm)} \leq r, \tag{2.2 .12}$$

system (2.2 .9)–(2.2 .11) admits a solution (u, p, X) with

$$u \in L^2(0, T; H^1(\Omega)), \quad u^\pm \in H^{2+\ell,1+\ell/2}((0, T) \times \Omega^\pm),$$
$$p \in L^2((0, T) \times \Omega), \quad \nabla p^\pm \in H^{\ell,\ell/2}((0, T) \times \Omega^\pm), \quad [p] \in H^{1/2+\ell,1/4+\ell/2}((0, T) \times \Gamma),$$
$$X \in L^2((0, T) \times \Omega), \quad X^\pm \in H^1(0, T; H^{2+\ell}(\Omega^\pm)) \cap H^{2+\ell/2}(0, T; L^2(\Omega^\pm)).$$

Furthermore, for all time $t \in (0, T)$, $X(t, \cdot)$ is invertible and if we set $u_s(t, x) = u(t, Y(t, x))$ and $p_s(t, x) = p(t, Y(t, x))$, where $Y(t, \cdot)$ is the inverse transformation of $X(t, \cdot)$, then (u_s, p_s, X) is a strong solution of (2.1 .1)–(2.1 .2).

2.2 .2.2 Existence of strong solutions for (2.2 .9)–(2.2 .11) in dimension $d = 3$

As in dimension 2, we have a first result of local in time existence of strong solutions for given data.

Theorem 2.2 .4. *Let*

$$d = 3, \quad \ell \in (1/2, 1), \quad \Gamma,\ \partial\Omega \in C^4,$$
$$f \in L^2_{loc}(0, +\infty; H^{1/2+\ell}(\Gamma)) \cap H^{1/4+\ell/2}_{loc}(0, +\infty; L^2(\Gamma)),$$
$$u_0 \in V(\Omega), \quad u_0^\pm \in H^{1+\ell}(\Omega^\pm).$$

We assume the compatibility condition

$$2\mu([Du_0]n_\Gamma)_\tau = f_\tau|_{t=0} \quad \textit{on } \Gamma, \tag{2.2 .13}$$

where the subscript τ stands for the tangential component and D is the symmetric gradient. Then there exists a time $T > 0$ such that system (2.2 .9)–(2.2 .11) admits a solution (u, p, X) with

$$u \in L^2(0,T; H^1(\Omega)), \quad u^\pm \in H^{2+\ell,1+\ell/2}((0,T)\times\Omega^\pm),$$
$$p \in L^2((0,T)\times\Omega), \quad \nabla p^\pm \in H^{\ell,\ell/2}((0,T)\times\Omega^\pm), \quad [p] \in H^{1/2+\ell,1/4+\ell/2}((0,T)\times\Gamma),$$
$$X \in L^2((0,T)\times\Omega), \quad X^\pm \in H^1(0,T; H^{2+\ell}(\Omega^\pm)) \cap H^{2+\ell/2}(0,T; L^2(\Omega^\pm)).$$

Furthermore, for all time $t \in (0,T)$, $X(t,\cdot)$ is invertible and if we set $u_s(t,x) = u(t, Y(t,x))$ and $p_s(t,x) = p(t, Y(t,x))$, where $Y(t,\cdot)$ is the inverse transformation of $X(t,\cdot)$, then (u_s, p_s, X) is a strong solution of (2.1 .1)–(2.1 .2).

We also have a second result of global in time existence of strong solutions for small data.

Theorem 2.2 .5. *Let*

$$d = 3, \quad \ell \in (1/2, 1), \quad \Gamma,\ \partial\Omega \in C^4.$$

For all time $T > 0$, there exists $r > 0$ such that, for all

$$f \in H^{1/2+\ell,1/4+\ell/2}((0,T)\times\Gamma),$$
$$u_0 \in V(\Omega), \quad u_0^\pm \in H^{1+\ell}(\Omega^\pm).$$

satisfying the compatibility condition

$$2\mu([Du_0]n_\Gamma)_\tau = f_\tau|_{t=0} \quad \text{on } \Gamma,$$

where the subscript τ stands for the tangential component, D is the symmetric gradient, and the smallness condition

$$\|f\|_{H^{1/2+\ell,1/4+\ell/2}((0,T)\times\Gamma)} + \sum_\pm \|u_0^\pm\|_{H^{1+\ell}(\Omega^\pm)} \le r,$$

system (2.2 .9)–(2.2 .11) admits a solution (u, p, X) with

$$u \in L^2(0,T; H^1(\Omega)), \quad u^\pm \in H^{2+\ell,1+\ell/2}((0,T)\times\Omega^\pm),$$
$$p \in L^2((0,T)\times\Omega), \quad \nabla p^\pm \in H^{\ell,\ell/2}((0,T)\times\Omega^\pm), \quad [p] \in H^{1/2+\ell,1/4+\ell/2}((0,T)\times\Gamma),$$
$$X \in L^2((0,T)\times\Omega), \quad X^\pm \in H^1(0,T; H^{2+\ell}(\Omega^\pm)) \cap H^{2+\ell/2}(0,T; L^2(\Omega^\pm)).$$

Furthermore, for all time $t \in (0,T)$, $X(t,\cdot)$ is invertible and if we set $u_s(t,x) = u(t, Y(t,x))$ and $p_s(t,x) = p(t, Y(t,x))$, where $Y(t,\cdot)$ is the inverse transformation of $X(t,\cdot)$, then (u_s, p_s, X) is a strong solution of (2.1 .1)–(2.1 .2).

2.2 .2.3 Remarks on the main results

We need the enhanced regularity by ℓ to handle the non linearities generated by the moving boundary. The condition $\ell > 0$ in dimension $d = 2$ and $\ell > 1/2$ in dimension $d = 3$ is necessary to estimate non linear terms by using various embedding theorems. On the other hand, the condition $\ell < 1/2$ in dimension $d = 2$ and $\ell < 1$ in dimension $d = 3$ is used for minimizing the number of compatibility conditions.

Concerning the compatibility condition (2.2 .13), first remark that the Lagrangian transformation (2.2 .11) is identity at time $t = 0$. Therefore, $F^\pm|_{t=0} = G^\pm|_{t=0} = H|_{t=0} = 0$. Furthermore, $(2.1\ .1)_{2,3}$ imposes $u_0 \in V(\Omega)$. In dimension 3, due to the regularity of u_0 and f, equation $(2.2\ .1)_5$ has a trace at $t = 0$. The tangential part of $(2.2\ .1)_5$ gives (2.2 .13). Notice that the normal part of $(2.2\ .1)_5$ gives

$$[p]|_{t=0} = (-f|_{t=0} + 2\mu([Du_0]n_\Gamma)) \cdot n_\Gamma \quad \text{on } \Gamma. \tag{2.2 .14}$$

This condition is not a compatibility condition on the data, but a transmission condition in the Cauchy problem which defines $p|_{t=0}$. In dimension 2, equation $(2.2\ .1)_5$ does not give any compatibility condition.

2.2 .3 Strategy of proof

The proofs of Theorems 2.2 .2, 2.2 .3, 2.2 .4, 2.2 .5 are based on a precise study of the linearized equations and a fixed point argument. In Section 2.3 , we studied the associated linear system: a Stokes equation with a force distributed on Γ. In Section 2.4 , we estimate the non linear terms F, G, H. The estimations are slightly different to establish Theorems 2.2 .2 and 2.2 .4 or Theorems 2.2 .3 and 2.2 .5. First, we introduce some notations.

2.2 .4 Notations and functional settings

We divide this subsection into two parts: in Section 2.2 .4.1, we present the general notations used in the article, and in Section 2.2 .4.2, we introduce the functional settings proper to our work.

2.2 .4.1 Notations

We equip $\mathbb{R}^d$ with the natural scalar product and its norm $|\cdot|$ and we set

$$\mathbb{R}^d_- = \{x \in \mathbb{R}^d \mid x_d < 0\}.$$

The matrix I denotes the identity matrix of size d.

When not mentioned otherwise, if $w^\pm$ is a function defined in $\Omega^\pm$, $w = \mathbb{1}_{\Omega^-} w^- + \mathbb{1}_{\Omega^+} w^+$. However, pay attention that generally $\sigma(\mathbb{1}_{\Omega^-} u^- + \mathbb{1}_{\Omega^+} u^+, \mathbb{1}_{\Omega^-} p^- + \mathbb{1}_{\Omega^+} p^+) \neq \mathbb{1}_{\Omega^-} \sigma^-(u^-, p^-) + \mathbb{1}_{\Omega^+} \sigma^+(u^+, p^+)$.

We will not distinguish scalar functions and vectorial functions. It will appear clearly from the context which type of functions we consider.

For a measurable set O, we equip $L^2(O)$ with its natural inner product $(\cdot,\cdot)_O$. For a Banach space E, $r = p + s$ with $p \in \mathbb{N}$ and $s \in (0,1)$, we equip the scalar space $H^r(O; E)$ with the following norm:

$$\|f\|_{H^r(O;E)} = \sum_{|\alpha| \le p} \|D^\alpha f\|_{L^2(O;E)} + \sum_{|\alpha| = p} \int_O \int_O \frac{\|D^\alpha f(x) - D^\alpha f(y)\|_E^2}{\|x - y\|_O^{n+2s}} dx dy,$$

where $\alpha = (\alpha^1, ..., \alpha^n) \in \mathbb{N}^n$, $D^\alpha f = \partial_1^{\alpha^1} ... \partial_n^{\alpha^n} f$ and

$$\|D^\alpha f\|_{L^2(O;E)} = \sqrt{\int_O \|D^\alpha f(y)\|_E^2 dy}.$$

We recall that for $r, s, T > 0$ we also define

$$\begin{gathered} V(\Omega) = \{u \in H^1(\Omega) \mid \nabla \cdot u = 0 \text{ and } u = 0 \text{ on } \partial\Omega\}, \\ H^{r,s}((0,T) \times \Omega^\pm) = H^s(0,T; L^2(\Omega^\pm)) \cap L^2(0,T; H^r(\Omega^\pm)), \\ H^{r,s}((0,T) \times \Gamma) = H^s(0,T; L^2(\Gamma)) \cap L^2(0,T; H^r(\Gamma)). \end{gathered}$$

Lastly, we make the usual identification

$$(L^2(\Omega))' = L^2(\Omega).$$

2.2 .4.2 Functional settings

We now introduce some notations more specific to our work.

Let

$$\ell \in \begin{cases} (0,1/2) & \text{if } d=2, \\ (1/2,1) & \text{if } d=3. \end{cases} \tag{2.2 .15}$$

For $T>0$, we introduce the following spaces

$$\begin{aligned}
E^\ell_{u,T} &= \{u \in L^2(0,T;H^1(\Omega)) \mid u^\pm \in H^{2+\ell,1+\ell/2}((0,T)\times\Omega^\pm)\}, \\
\hat{E}^\ell_{p,T} &= \left\{ p \in L^2((0,T)\times\Omega^\pm) \;\middle|\; \begin{array}{l} \nabla p^\pm \in H^{\ell,\ell/2}((0,T)\times\Omega^\pm), \\ \int_{\Omega^-} p^-(t)dy = 0 \end{array} \right\}, \\
E^\ell_{p,T} &= \{p \in \hat{E}^\ell_p \mid [p] \in H^{1/2+l,1/4+l/2}((0,T)\times\Gamma)\}, \\
\hat{E}^\ell_{H,T} &= \left\{ H \in L^2(0,T;H^{1/2+\ell}(\Gamma)) \;\middle|\; \begin{array}{l} H_\tau \in H^{1/4+\ell/2}(0,T;L^2(\Gamma)), \\ H_n \in H^{\ell/2}(0,T;H^{1/2}(\Gamma)) \end{array} \right\}, \\
E^\ell_{H,T} &= H^{1/2+\ell,1/4+\ell/2}((0,T)\times\Gamma), \qquad\qquad (2.2\ .16)\\
E^\ell_{F,T} &= \{F \in L^2((0,T)\times\Omega) \mid F^\pm \in H^{\ell,\ell/2}((0,T)\times\Omega^\pm)\}, \\
E^\ell_{G,T} &= \left\{ G \in L^2((0,T)\times\Omega) \;\middle|\; \begin{array}{l} G^\pm \in H^{1+\ell/2}(0,T;L^2(\Omega^\pm)), \\ \nabla\cdot G^\pm \in L^2(0,T;H^{1+\ell}(\Omega^\pm)), \\ G|_{\partial\Omega}\cdot n = 0, \\ G^+\cdot n_\Gamma = G^-\cdot n_\Gamma, \\ \nabla\cdot G^\pm|_{t=0} = 0 \end{array} \right\}, \\
E^\ell_0 &= \{u_0 \in V(\Omega) \mid u_0^\pm \in H^{1+\ell}(\Omega^\pm)\},
\end{aligned}$$

where H_τ and H_n are respectively the tangential component and the normal component of H.

We add some explanation to clarify the reading of the above spaces: the first subscript denotes the unknown or the data concerned by the spaces, the second subscript T denotes the time horizon, and the exponent ℓ stands for the regularity. They are two spaces that correspond to the unknown p and H: spaces $\hat{E}^\ell_p$ and $\hat{E}^\ell_H$ correspond to the natural spaces of the linear problem, whereas spaces E^ℓ_p and E^ℓ_H correspond to the regularity needed for the non linear problem.

So as to get estimates with constants independent of time T, we equip these spaces with the following norms:

$$\begin{aligned}
\|u^\pm\|_{E^\ell_{u,\pm,T}} &= \|u^\pm\|_{H^{2+\ell,1+\ell/2}((0,T)\times\Omega^\pm)} + \|u^\pm\|_{H^{\ell/2}(0,T;H^2(\Omega^\pm))} + \|u^\pm\|_{H^{1/2+\ell/2}(0,T;H^1(\Omega^\pm))} \\
&\quad + \|u^\pm\|_{H^{(1+\ell-\beta)/2}(0,T;H^{1+\beta}(\Omega^\pm))} + \|\nabla u^\pm\|_{H^{1/4+\ell/2}(0,T;L^2(\Gamma))}, \\
\|u\|_{E^\ell_{u,T}} &= \|u^-\|_{E^\ell_{u,-}} + \|u^+\|_{E^\ell_{u,+}}, \\
\|p\|_{\hat{E}^\ell_p} &= \|\nabla p^-\|_{H^{\ell,\ell/2}((0,T)\times\Omega^-)} + \|\nabla p^+\|_{H^{\ell,\ell/2}((0,T)\times\Omega^+)}, \\
\|p\|_{E^\ell_{p,T}} &= \|p\|_{\hat{E}^\ell_p} + \|[p]\|_{H^{1/2+\ell,1/4+\ell/2}((0,T)\times\Gamma)}, \\
\|H\|_{\hat{E}^\ell_{H,T}} &= \|f_\tau\|_{H^{1/2+\ell,1/4+\ell/2}((0,T)\times\Gamma)} + \|f_n\|_{L^2(0,T;H^{1/2+\ell}(\Gamma))\cap H^{\ell/2}(0,T;H^{1/2}(\Gamma))}, \\
\|H\|_{E^\ell_{H,T}} &= \|H\|_{H^{1/2+\ell,1/4+\ell/2}((0,T)\times\Gamma)} + \|H\|_{H^{\ell/2}(0,T;H^{1/2}(\Gamma))} + \delta_{d=3}\|H|_{t=0}\|_{H^{\ell-1/2}(\Gamma)}, \\
\|F\|_{E^\ell_{F,T}} &= \|F^-\|_{H^{\ell,\ell/2}((0,T)\times\Omega^-)} + \|F^+\|_{H^{\ell,\ell/2}((0,T)\times\Omega^+)}, \\
\|G\|_{E^\ell_{G,T}} &= \|G^-\|_{H^{1+\ell/2}(0,T;L^2(\Omega^-))} + \|\nabla\cdot G^-\|_{L^2(0,T;H^{1+\ell}(\Omega^-))} \\
&\quad + \|G^+\|_{H^{1+\ell/2}(0,T;L^2(\Omega^+))} + \|\nabla\cdot G^+\|_{L^2(0,T;H^{1+\ell}(\Omega^+))}, \\
\|u_0\|_{E^\ell_0} &= \|u_0^-\|_{H^{1+\ell}(\Omega^-)} + \|u_0^+\|_{H^{1+\ell}(\Omega^+)},
\end{aligned} \tag{2.2 .17}$$

where β, used in $(2.2\ .17)_1$, is defined as follows

$$\beta = \begin{cases} \dfrac{\ell}{2} & \text{if } d = 2, \\ \dfrac{\ell + 1/2}{2} & \text{if } d = 3. \end{cases} \tag{2.2 .18}$$

We would like to point out that the norm $(2.2\ .17)_6$ depends on dimension $d \in \{2,3\}$. Precisely, in dimension $d = 3$, we add the trace term $\|H|_{t=0}\|_{H^{\ell-1/2}(\Gamma)}$ so as to have norms which scales appropriately with respect to the time horizon T. It will appear explicitly in the proof of Lemma 2.4 .2.

Furthermore, in dimension 3, we also require some compatibility condition: for $T, K > 0$ and $(u_0, f) \in E^\ell_{u_0} \times E^\ell_{H,T}$, $R > 0$, we define

$$\begin{aligned} E^\ell_{p,T,(u_0,f)} &= \{p \in E^\ell_p \mid p \text{ satisfies } (2.2\ .14) \text{ if } d = 3\}, \\ \hat{E}^\ell_{H,T,u_0} &= \{H \in E^\ell_f \mid H \text{ satisfies } (2.2\ .13)\} \text{ if } d = 3\}, \\ E^\ell_{H,T,u_0} &= \{H \in E^\ell_H \mid H \text{ satisfies } (2.2\ .13)\}, \\ E^\ell_{u,T,u_0,K} &= \{u \in E^\ell_u \mid u|_{t=0} = u_0 \text{ and } \|u\|_{E^\ell_u} \leq K\}, \\ E^\ell_{p,T,(u_0,f),K} &= \{p \in E^\ell_{p,(u_0,f)} \mid \|p\|_{E^\ell_p} \leq K\}. \end{aligned}$$

The third subscript stands for the compatibility condition at time $t = 0$ and the last subscript denotes the radius of the ball of the corresponding space.

For the sake of simplicity, we will omit the exponent ℓ in the proofs.

In the following, we take a final time $T^* > 0$ and a maximum size of ball $K^* > 0$ that will be defined later. Lastly, C denotes a generic constant that may change from line to line. The dependence of this constant on the parameters of the problem appears only when it is necessary. In Section 2.3 , C is independent of time T and of the data f and u_0. In Section 2.4 .1, C may depend on f and u_0 but is independent of time T; and in Section 2.4 .2, C may depend on time T but is independent of the data f and u_0.

2.3 The linear problem

In this section, we are interested in

$$\begin{cases} \partial_t u - \nabla \cdot \sigma(u,p) = f\delta_\Gamma + F & \text{in } (0,T) \times \Omega, \\ \nabla \cdot u = \nabla \cdot G & \text{in } (0,T) \times \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \\ u|_{t=0} = u_0 & \text{in } \Omega. \end{cases} \tag{2.3 .1}$$

The purpose of this section is to prove the following result of existence, uniqueness and regularity of solutions to the linear system (2.3 .1).

Theorem 2.3 .1. *Let*

$$\ell \text{ as in } (2.2\ .15), \quad \Gamma \in C^4, \quad T \in (0, T^*),$$
$$(F, G, f, u_0) \in E^\ell_{F,T} \times E^\ell_{G,T} \times \hat{E}^\ell_{H,T} \times E^\ell_0.$$

We assume the compatibility condition (2.2 .13) in dimension $d = 3$. There exists a unique solution $(u,p) \in E^\ell_{u,T} \times \hat{E}^\ell_{p,T}$ of system (2.3 .1). Moreover,

$$\|(u,p)\|_{E^\ell_{u,T} \times \hat{E}^\ell_{p,T}} \leq C\|(F,G,f,u_0)\|_{E^\ell_{F,T} \times E^\ell_{G,T} \times \hat{E}^\ell_{H,T} \times E^\ell_0},$$

where C does not depend on time T. Furthermore, if we assume the regularity $f \in E^{\ell}_{H,T}$, then $p \in E^{\ell}_{p,T}$ and satisfies (2.2 .14), and we have the estimate

$$\|(u,p)\|_{E^{\ell}_{u,T}\times E^{\ell}_{p,T}} \leq C\|(F,G,f,u_0)\|_{E^{\ell}_{F,T}\times E^{\ell}_{G,T}\times E^{\ell}_{H,T}\times E^{\ell}_{0}},$$

where C does not depend on time T.

A similar result can be found in [19, Theorem 3.1] in the whole space $\mathbb{R}^d$, with lower regularity, and with a different regularity in the divergence term. Although the two results are close, the two proofs are technically different. The authors use a Dirichlet-to-Neumann operator whereas we prefer to use the lifting results in [13, Chapter 4, Theorem 2.3]. The main advantage of our proof is that it is easily adaptable to all regularities that would correspond to $\ell \in [0,+\infty)$ provided proper defined compatibility conditions. Another result which also deals with system (2.3 .1) is [25, Theorem 2.5], but it remains different in the integrability and the estimation.

The proof is based on liftings of the singularities induced by the boundary Γ. It is important to notice that $f\delta_\Gamma$ is not the only singularity induced by Γ. The non homogeneous term F and G and the initial data u_0 are regular only on $\Omega^\pm$ and not on the whole space Ω. We first lift u_0 to work with functions which value 0 at time $t=0$ so that we can easily get estimates with a constant independent of time T. Then we lift the divergence term $\nabla\cdot G$, the force term $f\delta_\Gamma$ and the singularity of F on Γ so as to get a distributed force with a maximal regularity given on the whole space Ω. Concerning the lifting of the force term $f\delta_\Gamma$, remark that system (2.3 .1) is equivalent to

$$\left\{\begin{array}{rcll} \partial_t u^\pm - \nabla\cdot\sigma(u^\pm,p^\pm) &=& F^\pm & \text{in } (0,T)\times\Omega^\pm,\\ \nabla\cdot u^\pm &=& \nabla\cdot G^\pm & \text{in } (0,T)\times\Omega^\pm,\\ u^+ &=& 0 & \text{on } (0,T)\times\partial\Omega,\\ u^- &=& u^+ & \text{on } (0,T)\times\Gamma,\\ [\sigma(u,p)]n_\Gamma &=& f & \text{on } (0,T)\times\Gamma,\\ u^\pm|_{t=0} &=& u_0^\pm & \text{in } \Omega^\pm. \end{array}\right. \tag{2.3 .2}$$

We therefore lift the force $f\delta_\Gamma$ constructing $(\tilde{u}^+,\tilde{p}^+)$ defined in Ω^+ such that $\sigma(\tilde{u}^+,\tilde{p}^+)n_\Gamma = f$.

The proof is organized as follows: in Section 2.3 .1, we lift the initial data u_0. In Section 2.3 .2, we lift the divergence $\nabla\cdot G$, the measure force $f\delta_\Gamma$ as explained above and the singularity on the distributed force F. Then, we prove Theorem 2.3 .1 in Section 2.3 .3. Lastly, Section 2.3 .4 is dedicated to prove technical Lemmas used in the second step to lift the divergence $\nabla\cdot G$, the measure force $f\delta_\Gamma$ and the singularity on the distributed force F.

2.3 .1 Lifting the initial data u_0

We are interested in constructing $\check{u} \in E_{u,T}$ such that

$$\left\{\begin{array}{rcll} \check{u} &=& 0 & \text{on } (0,T)\times\partial\Omega,\\ \check{u}|_{t=0} &=& u_0 & \text{in } \Omega, \end{array}\right. \tag{2.3 .3}$$

where $u_0 \in E_0$.

Theorem 2.3 .2. *Let $0 < T < T^*$ and $u_0 \in E^{\ell}_0$. There exists a solution $\check{u} \in E^{\ell}_{u,T}$ of (2.3 .3) which satisfies the estimate*

$$\|\check{u}\|_{E^{\ell}_{u,T}} \leq C\|u_0\|_{E^{\ell}_0},$$

where C does not depend on time T. In particular,

$$\partial_t\check{u} - 2\mu\nabla\cdot D(\check{u}) = \mathbb{1}_{\Omega^-}(\partial_t\check{u}^- - 2\mu\nabla\cdot D(\check{u}^-)) + \mathbb{1}_{\Omega^+}(\partial_t\check{u}^+ - 2\mu\nabla\cdot D(\check{u}^+)) + 2\mu[D(\check{u})n_\Gamma]\delta_\Gamma,$$

and

$$\check{u} \in E^{\ell}_{G,T},$$

with

$$\|\mathbb{1}_{\Omega^-}(\partial_t \check{u}^- - 2\mu\nabla\cdot D(\check{u}^-)) + \mathbb{1}_{\Omega^+}(\partial_t \check{u}^+ - 2\mu\nabla\cdot D(\check{u}^+))\|_{E^{\ell}_F} + \|[D(\check{u})n_\Gamma]\|_{E^{\ell}_{H,T}} + \|\check{u}\|_{E^{\ell}_{G,T}} \leq C\|u_0\|_{E^{\ell}_0},$$

where C does not depend on time T.

Proof of Theorem 2.3 .2. Using the result [13, Chapter 4, Theorem 2.3], there exists $\check{v}^- \in H^{2+\ell,1+\ell/2}((0,T^*)\times\Omega^-)$ such that

$$\check{v}^-|_{t=0} = u_0^- \quad \text{on } \Omega^-,$$

and which satisfies the estimate

$$\|\check{v}^-\|_{H^{2+\ell,1+\ell/2}((0,T^*)\times\Omega^-)} \leq C\|u_0^-\|_{H^{1+\ell}(\Omega^-)}.$$

Still with [13, Chapter 4, Theorem 2.3], there exists $\check{v}^+ \in H^{2+\ell,1+\ell/2}((0,T^*)\times\Omega^+)$ such that

$$\begin{cases} \check{v}^+ &= \check{v}^- \quad \text{on } (0,T^*)\times\Gamma, \\ \check{v}^+ &= 0 \quad \text{on } (0,T^*)\times\partial\Omega, \\ \check{v}^+(0) &= u_0^+ \quad \text{in } \Omega^+, \end{cases}$$

with

$$\|\check{v}^+\|_{H^{2+\ell,1+\ell/2}((0,T^*)\times\Omega^+)} \leq C(\|u_0^+\|_{H^{1+\ell}(\Omega^+)} + \|\check{v}^-|_\Gamma\|_{H^{3/2+\ell,3/4+\ell/2}((0,T^*)\times\Gamma)}) \leq C\|u_0\|_{E_0}.$$

Remark that $\check{v} = \mathbb{1}_{\Omega^-}\check{v}^- + \mathbb{1}_{\Omega^+}\check{v}^+ \in L^2(0,T^*;H^1(\Omega))$ with $\check{v}^\pm \in H^{2+\ell,1+\ell/2}((0,T^*)\times\Omega^\pm)$. The advantage of constructing $\check{v}$ on $(0,T^*)$ is that interpolation and trace estimates on $\check{v}^\pm$ do not depend on time T. Since $\nabla\cdot u_0 = 0$ and $\check{v} = 0$ on $\partial\Omega$, we have $\check{v} \in E_{G,T^*}$. Defining

$$\check{u} = \check{v}|_{(0,T)},$$

it is easily seen that $\check{u}$ satisfies the requirement of Theorem 2.3 .2. The interpolation and trace estimates do not depend on time T because they are done on $\check{v}$. □

2.3 .2 Lifting the divergence $\nabla\cdot G$ and the measure force f

The purpose of this section is to construct a lifting $(\tilde{u},\tilde{p})$ of all singularities created by Γ in system (2.3 .1). That way, the unknown $(\hat{u},\hat{p}) = (u-\check{u}-\tilde{u}, p-\tilde{p})$ will be a solution of a classical Stokes equation where Γ does not appear. It must be understood that Γ generates singularities with the measure force f but also in the regularities of u_0, G, F. In the previous subsection, we have lifted the initial data u_0. In this subsection, we lift the measure force f, the divergence term $\nabla\cdot G$ and also the singularity generated by Γ in F. Actually, in dimension $d=2$, there is no singularity on F: indeed, $F \in E^{\ell}_{F,T}$ with $\ell \in (0,1/2)$ so $F \in H^{\ell,\ell/2}((0,T)\times\Omega)$. However, in dimension $d=3$, $\ell \in (1/2,1)$ and to get $F \in H^{\ell,\ell/2}((0,T)\times\Omega)$, we must have $F^+ = F^-$ on Γ. This subsection is therefore divided into two subsection according to the dimension d.

2.3 .2.1 Lifting for d=2

We start with the case $d = 2$. We are interested in constructing a couple $(\tilde{u}, \tilde{p}) \in E_{u,T} \times \hat{E}_{p,T}$ such that

$$\begin{cases} \nabla \cdot \tilde{u}^{\pm} &= \nabla \cdot (G^{\pm} - \check{u}^{\pm}) & \text{in } (0,T) \times \Omega^{\pm}, \\ \tilde{u}^{-} &= \tilde{u}^{+} & \text{on } (0,T) \times \Gamma, \\ [\sigma(\tilde{u}, \tilde{p}) n_{\Gamma}] &= f - 2\mu [D(\check{u}) n_{\Gamma}] & \text{on } (0,T) \times \Gamma, \\ \tilde{u} &= 0 & \text{on } (0,T) \times \partial\Omega, \\ \tilde{u}|_{t=0} &= 0 & \text{in } \Omega, \end{cases} \tag{2.3 .4}$$

where $(f, u_0, G) \in \hat{E}_{H,T} \times E_0 \times E_G$ and $\check{u}$ is given by Theorem 2.3 .2. Remark that system (2.3 .4) is of the form

$$\begin{cases} \nabla \cdot \tilde{u}^{\pm} &= \nabla \cdot \tilde{G}^{\pm} & \text{in } (0,T) \times \Omega^{\pm}, \\ \tilde{u}^{-} &= \tilde{u}^{+} & \text{on } (0,T) \times \Gamma, \\ [\sigma(\tilde{u}, \tilde{p}) n_{\Gamma}] &= \tilde{f} & \text{on } (0,T) \times \Gamma, \\ \tilde{u} &= 0 & \text{on } (0,T) \times \partial\Omega, \\ \tilde{u}|_{t=0} &= 0 & \text{in } \Omega, \end{cases} \tag{2.3 .5}$$

where $(\tilde{f}, \tilde{G}) \in \hat{E}_{H,T} \times E_{G,T}$. Since we could only have regularity on $\Omega^{\pm}$, we construct $\tilde{u}$ on $\Omega^{\pm}$. The main idea is contained in the following Lemma.

Lemma 2.3 .3. *Let $d = 2$, $\ell \in (0, 1/2)$, $0 < T < T^*$, $(\tilde{G}, \tilde{h}, \tilde{f}) \in E^{\ell}_{G,T} \times H^{3/2+\ell, 3/4+\ell/2}((0,T) \times \Gamma) \times \hat{E}^{\ell}_{H}$. We assume that*

$$\begin{cases} \int_{\Gamma} \tilde{G} \cdot n_{\Gamma} dx &= \int_{\Gamma} \tilde{h} \cdot n_{\Gamma} dx, \\ (\tilde{G} - \tilde{h}) \cdot n_{\Gamma} &\in H^{1+\ell/2}(0,T; H^{-1/2}(\Gamma)), \\ \tilde{h}|_{t=0} &= 0 \quad \textit{on } \Gamma. \end{cases} \tag{2.3 .6}$$

Then, there exists $(\tilde{u}^{+}, \tilde{p}^{+}) \in H^{2+\ell, 1+\ell/2}((0,T) \times \Omega^{+}) \times (L^2(0,T; H^{1+\ell}(\Omega^{+}) \cap H^{\ell/2}(0,T; H^1(\Omega^{+})))$ such that

$$\begin{cases} \nabla \cdot \tilde{u}^{+} &= \nabla \cdot \tilde{G}^{+} & \textit{in } (0,T) \times \Omega^{+}, \\ \tilde{u}^{+} &= \tilde{h} & \textit{on } (0,T) \times \Gamma, \\ \sigma(\tilde{u}^{+}, \tilde{p}^{+}) n_{\Gamma} &= \tilde{f} & \textit{on } (0,T) \times \Gamma, \\ \tilde{u}^{+} &= 0 & \textit{on } (0,T) \times \partial\Omega, \\ \tilde{u}^{+}|_{t=0} &= 0 & \textit{in } \Omega^{+}. \end{cases} \tag{2.3 .7}$$

Moreover,

$$\begin{aligned} &\|(\tilde{u}^{+}, \tilde{p}^{+})\|_{H^{2+\ell,1+\ell/2}((0,T)\times\Omega^{+}) \times (L^2(0,T;H^{1+\ell}(\Omega^{+}) \cap H^{\ell/2}(0,T;H^1(\Omega^{+})))} \\ &\quad \leq C(\|(\tilde{G}, \tilde{h}, \tilde{f})\|_{E^{\ell}_{G,T} \times H^{3/2+\ell,3/4+\ell/2}((0,T)\times\Gamma) \times \hat{E}^{\ell}_{H,T}} + \|(\tilde{G}^{+} - \tilde{h}) \cdot n_{\Gamma}\|_{H^{1+\ell/2}(0,T;H^{-1/2}(\Gamma))}), \end{aligned} \tag{2.3 .8}$$

where C does not depend on time T.

Lemma 2.3 .3 is the key point of the linear problem. The proof is technical and is done in Section 2.3 .4. With this result, we are able to construct a couple $(\tilde{u}, \tilde{p}) \in E_{u,T} \times \hat{E}_{p,T}$ which satisfies (2.3 .5):

Theorem 2.3 .4. *Let $d = 2$, $\ell \in (0, 1/2)$, $0 < T < T^*$, $(\tilde{f}, \tilde{G}) \in \hat{E}^{\ell}_{H,T} \times E^{\ell}_{G,T}$. There exists a solution $(\tilde{u}, \tilde{p}) \in E^{\ell}_{u,T} \times \hat{E}^{\ell}_{p,T}$ of (2.3 .5) which satisfies the estimate*

$$\|(\tilde{u}, \tilde{p})\|_{E^{\ell}_{u,T} \times \hat{E}^{\ell}_{p,T}} \leq C \|(\tilde{f}, \tilde{G})\|_{\hat{E}^{\ell}_{H,T} \times E^{\ell}_{G,T}},$$

where C does not depend on time T. In particular,

$$\partial_t \tilde u - \nabla\cdot\sigma(\tilde u,\tilde p) = \mathbb{1}_{\Omega^-}(\partial_t\tilde u^- + \nabla\cdot\sigma(\tilde u^-,\tilde p^-)) + \mathbb{1}_{\Omega^+}(\partial_t\tilde u^+ + \nabla\cdot\sigma(\tilde u^+,\tilde p^+)) + \tilde f\delta_\Gamma,$$

with

$$\|\mathbb{1}_{\Omega^-}(\partial_t\tilde u^- + \nabla\cdot\sigma(\tilde u^-,\tilde p^-)) + \mathbb{1}_{\Omega^+}(\partial_t\tilde u^+ + \nabla\cdot\sigma(\tilde u^+,\tilde p^+))\|_{E^\ell_{F,T}} \le C\|(\tilde f,\tilde G)\|_{\hat E^\ell_{H,T}\times E^\ell_{G,T}},$$

where C does not depend on time T.

Proof of Theorem 2.3 .4. We look for $\tilde u^-$ of the form $\tilde u^- = \nabla\tilde\xi^-$ where, for all $t\in[0,T]$, $\tilde\xi^-(t)$ is solution of

$$\left\{\begin{array}{rcll} \Delta\tilde\xi^-(t) &=& \nabla\cdot\tilde G^-(t) & \text{in } \Omega^-,\\ \tilde\xi^-(t) &=& 0 & \text{on } \Gamma. \end{array}\right. \tag{2.3 .9}$$

For all $t\in[0,T]$, there exists a unique solution $\tilde\xi^-(t)\in H^1(\Omega^-)$ of (2.3 .9). Moreover, we have the estimate

$$\|\tilde\xi^-\|_{H^{1+\ell/2}(0,T;H^1(\Omega^-))\cap L^2(0,T;H^{3+\ell}(\Omega^-))} \le C\|\tilde G\|_{E_{G,T}},$$

The proof of this last statement can be found in [20, Lemma 4.1]. Note that to get the $H^{1+\ell/2}(0,T;H^1(\Omega^-))$ estimate, we use that

$$\left\{\begin{array}{rcll} \Delta(\partial_t\tilde\xi^-(t)) &=& \nabla\cdot\partial_t\tilde G^-(t) & \text{in } \Omega^-,\\ \partial_t\tilde\xi^-(t) &=& 0 & \text{on } \Gamma, \end{array}\right.$$

and therefore,

$$\left\{\begin{array}{rcll} \Delta(\partial_t\tilde\xi^-(t)-\partial_t\tilde\xi^-(s)) &=& \nabla\cdot(\partial_t\tilde G^-(t)-\partial_t\tilde G^-(s)) & \text{in } \Omega^-,\\ \partial_t\tilde\xi^-(t)-\partial_t\tilde\xi^-(s) &=& 0 & \text{on } \Gamma, \end{array}\right.$$

so that we have

$$\|\partial_t\tilde\xi^-(t)\|_{H^1(\Omega^-)} \le C\|\nabla\cdot\partial_t\tilde G^-(t)\|_{H^{-1}(\Omega^-)} \le C\|\partial_t\tilde G^-(t)\|_{L^2(\Omega^-)},$$

and

$$\|\partial_t\tilde\xi^-(t)-\partial_t\tilde\xi^-(s)\|_{H^1(\Omega^-)} \le C\|\partial_t\tilde G^-(t)-\partial_t\tilde G^-(s)\|_{L^2(\Omega^-)}.$$

Remark that $\nabla\cdot\tilde G^-(0)=0$ by definition of $E_{G,T}$ (see $(2.2\ .16)_7$), so the unique solution of (2.3 .9) at time $t=0$ is $\tilde\xi^-(0)=0$. Following the proof of [20, Lemma 4.3], we also have $(\tilde G^- - \nabla\tilde\xi^-)\cdot n_\Gamma \in H^{1+\ell/2}(0,T;H^{-1/2}(\Gamma))$ with

$$\|(\tilde G^- - \nabla\tilde\xi^-)\cdot n_\Gamma\|_{H^{1+\ell/2}(0,T;H^{-1/2}(\Gamma))} \le C\|\tilde G\|_{E_{G,T}}.$$

Moreover, by definition of $E_{G,T}$, $\tilde G^+\cdot n_\Gamma = \tilde G^-\cdot n_\Gamma$ so that

$$\int_{\Omega^+}\nabla\cdot\tilde G^+ dy = \int_\Gamma \nabla\tilde\xi^-\cdot(-n_\Gamma)ds.$$

The assumptions to apply Lemma 2.3 .3 are fulfilled: there exists $(\tilde u^+,\tilde p^+)\in H^{2+\ell,1+\ell/2}((0,T)\times\Omega^+)\times(L^2(0,T;H^{1+\ell}(\Omega^+))\cap H^{\ell/2}(0,T;H^1(\Omega^+)))$ such that

$$\left\{\begin{array}{rcll} \nabla\cdot\tilde u^+ &=& \nabla\cdot\tilde G^+ & \text{in } (0,T)\times\Omega^+,\\ \tilde u^+ &=& \tilde u^- & \text{on } (0,T)\times\Gamma,\\ \sigma(\tilde u^+,\tilde p^+)n_\Gamma &=& -\tilde f + 2\mu D(\tilde u^-)n_\Gamma & \text{on } (0,T)\times\Gamma,\\ \tilde u^+ &=& 0 & \text{on } (0,T)\times\partial\Omega,\\ \tilde u^+|_{t=0} &=& 0 & \text{in } \Omega^+, \end{array}\right. \tag{2.3 .10}$$

with the estimate

$$\|(\tilde{u}^+,\tilde{p}^+)\|_{H^{2+\ell,1+\ell/2}((0,T)\times\Omega^+)\times(L^2(0,T;H^{1+\ell}(\Omega^+))\cap H^{\ell/2}(0,T;H^1(\Omega^+)))} \leq C\|(\tilde{G},\tilde{f})\|_{E_{G,T}\times\hat{E}_{H,T}}.$$

Defining

$$\tilde{u} = \mathbb{1}_{\Omega^-}\tilde{u}^- + \mathbb{1}_{\Omega^+}\tilde{u}^+ \quad \text{and} \quad \tilde{p} = \mathbb{1}_{\Omega^+}\tilde{p}^+,$$

it is clear that $(\tilde{u},\tilde{p})$ satisfies the requirement of Theorem 2.3 .4. The interpolation and trace estimates do not depend on time T because $\tilde{u}|_{t=0} = 0$. □

2.3 .2.2 Lifting for d=3

The case $d = 3$ requires an additional equation: we are interested in constructing a couple $(\tilde{u},\tilde{p}) \in E_{u,T} \times \hat{E}_{p,T}$ such that

$$\begin{cases} \nabla\cdot\tilde{u}^\pm &= \nabla\cdot(G^\pm - \check{u}^\pm) & \text{in } (0,T)\times\Omega^\pm,\\ \tilde{u}^- &= \tilde{u}^+ & \text{on } (0,T)\times\Gamma,\\ [\sigma(\tilde{u},\tilde{p})n_\Gamma] &= f - 2\mu[D(\check{u})n_\Gamma] & \text{on } (0,T)\times\Gamma,\\ [\nabla\cdot\sigma(\tilde{u},\tilde{p})] &= [F + 2\mu D(\check{u})] & \text{on } (0,T)\times\Gamma,\\ \tilde{u} &= 0 & \text{on } (0,T)\times\partial\Omega,\\ \tilde{u}|_{t=0} &= 0 & \text{in } \Omega, \end{cases} \tag{2.3 .11}$$

where $(f,u_0,G,F) \in \hat{E}_{H,T}\times E_{u_0}\times E_{G,T}\times E_{F,T}$ satisfies the compatibility condition (2.2 .13) and $\check{u}$ is given by Theorem 2.3 .2. Remark that system (2.3 .4) is of the form

$$\begin{cases} \nabla\cdot\tilde{u}^\pm &= \nabla\cdot\tilde{G}^\pm & \text{in } (0,T)\times\Omega^\pm,\\ \tilde{u}^- &= \tilde{u}^+ & \text{on } (0,T)\times\Gamma,\\ [\sigma(\tilde{u},\tilde{p})n_\Gamma] &= \tilde{f} & \text{on } (0,T)\times\Gamma,\\ [\nabla\cdot\sigma(\tilde{u},\tilde{p})] &= \tilde{F} & \text{on } (0,T)\times\Gamma,\\ \tilde{u} &= 0 & \text{on } (0,T)\times\partial\Omega,\\ \tilde{u}|_{t=0} &= 0 & \text{in } \Omega, \end{cases} \tag{2.3 .12}$$

where $(\tilde{f},\tilde{G},\tilde{F}) \in \hat{E}_{H,T}\times E_{G,T}\times H^{\ell-1/2,\ell/2-1/4}((0,T)\times\Gamma)$ with the compatibility condition:

$$\tilde{f}_\tau|_{t=0} = 0.$$

The following Lemma is the key point of the linear part in dimension $d = 3$ and is the adaptation of Lemma 2.3 .3 for $\ell \in (1/2,1)$. Its proof is postponed to Section 2.3 .4.

Lemma 2.3 .5. *Let $d = 3$, $\ell \in (1/2,1)$, $0 < T < T^*$, $(\tilde{G},\tilde{F},\tilde{h},\tilde{f}) \in E^\ell_{G,T}\times H^{\ell-1/2,\ell/2-1/4}((0,T)\times\Gamma)\times H^{3/2+\ell,3/4+\ell/2}((0,T)\times\Gamma)\times\hat{E}^\ell_H$. We assume that*

$$\begin{cases} \int_\Gamma \tilde{G}\cdot n_\Gamma dx &= \int_\Gamma \tilde{h}\cdot n_\Gamma dx,\\ (\tilde{G}-\tilde{h})\cdot n_\Gamma &\in H^{1+\ell/2}(0,T;H^{-1/2}(\Gamma)),\\ \tilde{h}|_{t=0} &= 0 \quad \text{on } \Gamma,\\ \tilde{f}_\tau|_{t=0} &= 0 \quad \text{on } \Gamma. \end{cases} \tag{2.3 .13}$$

Then, there exists $(\tilde{u}^+,\tilde{p}^+) \in H^{2+\ell,1+\ell/2}((0,T)\times\Omega^+)\times(L^2(0,T;H^{1+\ell}(\Omega^+))\cap H^{\ell/2}(0,T;H^1(\Omega^+)))$ such that

$$\begin{cases} \nabla\cdot\tilde{u}^+ &= \nabla\cdot\tilde{G}^+ & \text{in } (0,T)\times\Omega^+,\\ \tilde{u}^+ &= \tilde{h} & \text{on } (0,T)\times\Gamma,\\ \sigma(\tilde{u}^+,\tilde{p}^+)n_\Gamma &= \tilde{f} & \text{on } (0,T)\times\Gamma,\\ \nabla\cdot\sigma(\tilde{u}^+,\tilde{p}^+) &= \tilde{F} & \text{on } (0,T)\times\Gamma,\\ \tilde{u}^+ &= 0 & \text{on } (0,T)\times\partial\Omega,\\ \tilde{u}^+|_{t=0} &= 0 & \text{in } \Omega^+. \end{cases} \tag{2.3 .14}$$

Moreover,

$$\begin{aligned}\|(\tilde u^+,\tilde p^+)\|_{H^{2+\ell,1+\ell/2}((0,T)\times\Omega^+)\times(L^2(0,T;H^{1+\ell}(\Omega^+)\cap H^{\ell/2}(0,T;H^1(\Omega^+)))}\\ \le C(\|(\tilde G,\tilde F,\tilde h,\tilde f)\|_{E^\ell_{G,T}\times H^{\ell-1/2,\ell/2-1/4}((0,T)\times\Gamma)\times H^{3/2+\ell,3/4+\ell/2}((0,T)\times\Gamma)\times\hat E^\ell_{H,T}}\\ +\|(\tilde G^+-\tilde h)\cdot n_\Gamma\|_{H^{1+\ell/2}(0,T;H^{-1/2}(\Gamma))}),\end{aligned} \quad (2.3\ .15)$$

where C does not depend on time T.

Proceeding as in dimension $d=2$, we can construct a couple $(\tilde u,\tilde p)\in E_{u,T}\times\hat E_{p,T}$ which satisfies (2.3 .12):

Theorem 2.3 .6. *Let $d=3$, $\ell\in(1/2,1)$, $0<T<T^*$, $(\tilde f,\tilde G,\tilde F)\in\hat E^\ell_{H,T}\times E^\ell_{G,T}\times H^{\ell-1/2,\ell/2-1/4}((0,T)\times\Gamma)$ such that $\tilde f_\tau|_{t=0}=0$. There exists a solution $(\tilde u,\tilde p)\in E^\ell_{u,T}\times\hat E^\ell_{p,T}$ of (2.3 .12) which satisfies the estimate*

$$\|(\tilde u,\tilde p)\|_{E^\ell_u\times\hat E^\ell_p}\le C\|(\tilde f,\tilde G,\tilde F)\|_{E^\ell_f\times E^\ell_G\times H^{\ell-1/2,\ell/2-1/4}((0,T)\times\Gamma)},$$

where C does not depend on time T. In particular,

$$\partial_t\tilde u-\nabla\cdot\sigma(\tilde u,\tilde p)=\mathbb{1}_{\Omega^-}(\partial_t\tilde u^-+\nabla\cdot\sigma(\tilde u^-,\tilde p^-))+\mathbb{1}_{\Omega^+}(\partial_t\tilde u^++\nabla\cdot\sigma(\tilde u^+,\tilde p^+))+\tilde f\delta_\Gamma,$$

with

$$\begin{aligned}\|\mathbb{1}_{\Omega^-}(\partial_t\tilde u^-+\nabla\cdot\sigma(\tilde u^-,\tilde p^-))+\mathbb{1}_{\Omega^+}(\partial_t\tilde u^++\nabla\cdot\sigma(\tilde u^+,\tilde p^+))\|_{E^\ell_{F,T}}\\ \le C\|(\tilde f,\tilde G,\tilde F)\|_{E^\ell_f\times E^\ell_G\times H^{\ell-1/2,\ell/2-1/4}((0,T)\times\Gamma)},\end{aligned}$$

where C does not depend on time T.

The proof of Theorem 2.3 .6 is essentially the same as the proof of Theorem 2.3 .4. The only difference is in the lifting in the system (2.3 .10). First, we have to check that $(-\tilde f+2\mu D(\tilde u^-)n_\Gamma)_\tau|_{t=0}=0$ on Γ which is immediate with the assumptions and $\tilde u^-=0$. Then, we have to add the line

$$\nabla\cdot\sigma(\tilde u^+,\tilde p^+)=\tilde F+\nabla\cdot\sigma(\tilde u^-,\tilde p^-)\quad\text{on }(0,T)\times\Gamma,$$

to system (2.3 .10). Lemma 2.3 .5 enables to conclude.

2.3 .3 Proof of Theorem 2.3 .1

We look for (u,p) of the form

$$u=\check u+\tilde u+\hat u\quad\text{and}\quad p=\tilde p+\hat p, \quad (2.3\ .16)$$

where $\check u$ is given by Theorem 2.3 .2 and $(\hat u,\hat p)$ by Theorem 2.3 .4 if $d=2$ and Theorem 2.3 .6 if $d=3$ and $(\hat u,\hat p)$ is solution of

$$\left\{\begin{array}{rcll}\partial_t\hat u-\mu\Delta\hat u+\nabla\hat p&=&F+\hat F&\text{in }(0,T)\times\Omega,\\ \nabla\cdot\hat u&=&0&\text{on }(0,T)\times\Omega,\\ \hat u&=&0&\text{on }(0,T)\times\partial\Omega,\\ \hat u|_{t=0}&=&0&\text{in }\Omega,\end{array}\right. \quad (2.3\ .17)$$

with $\hat F=\mathbb{1}_{\Omega^-}(-\partial_t(\check u^-+\tilde u^-)+2\mu\nabla\cdot D(\check u^-)+\nabla\cdot\sigma(\tilde u^-,\tilde p^-))+\mathbb{1}_{\Omega^+}(-\partial_t(\check u^++\tilde u^+)+2\mu\nabla\cdot D(\check u^+)+\nabla\cdot\sigma(\tilde u^+,\tilde p^+))\in E_{F,T}$. By construction of $(\tilde u,\tilde p)$ and using $[\partial_t(\check u+\hat u)]=0$, we have

$F+\hat{F}\in H^{\ell,\ell/2}((0,T)\times\Omega)$. With [13, Chapter 4, Theorem 5.3], there exists a unique solution $(\hat{u},\hat{p})\in E_{u,T}\times\hat{E}_{p,T}$ of (2.3 .17). Moreover we have

$$\|(\hat{u},\hat{p})\|_{E_{u,T}\times\hat{E}_{p,T}}\leq C\|F+\hat{F}\|_{H^{\ell,\ell/2}((0,T)\times\Omega)}\leq C\|(F,G,f,u_0)\|_{E_{F,T}\times E_{G,T}\times\hat{E}_{p,T}\times E_0}.$$

Then the couple (u,p) defined by (2.3 .16) is solution of (2.3 .1) and lies in $E_{u,T}\times\hat{E}_{p,T}$ and the corresponding estimate follows.

Uniqueness comes from uniqueness in classical Stokes equation (2.3 .17).

Lastly, equation $(2.3\ .2)_5\cdot n_\Gamma$ gives

$$[p]=-f_n+2\mu[D(u)]n_\Gamma\cdot n_\Gamma\quad\text{on }\Gamma.$$

Therefore, the regularity $f\in E_{H,T}$ implies that $p\in E_{p,T}$ and that p satisfies (2.2 .14). The corresponding estimate easily follows, the constant being independent of time T since $D(u)=D(\check{u})+D(\tilde{u})$ and the estimate of $D(\check{u})$ and $D(\tilde{u})$ in $E_{H,T}$ does not depend on time T.

2.3 .4 Proof of Lemmas 2.3 .3 and 2.3 .5

The proof is divided into four main steps. In Section 2.3 .4.1, we lift the divergence term $\nabla\cdot\tilde{G}^+$. In Section 2.3 .4.2, we lift the normal trace of $\tilde{h}\cdot n$. In Section 2.3 .4.3, we lift the tangential trace $\tilde{h}_\tau$ and the stress tensor $\tilde{f}$ in dimension $d=2$. And in Section 2.3 .4.4, we lift the tangential trace $\tilde{h}_\tau$, the stress tensor $\tilde{f}$ and the trace of the divergence of the stress tensor $\tilde{F}$ in dimension $d=3$. The two first steps are classical and are inspired from [20, Lemma 4.1 and Lemma 4.2], but the two last steps are more technical and constitute the key point of the analysis of the linear problem.

2.3 .4.1 Lifting the divergence term

Similarly as in the proof of Theorem 2.3 .4, for all $t\in[0,T]$, we are interested in solving

$$\left\{\begin{array}{rcll}\Delta\tilde{\xi}^+(t)&=&\nabla\cdot\tilde{G}^+(t)&\text{in }\Omega,\\ \tilde{\xi}^+(t)&=&0&\text{on }\Gamma,\\ \tilde{\xi}^+(t)&=&0&\text{on }\partial\Omega.\end{array}\right.\tag{2.3 .18}$$

We have already seen that there exists a unique solution $\tilde{\xi}^+(t)\in H^1(\Omega^+)$, that $\tilde{\xi}^+(0)=0$ and that we have the estimate

$$\begin{aligned}\|\tilde{\xi}^+\|_{H^{1+\ell/2}(0,T;H^1(\Omega^+))\cap L^2(0,T;H^{3+\ell}(\Omega^+))}+\|(\tilde{G}^+-\nabla\tilde{\xi}^+)\cdot n_\Gamma\|_{H^{1+\ell/2}(0,T;H^{-1/2}(\Gamma))}\\ +\|(\tilde{G}^+-\nabla\tilde{\xi}^+)\cdot n\|_{H^{1+\ell/2}(0,T;H^{-1/2}(\partial\Omega))}\leq C\|\tilde{G}\|_{E_{G,T}}.\end{aligned}\tag{2.3 .19}$$

2.3 .4.2 Lifting the normal trace

For all $t\in[0,T]$, we are interested in

$$\left\{\begin{array}{rcll}\Delta\tilde{\pi}^+(t)&=&0&\text{in }\Omega,\\ \partial_n\tilde{\pi}^+(t)&=&(\tilde{G}^+(t)-\nabla\tilde{\xi}^+(t))\cdot n_\Gamma+(\tilde{h}(t)-\tilde{G}^+(t))\cdot n_\Gamma&\text{on }\Gamma,\\ \partial_n\tilde{\pi}^+(t)&=&(\tilde{G}^+(t)-\nabla\tilde{\xi}^+(t))\cdot n&\text{on }\partial\Omega,\\ \displaystyle\int_\Omega\tilde{\pi}^+(t)dy&=&0.&\end{array}\right.\tag{2.3 .20}$$

Since the compatibility condition

$$\int_{\partial\Omega}(\tilde{G}^+-\nabla\tilde{\xi}^+(t))\cdot n ds-\int_\Gamma((\tilde{G}^+(t)-\nabla\tilde{\xi}^+(t))\cdot n_\Gamma+(\tilde{h}(t)-\tilde{G}^+(t))\cdot n_\Gamma)ds=0,$$

is fulfilled, there exists a unique solution $\tilde{\pi}^+(t) \in H^1(\Omega^+)$ of (2.3 .20). Moreover,

$$\|\tilde{\pi}^+\|_{H^{1+\ell/2}(0,T;H^1(\Omega^+))} \leq C\|(\tilde{G}, \tilde{G}^+ - \tilde{h}) \cdot n_\Gamma\|_{E_{G,T} \times H^{1+\ell/2}(0,T;H^{-1/2}(\Gamma))}, \tag{2.3 .21}$$

and

$$\|\tilde{\pi}^+\|_{L^2(0,T;H^{3+\ell}(\Omega^+))} \leq C\|(\tilde{G}, \tilde{h}) \cdot n_\Gamma\|_{E_{G,T} \times L^2(0,T;H^{3/2+\ell}(\Gamma))}. \tag{2.3 .22}$$

Lastly, since $\tilde{h}|_{t=0} = \partial_n \tilde{\xi}^+|_{t=0} = 0$, the unique solution of (2.3 .20) at time $t = 0$ is $\tilde{\pi}^+(0) = 0$.

Remark that we require the regularity C^4 for $\partial\Omega$ and Γ to get the estimate $\tilde{\xi}^+, \tilde{\pi}^+ \in L^2(0,T;H^{3+\ell}(\Omega^+))$.

2.3 .4.3 Lifting the tangential trace and the stress tensor in dimension $d = 2$

We are interested in the following problem

$$\left\{\begin{array}{rcll} \nabla \cdot w & = & 0 & \text{in } (0,T) \times \Omega^+, \\ w & = & \tilde{h} - \nabla(\tilde{\xi}^+ + \tilde{\pi}^+) & \text{on } (0,T) \times \Gamma, \\ w & = & -\nabla(\tilde{\xi}^+ + \tilde{\pi}^+) & \text{on } (0,T) \times \partial\Omega, \\ \sigma(w,p)n_\Gamma & = & \tilde{f} - 2\mu D(\nabla(\tilde{\xi}^+ + \tilde{\pi}^+))n_\Gamma & \text{on } (0,T) \times \Gamma, \\ \sigma(w,p)n & = & -2\mu D(\nabla(\tilde{\xi}^+ + \tilde{\pi}^+))n & \text{on } (0,T) \times \partial\Omega, \\ w|_{t=0} & = & 0 & \text{in } \Omega^+. \end{array}\right. \tag{2.3 .23}$$

Let $h = -\mathbb{1}_{\partial\Omega}\nabla(\tilde{\xi}^+ + \tilde{\pi}^+) + \mathbb{1}_\Gamma(\tilde{h} - \nabla(\tilde{\xi}^+ + \tilde{\pi}^+))$ and $f = -\mathbb{1}_{\partial\Omega}(2\mu D(\nabla(\tilde{\xi}^+ + \tilde{\pi}^+))n) + \mathbb{1}_\Gamma(\tilde{f} - 2\mu D(\nabla(\tilde{\xi}^+ + \tilde{\pi}^+))n_\Gamma)$. Then system (2.3 .23) rewrites

$$\left\{\begin{array}{rcll} \nabla \cdot w & = & 0 & \text{in } (0,T) \times \Omega^+, \\ w & = & h & \text{on } (0,T) \times \partial\Omega^+, \\ \sigma(w,p)n_\Gamma & = & f & \text{on } (0,T) \times \partial\Omega^+, \\ w|_{t=0} & = & 0 & \text{in } \Omega^+. \end{array}\right. \tag{2.3 .24}$$

with $h \in H^{3/2+\ell,3/4+\ell}((0,T)\times\Gamma)$ and $f \in L^2((0,T)\times\partial\Omega^+)$ such that $f_\tau \in H^{1/2+\ell,1/4+\ell/2}((0,T)\times\partial\Omega^+)$ and $f_n \in L^2(0,T;H^{1/2+\ell}(\partial\Omega^+)) \cap H^{\ell/2}(0,T;H^{1/2}(\partial\Omega^+))$ satisfying

$$\left\{\begin{array}{rcll} h \cdot n & = & 0 & \text{on } (0,T) \times \partial\Omega^+, \\ h|_{t=0} & = & 0 & \text{on } \partial\Omega^+. \end{array}\right.$$

The strategy to construct such a couple (w,p) is to localize the problem of lifting and to straighten the interface so as to apply [13, Chapter 4, Theorem 2.3].

The domain Ω^+ being C^2, the tangent and the outward normal of $\partial\Omega^+$ is well defined on each point. Let $s_0 \in \partial\Omega^+$ and (τ_0, n_0) the direct orthonormal basis composed of the tangent and the outward normal at point s_0. Since the domain is C^2, we can apply the implicit function theorem: there exists $a_{s_0} > 0$ and $g_{s_0} \in C^2([-a_{s_0}, a_{s_0}])$ such that $g_{s_0}(0) = g'_{s_0}(0) = 0$, $\forall x_s \in (-a_{s_0}, a_{s_0})$, $y_s = g_{s_0}(x_s)$ where (x_s, y_s) are the coordinates of the point s of $\partial\Omega^+$ in the basis (τ_0, n_0) and Ω^+ lies beneath the graph of g_{s_0}. We also introduce

$$\mathcal{O}_{s_0} = \{x_s\tau_0 + y_s n_0 \in \mathbb{R}^2 \mid x_s \in (-a_{s_0}, a_{s_0}), y_s \in (\min_{[-a_{s_0},a_{s_0}]} g_{s_0} - 1, \max_{[-a_{s_0},a_{s_0}]} g_{s_0} + 1)\}.$$

Remark that

$$\Omega^+ \cap \mathcal{O}_{s_0} \subset \{x_s\tau_0 + y_s n_0 \in \mathbb{R}^2 \mid x_s \in (-a_{s_0}, a_{s_0}),\ y_s < g_{s_0}(x_s))\}.$$

By construction, we have

$$\partial\Omega^+ \subset \bigcup_{s\in\Gamma} \mathcal{O}_s \text{ where } \mathcal{O}_s \text{ are open subset of } \mathbb{R}^2.$$

Since $\partial\Omega^+$ is a compact set, we can extract finite numbers of open subsets $(\mathcal{O}_i)_{i\in[\![1;q]\!]}$ with $q \in \mathbb{N}^*$ of $(\mathcal{O}_s)_{s\in\Gamma}$ such that

$$\partial\Omega^+ \subset \bigcup_{i\in[\![1;q]\!]} \mathcal{O}_i.$$

Let $(\varphi_j)_{j\in[\![1;q]\!]}$ a smooth partition of unity for $\bigcup_{i\in[\![1;q]\!]} \mathcal{O}_i$. We recall that

$$\begin{aligned}
&\varphi_j : \mathbb{R}^2 \to [0,1],\\
&\mathrm{Supp}(\varphi_j) \subset \mathcal{O}_j,\\
&\sum_{j=1}^{q} \varphi_j = 1 \quad \text{on a neighborhood of } \partial\Omega^+.
\end{aligned}$$

We reduce the problem of lifting on the subsets $\mathcal{O}_j$, $1 \le j \le q$. More precisely, we are going to prove the following Lemma:

Lemma 2.3 .7. *Let $0 < T < T^*$, $a > 0$, $g \in C^2([-a,a])$ with $g(0) = g'(0) = 0$. We define*

$$\begin{aligned}
&\gamma = \{(x,y) \in \mathbb{R}^2 \mid x \in (-a,a),\ y = g(x)\},\\
&F = \{(x,y) \in \mathbb{R}^2 \mid x \in (-a,a),\ \min_{[-a,a]} g - 1 < y \le g(x)\},\\
&U = Int(F).
\end{aligned}$$

Lastly, let $h \in H^{3/2+\ell,3/4+\ell/2}((0,T)\times\gamma)$, $f \in L^2((0,T)\times\gamma)$ such that $f_\tau \in H^{1/2+\ell,1/4+\ell/2}((0,T)\times\gamma)$, $f_n \in L^2(0,T;H^{1/2+\ell}(\gamma)) \cap H^{\ell/2}(0,T;H^{1/2}(\gamma))$. We assume

$$\begin{array}{rcl}
Supp(h) & \subset & [0,T]\times\gamma,\\
Supp(f) & \subset & [0,T]\times\gamma,\\
h\cdot n & = & 0 \quad on\ \gamma,\\
h|_{t=0} & = & 0 \quad on\ \gamma.
\end{array} \tag{2.3 .25}$$

Then there exists $(w,p) \in H^{2+\ell,1+\ell/2}((0,T)\times U) \times (L^2(0,T;H^{1+\ell}(U)) \cap H^{\ell/2}(0,T;H^1(U)))$ with

$$Supp(w) \subset [0,T]\times F \text{ and } Supp(p) \subset [0,T]\times F, \tag{2.3 .26}$$

and such that

$$\left\{\begin{array}{rcll}
\nabla\cdot w & = & 0 & in\ (0,T)\times U,\\
w & = & h & on\ (0,T)\times\gamma,\\
\sigma(w,p)n & = & f & on\ (0,T)\times\gamma,\\
w|_{t=0} & = & 0 & in\ U,
\end{array}\right. \tag{2.3 .27}$$

with the estimate

$$\begin{aligned}
&\|(w,p)\|_{H^{2+\ell,1+\ell/2}((0,T)\times U)\times(L^2(0,T;H^{1+\ell}(U))\cap H^{\ell/2}(0,T;H^1(U)))}\\
&\le C\|(h,f_\tau,f_n)\|_{H^{3/2+\ell,3/4+\ell/2}((0,T)\times\gamma)\times H^{1/2+\ell,1/4+\ell/2}((0,T)\times\gamma)\times(L^2(0,T;H^{1+\ell}(\gamma))\cap H^{\ell/2}(0,T;H^{1/2}(\gamma)))},
\end{aligned}$$

where C does not depend on time T.

Proof of Lemma 2.3 .7. We look for w of the form:

$$w = \nabla^{\perp} v = \begin{pmatrix} -\partial_y v \\ \partial_x v \end{pmatrix}, \tag{2.3 .28}$$

so that (2.3 .27)$_1$ is automatically fulfilled. The idea is to flatten γ. Let us introduce

$$\phi : \begin{array}{ccc} [-a,a]\times\mathbb{R} & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x, y-g(x)) = (x', y') \end{array}.$$

We work on the domain $\phi(U)$ and we introduce

$$\tilde{v} = v \circ \phi^{-1} \text{ and } \tilde{p} = p \circ \phi^{-1}. \tag{2.3 .29}$$

Computations of the derivatives lead to

$$w = \begin{pmatrix} -\partial_2 \tilde{v} \\ \partial_1 \tilde{v} - g'\partial_2 \tilde{v} \end{pmatrix} \circ \phi, \tag{2.3 .30}$$

$$\nabla w = \begin{pmatrix} -\partial_{12}\tilde{v} + g'\partial_{22}\tilde{v} & -\partial_{22}\tilde{v} \\ \partial_{11}\tilde{v} - 2g'\partial_{12}\tilde{v} + (g')^2\partial_{22}\tilde{v} - g''\partial_2\tilde{v} & \partial_{12}\tilde{v} - g'\partial_{22}\tilde{v} \end{pmatrix} \circ \phi. \tag{2.3 .31}$$

We also recall the expression of $\tau(x)$ the tangential vector and $n(x)$ the outward normal of γ at point x:

$$\tau(x) = \frac{1}{\sqrt{1+(g')^2(x)}} \begin{pmatrix} 1 \\ g'(x) \end{pmatrix} \quad \text{and} \quad n(x) = \frac{1}{\sqrt{1+(g')^2(x)}} \begin{pmatrix} -g'(x) \\ 1 \end{pmatrix}. \tag{2.3 .32}$$

We now translate conditions (2.3 .27)$_2$-(2.3 .27)$_3$ on our new unknowns $\tilde{v}$ et $\tilde{p}$. First, using (2.3 .30), assumption (2.3 .25)$_3$ and condition (2.3 .27)$_2 \cdot n$ give

$$\partial_1 \tilde{v} = 0 \quad \text{on } (0,T)\times(-a,a).$$

Condition (2.3 .27)$_2 \cdot \tau$ give

$$\partial_2 \tilde{v} = \frac{-1}{1+(g')^2}(h_1 + g'h_2)\circ\phi^{-1} \quad \text{on } (0,T)\times(-a,a).$$

Remark that (2.3 .25) gives

$$h_2 = g'h_1 \quad \text{on } (0,T)\times\gamma,$$

so that

$$\frac{-1}{1+(g')^2}(h_1 + g'h_2) = -h_1 \quad \text{on } (0,T)\times\gamma.$$

To obtain (2.3 .27)$_2$, it is sufficient that $\tilde{v}$ satisfies

$$\left\{ \begin{array}{rcll} \tilde{v} & = & 0 & \text{on } (0,T)\times(-a,a), \\ \partial_2\tilde{v} & = & -h_1\circ\phi^{-1} & \text{on } (0,T)\times(-a,a). \end{array} \right. \tag{2.3 .33}$$

Assume that $\tilde{v}$ satisfies (2.3 .33). Using (2.3 .31), (2.3 .32), (2.3 .33), condition (2.3 .27)$_3$ can be rewritten:

$$\begin{pmatrix} -(1+(g')^2) & g' \\ -g'(1+(g')^2) & -1 \end{pmatrix} \begin{pmatrix} \mu\partial_{22}\tilde{v} \\ \tilde{p} \end{pmatrix} = \sqrt{1+(g')^2} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \circ \phi^{-1} + b \quad \text{on } (0,T)\times(-a,a),$$

where

$$b = \mu \begin{pmatrix} -g''h_1 \\ g'g''h_1 + 2((g')^2+1)\partial_1 h_1 \end{pmatrix} \circ \phi^{-1}.$$

The matrix $\begin{pmatrix} -(1+(g')^2) & g' \\ -g'(1+(g')^2) & -1 \end{pmatrix}$ is invertible. Set

$$\hat{b} = \begin{pmatrix} -(1+(g')^2) & g' \\ -g'(1+(g')^2) & -1 \end{pmatrix}^{-1} b \in H^{1/2+\ell,1/4+\ell/2}((0,T)\times(-a,a)).$$

A straightforward computation leads to:

$$\begin{cases} \partial_{22}\tilde{v} &= \dfrac{-1}{\mu(1+(g')^2)^{3/2}}(f_1+g'f_2)\circ\phi^{-1} + \hat{b}_1 \quad \text{on } (0,T)\times(-a,a), \\ \tilde{p} &= \dfrac{1}{\sqrt{1+(g')^2}}(g'f_1 - f_2)\circ\phi^{-1} + \hat{b}_2 \qquad \text{on } (0,T)\times(-a,a), \end{cases} \tag{2.3 .34}$$

First remark that

$$f_\tau = f\cdot\tau = \frac{1}{\sqrt{1+(g')^2}}(f_1+g'f_2) \in H^{1/2+\ell,1/4+\ell/2}((0,T)\times\gamma),$$
$$f_n = f\cdot n = -\frac{1}{\sqrt{1+(g')^2}}(g'f_1-f_2) \in L^2(0,T;H^{1/2+\ell}(\gamma))\cap H^{\ell/2}(0,T;H^{1/2}(\gamma)).$$

The lifting of the tangential part of f is then performed by $\partial_{22}\tilde{v}$ and the lifting of the normal part of f by $\tilde{p}$.

So far, we have identified the lifting to apply to $(\tilde{v},\tilde{p})$ to obtain (2.3 .27). To use [13, Chapter 4, Theorem 2.3], we first extend the data on $\mathbb{R}$. We then construct a suitable cut off function to obtain compact supports for the lifting (see condition (2.3 .26)).

We first extend $h\circ\phi^{-1}$ and $f\circ\phi^{-1}$ by 0 on $\mathbb{R}_+^*\times\mathbb{R}$. With assumptions (2.3 .25), we have:

$$\|h\circ\phi^{-1}\|_{H^{3/2+\ell,3/4+\ell/2}(\mathbb{R}_+^*\times\mathbb{R})} \le C\|h\circ\phi^{-1}\|_{H^{3/2+\ell,3/4+\ell/2}((0,T)\times(-a,a))},$$
$$\|f_\tau\circ\phi^{-1}\|_{H^{1/2+\ell,1/4+\ell/2}(\mathbb{R}_+^*\times\mathbb{R})} \le C\|f_\tau\circ\phi^{-1}\|_{H^{1/2+\ell,1/4+\ell/2}((0,T)\times(-a,a))},$$
$$\|f_n\circ\phi^{-1}\|_{L^2(\mathbb{R}_+^*;H^{1/2+\ell}(\mathbb{R}))\cap H^{\ell/2}(\mathbb{R}_+^*;H^{1/2}(\mathbb{R}))} \le C\|f_n\circ\phi^{-1}\|_{L^2(0,T;H^{1/2+\ell}(-a,a))\cap H^{\ell/2}(0,T;H^{1/2}(-a,a))},$$

Using [13, Chapter 4, Theorem 2.3], there exists $\tilde{V}\in H^{3+\ell,3/2+\ell/2}((0,+\infty)\times\mathbb{R}_-^2)$ such that

$$\begin{cases} \tilde{V} &= 0 & \text{on } (0,+\infty)\times(\mathbb{R}\times\{0\}), \\ \partial_2\tilde{V} &= -h_1\circ\phi^{-1} & \text{on } (0,+\infty)\times(\mathbb{R}\times\{0\}), \\ \partial_{22}\tilde{V} &= \dfrac{-1}{\mu(1+(g')^2)^{3/2}}(f_1+g'f_2)\circ\phi^{-1}+\hat{b}_1, & \text{on } (0,+\infty)\times(\mathbb{R}\times\{0\}), \\ \tilde{V}|_{t=0} &= 0, & \text{in } \mathbb{R}_-^2, \end{cases} \tag{2.3 .35}$$

with the estimate

$$\|\tilde{V}\|_{H^{3+\ell,3/2+\ell/2}((0,+\infty)\times\mathbb{R}_-^2)} \le C(\|h\|_{H^{3/2+\ell,3/4+\ell/2}((0,T)\times\gamma)} + \|f_\tau\|_{H^{1/2+\ell,1/4+\ell/2}((0,T)\times\gamma)}).$$

Moreover using [12, Chapter 1, Theorem 9.4, page 41], there exists $\tilde{P}\in L^2(0,+\infty;H^{1+\ell}(\mathbb{R}_-^2))\cap H^{\ell/2}(0,+\infty;H^1(\mathbb{R}_-^2))$ such that

$$\tilde{P} = \frac{1}{\sqrt{1+(g')^2}}(g'f_1-f_2)\circ\phi^{-1}+\hat{b}_2 \quad \text{on } (0,+\infty)\times(\mathbb{R}\times\{0\}), \tag{2.3 .36}$$

satisfying the estimate

$$\|\tilde{P}\|_{L^2(0,+\infty;H^{1+\ell}(\mathbb{R}^2_-))\cap H^{\ell/2}(0,+\infty;H^1(\mathbb{R}^2_-))} \leq C(\|h\|_{H^{3/2+\ell,3/4+\ell/2}((0,T)\times\gamma)} + \|f_n\|_{L^2(0,T;H^{1/2+\ell}(\gamma))\cap H^{\ell/2}(0,T;H^1(\gamma))}).$$

We now introduce a cut off function η to construct a compact support lifting. Let $O_0, \mathcal{O}_0$ be two domains such that:

$$\mathrm{Supp}(h) \subset O_0 \subset\subset \mathcal{O}_0 \quad \text{and} \quad \mathrm{Supp}(f) \subset O_0 \subset\subset \mathcal{O}_0.$$

and let $\eta \in C^\infty(\mathbb{R}^2_-)$ a cut off function such that

$$\eta = \begin{cases} 1 & \text{on } \phi(O_0) \cap \mathbb{R}^2_-, \\ 0 & \text{on } \mathbb{R}^2_- \setminus (\phi(\mathcal{O}_0) \cap \mathbb{R}^2_-). \end{cases}$$

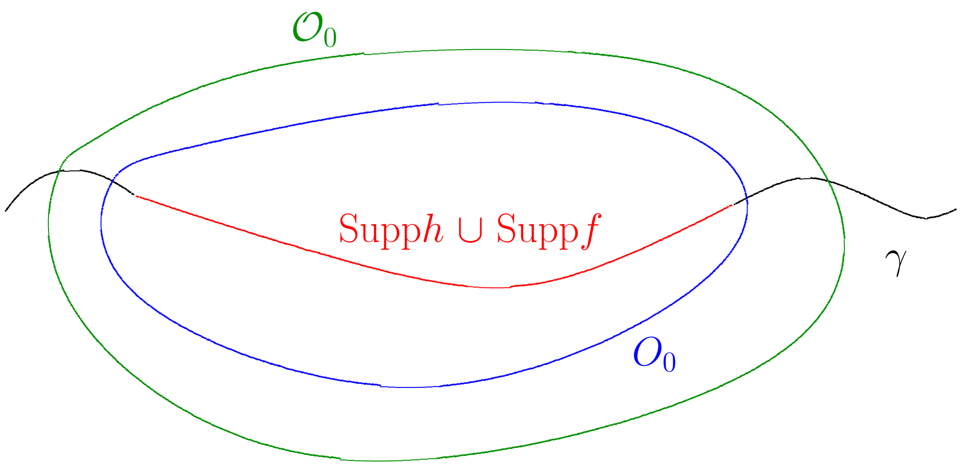

Figure 2.2: Construction of the cut-off η.

We define

$$\begin{aligned} \tilde{v} &= \eta \tilde{V}|_{(0,T)}, \\ \tilde{p} &= \eta \tilde{P}|_{(0,T)}. \end{aligned} \tag{2.3 .37}$$

With the construction of O_0, $\mathcal{O}_0$ and η, we have

$$\mathrm{Supp}(\tilde{v}) \subset [0,T] \times \phi(F) \text{ and } \mathrm{Supp}(\tilde{p}) \subset [0,T] \times \phi(F).$$

Furthermore, $(\tilde{v}, \tilde{p}) \in H^{3+\ell,3/2+\ell/2}((0,T)\times\phi(U))\times(L^2(0,T;H^{1+\ell}(\phi(U)))\cap H^{\ell/2}(0,T;H^1(\phi(U))))$ with the estimate:

$$\|(\tilde{v},\tilde{p})\|_{H^{3+\ell,3/2+\ell/2}((0,T)\times\phi(U))\times(L^2(0,T;H^{1+\ell}(\phi(U)))\cap H^{\ell/2}(0,T;H^1(\phi(U))))} \leq C\|(h, f_\tau, f_n)\|_{H^{3/2+\ell,3/4+\ell/2}((0,T)\times\gamma)\times H^{1/2+\ell,1/4+\ell/2}((0,T)\times\gamma)\times L^2(0,T;H^{1/2+\ell}(\gamma))\cap H^{\ell/2}(0,T;H^1(\gamma))}.$$

Since $\mathrm{Supp}(h) \subset O_0$ and $\mathrm{Supp}(f) \subset O_0$ and (v, P) satisfy (2.3 .35) and (2.3 .36), $(\tilde{v}, \tilde{p})$ satisfies (2.3 .33)–(2.3 .34). Defining w by (2.3 .30) and p by $p = \tilde{p} \circ \phi$, the pair (w, p) satisfies all the terms of Lemma 2.3 .7. $\square$

We define

$$\begin{array}{rcl} U_j & = & \mathcal{O}_j \cap \Omega^+, \\ \Gamma_j & = & \Gamma \cap \mathcal{O}_j, \\ F_j & = & U_j \cup \Gamma_j. \end{array}$$

With Lemma 2.3 .7, there exists $(w_j, p_j) \in H^{2+\ell,1+\ell/2}((0,T) \times U_j) \times (L^2(0,T; H^{1+\ell}(U_j)) \cap H^{\ell/2}(0,T; H^1(U_j)))$ such that

$$\mathrm{Supp}(w_j) \subset [0,T] \times F_j \text{ et } \mathrm{Supp}(p_j) \subset [0,T] \times F_j, \tag{2.3 .38}$$

and

$$\left\{ \begin{array}{rcll} \nabla \cdot w_j & = & 0 & \text{in } U_j, \\ w_j & = & \varphi_j h & \text{on } \Gamma_j, \\ \sigma(w_j, p_j) n & = & \varphi_j f & \text{on } \Gamma_j, \\ w_j|_{t=0} & = & 0 & \text{in } U_j, \end{array} \right.$$

with the estimate

$$\begin{aligned} \|(w_j,p_j)\|_{H^{2+\ell,1+\ell/2}((0,T)\times U_j)\times(L^2(0,T;H^{1+\ell}(U_j))\cap H^{\ell/2}(0,T;H^1(U_j)))} &\leq C(\|h\|_{H^{3/2+\ell,3/4+\ell/2}((0,T)\times\partial\Omega^+)} \\ &+ \|(f_\tau, f_n)\|_{H^{1/2+\ell,1/4+\ell/2}((0,T)\times\partial\Omega^+)\times(L^2(0,T;H^{1+\ell}(\partial\Omega^+))\cap H^{\ell/2}(0,T;H^{1/2}(\partial\Omega^+)))}). \end{aligned}$$

We extend w_j and p_j by 0 on Ω^+. We have $(w_j, p_j) \in H^{2+\ell,1+\ell/2}((0,T)\times\Omega^+)\times(L^2(0,T; H^{1+\ell}(\Omega^+))\cap H^{\ell/2}(0,T; H^1(\Omega^+)))$ by (2.3 .38). Define

$$\begin{array}{rcl} w & = & \displaystyle\sum_{j=1}^{q} w_j, \\ p & = & \displaystyle\sum_{j=1}^{q} p_j. \end{array}$$

We obviously have that $(w,p) \in H^{2+\ell,1+\ell/2}((0,T)\times\Omega^+)\times(L^2(0,T; H^{1+\ell}(\Omega^+))\cap H^{\ell/2}(0,T; H^1(\Omega^+)))$ satisfies (2.3 .24). Moreover, we have the estimate

$$\begin{aligned} \|(w,p)\|_{H^{2+\ell,1+\ell/2}((0,T)\times\Omega^+)\times(L^2(0,T;H^{1+\ell}(\Omega^+))\cap H^{\ell/2}(0,T;H^1(\Omega^+)))} &\leq C(\|h\|_{H^{3/2+\ell,3/4+\ell/2}((0,T)\times\partial\Omega^+)} \\ &+ \|(f_\tau, f_n)\|_{H^{1/2+\ell,1/4+\ell/2}((0,T)\times\partial\Omega^+)\times(L^2(0,T;H^{1+\ell}(\partial\Omega^+))\cap H^{\ell/2}(0,T;H^{1/2}(\partial\Omega^+)))}). \end{aligned} \tag{2.3 .39}$$

2.3 .4.4 Lifting the tangential trace and the stress tensor in dimension $d = 3$

We are interested in the following problem

$$\left\{ \begin{array}{rcll} \nabla \cdot w & = & 0 & \text{on } (0,T) \times \Omega^+, \\ w & = & \tilde{h} - \nabla(\tilde{\xi}^+ + \tilde{\pi}^+) & \text{in } (0,T) \times \Gamma, \\ w & = & -\nabla(\tilde{\xi}^+ + \tilde{\pi}^+) & \text{on } (0,T) \times \partial\Omega, \\ \sigma(w,p) n_\Gamma & = & \tilde{f} - 2\mu D(\nabla(\tilde{\xi}^+ + \tilde{\pi}^+)) n_\Gamma & \text{on } (0,T) \times \Gamma, \\ \sigma(w,p) n & = & -2\mu D(\nabla(\tilde{\xi}^+ + \tilde{\pi}^+)) n & \text{on } (0,T) \times \partial\Omega, \\ \nabla \cdot \sigma(w,p) & = & \tilde{F} - 2\mu \nabla \cdot D(\nabla(\tilde{\xi}^+ + \tilde{\pi}^+)) & \text{on } (0,T) \times \Gamma, \\ \nabla \cdot \sigma(w,p) & = & -2\mu \nabla \cdot D(\nabla(\tilde{\xi}^+ + \tilde{\pi}^+)) & \text{on } (0,T) \times \partial\Omega, \\ w|_{t=0} & = & 0 & \text{in } \Omega^+. \end{array} \right. \tag{2.3 .40}$$

Let $h = -\mathbb{1}_{\partial\Omega}\nabla(\tilde{\xi}^+ + \tilde{\pi}^+) + \mathbb{1}_\Gamma(\tilde{h} - \nabla(\tilde{\xi}^+ + \tilde{\pi}^+))$ and $f = -\mathbb{1}_{\partial\Omega}(2\mu D(\nabla(\tilde{\xi}^+ + \tilde{\pi}^+))n) + \mathbb{1}_\Gamma(\tilde{f} - 2\mu D(\nabla(\tilde{\xi}^+ + \tilde{\pi}^+))n_\Gamma)$ and $F = -\mathbb{1}_{\partial\Omega}(2\mu\nabla \cdot D(\nabla(\tilde{\xi}^+ + \tilde{\pi}^+))) + \mathbb{1}_\Gamma(\tilde{F} - 2\mu\nabla \cdot D(\nabla(\tilde{\xi}^+ + \tilde{\pi}^+)))$.

Then system (2.3 .40) rewrites

$$\begin{cases} \nabla \cdot w &= 0 \quad \text{in } (0,T) \times \Omega^+, \\ w &= h \quad \text{on } (0,T) \times \partial\Omega^+, \\ \sigma(w,p)n_\Gamma &= f \quad \text{on } (0,T) \times \partial\Omega^+, \\ \nabla \cdot \sigma(w,p) &= F \quad \text{on } (0,T) \times \partial\Omega^+, \\ w|_{t=0} &= 0 \quad \text{in } \Omega^+. \end{cases}$$

with $(F,h) \in H^{\ell-1/2,\ell/2-1/4}((0,T)\times\partial\Omega^+)\times H^{3/2+\ell,3/4+\ell}((0,T)\times\Gamma)$ and $f \in L^2((0,T)\times\partial\Omega^+)$ such that $f_\tau \in H^{1/2+\ell,1/4+\ell/2}((0,T)\times\partial\Omega^+)$ and $f_n \in L^2(0,T;H^{1/2+\ell}(\partial\Omega^+)) \cap H^{\ell/2}(0,T;H^{1/2}(\partial\Omega^+))$ satisfying

$$\begin{cases} h \cdot n &= 0 \quad \text{on } (0,T) \times \partial\Omega^+, \\ h|_{t=0} &= 0 \quad \text{on } \partial\Omega^+, \\ f_\tau|_{t=0} &= 0 \quad \text{on } \partial\Omega^+. \end{cases}$$

The idea remains the same as in dimension $d = 2$ but the computations are longer. The main differences are that function g_{s_0} which describes Γ depends on two parameters (x_s, y_s) and that we have added one condition on the lifting for (w,p). Following the computations of the 2 dimensional case, it is therefore sufficient to prove the following lemma:

Lemma 2.3 .8. *Let $0 < T < T^*$, $a, b > 0$, $g \in C^3([-a,a]\times[-b,b])$ with $g(0,0) = \nabla g(0,0) = 0$. We define*

$$\begin{aligned} &\gamma = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in (-a,a)\times(-b,b),\ z = g(x,y)\}, \\ &F = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in (-a,a)\times(-b,b),\ \min_{[-a,a]\times[-b,b]} g - 1 < z \le g(x,y)\}, \\ &U = Int(F). \end{aligned}$$

Lastly, let $(F,h) \in H^{\ell-1/2,\ell/2-1/4}((0,T)\times\gamma)\times H^{3/2+\ell,3/4+\ell/2}((0,T)\times\gamma)$, $f \in L^2((0,T)\times\gamma)$ such that $f_\tau \in H^{1/2+\ell,1/4+\ell/2}((0,T)\times\gamma)$, $f_n \in L^2(0,T;H^{1/2+\ell}(\gamma)) \cap H^{\ell/2}(0,T;H^{1/2}(\gamma))$. We assume

$$\begin{aligned} Supp(h) &\subset [0,T]\times\gamma, \\ Supp(f) &\subset [0,T]\times\gamma, \\ h\cdot n &= 0 \quad on\ \gamma, \\ h|_{t=0} &= 0 \quad on\ \gamma, \\ f_\tau|_{t=0} &= 0 \quad on\ \gamma. \end{aligned}$$

Then there exists $(w,p) \in H^{2+\ell,1+\ell/2}((0,T)\times U)\times(L^2(0,T;H^{1+\ell}(U)) \cap H^{\ell/2}(0,T;H^1(U)))$ with

$$Supp(w) \subset [0,T]\times F \text{ and } Supp(p) \subset [0,T]\times F,$$

and such that

$$\begin{cases} \nabla \cdot w &= 0 \quad in\ (0,T)\times U, \\ w &= h \quad on\ (0,T)\times\gamma, \\ \sigma(w,p)n &= f \quad on\ (0,T)\times\gamma, \\ \nabla\cdot\sigma(w,p) &= F \quad on\ (0,T)\times\gamma, \\ w|_{t=0} &= 0 \quad in\ U, \end{cases} \tag{2.3 .41}$$

with the estimate

$$\|(w,p)\|_{H^{2+\ell,1+\ell/2}((0,T)\times U)\times(L^2(0,T;H^{1+\ell}(U))\cap H^{\ell/2}(0,T;H^1(U)))} \le C(\|F\|_{H^{\ell-1/2,\ell/2-1/4}((0,T)\times\gamma)}$$
$$+ \|h\|_{H^{3/2+\ell,3/4+\ell/2}((0,T)\times\gamma)} + \|(f_\tau,f_n)\|_{H^{1/2+\ell,1/4+\ell/2}((0,T)\times\gamma)\times(L^2(0,T;H^{1+\ell}(\gamma))\cap H^{\ell/2}(0,T;H^{1/2}(\gamma)))}).$$

Proof of Lemma 2.3 .8. We only give the main idea to preserve the divergence free condition and the result of the main computations. For convenience, we set $\mathcal{U} = (-a, a) \times (-b, b)$. We look for w of the form

$$w = \mathrm{curl}(v).$$

To simplify the computations, we take $v_3 = 0$, which gives

$$w = \begin{pmatrix} -\partial_z v_1 \\ -\partial_z v_2 \\ \partial_x v_1 + \partial_y v_2 \end{pmatrix}, \tag{2.3 .42}$$

so that w satisfies $(2.3\ .41)_1$. We introduce

$$\phi : \begin{array}{rcl} [-a, a] \times [-b, b] \times \mathbb{R} & \to & \mathbb{R}^3 \\ (x, y, z) & \mapsto & (x, y, z - g(x, y)) = (x', y', z') \end{array},$$

and

$$\tilde{v}_1 = v_1 \circ \phi^{-1} \text{ and } \tilde{v}_2 = v_2 \circ \phi^{-1} \text{ and } \tilde{p} = p \circ \phi^{-1}. \tag{2.3 .43}$$

Computations show that system (2.3 .41) can be reduced to

$$\left\{\begin{array}{rll}
\tilde{v}_1 & = 0 & \text{on } (0,T) \times \mathcal{U}, \\
\tilde{v}_2 & = 0 & \text{on } (0,T) \times \mathcal{U}, \\
\partial_3 \tilde{v}_1 & = -h_1 \circ \phi^{-1} & \text{on } (0,T) \times \mathcal{U}, \\
\partial_3 \tilde{v}_2 & = -h_2 \circ \phi^{-1} & \text{on } (0,T) \times \mathcal{U}, \\
\partial_{33} \tilde{v}_1 & = \left(\dfrac{-1}{\mu(1+|\nabla g|^2)^{3/2}} (((\partial_y g)^2 + 1) f_1 - \partial_x g \partial_y g f_2 + \partial_x g f_3) + b^1 \right) \circ \phi^{-1} & \text{on } (0,T) \times \mathcal{U}, \\
\partial_{33} \tilde{v}_2 & = \left(\dfrac{1}{\mu(1+|\nabla g|^2)^{3/2}} (\partial_x g \partial_y g f_1 - ((\partial_x g)^2 + 1) f_2 - \partial_y g f_3) + b_2 \right) \circ \phi^{-1} & \text{on } (0,T) \times \mathcal{U}, \\
\tilde{p} & = \dfrac{1}{\sqrt{1+|\nabla g|^2}} (\partial_x g f_1 + \partial_y g f_2 - f_3) \circ \phi^{-1} + b^3 \circ \varphi^{-1} & \text{on } (0,T) \times \mathcal{U}, \\
\partial_3^3 \tilde{v}_1 & = \left(\dfrac{-1}{\mu(1+|\nabla g|^2)^2} (((\partial_y g)^2 + 1) F_1 - \partial_x g \partial_y g F_2 + \partial_x g F_3) + c^1 \right) \circ \phi^{-1} & \text{on } (0,T) \times \mathcal{U}, \\
\partial_3^3 \tilde{v}_2 & = \left(\dfrac{1}{\mu(1+|\nabla g|^2)^2} (\partial_x g \partial_y g F_1 - ((\partial_x g)^2 + 1) F_2 - \partial_y g F_3) + c^2 \right) \circ \phi^{-1} & \text{on } (0,T) \times \mathcal{U}, \\
\partial_3 \tilde{p} & = \left(\dfrac{1}{\sqrt{1+|\nabla g|^2}} (\partial_x g F_1 + \partial_y g F_2 - F_3) + c^3 \right) \circ \phi^{-1} & \text{on } (0,T) \times \mathcal{U}, \\
\tilde{v}_1|_{t=0} & = 0 & \text{on } \mathcal{U}, \\
\tilde{v}_2|_{t=0} & = 0 & \text{on } \mathcal{U}.
\end{array}\right.$$

where $b \in H^{1/2+\ell, 1/4+\ell/2}((0,T) \times \gamma)$ is given by

$$b = \begin{pmatrix} -(1+|\nabla g|^2) & 0 & \partial_x g \\ 0 & -(1+|\nabla g|^2) & \partial_y g \\ -\partial_x g(1+|\nabla g|^2) & -\partial_y g(1+|\nabla g|^2) & -1 \end{pmatrix} \begin{pmatrix} -\partial_{xx} g h_1 - \partial_{xy} g h_2 + \partial_y g \partial_2 h_1 - \partial_x g \partial_2 h_2 \\ -\partial_{xy} g h_1 - \partial_{yy} g h_2 + \partial_x g \partial_1 h_2 - \partial_y g \partial_1 h_1 \\ \tilde{b}_3 \end{pmatrix},$$

with

$$\begin{aligned} \tilde{b}_3 = {} & (\partial_x g \partial_{xx} g + \partial_y g \partial_{xy} g) h_1 + (\partial_x g \partial_{xy} g + \partial_y g \partial_{yy} g) h_2 + (2 + 2(\partial_x g)^2 + (\partial_y g)^2) \partial_1 h_1 \\ & + \partial_x g \partial_y g \partial_2 h_1 + \partial_x g \partial_y g \partial_1 h_2 + (2 + 2(\partial_y g)^2 + (\partial_x g)^2) \partial_2 h_2, \end{aligned}$$

and $c \in H^{\ell-1/2,\ell/2-1/4}((0,T)\times\gamma)$ is a three dimensional vector depending linearly on h, ∇h, $\nabla\nabla h$, f, ∇f. Notice that we need $g \in C^3$ in the computations of $\partial_3^3 \tilde{v}$. Remark also that the outward normal $n(x,y)$ of γ at point (x,y) writes

$$n(x,y) = \frac{1}{\sqrt{1+|\nabla g|^2(x,y)}} \begin{pmatrix} -\partial_x g \\ -\partial_x y \\ 1 \end{pmatrix} (x,y),$$

and that vectors

$$\tau_1(x,y) = \begin{pmatrix} (\partial_y g)^2+1 \\ -\partial_x g \partial_y g \\ \partial_x g \end{pmatrix} (x,y) \quad \text{and} \quad \tau_2(x,y) = \begin{pmatrix} \partial_x g \partial_y g \\ -((\partial_x g)^2+1) \\ -\partial_y g \end{pmatrix} (x,y),$$

are in the tangential space of γ at point (x,y). End of proof of Lemma 2.3 .8 follows exactly the same lines as Lemma 2.3 .7. Note that assertion $f_\tau|_{t=0} = 0$ enables us to keep $H^{1/2+\ell,1/4+\ell/2}$ integrability when we extend f_τ by 0 on $\mathbb{R}_+^* \times (\mathbb{R}^2 \times \{0\})$. □

The end of the construction of (w,p) in dimension $d=3$ is then exactly the same as in dimension $d=2$ and we get the estimate

$$\begin{aligned}
&\|(w,p)\|_{H^{2+\ell,1+\ell/2}((0,T)\times\Omega^+)\times(L^2(0,T;H^{1+\ell}(\Omega^+))\cap H^{\ell/2}(0,T;H^1(\Omega^+)))} \\
&\quad \le C(\|(f_\tau,f_n)\|_{H^{1/2+\ell,1/4+\ell/2}((0,T)\times\partial\Omega^+)\times(L^2(0,T;H^{1+\ell}(\partial\Omega^+))\cap H^{\ell/2}(0,T;H^{1/2}(\partial\Omega^+)))} \\
&\qquad + \|(F,h)\|_{H^{\ell-1/2,\ell/2-1/4}((0,T)\times\partial\Omega^+)\times H^{3/2+\ell,3/4+\ell/2}((0,T)\times\partial\Omega^+))}). \quad (2.3\ .44)
\end{aligned}$$

2.3 .4.5 End of Proof of Lemmas 2.3 .3 and 2.3 .5

To summarize, let $\tilde{\xi}^+$ the solution of (2.3 .18), let $\tilde{\pi}^+$ the solution of (2.3 .20) and let (w,p) a solution of (2.3 .23) in dimension $d=2$ and of (2.3 .40) in dimension $d=3$. Define

$$\tilde{u}^+ = \nabla\tilde{\xi}^+ + \nabla\tilde{\pi}^+ + w.$$

Then in dimension $d=2$, $(\tilde{u}^+,\tilde{p}^+)$ satisfies (2.3 .7), and estimate (2.3 .8) follows from the estimates on $\tilde{\xi}^+$ (2.3 .19), on $\tilde{\pi}^+$ (2.3 .21)-(2.3 .22) and on (w,p) (2.3 .39). And in dimension $d=3$, $(\tilde{u}^+,\tilde{p}^+)$ satisfies (2.3 .14) and estimate (2.3 .15) follows from (2.3 .19), (2.3 .21), (2.3 .22), (2.3 .44). Remark that the constructed $(\tilde{u}^+,\tilde{p}^+)$ also satisfies

$$\sigma(\tilde{u}^+,\tilde{p}^+)\cdot n = 0 \quad \text{on } (0,T)\times\partial\Omega,$$

and in dimension $d=3$,

$$\nabla\cdot\sigma(\tilde{u}^+,\tilde{p}^+) = 0 \quad \text{on } (0,T)\times\partial\Omega.$$

2.4 The nonlinear problem

Here, we are interested in the non linear problem (2.2 .9). We use a fixed point method so that we can apply the linear theory developed in the previous section. The main part of this section is dedicated to estimate the non linear terms. They are two different ways to control the non linear terms. In the first case, we use the smallness of T: estimates for small T and the proof of Theorems 2.2 .2 and 2.2 .4 are established in Section 2.4 .1. In the second case, we use the smallness of the data: estimates for small data and proof of Theorems 2.2 .3 and 2.2 .5 are established in Section 2.4 .2.

2.4 .1 The non linear problem in small time

We are interested in the non linear problem for small T. When T is small, X is expected to be closed to the identity so that it remains invertible. In a first step, we establish the estimates of X and the non linear terms. In a second step, we prove Theorems 2.2 .2 and 2.2 .4.

2.4 .1.1 Estimations of the non linear terms in small time

We get estimates on the non linear terms F, G, H. So as to apply Picard fixed point Theorem, we focus on Lipschitz estimates. The same kind of estimates can be found in [20, Section 6]. However, our estimates are much simpler, because we have replaced $J_{Y^\pm} \circ X^\pm$ by $\mathrm{cof} J_X^{\pm T}$ in the expression of our nonlinear terms (2.2 .3)-(2.2 .7). It is thus not necessary to get more involved estimates on the inverse $Y^\pm$. The same simplification can be done in [20].

We begin with a technical Lemma where we define the final time $T^* > 0$ for which $X(t,.)$ is invertible for all $t \in [0, T^*]$ and where we derive the estimates needed on $X^\pm$.

Lemma 2.4 .1. *Let $u_0 \in E_0^\ell$. For all $K > 0$, there exist $T^* \in (0,1)$ and $C = C(K) > 0$ such that for all $T \in (0, T^*)$, for all $u, v \in E^\ell_{u,T,u_0,K}$, $X(T,\cdot)$ is invertible in Ω and we have*

$$\begin{aligned} &\|\mathrm{cof}(J_{X^\pm}) - I\|_{H^1(0,T;H^{1+\ell}(\Omega^\pm)) \cap H^{3/2+\ell/2}(0,T;L^2(\Omega^\pm))} \le C\|u\|_{E^\ell_{u,T}}, \\ &\left\| \frac{\mathrm{cof}(J_{X^\pm})}{|\mathrm{cof}(J_{X^\pm}) n_\Gamma|} - I \right\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))} \le C\|u\|_{E^\ell_{u,T}}, \end{aligned} \tag{2.4 .1}$$

and the Lipschitz estimates:

$$\begin{aligned} &\|\mathrm{cof}(J_{X_u^\pm}) - \mathrm{cof}(J_{X_v^\pm})\|_{H^1(0,T;H^{1+\ell}(\Omega^\pm)) \cap H^{3/2+\ell/2}(0,T;L^2(\Omega^\pm))} \le C\|u - v\|_{E^\ell_{u,T}}, \\ &\left\| \frac{\mathrm{cof}(J_{X_u^\pm})}{|\mathrm{cof}(J_{X_u^\pm}) n_\Gamma|} - \frac{\mathrm{cof}(J_{X_v^\pm})}{|\mathrm{cof}(J_{X_v^\pm}) n_\Gamma|} \right\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))} \le C\|u - v\|_{E^\ell_{u,T}}. \end{aligned} \tag{2.4 .2}$$

Besides, $C = C(K)$ can be taken as an increasing function of K.

Spaces $H^1(0,T;H^{1+\ell}(\Omega^\pm)) \cap H^{3/2+\ell/2}(0,T;L^2(\Omega^\pm))$ and $H^1(0,T;H^{1/2+\ell}(\Gamma))$ are very interesting for two main reasons. First, they are algebras (see Lemmas 2.A.1 and 2.A.2). Second, they provide $L^\infty(0,T;H^{1+\ell}(\Omega^\pm))$ and $L^\infty(0,T;H^{1/2+\ell}(\Gamma))$ estimates with an explicit in time constant: $T^{1/2}$.

Proof of Lemma 2.4 .1. One can check that in every estimate written in the proof, when not mentioned explicitly, the constant C is independent of time T and is a polynomial of K.
We divide the proof in three steps.

Inversion of X.

Since

$$X^\pm(t,y) = y + \int_0^t u^\pm(s,y)ds,$$

we have

$$J_{X^\pm} = I + \int_0^t \nabla u^\pm(s,y)ds.$$

Using the continuous embedding $H^{1+\ell}(\Omega^\pm) \hookrightarrow C(\overline{\Omega^\pm})$, we get $J_{X^\pm} \in C([0,T] \times \overline{\Omega^\pm})$ with

$$\|J_{X^\pm} - I\|_{C([0,T]\times\overline{\Omega^\pm})} \le CT^{*1/2}\|u\|_{E_{u,T}} \le CT^{*1/2}K, \tag{2.4 .3}$$

where C is the constant of the embedding $H^{1+\ell}(\Omega^{\pm}) \hookrightarrow C(\overline{\Omega^{\pm}})$ and is independent of K, T and T^*. With estimate (2.4 .3), we are able to apply the inverse global theorem: for a given K, there exists $T^* > 0$ sufficiently small such that for all $t \in [0, T^*]$, $J_{X^{\pm}}(t, \cdot)$ is invertible on $\Omega^{\pm}$ and $X^{\pm}(t, \cdot)$ is invertible on $\Omega^{\pm}$. Furthermore, since $X^{\pm}(t, \cdot) \in C(\overline{\Omega^{\pm}})$, $X^{\pm}(t, \cdot)$ is in fact invertible on $\overline{\Omega^{\pm}}$. Let $Y^{\pm}(t, \cdot)$ be the inverse of $X^{\pm}(t, \cdot)$ on $\overline{\Omega^{\pm}}$. Since for $y \in \Gamma$, $X^+(t, y) = X^-(t, y) = X(t, y)$, then $Y^+(t, x) = Y^-(t, x)$ for $x \in X(t, \Gamma)$. Then $Y(t, \cdot) = \mathbb{1}_{\Omega^+} Y^+(t, \cdot) + \mathbb{1}_{\Omega^-} Y^-(t, \cdot)$ is continuous across $X(t, \Gamma)$ and is the inverse of $X(t, \cdot)$. And for all $t \in [0, T^*]$, $X(t, \cdot)$ is invertible.

Estimates of cof$(J_{X^{\pm}})$: (2.4 .1)$_1$ - (2.4 .2)$_1$.

Remark that

$$\mathrm{cof}(J_{X^{\pm}}) = I + B,$$

with $B = (b_{ij})_{1 \le i,j \le d}$ where b_{ij} is a sum of a product of at most $d - 1$ terms of the matrix

$$\int_0^t \nabla u^{\pm}(s, .)ds \in H^1(0, T; H^{1+l}(\Omega^{\pm})) \cap H^{3/2+\ell/2}(0, T; L^2(\Omega^{\pm})),$$

with by definition of the norm of E_u in (2.2 .17)$_1$-(2.2 .17)$_2$,

$$\left\| \int_0^t \nabla u^{\pm}(s, .)ds \right\|_{H^1(0,T;H^{1+\ell}(\Omega^{\pm})) \cap H^{3/2+\ell/2}(0,T;L^2(\Omega^{\pm}))} \le C\|u\|_{E_{u,T}},$$

where C does not depend on time T. In the case $d = 2$, the estimate (2.4 .1)$_1$ is then straightforward. And in the case $d = 3$, with Lemma 2.A.2 and the fact that $\int_0^t \nabla u^{\pm}(s, .)ds|_{t=0} = 0$, we have

$$\begin{aligned}
&\|\mathrm{cof}(J_{X^{\pm}}) - I\|_{H^1(0,T;H^{1+\ell}(\Omega^{\pm})) \cap H^{3/2+\ell/2}(0,T;L^2(\Omega^{\pm}))} \\
&\qquad = \|B\|_{H^1(0,T;H^{1+\ell}(\Omega^{\pm})) \cap H^{3/2+\ell/2}(0,T;L^2(\Omega^{\pm}))} \le C\|u\|_{E_{u,T}}.
\end{aligned}$$

The corresponding Lipschitz estimate (2.4 .2)$_1$ is derived exactly the same way.

Estimates of $\dfrac{\mathrm{cof}(J_{X^{\pm}})}{|\mathrm{cof}(J_{X^{\pm}})n_\Gamma|}$: (2.4 .1)$_2$ - (2.4 .2)$_2$.

With Lemma 2.A.2(ii), we have

$$\begin{aligned}
\left\| \frac{\mathrm{cof}(J_{X^{\pm}})}{|\mathrm{cof}(J_{X^{\pm}})n_\Gamma|} - I \right\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))} &\le C\Big(\|\mathrm{cof}(J_{X^{\pm}}) - I\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))} \\
&+ \left\| \frac{1}{|\mathrm{cof}(J_{X^{\pm}})n_\Gamma|} - 1 \right\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))} (\|\mathrm{cof}(J_{X^{\pm}})\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))} + 1) \Big).
\end{aligned}$$

The difficult point is to handle the term with $|\mathrm{cof}(J_{X^{\pm}})n_\Gamma|$. Remark that

$$|\mathrm{cof}(J_{X^{\pm}})n_\Gamma| = \sqrt{1 + r(u)}, \tag{2.4 .4}$$

where $r(u)$ is defined as follow

$$r(u)(t, y) = \sum_{i=1}^d \left[\left(2n_{\Gamma i} + \sum_{j=1}^d (n_{\Gamma j} \int_0^t \partial_j u_i^{\pm}(s, y)ds) \right) \sum_{j=1}^d (n_{\Gamma j} \int_0^t \partial_j u_i^{\pm}(s, y)ds) \right].$$

Since $n_\Gamma \in H^{1/2+\ell}(\Gamma)$, which is an algebra from Lemma 2.A.1 and since $\displaystyle\int_0^t \nabla u^\pm(s,.)ds\big|_{t=0} = 0$, we can use Lemma 2.A.2$(i)$ to get $r(u) \in H^1(0,T;H^{1/2+\ell}(\Gamma))$ with

$$\|r(u)\|_{H^1(0,T;H^{1/2+l}(\Gamma))} \leq C\|u\|_{E_{u,T}}.$$

We also have

$$\|r(u)\|_{C([0,T]\times\Gamma)} \leq CT^{*1/2}K(1+T^{*1/2}K), \tag{2.4 .5}$$

where C only depends the constant of the embedding $H^{1/2+\ell}(\Gamma) \hookrightarrow C(\Gamma)$ but is independent of K, T and T^*. Thus for T^* sufficiently small, $r(u) \in (-1/2, 1/2)$. We can then apply Lemma 2.A.4(ii) and we get

$$\left\|\frac{1}{|\mathrm{cof}(J_{X^\pm})n_\Gamma|} - 1\right\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))} \leq C\|u\|_{E_{u,T}}. \tag{2.4 .6}$$

The estimate easily follows.

The corresponding Lipschitz estimate is treated differently. First, with (2.4 .4) and Lemma 2.A.4(iii), we have

$$\||\mathrm{cof}(J_{X^\pm})n_\Gamma| - 1\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))} \leq C\|r(u)\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))},$$

and thus,

$$\||\mathrm{cof}(J_{X^\pm})n_\Gamma|\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))} \leq C. \tag{2.4 .7}$$

Moreover, using (2.4 .5), we immediately have

$$\||\mathrm{cof}(J_{X^\pm})n_\Gamma|\|^2_{C([0,T]\times\Gamma)} \geq 1 - CT^{*1/2}K(1+T^{*1/2}K), \tag{2.4 .8}$$

where C does not depend on K, T and T^*. Let $u, v \in E_{u,u_0}$. With Lemma 2.A.2(ii), we have

$$\begin{aligned}
&\||\mathrm{cof}(J_{X_u^\pm})n_\Gamma| - |\mathrm{cof}(J_{X_v^\pm})n_\Gamma|\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))},\\
&\quad\leq C\left\|\frac{|\mathrm{cof}(J_{X_u^\pm})n_\Gamma|^2 - |\mathrm{cof}(J_{X_v^\pm})n_\Gamma|^2}{|\mathrm{cof}(J_{X_u^\pm})n_\Gamma| + |\mathrm{cof}(J_{X_v^\pm})n_\Gamma|}\right\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))},\\
&\quad\leq C\|r(u)-r(v)\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))}\\
&\qquad\qquad\left(\left\|\frac{1}{|\mathrm{cof}(J_{X_u^\pm})n_\Gamma| + |\mathrm{cof}(J_{X_v^\pm})n_\Gamma|}\right\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))} + 1\right),
\end{aligned} \tag{2.4 .9}$$

Using (2.4 .8), we have $|\mathrm{cof}(J_{X_u^\pm})n_\Gamma| + |\mathrm{cof}(J_{X_v^\pm})n_\Gamma| \geq 1$ for T^* sufficiently small. We can apply Lemma 2.A.4(i) and with (2.4 .7) we get

$$\left\|\frac{1}{|\mathrm{cof}(J_{X_u^\pm})n_\Gamma| + |\mathrm{cof}(J_{X_v^\pm})n_\Gamma|}\right\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))} \leq C.$$

Moreover, using Lemma 2.A.2(i), we have

$$\|r(u)-r(v)\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))} \leq C\|u-v\|_{E_{u,T}}.$$

Using this estimate in (2.4 .9), we get

$$\||\mathrm{cof}(J_{u^\pm})n_\Gamma| - |\mathrm{cof}(J_{X_v^\pm})n_\Gamma|\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))} \leq C\|u-v\|_{E_{u,T}}.$$

Now with Lemma 2.A.2(ii) and (2.4 .6), we have

$$\begin{aligned}
&\left\|\frac{1}{|\mathrm{cof}(J_{X_u^\pm})n_\Gamma|}-\frac{1}{|\mathrm{cof}(J_{X_v^\pm})n_\Gamma|}\right\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))}\\
&\quad\leq C\left(\left\|\frac{1}{|\mathrm{cof}(J_{X_u^\pm})n_\Gamma|}\right\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))}+1\right)\left(\left\|\frac{1}{|\mathrm{cof}(J_{X_v^\pm})n_\Gamma|}\right\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))}+1\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad \big\||\mathrm{cof}(J_{X_v^\pm})n_\Gamma|-|\mathrm{cof}(J_{X_u^\pm})n_\Gamma|\big\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))},\\
&\quad\leq C\|u-v\|_{E_{u,T}}.
\end{aligned}$$

The Lipschitz estimate $(2.4\ .2)_2$ follows. $\square$

Lemma 2.4 .2. *Let $(u_0,f)\in E_0^\ell\times E_{H,T}^\ell$, $K>0$ and $T^*\in(0,1)$ be defined in Lemma 2.4 .1. There exists $C=C(K)>0$ such that for all $T\in(0,T^*)$, $(u,p),(v,q)\in E_{u,T,u_0,K}^\ell\times E_{p,T,(u_0,f),K}^\ell$, defining $F(u,p),G(u),H(u,p)$ by (2.2 .2)-(2.2 .7), (2.2 .10), we have $(F(u,p),G(u),H(u,p))\in E_{F,T}^\ell\times E_{G,T}^\ell\times E_{H,T}^\ell$ with $H(u,p)|_{t=0}=0$ and the estimates*

$$\|(F(u,p),H(u,p))\|_{E_{F,T}^\ell\times E_{H,T}^\ell}\leq CT^{1/2-\ell/2}\|(u,p)\|_{E_{u,T}^\ell\times E_{p,T}^\ell}\|(u_0,f,u,p)\|_{E_0^\ell\times E_{H,T}^\ell\times E_{u,T}^\ell\times E_{p,T}^\ell},$$
$$\|G(u)\|_{E_{G,T}^\ell}\leq C\|u\|_{E_{u,T}^\ell}(\|u_0\|_{E_0^\ell}+\|u\|_{E_{u,T}^\ell}),$$

and

$$\begin{aligned}
&\|(F(u,p),G(u),H(u,p))-(F(v,q),G(v),H(v,q))\|_{E_{F,T}^\ell\times E_{G,T}^\ell\times E_{H,T}^\ell}\\
&\leq CT^{1/2-\ell/2}(\|(u_0,f)\|_{E_0^\ell\times E_{H,T}^\ell}+\|(u,p)\|_{E_{u,T}^\ell\times E_{p,T}^\ell}+\|(v,q)\|_{E_{u,T}^\ell\times E_{p,T}^\ell})\|(u,p)-(v,q)\|_{E_{u,T}^\ell\times E_{p,T}^\ell}.
\end{aligned}$$

Besides, $C=C(K)$ can be taken as an increasing function of K.

Proof of Lemma 2.4 .2. We treat the non linear terms one by one. Except for G, we only prove the direct estimate, the lipschitz estimate being a straightforward adaptation.
One can check that in every estimate written in the proof, the constant C is independent of time T and is a polynomial of K.

Estimates of $F_1^\pm(u^\pm)$.

We recall that

$$F_1^\pm(u^\pm)=\mu\sum_{i,j=1}^d\partial_{x_j}\mathrm{cof}_{j,i}(J_{X^\pm})\partial_{y_i}u^\pm.$$

Remark that

$$\mathrm{cof}(J_{X^\pm})|_{t=0}=\mathrm{cof}(I)=I.$$

Thus $\partial_{x_j}\mathrm{cof}_{j,i}(J_{X^\pm})|_{t=0}=0$ and

$$\|\partial_{x_j}\mathrm{cof}_{j,i}(J_{X^\pm})\|_{L^\infty(0,T;H^\ell(\Omega^\pm))}\leq CT^{1/2}\|\partial_{x_j}\mathrm{cof}_{j,i}(J_{X^\pm})\|_{H^1(0,T;H^\ell(\Omega^\pm))},$$

where C does not depend on time T. Using Lemma 2.A.1 and Lemma 2.4 .1, we easily deduce

$$\|F_1^\pm(u^\pm)\|_{L^2(0,T;H^\ell(\Omega^\pm))}\leq CT^{1/2}\|u\|_{E_{u,T}}^2.$$

Using Lemma 2.A.2(i), we have

$$\|F_1^\pm(u^\pm)\|_{H^{\ell/2}(0,T;L^2(\Omega^\pm))}\leq C\|\nabla\mathrm{cof}(J_{X^\pm})\|_{H^{1/2+\ell/2}(0,T;H^\ell(\Omega^\pm))}\|\nabla u\|_{H^{\ell/2}(0,T;H^1(\Omega^\pm))}.$$

By definition of the norm of E_u in (2.2 .17)$_1$-(2.2 .17)$_2$, we have

$$\|\nabla u\|_{H^{\ell/2}(0,T;H^1(\Omega^\pm))} \leq \|u\|_{E_{u,T}}.$$

Furthermore, with Lemma 2.A.3 and the fact that $\nabla \mathrm{cof}(J_{X^\pm})|_{t=0} = 0$, we have

$$\|\nabla \mathrm{cof}(J_{X^\pm})\|_{H^{1/2+\ell/2}(0,T;H^\ell(\Omega^\pm))} \leq CT^{1/2-\ell/2}\|\nabla \mathrm{cof}(J_{X^\pm})\|_{H^1(0,T;H^\ell(\Omega^\pm))},$$

where C is independent of time T. We deduce

$$\|F_1^\pm(u^\pm)\|_{H^{\ell/2}(0,T;L^2(\Omega^\pm))} \leq CT^{1/2-\ell/2}\|u\|^2_{E_{u,T}}.$$

The corresponding Lipschitz estimates follow exactly the same lines:

$$\begin{aligned}
&\|F_1^\pm(u^\pm) - F_1^\pm(v^\pm)\|_{L^2(0,T;H^\ell(\Omega^\pm))} \leq CT^{1/2}\|(u,v)\|_{E_{u,T}\times E_{u,T}}\|u-v\|_{E_u},\\
&\|F_1^\pm(u^\pm) - F_1^\pm(v^\pm)\|_{H^{\ell/2}(0,T;L^2(\Omega^\pm))} \leq CT^{1/2-l/2}\|(u,v)\|_{E_{u,T}\times E_{u,T}}\|u-v\|_{E_u},
\end{aligned}$$

where C does not depend on time T.

Estimates of $F_2^\pm(u^\pm)$.

We recall that

$$F_2^\pm(u^\pm) = \mu \sum_{i,j=1}^d \big(\sum_{k=1}^d \mathrm{cof}_{k,j}(J_{X^\pm})\mathrm{cof}_{k,i}(J_{X^\pm}) - \delta_{ij}\big)\partial_{y_i y_j} u^\pm.$$

First remark that

$$\left(\sum_{k=1}^d \mathrm{cof}_{k,j}(J_{X^\pm})\mathrm{cof}_{k,i}(J_{X^\pm}) - \delta_{ij}\right)_{1\leq i,j\leq 2} = \mathrm{cof}(J_{X^\pm})^T\mathrm{cof}(J_{X^\pm}) - I \in H^1(0,T;H^{1+\ell}(\Omega^\pm)),$$

with $(\mathrm{cof}(J_{X^\pm})^T\mathrm{cof}(J_{X^\pm}) - I)|_{t=0} = 0$ and

$$\|\mathrm{cof}(J_{X^\pm})^T\mathrm{cof}(J_{X^\pm}) - I\|_{H^1(0,T;H^{1+\ell}(\Omega^\pm))} \leq C\|u\|_{E_{u,T}}.$$

The argument is exactly the same as for $F_1^\pm(u^\pm)$ and we have

$$\begin{aligned}
&\|F_2^\pm(u^\pm)\|_{L^2(0,T;H^\ell(\Omega^\pm))} \leq CT^{1/2}\|u\|^2_{E_{u,T}},\\
&\|F_2^\pm(u^\pm)\|_{H^{\ell/2}(0,T;L^2(\Omega^\pm))} \leq CT^{1/2-\ell/2}\|u\|^2_{E_{u,T}},
\end{aligned}$$

and the Lipschitz estimates:

$$\begin{aligned}
&\|F_2^\pm(u^\pm) - F_2^\pm(v^\pm)\|_{L^2(0,T;H^\ell(\Omega^\pm))} \leq CT^{1/2}\|(u,v)\|_{E_{u,T}\times E_{u,T}}\|u-v\|_{E_{u,T}},\\
&\|F_2^\pm(u^\pm) - F_2^\pm(v^\pm)\|_{H^{\ell/2}(0,T;L^2(\Omega^\pm))} \leq CT^{1/2-\ell/2}\|(u,v)\|_{E_{u,T}\times E_{u,T}}\|u-v\|_{E_{u,T}},
\end{aligned}$$

where C does not depend on time T.

Estimates of $F_3^{\pm}(u^{\pm},p^{\pm})$.

We recall that

$$F_3^{\pm}(u^{\pm},p^{\pm}) = (I - \mathrm{cof}(J_{X^{\pm}}))\nabla p^{\pm}.$$

Following the same lines as for $F_1^{\pm}(u^{\pm})$, we easily get

$$\begin{aligned}
&\|F_3^{\pm}(u^{\pm},p^{\pm})\|_{L^2(0,T;H^{\ell}(\Omega^{\pm}))} \leq CT^{1/2}\|u\|_{E_{u,T}}\|p\|_{E_{p,T}},\\
&\|F_3^{\pm}(u^{\pm},p^{\pm})\|_{H^{\ell/2}(0,T;L^2(\Omega^{\pm}))} \leq CT^{1/2-\ell/2}\|u\|_{E_{u,T}}\|p\|_{E_{p,T}},
\end{aligned}$$

and the Lipschitz estimate:

$$\begin{aligned}
&\|F_3^{\pm}(u^{\pm},p^{\pm}) - F_3^{\pm}(v^{\pm},q^{\pm})\|_{L^2(0,T;H^{\ell}(\Omega^{\pm}))}\\
&\qquad\qquad \leq CT^{1/2}(\|(u,p)\|_{E_{u,T}\times E_{p,T}} + \|(v,q)\|_{E_{u,T}\times E_{p,T}})\|(u-v,p-q)\|_{E_{u,T}\times E_{p,T}},\\
&\|F_3^{\pm}(u^{\pm},p^{\pm}) - F_3^{\pm}(v^{\pm},q^{\pm})\|_{H^{\ell/2}(0,T;L^2(\Omega^{\pm}))}\\
&\qquad\qquad \leq CT^{1/2-\ell/2}(\|(u,p)\|_{E_{u,T}\times E_{p,T}} + \|(v,q)\|_{E_{u,T}\times E_{p,T}})\|(u-v,p-q)\|_{E_{u,T}\times E_{p,T}},
\end{aligned}$$

where C does not depend on time T.

Estimates of $G^{\pm}(u^{\pm})$.

We recall that

$$G^{\pm}(u^{\pm}) = (I - \mathrm{cof}(J_{X^{\pm}})^T)u^{\pm}.$$

and with Piola identity

$$\nabla\cdot G^{\pm}(u^{\pm}) = \nabla u^{\pm} : (I - \mathrm{cof}(J_{X^{\pm}})).$$

Reasoning as for $F_3^{\pm}(u^{\pm},p^{\pm})$, we get

$$\begin{aligned}
&\|\nabla\cdot G^{\pm}(u^{\pm})\|_{L^2(0,T;H^{1+\ell}(\Omega^{\pm}))} \leq CT^{1/2}\|u\|^2_{E_{u,T}},\\
&\|\nabla\cdot G^{\pm}(u^{\pm}) - \nabla\cdot G^{\pm}(v^{\pm})\|_{L^2(0,T;H^{1+\ell}(\Omega^{\pm}))} \leq CT^{1/2}\|(u,v)\|_{E_{u,T}\times E_{u,T}}\|u-v\|_{E_{u,T}},
\end{aligned}$$

where C does not depend on time T. For the $L^2(0,T;L^2(\Omega^{\pm}))$ estimate on $G^{\pm}$, using Lemma 2.A.1 we have

$$\begin{aligned}
\|G^{\pm}(u^{\pm})\|_{L^2(0,T;L^2(\Omega^{\pm}))} \leq C\|I - \mathrm{cof}(J_{X^{\pm}})^T\|_{L^{\infty}(0,T;H^{1+l}(\Omega^{\pm}))}\|u^{\pm}\|_{L^2(0,T;L^2(\Omega^{\pm}))}&\\
\leq CT^{1/2}\|u^{\pm}\|^2_{E_{u,T}}.&
\end{aligned}$$

The corresponding Lipschitz estimate easily follows:

$$\|G^{\pm}(u^{\pm}) - G^{\pm}(v^{\pm})\|_{L^2(0,T;L^2(\Omega^{\pm}))} \leq CT^{1/2}\|(u,v)\|_{E_{u,T}\times E_{u,T}}\|u-v\|_{E_{u,T}}.$$

We are now interested in the $H^{\ell/2}(0,T;L^2(\Omega^{\pm}))$ norm of $\partial_t G^{\pm}(u^{\pm})$:

$$\partial_t G^{\pm}(u^{\pm}) = (I - \mathrm{cof}(J_{X^{\pm}})^T)\partial_t u^{\pm} - \partial_t\mathrm{cof}(J_{X^{\pm}})^T u^{\pm}.$$

Using Lemmas 2.A.1 and 2.A.2(i), we have

$$\begin{aligned}
&\|(I - \mathrm{cof}(J_{X^{\pm}})^T)\partial_t u^{\pm}\|_{H^{\ell/2}(0,T;L^2(\Omega^{\pm}))}\\
&\qquad\qquad \leq C\|I - \mathrm{cof}(J_{X^{\pm}})^T\|_{H^{1/2+\ell/2}(0,T;H^{1+\ell}(\Omega^{\pm}))}\|\partial_t u^{\pm}\|_{H^{\ell/2}(0,T;L^2(\Omega^{\pm}))},\\
&\qquad\qquad \leq CT^{1/2-\ell/2}\|u\|^2_{E_{u,T}}.
\end{aligned}$$

Using Lemmas 2.A.1 and 2.A.2(i), we also have

$$\begin{aligned}&\|\partial_t \mathrm{cof}(J_{X^\pm})^T u^\pm\|_{H^{\ell/2}(0,T;L^2(\Omega^\pm))}\\&\qquad\leq C(\|\partial_t \mathrm{cof}(J_{X^\pm})^T\|_{H^{1/2+\ell/2}(0,T;L^2(\Omega^\pm))}+\|\partial_t \mathrm{cof}(J_{X^\pm})^T|_{t=0}\|_{L^2(\Omega^\pm)})\|u^\pm\|_{H^{\ell/2}(0,T;H^2(\Omega^\pm))},\\&\qquad\leq C\|u\|_{E_{u,T}}(\|u_0\|_{E_0}+\|u\|_{E_{u,T}}).\end{aligned}$$

We point out that one does not have a constant which decreases in time in the estimate of $\|\partial_t G^\pm(u^\pm)\|_{H^{\ell/2}(0,T;L^2(\Omega^\pm))}$ because $(\partial_t \mathrm{cof}(J_{X^\pm}))|_{t=0}=\mathrm{cof}(\nabla u_0^\pm)$ is not necessarily trivial. This creates a difference in the corresponding Lipschitz estimate that we did not have so far. The term $(I-\mathrm{cof}(J_{X^\pm})^T)\partial_t u^\pm$ do no create any problem and we directly have

$$\begin{aligned}&\|(I-\mathrm{cof}(J_{X_u^\pm})^T)\partial_t u^\pm-(I-\mathrm{cof}(J_{X_v^\pm})^T)\partial_t v^\pm\|_{H^{\ell/2}(0,T;L^2(\Omega^\pm))}\\&\qquad\qquad\leq CT^{1/2-\ell/2}\|(u,v)\|_{E_{u,T}\times E_{u,T}}\|u-v\|_{E_{u,T}}.\end{aligned}$$

The Lipschitz estimate for the term $\partial_t \mathrm{cof}(J_{X^\pm})u^\pm$ is treated this way

$$\begin{aligned}&\|\partial_t \mathrm{cof}(J_{X_u^\pm})^T u^\pm-\partial_t \mathrm{cof}(J_{X_v^\pm})^T v^\pm\|_{H^{\ell/2}(0,T;L^2(\Omega^\pm))}\\&\qquad\leq \|(\partial_t \mathrm{cof}(J_{X_u^\pm})^T-\partial_t \mathrm{cof}(J_{X_v^\pm})^T)v^\pm\|_{H^{\ell/2}(0,T;L^2(\Omega^\pm))}\\&\qquad\qquad+\|\partial_t \mathrm{cof}(J_{X_u^\pm})^T(u^\pm-v^\pm)\|_{H^{\ell/2}(0,T;L^2(\Omega^\pm))}.\end{aligned}$$

Using Lemmas 2.A.1 and 2.A.2(i), we have

$$\begin{aligned}&\|\partial_t \mathrm{cof}(J_{X_u^\pm})^T(u^\pm-v^\pm)\|_{H^{\ell/2}(0,T;L^2(\Omega^\pm))}\\&\qquad\leq C(\|\partial_t \mathrm{cof}(J_{X_u^\pm})^T\|_{H^{1/2+\ell/2}(0,T;L^2(\Omega^\pm))}\|u^\pm-v^\pm\|_{H^{\ell/2}(0,T;H^{1+\beta}(\Omega^\pm))}\\&\qquad\qquad+\|\partial_t \mathrm{cof}(J_{X^\pm})^T|_{t=0}\|_{L^2(\Omega^\pm)}\|u^\pm-v^\pm\|_{H^{\ell/2}(0,T;H^{1+\beta}(\Omega^\pm))}).\end{aligned}$$

With Lemma 2.A.3, the fact that $\|\partial_t \mathrm{cof}(J_{X^\pm})|_{t=0}\|_{L^2(\Omega^\pm)}=\|\nabla u_0^\pm\|_{L^2(\Omega^\pm)}$ and $\|u-v\|_{E_{u,T}}\geq \|u^\pm-v^\pm\|_{H^{(1+\ell-\beta)/2}(0,T;H^{1+\beta}(\Omega^\pm))}$ (see the definition of the norm of $E_{u,T}$ and β in $(2.2\ .17)_1$-$(2.2\ .17)_2$ and $(2.2\ .18)$), we get

$$\begin{aligned}&\|\partial_t \mathrm{cof}(J_{X_u^\pm})^T(u^\pm-v^\pm)\|_{H^{\ell/2}(0,T;L^2(\Omega^\pm))}\\&\qquad\leq CT^{1/2-\beta/2}(\|u_0\|_{E_0}+\|u\|_{E_{u,T}})\|u^\pm-v^\pm\|_{H^{(1+\ell-\beta)/2}(0,T;H^{1+\beta}(\Omega^\pm))},\\&\qquad\leq CT^{1/2-\ell/2}(\|u_0\|_{E_0}+\|u\|_{E_{u,T}})\|u-v\|_{E_{u,T}}.\end{aligned}$$

With same type of arguments, we also have

$$\begin{aligned}&\|(\partial_t \mathrm{cof}(J_{X_u^\pm})^T-\partial_t \mathrm{cof}(J_{X_v^\pm})^T)v^\pm\|_{H^{\ell/2}(0,T;L^2(\Omega^\pm))}\\&\qquad\leq C(\|\partial_t \mathrm{cof}(J_{X_u^\pm})^T-\partial_t \mathrm{cof}(J_{X_v^\pm})^T\|_{H^{\ell/2}(0,T;L^2(\Omega^\pm))}\|v^\pm\|_{H^{(1+\ell-\beta)/2}(0,T;H^{1+\beta}(\Omega^\pm))}\\&\qquad\qquad+\|\partial_t \mathrm{cof}(J_{X_u^\pm})^T-\partial_t \mathrm{cof}(J_{X_v^\pm})^T\|_{H^{\ell/2}(0,T;L^2(\Omega^\pm))}\|u_0^\pm\|_{H^{1+\beta}(\Omega^\pm))}),\\&\qquad\leq CT^{1/2}\|\partial_t \mathrm{cof}(J_{X_u^\pm})^T-\partial_t \mathrm{cof}(J_{X_v^\pm})^T\|_{H^{1/2+\ell/2}(0,T;L^2(\Omega^\pm))}(\|v\|_{E_{u,T}}+\|u_0\|_{E_0}),\\&\qquad\leq CT^{1/2}(\|u_0\|_{E_0}+\|v\|_{E_{u,T}})\|u-v\|_{E_{u,T}}.\end{aligned}$$

We finally get

$$\begin{aligned}&\|G^\pm(u^\pm)\|_{H^{1+\ell/2}(0,T;L^2(\Omega^\pm))}\leq C\|u\|_{E_{u,T}}(\|u_0\|_{E_0}+\|u\|_{E_{u,T}}),\\&\|G^\pm(u^\pm)-G^\pm(v^\pm)\|_{H^{1+\ell/2}(0,T;L^2(\Omega^\pm))}\leq CT^{1/2-\ell/2}(\|u_0\|_{E_0}+\|(u,v)\|_{E_{u,T}\times E_{u,T}})\|u-v\|_{E_{u,T}},\end{aligned}$$

where C does not depend on time T. We also need an estimate of $\nabla(\nabla\cdot G^{\pm}(u^{\pm}))$ in $E_{F,T}$ for this term appears in $F^{\pm}(u^{\pm},p^{\pm})$. With the same ideas as in the estimates of $F_1^{\pm}(u^{\pm})$, one easily gets

$$\begin{aligned}
&\|\nabla(\nabla\cdot G^{\pm}(u^{\pm}))\|_{H^{\ell,\ell/2}((0,T)\times\Omega^{\pm})}\leq CT^{1/2-\ell/2}\|u\|^2_{E_{u,T}},\\
&\|\nabla(\nabla\cdot G^{\pm}(u^{\pm}))-\nabla(\nabla\cdot G^{\pm}(v^{\pm}))\|_{H^{\ell,\ell/2}((0,T)\times\Omega^{\pm})}\leq CT^{1/2-\ell/2}\|(u,v)\|_{E_{u,T}\times E_{u,T}}\|u-v\|_{E_{u,T}},
\end{aligned}$$

with C independent of time T. Remarking that $\mathrm{cof}(J_{X^-})n_\Gamma=\mathrm{cof}(J_{X^+})n_\Gamma$, it is then obvious that $G\in E_{G,T}$.

Estimates of $H(u,p)$.

We recall that

$$\begin{aligned}
H(u,p)=-\Big[&\mu((\nabla u)\mathrm{cof}(J_X)^T+\mathrm{cof}(J_X)(\nabla u)^T)\big(\frac{\mathrm{cof}(J_X)n_\Gamma}{|\mathrm{cof}(J_X)n_\Gamma|}-I\big)\\
&+\mu\nabla u(\mathrm{cof}(J_X)^T-I)+\mu(\mathrm{cof}(J_X)-I)(\nabla u)^T-p\big(\frac{\mathrm{cof}(J_X)n_\Gamma}{|\mathrm{cof}(J_X)n_\Gamma|}-n_\Gamma\big)\Big],
\end{aligned}$$

Remark that $n_\Gamma\in H^{1+\ell}(\Gamma)\cap C(\Gamma)$ and does not depend on time. Thus, using Lemma 2.A.1, it does not play a role in the estimates. With Lemma 2.A.1, we also have

$$\begin{aligned}
&\|(\nabla u^{\pm}\mathrm{cof}(J_{X^\pm})^T+\mathrm{cof}(J_{X^\pm})(\nabla u^{\pm})^T)\big(\frac{\mathrm{cof}(J_{X^\pm})}{|\mathrm{cof}(J_{X^\pm})n_\Gamma|}-I\big)\|_{L^2(0,T;H^{1/2+\ell}(\Gamma))}\\
&\quad\leq C\Big\|\frac{\mathrm{cof}(J_{X^\pm})}{|\mathrm{cof}(J_{X^\pm})n_\Gamma|}-I\Big\|_{L^\infty(0,T;H^{1/2+\ell}(\Gamma))}\|\mathrm{cof}(J_{X^\pm})\|_{L^\infty(0,T;H^{1/2+l}(\Gamma))}\|\nabla u^{\pm}\|_{L^2(0,T;H^{1/2+\ell}(\Gamma))},\\
&\quad\leq CT^{1/2}\|u\|^2_{E_{u,T}}.
\end{aligned}$$

For the estimate in time, we use Lemmas 2.A.2 and 2.A.3 and $\|\nabla u^{\pm}\|_{H^{1/4+\ell/2}(0,T;L^2(\Gamma))}\leq\|u\|_{E_{u,T}}$ (see the definition of the norm of $E_{u,T}$ (2.2 .17)$_1$-(2.2 .17)$_2$):

$$\begin{aligned}
&\|(\nabla u^{\pm}\mathrm{cof}(J_{X^\pm})^T+\mathrm{cof}(J_{X^\pm})(\nabla u^{\pm})^T)\big(\frac{\mathrm{cof}(J_{X^\pm})}{|\mathrm{cof}(J_{X^\pm})n_\Gamma|}-I\big)\|_{H^{1/4+\ell/2}(0,T;L^2(\Gamma))}\\
&\quad\leq C\Big\|\frac{\mathrm{cof}(J_{X^\pm})}{|\mathrm{cof}(J_{X^\pm})n_\Gamma|}-I\Big\|_{H^{1/2+\ell/2}(0,T;H^{1/2+\ell}(\Gamma))}(\|\mathrm{cof}(J_{X^\pm})\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))}+1)\\
&\qquad\qquad(\|\nabla u^{\pm}\|_{H^{1/4+\ell/2}(0,T;L^2(\Gamma))}+\delta_{d=3}\|\nabla u_0^{\pm}\|_{L^2(\Gamma)}),\\
&\quad\leq CT^{1/2-\ell/2}\|u\|_{E_{u,T}}(\|u_0\|_{E_0}+\|u\|_{E_{u,T}}),
\end{aligned}$$

where $\delta_{d=3}=0$ if $d=2$, and 1 if $d=3$. With the same arguments, we easily get

$$\begin{aligned}
&\|\nabla u^{\pm}(\mathrm{cof}(J_{X^\pm})^T-I)+(\mathrm{cof}(J_{X^\pm})-I)(\nabla u^{\pm})^T\|_{H^{1/2+\ell,1/4+\ell/2}((0,T)\times\Gamma)}\\
&\qquad\qquad\leq CT^{1/2-\ell/2}\|u\|_{E_{u,T}}(\|u_0\|_{E_0}+\|u\|_{E_{u,T}}),
\end{aligned}$$

where C does not depend on time T. For the last term with the pressure, remark that $\mathrm{cof}(J_{X^-})n_\Gamma=\mathrm{cof}(J_{X^+})n_\Gamma$, so that

$$\Big[\Big(I-\frac{\mathrm{cof}(J_{X^-})}{|\mathrm{cof}(J_{X^-})n_\Gamma|}\Big)p^--\Big(I-\frac{\mathrm{cof}(J_{X^+})}{|\mathrm{cof}(J_{X^+})n_\Gamma|}\Big)p^+\Big]n_\Gamma=\Big(I-\frac{\mathrm{cof}(J_{X^-})}{|\mathrm{cof}(J_{X^-})n_\Gamma|}\Big)[p]n_\Gamma.$$

Remark that in the case $d=3$, p satisfies the initial pressure compatibility condition (2.2 .14), so that we have with the definition of the norm of $E_{H,T}$ (see (2.2 .17)$_6$):

$$\|p|_{t=0}\|_{L^2(\Gamma)}\leq C(\|f|_{t=0}\|_{L^2(\Gamma)}+\|[D(u_0)]\|_{L^2(\Gamma)})\leq C\|(u_0,f)\|_{E_0\times E_{H,T}},$$

where C is independent of time T. Same type of arguments then lead to

$$\left\| \left(I - \frac{\mathrm{cof}(J_{X^-})}{|\mathrm{cof}(J_{X^-}) n_\Gamma|} \right) [p] n_\Gamma \right\|_{H^{1/2+\ell,1/4+\ell/2}((0,T)\times\Gamma)} \leq CT^{1/2-\ell/2} \|u\|_{E_{u,T}} (\|(u_0,f)\|_{E_0\times E_{H,T}} + \|p\|_{E_{p,T}}),$$

where C does not depend on the time T. The interpolated estimate in $H^{\ell/2}(0,T;H^{1/2}(\Gamma))$ is deduced by interpolation with a constant of interpolation independent of time T since $H(u,p)|_{t=0} = 0$. Lipschitz estimates follow exactly the same lines and do not create any difficulty:

$$\begin{aligned} &\|H(u,p) - H(v,q)\|_{E_{H,T}} \\ &\leq CT^{1/2-l/2} (\|(u_0,f)\|_{E_0\times E_{H,T}} + \|(u,p)\|_{E_{u,T}\times E_{p,T}} + \|(v,q)\|_{E_{u,T}\times E_{p,T}}) \|(u-v,p-q)\|_{E_{u,T}\times E_{p,T}}, \end{aligned}$$

where C does not depend on time T. □

2.4 .1.2 Proof of Theorems 2.2 .2 and 2.2 .4

Let $K > 0$, $T^* > 0$ given by Lemma 2.4 .1, $T \in (0,T^*)$ and $(u_0,f) \in E_0 \times E_{H,T}$ satisfying the compatibility condition (2.2 .13) in dimension $d = 3$. Estimates of Lemma 2.4 .2 indicate that for $(u,p) \in E_{u,T,u_0,K} \times E_{p,T,(u_0,f),K}$, non linear terms $F(\tilde{u},\tilde{p}), G(\tilde{u}), H(\tilde{u},\tilde{p})$ defined in (2.2 .2)-(2.2 .7) and (2.2 .10) satisfy $(F(\tilde{u},\tilde{p}), G(\tilde{u}), H(\tilde{u},\tilde{p})) \in E_{F,T} \times E_{G,T} \times E_{H,T}$ with $H(u,p)|_{t=0} = 0$. Moreover, Theorem 2.3 .1 claims that for $(F, G, f + H, u_0) \in E_{F,T} \times E_{G,T} \times E_{H,T} \times E_0$ such that the compatibility condition (2.2 .13) is fulfilled in dimension $d = 3$, there exists a unique solution $(u,p) \in E_{u,T,u_0} \times E_{p,T,(u_0,f)}$ of (2.3 .1). Consequently, we are able to construct a map

$$\mathcal{F} : \begin{array}{ccc} E_{u,T,u_0,K} \times E_{p,T,(u_0,f),K} & \to & E_{u,T,u_0} \times E_{p,T,(u_0,f)} \\ (\tilde{u},\tilde{p}) & \mapsto & (u,p) \end{array}, \tag{2.4 .10}$$

where (u,p) is the solution given by Theorem 2.3 .1 of the following system

$$\left\{ \begin{array}{rcll} \partial_t u - \nabla\cdot\sigma(u,p) & = & (f + H(\tilde{u},\tilde{p}))\delta_\Gamma + F(\tilde{u},\tilde{p}) & \text{in } (0,T)\times\Omega, \\ \nabla\cdot u & = & \nabla\cdot G(\tilde{u}) & \text{on } (0,T)\times\Omega, \\ u & = & 0 & \text{on } (0,T)\times\partial\Omega, \\ u|_{t=0} & = & u_0 & \text{in } \Omega. \end{array} \right.$$

with $F(\tilde{u},\tilde{p}), G(\tilde{u}), H(\tilde{u},\tilde{p})$ defined in (2.2 .2)-(2.2 .7) and (2.2 .10). Remark that the existence of a solution $(u,p) \in E_{u,T,u_0} \times E_{p,T,(u_0,f)}$ of (2.2 .9) is induced by the existence of a fixed point of $\mathcal{F}$. The space $(E_{u,T,u_0} \times E_{p,T,(u_0,f)}, \|\cdot\|_{E_{u,T}\times E_{p,T}})$ is a complete space. Therefore, our goal is to show that $\mathcal{F}$ enjoys the assumptions of Picard fixed point theorem for a set of the form $E_{u,T,u_0,K} \times E_{p,T,(u_0,f),K}$ for some $K, T > 0$ suitably chosen. Notice also that for $T^* > 0$, there exists $C_0 > 0$, such that for all K such that

$$K > C_0,$$

the space $E_{u,T^*,u_0,K} \times E_{p,T^*,(u_0,f),K}$ is not empty.

Step 1: We first prove that for $T > 0$ sufficiently small, $\mathcal{F}$ is a contraction mapping. Let $(\tilde{u},\tilde{p}), (\tilde{v},\tilde{q}) \in E_{u,T,u_0,K} \times E_{p,T,(u_0,f),K}$ and $(u,p) = \mathcal{F}(\tilde{u},\tilde{p})$, $(v,q) = \mathcal{F}(\tilde{v},\tilde{q})$. Then $(u-v, p-q)$ is a solution of

$$\left\{ \begin{array}{rll} \partial_t(u-v) - \nabla\cdot\sigma(u-v,p-q) & = (H(\tilde{u},\tilde{p}) - H(\tilde{v},\tilde{q}))\delta_\Gamma + F(\tilde{u},\tilde{p}) - F(\tilde{v},\tilde{q}) & \text{in } (0,T)\times\Omega, \\ \nabla\cdot(u-v) & = \nabla\cdot(G(\tilde{u}) - G(\tilde{v})) & \text{in } (0,T)\times\Omega, \\ u - v & = 0 & \text{on } (0,T)\times\partial\Omega, \\ (u-v)|_{t=0} & = 0 & \text{in } \Omega. \end{array} \right.$$

Applying the estimates of Theorem 2.3 .1, we have

$$\|(u,p)-(v,q)\|_{E_{u,T}\times E_{p,T}} \leq C\|(F(\tilde{u},\tilde{p}),G(\tilde{u}),H(\tilde{u},\tilde{p}))-(F(\tilde{v},\tilde{q}),G(\tilde{v}),H(\tilde{v},\tilde{q}))\|_{E_{F,T}\times E_{G,T}\times E_{H,T}},$$

with C independent of time T. We can use Lipschitz estimates of Lemma 2.4 .2 to obtain:

$$\|(u,p)-(v,q)\|_{E_{u,T}\times E_{p,T}} \leq C_1(K)T^{1/2-\ell/2}\|(\tilde{u},\tilde{p})-(\tilde{v},\tilde{q})\|_{E_{u,T}\times E_{p,T}}, \tag{2.4 .11}$$

where $C_1(K)$ does not depend on T and is an increasing function of K. So as $\mathcal{F}$ to be a contraction mapping, we require

$$T^{1/2-\ell/2} < \frac{1}{C_1(K)}.$$

Step 2: To apply Picard fixed point Theorem, as C_1 depends on K, we also need

$$\mathcal{F}(E_{u,T,u_0,K}\times E_{p,T,(u_0,f),K}) \subset E_{u,T,u_0,K}\times E_{p,T,(u_0,f),K}. \tag{2.4 .12}$$

Remark that direct estimates may cause problems because we do not have an estimate with a constant decreasing in time T for $G(u)$: we must rely on Lipschitz estimate. Actually, we work around a 'reference' solution. Remark that $(0,0)\notin E_{u,T,u_0}\times E_{p,T,(u_0,f)}$ so that we cannot work directly on $\mathcal{F}(0,0)$. Let $(\tilde{v},\tilde{q})\in E_{u,T,u_0}\times E_{p,T,(u_0,f)}$, be the solution given by Theorem 2.3 .1 of

$$\left\{\begin{array}{rcll} \partial_t\tilde{v}-\mu\Delta\tilde{v}+\nabla\tilde{q} &=& f\delta_\Gamma & \text{in } (0,T)\times\Omega, \\ \nabla\cdot\tilde{v} &=& 0 & \text{on } (0,T)\times\Omega, \\ \tilde{v} &=& 0 & \text{on } (0,T)\times\partial\Omega, \\ \tilde{v}|_{t=0} &=& u_0 & \text{in } \Omega \end{array}\right. \tag{2.4 .13}$$

Using the estimates of Theorem 2.3 .1, we have

$$\|(\tilde{v},\tilde{q})\|_{E_{u,T}\times E_{p,T}} \leq C_2. \tag{2.4 .14}$$

where C_2 only depends on the data f and u_0 and is independent of time T and of K. We take

$$K > C_2,$$

and T^* associated to K by Lemma 2.4 .1 so that we can define $(v,q)=\mathcal{F}(\tilde{v},\tilde{q})$. With Theorem 2.3 .1 and Lemma 2.4 .2, we have

$$\|(v,q)\|_{E_{u,T}\times E_{p,T}} \leq C_3. \tag{2.4 .15}$$

where C_3 does not depend on time T and K. We now have all the tools to prove that for well chosen $K>0$, there exists $T\in(0,T^*)$ sufficiently small to have (2.4 .12). Let $(\tilde{u},\tilde{p})\in E_{u,T,u_0,K}\times E_{p,T,(u_0,f),K}$ and $(u,p)=\mathcal{F}(\tilde{u},\tilde{p})$. Using the computations of (2.4 .11) and estimates (2.4 .14)-(2.4 .15), we get

$$\|(u,p)\|_{E_{u,T}\times E_{p,T}} \leq C_1(K)T^{1/2-\ell/2}(K+C_2)+C_3.$$

To get (2.4 .12), it is sufficient to have

$$K > C_3 \quad \text{and} \quad C_1(K)T^{1/2-l/2}(K+C_2) < K-C_3.$$

To summarize, let

$$K > \max\{C_0, C_2, C_3\},$$

and the final time T^* associated by Lemma 2.4 .1 and let

$$T < \min\left\{T^*, \left(\frac{1}{C_1(K)}\right)^{1/(1/2-\ell/2)}, \left(\frac{K-C_3}{(K+C_2)C_1(K)}\right)^{1/(1/2-\ell/2)}\right\}.$$

Then the map $\mathcal{F}$ satisfies the assumptions of Picard fixed point Theorem. The fixed point $(u,p)\in E_{u,T,u_0,K}\times E_{p,T,(u_0,f),K}$ of $\mathcal{F}$ is a solution of (2.2 .9).

Last step: We go back to the Eulerian variables. With Lemma 2.4 .1, we introduce for all $t \in [0,T]$, the inverse $Y(t,.)$ of the flow $X(t,.)$. Define

$$u_s(t,x) = u(t,Y(t,x)) \quad \text{and} \quad p_s(t,x) = p(t,Y(t,x)) \quad \text{for} \quad (t,x) \in (0,T)\times\Omega.$$

Computations of Section 2.2 .1 are justified and (u_s, p_s, X) is a strong solution of (2.1 .1)-(2.1 .2).

2.4 .2 The non linear problem with small data

We are interested in the non linear problem (2.2 .9) but this time, we consider small data instead of small time T. To apply Picard fixed point Theorem, we rely on the smallness of the norm of (u,p) to get a contraction. Section 2.4 .2.1 is dedicated to the estimates of the non linear terms and Section 2.4 .2.2 to the proof of Theorems 2.2 .3 and 2.2 .5.

2.4 .2.1 Estimates of the non linear terms with small data

Estimates for the non linear terms are exactly the same as the ones of Lemmas 2.4 .1 and 2.4 .2. Proofs are almost identical and are even easier for we do not take the time T into account. In Lemma 2.4 .1, to assure that X is invertible, we chose the final time T^* according to the size K of the ball of u, whereas here, we chose the maximal size K^* of the ball according to the final time T.

Lemma 2.4 .3. *For all $T>0$, there exist $K^*, C>0$ such that for all $K \in (0,K^*)$, $u_0 \in E_0^\ell$ and $u, v \in E^\ell_{u,T,u_0,K}$, $X(T,\cdot)$ is invertible on Ω and we have*

$$\|\mathrm{cof}(J_{X^\pm}) - I\|_{H^1(0,T;H^{1+\ell}(\Omega^\pm))\cap H^{3/2+\ell/2}(0,T;L^2(\Omega^\pm))} \leq C\|u\|_{E^\ell_{u,T}},$$
$$\left\|\frac{\mathrm{cof}(J_{X^\pm})}{|\mathrm{cof}(J_{X^\pm})n_\Gamma|} - I\right\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))} \leq C\|u\|_{E^\ell_{u,T}},$$

and the Lipschitz estimates:

$$\|\mathrm{cof}(J_{X_u^\pm}) - \mathrm{cof}(J_{X_v^\pm})\|_{H^1(0,T;H^{1+\ell}(\Omega^\pm))\cap H^{3/2+\ell/2}(0,T;L^2(\Omega^\pm))} \leq C\|u-v\|_{E^\ell_{u,T}},$$
$$\left\|\frac{\mathrm{cof}(J_{X_u^\pm})}{|\mathrm{cof}(J_{X_u^\pm})n_\Gamma|} - \frac{\mathrm{cof}(J_{X_v^\pm})}{|\mathrm{cof}(J_{X_v^\pm})n_\Gamma|}\right\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))} \leq C\|u-v\|_{E^\ell_{u,T}}.$$

The proof of Lemma 2.4 .3 follows the same lines as Lemma 2.4 .1 changing T^* by T and K by K^* in (2.4 .3), (2.4 .5), (2.4 .8) and is therefore left to the reader.
We can now state the estimates of the non linear terms.

Lemma 2.4 .4. *Let $T>0$, $K^*>0$ be defined in Lemma 2.4 .1. There exists $C>0$ such that for all $K \in (0,K^*)$, $(u_0,f) \in E_0^\ell \times E^\ell_{H,T}$, $(u,p),(v,q) \in E^\ell_{u,T,u_0,K} \times E^\ell_{p,T,(u_0,f),K}$. Defining $F(u,p), G(u), H(u,p)$ by (2.2 .2)-(2.2 .7) and (2.2 .10), we have $(F(u,p),G(u),H(u,p)) \in E^\ell_{F,T} \times E^\ell_{G,T} \times E^\ell_{H,T}$ with $H(u,p)|_{t=0} = 0$ and the estimates*

$$\|(F(u,p),G(u),H(u,p))\|_{E^\ell_{F,T}\times E^\ell_{H,T}} \leq C\|(u,p)\|^2_{E^\ell_{u,T}\times E^\ell_{p,T}},$$

and

$$\|(F(u,p),G(u),H(u,p)) - (F(v,q),G(v),H(v,q))\|_{E^\ell_{F,T}\times E^\ell_{G,T}\times E^\ell_{H,T}}$$
$$\leq C(\|(u,p)\|_{E^\ell_{u,T}\times E^\ell_{p,T}} + \|(v,q)\|_{E^\ell_{u,T}\times E^\ell_{p,T}})\|(u,p)-(v,q)\|_{E^\ell_{u,T}\times E^\ell_{p,T}}.$$

The proof of Lemma 2.4 .4 is similar to the one of Lemma 2.4 .2 and is therefore left to the reader.

2.4 .2.2 Proof of Theorems 2.2 .3 and 2.2 .5

Let $T > 0$, K^* defined in Lemma 2.4 .3, $K \in (0, K^*)$, $r > 0$, $(u_0, f) \in E_0 \times E_{H,T}$ satisfying the smallness condition (2.2 .12) and the compatibility condition (2.2 .13) in dimension $d = 3$. We take the same notations as in Section 2.4 .1.2.

For a given $K^* > 0$ and $T > 0$, there exists $r_0 > 0$ such that for all $(u_0, f) \in E_0 \times E_{H,T}$ such that

$$\|(u_0, f)\|_{E_0 \times E_{H,T}} < r_0,$$

the space $E_{u,T,u_0,K^*} \times E_{p,T,(u_0,f),K^*}$ is not empty.

Let $(\tilde{u}, \tilde{p}), (\tilde{v}, \tilde{q}) \in E_{u,T,u_0,K} \times E_{p,T,(u_0,f),K}$ and $(u, p) = \mathcal{F}(\tilde{u}, \tilde{p})$, $(v, q) = \mathcal{F}(\tilde{v}, \tilde{q})$. With Theorem 2.3 .1 and Lemma 2.4 .4, we have

$$\|(u, p)\|_{E_{u,T} \times E_{p,T}} \leq C'(K^2 + r),$$
$$\|(u, p) - (v, q)\|_{E_{u,T} \times E_{p,T}} \leq C'K \|(\tilde{u}, \tilde{p}) - (\tilde{v}, \tilde{q})\|_{E_{u,T} \times E_{p,T}},$$

where C' does not depend on K and r. Let $r, K > 0$ such that

$$C'(K^2 + r) < K \quad \text{and} \quad C'K < 1 \quad \text{and} \quad r < r_0. \tag{2.4 .16}$$

For example, take $r = K^2$ and $K < \min\left\{K^*, \sqrt{r_0}, \dfrac{1}{2C'}\right\}$. The map $\mathcal{F}$ then satisfies the assumptions of Picard fixed point Theorem and the fixed point of $\mathcal{F}$ is a solution of (2.2 .9). The end of the proof of Theorems 2.2 .3 and 2.2 .5 is the same as Theorems 2.2 .2 and 2.2 .4.

2.A Technical results

For the convenience of the reader, we collect here some technical results. The two first Lemmas can be found in [8] and [20]. We add them in our article for the purpose of completeness.

Lemma 2.A.1. *Let $\lambda, \mu, \omega \in \mathbb{R}$, Ω a smooth open subset of $\mathbb{R}^n$, $f \in H^{\lambda+\mu}(\Omega)$, $g \in H^{\lambda+\omega}(\Omega)$. Then,*

$$\|fg\|_{H^\lambda(\Omega)} \leq C \|f\|_{H^{\lambda+\mu}(\Omega)} \|g\|_{H^{\lambda+\omega}(\Omega)},$$

(i) when $\lambda + \mu + \omega \geq n/2$,

(ii) with $\mu \geq 0$, $\omega \geq 0$, $2\lambda \geq -\mu - \lambda$,

(iii) except that $\lambda + \mu + \omega > n/2$ if equality holds somewhere in (ii).

Lemma 2.A.1 is exactly [8, Proposition B.1].

Lemma 2.A.2. *Let $0 < T < T^*$, X, Y, Z three real Hilbert spaces and m a bounded bilinear mapping from $X \times Y$ into Z. Let us assume that $f \in H^{s_1}(0, T; X)$ and $g \in H^{s_2}(0, T; Y)$ for $s_1, s_2 \geq 0$.*

(i) If $s_1 \in (1/2, 3/2)$, $s_2 \in [0, 1/2)$, then $m(f, g) \in H^{s_2}(0, T; Z)$ with

$$\|m(f, g)\|_{H^{s_2}(0,T;Z)} \leq C(\|f\|_{H^{s_1}(0,T;X)} + \|f(0)\|_X) \|g\|_{H^{s_2}(0,T;Y)},$$

where the constant C does not depend on time T.

(ii) If $s_1, s_2 \in (1/2, 3/2)$, then $m(f, g) \in H^{s_2}(0, T; Z)$ with

$$\|m(f, g)\|_{H^{s_2}(0,T;Z)} \leq C(\|f\|_{H^{s_1}(0,T;X)} + \|f(0)\|_X)(\|g\|_{H^{s_2}(0,T;Y)} + \|g(0)\|_Y),$$

where the constant C does not depend on time T.

In particular, if one has $f, g \in H^1(0,T,H^{1+\ell}(\Omega^\pm)) \cap H^{3/2+\ell/2}(0,T;L^2(\Omega^\pm))$, then one gets $fg \in H^1(0,T,H^{1+\ell}(\Omega^\pm)) \cap H^{3/2+\ell/2}(0,T;L^2(\Omega^\pm))$ with

$$\begin{aligned}
&\|fg\|_{H^1(0,T,H^{1+\ell}(\Omega^\pm))\cap H^{3/2+\ell/2}(0,T;L^2(\Omega^\pm))} \\
&\qquad \le C\big(\|f\|_{H^1(0,T,H^{1+\ell}(\Omega^\pm))\cap H^{3/2+\ell/2}(0,T;L^2(\Omega^\pm))}\|g\|_{H^1(0,T,H^{1+\ell}(\Omega^\pm))\cap H^{3/2+\ell/2}(0,T;L^2(\Omega^\pm))} \\
&\qquad +\|f(0)\|_{H^{1+\ell}(\Omega^\pm)}(\|g\|_{H^1(0,T,H^{1+\ell}(\Omega^\pm))\cap H^{3/2+\ell/2}(0,T;L^2(\Omega^\pm))} + \|\partial_t g(0)\|_{L^2(\Omega^\pm)}) \\
&\qquad +\|g(0)\|_{H^{1+\ell}(\Omega^\pm)}(\|f\|_{H^1(0,T,H^{1+\ell}(\Omega^\pm))\cap H^{3/2+\ell/2}(0,T;L^2(\Omega^\pm))} + \|\partial_t f(0)\|_{L^2(\Omega^\pm)}) \\
&\qquad +\|f\|_{H^1(0,T;H^{1+\ell}(\Omega^\pm))}\|\partial_t g(0)\|_{L^2(\Omega^\pm)} + \|g\|_{H^1(0,T;H^{1+\ell}(\Omega^\pm))}\|\partial_t f(0)\|_{L^2(\Omega^\pm)} \\
&\qquad +\|f(0)\|_{H^{1+\ell}(\Omega^\pm)}\|g(0)\|_{H^{1+\ell}(\Omega^\pm)}\big),
\end{aligned}$$

where the constant C does not depend on time T.

Lemma 2.A.2 is completely contained in [20, Lemma A.1, Lemma A.4].

Lemma 2.A.3. *Let E be a Banach Space, $T > 0$, $b \in [0,1]$, $a \in [\max\{b, 1/2\}, 1]$, and $f \in H^a(0,T;E)$ such that $f(0) = 0$. Then for the norm defined in Section 2.2 .4.1, we have*

$$\|f\|_{H^b(0,T;E)} \le C \max\{T^{a-b}, T^a\}\|f\|_{H^a(0,T;E)},$$

where the constant C does not depend on time T.

Proof of Lemma 2.A.3. We divide the proof in several cases.

The case $b = 0$ and $a = 1$:

It is sufficient to remark that

$$\int_0^T \|f(t)\|_E^2 dt \le \int_0^T \left(\int_0^t \|\partial_t f(s)\|_E ds\right)^2 dt \le T^2 \int_0^T \|\partial_t f(s)\|_E^2 ds. \tag{2.A.1}$$

In the following we use a scaling argument mapping $(0,T)$ to $(0,1)$ through the transformation

$$\tilde{f}(s) = f(sT).$$

We thus have $\tilde{f} \in H^a(0,1;E)$ with $f(0) = 0$. The main advantage of working on $(0,1)$ instead of $(0,T)$ is that the constants of interpolation and of equivalent norms do not depend on time T.

The case $a = 1$:

Using interpolation estimate, Poincaré inequality and the equivalence of the interpolated norm and the norm defined in Section 2.2 .4.1, we have

$$\|\tilde{f}\|_{H^b(0,1;E)} \le C\|\tilde{f}\|_{L^2(0,1;E)}^{1-b}\|\partial_t \tilde{f}\|_{L^2(0,1;E)}^b,$$

with

$$\|\tilde{f}\|_{L^2(0,1;E)} = T^{-1/2}\|f\|_{L^2(0,T;E)} \quad \text{and} \quad \|\partial_t \tilde{f}\|_{L^2(0,1;E)} = T^{1/2}\|\partial_t f\|_{L^2(0,1;E)}. \tag{2.A.2}$$

Using (2.A.1), we obtain

$$\|\tilde{f}\|_{L^2(0,1;E)}^{1-b}\|\partial_t \tilde{f}\|_{L^2(0,1;E)}^b \le T^{1/2}\|f\|_{H^1(0,T;E)}.$$

Furthermore, by the change of variables $t = sT$ and $t' = s'T$,

$$\int_0^1 \int_0^1 \frac{\|\tilde{f}(s) - \tilde{f}(s')\|_E^2}{|s-s'|^{1+2b}} ds ds' = T^{2b-1} \int_0^T \int_0^T \frac{\|f(t) - f(t')\|_E^2}{|t-t'|^{1+2b}} dt dt'. \tag{2.A.3}$$

Together with (2.A.1), we finally get

$$\|f\|_{H^b(0,T;E)} \le C \max\{T, T^{1-b}\}\|f\|_{H^1(0,T;E)},$$

where C does not depend on time T.

The case $b = 0$:

Using Poincaré inequality, we have

$$\|\tilde{f}\|^2_{L^2(0,1;E)} \leq C \int_0^1 \int_0^1 \frac{\|\tilde{f}(s) - \tilde{f}(s')\|^2_E}{|s - s'|^{1+2a}} ds ds',$$

which directly gives with (2.A.2) and (2.A.3)

$$\|f\|_{L^2(0,T;E)} \leq CT^a \|f\|_{H^a(0,T;E)}, \tag{2.A.4}$$

where C does not depend on time T.

The case $b > 0$ and $a < 1$ (with $a > \max\{b, 1/2\}$):

We have

$$\int_0^1 \int_0^1 \frac{\|\tilde{f}(s) - \tilde{f}(s')\|^2_E}{|s - s'|^{1+2b}} ds ds' \leq \int_0^1 \int_0^1 \frac{\|\tilde{f}(s) - \tilde{f}(s')\|^2_E}{|s - s'|^{1+2a}} ds ds'.$$

Equality (2.A.3) gives

$$\int_0^T \int_0^T \frac{\|f(t) - f(t')\|^2_E}{|t - t'|^{1+2b}} dt dt' \leq T^{2(a-b)} \int_0^T \int_0^T \frac{\|f(t) - f(t')\|^2_E}{|t - t'|^{1+2a}} dt dt'.$$

Together with (2.A.4), we obtain

$$\|f\|_{H^b(0,T;E)} \leq C \max\{T^{a-b}, T^a\} \|f\|_{H^a(0,T;E)},$$

where C does not depend on time T. $\square$

Lemma 2.A.4. *Let $0 < T < T^*$, $f \in H^1(0,T; H^{1/2+\ell}(\Gamma))$ such that $\inf\limits_{[0,T]\times\Gamma} f \geq m > 0$.*

(i) Assume $m > 0$. Then $1/f \in H^1(0,T; H^{1/2+\ell}(\Gamma))$ and

$$\begin{aligned}&\|1/f\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))} \\ &\qquad \leq C(1 + \|f(0)\|_{H^{1/2+\ell}(\Gamma)} + \|f\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))})^4 \|f\|_{H^1(0,T;H^{1/2+\ell}(\Gamma)},\end{aligned}$$

where the constant C does not depend on time T.

(ii) Assume $m > -1$. Then $\dfrac{1}{\sqrt{1+f}} - 1 \in H^1(0,T; H^{1/2+\ell}(\Gamma))$ and

$$\begin{aligned}&\left\|\tfrac{1}{\sqrt{1+f}} - 1\right\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))} \\ &\qquad \leq C(1 + \|f(0)\|_{H^{1/2+\ell}(\Gamma)} + \|f\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))})^6 \|f\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))},\end{aligned}$$

where the constant C does not depend on time T.

(iii) Assume $m > -1$. Then $\sqrt{1+f} - 1 \in H^1(0,T; H^{1/2+\ell}(\Gamma))$ and

$$\begin{aligned}&\|\sqrt{1+f} - 1\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))} \\ &\qquad \leq C(1 + \|f(0)\|_{H^{1/2+l}(\Gamma)} + \|f\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))})^2 \|f\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))},\end{aligned}$$

where the constant C does not depend on time T.

Proof of Lemma 2.A.4. First remark that with the continuous embedding $H^1(0,T; H^{1/2+\ell}(\Gamma)) \hookrightarrow C([0,T] \times \Gamma)$, $f \in C([0,T] \times \Gamma)$ and $\inf\limits_{[0,T]\times\Gamma} f$ is a minimum since $[0,T] \times \Gamma$ is compact.

(i) Let G_1 be a C^∞ non-negative function such that $G_1(0) = 0$ and $G_1(r) = 1/r$ for $|r| \geq m$. Remark that all derivatives of G_1 are bounded. Using [21, Chapter 5.3.6, Theorem 1] if $\ell \in (0, 1/2)$ or [21, Chapter 5.3.6, Theorem 2] if $\ell \in (1/2, 1)$, we have

$$\|G_1(f(t))\|_{H^{1/2+\ell}(\Gamma)} \leq C(1 + \|f(t)\|_{L^\infty(\Gamma)})\|f(t)\|_{H^{1/2+\ell}(\Gamma)}.$$

Using the continuous embedding $H^{1/2+\ell}(\Gamma) \hookrightarrow L^\infty(\Gamma)$ and the fact that

$$\|f\|_{L^\infty(0,T;H^{1/2+\ell}(\Gamma))} \leq T^{1/2}\|f\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))} + \|f(0)\|_{H^{1/2+\ell}(\Gamma)},$$

we obtain

$$\|G_1(f)\|_{L^\infty(0,T;H^{1/2+\ell}(\Gamma))} \leq C(1 + \|f(0)\|_{H^{1/2+\ell}(\Gamma)} + \|f\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))})^2,$$
$$\|G_1(f)\|_{L^2(0,T;H^{1/2+\ell}(\Gamma))} \leq C(1 + \|f(0)\|_{H^{1/2+\ell}(\Gamma)} + \|f\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))})\|f\|_{L^2(0,T;H^{1/2+\ell}(\Gamma))},$$

where C does not depend on time T. Furthermore,

$$\partial_t G_1(f) = G_1'(f)\partial_t f.$$

Remark that since $f \geq m$, then $G_1'(f) = -G_1(f)^2$ and

$$\|G_1'(f)\|_{L^\infty(0,T;H^{1/2+\ell}(\Gamma))} \leq C(1 + \|f(0)\|_{H^{1/2+\ell}(\Gamma)} + \|f\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))})^4.$$

Thus

$$\|\partial_t G_1(f)\|_{L^2(0,T;H^{1/2+\ell}(\Gamma))} \leq C(1 + \|f(0)\|_{H^{1/2+\ell}(\Gamma)} + \|f\|_{H^1(0,T;H^{1/2+\ell}(\Gamma))})^4\|\partial_t f\|_{L^2(0,T;H^{1/2+\ell}(\Gamma))},$$

where C does not depend on time T.

(ii) Let G_2 be a C^∞ function such that $G_2(r) = \dfrac{1}{\sqrt{1+r}} - 1$ for $r \geq m$ and $G_2(r) = 0$ for $r \leq -1$. Remark that all derivatives of G_2 are bounded. We proceed exactly as in (i) remarking that $G_2'(f) = -\dfrac{1}{2}(G_2(f) + 1)^3$.

(iii) Let G_3 be a C^∞ function such that $G_3(r) = \sqrt{1+r} - 1$ for $r \geq m$ and $G_3(r) = 0$ for $r \leq -1$. Remark that G_3 is not bounded, but all its non trivial derivatives are bounded. We proceed exactly as in (i) remarking that $G_3'(f) = \dfrac{1}{2}(G_2(f) + 1)$. □

Bibliography

[1] K. M. Arthurs, L. C. Moore, C. S. Peskin, E. B. Pitman, and H. E. Layton. Modeling arteriolar flow and mass transport using the immersed boundary method. *J. Comput. Phys.*, 147(2):402–440, 1998.

[2] I. V. Denisova. Global solvability of a problem on two fluid motion without surface tension. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 348(Kraevye Zadachi Matematicheskoi Fiziki i Smezhnye Voprosy Teorii Funktsii. 38):19–39, 303, 2007.

[3] I. V. Denisova. Global L_2-solvability of a problem governing two-phase fluid motion without surface tension. *Port. Math.*, 71(1):1–24, 2014.

[4] R. H. Dillon and L. J. Fauci. An integrative model of internal axoneme mechanics and external fluid dynamics in ciliary beating. *Journal of Theoretical Biology*, 207(3):415 – 430, 2000.

[5] R. H. Dillon, L. J. Fauci, and C. Omoto. Mathematical modeling of axoneme mechanics and fluid dynamics in ciliary and sperm motility. *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, 10(5):745–757, 2003. Progress in partial differential equations (Pullman, WA, 2002).

[6] L. J. Fauci and C. S. Peskin. A computational model of aquatic animal locomotion. *J. Comput. Phys.*, 77(1):85–108, 1988.

[7] A. L. Fogelson. A mathematical model and numerical method for studying platelet adhesion and aggregation during blood clotting. *J. Comput. Phys.*, 56(1):111–134, 1984.

[8] G. Grubb and V. A. Solonnikov. Boundary value problems for the nonstationary Navier-Stokes equations treated by pseudo-differential methods. *Math. Scand.*, 69(2):217–290 (1992), 1991.

[9] Y. Hataya. Global solution of two-layer navier-stokes flow. *Nonlinear Analysis: Theory, Methods and Applications*, 63(5-7):1409–1420, 2005.

[10] M. Köhne, J. Prüss, and M. Wilke. On quasilinear parabolic evolution equations in weighted L_p-spaces. *J. Evol. Equ.*, 10(2):443–463, 2010.

[11] F.-H. Lin and J. Tong. Solvability of the Stokes Immersed Boundary Problem in Two Dimension. *ArXiv e-prints*, March 2017.

[12] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. I.* Springer-Verlag, New York-Heidelberg, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181.

[13] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. II.* Springer-Verlag, New York-Heidelberg, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 182.

[14] Y. Mori, A. Rodenberg, and D. Spirn. Well-posedness and global behavior of the Peskin problem of an immersed elastic filament in Stokes flow. *ArXiv e-prints*, April 2017.

[15] C. S. Peskin. Flow patterns around heart valves: A numerical method. *Journal of Computational Physics*, 10(2):252 – 271, 1972.

[16] C. S. Peskin. The immersed boundary method. *Acta Numer.*, 11:479–517, 2002.

[17] C. S. Peskin and D. M. McQueen. A three-dimensional computational method for blood flow in the heart. I. Immersed elastic fibers in a viscous incompressible fluid. *J. Comput. Phys.*, 81(2):372–405, 1989.

[18] C. S. Peskin and B. F. Printz. Improved volume conservation in the computation of flows with immersed elastic boundaries. *J. Comput. Phys.*, 105(1):33–46, 1993.

[19] J. Prüss and G. Simonett. On the two-phase Navier-Stokes equations with surface tension. *Interfaces Free Bound.*, 12(3):311–345, 2010.

[20] J.-P. Raymond and M. Vanninathan. A fluid-structure model coupling the Navier-Stokes equations and the Lamé system. *J. Math. Pures Appl. (9)*, 102(3):546–596, 2014.

[21] T. Runst and W. Sickel. *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, volume 3 of *de Gruyter Series in Nonlinear Analysis and Applications.* Walter de Gruyter & Co., Berlin, 1996.

[22] J. Stockie. Immersed boundary method: Recent developments in analysis, algorithms and applications, August 2015. Beijing, ICIAM.

[23] N. Tanaka. Global existence of two phase nonhomogeneous viscous incompressible fluid flow. *Comm. Partial Differential Equations*, 18(1-2):41–81, 1993.

[24] N. Tanaka. Two-phase free boundary problem for viscous incompressible thermocapillary convection. *Japan. J. Math. (N.S.)*, 21(1):1–42, 1995.

[25] B. A. Ton. Time-dependent Stokes equations with measure data. *Abstr. Appl. Anal.*, 2003(17):953–973, 2003.

[26] B. A. Ton. A free boundary problem for the Navier-Stokes equations with measure data. *Nonlinear Anal.*, 64(1):1–21, 2006.

[27] M. Wilke. Rayleigh-Taylor instability for the two-phase Navier-Stokes equations with surface tension in cylindrical domains. *ArXiv e-prints*, March 2017.

[28] L. Zhu and C. S. Peskin. Simulation of a flapping flexible filament in a flowing soap film by the immersed boundary method. *J. Comput. Phys.*, 179(2):452–468, 2002.

Chapter 3

Exponential stabilization at any rate of a bifluid interface with surface tension in a torus

This article deals with a model of two layers of viscous incompressible capillary immiscible fluids in a torus in dimension 2 and 3 without any external force. This problem has been proved well-posed and stable at exponential rate in [53]. We prove that we can exponentially stabilize the system at any decay rate by mean of a boundary control acting only in one fluid, the control being found in an integrator feedback form. The proof relies on classical semigroup arguments based on a careful definition of weak solutions for the stationary problem. We hope to extend this work to the case of non trivial external forces to stabilize Rayleigh-Taylor instability.

3.1 Introduction

Let $\Omega = \mathbb{T}_d \times (-1,1) \subset \mathbb{R}^d$ where $\mathbb{T}_d = \mathbb{R}^{d-1}/2\pi\mathbb{Z}^{d-1}$ stands for the torus in dimension $d-1$ with $d \in \{2,3\}$. Let $k_0 \in H^2(\mathbb{T}_d)$ such that $k_0 \in (-1,1)$, $\Gamma_0 = \{(x', k_0(x')) \in \overline{\Omega} \mid x' \in \mathbb{T}_d\}$ be the initial position of the interface, $\Omega_0^+ = \{(x', x_d) \in \Omega \mid x_d > k_0(x')\}$ be the domain initially filled with the fluid $+$, $\Omega_0^- = \{(x', x_d) \in \Omega \mid x_d < k_0(x')\}$ be the domain initially filled with the fluid $-$, $\Gamma_d^\pm = \mathbb{T} \times \{\pm 1\}$ be the upper and lower boundary of the fluids, $\Gamma_d = \Gamma_d^+ \cup \Gamma_d^-$ be the boundary of the domain Ω and finally $\Gamma_c \subset\subset \Gamma_d^+$ or $\Gamma_c \subset\subset \Gamma_d^-$ be the control zone located on one of the boundary Γ_d^+ or Γ_d^-. This initial configuration is summarized in Figure 3.1.

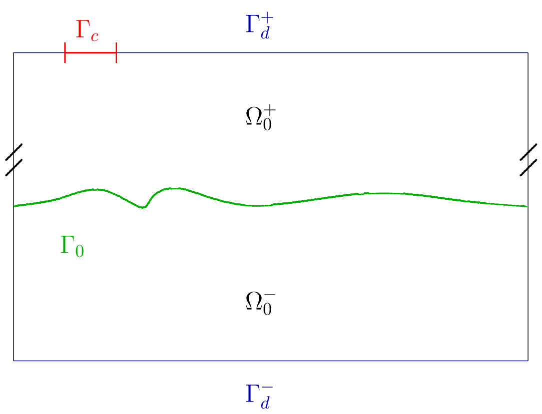

Figure 3.1: Initial configuration in dimension $d = 2$ with $\Gamma_c \subset \Gamma_d^+$.

We are interested in the stabilization of the following system:

$$\left\{\begin{array}{rcll} \rho^\pm\partial_t v^\pm + \rho^\pm(v^\pm\cdot\nabla)v^\pm &=& \mu^\pm\Delta v^\pm - \nabla q^\pm & \text{in } \Omega^\pm(t),\ t>0,\\ \nabla\cdot v^\pm &=& 0 & \text{in } \Omega^\pm(t),\ t>0,\\ v &=& g & \text{on } (0,+\infty)\times\Gamma_d,\\ v^- &=& v^+ & \text{on } \Gamma(t),\ t>0,\\ [\mathbb{S}(v,q)n_{\Gamma(t)}] &=& -\sigma\kappa n_{\Gamma(t)} & \text{on } \Gamma(t),\ t>0,\\ V(t,x') &=& v(t,x',k(t,x'))\cdot n_{\Gamma(t)} & \text{for } (t,x')\in(0,+\infty)\times\mathbb{T}_d,\\ v(0) &=& v_0 & \text{in } \Omega,\\ k(0) &=& k_0 & \text{on } \mathbb{T}_d, \end{array}\right. \tag{3.1 .1}$$

where the stress tensor $\mathbb{S}^\pm(v^\pm,q^\pm)$ is defined as follows

$$\mathbb{S}^\pm(v^\pm,q^\pm) = \mu^\pm D(v) - q^\pm I, \quad \text{with} \quad D(v) = \nabla v^\pm + (\nabla v^\pm)^T,$$

the jump $[w]$ across $\Gamma(t)$ is defined as follows

$$[w] = (w|_{\Omega^+(t)} - w|_{\Omega^-(t)})|_{\Gamma(t)},$$

and for $t\in\mathbb{R}_+^*$,

$$\begin{array}{rcl} \Gamma(t) &=& \{(x',k(t,x')) \mid x'\in\mathbb{T}_d\},\\ \Omega^+(t) &=& \{(x',x_d)\in\Omega \mid x_d > k(t,x')\},\\ \Omega^-(t) &=& \{(x',x_d)\in\Omega \mid x_d < k(t,x')\}. \end{array}$$

This system models the dynamic of a two phase fluid problem around a quasi flat interface in a periodic domain. The fluids are assumed to follow the incompressible Navier-Stokes equations in their domain. At the interface $\Gamma(t)$, we assume a non slip condition $(3.1\ .1)_4$ and a surface tension effect $(3.1\ .1)_5$. The interface $\Gamma(t)$ is assumed to be driven by the normal trace at the interface of the fluid flow $(3.1\ .1)_6$. The unknowns are the velocity of the fluids $v^\pm$, the pressure of the fluids $q^\pm$ and the height $k = k(t,x')\in\mathbb{R}$ of the interface $\Gamma(t)$. The densities $\rho^\pm>0$ and the viscosities $\mu^\pm>0$ of each fluid are assumed to be constant. The jump $\sigma\kappa n_{\Gamma(t)}$ represents the surface tension at the interface $\Gamma(t)$: the constant σ stands for the coefficient of the surface tension, $\kappa(t,\cdot)$ represents the mean curvature of $\Gamma(t)$ with respect to the normal $n_{\Gamma(t)}$ of $\Gamma(t)$ pointing outward $\Omega^-(t)$. We recall the formulations

$$n_{\Gamma(t)} = \frac{1}{\sqrt{1+|\nabla_{x'}k|^2}}\begin{pmatrix}-\nabla_{x'}k\\ 1\end{pmatrix},$$

and

$$\kappa = \nabla\cdot\left(\frac{\nabla_{x'}k}{\sqrt{1+|\nabla_{x'}k|^2}}\right).$$

The term $V(t,x')$ for $(t,x')\in(0,+\infty)\times\mathbb{T}_d$ represents the normal velocity of $\Gamma(t)$ and is given by

$$V(t,x') = \frac{\partial_t k(t,x')}{\sqrt{1+|\nabla_{x'}k(t,x')|^2}} \quad \text{for } (t,x')\in(0,+\infty)\times\mathbb{T}_d.$$

Lastly, g represents the boundary control and is supposed to be finite dimensional and act only on Γ_c:

$$\forall(t,x)\in(0,+\infty)\times\Gamma_d,\quad g(t,x) = \sum_{i=1}^{n_c} g_i(t)w_i(x) \quad \text{with } \mathbf{g} = (g_i)_{i\in[\![1,n_c]\!]}, \tag{3.1 .2}$$

and where the control functions $(w_i)_{i\in[\![1,n_c]\!]}$ will be defined later and are such that

$$\forall i \in [\![1, n_c]\!], \quad \mathrm{Supp}(w_i) \subset \Gamma_c \quad \text{and} \quad \int_{\Gamma_c} w_i \cdot n = 0. \tag{3.1 .3}$$

The stabilization question we would like to address is the following: given any decreasing rate $\omega > 0$, can we find a control g in the form (3.1 .2) such that there exists $C > 0$ such that for small enough initial data (v_0, k_0), the unique solution (v, q, k) of system (3.1 .1) satisfies

$$\forall t > 0, \quad \|(v, q, k)(t)\| \leq C\, \mathrm{e}^{-\omega t}\, \|(v_0, k_0)\|,$$

for some norm $\|\cdot\|$ to be defined ?

Previous works. To the best of the author knowledge, stabilization result for system (3.1 .1) does not exist. However, problem of free interfaces with viscous fluids have received a wide attention over the last decades. The pioneering work of Solonnikov [40, 41, 42, 43, 44] deals with a one phase problem of a finite isolated mass of incompressible viscous fluid bounded by a free boundary. At the beginning, he considered the problem without surface tension: he first proved the local solvability in Hölder space in [40] in dimension 3 without surface tension and a global solvability in the Sobolev space $W^{2,1}_p$ in dimension $d \in \{2, 3\}$ with $p > d$ in [43]. He then considered surface tension effect and obtained in dimension 3, local existence in Sobolev spaces $H^{r,r/2}$ with $r \in (5/2, 3)$ in [41] and global existence in the same space for small initial data and an initial free surface close to the ball in [42, 44]. At the same time, Beale [7, 8] considered the one phase problem of surface waves describing the motion of one layer of fluid having a non-compact free interface in dimension 3. He began by considering the model without surface tension in [7]. He proved three main results : local existence of solutions in Sobolev space $H^{r,r/2}$ with $r \in (3, 7/2)$, existence of solutions in large times for small data, and he also proved that one plausible extension of the second assertion to an infinite time interval is false. So far, these systems are solved only by means of Lagrangian coordinates. In [8], Beale used a different change of unknowns to deal with the surface waves problem including surface tension. He showed a small-data global existence and uniqueness of solution in $H^{r,r/2}$ with $r \in (3, 7/2)$ and proved that slightly after the initial time, the solution becomes arbitrarily smooth and satisfies the equations in a classical sense. Numerous works have followed the two pioneers approaches of Solonnikov and Beale. Beale and Nishida [9] proved that global solution of surface waves with surface tension of [8] decay algebraically in time: the free surface tends to equilibrum at a rate $t^{-1/2}$ and the velocity field at a rate t^{-1}. Sylvester [45] proved that if the initial data are sufficiently regular and small and satisfy some compatibility conditions, then there exists a global solution in $H^{r,r/2}$ with $r \in (9/2, 5)$ of the surface waves problem without surface tension of [7], the solution being defined on an infinite time interval and being unique on any finite time interval. Allain in [2], Tani in [48] and Tani and Tanaka in [49] worked in Beale's configuration of surface waves but with Sobolev approach that enables to get solutions in $H^{r,r/2}$ with $r \in (5/2, 3)$. In [2], Allain showed a local existence in dimension 2 taking the surface tension effect into account. This result was generalized in dimension 3 by Tani in [48]. In dimension 2, Tani and Tanaka [49] proved existence at any time $T > 0$ for small data with and without surface tension. In the case of surface waves with surface tension, Bae [5] proved small-data global well-posedness using energy methods rather than the Beale-Solonnikov framework. Nishida et al [33] and Hataya [28] considered the problem of surface waves in a periodic case. Nishida et al [33] took the surface tension effect into account and in the spirit of [8] and [9], they proved small-data global existence and uniqueness in Sobolev space $H^{r,r/2}$ with $r \in (3, 7/2)$ and they showed that slightly after the initial time, the solution becomes arbitrarily smooth and satisfies the equations in a classical sense. Lastly, they proved the exponential decay of the solution by means of an energy method.

Hataya [28] did not consider surface tension and proved a small-data global existence on an infinite time interval in $H^{r,r/2}$ with $r \in (5, 11/2)$. In the case of one layer without surface tension, Guo and Tice proved a local well-posedness for horizontally periodic or horizontally infinite domains in [26], a global well-posedness in infinite extent domain with an algebraic decay in time in [25] and a global well-posedness in periodic domain with an almost exponential decay in time in [24]. One can also find existence of weak solutions in [1, 11]. The literature on one phase free interface problems being very rich, the above list is subjective.

Results on two phase fluid models are more recent. In the spirit of the work of Solonnikov on the one phase problem of a finite isolated mass of incompressible viscous fluid bounded by a free boundary, Denisova [12, 14, 13] and Denisova and Solonnikov [17, 18, 19] have been interested in the problem in dimension 3 of a drop of incompressible viscous fluid in another non-miscible fluid assumed to be incompressible and viscous, taking surface tension effects into account. In [12, 17], they proved a priori estimates for solutions in Sobolev spaces which lead to a result of local existence and uniqueness in $H^{r,r/2}$ with $r \in (5/2, 3)$ in [14] . In [13, 18], they proved existence, uniqueness and estimates for the linear system in Hölder spaces which lead to a local existence and uniqueness result in Hölder spaces in [19]. Meanwhile, Tanaka [46, 47] has been interested in the same model but in a bounded domain. He proved local well-posedness in $H^{r,r/2}$ with $r \in (7/2, 4)$ in [47] and global well-posedness in Denisova's Sobolev framework and with non constant densities in [46]. Later, Denisova has been interested in the same problem without surface tension in a bounded domain, and proved local existence and uniqueness, and small-data global existence in Hölder space in [15] and in Sobolev space in $H^{r,r/2}$ with $r \in (5/2, 3)$ in [16]. Furthermore, she proved that the global solution decays exponentially in time.

There are a very few results on the two phase free boundary problem describing the motion of two layers of viscous incompressible immiscible fluids. Hataya [27] has been interested in a periodic free interface problem in a gravity field with surface tension around the plane Couette flow of two fluids. He showed existence at any time for small data in Denisova's Sobolev framework and proved that the solution is defined in an infinite time interval and decays exponentially in time if the viscosities are sufficiently large and their difference is sufficiently small. In the case of fluids filling all the domain $\mathbb{R}^d$, Prüss and Simonett [36] proved existence at all time for small data considered in L^p norm such that $p > d + 3$. Furthermore, they proved that the solution becomes instantly analytic and that the equations are satisfied in a classical sense. Considering a model of two layers of viscous immiscible fluids with surface tension effect and with a fixed bottom and a free surface at the upper boundary, Xu and Zhang [53] proved local well-posedness and small-data global well-posedness for data near the equilibrium state in Denisova's Sobolev framework. For the same geometry in a periodic case, taking gravity into account in the fluid equations, working with and without surface tension, Kim, Tice and Wang [51] proved a small data well-posedness at any time and a small-data global well-posedness if the lighter fluid lies on the heavier one. Furthermore, they proved that for a sufficiently large surface tension, the global well-posedness still hold true if the heavier fluid is above. The decay is almost exponential in time in the case without surface tension and exponential with surface tension.

The two layers model is also recently considered in the study of Rayleigh-Taylor instabilities. This situation arises when considering two fluids in the gravitational field and when the heavier fluid lies above the lighter one. Then, the trivial equilibrum ($v = k = 0$ and q constant) is unstable. Perturbing this unstable state creates Rayleigh-Taylor instabilities. One can wonder if the surface tension effect can stabilize Rayleigh-Taylor instabilities. This issue has generated interest in the last years. In the periodic case, Hataya [27] proved that for sufficiently large viscosities with sufficiently small difference, the trivial equilibrum is stable even if the heavier fluid is above. Kim, Tice and Wang [51] proved that given any viscosities, there exists a critical surface tension coefficient $\sigma_c > 0$ such that the trivial equilibrum is stable for $\sigma > \sigma_c$. Tice and

Wang [50] proved that the trivial equilibrum is unstable for $\sigma < \sigma_c$ and $[\rho] > 0$. In the case of the all space $\mathbb{R}^d$, Prüss and Simonett [35] proved that the trivial equilibrum with $[\rho] > 0$ is always unstable. Lastly, in the recent work [52], Wilke considers the Rayleigh-Taylor instability in cylindrical domains. He proved the existence of a critical surface tension value as for the periodic case and discusses the case of non trivial equilibra.

To the best of the author's knowledge, interface stabilization of viscous fluids is an open field. However, boundary stabilization of Navier-Stokes equations has been thoroughly studied in the recent years. The first works of Fursikov established the existence of a boundary stabilizing feedback depending on time in dimension 2 [21] and 3 [22]. Raymond [37] improved this work in dimension 2 obtaining a feedback that does not depend on time by solving an algebraic Riccati equation. However, in dimension 3, one can not avoid compatibility condition for a feedback boundary control (see the discussion in [3] for example). Raymond [38] then used a feedback that depends on time only about $t = 0$, allowing to satisfy the required compatibility condition with any initial data. Many others boundary stabilization results exist in dimension 3: using a tangential boundary control [6], an integrator feedback [3], a feedback with a restricted initial data set [4], a finite dimensional feedback [39]. This list is not exhaustive and the interest reader can refer to the bibliography of the cited articles.

3.2 Main results

Our main result is expressed in a fixed domain. This part is divided as follow: in Section 3.2 .1, we express system (3.1 .1) in a fixed domain; in Section 3.2 .2, the main result on system (3.2 .3) is stated; in Section 3.2 .3, we give our main result on the linear problem; in Section 3.2 .4, we present ideas of the proof and the main difficulties and novelties; lastly, in Section 3.2 .5, we introduce the notations used in the article.

3.2 .1 Change of unknowns

The main purpose of this change of unknowns is to express system (3.1 .1) in a fixed domain. Let first denote by k_e a lifting of k in Ω such that $k_e = 0$ on Γ_d that will be rigorously defined later (see Lemma 3.7 .2) and let

$$X(t,\cdot) : \begin{array}{ccc} \Omega & \to & \Omega \\ (y', y_d) & \mapsto & (y', k_e(t,y',y_d)(1-y_d^2) + y_d) \end{array}. \tag{3.2 .1}$$

We assume for the moment that for all $t > 0$, $X(t,\cdot)$ is invertible and we denote by $Y(t,\cdot)$ its inverse. The domain $Y(t,\Omega)$ is a fixed domain with a flat interface as drawn in Figure 3.2.

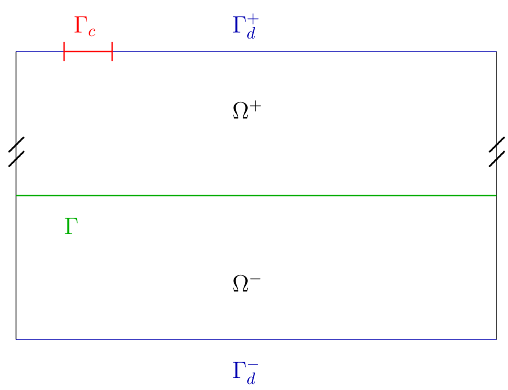

Figure 3.2: Fixed configuration in dimension $d = 2$ with $\Gamma_c \subset \Gamma_d^+$.

In the above figure, we have introduced

$$\begin{array}{ccccc} \Gamma & = & \mathbb{T}_d \times \{0\} & = & Y(t,\Gamma(t)), \\ \Omega^+ & = & \mathbb{T}_d \times (-1,0) & = & Y(t,\Omega^+(t)), \\ \Omega^- & = & \mathbb{T}_d \times (0,1) & = & Y(t,\Omega^-(t)), \end{array}$$

for all $t \in (0,+\infty)$.

We could transform directly the vector field by composition, but then the divergence free condition would be lost. A classical idea is to multiply the velocity by $\mathrm{cof}(\nabla X^\pm)^T$. This change of unknowns seems to be due to Inoue and Wakimoto [29]. Furthermore, as we want to study stabilization properties of (3.1 .1) at any rate exponential rate $\omega > 0$, we also multiply the variables by $\mathrm{e}^{\omega t}$. Let $\omega > 0$, we consider

$$u^\pm = \mathrm{e}^{\omega t}\,\mathrm{cof}(\nabla X^\pm)^T v^\pm(X^\pm) \quad \text{and} \quad p^\pm = \mathrm{e}^{\omega t}\, q(X^\pm) \quad \text{and} \quad h = \mathrm{e}^{\omega t}\, k. \tag{3.2 .2}$$

The other main advantage of this change of unknowns is that $(3.1\ .1)_6$ becomes

$$\partial_t(e^{-\omega t} h) = \mathrm{e}^{-\omega t}\, u_d \quad \text{on } (0,+\infty)\times\Gamma.$$

Remark that (3.2 .2) is equivalent to

$$v^\pm = \mathrm{e}^{-\omega t}\,\mathrm{cof}(\nabla Y^\pm)^T u^\pm(Y^\pm) \quad \text{and} \quad q^\pm = \mathrm{e}^{-\omega t}\, p^\pm(Y^\pm) \quad \text{and} \quad k = \mathrm{e}^{-\omega t}\, h.$$

Computations give

$$\begin{aligned}
\mathrm{e}^{\omega t}\,\partial_t v^\pm &= \partial_t \mathrm{cof}(\nabla Y^\pm)^T u^\pm(Y^\pm) + \mathrm{cof}(\nabla Y^\pm)^T(\partial_t u^\pm(Y^\pm) + \nabla u^\pm(Y^\pm)\partial_t Y^\pm) \\
&\quad -\omega \mathrm{cof}(\nabla Y^\pm)^T u^\pm(Y^\pm), \\
\mathrm{e}^{\omega t}\left((v^\pm\cdot\nabla)v^\pm\right)_i &= \sum_{j,k,l=1}^{2} \mathrm{cof}(\nabla Y^\pm)_{kl}\partial_l \mathrm{cof}(\nabla Y^\pm)_{ji} u_j^\pm(Y^\pm) u_k^\pm(Y^\pm) \\
&\quad + \det(\nabla Y^\pm)\sum_{j,k=1}^{2} \mathrm{cof}(\nabla Y^\pm)_{ki} u_j^\pm(Y^\pm)\partial_j u_k^\pm(Y^\pm), \\
\mathrm{e}^{\omega t}\,\nabla\cdot v^\pm &= \det(\nabla Y^\pm)(\nabla\cdot u^\pm)(Y^\pm), \\
\mathrm{e}^{\omega t}(\Delta v^\pm)_i &= \sum_{j=1}^{2}\Delta \mathrm{cof}(\nabla Y^\pm)_{ji} u_j^\pm(Y^\pm) + \sum_{j,k,l,m=1}^{2} \mathrm{cof}(\nabla Y^\pm)_{ji}\partial_l Y_m^\pm \partial_l Y_k^\pm \partial_{km} u_j^\pm(Y^\pm) \\
&\quad + \sum_{j,k,l=1}^{2} \mathrm{cof}(\nabla Y^\pm)_{ji}\partial_{ll} Y_k^\pm \partial_k u_j^\pm(Y^\pm) + 2\sum_{j,k,l=1}^{2} \partial_l \mathrm{cof}(\nabla Y^\pm)_{ji}\partial_l Y_k^\pm \partial_k u_j^\pm(Y^\pm), \\
\mathrm{e}^{\omega t}\,\nabla q^\pm &= (\nabla Y^\pm)^T p^\pm(Y^\pm), \\
\mathrm{e}^{\omega t}[\mathbb{S}(v,q)n_{\Gamma(t)}]_i &= \Big[\mu\Big(\mathrm{cof}(\nabla Y)^T\nabla u(Y)\nabla Y n_{\Gamma(t)} + (\nabla Y)^T\nabla u(Y)^T \mathrm{cof}(\nabla Y) n_{\Gamma(t)}\Big)_i \\
&\quad -p(Y) n_{\Gamma(t),i} + \mu\sum_{j=1}^{2}(\partial_j \mathrm{cof}(\nabla Y)_{ki} + \partial_i \mathrm{cof}(\nabla Y)_{kj}) u_k(Y) n_{\Gamma(t),i}\Big].
\end{aligned}$$

Therefore, the new unknowns (u,p,h) satisfy

$$\left\{\begin{array}{rcll} \partial_t u^\pm - \dfrac{1}{\rho^\pm}\nabla\cdot\mathbb{S}^\pm(u^\pm,p^\pm) - \omega u^\pm &=& F^\pm(u,p,h) & \text{in } (0,+\infty)\times\Omega^\pm,\\ \nabla\cdot u^\pm &=& 0 & \text{in } (0,+\infty)\times\Omega^\pm,\\ u &=& \mathrm{e}^{\omega t}\displaystyle\sum_{i=1}^{n_c} g_i w_i & \text{on } (0,+\infty)\times\Gamma_d,\\ u^- &=& u^+ & \text{on } (0,+\infty)\times\Gamma,\\ [\mathbb{S}(u,p)n_\Gamma] + (\sigma\Delta_{x'}h)n_\Gamma &=& H(u,p,h) & \text{on } (0,+\infty)\times\Gamma,\\ \partial_t h - u_d - \omega h &=& 0 & \text{on } (0,+\infty)\times\Gamma,\\ u(0) &=& u_0 & \text{in } \Omega,\\ h(0) &=& h_0 & \text{on } \Gamma, \end{array}\right. \tag{3.2 .3}$$

where

$$\begin{array}{rcl} F^\pm(u,p,h) &=& F_1^\pm(u,h) + F_2^\pm(u,h) + F_3^\pm(u,h) + F_4^\pm(u,h) + F_5^\pm(p,h),\\ F_1^\pm(u,h) &=& (I - \mathrm{cof}(\nabla Y^\pm)^T(X^\pm))\partial_t u^\pm - \omega(I-\mathrm{cof}(\nabla Y^\pm)^T(X^\pm))u^\pm,\\ F_2^\pm(u,h) &=& \partial_t \mathrm{cof}(\nabla Y^\pm)^T(X^\pm)u^\pm - \mathrm{cof}(\nabla Y^\pm)^T(X^\pm)\nabla u^\pm \partial_t Y^\pm(X^\pm),\\ \big(F_3^\pm(u,h)\big)_i &=& \displaystyle\sum_{j,k,l=1}^{d} \mathrm{cof}(\nabla Y^\pm)_{kl}(X^\pm)\partial_l \mathrm{cof}(\nabla Y^\pm)_{ji}(X^\pm)u_j^\pm u_k^\pm\\ && +\det(\nabla Y^\pm)(X^\pm)\displaystyle\sum_{j,k=1}^{d}\mathrm{cof}(\nabla Y^\pm)_{ki}(X^\pm)u_j^\pm\partial_j u_k^\pm,\\ \big(F_4^\pm(u,h)\big)_i &=& \dfrac{\mu^\pm}{\rho^\pm}\Bigg(\displaystyle\sum_{j=1}^{d}\Delta\mathrm{cof}(\nabla Y^\pm)_{ji}(X^\pm)u_j^\pm\\ && +2\displaystyle\sum_{j,k,l=1}^{d}\partial_l\mathrm{cof}(\nabla Y^\pm)_{ji}(X^\pm)\partial_l Y_k^\pm(X^\pm)\partial_k u_j^\pm\\ && +\displaystyle\sum_{j,k=1}^{d}\mathrm{cof}(\nabla Y^\pm)_{ji}(X^\pm)\Delta Y_k^\pm(X^\pm)\partial_k u_j^\pm\\ && +\displaystyle\sum_{j,k,l,m=1}^{d}\mathrm{cof}(\nabla Y^\pm)_{ji}(X^\pm)\partial_l Y_m^\pm(X^\pm)\partial_l Y_k^\pm(X^\pm)\partial_{km}u_j^\pm - \Delta u_i^\pm\Bigg),\\ F_5^\pm(p,h) &=& \dfrac{1}{\rho^\pm}\big(I - \nabla Y^\pm(X^\pm)\big)^T\nabla p^\pm, \end{array} \tag{3.2 .4}$$

and

$$\begin{array}{rcl} H(u,p,h) &=& H_1(u,h) + H_2(p,h) + H_3(u,h) + \sigma H_4(h) + \sigma H_5(h),\\ H_1(u,h) &=& \Big[\mu((\nabla u + (\nabla u)^T)n_\Gamma - \mathrm{cof}(\nabla Y)^T(X)\nabla u\nabla Y(X)n_{\Gamma(t)}\\ && -(\nabla Y)^T(X)\nabla u^T\mathrm{cof}(\nabla Y)(X)n_{\Gamma(t)})\Big],\\ H_2(p,h) &=& [p(n_{\Gamma(t)} - n_\Gamma)],\\ \big(H_3(u,h)\big)_i &=& -\Big[\mu\displaystyle\sum_{j=1}^{d}(\partial_j\mathrm{cof}(\nabla Y)_{ki}(X) + \partial_i\mathrm{cof}(\nabla Y)_{kj}(X))u_k n_{\Gamma(t),i}\Big],\\ H_4(h) &=& \Delta h\Bigg(n_\Gamma - \dfrac{n_{\Gamma(t)}}{\sqrt{1+\mathrm{e}^{-2\omega t}\,|\nabla_{x'}h|^2}}\Bigg) + \dfrac{\mathrm{e}^{-2\omega t}(\nabla_{x'}^2 h\nabla_{x'}h)\cdot\nabla_{x'}h}{(1+\mathrm{e}^{-2\omega t}\,|\nabla_{x'}h|^2)^{3/2}}n_{\Gamma(t)}, \end{array} \tag{3.2 .5}$$

where

$$n_{\Gamma(t)} = \frac{1}{\sqrt{1+e^{-2\omega t}|\nabla_{x'}h|^2}}\begin{pmatrix} -\,\mathrm{e}^{-\omega t}\,\nabla_{x'}h\\ 1\end{pmatrix}$$

and lastly

$$u_0 = \mathrm{cof}(\nabla X(0))^T v_0(X(0)) \quad \text{and} \quad h_0 = k_0.$$

3.2 .2 The non linear theorem

We first introduce the following spaces for $\gamma, r, s > 0$, $T \in (0, +\infty]$:

$$\begin{aligned} V^1_{\Gamma_d}(\Omega) &= \{u \in H^1(\Omega) \mid \nabla \cdot u = 0 \text{ and } u = 0 \text{ on } \Gamma_d\}, \\ H^\gamma_m(\Gamma) &= \{h \in H^\gamma(\Gamma) \mid \int_\Gamma h = 0\}, \\ H^{r,s}((0,T) \times \Omega^\pm) &= H^s(0,T; L^2(\Omega^\pm)) \cap L^2(0,T; H^r(\Omega^\pm)), \\ H^{r,s}((0,T) \times \Gamma) &= H^s(0,T; L^2(\Gamma)) \cap L^2(0,T; H^r(\Gamma)). \end{aligned}$$

In the following, we identify Γ with the torus $\mathbb{T}^{d-1}$ and we omit the subscript x' on $\nabla_{x'} h$ and $\Delta_{x'} h$.

Theorem 3.2 .1. *Let*

$$\ell \in \begin{cases} (0, 1/2) & \textit{pour } d = 2, \\ (1/2, 1) & \textit{pour } d = 3. \end{cases} \tag{3.2 .6}$$

Given any decreasing rate $\omega > 0$, there exists $\delta > 0$ such that for all $(u_0, h_0) \in V^1_{\Gamma_d}(\Omega) \times H^{2+\ell}_m(\Gamma)$ satisfying $u_0^\pm \in H^{1+\ell}(\Omega^\pm)$, the smallness condition

$$\|(u_0^\pm, h_0)\|_{H^{1+\ell}(\Omega^\pm) \times H^{2+\ell}(\Gamma)} \leq \delta, \tag{3.2 .7}$$

and the compatibility condition in dimension 3 :

$$\left[\mu \left(\nabla \left(\mathrm{cof}(X_0^{-1})^T u_0(X_0^{-1})\right) + \left(\nabla \left(\mathrm{cof}(X_0^{-1})^T u_0(X_0^{-1})\right)\right)^T\right) n_{\Gamma_0}\right]_\tau = 0 \quad \textit{on } \Gamma_0, \tag{3.2 .8}$$

where X_0 is defined in (3.2 .1) for h_0 and the subscript τ stands for the tangential component along Γ_0, there exists a finite dimensional control $g \in H^{1+\ell}(0, +\infty; C^\infty(\Gamma_d))$ of the form (3.1 .2) such that system (3.2 .3) admits a solution (u, p, h) which satisfies

$$\begin{aligned} &\|u^\pm\|_{H^{2+\ell, 1+\ell/2}((0,+\infty)\times\Omega^\pm)} + \|h\|_{H^{7/4+\ell/2}(0,+\infty; L^2_m(\Gamma)) \cap H^1(0,+\infty; H^{3/2+\ell}_m(\Gamma)) \cap L^2(0,+\infty; H^{5/2+\ell}_m(\Gamma))} \\ &\quad + \|\nabla p^\pm\|_{H^{\ell,\ell/2}((0,+\infty)\times\Omega^\pm)} + \|[p]\|_{H^{1/2+\ell, 1/4+\ell/2}((0,+\infty)\times\Gamma)} \leq C \|(u_0^\pm, h_0)\|_{H^{1+\ell}(\Omega^\pm) \times H^{2+\ell}_m(\Gamma)}. \end{aligned} \tag{3.2 .9}$$

Moreover, for all $t > 0$, the transformation $X(t, \cdot)$ defined in (3.2 .1) for $k(t, \cdot) = \mathrm{e}^{-\omega t} h(t, \cdot)$, is invertible and $(v, q, k) = \mathrm{e}^{-\omega t}(\mathrm{cof}(\nabla Y)^T u(Y), p(Y), h)$, where $Y(t, .)$ is the inverse transformation of $X(t, \cdot)$, is solution of (3.1 .1) and satisfies for all $t > 0$,

$$\|(v^\pm(t), k(t))\|_{H^{1+\ell}(\Omega^\pm(t)) \times H^{2+\ell}_m(\Gamma)} \leq C\, \mathrm{e}^{-\omega t} \|(v_0^\pm, k_0)\|_{H^{1+\ell}(\Omega_0^\pm) \times H^{2+\ell}_m(\Gamma)}.$$

Remark 3.2 .2. *The parameter $\delta > 0$ of (3.2 .7) will be chosen small enough to guarantee that X_0 defined in (3.2 .1) for $k_e = h_0$ is invertible, so that (3.2 .8) makes sense.*

Remark 3.2 .3. *In our mathematical framework, the control $\mathbf{g}$ cannot be found in a feedback form without introducing compatibility conditions between the feedback and the initial data (u_0, h_0). However, even in weaker regularity settings, these compatibility conditions cannot be avoided in the 3 dimensional case as discussed for example in [3]. Therefore, the control $\mathbf{g}$ is found in the form of an integrator*

$$\mathbf{g}' + \Lambda \mathbf{g} = \mathbf{f}, \quad \mathbf{g}(0) = 0, \tag{3.2 .10}$$

where

$$\Lambda \text{ is a diagonal matrix with non negative coefficients,}$$

and $\mathbf{f}$ *will be found in a feedback form*

$$\mathbf{f}(t) = K(u(t), p(t), h(t), \mathbf{g}(t)).$$

The feedback K *is linear for the unknowns* $(u, p, h, \mathbf{g})$*, but give a nonlinear feedback for the unknowns* (v, g, k) *of system* (3.1 .1)*. This is due to the multiplication of* u *by* $cof(\nabla Y)^T$ *in* (3.2 .2)*.*

The choice of Λ *is arbitrary. In the present work, one can choose* $\Lambda = 0$ *for convenience. But, it can be useful to have the choice of coefficients in the perspective of simulation. That is why we choose to treat the case of a general diagonal matrix with non negative coefficients.*

Remark 3.2 .4. *The regularity of* $\Gamma(t)$ *is given by the regularity of its height* h*. With the injection* $H^1(0, +\infty; H^{3/2+\ell}(\Gamma)) \cap L^2(0, +\infty; H^{5/2+\ell}(\Gamma)) \hookrightarrow C([0, +\infty], H^{2+\ell}(\Gamma))$*,* $\Gamma(t)$ *is of class* $H^{2+\ell}$ *which corresponds to the initial regularity of* Γ_0*. There is much to say about this remark on the regularity of* $\Gamma(t)$*:*

- *first, one may not expect such regularity without surface tension effect. It is indeed the coupling on the interface of equation* $(3.2\ .3)_5$ *which regularizes* h*. In the case where surface tension is neglected, the regularity of* h *is given by* $(3.2\ .3)_6$ *and may be expected in* $H^{7/4+\ell/2}(0, +\infty; L^2(\Gamma)) \cap H^1(0, +\infty; H^{3/2+\ell}(\Gamma)) \hookrightarrow C([0, +\infty], H^{3/2+\ell}(\Gamma))$*. The natural space for* h_0 *would therefore be* $H^{3/2+\ell}(\mathbb{T}_d)$ *which corresponds to the regularity found in the literature (see [16, 49]).*

- *secondly, this property of conservation of regularity at any time is a good indicator of the well-posedness in Hadamard's sense of system* (3.1 .1)*. In particular, one may expect uniqueness of solution given in Theorem 3.2 .1.*

- *lastly, in the literature, most of the authors deal with system* (3.1 .1) *by means of Lagrangian coordinates [2, 14, 48, 49]. Therein, the initial regularity for* Γ_0 *is* $H^{5/2+\ell}$*, which is slightly higher than ours. This causes problem to prove well-posedness since a priori* $\Gamma(t) \notin H^{5/2+\ell}$*. This issue is discussed in [36].*

Remark 3.2 .5. *We need the enhanced regularity by* ℓ *to handle the non linearities generated by the moving boundary. The condition* $\ell > 0$ *in dimension* $d = 2$ *and* $\ell > 1/2$ *in dimension* $d = 3$ *is necessary to estimate non linear terms by using various embedding theorems. On the other hand, the condition* $\ell < 1/2$ *in dimension* $d = 2$ *and* $\ell < 1$ *in dimension* $d = 3$ *is used for minimizing the number of compatibility conditions.*

The compatibility condition (3.2 .8) originates from the tangential component of $(3.1\ .1)_5$. In dimension $d = 2$, $\ell \in (0, 1/2)$, so the time regularity equation $(3.1\ .1)_5$ does not have a trace at time $t = 0$. However, in dimension $d = 3$, $\ell \in (1/2, 1)$ and the trace of $(3.1\ .1)_5$ imposes the compatibility condition (3.2 .8).

The proof of Theorem 3.2 .1 is based on the stabilization of an auxiliary linear system.

3.2 .3 The main linear theorem

The linear system we aim at studying is the following

$$\left\{\begin{array}{rcll} \partial_t u^\pm - \dfrac{1}{\rho^\pm}\nabla\cdot\mathbb{S}^\pm(u^\pm,p^\pm) - \omega u^\pm & = & f^\pm & \text{in } (0,+\infty)\times\Omega^\pm, \\ \nabla\cdot u^\pm & = & 0 & \text{in } (0,+\infty)\times\Omega^\pm, \\ u & = & \displaystyle\sum_{i=1}^{n_c} g_i w_i & \text{on } (0,+\infty)\times\Gamma_d, \\ u^+ & = & u^- & \text{on } (0,+\infty)\times\Gamma, \\ [\mathbb{S}(u,p)n_\Gamma] + (\sigma\Delta h)n_\Gamma & = & f_{jump} & \text{on } (0,+\infty)\times\Gamma, \\ \partial_t h - u_d - \omega h & = & 0 & \text{on } (0,+\infty)\times\Gamma, \\ \mathbf{g}' + \Lambda\mathbf{g} - \omega\mathbf{g} & = & \mathbf{f}_g, & \\ u(0) & = & u_0 & \text{in } \Omega, \\ h(0) & = & h_0 & \text{on } \Gamma, \\ \mathbf{g}(0) & = & 0. & \end{array}\right. \tag{3.2 .11}$$

The main result on the linear system (3.2 .11) is the following:

Theorem 3.2 .6. *Let $(f, f_{jump}) \in L^2((0,+\infty)\times\Omega)\times H^{1/2,1/4}((0,+\infty)\times\Gamma)$ and $(u_0,h_0)\in V^1_{\Gamma_d}(\Omega)\times H^2_m(\Gamma)$. Then there exists a family of control functions $(w_i)_{i\in[\![1,n_c]\!]}\in C^\infty(\Gamma_d)$ satisfying (3.1 .3) and a linear feedback $K\in\mathscr{L}(L^2(\Omega)\times H^{3/2}_m(\Gamma)\times\mathbb{R}^{n_c},\mathbb{R}^{n_c})$ such that system (3.2 .11) with $\mathbf{f}_g = K(u,h,\mathbf{g})$ admits a unique solution $(u,p,h,\mathbf{g})$ such that*

$$\begin{aligned} &\|u^\pm\|_{H^{2,1}((0,+\infty)\times\Omega^\pm)} + \|\nabla p^\pm\|_{L^2((0,+\infty)\times\Omega^\pm)} + \|[p]\|_{H^{1/2,1/4}((0,+\infty)\times\Gamma)} \\ &\qquad + \|h\|_{H^{7/4}(0,+\infty;L^2_m(\Gamma))\cap H^1(0,+\infty;H^{3/2}_m(\Gamma))\cap L^2(0,+\infty;H^{5/2}_m(\Gamma))} \\ &\qquad \le C\|(f,f_{jump},u_0,h_0)\|_{L^2((0,+\infty)\times\Omega)\times H^{1/2,1/4}((0,+\infty)\times\Gamma)\times H^1(\Omega)\times H^2_m(\Gamma)}. \end{aligned} \tag{3.2 .12}$$

This theorem is a lighter form of Theorem 3.5 .5.

3.2 .4 Main steps of the proof

3.2 .4.1 Main steps of the proof of Theorem 3.2 .6

We follow a classical strategy to stabilize the linear system (3.2 .11) based on the semigroup theory (see [10, 34, 39] for example). We first write system (3.2 .11) in a semigroup form. The main novelty lies in the proof of the analyticity of the semigroup and the compactness of the resolvent of its infinitesimal generator. These properties induce that there is a finite number of unstable modes. We then construct the control functions w_i to stabilize the unstable modes. With a Hautus criterion [10, Part V, Chapter 1, Proposition 3.3, page 492] and a unique continuation property based on [20, Proposition 1.1, page 3], we then prove that system (3.2 .11) is stabilizable.

The proof of Theorem 3.2 .6 is then divided as follow: Section 3.3 studies the stationary Stokes system which constitutes one of the main novelty of the article; Section 3.4 is based on the previous section and states the results on the extended stationary system of (3.2 .11); in Section 3.5 , the control functions w_i are constructed and Theorem 3.2 .6 is proved.

3.2 .4.2 Main steps of the proof of Theorem 3.2 .1

We first extend the stabilization result of Theorem 3.2 .6 in the regularity settings of Theorem 3.2 .1 in Section 3.6 . This constitutes another delicate point of the work. Lastly, we derive estimates on the non linear terms and stabilize the non linear system by means of Picard fixed point theorem in Section 3.7 .

3.2 .4.3 Main difficulty

System (3.2 .11) has a natural energy. In the lighter case $\omega = \mathbf{g} = f_{jump} = 0$, the energy balance reads

$$\frac{d}{dt}\left(\int_\Omega \rho|u|^2 + \sigma\int_\Gamma |\nabla h|^2\right)(t) + \int_\Omega \mu|D(u)|^2(t) = \int_\Omega \rho f\cdot u. \tag{3.2 .13}$$

In this energy equality, the regularizing effect on h of system (3.2 .11) does not appear. The dissipation of system (3.2 .11) is then strictly stronger than its energy. Many authors have used this property to show global well-posedness and exponential decay of the energy [5, 33, 51].

However, it represents a real difficulty for energy or duality methods may not be optimal in regularity. This difficulty is treated in the study of the stationary problem by a careful definition of weak solutions which is one of the main novelty of this work.

3.2 .5 Notations and functional settings

In the following, we shall often use the notations

$$\rho = \mathbb{1}_{\Omega^-}\rho^- + \mathbb{1}_{\Omega^+}\rho^+ \quad \text{and} \quad \mu = \mathbb{1}_{\Omega^-}\mu^- + \mathbb{1}_{\Omega^+}\mu^+.$$

When not mentioned otherwise, if $w^\pm$ is a function defined in $\Omega^\pm$, $w = \mathbb{1}_{\Omega^-}w^- + \mathbb{1}_{\Omega^+}w^+$. However, we emphasize that generally $\mathbb{S}(\mathbb{1}_{\Omega^-}u^- + \mathbb{1}_{\Omega^+}u^+, \mathbb{1}_{\Omega^-}p^- + \mathbb{1}_{\Omega^+}p^+) \neq \mathbb{1}_{\Omega^-}\mathbb{S}^-(u^-,p^-) + \mathbb{1}_{\Omega^+}\mathbb{S}^+(u^+,p^+)$.

The notation $(\cdot,\cdot)$ is the $L^2(\Omega)$ inner-product, $(\cdot,\cdot)_{\Omega^\pm}$ the $L^2(\Omega^\pm)$ inner-product and $(\cdot,\cdot)_\Gamma$ the $L^2(\Gamma)$ inner-product.

In the following, we identify Γ with the torus $\mathbb{T}^{d-1}$ and we omit the subscript x' on $\nabla_{x'}h$ and $\Delta_{x'}h$.

We recall that for $r, s > 0$, $T \in (0, +\infty]$:

$$\begin{aligned}
H^{r,s}((0,T)\times\Omega^\pm) &= H^s(0,T;L^2(\Omega^\pm)) \cap L^2(0,T;H^r(\Omega^\pm)),\\
H^{r,s}((0,T)\times\Gamma) &= H^s(0,T;L^2(\Gamma)) \cap L^2(0,T;H^r(\Gamma)).
\end{aligned}$$

3.2 .5.1 Settings for the stationary study of Sections 3.3 and 3.4

We set for $\gamma > 0$,

$$\begin{aligned}
V^0_{n,\Gamma_d^\pm}(\Omega^\pm) &= \{u^\pm \in L^2(\Omega^\pm) \mid \nabla\cdot u^\pm = 0 \text{ in } \Omega^\pm,\ u^\pm\cdot n = 0 \text{ on } \Gamma_d^\pm\},\\
V^0_{n,\Gamma_d}(\Omega) &= \{u \in L^2(\Omega) \mid \nabla\cdot u = 0 \text{ in } \Omega,\ u\cdot n = 0 \text{ on } \Gamma_d\},\\
\mathscr{V}^0_{n,\Gamma_d}(\Omega) &= \{u \in L^2(\Omega) \mid u^\pm \in V^0_{n,\Gamma_d^\pm}(\Omega^\pm)\},\\
V^1_{\Gamma_d}(\Omega) &= H^1_0(\Omega) \cap V^0_{n,\Gamma_d}(\Omega),\\
Z &= \mathscr{V}^0_{n,\Gamma_d}(\Omega) \times H^{3/2}_m(\Gamma),\\
Z_e &= Z \times \mathbb{R}^{n_c},\\
U(\Omega) &= \{u \in H^1(\Omega) \mid \nabla\cdot u = 0 \text{ and } u^\pm \in H^2(\Omega^\pm)\},\\
U_0(\Omega) &= U(\Omega) \cap V^1_{\Gamma_d}(\Omega),\\
L^2_m(\Omega) &= \{p \in L^2(\Omega) \mid \int_\Omega p = 0\},\\
P(\Omega) &= \{p \in L^2_m(\Omega) \mid p^\pm \in H^1(\Omega^\pm)\},\\
H^\gamma_m(\Gamma) &= \{h \in H^\gamma(\Gamma) \mid \int_\Gamma h = 0\},\\
H &= \mathscr{V}^0_{n,\Gamma_d}(\Omega) \times H^1_m(\Gamma).
\end{aligned}$$

We make the following identification

$$H' = H, \tag{3.2 .14}$$

and we define the scalar product on H by

$$((u,h),(v,k))_H = (\rho u, v) + \sigma(\partial_x h, \partial_x k)_\Gamma. \tag{3.2 .15}$$

Therefore, we point out that

$$Z' = \mathscr{V}^0_{n,\Gamma_d}(\Omega) \times H^{1/2}_m(\Gamma) \quad \text{and} \quad Z'_e = Z' \times \mathbb{R}^{n_c}. \tag{3.2 .16}$$

3.2 .5.2 Time-dependent spaces

For $T \in (0, +\infty]$, $r > 0$, we define

$$\begin{array}{rcl}
E^r_{u,T} & = & \{u \in L^2(0,T;H^1(\Omega)) \mid \nabla \cdot u = 0 \text{ in } \ \Omega,\ u^\pm \in H^{2+r,1+r/2}((0,T)\times\Omega^\pm)\},\\
E^r_{p,T} & = & \left\{ p \in L^2(0,T;L^2_m(\Omega)) \;\middle|\; \begin{array}{l} \nabla p^\pm \in H^{r,r/2}((0,T)\times\Omega^\pm), \\ [p] \in H^{1/2+r,1/4+r/2}((0,T)\times\Gamma) \end{array} \right\},\\
E^r_{h,T} & = & H^{7/4+r/2}(0,T;L^2_m(\Gamma)) \cap H^1(0,T;H^{3/2+r}_m(\Gamma)) \cap L^2(0,T;H^{5/2+r}_m(\Gamma)),\\
E^r_{\mathbf{g},T} & = & H^{1+r}(0,T;\mathbb{R}^{n_c}),\\
E^r_{f,T} & = & \{f \in L^2((0,T)\times\Omega) \mid f^\pm \in H^{r,r/2}((0,T)\times\Omega^\pm)\},\\
E^r_{f_{jump},T} & = & H^{1/2+r,1/4+r/2}((0,+\infty)\times\Gamma),\\
E^r_{u_0} & = & \{u_0 \in H^1_0(\Omega) \mid \nabla\cdot u_0 = 0,\ u_0^\pm \in H^{1+r}(\Omega^\pm)\},\\
E^r_{h_0} & = & H^{2+r}_m(\Gamma).
\end{array}$$

We add some explanation to clarify the reading of the above spaces. The first subscript denotes the unknown or the data concerned by the spaces. The second subscript T denotes the time horizon. Except in Theorem 3.5 .5 and in the proof of Theorem 3.6 .3, we always consider an infinite time horizon: $T = +\infty$. Lastly, the exponent r stands for the regularity. We are concerned with three degrees of regularity: $r = 0$ to prove Theorem 3.2 .6 in Section 3.5 , $r = 2$ to enhance the linear result of Theorem 3.2 .6 in Section 3.6 and $r = \ell$, ℓ being defined in (3.2 .6), to study the non linear system in Section 3.7 . In this last section, we will also use further notations.

3.2 .5.3 Settings for the non linear system in Section 3.7

We first introduce some compatibility conditions in dimension $d = 3$. Let $(u_0, h_0, f_{jump}, p) \in E^\ell_{u_0} \times E^\ell_{h_0} \times E^\ell_{f_{jump},\infty} \times E^\ell_{p,\infty}$. We introduce

$$[\mu D(u_0) n_\Gamma]_\tau = f_\tau(0) \quad \text{on } \Gamma, \tag{3.2 .17}$$

where the subscript τ stands for the tangential component along Γ, and

$$[\mathbb{S}(v_0, q_0) n_{\Gamma_0}] \cdot n_{\Gamma_0} = -\sigma \nabla \cdot \left(\frac{\nabla h_0}{\sqrt{1 + |\nabla h_0|^2}} \right) \quad \text{on } \Gamma_0, \tag{3.2 .18}$$

where

$$v_0 = \mathrm{cof}(\nabla Y_0)^T u_0(Y_0) \quad \text{and} \quad q_0 = p(0, Y_0), \tag{3.2 .19}$$

and Y_0 is the inverse transformation of X_0 defined in (3.2 .1) for h_0. It will be proved rigorously that for h_0 sufficiently small in $E^r_{h_0}$, X_0 is invertible (see Lemma 3.7 .2). For $(u_0, h_0) \in E^r_{u_0} \times E^r_{h_0}$, $R > 0$, we define

$$
\begin{array}{rcl}
E^\ell_{u,\infty,u_0} & = & \{u \in E_{u,\infty} \mid u(0) = u_0\}, \\
E^\ell_{p,\infty,(u_0,h_0)} & = & \{p \in E^\ell_{p,\infty} \mid ([p(0)], u_0, h_0) \text{ satisfies } (3.2\ .18) \text{ if } d = 3\}, \\
E^\ell_{h,\infty,h_0} & = & \{h \in E^\ell_{h,\infty} \mid h(0) = h_0\}, \\
E^\ell_{f_{jump},\infty,u_0} & = & \{f_{jump} \in E_{f_{jump},\infty} \mid (f_{jump}, u_0) \text{ satisfies } (3.2\ .17) \text{ if } d = 3\}, \\
E^\ell_{u,\infty,u_0,R} & = & \{u \in E^\ell_{u,\infty,u_0} \mid \|u\|_{E^\ell_{u,\infty}} \le R\}, \\
E^\ell_{p,\infty,(u_0,h_0),R} & = & \{p \in E^\ell_{p,\infty,u_0,h_0} \mid \|p\|_{E^\ell_{p,\infty}} \le R\}, \\
E^\ell_{h,\infty,R} & = & \{h \in E^\ell_{h,\infty} \mid \|h\|_{E^\ell_{h,\infty}} \le R\}, \\
E^\ell_{h,\infty,h_0,R} & = & \{h \in E^\ell_{h,\infty,h_0} \mid \|h\|_{E^\ell_{h,\infty}} \le R\},, \\
E^\ell_{\mathbf{g},\infty,R} & = & \{\mathbf{g} \in E^\ell_{\mathbf{g},\infty} \mid \|\mathbf{g}\|_{E^\ell_{\mathbf{g},\infty}} \le R\},
\end{array}
$$

Except for $E^\ell_{h,\infty,R}$ and $E^\ell_{\mathbf{g},\infty,R}$, the third subscript stands for the compatibility condition at time $t = 0$. The last subscript denotes the radius of the balls of the corresponding spaces.

3.3 The stationary surface tension Stokes system

The notations used through this section are defined in Section 3.2 .5.1.

We first focus on the following sationary system for $\lambda \in \mathbb{C}$:

$$
\left\{
\begin{array}{rcll}
\lambda u^\pm - \dfrac{1}{\rho^\pm} \nabla \cdot \mathbb{S}^\pm(u^\pm, p^\pm) & = & f^\pm & \text{in } \Omega^\pm, \\
\nabla \cdot u^\pm & = & 0 & \text{in } \Omega^\pm, \\
u & = & 0 & \text{on } \Gamma_d, \\
u^+ & = & u^- & \text{on } \Gamma, \\
[\mathbb{S}(u,p) n_\Gamma] + (\sigma \Delta h) n_\Gamma & = & 0 & \text{on } \Gamma, \\
\lambda h - u_d & = & f_h & \text{on } \Gamma,
\end{array}
\right.
\tag{3.3 .1}
$$

The purpose of this section is to write system (3.3 .1) in an operator form and to study properties of the operator. In Section 3.3 .1, we derive a weak formulation for (3.3 .1) and we prove existence and uniqueness of weak solutions and regularity results for $\Re e(\lambda) > 0$. Section 3.3 .1 constitutes a novelty on the approach of system (3.3 .1) as our analysis does not not rely on an explicit description of the Dirichlet-to-Neumann operator map as it has been previously made in the literature (see for example [33, 36]). In Section 3.3 .2, we write system (3.3 .1) in an operator form and we state a result of compact resolvent and a result of analyticity for the associated semigroup. Section 3.3 .3 is dedicated to the study of the adjoint operator. Lastly, in Section 3.3 .4, we study the non homogeneous stationary surface tension Stokes system.

3.3 .1 Weak formulation of system (3.3 .1)

This subsection is dedicated to derive a weak formulation of system (3.3 .1) and to study existence, uniqueness and regularity of weak solutions. The complete study of the weak formulation is one of the main novelty of this work.

First of all, remark that a classical energy balance on (3.3 .1) leads to

$$
\lambda \|\sqrt{\rho} u\|^2_{L^2(\Omega)} + \frac{1}{2} \|\sqrt{\mu} D(u)\|^2_{L^2(\Omega)} + \sigma \bar{\lambda} \|\nabla h\|^2_{L^2(\Gamma)} = (\rho f, u) + \sigma (\nabla h, \nabla f_h)_\Gamma. \tag{3.3 .2}
$$

Reading this equation, one may believe that the optimal spaces are $(u,h) \in H^1(\Omega) \times H^1_m(\Gamma)$ for $(\rho f, f_h) \in H^{-1}(\Omega) \times H^1_m(\Gamma)$. Actually, this energy balance does not make appear the regularization effect on h of system (3.3 .1). Indeed, the regularizing effect appears in the coupling $(3.3\ .1)_4$: for $u \in H^1(\Omega)$ satisfying $(3.3\ .1)_1$, we can define $[\mathbb{S}(u,p)n_\Gamma] \in H^{-1/2}(\Gamma)$ so that $h \in H^{3/2}_m(\Gamma)$. Therefore, $f_h \in H^{1/2}(\Gamma)$ is sufficient to give a sense to $(\nabla h, \nabla f_h)_\Gamma$. We shall therefore introduce a weak formulation that allows f_h to be $H^{1/2}_m(\Gamma)$. Our idea is to lift the nonhomogeneous term f_h with a velocity field so as to handle the H^1 norm of the lifing with the H^1 norm of the velocity. This corresponds to make the coupling equation $(3.3\ .1)_4$ appear in the weak formulation.

We begin with introducing

$$\begin{aligned}
\mathscr{V}_\lambda(\Omega) &= \{(\varphi,\psi) \in V^1_{\Gamma_d}(\Omega) \times H^1_m(\Gamma) \mid \varphi_d = \lambda\psi \text{ on } \Gamma\},\\
H^{-1}_\rho(\Omega) &= \{f \in \mathscr{D}'(\Omega) \mid \rho f \in H^{-1}(\Omega)\}.
\end{aligned}$$

Definition 3.3 .1. *Let $(f,f_h) \in H^{-1}_\rho(\Omega) \times H^{1/2}_m(\Gamma)$. We say that $(u,h) \in V^1_{\Gamma_d}(\Omega) \times H^1_m(\Gamma)$ is a* weak solution *of system* (3.3 .1) *if $u = v + w$ where $v \in V^1_{\Gamma_d}(\Omega)$ is such that $v_d = -f_h$ on Γ, and $(w,h) \in \mathscr{V}_\lambda(\Omega)$ is such that for all $(\varphi,\psi) \in \mathscr{V}_\lambda(\Omega)$,*

$$\begin{aligned}
&\lambda(\rho w,\varphi) + \frac{1}{2}(\mu D(w), D(\varphi)) + \sigma\bar{\lambda}(\nabla h, \nabla\psi)_\Gamma \\
&\qquad\qquad = \langle \rho f, \varphi\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} - \lambda(\rho v,\varphi) - \frac{1}{2}(\mu D(v), D(\varphi)). \quad (3.3\ .3)
\end{aligned}$$

Proposition 3.3 .2. *Let $\Re e(\lambda) > 0$, $(f,f_h) \in H^{-1}_\rho(\Omega) \times H^{1/2}_m(\Gamma)$. There exists a unique* weak solution *of system* (3.3 .1).

Proof of Proposition 3.3 .2. The existence is a direct consequence of Lax-Milgram theorem.

Concerning the uniqueness, remark that if $(u,h) \in V^1_{\Gamma_d}(\Omega) \times H^1_m(\Gamma)$ is a *weak solution*, then for all $(\varphi,\psi) \in \mathscr{V}_\lambda(\Omega)$,

$$\lambda(\rho u,\varphi) + \frac{1}{2}(\mu D(u), D(\varphi)) + \sigma\bar{\lambda}(\nabla h, \nabla\psi)_\Gamma = \langle \rho f, \varphi\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}.$$

So if $(u,h), (\tilde{u},\tilde{h}) V^1_{\Gamma_d}(\Omega) \times H^1_m(\Gamma)$ are two *weak solutions*, then for all $(\varphi,\psi) \in \mathscr{V}_\lambda(\Omega)$,

$$\lambda(\rho(u-\tilde{u}),\varphi) + \frac{1}{2}(\mu D(u-\tilde{u}), D(\varphi)) + \sigma\bar{\lambda}(\nabla(h-\tilde{h}), \nabla\psi)_\Gamma = 0.$$

Furthermore, $(u,h) - (\tilde{u},\tilde{h}) \in \mathscr{V}_\lambda(\Omega)$. The uniqueness is then straightforward. □

For the moment, remark that one does not have the equivalence between *weak solutions* as defined in Definition 3.3 .1 and solutions of system (3.3 .1) because the term $\rho f \in H^{-1}(\Omega)$ does not able to justify integration by parts. However, for $f \in L^2(\Omega)$, one has the following:

Proposition 3.3 .3. *Let $(f,f_h) \in L^2(\Omega) \times H^{1/2}_m(\Gamma)$. Then $(u,h) \in V^1_{\Gamma_d}(\Omega) \times H^1_m(\Gamma)$ is a* weak solution *of system* (3.3 .1) *if and only if there exists a unique $p \in L^2_m(\Omega)$ such that (u,p,h) is a solution of system* (3.3 .1) *in the $H^{-1/2}(\Gamma)$ sense for the fifth equation and in the L^2 sense for the others.*

Remark 3.3 .4. *We would like to point out that for $f \in L^2(\Omega)$, $[\mathbb{S}(u,p)n_\Gamma]$ is defined in $H^{-1/2}(\Gamma)$ the following way: let $\varphi \in H^{1/2}(\Gamma)$ and denote by $\phi \in V^1_{\Gamma_d}(\Omega)$ a lifting of φ in Ω, then*

$$\langle [\mathbb{S}(u,p)n_\Gamma], \varphi\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} = (\rho f,\phi) + \frac{1}{2}(\mu D(u), D(\phi)) + \lambda(\rho u,\phi).$$

With Proposition 3.3 .3 and Remark 3.3 .4, we deduce from (3.3 .1)$_5$:

Corollary 3.3 .5. *Let* $\Re e(\lambda) > 0$, $(f, f_h) \in L^2(\Omega) \times H^{1/2}_m(\Gamma)$. *The unique* weak solution $(u, h) \in V^1_{\Gamma_d}(\Omega) \times H^1_m(\Gamma)$ *of system* (3.3 .1) *is such that* $h \in H^{3/2}_m(\Gamma)$.

The following Proposition is a crucial point of this work as it gives the estimate allowing to prove the analyticity of the semigroup associated to the Stokes operator with a surface tension interface.

Proposition 3.3 .6. *Let* $\Re e(\lambda) > 0$ *and* $(f, f_h) \in L^2(\Omega) \times H^{3/2}_m(\Gamma)$. *Then the* weak solution (u, h) *of* (3.3 .1) *is in* $U_0(\Omega) \times H^{5/2}_m(\Gamma)$ *and*

$$\|(u^\pm, h)\|_{H^2(\Omega^\pm)\times H^{5/2}_m(\Gamma)} + |\lambda| \|(u, h)\|_{L^2(\Omega)\times H^{3/2}_m(\Gamma)} \leq C\|(f, f_h)\|_{L^2(\Omega)\times H^{3/2}_m(\Gamma)}, \tag{3.3 .4}$$

where C *is independent of* λ.

Our approach below is based on the weak formulation given in Definition 3.3 .1 and shortcuts any precise expansion of the Dirichlet-to-Neumann map.

Proof of Proposition 3.3 .6. The proof relies on the use of both the variational formulation (3.3 .3) and the energy balance (3.3 .2). The variational formulation (3.3 .3) gives more importance to the coupling equation (3.3 .1)$_5$ than to the equation of h (3.3 .1)$_6$. This enables to get optimal regularity estimates:

$$\|(u^\pm, h)\|_{H^2(\Omega^\pm)\times H^{5/2}_m(\Gamma)} \leq C\|(f, f_h)\|_{L^2(\Omega)\times H^{3/2}_m(\Gamma)}.$$

On the contrary, the energy balance (3.3 .2) gives more importance to the equation of h (3.3 .1)$_6$ than to the coupling equation (3.3 .1)$_5$. The energy balance is therefore more adapted to get the analyticity estimate on the resolvent

$$\|(u, h)\|_{L^2(\Omega)\times H^{3/2}_m(\Gamma)} \leq \frac{C}{\lambda}\|(f, f_h)\|_{L^2(\Omega)\times H^{3/2}_m(\Gamma)}.$$

The proof is divided into two steps. In the first step, we prove the estimate on $|\lambda|\|u\|_{L^2(\Omega)}$. This constitutes the delicate point of the proof. In the second step, we get the other estimates in a more classical way.

<u>*Step 1:*</u> In this step, we prove

$$\|u\|_{L^2(\Omega)} \leq \frac{C}{|\lambda|}\|(f, f_h)\|_{L^2(\Omega)\times H^{3/2}_m(\Gamma)}.$$

Let $v \in U_0(\Omega)$ such that

$$v_d = -f_h \text{ on } \Gamma \quad \text{with } \|v^\pm\|_{H^2(\Omega^\pm)} \leq C\|f_h\|_{H^{3/2}_m(\Gamma)},$$

and $(w, h) \in \mathscr{V}_\lambda(\Omega)$ the solution given by Lax-Milgram of (3.3 .3). Then $(u = v + w, h) \in V^1_{\Gamma_d}(\Omega) \times H^1_m(\Gamma)$ is the unique *weak solution* of (3.3 .1). From Corollary 3.3 .5, we also have $h \in H^{3/2}_m(\Gamma)$.
First, taking $(\varphi, \psi) = (w, h)$ in (3.3 .3) and using $u = w + v$, one has

$$\lambda\|\sqrt{\rho}u\|^2_{L^2(\Omega)} + \frac{1}{2}\|\sqrt{\mu}D(u)\|^2_{L^2(\Omega)} + \sigma\bar{\lambda}\|\nabla h\|^2_{L^2(\Gamma)} \\ = (\rho f, u) - \lambda(\rho u, v) - \frac{1}{2}(\mu D(u), D(v)) - (\rho f, v). \tag{3.3 .5}$$

In fact, this energy equality is not optimal to get precise estimates with constant independent of λ. This is due to the term $\lambda(\rho u, v)$. To overcome the problem, we use Remark 3.3 .4 and the fact that (u, h) is solution to (3.3 .1)$_5$:

$$-\frac{1}{2}(\mu D(u), D(v)) - \lambda(\rho u, v) - (\rho f, v) = \sigma(\nabla h, \nabla f_h)_\Gamma,$$

so that the energy equality (3.3 .5) can be rewritten as

$$\lambda\|\sqrt{\rho}u\|^2_{L^2(\Omega)} + \frac{1}{2}\|\sqrt{\mu}D(u)\|^2_{L^2(\Omega)} + \sigma\bar{\lambda}\|\nabla h\|^2_{L^2(\Gamma)} = (\rho f, u) + \sigma(\nabla h, \nabla f_h)_\Gamma. \tag{3.3 .6}$$

On this energy estimate, λ does not appear on the right hand side. With Young inequality, one has

$$|\lambda|\|u\|_{L^2(\Omega)} \leq C(\|(f, f_h)\|_{L^2(\Omega)\times H^{3/2}_m(\Gamma)} + |\lambda|\|h\|_{H^{3/2}_m(\Gamma)}). \tag{3.3 .7}$$

We now need to estimate $|\lambda|\|h\|_{H^{3/2}_m(\Gamma)}$. Since $\lambda h = u_d + f_h$, we first estimate $\|u_d\|_{H^2(\Omega)}$.

Let $i = 1$ for $d = 2$ or $i \in \{1, 2\}$ for $d = 3$. Using a translation method on the variational formulation (3.3 .3), one has that $(\partial_i w, \partial_i h) \in \mathscr{V}_\lambda(\Omega)$ satisfies: $\forall(\varphi, \psi) \in \mathscr{V}_\lambda(\Omega)$,

$$\begin{aligned}\lambda(\rho\partial_i w, \varphi) + \frac{1}{2}(\mu D(\partial_i w), D(\varphi)) + \sigma\bar{\lambda}(\nabla(\partial_i h), \nabla\psi)_\Gamma \\ = \langle \rho\partial_i f, \varphi\rangle_{H^{-1}(\Omega), H^1_0(\Omega)} - \lambda(\rho\partial_i v, \varphi) - \frac{1}{2}(\mu D(\partial_i v), D(\varphi)).\end{aligned}$$

Taking $(\varphi, \psi) = (\partial_i w, \partial_i h)$ and using $u = w + v$, one has

$$\begin{aligned}\lambda\|\sqrt{\rho}\partial_i u\|^2_{L^2(\Omega)} + \frac{1}{2}\|\sqrt{\mu}D(\partial_i u)\|^2_{L^2(\Omega)} + \sigma\bar{\lambda}\|\nabla\partial_i h\|^2_{L^2(\Gamma)} \\ = -(\rho f, \partial_{ii} u) + \lambda(\rho u, \partial_{ii} v) - \frac{1}{2}(\mu D(\partial_i u), D(\partial_i v)) + (\rho f, \partial_{ii} v).\end{aligned} \tag{3.3 .8}$$

Taking the real part and using Korn and Young inequalities, we obtain the following estimate

$$\|\partial_i u\|_{H^1(\Omega)} \leq C_\varepsilon\|(f, f_h)\|_{L^2(\Omega)\times H^{3/2}_m(\Gamma)} + \varepsilon|\lambda|\|u\|_{L^2(\Omega)}, \tag{3.3 .9}$$

where $\varepsilon > 0$ is independent of λ and will be fixed later. The divergence condition then gives

$$\|u_d\|_{H^2(\Omega)} \leq C_\varepsilon\|(f, f_h)\|_{L^2(\Omega)\times H^{3/2}_m(\Gamma)} + C\varepsilon|\lambda|\|u\|_{L^2(\Omega)}. \tag{3.3 .10}$$

Using $\lambda h = u_d + f_h$ on Γ,

$$|\lambda|\|h\|_{H^{3/2}_m(\Gamma)} \leq C(\|u_d\|_{H^2(\Omega)} + \|f_h\|_{H^{3/2}_m(\Gamma)}) \leq C_\varepsilon\|(f, f_h)\|_{L^2(\Omega)\times H^{3/2}_m(\Gamma)} + C\varepsilon|\lambda|\|u\|_{L^2(\Omega)}. \tag{3.3 .11}$$

Putting (3.3 .11) in (3.3 .7) and taking ε sufficiently small, we get

$$|\lambda|\|u\|_{L^2(\Omega)} \leq C\|(f, f_h)\|_{L^2(\Omega)\times H^{3/2}_m(\Gamma)}. \tag{3.3 .12}$$

Step 2: In this step, we get the other estimates

$$\|(u^\pm, h)\|_{H^2(\Omega^\pm)\times H^{5/2}_m(\Gamma)} + |\lambda|\|h\|_{H^{3/2}_m(\Gamma)} \leq C\|(f, f_h)\|_{L^2(\Omega)\times H^{3/2}_m(\Gamma)}.$$

Plugging (3.3 .12) in (3.3 .9), (3.3 .10), (3.3 .11), we obtain

$$\|\partial_i u\|_{H^1(\Omega)} + \|u_d\|_{H^2(\Omega)} + |\lambda|\|h\|_{H^{3/2}_m(\Gamma)} \leq C\|(f, f_h)\|_{L^2(\Omega)\times H^{3/2}_m(\Gamma)}.$$

Moreover, taking the real part of (3.3 .5) and using Korn and Young inequalities, we also have

$$\|u\|_{H^1(\Omega)} \leq C\|(f, f_h)\|_{L^2(\Omega)\times H^{3/2}_m(\Gamma)}.$$

It remains to estimate $\partial_{dd}u_i$ in $L^2(\Omega)$ and Δh in $H^{1/2}(\Gamma)$. To do that, we first estimate the pressure p. Using the equation $(3.3\ .1)_1$, we have

$$\begin{cases} -\mu^\pm \partial_{dd} u_i^\pm + \partial_i p^\pm &= \rho^\pm f_i^\pm - \lambda\rho^\pm u_i^\pm + \mu^\pm \Delta_\Gamma u_i^\pm \quad \text{on } \Omega^\pm, \\ \partial_d p^\pm &= \rho^\pm f_d^\pm - \rho^\pm \lambda u_d^\pm + \mu^\pm \Delta u_2^\pm \quad \text{on } \Omega^\pm, \end{cases} \tag{3.3 .13}$$

where $\Delta_\Gamma = \Delta - \partial_{dd}$. The second line of (3.3 .13) gives $\partial_d p^\pm \in L^2(\Omega^\pm)$ with

$$\|\partial_d p^\pm\|_{L^2(\Omega^\pm)} \leq C\|(f, f_h)\|_{L^2(\Omega)\times H^{3/2}_m(\Gamma)}.$$

The first line of (3.3 .13) gives $\partial_i p^\pm \in H^{-1}(\Omega^\pm)$ for $u_i^\pm \in H^1(\Omega)$. Differentiating this first line in the tangential direction of Γ, we obtain $\nabla\partial_i p^\pm \in H^{-1}(\Omega^\pm)$ as $\partial_i u^\pm \in H^1(\Omega)$. Using Necas inequality, we obtain $\partial_i p^\pm \in L^2(\Omega^\pm)$ with

$$\|\partial_i p^\pm\|_{L^2(\Omega^\pm)} \leq C(\|\partial_i p^\pm\|_{H^{-1}(\Omega^\pm)} + \|\nabla\partial_i p^\pm\|_{H^{-1}(\Omega^\pm)}) \leq C\|(f, f_h)\|_{L^2(\Omega)\times H^{3/2}_m(\Gamma)}.$$

Now, using $(3.3\ .13)_1$, we get $\partial_{dd}u_1^\pm \in L^2(\Omega^\pm)$ with

$$\|\partial_{dd}u_1^\pm\|_{L^2(\Omega^\pm)} \leq C\|(f, f_h)\|_{L^2(\Omega)\times H^{3/2}_m(\Gamma)}.$$

We then deduce that $[\mathbb{S}(u,p)n_\Gamma] \in H^{1/2}(\Gamma)$ and using $(3.3\ .1)_5$, we deduce $h \in H^{5/2}_m(\Gamma)$ with

$$\|h\|_{H^{5/2}_m(\Gamma)} \leq C\|(f, f_h)\|_{L^2(\Omega)\times H^{3/2}_m(\Gamma)}.$$

The proof of Proposition 3.3 .6 is complete. □

To the best of the author's knowledge, this is the first time that such an estimate is proved in the case of a two-phase fluid model with surface tension. In the literature on free surface, Beale in [8, Section 3, Theorem 1, page 318] and Nishida et al in [33, Proposition 5.1, page 302] obtain similar results. In [8], Beale develops a same kind of reasoning than in our proof but works with $f_h = 0$ and in higher regularity spaces. Nishida et al in [33] obtains exactly the estimate (3.3 .4) but develops another approach based on Fourier transform and complicated estimates on Fourier coefficients. The main advantage of our approach is to provide a simpler proof by avoiding long Fourier estimates.

Following the above proof, we can gain regularity:

Proposition 3.3 .7. *Let $s \geq 0$, $\Re e(\lambda) > 0$ and $(f, f_h) \in L^2(\Omega) \times H^{s+3/2}_m(\Gamma)$ such that $f^\pm \in H^s(\Omega^\pm)$. Then system (3.3 .1) has a unique strong solution $(u, p, h) \in U_0(\Omega) \times P(\Omega) \times H^{5/2}_m(\Gamma)$. This solution enjoys the regularity $(u^\pm, \nabla p^\pm, h) \in H^{s+2}(\Omega^\pm) \times H^s(\Omega^\pm) \times H^{s+5/2}_m(\Gamma)$ with*

$$\|(u^\pm, \nabla p^\pm, h)\|_{H^{s+2}(\Omega^\pm)\times H^s(\Omega^\pm)\times H^{s+5/2}_m(\Gamma)} \leq C(\lambda)\|(f^\pm, f_h)\|_{H^s(\Omega^\pm)\times H^{s+3/2}_m(\Gamma)}. \tag{3.3 .14}$$

Proof of Proposition 3.3 .7. We first prove the results for the integers $k \in \mathbb{N}$, $k \geq 2$, and we deduce the estimates for $s \geq 2$ by interpolation.

The proof for integers are done by induction. The case $k = 2$ is a consequence of Propositions 3.3 .3 and 3.3 .6. Suppose that (3.3 .14) is true for $k \geq 2$ and let $(f, f_h) \in L^2(\Omega) \times H^{k+5/2}_m(\Gamma)$ such that $f^\pm \in H^{k+1}(\Omega^\pm)$. With a translation method and the induction hypothesis, we have $(\partial_i u^\pm, \partial_i \nabla p^\pm, \partial_i h) \in H^{k+2}(\Omega^\pm) \times H^{k+2}(\Omega^\pm) \times H^{k+5/2}_m(\Gamma)$, $i = 1$ for $d = 2$ and $i \in \{2, 3\}$ for $d = 3$. With the incompressibility equation, we deduce $u_d^\pm \in H^{k+3}(\Omega^\pm)$. Then with the equation $(3.3\ .1)_1$, we deduce $\partial_d p \in H^{k+1}(\Omega^\pm)$ and $\partial_{dd}u_i^\pm \in H^{k+1}(\Omega^\pm)$. Putting all together, we then have $(u^\pm, \nabla p^\pm, h) \in H^{s+2}(\Omega^\pm) \times H^{s+1}(\Omega^\pm) \times H^{s+5/2}_m(\Gamma)$. □

Remark 3.3 .8. *The condition $\Re e(\lambda) > 0$ of Proposition 3.3 .7 enables to guarantee the existence of a solution of system (3.3 .1). However, if $(u, p, h) \in U_0(\Omega) \times P(\Omega) \times H^{5/2}_m(\Gamma)$ is a solution of (3.3 .1) for $(f, f_h) \in L^2(\Omega) \times H^{s+3/2}_m(\Gamma)$ such that $f^\pm \in H^s(\Omega^\pm)$ and with $\lambda \in \mathbb{C}$ not necessarily of positive real part, we also have $(u^\pm, \nabla p^\pm, h) \in H^{s+2}(\Omega^\pm) \times H^s(\Omega^\pm) \times H^{s+5/2}_m(\Gamma)$. Indeed, write system (3.3 .1):*

$$\left\{\begin{array}{rcll} u^\pm - \dfrac{1}{\rho^\pm}\nabla\cdot\mathbb{S}^\pm(u^\pm, p^\pm) &=& f^\pm + (1-\lambda)u & \text{in } \Omega^\pm, \\ \nabla\cdot u^\pm &=& 0 & \text{in } \Omega^\pm, \\ u &=& 0 & \text{on } \Gamma_d, \\ u^+ &=& u^- & \text{on } \Gamma, \\ [\mathbb{S}(u,p)n_\Gamma] + (\sigma\Delta h)n_\Gamma &=& 0 & \text{on } \Gamma, \\ h - u_d &=& f_h + (1-\lambda u) & \text{on } \Gamma. \end{array}\right.$$

Then, we can combine Proposition 3.3 .7 to a bootstrap argument to obtain $(u^\pm, \nabla p^\pm, h) \in H^{s+2}(\Omega^\pm) \times H^s(\Omega^\pm) \times H^{s+5/2}_m(\Gamma)$.

In particular, any eigenvector (u, p, h) of system (3.3 .1) is such that $(u^\pm, p^\pm, h) \in C^\infty(\Omega^\pm) \times C^\infty(\Omega^\pm) \times C^\infty(\Gamma)$.

3.3 .2 Operator formulation for system (3.3 .1)

In this subsection, we write system (3.3 .1) in an operator form and we state a result of compact resolvent and a result of analyticity for the associated semigroup.

Before stating a semigroup formulation for (3.3 .1), we need to introduce a well chosen projector:

Proposition 3.3 .9. *We have the following decomposition*

$$L^2(\Omega) = \mathscr{V}^0_{n,\Gamma_d}(\Omega) \overset{\perp}{\oplus} \nabla\mathscr{H}^1(\Omega),$$

where $L^2(\Omega)$ is endowed with its natural inner-product $(\cdot,\cdot)$ and

$$\mathscr{H}^1(\Omega) = \{p \in H^1(\Omega) \mid p = 0 \ \text{ on } \Gamma\}.$$

We denote by $P : L^2(\Omega) \to \mathscr{V}^0_{n,\Gamma_d}(\Omega)$ the orthogonal projector of $L^2(\Omega)$ on $\mathscr{V}^0_{n,\Gamma_d}(\Omega)$.
Let $u \in L^2(\Omega)$, then $(I-P)u = \mathbb{1}_{\Omega^+}(\nabla\check{q}^+ + \nabla\hat{q}^+) + \mathbb{1}_{\Omega^-}(\nabla\check{q}^- + \nabla\hat{q}^-)$, where $(\check{q}^\pm, \hat{q}^\pm) \in H^1_0(\Omega^\pm) \times H^1(\Omega^\pm)$ are solutions of

$$\Delta\check{q}^\pm = \nabla\cdot u^\pm \quad \text{in } \Omega^\pm,$$
$$\Delta\hat{q}^\pm = 0 \quad \text{in } \Omega^\pm, \quad \partial_n\hat{q} = (u - \nabla\check{q})\cdot n \quad \text{on } \Gamma^\pm_d, \quad \hat{q}^\pm = -\check{q}^\pm \quad \text{on } \Gamma.$$

Let $u \in H^1(\Omega)$, then $(Pu)^\pm \in H^1(\Omega^\pm)$ with

$$\|(Pu)^\pm\|_{H^1(\Omega^\pm)} \le C\|u\|_{H^1(\Omega)}.$$

Let $u \in H^1(\Omega)$ with $u^\pm \in H^2(\Omega^\pm)$, then $(Pu)^\pm \in H^2(\Omega^\pm)$ with

$$\|(Pu)^\pm\|_{H^2(\Omega^\pm)} \le C\|u^\pm\|_{H^2(\Omega^\pm)}.$$

Proof of Proposition 3.3 .9. It is easily seen that $\mathscr{V}^0_{n,\Gamma_d}(\Omega) \perp \nabla \mathscr{H}^1(\Omega)$ and that $u - \mathbb{1}_{\Omega^+}(\nabla \check{q}^+ + \nabla \hat{q}^+) + \mathbb{1}_{\Omega^-}(\nabla \check{q}^- + \nabla \hat{q}^-) \in \mathscr{V}^0_{n,\Gamma_d}(\Omega)$. If $u \in H^1(\Omega)$, then $(I-P)u = \nabla q$, where $q = \mathbb{1}_{\Omega^+} q^+ + \mathbb{1}_{\Omega^-} q^-$ where $q^\pm \in H^1(\Omega^\pm)$ is solution of

$$\Delta q^\pm = \nabla \cdot u^\pm \quad \text{in } \Omega^\pm, \quad \partial_n^\pm q = u^\pm \cdot n \quad \text{on } \Gamma_d^\pm, \quad q^\pm = 0 \quad \text{on } \Gamma. \tag{3.3 .15}$$

The solution $q^\pm$ lies in $H^2(\Omega^\pm)$. If $u \in H^1(\Omega)$ with $u^\pm \in H^2(\Omega^\pm)$, then the solution of (3.3 .15) is in $H^3(\Omega^\pm)$. ☐

Remark 3.3 .10. *The operator P could have been defined by $P = \mathbb{1}_{\Omega^+} P^+ + \mathbb{1}_{\Omega^-} P^-$ where $P^\pm$ is the Leray orthogonal projector of $L^2(\Omega^\pm)$ on $V^0_{n,\Gamma_d^\pm}(\Omega^\pm)$.*

This operator of projection P is different from the classical Leray projector P_L of $L^2(\Omega)$ on $V^0_{n,\Gamma_d}(\Omega)$. Besides, since $V^0_{n,\Gamma_d}(\Omega) \subset \mathscr{V}^0_{n,\Gamma_d}(\Omega)$ and the operators P and P_L are orthogonal projectors, one has $P_L P = P_L$.

Remark that since ρ is constant on Ω^+ and on Ω^-, we have

$$\forall \varphi \in L^2(\Omega), \quad P(\rho\varphi) = \rho P\varphi.$$

The operator P_L does not fulfill this property and that is the reason why it is not well adapted to our problem of transmission.

We now work on

$$Z = \mathscr{V}^0_{n,\Gamma_d}(\Omega) \times H^{3/2}_m(\Gamma).$$

First, introduce the following lifting projector: for $h \in H^{5/2}_m(\Gamma)$, let $L(h) \in U_0(\Omega)$ be the velocity component of the solution of

$$\left\{ \begin{array}{rcll} \nabla \cdot \mathbb{S}^\pm(u^\pm, p^\pm) & = & 0 & \text{in } \Omega^\pm, \\ \nabla \cdot u^\pm & = & 0 & \text{in } \Omega^\pm, \\ u & = & 0 & \text{on } \Gamma_d, \\ u^+ & = & u^- & \text{on } \Gamma, \\ [\mathbb{S}(u,p) n_\Gamma] & = & -\Delta h & \text{on } \Gamma. \end{array} \right. \tag{3.3 .16}$$

We now define the Stokes operator for our problem

$$A_s u = \frac{1}{\rho} P \nabla \cdot \mathbb{S}(u,p),$$

which is an unbounded operator in $\mathscr{V}^0_{n,\Gamma_d}(\Omega)$ with domain

$$D(A_s; \mathscr{V}^0_{n,\Gamma_d}(\Omega)) = \{u \in U_0(\Omega) \mid \exists p \in P(\Omega),\ \nabla \cdot \mathbb{S}(u,p) \in L^2(\Omega)\}.$$

Remark that $u \in D(A_s; \mathscr{V}^0_{n,\Gamma_d}(\Omega))$ implies $[\mathbb{S}(u,p) n_\Gamma] = 0$ on Γ. We also introduce

$$\gamma_2(v) = v_2|_\Gamma.$$

Lastly, we introduce the operator

$$A(u,h) = (A_s(u - \sigma L(h)), \gamma_2 u),$$

with domain

$$D(A; Z) = \{(u,h) \in U_0(\Omega) \times H^{5/2}_m(\Gamma) \mid u - \sigma L(h) \in D(A_s; \mathscr{V}^0_{n,\Gamma_d}(\Omega))\},$$

where L is defined in (3.3 .16).

One easily checks that this operator corresponds to the one appearing in system (3.3 .1), namely:

Proposition 3.3 .11. *Let* $(f, f_h) \in L^2(\Omega) \times H_m^{3/2}(\Gamma)$, $(u, h) \in U_0(\Omega) \times H_m^{5/2}(\Gamma)$. *Then*

$$(u, h) \in D(A; Z) \quad \textit{and} \quad (\lambda I - A)(u, h) = (Pf, f_h),$$

if and only if there exists $p \in P(\Omega)$ *such that* (u, p, h) *is solution of* (3.3 .1).

As a consequence of Propositions 3.3 .3 and 3.3 .6 and [34, Chapter 2, Theorem 5.2, page 61], we have the following result:

Theorem 3.3 .12. *The operator* $(A, D(A; Z))$ *generates an analytic semigroup on* Z *and its resolvent is compact.*

Remark 3.3 .13. *We could have defined* A *on* Z' *(see* (3.2 .16) *for the formulation of* Z'*):*

$$A(u, h) = \left(\frac{1}{\rho} P\nabla \cdot \mathbb{S}(u, p), \gamma_2 u\right),$$

with domain

$$D(A; Z') = \left\{ (u, h) \in V^1_{\Gamma_d}(\Omega) \times H_m^{3/2}(\Gamma) \mid \exists p \in L^2_m(\Omega), \begin{array}{l} \nabla \cdot \mathbb{S}(u, p) \in L^2(\Omega), \\ [\mathbb{S}(u, p) n_\Gamma] = -\sigma \Delta h \ \textit{on} \ \Gamma \end{array} \right\}.$$

This will be useful when dealing with the adjoint problem. Besides, we have the following result:

Proposition 3.3 .14. *Let* $(f, f_h) \in L^2(\Omega) \times H_m^{1/2}(\Gamma)$, $(u, h) \in V^1_{\Gamma_d}(\Omega) \times H_m^{3/2}(\Gamma)$. *Then*

$$(u, h) \in D(A; Z') \quad \textit{and} \quad (\lambda I - A)(u, h) = (Pf, f_h),$$

if and only if (u, h) *is a* weak solution *of system* (3.3 .1).

3.3 .3 Adjoint of the operator $(A, D(A; Z))$

The adjoint operator is computed with the identification (3.2 .14) ($H = H'$) and the scalar product (3.2 .15) ($L^2(\Omega)$ weighted by ρ and $H^1(\Gamma)$ weighted by σ). It corresponds to the identification of the energy space as one can see in (3.3 .2). Therefore, the adjoint operator is defined on Z' where

$$Z' = \mathscr{V}^0_{n,\Gamma_d}(\Omega) \times H_m^{1/2}(\Gamma).$$

Proposition 3.3 .15. *The adjoint operator of* A *is*

$$A^*(v, k) = \left(\frac{1}{\rho} P\nabla \cdot \mathbb{S}(v, q), -\gamma_2 v\right),$$

with domain

$$D(A^*; Z') = \left\{ (v, k) \in V^1_{\Gamma_d}(\Omega) \times H_m^{3/2}(\Gamma) \mid \exists q \in L^2_m(\Omega), \begin{array}{l} \nabla \cdot \mathbb{S}(v, q) \in L^2(\Omega), \\ [\mathbb{S}(v, q) n_\Gamma] = \sigma \Delta k \ \textit{on} \ \Gamma \end{array} \right\}.$$

Let $(\varphi, \varphi_k) \in Z'$, $(v, k) \in V^1_{\Gamma_d}(\Omega) \times H_m^{3/2}(\Gamma)$. *Then*

$$(v, k) \in D(A^*; Z') \quad \textit{and} \quad (\lambda I - A^*)(v, k) = (P\varphi, \varphi_k),$$

if and only if there exists $q \in L^2_m(\Omega)$ *such that* (v, q, k) *is solution of*

$$\left\{\begin{array}{rcll} \lambda v^\pm - \dfrac{1}{\rho^\pm}\nabla\cdot\mathbb{S}^\pm(v^\pm, q^\pm) & = & \varphi^\pm & \text{in } \Omega^\pm, \\ \nabla\cdot v^\pm & = & 0 & \text{in } \Omega^\pm, \\ v & = & 0 & \text{on } \Gamma_d, \\ v^+ & = & v^- & \text{on } \Gamma, \\ [\mathbb{S}(v,q)n_\Gamma] - (\sigma\Delta k)n_\Gamma & = & 0 & \text{on } \Gamma, \\ \lambda k + v_d & = & \varphi_k & \text{on } \Gamma, \end{array}\right. \tag{3.3 .17}$$

where the fifth line is in the $H^{-1/2}(\Gamma)$ *sense introduced in Remark 3.3 .4.*

Let $\Re e(\lambda) > 0$, $(\varphi, \varphi_k) \in Z'$. *System (3.3 .17) admits a unique solution* $(v, q, k) \in V^1_{\Gamma_d}(\Omega) \times L^2_m(\Omega) \times H^{3/2}_m(\Gamma)$.

Note that system (3.3 .17) is very similar to system (3.3 .1). But here, the natural space for (φ, φ_k) is $\mathscr{V}^0_{n,\Gamma_d}(\Omega) \times H^{1/2}_m(\Gamma)$ while it was $(f, f_h) \in \mathscr{V}^0_{n,\Gamma_d}(\Omega) \times H^{3/2}_m(\Gamma)$ in system (3.3 .1). Nevertheless, we pointed out in Proposition 3.3 .3 that our concept of *weak solution* introduced in Definition 3.3 .1 can be used also for $(f, f_h) \in \mathscr{V}^0_{n,\Gamma_d}(\Omega) \times H^{1/2}_m(\Gamma)$. Therefore, similarly as in Proposition 3.3 .3, a concept of weak solutions can be developed for (3.3 .17) as well for $(\varphi, \varphi_k) \in \mathscr{V}^0_{n,\Gamma_d}(\Omega) \times H^{1/2}_m(\Gamma)$.

Proof of Proposition 3.3 .15. Introduce the operator

$$A^\sharp(v, k) = \left(\frac{1}{\rho}P\nabla\cdot\mathbb{S}(v, q), -\gamma_2 v\right),$$

with domain

$$D(A^\sharp; Z') = \left\{(v, k) \in V^1_{\Gamma_d}(\Omega) \times H^{3/2}_m(\Gamma) \mid \exists q \in L^2_m(\Omega), \begin{array}{l} \nabla\cdot\mathbb{S}(v,q) \in L^2(\Omega), \\ [\mathbb{S}(v,q)n_\Gamma] = \sigma\Delta k \text{ on } \Gamma \end{array}\right\}.$$

First, it is straightforward that: $((v, k) \in D(A^\sharp; Z')$ and $(\lambda I - A^\sharp)(v, k) = (P\varphi, \varphi_k))$ is equivalent to system (3.3 .17). Moreover, let $\Re e(\lambda) > 0$ and $(\varphi, \varphi_k) \in Z'$. With Proposition 3.3 .2 and 3.3 .3, it is easily seen that system (3.3 .17) admits a unique solution $(v, q, k) \in V^1_{\Gamma_d}(\Omega) \times L^2_m(\Omega) \times H^{3/2}_m(\Gamma)$.

Besides, using the scalar production on H (3.2 .15), a direct computation shows that

$$\forall ((v, k), (u, h)) \in D(A^\sharp; Z') \times D(A; Z), \quad \langle (v, k), A(u, h)\rangle_{Z',Z} = \langle A^\sharp(v, k), (u, h)\rangle_{Z',Z}.$$

This equality enables to say that $D(A^\sharp; Z') \subset D(A^*; Z')$ and $A^*|_{D(A^\sharp;Z')} = A^\sharp$. It remains to prove that $D(A^*; Z') \subset D(A^\sharp; Z')$. Let $z \in D(A^*; Z')$. By definition of $D(A^*; Z')$, $(I - A^*)z \in Z'$. Since $I - A^\sharp : D(A^\sharp; Z') \to Z'$ is onto, there exists $z^\sharp \in D(A^\sharp; Z')$ such that $(I - A^\sharp)z^\sharp = (I - A^*)z$. Then

$$\forall y \in D(A; Z), \quad \langle z, (I - A)y\rangle_{Z',Z} = \langle z^\sharp, (I - A)y\rangle_{Z',Z}.$$

Since $I - A : D(A; Z) \to Z$ is onto, we deduce that $z = z^\sharp \in D(A^\sharp; Z')$ which concludes the proof. □

3.3 .4 Non homogeneous stationary surface tension Stokes system

We consider the following system for $\lambda \in \mathbb{C}$:

$$\begin{cases} \lambda u^{\pm} - \dfrac{1}{\rho^{\pm}} \nabla \cdot \mathbb{S}^{\pm}(u^{\pm}, p^{\pm}) &=& f^{\pm} & \text{in } \Omega^{\pm}, \\ \nabla \cdot u^{\pm} &=& f^{\pm}_{div} & \text{in } \Omega^{\pm}, \\ u &=& g & \text{on } \Gamma_d, \\ u^{+} &=& u^{-} & \text{on } \Gamma, \\ [\mathbb{S}(u,p)n_{\Gamma}] + (\sigma \Delta h) n_{\Gamma} &=& f_{jump} & \text{on } \Gamma, \\ \lambda h - u_d &=& f_h & \text{on } \Gamma, \end{cases} \tag{3.3 .18}$$

Proposition 3.3 .16. *Let $\Re e(\lambda) > 0$, $(f, f_{div}, g, f_{jump}, f_h) \in L^2(\Omega) \times L^2(\Omega) \times H^{3/2}(\Gamma_d) \times H^{1/2}(\Gamma) \times H^{3/2}(\Gamma)$ with $f^{\pm}_{div} \in H^1(\Omega^{\pm})$. We assume the compatibility conditions:*

$$\begin{aligned} \int_{\Omega^+} f^{+}_{div} &= \int_{\Gamma_d^+} g^{+} \cdot n + \int_{\Gamma} f_h, \\ \int_{\Omega^-} f^{-}_{div} &= \int_{\Gamma_d^-} g^{-} \cdot n - \int_{\Gamma} f_h. \end{aligned}$$

There exists a unique strong solution $(u, p, h) \in \{u \in H^1(\Omega) \mid u^{\pm} \in H^2(\Omega^{\pm})\} \times P(\Omega) \times H^{5/2}(\Gamma)$ of system (3.3 .1). Moreover, we have

$$\|(u,p,h)\|_{U(\Omega)\times P(\Omega)\times H^{5/2}(\Gamma)} \leq C \|(f, f^{\pm}_{div}, g, f_{jump}, f_h)\|_{L^2(\Omega)\times H^1(\Omega^{\pm})\times H^{3/2}(\Gamma_d)\times H^{1/2}(\Gamma)\times H^{3/2}(\Gamma)}.$$

Proof of Proposition 3.3 .16. Let $(v, q) \in \{u \in H^1(\Omega) \mid u^{\pm} \in H^2(\Omega^{\pm})\} \times P(\Omega)$ be the solution given by Lemma 3.3 .17 of

$$\begin{cases} \nabla \cdot v^{\pm} &=& f^{\pm}_{div} & \text{in } \Omega^{\pm}, \\ v &=& g & \text{on } \Gamma_d, \\ v^{+} &=& v^{-} & \text{on } \Gamma, \\ [\mathbb{S}(v,q)n_{\Gamma}] &=& f_{jump} & \text{on } \Gamma, \\ v_d &=& -f_h & \text{on } \Gamma. \end{cases}$$

and $(w, r, h) \in U_0(\Omega) \times P(\Omega) \times H^{5/2}_m(\Gamma)$ the unique solution given by the previous subsection of

$$\begin{cases} \lambda w^{\pm} - \dfrac{1}{\rho^{\pm}} \nabla \cdot \mathbb{S}^{\pm}(w^{\pm}, r^{\pm}) &=& f^{\pm} + \dfrac{1}{\rho^{\pm}} \nabla \cdot \mathbb{S}^{\pm}(v^{\pm}, q^{\pm}) - \lambda v^{\pm} & \text{in } \Omega^{\pm}, \\ \nabla \cdot w^{\pm} &=& 0 & \text{in } \Omega^{\pm}, \\ w &=& 0 & \text{on } \Gamma_d, \\ w^{+} &=& w^{-} & \text{on } \Gamma, \\ [\mathbb{S}(u,p)n_{\Gamma}] + (\sigma \Delta h) n_{\Gamma} &=& 0 & \text{on } \Gamma, \\ \lambda h - w_d &=& 0 & \text{on } \Gamma, \end{cases}$$

Then $(u, p, h) = (v, q, 0) + (w, r, h) \in \{u \in H^1(\Omega) \mid u^{\pm} \in H^2(\Omega^{\pm})\} \times P(\Omega) \times H^{5/2}(\Gamma)$ is solution of (3.3 .1) and satfisfies the desired estimate. □

Lemma 3.3 .17. *Let $f_{div} \in L^2(\Omega)$ with $f^{\pm}_{div} \in H^1(\Omega^{\pm})$, $g \in H^{3/2}(\Gamma_d)$, $f_{jump} \in H^{1/2}(\Gamma)$, $f_h \in H^{3/2}(\Gamma)$. We assume the compatibility conditions:*

$$\begin{aligned} \int_{\Omega^+} f^{+}_{div} &= \int_{\Gamma_d^+} g^{+} \cdot n + \int_{\Gamma} f_h, \\ \int_{\Omega^-} f^{-}_{div} &= \int_{\Gamma_d^-} g^{-} \cdot n - \int_{\Gamma} f_h. \end{aligned}$$

There exists a strong solution $(v, q) \in \{u \in H^1(\Omega) \mid u^\pm \in H^2(\Omega^\pm)\} \times P(\Omega)$ *of system*

$$\left\{\begin{array}{rcll} \nabla\cdot v^\pm &=& f_{div}^\pm & \text{in } \Omega^\pm,\\ v &=& g & \text{on } \Gamma_d,\\ v^+ &=& v^- & \text{on } \Gamma,\\ [\mathbb{S}(v,q)n_\Gamma] &=& f_{jump} & \text{on } \Gamma,\\ v_d &=& -f_h & \text{on } \Gamma. \end{array}\right.$$

Moreover we have

$$\|(v,q)\|_{U(\Omega)\times P(\Omega)} \leq C\|(f_{div}^\pm, g, f_{jump}, f_h)\|_{H^1(\Omega^\pm)\times H^{3/2}(\Gamma_d)\times H^{1/2}(\Gamma)\times H^{3/2}(\Gamma)}.$$

Proof of Lemma 3.3 .17. *Step 1:* we lift the divergence and the normal trace term. We look for $\tilde{v}^\pm = \nabla\phi^\pm$ and we are interested in the following systems, written separately on Ω^+ and Ω^-:

$$\left\{\begin{array}{rcll} \Delta\phi^\pm &=& f_{div}^\pm & \text{in } \Omega^\pm,\\ \partial_n\phi^\pm &=& g^\pm\cdot n & \text{on } \Gamma_d^\pm,\\ \partial_n\phi^\pm &=& -f_h & \text{on } \Gamma. \end{array}\right.$$

There exists a solution $\phi^\pm \in H^3(\Omega^\pm)$ which satisfies the estimate

$$\|\phi^\pm\|_{H^3(\Omega^\pm)} \leq C\|(f_{div}^\pm, g, f_h)\|_{H^1(\Omega^\pm)\times H^{3/2}(\Gamma_d)\times H^{3/2}(\Gamma)}.$$

Step 2: we lift the tangential trace and the jump of the stress tensor. We look for $\hat{v}^\pm$ solution of

$$\left\{\begin{array}{rcll} \nabla\cdot \hat{v}^\pm &=& 0 & \text{in } \Omega^\pm,\\ \hat{v} &=& g-\tilde{v} & \text{on } \Gamma_d,\\ \hat{v}^+ &=& \hat{v}^- & \text{on } \Gamma,\\ [\mathbb{S}(\hat{v},\hat{q})n_\Gamma] &=& f_{jump} - [\mu D(v)n_\Gamma] & \text{on } \Gamma,\\ \hat{v}_d &=& 0 & \text{on } \Gamma. \end{array}\right. \tag{3.3 .19}$$

Let $\tilde{g} = g - \tilde{v}$ and $\tilde{f}_{jump} = f_{jump} - [\mu(\nabla\tilde{v} + (\nabla\tilde{v})^T)n_\Gamma]$. Remark that $\tilde{g}\cdot n = 0$.

First case: $d = 2$**.** We look for $\hat{v}^\pm = (\nabla\varphi^\pm)^\perp = (-\partial_2\varphi^\pm, \partial_1\varphi^\pm)$. The first equation of (3.3 .19) is fulfilled due to the form of $\hat{v}$. System (3.3 .19) can be reduced to

$$\left\{\begin{array}{rcll} \varphi^\pm &=& 0 & \text{on } \Gamma_d^\pm,\\ \partial_2\varphi^\pm &=& \pm\tilde{g}_1^\pm & \text{on } \Gamma_d^\pm,\\ \varphi^\pm &=& 0 & \text{on } \Gamma,\\ \partial_2\varphi^\pm &=& 0 & \text{on } \Gamma,\\ \partial_{22}\varphi^+ &=& -\tilde{f}_{jump,1}/\mu^+ & \text{on } \Gamma,\\ \partial_{22}\varphi^- &=& 0 & \text{on } \Gamma,\\ q^+ &=& -\tilde{f}_{jump,2} & \text{on } \Gamma,\\ q^- &=& 0 & \text{on } \Gamma. \end{array}\right. \tag{3.3 .20}$$

Using [31, Chapter 1, Theorem 9.4, page 41], there exists a solution $(\varphi^\pm, q^\pm)\times H^3(\Omega^\pm) \in H^1(\Omega^\pm)$ satisfying the estimate

$$\|(\varphi^\pm, q^\pm)\|_{H^3(\Omega^\pm)\times H^1(\Omega^\pm)} \leq C\|(\tilde{g}, \tilde{f}_{jump})\|_{H^{3/2}(\Gamma_d)\times H^{1/2}(\Gamma)}.$$

Second case: $d = 3$. We look for $\hat{v}^{\pm} = (-\partial_3\varphi_1^{\pm}, -\partial_3\varphi_2^{\pm}, \partial_1\varphi_1^{\pm} + \partial_2\varphi_2^{\pm})$ with $\varphi^{\pm} = (\varphi_1^{\pm}, \varphi_2^{\pm}, 0)$. The first equation of (3.3 .19) is fulfilled due to the form of $\hat{v}$. System (3.3 .19) can be reduced to

$$\left\{\begin{array}{rcll} \varphi_i^{\pm} & = & 0 & \text{on } \Gamma_d^{\pm}, \\ \partial_3\varphi_i^{\pm} & = & \pm\tilde{g}_i^{\pm} & \text{on } \Gamma_d^{\pm}, \\ \varphi_i^{\pm} & = & 0 & \text{on } \Gamma, \\ \partial_3\varphi_i^{\pm} & = & 0 & \text{on } \Gamma, \\ \partial_{33}\varphi_i^{+} & = & -\tilde{f}_{jump,i}/\mu^{+} & \text{on } \Gamma, \\ \partial_{33}\varphi_i^{-} & = & 0 & \text{on } \Gamma, \\ q^{+} & = & -\tilde{f}_{jump,3} & \text{on } \Gamma, \\ q^{-} & = & 0 & \text{on } \Gamma. \end{array}\right. \tag{3.3 .21}$$

Using [31, Chapter 1, Theorem 9.4, page 41], there exists a solution $(\varphi^{\pm}, q^{\pm}) \in H^3(\Omega^{\pm}) \in H^1(\Omega^{\pm})$ satisfying the estimate

$$\|(\varphi^{\pm}, q^{\pm})\|_{H^3(\Omega^{\pm})\times H^1(\Omega^{\pm})} \leq C\|(\tilde{g}, \tilde{f}_{jump})\|_{H^{3/2}(\Gamma_d)\times H^{1/2}(\Gamma)}.$$

<u>*Conclusion:*</u> the couple $(v, q) = \mathbb{1}_{\Omega^+}(\tilde{v}^+ + \hat{v}^+, q^+) + \mathbb{1}_{\Omega^-}(\tilde{v}^- + \hat{v}^-, q^-) \in \{u \in H^1(\Omega) \mid u^{\pm} \in H^{\pm}(\Omega^{\pm})\} \times P(\Omega)$ fulfils the requirements of Lemma 3.3 .17. $\square$

3.4 The extended stationary system

The notations used through this section are defined in Section 3.2 .5.1.

As explained in the introduction, we are interested in a dynamical control that follows the equation (3.2 .10). The stationary system associated to the evolution system (3.2 .11) is

$$\left\{\begin{array}{rcll} \lambda u^{\pm} - \dfrac{1}{\rho^{\pm}}\nabla\cdot\mathbb{S}^{\pm}(u^{\pm}, p^{\pm}) & = & f^{\pm} & \text{in } \Omega^{\pm}, \\ \nabla\cdot u^{\pm} & = & 0 & \text{in } \Omega^{\pm}, \\ u & = & \displaystyle\sum_{i=1}^{n_c} g_i w_i & \text{on } \Gamma_d, \\ u^{+} & = & u^{-} & \text{on } \Gamma, \\ [\mathbb{S}(u,p)n_\Gamma] + (\sigma\Delta h)n_\Gamma & = & 0 & \text{on } \Gamma, \\ \lambda h - u_d & = & f_h & \text{on } \Gamma, \\ \lambda\mathbf{g} + \Lambda\mathbf{g} & = & \mathbf{f}_g, & \end{array}\right. \tag{3.4 .1}$$

where $\lambda \in \mathbb{C}$, Λ is a diagonal matrix with non negative coefficients, $\mathbf{g} = (g_i)_{i\in[\![1,n_c]\!]}$ and the functions w_i will be defined later (see Section 3.5 .2) and satisfy $\mathrm{Supp}(w_i) \subset \Gamma_c$ and $\displaystyle\int_{\Gamma_c} w_i \cdot n = 0$.

In this section, we write system (3.4 .1) in an operator form and we study properties of the operator. This section is based on the results and tools introduced in Section 3.3 . In Section 3.4 .1, we prove existence and uniqueness of strong solutions for system (3.4 .1) for $\Re e(\lambda) > 0$. Then, in Section 3.4 .2, we give an operator formulation for system (3.4 .1) and state a result of compact resolvent and a result of analyticity for the associated semigroup. Lastly, we study the adjoint operator in Section 3.4 .3.

3.4 .1 Existence and uniqueness for system (3.4 .1)

We first introduce the following lifting operator: for any vector $\mathbf{g} = (g_i)_{i\in[\![1;p]\!]} \in \mathbb{R}^{n_c}$, let $D\mathbf{g} = (D_u\mathbf{g}, D_p\mathbf{g}, D_h\mathbf{g}) = (u, p, h) \in U(\Omega) \times P(\Omega) \times H_m^{5/2}(\Gamma)$ be the solution given by Proposition

3.3 .16 of

$$\left\{\begin{array}{rcll} u^{\pm}-\dfrac{1}{\rho^{\pm}}\nabla\cdot\mathbb{S}^{\pm}(u^{\pm},p^{\pm}) &=& 0 & \text{in } \Omega^{\pm},\\ \nabla\cdot u^{\pm} &=& 0 & \text{in } \Omega^{\pm},\\ u &=& \displaystyle\sum_{i=1}^{n_c} g_i w_i & \text{on } \Gamma_d,\\ u^{+} &=& u^{-} & \text{on } \Gamma,\\ [\mathbb{S}(u,p)n_\Gamma]+(\sigma\Delta h)n_\Gamma &=& 0 & \text{on } \Gamma,\\ h-u_d &=& 0 & \text{on } \Gamma. \end{array}\right. \tag{3.4 .2}$$

Proposition 3.4 .1. *Let $\Re e(\lambda)>0$. For $(f,f_h,\mathbf{f}_g)\in L^2(\Omega)\times H^{3/2}_m(\Gamma)\times\mathbb{C}^{n_c}$, there exists a unique solution $(u,p,h,g)\in U(\Omega)\times P(\Omega)\times H^{5/2}_m(\Gamma)\times\mathbb{C}^{n_c}$ of system (3.4 .1). Moreover, we have*

$$\|(u^{\pm},\nabla p^{\pm},h)\|_{H^2(\Omega^{\pm})\times L^2(\Omega^{\pm})\times H^{5/2}_m(\Gamma)}+|\lambda|\|(u,h,\mathbf{g})\|_{L^2(\Omega)\times H^{3/2}_m(\Gamma)\times\mathbb{C}^{n_c}} \\ \leq C\|(f,f_h,\mathbf{f}_g)\|_{L^2(\Omega)\times H^{3/2}_m(\Gamma)\times\mathbb{C}^{n_c}},$$

where C is independent of λ.

Proof of Proposition 3.4 .1. The existence and uniqueness easily follows from Proposition 3.3 .16 since we can solve the equation of $\mathbf{g}$ independently. It remains to get the estimate. First, the equation of $\mathbf{g}$ gives

$$\|\mathbf{g}\|_{\mathbb{C}^{n_c}}\leq\frac{\|\mathbf{f}_g\|_{\mathbb{C}^{n_c}}}{|\lambda|},$$

since every coefficient of the diagonal matrix Λ is non negative. Then, with the definition of $D\mathbf{g}$ in (3.4 .2) and the estimate of Proposition 3.3 .16, we have

$$\|D\mathbf{g}\|_{H^2(\Omega^{\pm})\times H^1(\Omega^{\pm})\times H^{5/2}_m(\Gamma)}\leq C\|\mathbf{g}\|_{\mathbb{C}^{n_c}}\leq C\frac{\|\mathbf{f}_g\|_{\mathbb{C}^{n_c}}}{|\lambda|},$$

Then $(\tilde u,\tilde p,\tilde h)=(u,p,h)-D\mathbf{g}$ is solution of

$$\left\{\begin{array}{rcll} \lambda\tilde u^{\pm}-\dfrac{1}{\rho^{\pm}}\nabla\cdot\mathbb{S}^{\pm}(\tilde u^{\pm},\tilde p^{\pm}) &=& f^{\pm}+(1-\lambda)(D_u\mathbf{g})^{\pm} & \text{in } \Omega^{\pm},\\ \nabla\cdot\tilde u^{\pm} &=& 0 & \text{in } \Omega^{\pm},\\ \tilde u &=& 0 & \text{on } \Gamma_d,\\ \tilde u^{+} &=& \tilde u^{-} & \text{on } \Gamma,\\ [\mathbb{S}(\tilde u,\tilde p)n_\Gamma]+(\sigma\Delta\tilde h)n_\Gamma &=& 0 & \text{on } \Gamma,\\ \lambda\tilde h-\tilde u_d &=& f_h+(1-\lambda)D_h\mathbf{g} & \text{on } \Gamma. \end{array}\right.$$

Using Proposition 3.3 .6 and the estimate on $D\mathbf{g}$, we deduce that

$$\|(\tilde u^{\pm},\nabla\tilde p,\tilde h)\|_{H^2(\Omega^{\pm})\times L^2(\Omega^{\pm})\times H^{5/2}_m(\Gamma)}+|\lambda|\|(\tilde u,\tilde h)\|_{L^2(\Omega)\times H^{3/2}_m(\Gamma)} \\ \leq C\|(f,f_h,\mathbf{f}_g)\|_{L^2(\Omega)\times H^{3/2}_m(\Gamma)\times\mathbb{C}^{n_c}},$$

where C is independent of λ. □

3.4 .2 Operator formulation for (3.4 .1)

In what follows, the subscript ’e’ stands for ’extended’ as (3.4 .1) corresponds to an extended version of system (3.3 .1).

We work on

$$Z_e = Z \times \mathbb{R}^{n_c}.$$

We introduce the operator

$$A_e = \begin{pmatrix} A & (I-A)(PD_u, D_h) \\ 0 & -\Lambda \end{pmatrix}, \tag{3.4 .3}$$

with domain

$$D(A_e; Z_e) =$$
$$\{(u,h,\mathbf{g}) \in (U(\Omega) \cap \mathscr{V}^0_{n,\Gamma_d}(\Omega)) \times H^{5/2}_m(\Omega) \times \mathbb{R}^{n_c} \mid (u,h) - (PD_u\mathbf{g}, D_h\mathbf{g}) \in D(A;Z)\}, \tag{3.4 .4}$$

where $D\mathbf{g}$ is defined in (3.4 .2).

Proposition 3.4 .2. *Let $(f, f_h, \mathbf{f}_g) \in L^2(\Omega) \times H^{3/2}_m(\Gamma) \times \mathbb{C}^{n_c}$, $(u,h,\mathbf{g}) \in U(\Omega) \times H^{5/2}_m(\Gamma) \times \mathbb{C}^{n_c}$. Then*

$$(Pu,h,\mathbf{g}) \in D(A_e;Z_e) \quad \text{and} \quad \left\{ \begin{array}{rcl} (\lambda I - A_e)(Pu,h,\mathbf{g}) & = & (Pf, f_h, \mathbf{f}_g), \\ (I-P)u & = & (I-P)D_u\mathbf{g}, \end{array} \right.$$

if and only if there exists $p \in P(\Omega)$ such that $(u,p,h,\mathbf{g})$ is solution of (3.4 .1).

This result is a direct consequence of Proposition 3.3 .11.

Theorem 3.4 .3. *The operator $(A_e, D(A_e;Z_e))$ generates an analytic semigroup on Z_e and its resolvent is compact.*

This is a direct consequence of Propositions 3.4 .1 and 3.4 .2 and [34, Chapter 2, Theorem 5.2, page 61].

3.4 .3 Adjoint of the operator $(A_e, D(A_e;Z_e))$

The adjoint operator is computed with the identification (3.2 .14). Therefore, the adjoint operator is defined on Z'_e where

$$Z'_e = Z' \times \mathbb{R}^{n_c}.$$

Proposition 3.4 .4. *The adjoint operator of A_e in Z'_e is*

$$A^*_e = \begin{pmatrix} A^* & 0 \\ ((I-A)(PD_u, D_h))^* & -\Lambda \end{pmatrix},$$

with domain

$$D(A^*_e; Z'_e) = D(A^*;Z') \times \mathbb{R}^{n_c}.$$

Let $(\varphi, \varphi_k, \varphi_{\tilde g}) \in L^2(\Omega) \times H^{1/2}_m(\Gamma) \times \mathbb{C}^{n_c}$, $(v,k,\tilde{\mathbf{g}}) \in V^1_{\Gamma_d}(\Omega) \times H^{3/2}_m(\Omega) \times \mathbb{C}^{n_c}$. Then

$$(v,k,\tilde{\mathbf{g}}) \in D(A^*_e;Z'_e) \quad \text{and} \quad (\lambda I - A^*_e)(v,k,\tilde{\mathbf{g}}) = (P\varphi, \varphi_k, \varphi_{\tilde g}),$$

if and only if there exists $q \in L^2_m(\Omega)$ such that $(v,p,k,\tilde{\mathbf{g}})$ is solution of

$$\left\{ \begin{array}{rcll} \lambda v^\pm - \dfrac{1}{\rho^\pm}\nabla\cdot\mathbb{S}^\pm(v^\pm,q^\pm) & = & \varphi^\pm & \text{in } \Omega^\pm, \\ \nabla\cdot v^\pm & = & 0 & \text{in } \Omega^\pm, \\ v & = & 0 & \text{on } \Gamma_d, \\ v^+ & = & v^- & \text{on } \Gamma, \\ [\mathbb{S}(v,q)n_\Gamma] - (\sigma\Delta k)n_\Gamma & = & 0 & \text{on } \Gamma, \\ \lambda k + v_d & = & \varphi_k & \text{on } \Gamma, \\ \lambda\tilde{\mathbf{g}} + \Lambda\tilde{\mathbf{g}} + \left(\langle \mathbb{S}(v,q)n, w_i\rangle_{H^{-1/2}(\Gamma_d), H^{1/2}(\Gamma_d)}\right)_{i\in[\![1,p]\!]} & = & \varphi_{\tilde g}, & \end{array} \right. \tag{3.4 .5}$$

where the fifth line is in the $H^{-1/2}(\Gamma)$ sense introduced in Remark 3.3 .4.

Let $\Re e(\lambda) > 0$ *and* $(\varphi, \varphi_k, \varphi_{\tilde{g}}) \in Z'_e$*. System* (3.4 .5) *admits a unique solution* $(v, q, k, \tilde{\mathbf{g}}) \in V^1_{\Gamma_d}(\Omega) \times L^2_m(\Omega) \times H^{3/2}_m(\Gamma) \times \mathbb{C}^{n_c}$.

Proof. It is in fact sufficient to compute $\left((I-A)(PD_u, D_h)\right)^*$. Since $(PD_u, D_h) \in \mathscr{L}(\mathbb{R}^{n_c}, Z)$ and A can be extended to an operator in $(D(A^*; Z'))'$ with domain $D(A, (D(A^*; Z'))') = Z$, we have the identity $\left((I-A)(PD_u, D_h)\right)^* = (D_u^* P^*, D_h^*)(I-A)^* \in \mathscr{L}(D(A^*; Z'), \mathbb{R}^{n_c})$. Let $(\varphi, \varphi_k) \in Z'$ and $(v, q, k) \in V^1_{\Gamma_d}(\Omega) \times L^2_m(\Omega) \times H^{3/2}_m(\Gamma)$ be the unique solution of system (3.3 .17) for $\lambda = 1$. A straightforward computation shows that

$$\langle (\varphi, \varphi_k), (PD_u\mathbf{g}, D_h\mathbf{g}) \rangle_{Z', Z} = -\left(\langle \mathbb{S}(v, q)n, w_i \rangle_{H^{-1/2}(\Gamma_d), H^{1/2}(\Gamma_d)}, \mathbf{g} \right)_{\mathbb{R}^{n_c}}.$$

We deduce that

$$(D_u^* P^*, D_h^*)(\varphi, \varphi_k) = -\left(\langle \mathbb{S}(v, q)n, w_i \rangle_{H^{-1/2}(\Gamma_d), H^{1/2}(\Gamma_d)} \right)_{i \in [\![1, p]\!]}.$$

And therefore, for all $(v, k) \in D(A^*; Z')$,

$$(D_u^* P^*, D_h^*)(I - A^*)(v, k) = -\left(\langle \mathbb{S}(v, q)n, w_i \rangle_{H^{-1/2}(\Gamma_d), H^{1/2}(\Gamma_d)} \right)_{i \in [\![1, p]\!]},$$

where $(v, q, k) \in V^1_{\Gamma_d}(\Omega) \times L^2_m(\Omega) \times H^{3/2}_m(\Gamma)$ is the unique solution of system (3.3 .17) with $\lambda = 1$ and right hand side $(\varphi, \varphi_k) = (I - A^*)(v, k)$.

The fact that for $\Re e(\lambda) > 0$ and $(\varphi, \varphi_k, \varphi_{\tilde{g}}) \in Z'_e$, system (3.4 .5) admits a unique solution $(v, q, k, \tilde{\mathbf{g}}) \in V^1_{\Gamma_d}(\Omega) \times L^2_m(\Omega) \times H^{3/2}_m(\Gamma) \times \mathbb{C}^{n_c}$ is straightforward with Proposition 3.3 .15. □

3.5 Stabilization of system (3.2 .11)

The notations used through this section are defined in Section 3.2 .5.2.

The main purpose of this section is to prove Theorem 3.2 .6. Our approach to stabilize system (3.2 .11) by mean of a feedback law follows a classical strategy. In Section 3.5 .1, we formulate system (3.2 .11) in a semigroup form. In Section 3.5 .2, we construct the control functions w_i. We then prove a unique continuation property in Section 3.5 .3 and proves that the linear system (3.2 .11) is stabilizable in Z_e in Section 3.5 .4. In Section 3.5 .5, we construct a feedback law stabilizing the linear system (3.2 .11) in Z_e by solving an algebraic Riccati equation. Lastly, Section 3.5 .6 is dedicated to prove that the constructed feedback stabilizes the linear system (3.2 .11) in the settings of Theorem 3.2 .6.

3.5 .1 Semigroup formulation for system (3.2 .11)

Proposition 3.5 .1. *Let* $(f, u_0, h_0) \in E^0_{F,\infty} \times E^0_{u_0} \times E^0_{h_0}$, $f_{jump} = 0$, $(u, h, \mathbf{g}) \in E^0_{u,\infty} \times E^0_{h,\infty} \times E^0_{\mathbf{g},\infty}$. *There exists* $p \in E^0_{p,\infty}$ *such that* $(u, p, h, \mathbf{g})$ *is solution of system* (3.2 .11) *if and only if*

$$\begin{cases} \dfrac{d}{dt}(Pu, h, \mathbf{g}) &=& (A_e + \omega I)(Pu, h, \mathbf{g}) + B\mathbf{f}_g + (Pf, 0, 0), \\ (I-P)u &=& (I-P)D_u\mathbf{g}, \\ (Pu, h, \mathbf{g})(0) &=& (u_0, h_0, 0), \end{cases}$$

where $B \in \mathscr{L}(\mathbb{R}^{n_c}, Z_e)$ *is defined by*

$$B\mathbf{f}_g = (0, 0, \mathbf{f}_g).$$

It is a direct consequence of Proposition 3.4 .2. It is easily seen that $B^* \in \mathscr{L}(Z_e, \mathbb{R}^{n_c})$ is given by

$$B^*(v, k, \tilde{\mathbf{g}}) = \tilde{\mathbf{g}}. \tag{3.5 .1}$$

3.5 .2 Construction of the control space

The control functions $(w_i)_{i\in[\![1,n_c]\!]}$ are constructed so that the pair $(A_e+\omega I, B)$ fulfills the Hautus Criterion (see [10, Part V, Chapter 1, Proposition 3.3, page 492]).

From now on, we make the following assumption:

Assumption 3.5 .2. *The decreasing rate* $\omega \notin \Re e(\sigma(A))$.

Note that this assumption can be made without loss of generality as the spectrum of A is discrete.

We introduce the set

$$J_u = \{\lambda \in \sigma(A) \mid \Re e(\lambda) \geq -\omega\}.$$

Since $(A, D(A,Z))$ generates an analytic semigroup on Z, this set is finite. We consider the eigenvalue problem

$$\left\{\begin{array}{rcll} \lambda v^\pm - \dfrac{1}{\rho^\pm}\nabla\cdot\mathbb{S}^\pm(v^\pm,q^\pm) &=& 0 & \text{in } \Omega^\pm,\\ \nabla\cdot v^\pm &=& 0 & \text{in } \Omega^\pm,\\ v &=& 0 & \text{on } \Gamma_d,\\ v^+ &=& v^- & \text{on } \Gamma,\\ [\mathbb{S}(v,q)n_\Gamma] - (\sigma\Delta k)n_\Gamma &=& 0 & \text{on } \Gamma,\\ \lambda k + v_d &=& 0 & \text{on } \Gamma, \end{array}\right. \tag{3.5 .2}$$

which is the problem of eigenvalues for A^* by Proposition 3.3 .15. For $\lambda \in J_u$, we define

$$E(\lambda) = \{(v,q,k) \in V^1_{\Gamma_d}(\Omega)\times L^2_m(\Omega)\times H^{3/2}_m(\Gamma) \mid (v,q,k) \text{ is solution of } (3.5\ .2)\},$$

We now introduce a cut-off for the control: let Γ_c' be a non empty open subset of Γ_c and $\chi \in C^\infty(\Gamma_d;[0,1])$ such that

$$\chi = \left\{\begin{array}{ll} 1 & \text{on } \Gamma_c',\\ 0 & \text{on } \Gamma_d\setminus\Gamma_c. \end{array}\right.$$

We now introduce the set of controls

$$W = \mathrm{Vect}\Big\{\chi\Re e(\mathbb{S}(v,q)n|_{\Gamma_d}), \chi\Im m(\mathbb{S}(v,q)n|_{\Gamma_d}) \mid (v,q,k)\in E(\lambda),\ \lambda\in J_u\Big\},$$

Since $E(\lambda)$ is of finite dimension and J_u is finite, W is of finite dimension. We now construct the family $(\tilde{w}_i)_{i\in[\![1,n_c]\!]}$:

$$(\tilde{w}_i)_{i\in[\![1,n_c]\!]} \text{ is chosen as a basis of } W.$$

As a direct consequence of Propositions 3.3 .3, 3.3 .6, 3.3 .7 and Remark 3.3 .8, the constructed $\tilde{w}_i$ are in $C^\infty(\Gamma_d)$. The family $(\tilde{w}_i)_{i\in[\![1,n_c]\!]}$ do not necessarily satisfy $\displaystyle\int_{\Gamma_c}\tilde{w}_i^\pm\cdot n = 0$. Therefore, we construct the family $(w_i)_{i\in[\![1,n_c]\!]}$ from the family $(\tilde{w}_i)_{i\in[\![1,n_c]\!]}$ as follow

$$w_i = \tilde{w}_i - \left(\int_{\Gamma_c}\tilde{w}_i\cdot n\right)\frac{\chi}{\displaystyle\int_{\Gamma_c}\chi}n. \tag{3.5 .3}$$

The functions $(w_i)_{i\in[\![1,n_c]\!]}$ satisfy $w_i \in C^\infty(\Gamma_d)$, $\mathrm{Supp}(w_i)\subset\Gamma_c$ and $\displaystyle\int_{\Gamma_c} w_i\cdot n = 0$.

We now fix these functions $(w_i)_{i\in[\![1,n_c]\!]}$.

3.5 .3 Unique continuation property

Before stating the result of stability, we prove the following unique continuation property for our system:

Proposition 3.5 .3. *Let $\lambda \in \mathbb{C}$ with $\Re e(\lambda) \geq -\omega$. If $(v, q, k) \in V^1_{\Gamma_d}(\Omega) \times L^2_m(\Omega) \times H^{3/2}_m(\Gamma)$ is solution of (3.5 .2) and satisfies*

$$\forall i \in [\![1, n_c]\!], \int_{\Gamma_d} S(v, q)n \cdot w_i = 0, \tag{3.5 .4}$$

then $(v, q, k) = 0$.

Proof of Proposition 3.5 .3. Assume that $\Gamma'_c \subset \Gamma^+_d$, the case $\Gamma'_c \subset \Gamma^-_d$ being a straightforward adaptation. The proof is divided into two main steps. First we prove that $\mathbb{S}(v, q)n = Cn$ on Γ'_c. Then, we conclude that $(v, q, k) = 0$ with [20, Proposition 1.1, page 3].

Step 1: By definition of w_i and $\tilde{w}_i$, (3.5 .4) gives

$$\int_{\Gamma_c} \chi |S(v, q)n|^2 - \frac{1}{\displaystyle\int_{\Gamma_c} \chi} \left| \int_{\Gamma_c} \chi S(v, q)n \cdot n \right|^2 = 0$$

Let $w = S(v, q)n$ and write

$$\int_{\Gamma_c} \chi |w_\tau|^2 + \left(\int_{\Gamma_c} \chi |w \cdot n|^2 - \frac{1}{\displaystyle\int_{\Gamma_c} \chi} \left| \int_{\Gamma_c} \chi w \cdot n \right|^2 \right) = 0.$$

With Cauchy-Schwarz inequality, we have

$$\left(\int_{\Gamma_c} \chi \right) \left(\int_{\Gamma_c} \chi |w \cdot n|^2 \right) - \left| \int_{\Gamma_c} \chi w \cdot n \right|^2 \geq 0.$$

We then deduce that

$$\int_{\Gamma_c} \chi |w_\tau|^2 = 0 \quad \text{and} \quad \left(\int_{\Gamma_c} \chi \right) \left(\int_{\Gamma_c} \chi |w \cdot n|^2 \right) = \left| \int_{\Gamma_c} \chi w \cdot n \right|^2.$$

The first equality gives $w_\tau = 0$ on Γ'_c. The second is a case of equality in Cauchy-Schwartz inequality. Then, there exists $C \in \mathbb{C}$ such that $\sqrt{\chi} w \cdot n = C\sqrt{\chi}$ on Γ_c. In particular,

$$w \cdot n = \mathbb{S}(v, q) \cdot n = C \quad \text{on } \Gamma'_c.$$

Step 2: *Case $\lambda \neq 0$:* We have

$$\begin{cases} \lambda v^+ - \dfrac{1}{\rho^+} \nabla \cdot \mathbb{S}^+(v^+, q^+) &= 0 & \text{in } \Omega^+, \\ \nabla \cdot v^+ &= 0 & \text{in } \Omega^+, \\ v^+ &= 0 & \text{on } \Gamma^+_d, \\ \mathbb{S}^+(v^+, q^+)n &= C^+ n & \text{on } \Gamma'_c. \end{cases}$$

In particular, $\partial_n v^+ = 0$ and $q^+ = C^+$ on Γ'_c. Using [20, Proposition 1.1, page 3], we deduce that $v^+ = 0$ and $q^+ = C^+$ in Ω^+. Since $\lambda \neq 0$, we also deduce $k = 0$ in Γ from $(3.5\ .2)_6$. With $(3.5\ .2)_4$ and $(3.5\ .2)_5$, we get on Ω^-,

$$\begin{cases} \lambda v^- - \dfrac{1}{\rho^-} \nabla \cdot \mathbb{S}^-(v^-, q^-) &= 0 & \text{in } \Omega^-, \\ \nabla \cdot v^- &= 0 & \text{in } \Omega^-, \\ v^- &= 0 & \text{on } \Gamma, \\ \mathbb{S}^-(v^-, q^-)n_\Gamma &= C^+ n_\Gamma & \text{on } \Gamma. \end{cases}$$

Using [20, Proposition 1.1, page 3], we deduce that $v^- = 0$ and $q^- = C^+$ on Ω^-. Since $\int_\Omega p = 0$, we finally have $C^+ = 0$ and $q = 0$.

Case $\lambda = 0$: A classical energy balance leads to

$$\|\mu D(v)\|_{L^2(\Omega)} = 0.$$

We deduce that $v = p = 0$ in Ω and that $\Delta k = 0$ in Γ. Using $\int_\Gamma k = 0$, one has $k = 0$ in Γ. □

3.5 .4 Stabilizability of the pair $(A_e + \omega I, B)$

Proposition 3.5 .4. *The pair $(A_e + \omega I, B)$ is stabilizable.*

Proof. The proof is based on the Hautus Criterion [10, Part V, Chapter 1, Proposition 3.3, page 492]: the pair $(A_e + \omega I, B)$ is stabilizable if and only if

$$\mathrm{Ker}(\lambda I - A_e^*) \cap \mathrm{Ker}(B^*) = 0 \quad \text{for } \Re e(\lambda) \geq -\omega.$$

Let λ such that $\Re e(\lambda) \geq -\omega$ and $(v, k, \tilde{\mathbf{g}}) \in \mathrm{Ker}(\lambda I - A_e^*) \cap \mathrm{Ker}(B^*)$. If $\lambda \notin \sigma(A_e) = \sigma(A_e^*)$, then $(v, k, \tilde{\mathbf{g}}) = 0$. It remains to treat the case $\lambda \in \sigma(A_e)$. Using $(v, k, \tilde{\mathbf{g}}) \in \mathrm{Ker}(\lambda I - A_e^*)$ and Proposition 3.4 .4, there exists a unique $q \in P(\Omega)$ such that

$$\left\{\begin{array}{rcll} \lambda v^\pm - \dfrac{1}{\rho^\pm}\nabla \cdot \mathbb{S}^\pm(v^\pm, q^\pm) & = & 0 & \text{in } \Omega^\pm, \\ \nabla \cdot v^\pm & = & 0 & \text{in } \Omega^\pm, \\ v & = & 0 & \text{on } \Gamma_d, \\ v^+ & = & v^- & \text{on } \Gamma, \\ [\mathbb{S}(v, q) n_\Gamma] - (\sigma \Delta k) n_\Gamma & = & 0 & \text{on } \Gamma, \\ \lambda k + v_d & = & 0 & \text{on } \Gamma, \\ \lambda \tilde{\mathbf{g}} + \Lambda \tilde{\mathbf{g}} - \left(\langle \mathbb{S}(v, q) n, w_i \rangle_{H^{-1/2}(\Gamma_d), H^{1/2}(\Gamma_d)}\right)_{i \in [\![1,p]\!]} & = & 0. & \end{array}\right.$$

With Propositions 3.3 .3, 3.3 .6, 3.3 .7 and Remark 3.3 .8, (v, p, h) are smooth functions. Using $(v, k, \tilde{\mathbf{g}}) \in \mathrm{Ker}(B^*)$ and (3.5 .1), we have

$$\tilde{\mathbf{g}} = 0.$$

And therefore, for all $i \in [\![1, n_c]\!]$,

$$\int_{\Gamma_d} \mathbb{S}(v, q) n \cdot w_i = 0.$$

Using the unique continuation property in Proposition 3.5 .3, we deduce that

$$(v, k) = 0.$$

□

3.5 .5 Feedback law

We consider the following degenerate Riccati equation:

$$G_\omega \in \mathscr{L}(Z_e), \quad G_\omega^* = G_\omega > 0, \quad G_\omega(A_e + \omega I) + (A_e + \omega I)^* G_\omega - G_\omega B B^* G_\omega = 0.$$

Using [30, Theorem 3, page 12], there exists a solution to the Riccati equation G_ω. Furthermore, the operator $K = -B^* G_\omega \in \mathscr{L}(Z_e, \mathbb{R}^{n_c})$ provides a stabilizing feedback for $(A_e + \omega I, B)$. The operator $A_e + \omega I + BK$ with domain

$$D(A_e + \omega I + BK; Z_e) = D(A_e; Z_e).$$

is the generator of an analytic semigroup exponentially stable on Z_e.

3.5 .6 Regularity of the closed loop extended linear system: proof of Theorem 3.2 .6.

We are interested in the following system

$$\left\{\begin{array}{rcll} \partial_t u^\pm - \dfrac{1}{\rho^\pm}\nabla\cdot\mathbb{S}^\pm(u^\pm,p^\pm) - \omega u^\pm &=& f^\pm & \text{in } (0,+\infty)\times\Omega^\pm,\\ \nabla\cdot u^\pm &=& 0 & \text{in } (0,+\infty)\times\Omega^\pm,\\ u &=& \displaystyle\sum_{i=1}^{n_c} g_i w_i & \text{on } (0,+\infty)\times\Gamma_d,\\ u^+ &=& u^- & \text{on } (0,+\infty)\times\Gamma,\\ [\mathbb{S}(u,p)n_\Gamma] + (\sigma\Delta h)n_\Gamma &=& f_{jump} & \text{on } (0,+\infty)\times\Gamma,\\ \partial_t h - u_d - \omega h &=& 0 & \text{on } (0,+\infty)\times\Gamma,\\ \mathbf{g}' + \Lambda\mathbf{g} - \omega\mathbf{g} &=& K(Pu,h,\mathbf{g}), & \\ u(0) &=& u_0 & \text{in } \Omega,\\ h(0) &=& h_0 & \text{on } \Gamma,\\ \mathbf{g}(0) &=& 0. & \end{array}\right. \tag{3.5 .5}$$

Our aim is to prove the following Theorem which is slightly stronger than Theorem 3.2 .6:

Theorem 3.5 .5. *Let $(f,f_{jump},u_0,h_0)\in E^0_{f,\infty}\times E^0_{f_{jump},\infty}\times E^0_{u_0}\times E^0_{h_0}$. Then there exists a unique solution $(u,p,h,\mathbf{g})\in E^0_{u,\infty}\times E^0_{p,\infty}\times E^0_{h,\infty}\times E^0_{\mathbf{g},\infty}$ to system (3.5 .5). Furthermore, we have the estimate*

$$\|(u,p,h,\mathbf{g})\|_{E^0_{u,\infty}\times E^0_{p,\infty}\times E^0_{h,\infty}\times E^0_{\mathbf{g},\infty}} \le C\|(f,f_{jump},u_0,h_0)\|_{E^0_{f,\infty}\times E^0_{f_{jump},\infty}\times E^0_{u_0}\times E^0_{h_0}}.$$

Besides, we have uniqueness in the class of functions $E^0_{u,T}\times E^0_{p,T}\times E^0_{h,T}\times E^0_{\mathbf{g},T}$ for all $T>0$.

Proof of Theorem 3.5 .5. We begin with lifting the nonhomogeneous term f_{jump}: using Lemma 3.5 .7 below, there exists $(\check{u},\check{p})\in E^0_{u,\infty}\times E^0_{p,\infty}$ such that

$$\left\{\begin{array}{rcll} \nabla\cdot\check{u}^\pm &=& 0 & \text{in } (0,+\infty)\times\Omega^\pm,\\ \check{u} &=& 0 & \text{on } (0,+\infty)\times\Gamma_d,\\ \check{u}^+ &=& \check{u}^- & \text{on } (0,+\infty)\times\Gamma,\\ [\mathbb{S}(\check{u},\check{p})n_\Gamma] &=& f_{jump} & \text{on } (0,+\infty)\times\Gamma,\\ \check{u}_d &=& 0 & \text{on } (0,+\infty)\times\Gamma, \end{array}\right.$$

with

$$\|(\check{u},\check{p})\|_{E^0_{u,\infty},E^0_{p,\infty}} \le C\|f_{jump}\|_{E^0_{f_{jump},\infty}}.$$

The new unknown

$$(\tilde{u},\tilde{p},h,\mathbf{g}) = (u,p,h,\mathbf{g}) - (\check{u},\check{p},0,0),$$

satisfies the system

$$\begin{cases} \partial_t \tilde{u}^\pm - \dfrac{1}{\rho^\pm}\nabla\cdot\mathbb{S}^\pm(\tilde{u}^\pm,\tilde{p}^\pm) - \omega\tilde{u}^\pm &= \tilde{f}^\pm & \text{in } (0,+\infty)\times\Omega^\pm, \\ \nabla\cdot\tilde{u}^\pm &= 0 & \text{in } (0,+\infty)\times\Omega^\pm, \\ \tilde{u} &= \displaystyle\sum_{i=1}^{n_c} g_i w_i & \text{on } (0,+\infty)\times\Gamma_d, \\ \tilde{u}^+ &= \tilde{u}^- & \text{on } (0,+\infty)\times\Gamma, \\ [\mathbb{S}(\tilde{u},\tilde{p})n_\Gamma] + (\sigma\Delta h)n_\Gamma &= 0 & \text{on } (0,+\infty)\times\Gamma, \\ \partial_t h - \tilde{u}_d - \omega h &= 0 & \text{on } (0,+\infty)\times\Gamma, \\ \mathbf{g}' + \Lambda\mathbf{g} - \omega\mathbf{g} &= K(P\tilde{u},h,\mathbf{g}) + K(P\check{u},h,\mathbf{g}), & \\ \tilde{u}(0) &= \tilde{u}_0 & \text{in } \Omega, \\ h(0) &= h_0 & \text{on } \Gamma, \\ \mathbf{g}(0) &= 0. & \end{cases}$$

where

$$\tilde{f}^\pm = f^\pm - \partial_t\check{u}^\pm + \frac{1}{\rho^\pm}\nabla\cdot\mathbb{S}^\pm(\check{u}^\pm,\check{p}^\pm) + \omega\check{u}^\pm \in L^2((0,+\infty)\times\Omega^\pm), \quad \tilde{u}_0 = u_0 - \check{u}(0) \in V^1_{\Gamma_d}(\Omega).$$

With Proposition (3.5 .1), this system is equivalent to

$$\begin{cases} z' &= (A_e + \omega I + BK)z + BK(P\check{u},0,0) + (P\tilde{f},0,0), \\ (I-P)u &= (I-P)D_u\mathbf{g}, \\ z(0) &= z_0, \end{cases} \tag{3.5 .6}$$

where $z = (P\tilde{u},h,\mathbf{g})$ and $z_0 = (P\tilde{u}_0,h_0,0)$. Using [10, Part II, Chapter 1, Theorem 3.1 (i), page 143], we know that there exists a unique solution $z \in H^1(0,+\infty;Z_e)\cap L^2(0,+\infty;D(A_e+\omega I+BK;Z_e))$ to $(3.5\ .6)_{1,3}$. Furthermore, one has

$$\|z\|_{H^1(0,+\infty;Z_e)\cap L^2(0,+\infty;D(A_e+\omega I+BK;Z_e))} \le C\|(f,f_{jump},u_0,h_0)\|_{E^0_{f,\infty}\times E^0_{f_{jump},\infty}\times E^0_{u_0}\times E^0_{h_0}}.$$

We deduce that $\mathbf{g}\in E^0_{\mathbf{g},\infty}$ so that $D_u\mathbf{g}\in H^1(0,+\infty;U(\Omega))$ with

$$\|D_u\mathbf{g}\|_{H^1(0,+\infty;U(\Omega))} \le C\|\mathbf{g}\|_{E^0_{\mathbf{g},\infty}}.$$

We deduce that $(I-P)D_u\mathbf{g}\in H^1(0,+\infty;U(\Omega))$. Combining the last estimate and $(3.5\ .6)_2$, one has

$$\|(I-P)u\|_{H^1(0,+\infty;U(\Omega))} \le C\|\mathbf{g}\|_{E^0_{\mathbf{g},\infty}}.$$

Lastly, the estimate $[p]\in H^{1/2,1/4}((0,+\infty)\times\Gamma)$ is obtained with $(3.5\ .5)_5$, and the estimate $h\in H^{7/4}(0,+\infty;L^2_m(\Gamma))$ with $(3.5\ .5)_6$. Therefore, $(u,p,h,\mathbf{g})\in E^0_{u,\infty}\times E^0_{p,\infty}\times E^0_{h,\infty}\times E^0_{\mathbf{g},\infty}$.

The uniqueness in the class of functions $E^0_{u,T}\times E^0_{p,T}\times E^0_{H,T}\times E^0_{\mathbf{g},T}$ for all $T>0$ is a consequence of [10, Part II, Chapter 1, Theorem 3.1 (ii), page 43]. □

Remark 3.5 .6. *The compatibility conditions in Theorem 3.5 .5 are hidden in the space and the particular choice that $\mathbf{g}(0)=0$. But if we had taken $(u_0,\mathbf{g}_0)\in L^2(\Omega)\times\mathbb{R}^{n_c}$ with $u_0^\pm\in H^1(\Omega^\pm)$, the compatibility conditions would have read*

$$\begin{cases} \nabla\cdot u_0^\pm &= 0 & \text{on } \Omega^\pm, \\ u_0^+ &= u_0^- & \text{on } \Gamma, \\ u_0 &= \displaystyle\sum_{i=1}^{n_c} g_{0,i}w_i & \text{on } \Gamma_d. \end{cases}$$

The following lifting result is used in the above proof.

Lemma 3.5 .7. *Let $f_{jump} \in E^0_{f_{jump},\infty}$. There exists $(v,q) \in E^0_{u,\infty} \times E^0_{p,\infty}$ such that*

$$\begin{cases} \nabla\cdot v^\pm &= 0 & \text{in } (0,+\infty)\times\Omega^\pm, \\ v &= 0 & \text{on } (0,+\infty)\times\Gamma_d, \\ v^+ &= v^- & \text{on } (0,+\infty)\times\Gamma, \\ [\mathbb{S}(v,q)n_\Gamma] &= f_{jump} & \text{on } (0,+\infty)\times\Gamma. \end{cases}$$

Moreover we have

$$\|(v,q)\|_{E^0_{u,\infty}\times E^0_{p,\infty}} \leq C\|f_{jump}\|_{E^0_{f_{jump},\infty}}.$$

Proof of Lemma 3.5 .7. The proof follows the same lines as in *Step 2* of in the proof of Lemma 3.3 .17 except that we lift φ in both the time variable and the space variable with [32, Chapter 4, Theorem 2.3, page 18].

□

3.6 Higher regularity estimates for the closed loop extended linear system (3.5 .5)

The notations used through this section are defined in Section 3.2 .5.2.

So as to be in the settings of the non linear problem of Theorem 3.2 .1, we have to enlarge the regularity of Theorem 3.2 .6. The purpose of this section is to prove the following Theorem:

Theorem 3.6 .1. *Let $(f, f_{jump}, u_0, h_0) \in E^\ell_{f,\infty} \times E^\ell_{f_{jump},\infty} \times E^\ell_{u_0} \times E^\ell_{h_0}$ where ℓ is defined in (3.2 .6). In dimension $d = 3$, we further assume the following compatibility condition*

$$[\mu D(u_0)n_\Gamma]_\tau = f_{jump,\tau}(0) \quad \text{on } \Gamma, \tag{3.6 .1}$$

where the subscript τ stands for the tangential component along Γ. Then there exists a unique solution $(u,p,h,\mathbf{g}) \in E^\ell_{u,\infty} \times E^\ell_{p,\infty} \times E^\ell_{h,\infty} \times E^\ell_{\mathbf{g},\infty}$ to system (3.5 .5) such that

$$\|(u,p,h,\mathbf{g})\|_{E^\ell_{u,\infty}\times E^\ell_{p,\infty}\times E^\ell_{h,\infty}\times E^\ell_{\mathbf{g},\infty}} \leq C\|(f,f_{jump},u_0,h_0))\|_{E^\ell_{f,\infty}\times E^\ell_{f_{jump},\infty}\times E^\ell_{u_0}\times E^\ell_{h_0}}.$$

To reach such regularity, we proceed by interpolation working with higher integer regularity, namely with the spaces of Section 3.2 .5.2 with $r = 2$, which corresponds to $u^\pm \in H^{4,2}((0,+\infty)\times\Omega^\pm)$. We use a classical strategy which consists in differentiating system (3.5 .5) in time and applying the first regularity result of Theorem 3.5 .5 on the differentiated system. The first delicate point is to introduce proper compatibility conditions to apply the mentioned strategy. Another subtlety raises in the gain of space regularity as will be specified in the proof of Theorem 3.6 .3.

The Section is divided as follow: in Section 3.6 .1, we introduce proper compatibility conditions; in Section 3.6 .2, we prove a higher integer regularity estimate; lastly, in Section 3.6 .3, we prove Theorem 3.6 .1.

3.6 .1 Compatibility conditions

The first problem to introduce accurate compatibility condition for system (3.5 .5) in higher regularity settings is to compute our initial pressure p_0.

Lemma 3.6 .2. *Let* $(u_0, h_0, f_0^\pm, g_0, f_{jump,0}) \in E^2_{u_0} \times E^2_{h_0} \times H^1(\Omega^\pm) \times H^{1/2}(\Gamma_d) \times H^{3/2}(\Gamma)$. *Then system*

$$\left\{\begin{array}{rcll} \Delta p_0^\pm &=& \rho^\pm \nabla\cdot f_0^\pm & \text{in } \Omega^\pm, \\ \partial_n p_0 &=& \left(\mu\nabla\cdot D(u_0) + \rho(f_0 - \sum_{i=1}^{n_c} K(u_0,h_0,0)_i w_i)\right)\cdot n & \text{on } \Gamma_d, \\ [p_0] &=& \sigma\Delta h_0 + ([\mu D(u_0)n_\Gamma] - f_{jump,0})\cdot n_\Gamma & \text{on } \Gamma, \\ \left[\frac{1}{\rho}\partial_n p_0\right] &=& \left[f_0 + \frac{\mu}{\rho}\nabla\cdot D(u_0)\right]\cdot n_\Gamma & \text{on } \Gamma, \end{array}\right. \tag{3.6 .2}$$

admits a unique solution $p_0 \in P^2_0(\Omega)$ *where*

$$P^2_0(\Omega) = \{p_0 \in L^2_m(\Omega) \mid p_0^\pm \in H^2(\Omega^\pm)\}.$$

Furthermore, we have the estimate

$$\|p_0\|_{P^2_0(\Omega)} \le C\|(u_0, h_0, f_0^\pm, g_0, f_{jump,0})\|_{E^2_{u_0}\times E^2_{h_0}\times H^1(\Omega^\pm)\times H^{1/2}(\Gamma_d)\times H^{3/2}(\Gamma)}.$$

Lastly, defining

$$v_0 = \frac{1}{\rho}(\nabla\cdot\mathbb{S}(u_0,p_0) + ((\sigma\Delta h_0)n_\Gamma - f_{jump,0})\delta_\Gamma) + \omega u_0 + f_0,$$

we have

$$v_0 \in H^1(\Omega) \quad \text{and} \quad v_0 = \sum_{i=1}^{n_c} K(u_0,h_0,0)_i w_i \ \text{ on } \Gamma_d, \tag{3.6 .3}$$

if and only if

$$\left\{\begin{array}{rcll} [\mu D(u_0)n_\Gamma]_\tau &=& f_{jump,0,\tau} & \text{on } \Gamma, \\ \left[\frac{1}{\rho}\nabla\cdot\mathbb{S}(u_0,p_0) + f_0\right]_\tau &=& 0 & \text{on } \Gamma, \\ \left(\frac{1}{\rho}\nabla\cdot\mathbb{S}(u_0,p_0) + f_0\right)_\tau &=& \sum_{i=1}^{n_c} K(u_0,h_0,0)_i w_{i,\tau} & \text{on } \Gamma_d, \end{array}\right. \tag{3.6 .4}$$

where the subscript τ *stands for the component of the tangential space of* Γ *or* Γ_d.

Proof of Lemma 3.6 .2. The proof is done in two steps: in the first step, we prove existence and uniqueness of a solution $p_0 \in P^2_0(\Omega)$ of system (3.6 .2), and in the second step, we prove the equivalence between (3.6 .3) and (3.6 .4).

Step 1: We focus on existence and uniqueness of a solution $p_0 \in P^2_0(\Omega)$ of system (3.6 .2). First, we derive a variational formulation for the following system

$$\left\{\begin{array}{rcll} \Delta\tilde{p}_0^\pm &=& \rho^\pm\nabla\cdot\tilde{f}_0^\pm & \text{in } \Omega^\pm, \\ \partial_n\tilde{p}_0 &=& 0 & \text{on } \Gamma_d, \\ [\tilde{p}_0] &=& 0 & \text{on } \Gamma, \\ \left[\frac{1}{\rho}\partial_n\tilde{p}_0\right] &=& 0 & \text{on } \Gamma, \end{array}\right. \tag{3.6 .5}$$

with

$$\tilde{f}_0 \in \left\{\tilde{f}_0 \in L^2(\Omega) \mid \tilde{f}_0^\pm \in H^1(\Omega^\pm),\ \int_{\Gamma_d}\tilde{f}_0\cdot n + \int_\Gamma [\tilde{f}_0\cdot n_\Gamma] = 0\right\}. \tag{3.6 .6}$$

Multiplying (3.6 .5)$_1$ by $\varphi \in H^1(\Omega) \cap L^2_m(\Omega)$ and integrating by parts, one has

$$\sum_{\pm} \frac{1}{\rho^\pm}(\nabla \tilde{p}_0^\pm, \nabla \varphi^\pm)_{\Omega^\pm} = (\tilde{f}_0, \nabla\varphi)_\Omega - (\tilde{f}_0 \cdot n, \varphi)_{\Gamma_d} + ([\tilde{f}_0 \cdot n_\Gamma], \varphi)_\Gamma. \tag{3.6 .7}$$

Remark that the variational formulation (3.6 .7) fulfills the assumptions of Lax Milgram theorem on $H^1(\Omega) \cap L^2_m(\Omega)$ with

$$\tilde{f}_0 \in \left\{ \tilde{f}_0 \in L^2(\Omega) \;\middle|\; \begin{array}{l} \tilde{f}_0 \cdot n \in H^{-1/2}(\Gamma_d), \\ [\tilde{f}_0 \cdot n_\Gamma] \in H^{-1/2}(\Gamma_d), \\ \langle [\tilde{f}_0 \cdot n_\Gamma], 1\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} - \langle \tilde{f}_0 \cdot n, 1\rangle_{H^{-1/2}(\Gamma_d), H^{1/2}(\Gamma_d)} = 0 \end{array} \right\}. \tag{3.6 .8}$$

Therefore, for every $\tilde{f}_0$ in the space (3.6 .8), there exists a unique solution $\tilde{p}_0 \in H^1(\Omega) \cap L^2_m(\Omega)$ of (3.6 .7). If $\tilde{f}_0$ is in the space (3.6 .6), one can apply a translation method to obtain $\partial_i \tilde{p}_0 \in H^1(\Omega) \cap L^2_m(\Omega)$ where $i = 1$ if $d = 2$ and $i \in \{1, 2\}$ for $d = 3$. From the variational formulation (3.6 .7), one has $\Delta p^\pm = \nabla \cdot \tilde{f}_0^\pm$ in the distribution sense. Since $\nabla \cdot \tilde{f}_0^\pm \in L^2(\Omega^\pm)$, we deduce $\partial_{dd} \tilde{p}_0^\pm \in L^2(\Omega^\pm)$ and then $\tilde{p}_0^\pm \in H^2(\Omega^\pm)$. Given this regularity, one can easily prove that the variational formulation (3.6 .7) is equivalent to system (3.6 .5). Therefore, for every $\tilde{f}_0$ given in the space (3.6 .6), there exists a unique solution $\tilde{p}_0 \in P^2_0(\Omega)$ of (3.6 .5).

To deal with system (3.6 .2), we first introduce a lifting $\check{p}_0 \in P^2_0(\Omega)$ such that

$$\left\{ \begin{array}{rcll} \partial_n \check{p}_0 & = & \left(\mu\nabla \cdot D(u_0) + \rho(f_0 - \displaystyle\sum_{i=1}^{n_c} K(u_0, h_0, 0)_i w_i)\right) \cdot n & \text{on } \Gamma_d, \\ [\check{p}_0] & = & \sigma\Delta h_0 + ([\mu D(u_0) n_\Gamma] - f_{jump,0}) \cdot n_\Gamma & \text{on } \Gamma, \\ \left[\dfrac{1}{\rho}\partial_n \check{p}_0\right] & = & \left[f_0 + \dfrac{\mu}{\rho}\nabla \cdot D(u_0)\right] \cdot n_\Gamma & \text{on } \Gamma. \end{array} \right.$$

Then, $\tilde{p}_0 = p_0 - \check{p}_0$ satisfies (3.6 .5) where $\tilde{f}_0$ defined by $\tilde{f}_0{}^\pm = f_0^\pm - \dfrac{1}{\rho^\pm}\nabla \check{p}_0^\pm$, lies in the space (3.6 .6). We have seen that there exists a unique solution $\tilde{p}_0 \in P^2_0(\Omega)$ of (3.6 .5). This concludes *Step 1.*

Step 2: We are now interested in the equivalence between (3.6 .3) and (3.6 .4). First, one has $v_0 \in H^{-1}(\Omega)$ with $v_0^\pm \in H^1(\Omega^\pm)$ with $\nabla \cdot v_0^\pm = 0$ on $\Omega^\pm$.

Suppose (3.6 .4). We deduce (3.6 .3)$_2$ from (3.6 .4)$_3$ and (3.6 .2)$_2$. With (3.6 .4)$_1$ and (3.6 .2)$_3$, we have $[\mathbb{S}(u_0, p_0) n_\Gamma] + (\sigma\Delta h_0) n_\Gamma = f_{jump,0}$ on Γ. Therefore $\nabla \cdot \mathbb{S}(u_0, p_0) + ((\sigma\Delta h_0) n_\Gamma - f_{jump,0})\delta_\Gamma \in L^2(\Omega)$ and then $v_0 \in L^2(\Omega)$. With (3.6 .4)$_2$ and (3.6 .2)$_4$, we deduce that $\left[\dfrac{1}{\rho}\nabla \cdot \mathbb{S}(u_0, p_0) + f_0 + \omega u_0\right] = 0$ on Γ. Therefore, $v_0^+ = v_0^-$ on Γ and then $v_0 \in H^1(\Omega)$. We thus have proved that (3.6 .4) implies (3.6 .3).

The proof of the inverse statement uses exactly the same argument.

□

3.6 .2 Higher integer regularity: $(u, p, h, \mathbf{g}) \in E^2_{u,\infty} \times E^2_{p,\infty} \times E^2_{h,\infty} \times E^2_{\mathbf{g},\infty}$.

We can now define properly the compatibility conditions to state the following regularity result for the closed loop system (3.5 .5):

Theorem 3.6 .3. *Let* $(f, f_{jump}, u_0, h_0) \in E^2_{f,\infty} \times E^2_{f_{jump},\infty} \times E^2_{u_0} \times E^2_{h_0}$ *and* $p_0 \in P^2_0(\Omega)$ *the solution of* (3.6 .2) *with* $(f_0, f_{jump,0}) = (f, f_{jump})(0)$. *We assume the following compatibility conditions*

$$\begin{cases} [\mu D(u_0)n_\Gamma]_\tau = f_{jump,\tau}(0) & \text{on } \Gamma, \\ \left[\frac{1}{\rho}\nabla\cdot\mathbb{S}(u_0,p_0) + f(0)\right]_\tau = 0 & \text{on } \Gamma, \\ \left(\frac{1}{\rho}\nabla\cdot\mathbb{S}(u_0,p_0) + f(0)\right)_\tau = \sum_{i=1}^{n_c} K(u_0,h_0,0)w_{i,\tau} & \text{on } \Gamma_d. \end{cases}$$

Then the unique solution $(u, p, h, \mathbf{g})$ *to system* (3.5 .5) *given by Theorem 3.5 .5 lies in* $E^2_{u,\infty} \times E^2_{p,\infty} \times E^2_{h,\infty} \times E^2_{\mathbf{g},\infty}$ *with*

$$\|(u,p,h,\mathbf{g})\|_{E^2_{u,\infty}\times E^2_{p,\infty}\times E^2_{h,\infty}\times E^2_{\mathbf{g},\infty}} \le C\|(f,f_{jump},u_0,h_0)\|_{E^2_{f,\infty}\times E^2_{f_{jump},\infty}\times E^2_{u_0}\times E^2_{h_0}}.$$

Proof of Theorem 3.6 .3. As explained at the beginning of the section, the strategy is to differentiate (3.5 .5) in time and apply the regularity result of Theorem 3.5 .5 to the differentiated system. To gain space regularity, a classical strategy consists in applying the stationary regularity result of Proposition 3.3 .7 regarding $\partial_t(u,h)$ as a non homogeneous term. However, in our case, the gain of regularity between Z and $D(A)$ is 2 for the velocity field and 1 for the height. Therefore, the regularity result given by Theorem 3.5 .5 applied to the differentiated system gives a spatial regularity for $\partial_t(u,h)$ which is not optimal to apply stationary regularity result of Proposition 3.3 .7 (precisely, $\partial_t(u^\pm,h) \in L^2(0,+\infty; H^2(\Omega^\pm)) \times L^2(0,+\infty; H^{5/2}_m(\Gamma))$). To be in a suitable case, we need to gain one more space regularity for $\partial_t h$ (that is $\partial_t h \in L^2(0,+\infty; H^{7/2}_m(\Gamma))$). To get this desired regularity, our strategy is to lift the non homogeneous term $\left(f_{jump}, \sum_{i=1}^{n_c} g_i w_i\right)$, to differentiate the obtained system two times in the tangential direction and to use [10, Part II, Chapter 1, Theorem 3.1 (i), page 143].

To make this strategy rigorous, we divide the proof in four steps. In the first step, we gain regularity in time by formally differentiating (3.5 .5) in time. In the second step, we lift the nonhomogeneous terms $\left(f_{jump}, \sum_{i=1}^{n_c} g_i w_i\right)$. In the third step, we differentiate (3.5 .5) two times in the tangential direction of Γ and we apply [10, Part II, Chapter 1, Theorem 3.1 (i), page 143] to get $\partial_t h$ in the desired space. In the fourth step, we use the stationary regularity in Proposition 3.3 .7 to conclude.

Step 1: First, we formally differentiate (3.5 .5) in time:

$$\begin{cases} \partial_t v^\pm - \frac{1}{\rho^\pm}\nabla\cdot\mathbb{S}^\pm(v^\pm,q^\pm) - \omega v^\pm = \partial_t f^\pm & \text{in } (0,+\infty)\times\Omega^\pm, \\ \nabla\cdot v^\pm = 0 & \text{in } (0,+\infty)\times\Omega^\pm, \\ v = \sum_{i=1}^{n_c} \tilde{g}_i w_i & \text{on } (0,+\infty)\times\Gamma_d, \\ v^+ = v^- & \text{on } (0,+\infty)\times\Gamma, \\ [\mathbb{S}(v,q)n_\Gamma] + (\sigma\Delta k)n_\Gamma = \partial_t f_{jump} & \text{on } (0,+\infty)\times\Gamma, \\ \partial_t k - v_d - \omega k = 0 & \text{on } (0,+\infty)\times\Gamma, \\ \tilde{\mathbf{g}}' + \Lambda\tilde{\mathbf{g}} - \omega\tilde{\mathbf{g}} = K(Pv,k,\tilde{\mathbf{g}}), & \\ v(0) = v_0 & \text{in } \Omega, \\ k(0) = u_{0,d} + \omega h_0 & \text{on } \Gamma, \\ \tilde{\mathbf{g}}(0) = K(u_0,h_0,0), & \end{cases} \tag{3.6 .9}$$

where $u_{0,d} + \omega h_0 \in H^{5/2}_m(\Gamma)$ and

$$v_0 = \frac{1}{\rho}(\nabla \cdot \mathbb{S}(u_0, p_0) + ((\sigma\Delta h_0)n_\Gamma - f_{jump}(0))\delta_\Gamma) + \omega u_0 + f(0).$$

With Lemma 3.6 .2, we have that $v_0 \in H^1(\Omega)$ with $\nabla \cdot v_0 = 0$ satisfying the compatibility condition

$$v_0 = \sum_{i=1}^{n_c} K(u_0, h_0, 0)_i w_i.$$

With Remark 3.5 .6, Theorem 3.5 .5 applies: there exists a solution $(v, q, k, \tilde{\mathbf{g}}) \in E^0_{u,\infty} \times E^0_{p,\infty} \times E^0_{h,\infty} \times E^0_{\mathbf{g},\infty}$ to system (3.6 .9). Furthermore, we have the estimate

$$\|(v, q, k, \tilde{\mathbf{g}})\|_{E^0_{u,\infty}\times E^0_{p,\infty}\times E^0_{h,\infty}\times E^0_{\mathbf{g},\infty}} \leq C\|(f, f_{jump}, u_0, h_0)\|_{E^2_{f,\infty}\times E^2_{f_{jump},\infty}\times E^2_{u_0}\times E^2_{h_0}}.$$

Define

$$(u, p, h, \mathbf{g})(t) = \int_0^t (v, q, h, \tilde{\mathbf{g}}) + (u_0, p_0, h_0, 0).$$

Then $(u, p, h, \mathbf{g}) \in E^0_{u,T} \times E^0_{p,T} \times E^0_{h,T} \times E^0_{\mathbf{g},T}$ for all $T > 0$ and is solution of (3.5 .5). With Theorem 3.5 .5, we have $(u, p, h, \mathbf{g}) \in E^0_{u,\infty} \times E^0_{p,\infty} \times E^0_{h,\infty} \times E^0_{\mathbf{g},\infty}$. For the moment, we have the following regularity

$$\begin{aligned}
u^\pm &\in H^2(0, +\infty; L^2(\Omega^\pm)) \cap H^1(0, +\infty; H^2(\Omega^\pm)),\\
p^\pm &\in H^1(0, +\infty; H^1(\Omega^\pm)) \quad \text{with} \quad [p] \in H^{5/4}(0, +\infty; L^2(\Gamma)) \cap H^1(0, +\infty; H^{1/2}(\Gamma)),\\
h &\in H^{11/4}(0, +\infty; L^2_m(\Gamma)) \cap H^2(0, +\infty; H^{3/2}_m(\Gamma)) \cap H^1(0, +\infty; H^{5/2}_m(\Gamma)),\\
\mathbf{g} &\in H^2(0, +\infty; \mathbb{R}^{n_c}).
\end{aligned}$$

In particular, $\partial_t u^\pm \in L^2(0, +\infty; H^2(\Omega^\pm))$ and $\partial_t h \in L^2(0, +\infty; H^{5/2}_m(\Gamma))$, which is not an optimal regularity to apply the stationary estimates of Proposition 3.3 .7 as said earlier. From now on, the purpose is to obtain $\partial_t h \in L^2(0, +\infty; H^{7/2}_m(\Gamma))$ to be in a suitable case.

Step 2: In this step, we lift the terms $\left(f_{jump}, \sum_{i=1}^{n_c} g_i w_i\right) \in E^2_{f_{jump},\infty} \times H^2(0, +\infty; C^\infty(\Gamma_d))$. Since $\int_{\Gamma_d^\pm} w_i^\pm \cdot n = 0$, one can apply Lemma 3.6 .4 below: there exists $(\check{u}, \check{p}) \in E^2_{u,\infty} \times E^2_{p,\infty}$ such that

$$\left\{\begin{array}{rcll}
\nabla \cdot \check{u}^\pm &=& 0 & \text{in } (0, +\infty) \times \Omega^\pm,\\
\check{u} &=& \sum_{i=1}^{n_c} g_i w_i & \text{on } (0, +\infty) \times \Gamma_d,\\
\check{u}^+ &=& \check{u}^- & \text{on } (0, +\infty) \times \Gamma,\\
[\mathbb{S}(\check{u}, \check{p})n_\Gamma] &=& f_{jump} & \text{on } (0, +\infty) \times \Gamma,\\
\check{u}_d &=& 0 & \text{on } (0, +\infty) \times \Gamma,
\end{array}\right.$$

with

$$\|(\check{u}, \check{p})\|_{E^2_{u,\infty}, E^2_{p,\infty}} \leq C\|(f, f_{jump}, u_0, h_0)\|_{E^2_{f,\infty}\times E^2_{f_{jump},\infty}\times E^2_{u_0}\times E^2_{h_0}}.$$

We now consider the new unkown $(\hat{u}, \hat{p}, h) = (u, p, h) - (\check{u}, \check{p}, 0)$ which satisfies the system

$$\left\{\begin{array}{rcll} \partial_t \hat{u}^\pm - \frac{1}{\rho^\pm}\nabla\cdot\mathbb{S}^\pm(\hat{u}^\pm, \hat{p}^\pm) - \omega\hat{u}^\pm &=& \hat{f}^\pm & \text{in } (0,+\infty)\times\Omega^\pm,\\ \nabla\cdot\hat{u}^\pm &=& 0 & \text{in } (0,+\infty)\times\Omega^\pm,\\ \hat{u} &=& 0 & \text{on } (0,+\infty)\times\Gamma_d,\\ \hat{u}^+ &=& \hat{u}^- & \text{on } (0,+\infty)\times\Gamma,\\ [\mathbb{S}(\hat{u},\hat{p})n_\Gamma] + (\sigma\Delta h)n_\Gamma &=& 0 & \text{on } (0,+\infty)\times\Gamma,\\ \partial_t h - \hat{u}_d - \omega h &=& 0 & \text{on } (0,+\infty)\times\Gamma,\\ \hat{u}(0) &=& \hat{u}_0 & \text{in } \Omega,\\ h(0) &=& h_0 & \text{on } \Gamma, \end{array}\right. \tag{3.6 .10}$$

where

$$\hat{f}^\pm = f^\pm - \partial_t\check{u}^\pm + \frac{1}{\rho^\pm}\nabla\cdot\mathbb{S}^\pm(\check{u}^\pm,\check{p}^\pm) + \omega\check{u}^\pm \in H^{2,1}((0,+\infty)\times\Omega^\pm), \quad \hat{u}_0 = u_0 - \hat{u}(0) \in E^2_{u_0}(\Omega).$$

Applying the stationary regularity of Proposition 3.3 .7 with $s=1$, we deduce that $(\hat{u}^\pm, \hat{p}^\pm, h) \in L^2(0,+\infty; H^3(\Omega^\pm)) \times L^2(0,+\infty; H^2(\Omega^\pm)) \times L^2(0,+\infty; H^{7/2}_m(\Gamma))$ with

$$\begin{aligned} &\|(\hat{u}^\pm,\hat{p}^\pm,h)\|_{L^2(0,+\infty;H^3(\Omega^\pm))\times L^2(0,+\infty;H^2(\Omega^\pm))\times L^2(0,+\infty;H^{7/2}_m(\Gamma))}\\ &\qquad\qquad\qquad\qquad \le C\|(f, f_{jump}, u_0, h_0)\|_{E^2_{f,\infty}\times E^2_{f_{jump},\infty}\times E^2_{u_0}\times E^2_{h_0}}. \end{aligned} \tag{3.6 .11}$$

So far, we have the regularity results:

$$\begin{array}{r} \hat{u}^\pm \in H^2(0,+\infty; L^2(\Omega^\pm)) \cap H^1(0,+\infty; H^2(\Omega^\pm)) \cap L^2(0,+\infty; H^3(\Omega^\pm)),\\ \hat{p}^\pm \in H^1(0,+\infty; H^1(\Omega^\pm)) \cap L^2(0,+\infty; H^2(\Omega^\pm)),\\ [\hat{p}] \in H^{5/4}(0,+\infty; L^2(\Gamma)) \cap H^1(0,+\infty; H^{1/2}(\Gamma)) \cap L^2(0,+\infty; H^{3/2}(\Omega^\pm)),\\ \hat{h} \in H^{11/4}(0,+\infty; L^2_m(\Gamma)) \cap H^1(0,+\infty; H^{5/2}_m(\Gamma)) \cap L^2(0,+\infty; H^{7/2}_m(\Gamma)),\\ \mathbf{g} \in H^2(0,+\infty;\mathbb{R}^{n_c}). \end{array} \tag{3.6 .12}$$

Step 3: The purpose of this step is to get the estimate $h \in H^1(0,+\infty; H^{7/2}_m(\Gamma))$. We differentiate system (3.6 .10) two times in the tangential direction of Γ. Let $i,j = 1$ for $d=2$ or $i,j \in \{1,2\}$ for $d=3$ and apply ∂_{ij} to system (3.6 .10):

$$\left\{\begin{array}{rcll} \partial_t\partial_{ij}\hat{u}^\pm - \frac{1}{\rho^\pm}\nabla\cdot\mathbb{S}^\pm(\partial_{ij}\hat{u}^\pm, \partial_{ij}\hat{p}^\pm) + \partial_{ij}\hat{u}^\pm &=& \partial_{ij}\tilde{f}^\pm + (1+\omega)\partial_{ij}\hat{u}^\pm & \text{in } (0,+\infty)\times\Omega^\pm,\\ \nabla\cdot\partial_{ij}\hat{u}^\pm &=& 0 & \text{in } (0,+\infty)\times\Omega^\pm,\\ \partial_{ij}\hat{u} &=& 0 & \text{on } (0,+\infty)\times\Gamma_d,\\ \partial_{ij}\hat{u}^+ &=& \partial_{ij}\hat{u}^- & \text{on } (0,+\infty)\times\Gamma,\\ [\mathbb{S}(\partial_{ij}\hat{u},\partial_{ij}\hat{p})n_\Gamma] + (\sigma\Delta\partial_{ij}h)n_\Gamma &=& 0 & \text{on } (0,+\infty)\times\Gamma,\\ \partial_t\partial_{ij}h - \partial_{ij}\hat{u}_d + \partial_{ij}h &=& (1+\omega)\partial_{ij}h & \text{on } (0,+\infty)\times\Gamma,\\ \partial_{ij}\hat{u}(0) &=& \partial_{ij}\hat{u}_0 & \text{in } \Omega,\\ \partial_{ij}h(0) &=& \partial_{ij}h_0 & \text{on } \Gamma, \end{array}\right.$$

With the regularity (3.6 .12), we have $\partial_{ij}(\tilde{u}, h) \in L^2(0,\infty; D(A;Z'))$. Therefore, with Proposition 3.3 .14, this system is equivalent to

$$\left\{\begin{array}{rcl} z'_{ij} &=& (A+I)z_{ij} + (\partial_{ij}\tilde{f}^\pm + (1+\omega)\partial_{ij}\tilde{u}^\pm, (1+\omega)\partial_{ij}h),\\ z_{ij}(0) &=& z_{ij,0}, \end{array}\right. \tag{3.6 .13}$$

where $z_{ij} = \partial_{ij}(\hat{u}, h)$, $z_{ij,0} = \partial_{ij}(\hat{u}_0, h_0)$ and the operator A is considered on its domain $D(A; Z')$. But one can look at system (3.6 .13) in Z since $(\partial_{ij}\tilde{f} + (1+\omega)\partial_{ij}\hat{u}, (1+\omega)\partial_{ij}h) \in L^2(0,+\infty; Z)$ with the regularity (3.6 .12) and consider the operator $(A, D(A; Z))$ instead. Using [10, Part II, Chapter 1, Theorem 3.1 (i), page 143], $z_{ij} \in H^1(0,+\infty; Z) \cap L^2(0,+\infty; D(A; Z))$. In particular, $\partial_t\partial_{ij}h \in H^1(0,+\infty; H^{3/2}_m(\Gamma))$ and therefore $\partial_t h \in H^1(0,+\infty; H^{7/2}_m(\Gamma))$.

At the end of this step, we have the regularity results:

$$
\begin{gathered}
\hat{u}^\pm \in H^2(0,+\infty; L^2(\Omega^\pm)) \cap H^1(0,+\infty; H^2(\Omega^\pm)) \cap L^2(0,+\infty; H^3(\Omega^\pm)),\\
\hat{p}^\pm \in H^1(0,+\infty; H^1(\Omega^\pm)) \cap L^2(0,+\infty; H^2(\Omega^\pm)),\\
[\hat{p}] \in H^{5/4}(0,+\infty; L^2(\Gamma)) \cap H^1(0,+\infty; H^{1/2}(\Gamma)) \cap L^2(0,+\infty; H^{3/2}(\Omega^\pm)),\\
\hat{h} \in H^{11/4}(0,+\infty; L^2_m(\Gamma)) \cap H^1(0,+\infty; H^{7/2}_m(\Gamma)),\\
\mathbf{g} \in H^2(0,+\infty; \mathbb{R}^{n_c}),
\end{gathered}
$$

with a slightly better estimate for $\hat{u}$ that is not needed in the last argument.

<u>*Step 4:*</u> Now that we have $(\partial_t\hat{u}^\pm, \partial_t h) \in L^2(0,+\infty; H^2(\Omega^\pm)) \times L^2(0,+\infty; H^{7/2}_m(\Gamma))$, we can apply the stationary regularity result of Proposition 3.3 .7 to system (3.6 .10) with $s = 2$ and get $(\hat{u}^\pm, \hat{p}^\pm, h) \in L^2(0,+\infty; H^4(\Omega^\pm)) \times L^2(0,+\infty; H^3(\Omega^\pm)) \times L^2(0,+\infty; H^{9/2}_m(\Gamma))$ with the estimate

$$
\begin{aligned}
&\|(\hat{u}^\pm, \hat{p}^\pm, h)\|_{L^2(0,+\infty;H^4(\Omega^\pm))\times L^2(0,+\infty:H^3(\Omega^\pm))\times L^2(0,+\infty;H^{9/2}_m(\Gamma))}\\
&\qquad\qquad\qquad\qquad \le C\|(f, f_{jump}, u_0, h_0)\|_{E^2_{f,\infty}\times E^2_{f_{jump},\infty}\times E^2_{u_0}\times E^2_{h_0}}.
\end{aligned}
$$

The last estimate $[\hat{p}] \in L^2(0,+\infty; H^{5/2}_m(\Gamma))$ can be read on the equation $(3.5\ .5)_5$. □

The following lifting result has been used in the above proof.

Lemma 3.6 .4. *Let* $(g, f_{jump}) \in (H^{7/2,7/4}((0,+\infty)\times\Gamma_d) \cap H^2(0,+\infty; H^{-1/2}(\Gamma_d))) \times E^2_{f_{jump},\infty}$. *We assume the compatibility condition:*

$$\int_{\Gamma_d^\pm} g^\pm \cdot n = 0.$$

There exists $(v, q) \in E^2_{u,\infty} \times E^2_{p,\infty}$ *such that*

$$
\begin{cases}
\nabla\cdot v^\pm = 0 & \text{in } (0,+\infty)\times\Omega^\pm,\\
v = g & \text{on } (0,+\infty)\times\Gamma_d,\\
v^+ = v^- & \text{on } (0,+\infty)\times\Gamma,\\
[\mathbb{S}(v,q)n_\Gamma] = f_{jump} & \text{on } (0,+\infty)\times\Gamma,
\end{cases}
$$

Moreover we have

$$\|(v,q)\|_{E^2_{u,\infty}\times E^2_{p,\infty}} \le C\|(g, f_{jump})\|_{(H^{7/2,7/4}((0,+\infty)\times\Gamma_d)\cap H^2(0,+\infty;H^{-1/2}(\Gamma_d)))\times E^2_{f_{jump},\infty}}.$$

Proof of Lemma 3.5 .7. We proceed exactly as in the proof of Lemma 3.3 .17 except that we lift in both the time variable and the space variable with [32, Chapter 4, Theorem 2.3, page 18].

<u>*Step 1:*</u> we lift the normal trace of g. We look for $\tilde{v}^\pm = \nabla\phi^\pm$ and we are interested in the following systems, written separately on Ω^+ and Ω^-:

$$
\begin{cases}
\Delta\phi^\pm(t) = 0 & \text{in } (0,+\infty)\times\Omega^\pm,\\
\partial_n\phi^\pm(t) = g^\pm(t)\cdot n & \text{on } (0,+\infty)\times\Gamma_d^\pm,\\
\partial_n\phi^\pm(t) = 0 & \text{on } (0,+\infty)\times\Gamma.
\end{cases}
$$

For $g(t) \in H^{-1/2}(\Gamma_d)$, there exists a unique solution $\varphi^{\pm}(t) \in H^1(\Omega^{\pm})$. Concerning the space regularity, since $g \in L^2(0,+\infty; H^{7/2}(\Gamma_d))$, $\phi^{\pm} \in L^2(0,+\infty; H^5(\Omega^{\pm}))$ with

$$\|\phi^{\pm}\|_{L^2(0,+\infty;H^5(\Omega^{\pm}))} \leq C\|g\|_{L^2(0,+\infty;H^{7/2}(\Gamma_d))}.$$

For the time regularity, remark that the operator $g(t) \in H^{-1/2}(\Gamma_d) \mapsto \varphi^{\pm}(t) \in H^1(\Omega^{\pm})$ is linear continuous. Since $g \in H^2(0,+\infty; H^{-1/2}(\Gamma_d))$, $\phi^{\pm} \in H^2(0,+\infty; H^1(\Omega^{\pm}))$ with

$$\|\phi^{\pm}\|_{H^2(0,+\infty;H^1(\Omega^{\pm}))} \leq C\|g\|_{H^2(0,+\infty;H^{-1/2}(\Gamma_d))}.$$

<u>*Step 2:*</u> we lift the tangential trace of g and the jump of the stress tensor. We follow the same lines as in the *Step 2* of the proof of Lemma 3.3 .17 and the proof of Lemma 3.5 .7. In (3.3 .20) and (3.3 .21), since $(g, f_{jump}) \in H^{7/2,7/4}((0,+\infty)\times\Gamma_d)\times E^2_{f_{jump},\infty}$, we have a lifting $\varphi^{\pm} \in H^{4,2}((0,+\infty)\times\Omega^{\pm})$ and $q^{\pm} \in H^{3,3/2}((0,+\infty)\times\Omega^{\pm})$ with [32, Chapter 4, Theorem 2.3, page 18]. This is even slightly better than the expected estimate for q since $H^{3,3/2}((0,+\infty)\times\Omega^{\pm}) \hookrightarrow H^1(0,+\infty; H^1(\Omega^{\pm}))$. □

3.6 .3 Proof of Theorem 3.6 .1

Theorem 3.6 .1 is obtained by interpolation using results of Theorems 3.5 .5 and 3.6 .3.

About the compatibility condition: in dimension $d = 2$, as $\ell \in (0, 1/2)$, the compatibility condition (3.6 .1) of Theorem 3.6 .1 disappears and only the ones of Theorem 3.5 .5 remains; in dimension $d = 3$, as $\ell \in (1/2, 1)$, one should keep the compatibility condition (3.6 .1).

3.7 Stabilization of the non linear system

The notations used through this section are defined in Section 3.2 .5.3.

The purpose of this section is to prove Theorem 3.2 .1. This Section is divided as follow: in Section 3.7 .1, we defined properly the transformation $X(t,\cdot)$ and we get estimates on the non linear terms F and H defined in (3.2 .4)–(3.2 .5); in Section 3.7 .2, we prove Theorem 3.2 .1 by mean of Picard fixed point theorem.

3.7 .1 Estimates on the non linear terms

3.7 .1.1 Preliminary Lemma

We need a preliminary technical lemma.

Lemma 3.7 .1. *The space $H^1(0,+\infty; H^{1+\ell}(\Omega)) \cap L^2(0,+\infty; H^{2+\ell}(\Omega))$ is a normed algebra and the space $H^{1+\ell/2}(0,+\infty; H^1(\Omega)) \cap H^1(0,+\infty; H^{1+\ell}(\Omega))$ is a normed algebra.*

Proof of Lemma 3.7 .1. The proof is based upon the following result [23, Proposition B.1, page 67].

<u>*Step 0:*</u> First, notice that $H^1(0,+\infty; H^{1+\ell}(\Omega))$ is a normed algebra.

<u>*Step 1:*</u> We show that $H^1(0,+\infty; H^{1+\ell}(\Omega)) \cap L^2(0,+\infty; H^{2+\ell}(\Omega))$ is a normed algebra.
Let $f, g \in H^1(0,+\infty; H^{1+\ell}(\Omega)) \cap L^2(0,+\infty; H^{2+\ell}(\Omega))$. Since $f \in H^1(0,+\infty; H^{1+\ell}(\Omega))$ and $g \in L^2(0,+\infty; H^{2+\ell}(\Omega))$, we have $fg \in L^2(0,+\infty; H^{1+\ell}(\Omega))$. A direct computation shows that

$$\partial_i(fg) = \partial_i f g + f\partial_i g.$$

Since we have $\partial_i f \in L^2(0,+\infty; H^{1+\ell}(\Omega))$ and $g \in H^1(0,+\infty; H^{1+\ell}(\Omega))$, we deduce that $\partial_i f g \in L^2(0,+\infty; H^{1+\ell}(\Omega))$. This proves that $fg \in L^2(0,+\infty; H^{2+\ell}(\Omega))$ with

$$\|fg\|_{L^2(0,+\infty;H^{2+\ell}(\Omega))} \\ \leq C\|f\|_{H^1(0,+\infty;H^{1+\ell}(\Omega))\cap L^2(0,+\infty;H^{2+\ell}(\Omega))}\|g\|_{H^1(0,+\infty;H^{1+\ell}(\Omega))\cap L^2(0,+\infty;H^{2+\ell}(\Omega))}.$$

Step 2: We show that $H^{1+\ell/2}(0,+\infty; H^1(\Omega)) \cap H^1(0,+\infty; H^{1+\ell}(\Omega))$ is a normed algebra.

Let $f, g \in H^{1+\ell/2}(0,+\infty; H^1(\Omega)) \cap H^1(0,+\infty; H^{1+\ell}(\Omega))$. We have proved that $fg \in H^1(0,+\infty; H^{1+\ell}(\Omega))$. Differentiate in time

$$\partial_t(fg) = \partial_t f g + f\partial_t g.$$

We have $\partial_t f \in H^{\ell/2}(0,+\infty; H^1(\Omega))$ and $g \in H^1(0,+\infty; H^{1+\ell}(\Omega))$. Thus, we deduce that $\partial_t f g \in H^{\ell/2}(0,+\infty; H^1(\Omega))$. This proves that $fg \in H^{1+\ell/2}(0,+\infty; H^1(\Omega))$ with

$$\|fg\|_{H^{1+\ell/2}(0,+\infty;H^1(\Omega))} \\ \leq C\|f\|_{H^{1+\ell/2}(0,+\infty;H^1(\Omega))\cap H^1(0,+\infty;H^{1+\ell}(\Omega))}\|g\|_{H^{1+\ell/2}(0,+\infty;H^1(\Omega))\cap H^1(0,+\infty;H^{1+\ell}(\Omega))}.$$

□

3.7 .1.2 Estimates on the transformations X and Y

We are now able to define properly the transformation X.

Lemma 3.7 .2. *There exists a continuous linear operator*

$$L_e : \begin{array}{ccc} E^\ell_{h,\infty} & \to & H^{1+\ell/2}(0,+\infty; H^2(\Omega)) \cap H^1(0,+\infty; H^{2+\ell}(\Omega)) \cap L^2(0,+\infty; H^{3+\ell}(\Omega)) \\ h & \mapsto & h_e \end{array},$$

where

$$h_e = h \quad on\ \Gamma \quad and \quad h_e = 0 \quad on\ \Gamma_d.$$

Moreover, there exists $R^* > 0$ such that for all $h \in E^\ell_{h,\infty,R^*}$, the transformation

$$X(t,\cdot) : \begin{array}{ccc} \Omega & \to & \Omega \\ (y', y_d) & \mapsto & (y', \mathrm{e}^{-\omega t}\, h_e(t,y',y_d)(1-y_d^2) + y_d) \end{array}, \tag{3.7 .1}$$

has an inverse $Y(t,\cdot)$ for all $t > 0$. Furthermore, we have $X \in H^{1+\ell/2}(0,+\infty; H^2(\Omega)) \cap H^1(0,+\infty; H^{2+\ell}(\Omega)) \cap L^2(0,+\infty; H^{3+\ell}(\Omega))$ and the estimates

$$\begin{aligned}
&\|\nabla Y(X) - I\|_{H^{1+\ell/2}(0,+\infty;H^1(\Omega))\cap H^1(0,+\infty;H^{1+\ell}(\Omega))\cap L^2(0,+\infty;H^{2+\ell}(\Omega))} \leq C\|h\|_{E^\ell_{h,\infty}},\\
&\|\mathrm{cof}(\nabla Y)(X) - I\|_{H^{1+\ell/2}(0,+\infty;H^1(\Omega))\cap H^1(0,+\infty;H^{1+\ell}(\Omega))\cap L^2(0,+\infty;H^{2+\ell}(\Omega))} \leq C\|h\|_{E^\ell_{h,\infty}},\\
&\|\det Y(X) - I\|_{H^{1+\ell/2}(0,+\infty;H^1(\Omega))\cap H^1(0,+\infty;H^{1+\ell}(\Omega))\cap L^2(0,+\infty;H^{2+\ell}(\Omega))} \leq C\|h\|_{E^\ell_{h,\infty}},\\
&\|\partial_t Y(X)\|_{H^{\ell/2}(0,+\infty;H^{1+\ell}(\Omega))} \leq C\|h\|_{E^\ell_{h,\infty}},\\
&\|\partial_t \mathrm{cof}(\nabla Y)(X)\|_{H^{\ell/2}(0,+\infty;H^{\ell}(\Omega))} \leq C\|h\|_{E^\ell_{h,\infty}},\\
&\|\partial_l \mathrm{cof}(\nabla Y)(X)\|_{H^1(0,+\infty;H^{\ell}(\Omega))\cap L^2(0,+\infty;H^{1+\ell}(\Omega))} \leq C\|h\|_{E^\ell_{h,\infty}},\\
&\|\Delta Y(X)\|_{H^1(0,+\infty;H^{\ell}(\Omega))\cap L^2(0,+\infty;H^{1+\ell}(\Omega))} \leq C\|h\|_{E^\ell_{h,\infty}},\\
&\|\Delta \mathrm{cof}(\nabla Y)(X)\|_{H^{\ell,\ell}((0,+\infty)\times\Omega)} \leq C\|h\|_{E^\ell_{h,\infty}},\\
&\|n_{\Gamma(t)} - n_\Gamma\|_{H^1(0,+\infty;H^{1/2+\ell}(\Gamma))} \leq C\|h\|_{E^\ell_{h,\infty}}.
\end{aligned} \tag{3.7 .2}$$

And for all $h, \tilde{h} \in E^{\ell}_{h,\infty,R^*}$, we have the following lipschitz estimates

$$
\begin{aligned}
&\|\nabla Y(X) - \nabla \tilde{Y}(\tilde{X})\|_{H^{1+\ell/2}(0,+\infty;H^1(\Omega))\cap H^1(0,+\infty;H^{1+\ell}(\Omega))\cap L^2(0,+\infty;H^{2+\ell}(\Omega))} \\
&\qquad\qquad\qquad\qquad \le C\|h-\tilde{h}\|_{E^{\ell}_{h,\infty}}, \\
&\|\mathrm{cof}(\nabla Y)(X) - \mathrm{cof}(\nabla \tilde{Y})(\tilde{X})\|_{H^{1+\ell/2}(0,+\infty;H^1(\Omega))\cap H^1(0,+\infty;H^{1+\ell}(\Omega))\cap L^2(0,+\infty;H^{2+\ell}(\Omega))} \\
&\qquad\qquad\qquad\qquad \le C\|h-\tilde{h}\|_{E^{\ell}_{h,\infty}}, \\
&\|\det Y(X) - \det \tilde{Y}(\tilde{X})\|_{H^{1+\ell/2}(0,+\infty;H^1(\Omega))\cap H^1(0,+\infty;H^{1+\ell}(\Omega))\cap L^2(0,+\infty;H^{2+\ell}(\Omega))} \\
&\qquad\qquad\qquad\qquad \le C\|h-\tilde{h}\|_{E^{\ell}_{h,\infty}}, \\
&\|\partial_t Y(X) - \partial_t \tilde{Y}(\tilde{X})\|_{H^{\ell/2}(0,+\infty;H^{1+\ell}(\Omega))} \le C\|h-\tilde{h}\|_{E^{\ell}_{h,\infty}}, \\
&\|\partial_t \mathrm{cof}(\nabla Y)(X) - \partial_t \mathrm{cof}(\nabla \tilde{Y})(\tilde{X})\|_{H^{\ell/2}(0,+\infty;H^{\ell}(\Omega))} \le C\|h-\tilde{h}\|_{E^{\ell}_{h,\infty}}, \\
&\|\partial_l \mathrm{cof}(\nabla Y)(X) - \partial_l \mathrm{cof}(\nabla \tilde{Y})(\tilde{X})\|_{H^{1}(0,+\infty;H^{\ell}(\Omega))\cap L^2(0,+\infty;H^{1+\ell}(\Omega))} \le C\|h-\tilde{h}\|_{E^{\ell}_{h,\infty}}, \\
&\|\Delta Y(X) - \Delta \tilde{Y}(\tilde{X})\|_{H^{1}(0,+\infty;H^{\ell}(\Omega))\cap L^2(0,+\infty;H^{1+\ell}(\Omega))} \le C\|h-\tilde{h}\|_{E^{\ell}_{h,\infty}}, \\
&\|\Delta \mathrm{cof}(\nabla Y)(X) - \Delta \mathrm{cof}(\nabla \tilde{Y})(\tilde{X})\|_{H^{\ell,\ell}((0,+\infty)\times\Omega)} \le C\|h-\tilde{h}\|_{E^{\ell}_{h,\infty}}, \\
&\|n_{\Gamma(t)} - \tilde{n}_{\Gamma(t)}\|_{H^1(0,+\infty;H1/2+\ell(\Gamma))} \le C\|h-\tilde{h}\|_{E^{\ell}_{h,\infty}}.
\end{aligned}
\tag{3.7 .3}
$$

Remark 3.7 .3. *One can have a better regularity in time for h_e. But the regularity given in Lemma 3.7 .2 is sufficient to estimates the non linear terms conveniently.*

Proof of Lemma 3.7 .2. The proof is divided as follow: we first construct the lifting operator L_e and the ball E^{ℓ}_{h,∞,R^*} so that $X(t,\cdot)$ defined by (3.7 .1) is invertible. Then, we get all the estimates one by one. We only prove the direct estimates, the lipschitz estimates being a straightforward adaptation. The result [23, Proposition B.1, page 67] is an essential tool to derive our estimates.

Construction of L_e.
Let $h \in E^{\ell}_{h,\infty}$. Remark that $E^{\ell}_{h,\infty} \hookrightarrow H^{1+\ell/2}(0,+\infty;H^{3/2}_m(\Gamma))$. Using [31, Chapter 1, Theorem 9.4, page 41], there exists a lifting $\tilde{h}_e \in H^{1+\ell/2}(0,+\infty;H^2(\Omega)) \cap H^1(0,+\infty;H^{2+\ell}(\Omega)) \cap L^2(0,+\infty;H^{3+\ell}(\Omega))$ of h in Ω such that

$$\|\tilde{h}_e\|_{H^{1+\ell/2}(0,+\infty;H^2(\Omega))\cap H^1(0,+\infty;H^{2+\ell}(\Omega))\cap L^2(0,+\infty;H^{3+\ell}(\Omega))} \le C\|h\|_{E^{\ell}_{h,\infty}},$$

We introduce a smooth cut-off function $\eta \in C^\infty([-1;1];[0;1])$ such that

$$\eta(x_d) = \begin{cases} 0 & \text{for } x_d \in [-1;-1/2]\cup[1/2;1], \\ 1 & \text{for } x_d \in [-1/3;1/3]. \end{cases}$$

Define $h_e = \eta\tilde{h}_e \in H^{1+\ell/2}(0,+\infty;H^2(\Omega)) \cap H^1(0,+\infty;H^{2+\ell}(\Omega)) \cap L^2(0,+\infty;H^{3+\ell}(\Omega))$. This concludes the construction of the lifting operator L_e.

Definition and inversion of X.
Using the continuous injection $H^1(0,+\infty;H^{2+\ell}(\Omega)) \hookrightarrow C(\mathbb{R}_+\times\Omega)$, we have

$$\|h_e\|_{C(\mathbb{R}_+\times\Omega)} \le C_0\|h\|_{E^{\ell}_{h,\infty}},$$

where C_0 is independent of h. Let

$$R^* < \min\left\{\frac{1}{2}, \frac{1}{2C_0}\right\},$$

and $h \in E^{\ell}_{h,\infty,R^*}$. The extension h_e defined above is such that

$$\|h_e\|_{C(\mathbb{R}_+\times\Omega)\cap H^{1+\ell/2}(0,+\infty;H^2(\Omega))\cap H^1(0,+\infty;H^{2+\ell}(\Omega))\cap L^2(0,+\infty;H^{3+\ell}(\Omega))} < \frac{1}{2}.$$

For all $t > 0$, we define

$$X(t,\cdot): \begin{array}{ccc} \Omega & \to & \mathbb{R}^2 \\ (y', y_d) & \mapsto & (y', \mathrm{e}^{-\omega t}(1-y_d^2)h_e(t,y',y_d) + y_d) \end{array}.$$

Remark that by construction of h_e, we have actually that $X(t,\Omega) = \Omega$. Indeed, if $|y_d| < 1/2$, then $|\,\mathrm{e}^{-\omega t}\, h_e(t,y',y_d)(1-y_d^2) + y_d| \leq \|h_e\|_{C(\mathbb{R}_+\times\Omega)} + 1/2 < 1$, and if $|y_d| > 1/2$, then $\mathrm{e}^{-\omega t}\, h_e(t,y',y_d)(1-y_d^2)+y_d = y_d$. Therefore, $X(t,\Omega) \subset \Omega$. Furthermore, for all $y_d \in \mathbb{T}_d$, the map $f: y_d \mapsto \mathrm{e}^{-\omega t}(1-y_d^2)h_e(t,y',y_d) + y_d$ is continuous and $f(-1) = -1$ and $f(1) = 1$. Therefore, $\Omega \subset X(t,\Omega)$.

Besides, a straightforward computation gives

$$\nabla X = \begin{pmatrix} I_{d-1} & 0 \\ \mathrm{e}^{-\omega t}(1-y_d^2)(\nabla_{y'} h_e)^T & 1 + \mathrm{e}^{-\omega t}\left((1-y_d^2)\partial_d h_e - 2y_d h_e\right) \end{pmatrix}.$$

Since the injection $H^1(0,+\infty;H^{1+\ell}(\Omega)) \hookrightarrow C(\mathbb{R}_+\times\Omega)$ is continuous, we have

$$\|I - \nabla X\|_{C(\mathbb{R}_+\times\Omega)} \leq C_1 \|h\|_{E^{\ell}_{h,\infty}},$$

where C_1 is independent of h. Let

$$R^* < \min\left\{\frac{1}{2}, \frac{1}{2C_0}, \frac{1}{2C_1}\right\},$$

and $h \in E^{\ell}_{h,\infty,R^*}$. The transformation $X(t,\cdot)$ satisfies the assumptions of the inversion function theorem and is globally invertible from Ω to $X(t,\Omega) = \Omega$ for all $t > 0$.

Estimates on $\nabla Y(X)$, cof$(\nabla Y)(X)$, det$(\nabla Y)(X)$: (3.7 .2)$_{1,2,3}$ - (3.7 .3)$_{1,2,3}$.
We use the following identities

$$\nabla Y(X) = \frac{\mathrm{cof}(\nabla X)^T}{\det(\nabla X)} \quad \text{and} \quad \mathrm{cof}(\nabla Y)(X) = \frac{\nabla X}{\det(\nabla X)^T}, \quad \text{and} \quad \det(\nabla Y)(X) = \frac{1}{\det(\nabla X)}.$$

Straightforward computations show that

$$\det(\nabla X) = 1 + \mathrm{e}^{-\omega t}\left((1-y_d^2)\partial_d h_e - 2y_d h_e\right).$$

By construction of h_e and R, we have $\mathrm{e}^{-\omega t}\left((1-y_d^2)\partial_d h_e - 2y_d h_e\right) \in H^{1+\ell/2}(0,+\infty;H^1(\Omega)) \cap H^1(0,+\infty;H^{1+\ell}(\Omega)) \cap L^2(0,+\infty;H^{2+\ell}(\Omega))$ with

$$\|\,\mathrm{e}^{-\omega t}\left((1-y_d^2)\partial_d h_e - 2y_d h_e\right\|_{H^{1+\ell/2}(0,+\infty;H^1(\Omega))\cap H^1(0,+\infty;H^{1+\ell}(\Omega))\cap L^2(0,+\infty;H^{2+\ell}(\Omega))} \leq \frac{1}{2}.$$

Therefore, we can write

$$\det(\nabla Y)(X) = \sum_{j=0}^{+\infty} e^{-j\omega t}\left((y_d^2-1)\partial_d h_e + 2y_d h_e\right)^j.$$

and since $H^{1+\ell/2}(0,+\infty;H^1(\Omega))\cap H^1(0,+\infty;H^{1+\ell}(\Omega))\cap L^2(0,+\infty;H^{2+\ell}(\Omega))$ is an algebra with Lemma 3.7 .1, we have

$$\|\det(\nabla Y)(X)-1\|_{H^{1+\ell/2}(0,+\infty;H^1(\Omega))\cap H^1(0,+\infty;H^{1+\ell}(\Omega))\cap L^2(0,+\infty;H^{2+\ell}(\Omega))}\leq C\|h\|_{E^\ell_{h,\infty}}.$$

Furthermore, one has

$$\|\nabla X-I\|_{H^{1+\ell/2}(0,+\infty;H^1(\Omega))\cap H^1(0,+\infty;H^{1+\ell}(\Omega))\cap L^2(0,+\infty;H^{2+\ell}(\Omega))}\leq C\|h\|_{E^\ell_{h,\infty}}.$$

Lastly,

$$\mathrm{cof}(\nabla X)^T=\begin{pmatrix}(1+\mathrm{e}^{-\omega t}\left((1-y_d^2)\partial_d h_e-2y_dh_e\right))I_{d-1} & 0\\ -\mathrm{e}^{-\omega t}(1-y_d^2)(\nabla_{y'}h_e)^T & 1\end{pmatrix},$$

which gives

$$\|\mathrm{cof}(\nabla X)^T-I\|_{H^{1+\ell/2}(0,+\infty;H^1(\Omega))\cap H^1(0,+\infty;H^{1+\ell}(\Omega))\cap L^2(0,+\infty;H^{2+\ell}(\Omega))}\leq C\|h\|_{E^\ell_{h,\infty}}.$$

Putting all together, we get $(3.7\ .2)_{1,2,3}$.

Estimates on $\partial_t Y(X)$: $(3.7\ .2)_4$ - $(3.7\ .3)_4$.
Differentiating in time the identity $Y(t,X(t,\cdot))=Id$, one gets

$$\partial_t Y(X)=-\nabla Y(X)\partial_t X.$$

Remark that we have $\partial_t X\in H^{\ell/2}(0,+\infty;H^2(\Omega))\cap L^2(0,+\infty;H^{2+\ell}(\Omega))$ with

$$\|\partial_t X\|_{H^{\ell/2}(0,+\infty;H^2(\Omega))\cap L^2(0,+\infty;H^{2+\ell}(\Omega))}\leq C\|h\|_{E^\ell_{h,\infty}}.$$

Since $\nabla Y(X)\in H^1(0,+\infty;H^{1+\ell}(\Omega))$, we easily deduce $\partial_t Y(X)\in H^{\ell/2}(0,+\infty;H^{1+\ell}(\Omega))$ and estimate $(3.7\ .2)_4$ follows.

Estimates on ∂_tcof$(\nabla Y)(X)$: $(3.7\ .2)_5$ - $(3.7\ .3)_5$.
We have

$$\partial_t\mathrm{cof}(\nabla Y)(X)=\partial_t(\mathrm{cof}(\nabla Y)(X))+\sum_{l=1}^d\partial_l(\mathrm{cof}(\nabla Y)(X))\partial_t Y_l(X).$$

Remark that $\partial_t(\mathrm{cof}(\nabla Y)(X))\in H^{\ell/2}(0,+\infty;H^1(\Omega))\cap L^2(0,+\infty;H^{1+\ell}(\Omega))$ with

$$\|\partial_t(\mathrm{cof}(\nabla Y)(X))\|_{H^{\ell/2}(0,+\infty;H^1(\Omega))\cap L^2(0,+\infty;H^{1+\ell}(\Omega))}\leq C\|h\|_{E^\ell_{h,\infty}}.$$

Since we have $\partial_l(\mathrm{cof}(\nabla Y)(X))\in H^1(0,+\infty;H^\ell(\Omega))\cap L^2(0,+\infty;H^{1+\ell}(\Omega))$ and $\partial_t Y(X)\in H^{\ell/2}(0,+\infty;H^{1+\ell}(\Omega))$, we get $\partial_l(\mathrm{cof}(\nabla Y)(X))\partial_t Y_l(X)\in H^{\ell/2}(0,+\infty;H^\ell(\Omega))$ with

$$\|\partial_l(\mathrm{cof}(\nabla Y)(X))\partial_t Y_l(X)\|_{H^{\ell/2}(0,+\infty;H^\ell(\Omega))}\leq C\|h\|_{E^\ell_{h,\infty}}.$$

Estimate $(3.7\ .2)_5$ easily follows.

Estimates on ∂_lcof$(\nabla Y)(X)$: $(3.7\ .2)_6$ - $(3.7\ .3)_6$.
We have

$$\partial_l\mathrm{cof}(\nabla Y)(X)=\sum_{j=1}^d\partial_j(\mathrm{cof}(\nabla Y)(X))\partial_l Y_j(X).$$

Since $\partial_j(\mathrm{cof}(\nabla Y)(X))\in H^1(0,+\infty;H^\ell(\Omega))\cap L^2(0,+\infty;H^{1+\ell}(\Omega))$ with

$$\|\partial_j(\mathrm{cof}(\nabla Y)(X))\|_{H^1(0,+\infty;H^\ell(\Omega))\cap L^2(0,+\infty;H^{1+\ell}(\Omega))}\leq C\|h\|_{E^\ell_{h,\infty}},$$

and since $\nabla Y(X)\in H^1(0,+\infty;H^{1+\ell}(\Omega))$, we therefore obtain $\partial_j(\mathrm{cof}(\nabla Y)(X))\partial_l Y_j(X)\in H^1(0,+\infty;H^\ell(\Omega))\cap L^2(0,+\infty;H^{1+\ell}(\Omega))$ and estimate $(3.7\ .2)_6$ follows.

Estimates on $\Delta Y(X)$: (3.7 .2)$_7$ - (3.7 .3)$_7$.
We have

$$\partial_l \nabla Y(X) = \sum_{j=1}^{d} \partial_j(\nabla Y(X))\partial_l Y_j(X).$$

Since $\partial_j(\mathrm{cof}(\nabla Y)(X)) \in H^1(0,+\infty; H^\ell(\Omega)) \cap L^2(0,+\infty; H^{1+\ell}(\Omega))$ with

$$\|\partial_j(\nabla Y(X))\|_{H^1(0,+\infty;H^\ell(\Omega))\cap L^2(0,+\infty;H^{1+\ell}(\Omega))} \leq C\|h\|_{E^\ell_{h,\infty}},$$

and since $\nabla Y(X) \in H^1(0,+\infty; H^{1+\ell}(\Omega))$, we therefore obtain $\partial_j(\mathrm{cof}(\nabla Y)(X))\partial_l Y_j(X) \in H^1(0,+\infty; H^\ell(\Omega)) \cap L^2(0,+\infty; H^{1+\ell}(\Omega))$ and estimate (3.7 .2)$_7$ follows.

Estimates on $\Delta\mathrm{cof}(\nabla Y)(X)$: (3.7 .2)$_8$ - (3.7 .3)$_8$.
We have

$$\partial_{ll}\mathrm{cof}(\nabla Y)(X) = \sum_{j=1}^{d} \partial_j(\partial_l \mathrm{cof}(\nabla Y)(X))\partial_l Y_j(X).$$

With the continuous injection $H^1(0,+\infty; H^\ell(\Omega)) \cap L^2(0,+\infty; H^{1+\ell}(\Omega)) \hookrightarrow H^\ell(0,+\infty; H^1(\Omega))$, we have $\partial_l\mathrm{cof}(\nabla Y)(X) \in H^\ell(0,+\infty; H^1(\Omega)) \cap L^2(0,+\infty; H^{1+\ell}(\Omega))$. Then, $\partial_j(\partial_l\mathrm{cof}(\nabla Y)(X)) \in H^{\ell,\ell}((0,+\infty)\times\Omega)$. Since $\nabla Y(X) \in H^1(0,+\infty; H^{1+\ell}(\Omega))$, we have $\partial_j(\partial_l\mathrm{cof}(\nabla Y)(X))\partial_l Y_j(X) \in H^{\ell,\ell}((0,+\infty)\times\Omega)$ and estimate (3.7 .2)$_7$ follows.

Estimates on $n_{\Gamma(t)}$: (3.7 .2)$_9$ - (3.7 .3)$_9$.
We recall

$$n_{\Gamma(t)} = \frac{1}{\sqrt{1+\mathrm{e}^{-2\omega t}\,|\nabla h|^2}}\begin{pmatrix} -\mathrm{e}^{-\omega t}\,\nabla h \\ 1 \end{pmatrix}.$$

First remark that $\nabla h \in H^1(0,+\infty; H^{1/2+\ell}(\Gamma)) \hookrightarrow C(\mathbb{R}_+\times\Gamma)$. Taking $R^* > 0$ sufficiently small, we can assume $\|\nabla h\|_{H^1(0,+\infty;H^{1/2+\ell}(\Gamma))\cap C(\mathbb{R}_+\times\Gamma)} < 1/2$. Therefore, we can write

$$\frac{1}{\sqrt{1+\mathrm{e}^{-\omega t}\,|\nabla h|^2}} = \sum_{j=0}^{+\infty} \frac{(-1)^j(2j)!}{2^{2j}(j!)^2}\,\mathrm{e}^{-2j\omega t}\,|\nabla h|^{2j}.$$

Since $H^1(0,+\infty; H^{1/2+\ell}(\Gamma))$ is an algebra, we deduce $\dfrac{1}{\sqrt{1+\mathrm{e}^{-\omega t}\,|\nabla h|^2}} \in H^1(0,+\infty; H^{1/2+\ell}(\Gamma))$ with

$$\left\|\frac{1}{\sqrt{1+\mathrm{e}^{-\omega t}\,|\nabla h|^2}} - 1\right\|_{H^1(0,+\infty;H^{1/2+\ell}(\Gamma))} \leq C\|h\|_{E^\ell_{h,\infty}}. \tag{3.7 .4}$$

Estimate (3.7 .2)$_9$ follows. □

3.7 .1.3 Estimates on (F,H) defined in (3.2 .4)–(3.2 .5)

We now get the estimates on the nonlinear terms.

Lemma 3.7 .4. *Let $R^* > 0$ be given in Lemma 3.7 .2 and $(u_0,h_0) \in E^\ell_{u_0}\times E^\ell_{h_0}$ satisfying (3.2 .8). For all $((u,p,h),(\tilde u,\tilde p,\tilde h)) \in E^\ell_{u,\infty,u_0,R^*}\times E^\ell_{p,\infty,(u_0,h_0),R^*}\times E^\ell_{h,\infty,h_0,R^*}$, the non linear term (F,H) given in (3.2 .4)-(3.2 .5) belongs to $E^\ell_{f,\infty}\times E^\ell_{f_{jump},\infty,u_0}$ and satisfies*

$$\|(F(u,p,h),H(u,p,h))\|_{E^\ell_{f,\infty}\times E^\ell_{f_{jump},\infty}} \leq C\|(u,p,h)\|^2_{E^\ell_{u,\infty}\times E^\ell_{p,\infty}\times E^\ell_{h,\infty}},$$

and the lipschitz estimate

$$\begin{aligned}\|(F(u,p,h),H(u,p,h))-(F(\tilde{u},\tilde{p},\tilde{h}),H(\tilde{u},\tilde{p},\tilde{h}))\|_{E^\ell_{f,\infty}\times E^\ell_{f_{jump},\infty}}\\ \leq C\|((u,p,h),(\tilde{u},\tilde{p},\tilde{h}))\|_{(E^\ell_{u,\infty}\times E^\ell_{p,\infty}\times E^\ell_{h,\infty})^2}\|(u,p,h)-(\tilde{u},\tilde{p},\tilde{h})\|_{E^\ell_{u,\infty}\times E^\ell_{p,\infty}\times E^\ell_{h,\infty}}.\end{aligned}$$

Proof of Lemma 3.7 .4. We only prove the direct estimate, the lipschitz estimate being a straightforward adaptation. We treat each term separately. The result [23, Proposition B.1, page 67] is our main tool for the estimates. We point out that we do not require our constant C to be independent of R^*.

Estimate of $F_1^{\pm}(u,h)$.
We recall

$$F_1^{\pm}(u,h)=(I-\mathrm{cof}(\nabla Y^{\pm})^T(X^{\pm}))\partial_t u^{\pm}-\omega(I-\mathrm{cof}(\nabla Y^{\pm})^T(X^{\pm}))u^{\pm}.$$

Since $\partial_t u^{\pm}\in H^{\ell,\ell/2}((0,+\infty)\times\Omega^{\pm})$ and $(I-\mathrm{cof}(\nabla Y)^T(X))\in H^1(0,+\infty;H^{1+\ell}(\Omega))$, we have $(I-\mathrm{cof}(\nabla Y^{\pm})^T(X^{\pm}))\partial_t u^{\pm}\in H^{\ell,\ell/2}((0,+\infty)\times\Omega^{\pm})$ with

$$\|(I-\mathrm{cof}(\nabla Y^{\pm})^T(X^{\pm}))\partial_t u^{\pm}\|_{H^{\ell,\ell/2}((0,+\infty)\times\Omega^{\pm})}\leq C\|h\|_{E^\ell_{h,\infty}}\|u\|_{E^\ell_{u,\infty}}.$$

It is straightforward that $(I-\mathrm{cof}(\nabla Y^{\pm})^T(X^{\pm}))u^{\pm}\in H^{\ell,\ell/2}((0,+\infty)\times\Omega^{\pm})$ with the same type of estimate. Therefore,

$$\|F_1^{\pm}(u,h)\|_{H^{\ell,\ell/2}((0,+\infty)\times\Omega^{\pm})}\leq C\|h\|_{E^\ell_{h,\infty}}\|u\|_{E^\ell_{u,\infty}}.$$

Estimate of $F_2^{\pm}(u,h)$.
We recall

$$F_2^{\pm}(u,h)=\partial_t\mathrm{cof}(\nabla Y^{\pm})^T(X^{\pm})u^{\pm}-\mathrm{cof}(\nabla Y^{\pm})^T(X^{\pm})\nabla u^{\pm}\partial_t Y^{\pm}(X^{\pm}).$$

We have $\partial_t\mathrm{cof}(\nabla Y)(X)\in H^{\ell/2}(0,+\infty;H^{\ell}(\Omega))$ and the injection $u^{\pm}\in H^{2+\ell,1+\ell/2}((0,+\infty)\times\Omega^{\pm})\hookrightarrow L^\infty(0,+\infty;H^{1+\ell}(\Omega^{\pm}))\cap H^{1/2+\ell/2}(0,+\infty;H^1(\Omega^{\pm}))$. We deduce $\partial_t\mathrm{cof}(\nabla Y^{\pm})^T(X^{\pm})u^{\pm}\in H^{\ell,\ell/2}((0,+\infty)\times\Omega^{\pm})$ with

$$\|\partial_t\mathrm{cof}(\nabla Y^{\pm})^T(X^{\pm})u^{\pm}\|_{H^{\ell,\ell/2}((0,+\infty)\times\Omega^{\pm})}\leq C\|h\|_{E^\ell_{h,\infty}}\|u\|_{E^\ell_{u,\infty}}.$$

We have $\partial_t Y(X)\in H^{\ell/2}(0,+\infty;H^{1+\ell}(\Omega))$ and the injection $\nabla u^{\pm}\in H^{1+\ell,1/2+\ell/2}((0,+\infty)\times\Omega^{\pm})\hookrightarrow L^\infty(0,+\infty;H^{\ell}(\Omega^{\pm}))$. Therefore, $\nabla u^{\pm}\partial_t Y^{\pm}(X^{\pm})\in H^{\ell,\ell/2}((0,+\infty)\times\Omega^{\pm})$. Furthermore, $\mathrm{cof}(\nabla Y)(X)\in H^1(0,+\infty;H^{1+\ell}(\Omega))$, then $\mathrm{cof}(\nabla Y^{\pm})^T(X^{\pm})\nabla u^{\pm}\partial_t Y^{\pm}(X^{\pm})\in H^{\ell,\ell/2}((0,+\infty)\times\Omega^{\pm})$ with

$$\|\mathrm{cof}(\nabla Y^{\pm})^T(X^{\pm})\nabla u^{\pm}\partial_t Y^{\pm}(X^{\pm})\|_{H^{\ell,\ell/2}((0,+\infty)\times\Omega^{\pm})}\leq C(1+\|h\|_{E^\ell_{h,\infty}})\|h\|_{E^\ell_{h,\infty}}\|u\|_{E^\ell_{u,\infty}}.$$

We then deduce

$$\|F_2^{\pm}(u,h)\|_{H^{\ell,\ell/2}((0,+\infty)\times\Omega^{\pm})}\leq C(1+\|h\|_{E^\ell_{h,\infty}})\|h\|_{E^\ell_{h,\infty}}\|u\|_{E^\ell_{u,\infty}}.$$

Estimate of $F_3^\pm(u,h)$.
We recall

$$\big(F_3^\pm(u,h)\big)_i = \sum_{j,k,l=1}^d \mathrm{cof}(\nabla Y^\pm)_{kl}(X^\pm)\partial_l \mathrm{cof}(\nabla Y^\pm)_{ji}(X^\pm)u_j^\pm u_k^\pm$$
$$+ \det(\nabla Y^\pm)(X^\pm)\sum_{j,k=1}^d \mathrm{cof}(\nabla Y^\pm)_{ki}(X^\pm)u_j^\pm \partial_j u_k^\pm.$$

We have $\partial_l \mathrm{cof}(\nabla Y)(X) \in H^1(0,+\infty;H^\ell(\Omega))$ and the injection $u^\pm \in H^{2+\ell,1+\ell/2}((0,+\infty)\times\Omega^\pm) \hookrightarrow H^{1/2}(0,+\infty;H^{1+\ell}(\Omega))$. Therefore, $\partial_l\mathrm{cof}(\nabla Y^\pm)_{ji}(X^\pm)u_j^\pm \in H^{\ell/2}(0,+\infty;H^\ell(\Omega^\pm))$. Furthermore, since $u^\pm \in H^{2+\ell,1+\ell/2}((0,+\infty)\times\Omega^\pm) \hookrightarrow L^\infty(0,+\infty;H^{1+\ell}(\Omega^\pm)) \cap H^{1/2+\ell/2}(0,+\infty;H^1(\Omega^\pm))$, we deduce that $\partial_l\mathrm{cof}(\nabla Y^\pm)_{ji}(X^\pm)u_j^\pm u_k^\pm \in H^{\ell,\ell/2}((0,+\infty)\times\Omega^\pm)$. Lastly, with $\mathrm{cof}(\nabla Y)(X) \in H^1(0,+\infty;H^{1+\ell}(\Omega))$, we have $\mathrm{cof}(\nabla Y^\pm)_{kl}(X^\pm)\partial_l\mathrm{cof}(\nabla Y^\pm)_{ji}(X^\pm)u_j^\pm u_k^\pm \in H^{\ell,\ell/2}((0,+\infty)\times\Omega^\pm)$ with

$$\|\mathrm{cof}(\nabla Y^\pm)_{kl}(X^\pm)\partial_l\mathrm{cof}(\nabla Y^\pm)_{ji}(X^\pm)u_j^\pm u_k^\pm\|_{H^{\ell,\ell/2}((0,+\infty)\times\Omega^\pm)} \leq C(1+\|h\|_{H^\infty})\|h\|_{H^\infty}\|u\|_{U^\infty}^2.$$

Since $u^\pm \in H^{2+\ell,1+\ell/2}((0,+\infty)\times\Omega^\pm) \hookrightarrow L^\infty(0,+\infty;H^{1+\ell}(\Omega^\pm)) \cap H^{\ell/2}(0,+\infty;H^2(\Omega^\pm))$ and $\nabla u^\pm \in H^{1+\ell,1/2+\ell/2}((0,+\infty)\times\Omega^\pm)$, we have $u_j^\pm\partial_j u_k^\pm \in H^{\ell,\ell/2}((0,+\infty)\times\Omega^\pm)$. Furthermore, since we have $\det(\nabla Y)(X)$, $\mathrm{cof}(\nabla Y)(X) \in H^1(0,+\infty;H^{1+\ell}(\Omega))$, we then deduce that $\det(\nabla Y^\pm)(X^\pm)\mathrm{cof}(\nabla Y^\pm)_{ki}(X^\pm)u_j^\pm\partial_j u_k^\pm \in H^{\ell,\ell/2}((0,+\infty)\times\Omega^\pm)$ with

$$\|\det(\nabla Y^\pm)(X^\pm)\mathrm{cof}(\nabla Y^\pm)_{ki}(X^\pm)u_j^\pm\partial_j u_k^\pm\|_{H^{\ell,\ell/2}((0,+\infty)\times\Omega^\pm)} \leq C(1+\|h\|_{E^\ell_{h,\infty}})^2\|u\|^2_{E^\ell_{u,\infty}}.$$

Finally,
$$\|F_3^\pm(u,h)\|_{H^{\ell,\ell/2}((0,+\infty)\times\Omega^\pm)} \leq C(1+\|h\|_{E^\ell_{h,\infty}})^2\|u\|^2_{E^\ell_{u,\infty}}.$$

Estimate of $F_4^\pm(u,h)$.
We recall

$$\big(F_4^\pm(u,h)\big)_i = \frac{\mu^\pm}{\rho^\pm}\Big(\sum_{j,k=1}^d \mathrm{cof}(\nabla Y^\pm)_{ji}(X^\pm)\Delta Y_k^\pm(X^\pm)\partial_k u_j^\pm + \sum_{j=1}^d \Delta\mathrm{cof}(\nabla Y^\pm)_{ji}(X^\pm)u_j^\pm$$
$$+ \sum_{j,k,l,m=1}^d \mathrm{cof}(\nabla Y^\pm)_{ji}(X^\pm)\partial_l Y_m^\pm(X^\pm)\partial_l Y_k^\pm(X^\pm)\partial_{km}u_j^\pm - \Delta u_i^\pm$$
$$+ 2\sum_{j,k,l=1}^d \partial_l\mathrm{cof}(\nabla Y^\pm)_{ji}(X^\pm)\partial_l Y_k^\pm(X^\pm)\partial_k u_j^\pm\Big).$$

Since $\nabla u^\pm \in H^{1+\ell,1/2+\ell/2}((0,+\infty)\times\Omega^\pm) \hookrightarrow L^\infty(0,+\infty;H^\ell(\Omega^\pm)) \cap H^{1/2+\ell/4}(0,+\infty;H^{\ell/2}(\Omega^\pm))$ and $\Delta Y(X) \in H^1(0,+\infty;H^\ell(\Omega)) \cap L^2(0,+\infty;H^{1+\ell}(\Omega)) \hookrightarrow H^{\ell/2}(0,+\infty;H^{1+\ell/2}(\Omega))$, we get $\Delta Y_k^\pm(X^\pm)\partial_k u_j^\pm \in H^{\ell,\ell/2}((0,+\infty)\times\Omega^\pm)$. Furthermore, since $\mathrm{cof}(\nabla Y)(X) \in H^1(0,+\infty;H^{1+\ell}(\Omega))$, we deduce that the term $\mathrm{cof}(\nabla Y^\pm)_{ji}(X^\pm)\Delta Y_k^\pm(X^\pm)\partial_k u_j^\pm \in H^{\ell,\ell/2}((0,+\infty)\times\Omega^\pm)$ with

$$\|\mathrm{cof}(\nabla Y^\pm)_{ji}(X^\pm)\Delta Y_k^\pm(X^\pm)\partial_k u_j^\pm\|_{H^{\ell,\ell/2}((0,+\infty)\times\Omega^\pm)} \leq C(1+\|h\|_{E^\ell_{h,\infty}})\|h\|_{E^\ell_{h,\infty}}\|u\|_{E^\ell_{u,\infty}}.$$

With the same type of arguments, we get $\partial_l\mathrm{cof}(\nabla Y^\pm)_{ji}(X^\pm)\partial_l Y_k^\pm(X^\pm)\partial_k u_j^\pm \in H^{\ell,\ell/2}((0,+\infty)\times\Omega^\pm)$ with the same estimate.

Since $\Delta\mathrm{cof}(\nabla Y)(X) \in H^{\ell,\ell}((0,+\infty)\times\Omega) \hookrightarrow H^{\ell/2}(0,+\infty;H^{\ell/2}(\Omega))$ and $u^\pm \in H^{2,1}((0,+\infty)\times\Omega^\pm) \hookrightarrow L^\infty(0,+\infty;H^{1+\ell}(\Omega^\pm)) \cap H^{1/2+\ell/4}(0,+\infty;H^{1+\ell/2}(\Omega^\pm))$, we get $\Delta\mathrm{cof}(\nabla Y^\pm)_{ji}(X^\pm)u_j^\pm \in H^{\ell,\ell/2}((0,+\infty)\times\Omega^\pm)$ with

$$\|\Delta\mathrm{cof}(\nabla Y^\pm)_{ji}(X^\pm)u_j^\pm\|_{H^{\ell,\ell/2}((0,+\infty)\times\Omega^\pm)} \leq C\|h\|_{E^\ell_{h,\infty}}\|u\|_{E^\ell_{u,\infty}}.$$

For the remaining term, write

$$\begin{aligned}\mathrm{cof}(\nabla Y^\pm)_{ji}(X^\pm)\partial_l Y_m^\pm(X^\pm)\partial_l Y_k^\pm(X^\pm)\partial_{km}u_j^\pm - \Delta u_i^\pm \\ = \left(\nabla Y^\pm(X^\pm)(\nabla Y^\pm)^T(X^\pm)\right)_{km}\left(\mathrm{cof}(\nabla Y^\pm)^T(X^\pm) - I)\partial_{km}u^\pm\right)_i \\ + \left(\nabla Y^\pm(X^\pm)(\nabla Y^\pm)^T(X^\pm) - I\right)_{km}\partial_{km}u_i^\pm.\end{aligned}$$

Remark that

$$\nabla Y(X)(\nabla Y)^T(X) - I = \nabla Y(X)((\nabla Y)^T(X) - I) + \nabla Y(X) - I,$$

so that we have $\nabla Y(X)(\nabla Y)^T(X) - I \in H^1(0,+\infty;H^{1+\ell}(\Omega))$ with

$$\|\nabla Y(X)(\nabla Y)^T(X) - I\|_{H^1(0,+\infty;H^{1+\ell}(\Omega))} \leq C(1+\|h\|_{E^\ell_{h,\infty}})\|h\|_{E^\ell_{h,\infty}}.$$

Since $\partial_{km}u^\pm \in H^{\ell,\ell/2}((0,+\infty)\times\Omega^\pm)$, then $\mathrm{cof}(\nabla Y^\pm)_{ji}(X^\pm)\partial_l Y_m^\pm(X^\pm)\partial_l Y_k^\pm(X^\pm)\partial_{km}u_j^\pm - \Delta u_i^\pm \in H^{\ell,\ell/2}((0,+\infty)\times\Omega^\pm)$ with

$$\begin{aligned}\|\mathrm{cof}(\nabla Y^\pm)_{ji}(X^\pm)\partial_l Y_m^\pm(X^\pm)\partial_l Y_k^\pm(X^\pm)\partial_{km}u_j^\pm - \Delta u_i^\pm\|_{H^{\ell,\ell/2}((0,+\infty)\times\Omega^\pm)} \\ \leq C(1+\|h\|_{E^\ell_{h,\infty}})^2\|h\|_{E^\ell_{h,\infty}}\|u\|_{E^\ell_{u,\infty}}.\end{aligned}$$

Finally,

$$\|F_4^\pm(u,h)\|_{H^{\ell,\ell/2}((0,+\infty)\times\Omega^\pm)} \leq C(1+\|h\|_{E^\ell_{h,\infty}})^2\|h\|_{E^\ell_{h,\infty}}\|u\|_{E^\ell_{u,\infty}}.$$

Estimate of $F_5^\pm(p,h)$.
We recall

$$F_5^\pm(p,h) = \frac{1}{\rho^\pm}\left(I - \nabla Y^\pm(X^\pm)\right)^T\nabla p^\pm.$$

Since $\nabla p^\pm \in H^{\ell,\ell/2}((0,+\infty)\times\Omega^\pm)$ and $\nabla Y(X) \in H^1(0,+\infty;H^{1+\ell}(\Omega))$, we have

$$\|F_5^\pm(p,h)\|_{H^{\ell,\ell/2}((0,+\infty)\times\Omega^\pm)} \leq C\|h\|_{E^\ell_{h,\infty}}\|p\|_{E^\ell_{p,\infty}}.$$

Estimate of $H_1(u,h)$.
We recall

$$\begin{aligned}H_1(u,h) = \Big[\mu((\nabla u + (\nabla u)^T)n_\Gamma - \mathrm{cof}(\nabla Y)^T(X)\nabla u\nabla Y(X)n_{\Gamma(t)} \\ - (\nabla Y)^T(X)\nabla u^T\mathrm{cof}(\nabla Y)(X)n_{\Gamma(t)})\Big].\end{aligned}$$

First write

$$\begin{aligned}\Big[\mu(\nabla u\, n_\Gamma - \mathrm{cof}(\nabla Y)^T(X)\nabla u\nabla Y(X)n_{\Gamma(t)})\Big] = \Big[\mu((I - \mathrm{cof}(\nabla Y)^T(X))\nabla u\, n_\Gamma \\ + \mathrm{cof}(\nabla Y)^T(X)\nabla u(I - \nabla Y(X))n_\Gamma + \mathrm{cof}(\nabla Y)^T(X)\nabla u\nabla Y(X)(n_\Gamma - n_{\Gamma(t)}))\Big].\end{aligned}$$

We have $I-(\nabla Y)^T(X), (I-\mathrm{cof}(\nabla Y)^T(X)), n_{\Gamma(t)}-n_\Gamma \in H^1(0,+\infty;H^{1/2+\ell}(\Gamma))$ with

$$\|I-(\nabla Y)^T(X)\|_{H^1(0,+\infty;H^{1/2+\ell}(\Gamma))}+\|I-\mathrm{cof}(\nabla Y)^T(X)\|_{H^1(0,+\infty;H^{1/2+\ell}(\Gamma))} \\ +\|n_{\Gamma(t)}-n_\Gamma\|_{H^1(0,+\infty;H^{1/2+\ell}(\Gamma))}\leq C\|h\|_{E^\ell_{h,\infty}}. \quad (3.7\ .5)$$

Since $\nabla u^\pm \in H^{1/2+\ell,1/4+\ell/2}((0,+\infty)\times\Gamma)$, we deduce $\Big[\nabla u\ n_\Gamma - \mathrm{cof}(\nabla Y)^T(X)\nabla u \nabla Y(X) n_{\Gamma(t)}\Big]\in H^{1/2+\ell,1/4+\ell/2}((0,+\infty)\times\Gamma)$ with

$$\Big\|\Big[\nabla u\ n_\Gamma - \mathrm{cof}(\nabla Y)^T(X)\nabla u \nabla Y(X) n_{\Gamma(t)}\Big]\Big\|_{H^{1/2+\ell,1/4+\ell/2}((0,+\infty)\times\Gamma)} \\ \leq C(1+\|h\|_{E^\ell_{h,\infty}})^2\|h\|_{E^\ell_{h,\infty}}\|u\|_{E^\ell_{u,\infty}}.$$

The term $\Big[\mu((\nabla u)^T n_\Gamma - (\nabla Y)^T(X)\nabla u^T \mathrm{cof}(\nabla Y)(X) n_{\Gamma(t)})\Big]$ can be handled exactly the same way. Finally,

$$\|H_1(u,h)\|_{H^{1/2+\ell,1/4+\ell/2}((0,+\infty)\times\Gamma)}\leq C(1+\|h\|_{E^\ell_{h,\infty}})^2\|h\|_{E^\ell_{h,\infty}}\|u\|_{E^\ell_{u,\infty}}.$$

Estimate of $H_2(p,h)$.
We recall

$$H_2(p,h)=[p(n_{\Gamma(t)}-n_\Gamma)]=[p](n_{\Gamma(t)}-n_\Gamma).$$

Since $[p]\in H^{1/2+\ell,1/4+\ell/2}((0,+\infty)\times\Gamma)$ and $n_{\Gamma(t)}-n_\Gamma\in H^1(0,+\infty;H^{1/2+\ell}(\Gamma))$, we easily have

$$\|H_2(p,h)\|_{H^{1/2+\ell,1/4+\ell/2}((0,+\infty)\times\Gamma)}\leq C\|h\|_{E^\ell_{h,\infty}}\|p\|_{E^\ell_{p,\infty}}.$$

Estimate of $H_3(u,h)$.
We recall

$$\big(H_3(u,h)\big)_i=-\Big[\mu\sum_{j=1}^{2}\big(\partial_j\mathrm{cof}(\nabla Y)_{ki}(X)+\partial_i\mathrm{cof}(\nabla Y)_{kj}(X)\big)u_k n_{\Gamma(t),i}\Big].$$

With the continuous embedding $\partial_j\mathrm{cof}(\nabla Y)(X)\in H^1(0,+\infty;H^\ell(\Omega))\cap L^2(0,+\infty;H^{1+\ell}(\Omega))\hookrightarrow H^{1/4+\ell/2}(0,+\infty;H^{3/4+\ell/2}(\Omega))$, then its trace on Γ belongs to $H^{1/4+\ell/2}(0,+\infty;H^{1/4+\ell/2}(\Omega))\cap L^2(0,+\infty;H^{1/2+\ell}(\Omega))$ with the estimate

$$\|\partial_j\mathrm{cof}(\nabla Y)(X)\|_{H^{1/4+\ell/2}(0,+\infty;H^{1/4+\ell/2}(\Omega))\cap L^2(0,+\infty;H^{1/2+\ell}(\Omega))}\leq C\|h\|_{E^\ell_{h,\infty}}.$$

With $u^\pm\in H^{3/2+\ell,3/4+\ell/2}((0,+\infty)\times\Gamma)\hookrightarrow L^\infty(0,+\infty;H^{1/2+\ell}(\Gamma))\cap H^{1/2+\ell/4}(0,+\infty;H^{1/2+\ell/2}(\Gamma))$, then $\partial_j\mathrm{cof}(\nabla Y^\pm)_{ki}(X^\pm)u^\pm_k\in H^{1/2+\ell,1/4+\ell/2}((0,+\infty)\times\Gamma)$. Since $n_{\Gamma(t)}\in H^1(0,+\infty;H^{1/2+\ell}(\Gamma))$, we finally get

$$\|H_3(u,h)\|_{H^{1/2+\ell,1/4+\ell/2}((0,+\infty)\times\Gamma)}\leq C(1+\|h\|_{E^\ell_{h,\infty}})\|h\|_{E^\ell_{h,\infty}}\|u\|_{E^\ell_{u,\infty}}.$$

Estimate of $H_4(h)$.
We recall

$$H_4(h) = \Delta h\Big(n_\Gamma - \frac{n_{\Gamma(t)}}{\sqrt{1+\mathrm{e}^{-2\omega t}\,|\nabla h|^2}}\Big) + \frac{\mathrm{e}^{-2\omega t}(\nabla^2 h\nabla h)\cdot\nabla h}{(1+\mathrm{e}^{-2\omega t}\,|\nabla h|^2)^{3/2}} n_{\Gamma(t)}.$$

With estimate (3.7 .4) and since $H^1(0,+\infty;H^{1/2+\ell}(\Gamma))$ is an algebra, we then deduce that $n_\Gamma - \dfrac{n_{\Gamma(t)}}{\sqrt{1+\mathrm{e}^{-2\omega t}\,|\nabla h|^2}} \in H^1(0,+\infty;H^{1/2+\ell}(\Gamma))$ with

$$\Big\|n_\Gamma - \frac{n_{\Gamma(t)}}{\sqrt{1+\mathrm{e}^{-2\omega t}\,|\nabla h|^2}}\Big\|_{H^1(0,+\infty;H^{1/2+\ell}(\Gamma))} \le C(1+\|h\|_{E^\ell_{h,\infty}})\|h\|_{E^\ell_{h,\infty}}.$$

With the injection $H^1(0,+\infty;H^{3/2+\ell}(\Gamma))\cap L^2(0,+\infty;H^{5/2+\ell}(\Gamma)) \hookrightarrow H^{1/2+\ell}(0,+\infty;H^2(\Gamma))$, we have $\Delta h \in H^{1/2+\ell,1/2+\ell}((0,+\infty)\times\Gamma)$. Therefore, we deduce $\Delta h\Big(n_\Gamma - \dfrac{n_{\Gamma(t)}}{\sqrt{1+\mathrm{e}^{-2\omega t}\,|\nabla h|^2}}\Big) \in H^{1/2+\ell,1/4+\ell/2}((0,+\infty)\times\Gamma)$ with

$$\Big\|\Delta h\Big(n_\Gamma - \frac{n_{\Gamma(t)}}{\sqrt{1+\mathrm{e}^{-2\omega t}\,|\nabla h|^2}}\Big)\Big\|_{H^{1/2+\ell,1/4+\ell/2}((0,+\infty)\times\Gamma)} \le C(1+\|h\|_{E^\ell_{h,\infty}})\|h\|^2_{E^\ell_{h,\infty}}.$$

With the same type of argument, one obtains $\dfrac{\mathrm{e}^{-2\omega t}(\nabla^2 h\nabla h)\cdot\nabla h}{(1+\mathrm{e}^{-2\omega t}\,|\nabla h|^2)^{3/2}} n_{\Gamma(t)} \in H^{1/2+\ell,1/4+\ell/2}((0,+\infty)\times\Gamma)$ with the estimate

$$\Big\|\frac{\mathrm{e}^{-2\omega t}(\nabla^2 h\nabla h)\cdot\nabla h}{(1+\mathrm{e}^{-2\omega t}\,|\nabla h|^2)^{3/2}} n_{\Gamma(t)}\Big\|_{H^{1/2+\ell,1/4+\ell/2}((0,+\infty)\times\Gamma)} \le C(1+\|h\|_{E^\ell_{h,\infty}})^4\|h\|^3_{E^\ell_{h,\infty}}.$$

Therefore,

$$\|H_4(h)\|_{H^{1/2+\ell,1/4+\ell/2}((0,+\infty)\times\Gamma)} \le C(1+\|h\|_{E^\ell_{h,\infty}})^4\|h\|^3_{E^\ell_{h,\infty}}.$$

Proof of $H(u,p,h) \in E^\ell_{f_{jump},\infty,u_0}$.
We already have $H(u,p,h) \in E^\ell_{f_{jump},\infty}$. It remains to prove that $H(u,p,h)$ satisfies the compatibility condition (3.2 .17) when $d=3$. With the calculation made in Section 3.2 .1, we know that

$$H(u,p,h)(0) = [\mathbb{S}(u_0,p(0))n_\Gamma] + \sigma(\Delta h_0)n_\Gamma - [\mathbb{S}(v_0(X(0)),q_0(X(0))n_{\Gamma_0}] - \sigma\nabla\cdot\Big(\frac{\nabla h_0}{\sqrt{1+|\nabla h_0|^2}}\Big)n_{\Gamma_0}, \tag{3.7 .6}$$

where v_0, q_0 are defined in (3.2 .19). Since $(u_0,p(0),h_0)$ satisfies (3.2 .8) and (3.2 .18), we have

$$[\mathbb{S}(v_0(X(0)),q_0(X(0))n_{\Gamma_0}] = -\sigma\nabla\cdot\Big(\frac{\nabla h_0}{\sqrt{1+|\nabla h_0|^2}}\Big).$$

Then, regarding the tangential part of (3.7 .6), we obtain

$$H(u,p,h)_\tau(0) = [\mu D(u_0)n_\Gamma]_\tau.$$

Therefore, $H(u,p,h)$ satisfies (3.2 .17) and belongs to $E^\ell_{f_{jump},\infty,u_0}$. □

3.7 .2 Proof of Theorem 3.2 .1

Let $(u_0, h_0) \in E^\ell_{u_0} \times E^\ell_{h_0}$ satisfying the compatibility condition (3.2 .8) and the smallness condition (3.2 .7) with $\delta > 0$ to be defined.

Before defining the fixed point map, we first would like to prove the following: take $(F(\tilde{u}, \tilde{p}, \tilde{h}), H(\tilde{u}, \tilde{p}, \tilde{h})) \in E^\ell_{f,\infty} \times E^\ell_{f_{jump},\infty,u_0}$ with $(\tilde{u}, \tilde{p}, \tilde{h}) \in E^\ell_{u,\infty,u_0} \times E^\ell_{p,\infty,(u_0,h_0)} \times E^\ell_{h,\infty,h_0}$ and let $(u, p, h, \mathbf{g}) \in E^\ell_{u,\infty,u_0} \times E^\ell_{p,\infty} \times E^\ell_{h,\infty,h_0} \times E^\ell_{\mathbf{g},\infty}$ be the unique solution of (3.5 .5) given by Theorem 3.6 .1, then $p \in E^\ell_{p,\infty,(u_0,h_0)}$.

Indeed, since we already have $p \in E^\ell_{p,\infty}$, it remains to prove that p satisfies the compatibility condition (3.2 .18) in dimension $d = 3$. With the calculation made in the proof of Lemma 3.7 .4, we have

$$H(\tilde{u}, \tilde{p}, \tilde{h})(0) = [\mathbb{S}(u_0, \tilde{p}(0))n_\Gamma] + \sigma(\Delta h_0)n_\Gamma,$$

since $(u_0, \tilde{p}(0), h_0)$ satisfies (3.2 .8) and (3.2 .18). But reading $(3.5\ .5)_5$ at time $t = 0$, we also have

$$[\mathbb{S}(u_0, p(0))n_\Gamma] + (\sigma\Delta h_0)n_\Gamma = H(\tilde{u}, \tilde{p}, \tilde{h})(0).$$

From that two equalities, we deduce that $p(0) = \tilde{p}(0)$ and since $\tilde{p}$ satisfies (3.2 .18), we have that $\tilde{p}$ satisfies (3.2 .18) and therefore, $p \in E^\ell_{p,\infty,(u_0,h_0)}$.

We now define a map

$$\mathcal{F} = \begin{array}{ccc} E^\ell_{u,\infty,u_0} \times E^\ell_{p,\infty,(u_0,h_0)} \times E^\ell_{h,\infty,h_0} \times E^\ell_{\mathbf{g},\infty} & \to & E^\ell_{u,\infty,u_0} \times E^\ell_{p,\infty,(u_0,h_0)} \times E^\ell_{h,\infty,h_0} \times E^\ell_{\mathbf{g},\infty} \\ (\tilde{u}, \tilde{p}, \tilde{h}, \tilde{\mathbf{g}}) & \mapsto & (u, p, h, \mathbf{g}) \end{array},$$

where $(u, p, h, \mathbf{g})$ is the unique solution of system (3.5 .5) given by Theorem 3.2 .6 with the right hand side $(f, f_h) = (F(\tilde{u}, \tilde{p}, \tilde{h}), H(\tilde{u}, \tilde{p}, \tilde{h})) \in E^\ell_{f,\infty} \times E^\ell_{f_{jump},\infty,u_0}$ given in (3.2 .4)–(3.2 .5).

The purpose is to apply Picard fixed point theorem to $\mathcal{F}$. Therefore, we will show that there exists a radius $\delta > 0$ for the initial data $(u_0, h_0) \in E^\ell_{u_0} \times E^\ell_{h_0}$, and a radius $0 < R < R^*$ for the space $E^\ell_{u,\infty,u_0} \times E^\ell_{p,\infty,(u_0,h_0)} \times E^\ell_{h,\infty,h_0} \times E^\ell_{\mathbf{g},\infty}$, R^* given in Lemma 3.7 .2, such that $\mathcal{F}$ maps the ball $E^\ell_{u,\infty,u_0,R} \times E^\ell_{p,\infty,(u_0,h_0),R} \times E^\ell_{h,\infty,h_0,R} \times E^\ell_{\mathbf{g},\infty,R}$ into the same ball and is contractant on this ball.

First, let $(\tilde{u}, \tilde{p}, \tilde{h}, \tilde{\mathbf{g}}) \in E^\ell_{u,\infty,u_0,R} \times E^\ell_{p,\infty,(u_0,h_0),R} \times E^\ell_{h,\infty,h_0,R} \times E^\ell_{\mathbf{g},\infty,R}$, then with Theorem 3.6 .1,

$$\begin{aligned} &\|(u, p, h, \mathbf{g})\|_{E^\ell_{u,\infty} \times E^\ell_{p,\infty} \times E^\ell_{h,\infty} \times E^\ell_{\mathbf{g},\infty}} \\ &\qquad \le C(\|(F(\tilde{u}, \tilde{p}, \tilde{h}), H(\tilde{u}, \tilde{p}, \tilde{h}))\|_{E^\ell_{f,\infty} \times E^\ell_{f_{jump},\infty}} + \|(u_0, h_0)\|_{E^\ell_{u_0} \times E^\ell_{h_0}}). \end{aligned}$$

With Lemma 3.7 .4, we have

$$\|(F(\tilde{u}, \tilde{p}, \tilde{h}), H(\tilde{u}, \tilde{p}, \tilde{h}))\|_{E^\ell_{f,\infty} \times E^\ell_{f_{jump},\infty}} \le C\|(\tilde{u}, \tilde{p}, \tilde{h}, \tilde{\mathbf{g}})\|^2_{E^\ell_{u,\infty} \times E^\ell_{p,\infty} \times E^\ell_{h,\infty} \times E^\ell_{\mathbf{g},\infty}}.$$

Therefore,

$$\|(u, p, h, \mathbf{g})\|_{E^\ell_{u,\infty} \times E^\ell_{p,\infty} \times E^\ell_{h,\infty} \times E^\ell_{\mathbf{g},\infty}} \le C_0(R^2 + \delta_0),$$

where

$$\delta_0 = \|(u_0, h_0)\|_{E^\ell_{u_0} \times E^\ell_{h_0}} < \delta,$$

and C_0 is independent of δ_0, δ and R. Therefore, we first require

$$C_0(R^2 + \delta) < R. \tag{3.7 .7}$$

Let $(\tilde{u}, \tilde{p}, \tilde{h}, \tilde{\mathbf{g}}), (\hat{u}, \hat{p}, \hat{h}, \hat{\mathbf{g}}) \in E^{\ell}_{u,\infty,u_0,R} \times E^{\ell}_{p,\infty,(u_0,h_0),R} \times E^{\ell}_{h,\infty,h_0,R} \times E^{\ell}_{\mathbf{g},\infty,R}$. Combining Theorem 3.6 .1 and the lipschitz estimates of Lemma 3.7 .4, we have

$$\|\mathcal{F}(\tilde{u}, \tilde{p}, \tilde{h}, \tilde{\mathbf{g}}) - \mathcal{F}(\hat{u}, \hat{p}, \hat{h}, \hat{\mathbf{g}})\|_{E^{\ell}_{u,\infty} \times E^{\ell}_{p,\infty} \times E^{\ell}_{h,\infty} \times E^{\ell}_{\mathbf{g},\infty}} \\ \leq C_1 R^2 \|(\tilde{u}, \tilde{p}, \tilde{h}, \tilde{\mathbf{g}}) - (\hat{u}, \hat{p}, \hat{h}, \hat{\mathbf{g}})\|_{E^{\ell}_{u,\infty} \times E^{\ell}_{p,\infty} \times E^{\ell}_{h,\infty} \times E^{\ell}_{\mathbf{g},\infty}}.$$

where $C_1 > 0$ is independent of R. Therefore, we require

$$C_1 R^2 < 1. \tag{3.7 .8}$$

Lastly, so as to fullfill estimate (3.2 .9), we also require

$$R \leq C_2 \delta_0, \tag{3.7 .9}$$

for some constant $C_2 > 0$ of our choice.

With

$$C_2 = 2C_0 \quad \text{and} \quad R = C_2\delta_0 \quad \text{and} \quad \delta = \frac{1}{C_2} \min\left\{\frac{R^*}{2}, \frac{1}{4C_0}, \frac{1}{2\sqrt{C_1}}\right\},$$

inequalities $0 < R < R^*$, (3.7 .7), (3.7 .8) and (3.7 .9) are satisfied. Picard fixed point theorem applies and concludes the proof of Theorem 3.2 .1.

Bibliography

[1] H. Abels. On generalized solutions of two-phase flows for viscous incompressible fluids. *Interfaces Free Bound.*, 9(1):31–65, 2007.

[2] G. Allain. Small-time existence for the Navier-Stokes equations with a free surface. *Appl. Math. Optim.*, 16(1):37–50, 1987.

[3] M. Badra. Feedback stabilization of the 2-D and 3-D Navier-Stokes equations based on an extended system. *ESAIM Control Optim. Calc. Var.*, 15(4):934–968, 2009.

[4] M. Badra. Lyapunov function and local feedback boundary stabilization of the Navier-Stokes equations. *SIAM J. Control Optim.*, 48(3):1797–1830, 2009.

[5] H. Bae. Solvability of the free boundary value problem of the Navier-Stokes equations. *Discrete Contin. Dyn. Syst.*, 29(3):769–801, 2011.

[6] V. Barbu, I. Lasiecka, and R. Triggiani. Tangential boundary stabilization of Navier-Stokes equations. *Mem. Amer. Math. Soc.*, 181(852):x+128, 2006.

[7] J. T. Beale. The initial value problem for the Navier-Stokes equations with a free surface. *Comm. Pure Appl. Math.*, 34(3):359–392, 1981.

[8] J. T. Beale. Large-time regularity of viscous surface waves. *Arch. Rational Mech. Anal.*, 84(4):307–352, 1983/84.

[9] J. T. Beale and T. Nishida. Large-time behavior of viscous surface waves. In *Recent topics in nonlinear PDE, II (Sendai, 1984)*, volume 128 of *North-Holland Math. Stud.*, pages 1–14. North-Holland, Amsterdam, 1985.

[10] A. Bensoussan, G. Da Prato, M. C. Delfour, and S. K. Mitter. *Representation and control of infinite dimensional systems.* Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA, second edition, 2007.

[11] D. Coutand and S. Shkoller. Unique solvability of the free-boundary Navier-Stokes equations with surface tension. *ArXiv Mathematics e-prints*, December 2002.

[12] I. V. Denisova. A priori estimates for the solution of the linear nonstationary problem connected with the motion of a drop in a liquid medium. *Trudy Mat. Inst. Steklov.*, 188:3–21, 191, 1990. Translated in Proc. Steklov Inst. Math. **1**991, no. 3, 1–24, Boundary value problems of mathematical physics, 14 (Russian).

[13] I. V. Denisova. Solvability in Hölder spaces of a linear problem on the motion of two fluids separated by a closed surface. *Algebra i Analiz*, 5(4):122–148, 1993.

[14] I. V. Denisova. Problem of the motion of two viscous incompressible fluids separated by a closed free interface. *Acta Appl. Math.*, 37(1-2):31–40, 1994. Mathematical problems for Navier-Stokes equations (Centro, 1993).

[15] I. V. Denisova. Global solvability of a problem on two fluid motion without surface tension. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 348(Kraevye Zadachi Matematicheskoi Fiziki i Smezhnye Voprosy Teorii Funktsii. 38):19–39, 303, 2007.

[16] I. V. Denisova. Global L_2-solvability of a problem governing two-phase fluid motion without surface tension. *Port. Math.*, 71(1):1–24, 2014.

[17] I. V. Denisova and V. A. Solonnikov. Solvability of a linearized problem on the motion of a drop in a fluid flow. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 171(Kraev. Zadachi Mat. Fiz. i Smezh. Voprosy Teor. Funktsii. 20):53–65, 184, 1989.

[18] I. V. Denisova and V. A. Solonnikov. Solvability in Hölder spaces of a model initial-boundary value problem generated by a problem on the motion of two fluids. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 188(Kraev. Zadachi Mat. Fiz. i Smezh. Voprosy Teor. Funktsiĭ. 22):5–44, 186, 1991.

[19] I. V. Denisova and V. A. Solonnikov. Classical solvability of the problem of the motion of two viscous incompressible fluids. *Algebra i Analiz*, 7(5):101–142, 1995.

[20] C. Fabre and G. Lebeau. Prolongement unique des solutions de l'equation de Stokes. *Comm. Partial Differential Equations*, 21(3-4):573–596, 1996.

[21] A. V. Fursikov. Stabilizability of two-dimensional Navier-Stokes equations with help of a boundary feedback control. *J. Math. Fluid Mech.*, 3(3):259–301, 2001.

[22] A. V. Fursikov. Stabilization for the 3D Navier-Stokes system by feedback boundary control. *Discrete Contin. Dyn. Syst.*, 10(1-2):289–314, 2004. Partial differential equations and applications.

[23] G. Grubb and V. A. Solonnikov. Boundary value problems for the nonstationary Navier-Stokes equations treated by pseudo-differential methods. *Math. Scand.*, 69(2):217–290 (1992), 1991.

[24] Y. Guo and I. Tice. Almost exponential decay of periodic viscous surface waves without surface tension. *Arch. Ration. Mech. Anal.*, 207(2):459–531, 2013.

[25] Y. Guo and I. Tice. Decay of viscous surface waves without surface tension in horizontally infinite domains. *Anal. PDE*, 6(6):1429–1533, 2013.

[26] Y. Guo and I. Tice. Local well-posedness of the viscous surface wave problem without surface tension. *Anal. PDE*, 6(2):287–369, 2013.

[27] Y. Hataya. Global solution of two-layer navier-stokes flow. *Nonlinear Analysis: Theory, Methods and Applications*, 63(5-7):1409–1420, 2005.

[28] Y. Hataya. Decaying solution of a Navier-Stokes flow without surface tension. *J. Math. Kyoto Univ.*, 49(4):691–717, 2009.

[29] A. Inoue and M. Wakimoto. On existence of solutions of the Navier-Stokes equation in a time dependent domain. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 24(2):303–319, 1977.

[30] S. Kesavan and J.-P. Raymond. On a degenerate Riccati equation. *Control Cybernet.*, 38(4B):1393–1410, 2009.

[31] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. I.* Springer-Verlag, New York-Heidelberg, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181.

[32] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. II.* Springer-Verlag, New York-Heidelberg, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 182.

[33] T. Nishida, Y. Teramoto, and H. Yoshihara. Global in time behavior of viscous surface waves: horizontally periodic motion. *J. Math. Kyoto Univ.*, 44(2):271–323, 2004.

[34] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[35] J. Prüss and G. Simonett. On the Rayleigh-Taylor instability for the two-phase Navier-Stokes equations. *Indiana Univ. Math. J.*, 59(6):1853–1871, 2010.

[36] J. Prüss and G. Simonett. On the two-phase Navier-Stokes equations with surface tension. *Interfaces Free Bound.*, 12(3):311–345, 2010.

[37] J.-P. Raymond. Feedback boundary stabilization of the two-dimensional Navier-Stokes equations. *SIAM J. Control Optim.*, 45(3):790–828, 2006.

[38] J.-P. Raymond. Feedback boundary stabilization of the three-dimensional incompressible Navier-Stokes equations. *J. Math. Pures Appl. (9)*, 87(6):627–669, 2007.

[39] J.-P. Raymond and L. Thevenet. Boundary feedback stabilization of the two dimensional Navier-Stokes equations with finite dimensional controllers. *Discrete Contin. Dyn. Syst.*, 27(3):1159–1187, 2010.

[40] V. A. Solonnikov. Solvability of the problem of the motion of a viscous incompressible fluid that is bounded by a free surface. *Izv. Akad. Nauk SSSR Ser. Mat.*, 41(6):1388–1424, 1448, 1977.

[41] V. A. Solonnikov. Solvability of the problem of evolution of an isolated amount of a viscous incompressible capillary fluid. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 140:179–186, 1984. Mathematical questions in the theory of wave propagation, 14.

[42] V. A. Solonnikov. Unsteady flow of a finite mass of a fluid bounded by a free surface. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 152(Kraev. Zadachi Mat. Fiz. i Smezhnye Vopr. Teor. Funktsiĭ18):137–157, 183–184, 1986.

[43] V. A. Solonnikov. Unsteady motion of an isolated volume of a viscous incompressible fluid. *Izv. Akad. Nauk SSSR Ser. Mat.*, 51(5):1065–1087, 1118, 1987.

[44] V. A. Solonnikov. Unsteady motions of a finite isolated mass of a self-gravitating fluid. *Algebra i Analiz*, 1(1):207–249, 1989.

[45] D. L. G. Sylvester. Large time existence of small viscous surface waves without surface tension. *Comm. Partial Differential Equations*, 15(6):823–903, 1990.

[46] N. Tanaka. Global existence of two phase nonhomogeneous viscous incompressible fluid flow. *Comm. Partial Differential Equations*, 18(1-2):41–81, 1993.

[47] N. Tanaka. Two-phase free boundary problem for viscous incompressible thermocapillary convection. *Japan. J. Math. (N.S.)*, 21(1):1–42, 1995.

[48] A. Tani. Small-time existence for the three-dimensional Navier-Stokes equations for an incompressible fluid with a free surface. *Arch. Rational Mech. Anal.*, 133(4):299–331, 1996.

[49] A. Tani and N. Tanaka. Large-time existence of surface waves in incompressible viscous fluids with or without surface tension. *Arch. Rational Mech. Anal.*, 130(4):303–314, 1995.

[50] Y. Wang and I. Tice. The viscous surface-internal wave problem: nonlinear Rayleigh-Taylor instability. *Comm. Partial Differential Equations*, 37(11):1967–2028, 2012.

[51] Y. Wang, I. Tice, and C. Kim. The viscous surface-internal wave problem: global well-posedness and decay. *Arch. Ration. Mech. Anal.*, 212(1):1–92, 2014.

[52] M. Wilke. Rayleigh-Taylor instability for the two-phase Navier-Stokes equations with surface tension in cylindrical domains. *ArXiv e-prints*, March 2017.

[53] L. Xu and Z. Zhang. On the free boundary problem to the two viscous immiscible fluids. *J. Differential Equations*, 248(5):1044–1111, 2010.

Analyse et contrôle de modèles d'écoulements fluides

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Résumé : Dans cette thèse, nous étudions le caractère bien posé, le contrôle et la stabilisation de quelques modèles d'écoulements fluides.

Dans la première partie, on s'intéresse aux équations de Navier-Stokes compressibles 1D. Un résultat de contrôlabilité locale aux trajectoires par contrôle frontière est établi sous l'hypothèse géométrique de vidage du domaine par le flot de la trajectoire cible. La principale nouveauté de ce travail est que les trajectoires cibles peuvent être choisies non constantes.

Dans la deuxième partie, nous travaillons sur un modèle de frontière immergée dans un fluide visqueux incompressible en 2D et 3D. Contrairement à la méthode des frontières immergées de Peskin où la force générée par la structure dépend de ses propriétés élastiques et géométriques, nous considérons que la force de la structure est une donnée du système. Nous montrons alors des résultats d'existence locale en temps et en tout temps à données petites de solutions fortes. Ce travail est un premier pas vers l'analyse mathématique de la méthode des frontières immergées de Peskin.

Dans la dernière partie, nous étudions la stabilisation d'une interface entre deux couches de fluides visqueux non miscibles soumis à l'effet de tension de surface en 2D et 3D. Nous montrons qu'au moyen d'un contrôle de dimension finie agissant sur une partie de la frontière d'un seul des deux fluides, le système est exponentiellement stabilisable à tout taux de décroissance autour de la configuration plate avec fluides au repos. Ce travail est une première étape dans l'étude de la stabilisation des instabilités de Rayleigh-Taylor.

Mots-clés : équations de Navier-Stokes, frontières immergées, tension de surface, contrôlabilité locale aux trajectoires, stabilisation d'interface fluide.

Abstract : In this work we study the well-posedness, the control and the stabilization of some fluid flow models.

First, we focus on the 1D compressible Navier-Stokes equations. Under a geometric assumption on the flow of the target velocity corresponding to the possibility of emptying the domain under the action of the flow, we prove the local exact boundary controllability to trajectory. The main novelty of this work is that the target trajectory can now depend on time and space.

In the second part, we study a model of an immersed boundary in an incompressible viscous fluid in 2D and 3D. Contrary to Peskin's Immersed Boundary Method where the boundary force depends on the elastic properties of the structure and its geometry, we consider that the boundary force is a given data. Two results are established: a local in time existence of strong solutions and an existence of strong solutions for all time with small data. This work is a first step on the mathematical analysis of Peskin's Immersed Boundary Method.

Finally, we are interested in the stabilization of the interface between two fluid layers coupled through surface tension effect in 2D and 3D. We prove that the system is exponentially stabilizable at any rate around a flat configuration with fluids at rest using a control of finite dimension acting locally at one fluid boundary. This work is a first step in the study of the stabilization of Rayleigh-Taylor instabilities.

Keywords : Navier-Stokes equations, immersed boundary, surface tension, local controllability to trajectory, bifluide interface stabilization.