Relaxation of two-phase flows via the Born-Infeld system

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Abstract The aim of this paper is to call the reader's attention to an original scheme for solving special scalar conservation laws and drift-flux two-phase flow models. The scalar conservation law is $u_t + f(u)_x = 0$ in which $f(u) = u(1-u)g(u)$. The new scheme is based on a relaxation strategy, but unlike Xin and Jin's approach [69], it is more "natural" insofar as there is no free parameter involved. Instead, it makes use of the 2×2 Born-Infeld system [12, 59] and enables one to achieve a better control of the velocity variables $w = (1-u)g$ and $z = -ug$. For the scalar conservation law, stability and monotonicity are established under appropriate conditions on g. Several numerical experiments are shown.

D.1 Introduction

Initially, we were interested in relaxation methods for solving a two-phase flow model. We devised an explicit scheme [6] and its semi-implicit version [7]. When we tried to extend these schemes to the multi-component case, a major difficulty arised, the essence of which boils down to the following prototype.

Let us consider the scalar conservation law

$$u_t + \{u(1-u)g(u)\}_x = 0, \tag{D.1}$$

where $u \in [0,1]$ represents a phase fraction and g is a C^1 function of u, bounded at $u = 0$ and $u = 1$. Subscripts t and x denote partial derivatives with respect to time and space. For convenience, introduce the flux

$$f = u(1-u)g \tag{D.2}$$

as well as the phase velocities

$$w = (1-u)g = \frac{f}{u} \qquad \text{and} \qquad z = -ug = -\frac{f}{1-u}. \tag{D.3}$$

Equation (D.1) can be thought of as

$$u_t + \{uw\}_x = 0, \tag{D.4}$$

which highlights w as the convecting speed for the phase u. Likewise, multiplying (D.1) by -1, we obtain

$$(1-u)_t + \{(1-u)z\}_x = 0, \tag{D.5}$$

which evidences z as the convecting speed for the other phase $1-u$. Understandably, $g = w - z$ is called slip velocity between the two phases.

Xin and Jin's [69] relaxation system for (D.1), namely

$$\left| \begin{array}{l} u_t + \quad F_x = 0 \\ F_t + a^2 u_x = \lambda[f(u) - F], \end{array} \right. \tag{D.6}$$

involves two free parameters a and λ. It is classically solved, at each iteration, by setting $\lambda = 0$ first and by projecting afterward, i.e., $F \equiv f(u)$ (in [69], this scheme is called the "relaxed scheme"). The technical requirement to ensure stability and monotonicity is the Whitham condition

$$a > \max_{u \in [u_L, u_R]} |f'(u)|, \tag{D.7}$$

where f' is the derivative of f with respect to u. In practice, the right-hand side of (D.7) is not always available. Therefore, one has to content oneself with the weak condition

$$a > \max_{u \in \{u_L; u_R\}} |f'(u)|, \tag{D.8}$$

where u_L and u_R are the left and right state of a local Riemann problem.

Xin and Jin's method is what we refer to as the "traditional" or "brute-force" relaxation strategy. Unfortunately, when extended to our initial multi-component system, the traditional relaxation is spurred with a division-by-zero problem for the computation of the variables $W^\star$ and $Z^\star$ defined by

$$W^\star = \frac{F^\star}{u^\star} \qquad \text{and} \qquad Z^\star = -\frac{F^\star}{1-u^\star}. \tag{D.9}$$

Another relaxation strategy for (D.1) must be worked out, which avoids the disadvantage of dividing by $u^\star$ or $1-u^\star$.

In this paper, we would like to put forward a new relaxation approach to (D.1), which makes explicit use of the velocity variables w et z. The outline is as follows. First, we explain how —to the astonishment of ourselves— the quasi-linear Born-Infeld equations

$$\left| \begin{array}{l} w_t + z w_x = 0 \\ z_t \ + w z_x = 0 \end{array} \right. \tag{D.10}$$

can be interpreted as a relaxation system for (D.1). Then, we elaborate on various properties of this unexpected and non-standard system. In particular, the Riemann problem associated with it is readily solved, which provides us with bounded values for $w^\star$ and $z^\star$. This feature is a tremendous advantage of (D.10) over (D.6). It is important to notice that the characteristic fields of the system D.10 are linearly degenerate, which is a central property, shared with (D.6). This advantage is one of a series of other enjoyable properties that will be reviewed afterward.

This strategy has been applied on the drift-flux model which is a conservative two-phase flow model. We will verify that the classical Zuber-Findlay slip law obeys to a Whitham-like stability condition. In pratice, the relaxation scheme behave well on Riemann problems for several slip laws that have a physical meaning which will be highlighted by the numerical experiments.

This paper is organized as follows. Sections 2 and 3 are devoted to the scalar conservation laws while sections 4 and 5 are devoted to the drift-flux model. In section 2, we present the properties of the numerical scheme for the scalar conservation law, including the solution of the Riemann problem associated and the stability properties. Section 3 is devoted to the numerical results for the scalar conservation law. In section 4, we show how to construct a relaxation method on the drift-flux model and the numerical experiments for this model are shown in section 5. Section 6 concludes.

D.2 Relaxation scheme for the scalar conservation law

D.2.1 Basic ideas

A most natural way to ensure finite values for $W^\star$ and $Z^\star$ is to consider a system in which the primitive variables are the phase velocities (w, z). In what follows, we describe the road toward such a system.

First, note that by inverting (D.3), we have

$$u = \frac{z}{z-w} \qquad \text{and} \qquad f = \frac{wz}{z-w}, \tag{D.11}$$

provided that $w \neq z$. Here, f is a function of u, so that (u, f) are not independent variables. Likewise, since (w, z) are both functions of u, they are not independent from each other.

In order to avoid ambiguity between dependent variables and independent variables, we switch to the notation (u, F) and call the components of this pair relaxed phase variables. Note that, although F embodies the relaxed value of the flux f, it should not be confused with its Xin-Jin counterpart in (D.6). We also consider the pair (W, Z), whose components are named relaxed velocity variables. The transformation $(u, F) \longmapsto (W, Z)$ is given by

$$W(u, F) = \frac{F}{u} \qquad \text{and} \qquad Z(u, F) = -\frac{F}{1-u}, \tag{D.12}$$

while the inverse transformation $(W, Z) \longmapsto (u, F)$ is given by

$$u(W, Z) = \frac{Z}{Z-W} \qquad \text{and} \qquad F(W, Z) \equiv \frac{WZ}{Z-W}. \tag{D.13}$$

We insist on the fact that now (u, F) are independent variables, as well as (W, Z). The only states for which $F = f(u)$, $W = w(u)$ and $Z = z(u)$ are equilibrium states.

D.2.1.1 Non-conservative form

We look for a quasi-linear system under its most general form

$$\begin{pmatrix} W \\ Z \end{pmatrix}_t + \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} W \\ Z \end{pmatrix}_x = \begin{pmatrix} r \\ s \end{pmatrix}, \tag{D.14}$$

where $(\alpha, \beta, \gamma, \delta)$ and (r, s) are functions of (W, Z) such that, by a suitable combination of equations in (D.14), we are in a position to obtain

$$u_t + F_x = 0. \tag{D.15}$$

If successful, we will say that (D.14) is a relaxation system for (D.15).

Lemma D.2.1 *(D.14) is a relaxation system for (D.15) if and only if*

$$Z\alpha - W\gamma = Z^2, \quad Z\beta - W\delta = -W^2 \quad \textit{and} \quad Zr - Ws = 0. \tag{D.16}$$

PROOF- Let us multiply both sides of (D.14) by (u_W, u_Z). This leads us to

$$u_t + (u_W\alpha + u_Z\gamma)W_x + (u_W\beta + u_Z\delta)Z_x = u_W r + u_Z s. \tag{D.17}$$

The left-hand side is equal to $u_t + F_x$ if and only if $F_W = u_W\alpha + u_Z\gamma$ and $F_Z = u_W\beta + u_Z\delta$, whereas the right-hand side vanishes if and only if $u_W\, r + u_Z\, s = 0$. Using the Jacobian matrix

$$\begin{pmatrix} u_W & u_Z \\ F_W & F_Z \end{pmatrix} = \frac{1}{(Z-W)^2}\begin{pmatrix} Z & -W \\ Z^2 & -W^2 \end{pmatrix} \tag{D.18}$$

and simplifying by $(W-Z)^{-2}$, we obtain the desired result. □

There are infinitely many solutions to (D.16). The most obvious one, though, corresponds to

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} r \\ s \end{pmatrix} = \mu \begin{pmatrix} W \\ Z \end{pmatrix} \tag{D.19}$$

where μ is any function of (W, Z). The exact value for μ will be supplied later on. For the time being, let us work with $\mu = 0$, so that the resulting relaxation system reads

$$\left| \begin{array}{l} W_t + ZW_x = 0 \\ Z_t + WZ_x = 0. \end{array} \right. \tag{D.20}$$

We propose to call (D.20) the Born-Infeld inner relaxation associated with equation (D.15).

Serre [59] mentioned several properties of (D.20). Historically, the Born-Infeld equations [12, 64] stem from quantum mechanics and are written in a slightly more sophisticated form. To our knowledge, this is the first time that :

- the Born-Infeld system, at least under Serre's form, appears in the design of relaxation schemes, a context totally unrelated to field theory ;
- a non-linear combination of the relaxation equations (D.20) must be used in order to recover the original equation (D.15).

Proposition 1 *The eigenvalues (Z, W) of (D.20), both linearly degenerate, are respectively associated with the strong Riemann invariants W and Z.*

PROOF- The eigenfield Z is associated with the eigenvector $(1, 0)^T$. This vector is orthogonal to $\nabla Z = (0, 1)$, hence Z is linearly degenerate. Same argument for W. As for the strong Riemann invariants, these can be seen directly from the equations. □

As was already the case for other relaxation schemes [15, 17, 20, 36], linear degeneracy of the eigenfields is a crucial property. Among other nice features, it makes Riemann problems easy to solve and guarantees equivalence between weak solutions to various conservative and non-conservative forms of the system.

D.2.1.2 Range of variables

Before tackling with Riemann problems, let us clarify the range of the variables involved.

Proposition 2 $u \in [0, 1]$ *if and only if $WZ \leq 0$. In other words, the relaxed velocity variables W and Z must be of opposite signs.*

PROOF- According to (D.12), we have

$$WZ \times u(1-u) = -F^2 \leq 0. \tag{D.21}$$

Hence, $u \in [0,1]$ if and only if $WZ \leq 0$. □

In the (W,Z)-plane, we consider the second and fourth quadrants

$$\Omega_2 = \{W \leq 0,\ Z \geq 0\} \qquad \text{and} \qquad \Omega_4 = \{W \geq 0,\ Z \leq 0\}. \tag{D.22}$$

The set of admissible states is then $\Omega = \Omega_2 \cup \Omega_4$. Over Ω, the equality $W = Z$ occurs at $(W,Z) = (0,0)$ alone. As a consequence, formulas (D.13) are well-defined over $\Omega \setminus (0,0)$.

Lemma D.2.2 *In the (W,Z)-plane,*
- *the isoline of $u \in [0,1]$ is the straightline $\mathcal{L}_u$ passing through the origin $(0,0)$ and directed by the vector $(1-u,-u)$;*
- *the isoline of $F \in \mathbb{R}$ is the hyperbola $\mathcal{H}_F$ passing through the origin $(0,0)$ and having for asymptotes the lines $W = F$, $Z = -F$. The hyperbola degenerates into the two axes $W = 0$ and $Z = 0$ if and only if $F = 0$.*

Proof- Left to the reader (see figure D.2.1.2). □

Since $\mathcal{L}_u$ contain $(0,0)$ for any $(u,F) \in [0,1] \times \mathbb{R}$, the state u, computed from (D.13), cannot be given a sense at the origin by a kind of continuity argument. On the contrary, the analysis of the hyperbola $\mathcal{H}_F$ shows that F can be defined on $(0,0)$ because the relaxed variable (W,Z) lie in the subdomain Ω :

$$\lim_{(W,Z)\in\Omega\to 0} F(W,Z) = 0. \tag{D.23}$$

Therefore, we will set $F(0,0) = 0$.

D.2.1.3 Riemann problem

A generic point in Ω is designated by $\mathbf{v} \equiv (W,Z)$. Let $\mathbf{v}_L \equiv (W_L, Z_L)$ and $\mathbf{v}_R \equiv (W_R, Z_R)$ be two given states, referred to as *L*eft and *R*ight. We wish to express the solution of the Riemann problem associated with these states. More specifically, we seek the solution over $x \in \mathbb{R}$, at $t > 0$, of (D.20) when

$$\mathbf{v}(t=0;x) = \begin{cases} \mathbf{v}_L & \text{if } x < 0 \\ \mathbf{v}_R & \text{if } x > 0 \end{cases} \tag{D.24}$$

Theorem 1 *Let $\mathbf{v}_L = (W_L, Z_L) \in \Omega$ and $\mathbf{v}_R = (W_R, Z_R) \in \Omega$. Consider the two velocities*

$$\sigma_L = \min(W_L, Z_L) \qquad \textit{and} \qquad \sigma_R = \max(W_R, Z_R). \tag{D.25}$$

Then,
1. $\boxed{\text{Case 1}}$ *if $W_L W_R \geq 0$ and $Z_L Z_R \geq 0$, the Riemann solution to (D.20) consists of the 3 states*

$$\mathbf{v}(t,x) = \begin{cases} \mathbf{v}_L & \textit{if } x/t < \sigma_L \\ \widetilde{\mathbf{v}} & \textit{if } x/t \in]\sigma_L, \sigma_R[\\ \mathbf{v}_R & \textit{if } x/t > \sigma_R \end{cases} \tag{D.26}$$

with $\widetilde{\mathbf{v}} \equiv (\widetilde{W}, \widetilde{Z}) = (W_L^- + W_R^+, Z_L^- + Z_R^+)$;

2. $\boxed{\text{Case 2}}$ *if $W_L W_R < 0$ or $Z_L Z_R < 0$, the Riemann solution to (D.20) consists of the 4 states*

$$\mathbf{v}(t,x) = \begin{cases} \mathbf{v}_L & \textit{if } x/t < \sigma_L \\ \widetilde{\mathbf{v}}_L & \textit{if } x/t \in]\sigma_L, 0[\\ \widetilde{\mathbf{v}}_R & \textit{if } x/t \in]0, \sigma_R[\\ \mathbf{v}_R & \textit{if } x/t > \sigma_R \end{cases} \tag{D.27}$$

with $\widetilde{\mathbf{v}}_L \equiv (\widetilde{W}_L, \widetilde{Z}_L) = (W_L^-, Z_L^-)$ and $\widetilde{\mathbf{v}}_R \equiv (\widetilde{W}_R, \widetilde{Z}_R) = (W_R^+, Z_R^+)$.

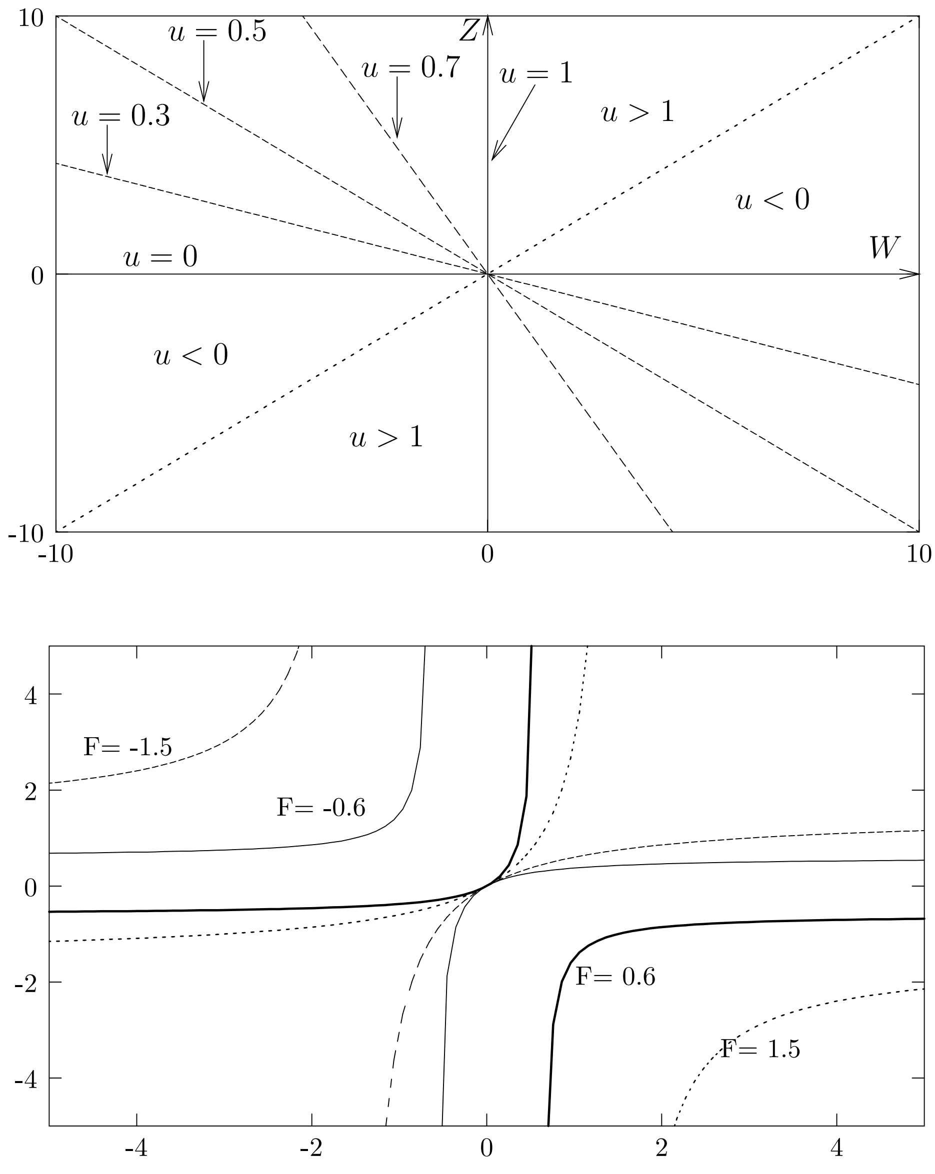

FIG. D.1 – Isolines of u and F

The notations $v^- = \min(v, 0)$ and $v^+ = \max(v, 0)$ stand for the negative and positive parts of any real number $v \in \mathbb{R}$.

PROOF- By virtue of linear degeneracy, shock curves coincide with rarefaction curves. In the (W, Z)-plane, W-curves are vertical lines, while Z-curves are horizontal lines. Riemann problems can thus be solved "by hand," depending on the positions of the two initial states, as depicted in Fig. D.2 and D.3. There are 4 different cases, which can be divided into 2 major cases (1-2), each having 2 subcases (a-b). For each major case, a synthetic formula can be found to encapsulate the 2 subcases. □

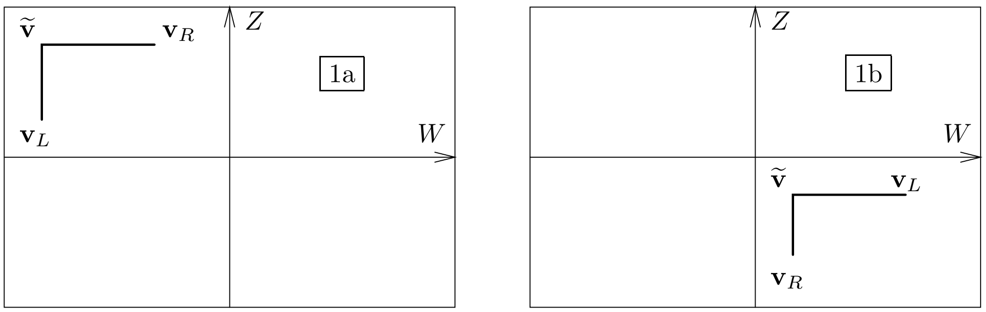

FIG. D.2 – Solution to inner Riemann problem (D.20) in Case 1.

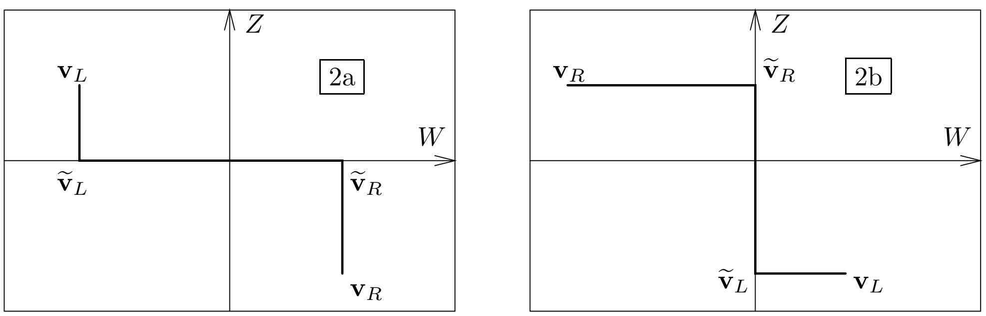

FIG. D.3 – Solution to inner Riemann problem (D.20) in Case 2.

D.2.1.4 Numerical flux

We now concentrate on the flux that is to be plugged in the numerical scheme. In this respect, it is well known [32] that only the constant values

$$\left| \begin{array}{l} W^\star = W(t > 0, x = 0) \\ Z^\star = Z(t > 0, x = 0) \end{array} \right. \tag{D.28}$$

need to be extracted from the full Riemann solution.

Let us assume that $\mathbf{v}^\star \equiv (W^\star, Z^\star)$ has been found by means of some procedure and that $\mathbf{v}^\star \in \Omega \setminus (0, 0)$. To emphasis the dependency of these variables on the initial data, we compel ourselves to write

$$W^\star = W^\star(\mathbf{v}_L, \mathbf{v}_R) \qquad \text{and} \qquad Z^\star = Z^\star(\mathbf{v}_L, \mathbf{v}_R). \tag{D.29}$$

Then, the numerical flux for the Born-Infeld relaxation scheme is defined as

$$F^\star \equiv F(W^\star, Z^\star) = \frac{W^\star Z^\star}{Z^\star - W^\star}, \tag{D.30}$$

a formula which makes sense because $\mathbf{v}^\star \in \Omega \setminus (0,0)$. Again, to emphasis the dependency of the flux on the initial data, we will occasionally resort to the notation

$$F^\star \equiv \mathcal{F}(\mathbf{v}_L, \mathbf{v}_R) = F(W^\star(\mathbf{v}_L, \mathbf{v}_R), Z^\star(\mathbf{v}_L, \mathbf{v}_R)). \tag{D.31}$$

Using this notation, the first-order explicit scheme to update u in (D.1) over a regular grid with a space meshing equal to Δx reads

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}[\mathcal{F}(\mathbf{v}_i^n, \mathbf{v}_{i+1}^n) - \mathcal{F}(\mathbf{v}_{i-1}^n, \mathbf{v}_i^n)], \tag{D.32}$$

the $\mathbf{v}$'s being evaluated in each cell at the equilibrium state, e.g.,

$$\mathbf{v}_i^n \equiv (w(u_i^n), z(u_i^n)) = ((1-u_i^n)g(u_i^n), -u_i^n g(u_i^n)). \tag{D.33}$$

The question to be addressed is whether or not the variables enumerated in (D.28) can be always defined in an unambiguous way ?

Proposition 3 *Let $\mathbf{v}_L \neq \mathbf{v}_R \in \Omega$. Then,*

1. *in Case 1,*

if	$W^\star$	$Z^\star$
$\widetilde{W} \neq 0$ & $\widetilde{Z} \neq 0$	$\widetilde{W}$	$\widetilde{Z}$
$\widetilde{W} = 0$ & $\widetilde{Z} \neq 0$	0	$\nexists$
$\widetilde{W} \neq 0$ & $\widetilde{Z} = 0$	$\nexists$	0
$\widetilde{W} = 0$ & $\widetilde{Z} = 0$	$\nexists$	$\nexists$

(D.34)

2. *in Case 2,*

if	$W^\star$	$Z^\star$
$\widetilde{W}_L\widetilde{W}_R \neq 0$ & $\widetilde{Z}_L = \widetilde{Z}_R = 0$	$\nexists$	0
$\widetilde{Z}_L\widetilde{Z}_R \neq 0$ & $\widetilde{W}_L = \widetilde{W}_R = 0$	0	$\nexists$
otherwise	$\nexists$	$\nexists$

(D.35)

Proof- The proof relies upon thorough examination of Fig. D.2, as well as the vanishing cases of one or more speeds. □

Even when $\mathbf{v}^\star$ is not well-defined, it is allways possible to attribute a value to $F^\star$. After all, this is what we are most concerned about, from the standpoint of (D.32).

Theorem 2 *Let $(\mathbf{v}_L, \mathbf{v}_R) \in \Omega^2$, the two components of each pair being not necessarily distinct. Then,*

1. *in Case 1,*

if	$F^\star$
$\widetilde{W} \neq 0$ & $\widetilde{Z} \neq 0$	$F(\widetilde{W}, \widetilde{Z})$
$\widetilde{W} = 0$ & $\widetilde{Z} \neq 0$	0
$\widetilde{W} \neq 0$ & $\widetilde{Z} = 0$	0
$\widetilde{W} = 0$ & $\widetilde{Z} = 0$	0

(D.36)

2. *in Case 2,*

if	$F^\star$
$\widetilde{W}_L\widetilde{W}_R \neq 0$ & $\widetilde{Z}_L = \widetilde{Z}_R = 0$	0
$\widetilde{Z}_L\widetilde{Z}_R \neq 0$ & $\widetilde{W}_L = \widetilde{W}_R = 0$	0
otherwise	0

(D.37)

PROOF- If $(W^\star, Z^\star) = (0, \nexists)$ or $(\nexists, 0)$, the unknown $\nexists$ component has to be considered as arbitrary : there is no reason for it to coincide with 0. This means that, in the (W, Z)-plane, $\mathbf{v}^\star$ belongs to one of the coordinate axes, excluding the origin. According to Lemma D.2.2, this is the locus of $F = 0$. Therefore, $F^\star = 0$. □

D.2.2 Further properties

From (D.3), we already said that $g = w - z$ acts as a velocity slip. This suggests to introduce $G \equiv W - Z$ for the relaxed version of g. From now on, we choose (u, G) as independent variables, in substitution for (u, F). The mapping $(u, G) \longmapsto (W, Z)$ is given by

$$W(u, G) = (1 - u)G \qquad \text{and} \qquad Z(u, G) = -uG, \tag{D.38}$$

while the inverse transformation is

$$u(W, Z) = \frac{Z}{Z - W} \qquad \text{and} \qquad G(W, Z) = W - Z. \tag{D.39}$$

D.2.2.1 Various forms with RHS

Let us look back at equation (D.19). So far, we have worked with $\mu = 0$. It is now time to indicate the appropriate right-hand side (RHS) of the Born-Infeld inner relaxation system. The easiest way to lay ideas out is to disclose the final result right away.

Consider (D.19) with

$$\mu = \frac{\lambda}{W - Z}[g(u) - G], \tag{D.40}$$

where $\lambda \in \mathbb{R}^+$. Then, the corresponding Born-Infeld inner system with RHS is[1]

$$\left| \begin{array}{l} W_t + ZW_x = \lambda(1 - u)[g(u) - G] \\ Z_t + WZ_x = \quad \lambda(-u)[g(u) - G]. \end{array} \right. \tag{D.41}$$

Of course, u and G are defined by (D.39) for $(W, Z) \in \Omega \setminus (0, 0)$. When $\lambda = 0$, we obtain the former system (D.20). When $\lambda = \infty$, both equations of (D.41) degenerate to $G = g(u)$, which is quite annoying because we do not see how (D.41) is a relaxation system to (D.1). However, matters can be made clear by the following result.

Theorem 3 *System (D.41) is equivalent to the* conservative *form*

$$\left| \begin{array}{r} u_t + \ \{u(1 - u)G\}_x = 0 \\ \{(1 - 2u)G\}_t - \{u(1 - u)G^2\}_x = \lambda(1 - 2u)[g(u) - G]. \end{array} \right. \tag{D.42}$$

PROOF- First, we assume that all solutions are smooth. Taking the dot product of the left-hand side of (D.41) with (u_W, u_Z), we end up with the first equation of (D.42), as was seen in the Section 1, because $F = u(1 - u)G$.

Let $I \equiv (1 - 2u)G$ and $J \equiv -u(1 - u)G^2$. In terms of (W, Z), we have $I = W + Z$ and $J = WZ$. Adding the two equations of (D.41) together, we obtain

$$I_t + ZW_x + WZ_x = \lambda\{(1 - u) - u\}[g(u) - G]. \tag{D.43}$$

[1]If we want to be very rigorous, we should be writing (W^λ, Z^λ) to remind ourselves that the solution does depend on λ.

Since $ZW_x + WZ_x = \{WZ\}_x$, the above equation reduces to

$$I_t + J_x = \lambda(1-2u)[g(u) - G]. \tag{D.44}$$

To prove that (D.42) implies (D.41), we proceed in a similar way. Now, if we have only weak solutions, equivalence between (D.42) and (D.41) is still valid because of the linear degeneracy of the eigenfields. □

The conservative form (D.42) is definitely the system actually solved by the scheme (D.32), equipped with the flux (D.30). The relaxed slip variable G comes in handier than the flux F. Had we kept working with F instead of G, the writing of the system would have been more cumbersome.

Corollary 1 *G is governed by the unambiguous equation*

$$G_t + G^2 u_x = \lambda[g(u) - G]. \tag{D.45}$$

Proof- Expanding the second equation of (D.42), we have

$$-2u_t G + (1-2u)G_t - (1-2u)G^2 u_x - u(1-u)2GG_x = \lambda(1-2u)[g(u) - G] \tag{D.46}$$

The left-hand side of (D.46) can be cast into

$$-2G[u_t + (1-2u)Gu_x + u(1-u)G_x] + (1-2u)[G_t + G^2 u_x]. \tag{D.47}$$

The first bracket is none other than $u_t + \{u(1-u)G\}_x$, which cancels out by virtue of the first equation of (D.42). The remaining terms in (D.46) are all multiple of $1-2u$, so that after simplification by $1-2u$, we arrive at (D.45). □

Proposition 4 *G is not an entropy for (D.41), i.e., there is no combination of equations in (D.41) which gives rise to a conservation law of the form $G_t + K_x = \dots$*

Proof- It is known [59] that any entropy $\phi(W,Z)$ of the Born-Infeld system must satisfy the Goursat equation

$$\phi_W - \phi_Z + (Z-W)\phi_{WZ} = 0. \tag{D.48}$$

As far as G is concerned, we have $G_W = 1$, $G_Z = -1$ and $G_{WZ} = 0$, and the Goursat condition is plainly violated. □

Using (D.48), we can show that F is not an entropy for (D.41) either. But G^{-1}, defined for $(W,Z) \in \Omega \setminus (0,0)$, is an entropy. Indeed, it can be inferred from (D.45) that

$$\{G^{-1}\}_t - u_x = -\lambda G^{-2}[g(u) - G]. \tag{D.49}$$

D.2.2.2 Chapman-Enskog analysis

We now proceed to carry out asymptotic expansion of various variables with respect to $\lambda \to \infty$. As is the case for all relaxation methods, our purpose is to obtain an equivalent equation for u and to see under which conditions the relaxation system (D.42) is a good approximation to the original equation (D.1).

Theorem 4 *At the first order approximation in λ, the solution u to relaxation system (D.42) satisfies the* equivalent *equation*

$$u_t + f(u)_x = \lambda^{-1}\{-[f'(u) - w][f'(u) - z]u_x\}_x \tag{D.50}$$

in which $w = (1-u)g(u)$ and $z = -ug(u)$.

Proof- Inserting the standard Chapman-Enskog form

$$G = g(u) + \lambda^{-1} g_1 + O(\lambda^{-2}) \tag{D.51}$$

into (D.45) and keeping leading terms only, we obtain

$$-g_1 = g(u)_t + g^2(u) u_x. \tag{D.52}$$

The right-hand side can be transformed into

$$\begin{aligned} -g_1 &= g'(u)u_t + g^2(u)u_x = -g'(u)f(u)_x + g^2(u)u_x && \text{(D.53)} \\ &= [g^2(u) - g'(u)f'(u)]u_x, && \text{(D.54)} \end{aligned}$$

where $u_t = -f(u)_x$ is valid at the zeroth-order. Now, plugging (D.51) into the first equation of (D.42), moving first-order terms to the right-hand side and using (D.53), we end up with

$$\begin{aligned} u_t + f(u)_x &= \lambda^{-1}\{-u(1-u)g_1\}_x && \text{(D.55)} \\ &= \lambda^{-1}\{u(1-u)[g^2(u) - g'(u)f'(u)]u_x\}_x. && \text{(D.56)} \end{aligned}$$

Let us rearrange the kernel

$$\mathcal{D}(u) = u(1-u)[g^2(u) - g'(u)f'(u)]. \tag{D.57}$$

On one hand,

$$u(1-u)g^2(u) = -wz. \tag{D.58}$$

On the other hand, multiplying the equality $f' = (1-2u)g + u(1-u)g'$ by f' and isolating the product $g'f'$, we have

$$\begin{aligned} u(1-u)g'(u)f'(u) &= [f'(u)]^2 - (1-2u)g(u)f(u) && \text{(D.59)} \\ &= [f'(u)]^2 - (w+z)f(u) && \text{(D.60)} \end{aligned}$$

Reporting (D.58) and (D.59) into (D.57) yields

$$\mathcal{D}(u) = -wz + (w+z)f(u) - [f'(u)]^2 = -[f'(u) - w][f'(u) - z], \tag{D.61}$$

which completes the proof. □

Corollary 2 *A sufficient condition for the relaxation system (D.42) to be stable is that the* Whitham inequalities

$$\min(w,z) < f'(u) < \max(w,z) \tag{D.62}$$

hold over some appropriate range of u, which depends on the problem at hand.

Proof- At the continuous level, a sufficient stability condition requires that the kernel $\mathcal{D}(u)$ in the equivalent equation (D.50) be a strictly diffusive kernel, i.e., $\mathcal{D}(u) > 0$. This amounts to demanding

$$[f'(u) - w][f'(u) - z] < 0, \tag{D.63}$$

hence (D.62). □

Unsurprisingly, the Whitham conditions demand that the eigenvalue of the original equation be enclosed by the two eigenvalues of the relaxation system. The novel feature is that, in the present method, conditions (D.62) turn out to be quite severe : only a small class of flux functions f are eligible for the Born-Infeld relaxation.

D.2.2.3 Eligible flux functions

Let us investigate further into the class of flux functions that meet the Whitham conditions (D.62). First, we observe that strict inequalities cannot occur at monophasic states $u \in \{0,1\}$, because $f'(0) = w(0) = g(0)$ and $f'(1) = z(1) = -g(1)$. This is why, unless otherwise indicated, we take it for granted that $u \notin \{0,1\}$.

Second, since $f(0) = f(1) = 0$, we can write

$$w = \frac{f(u) - f(0)}{u - 0} \qquad \text{and} \qquad z = \frac{f(1) - f(u)}{1 - u}, \tag{D.64}$$

which leads to an immediate geometrical interpretation, depicted in Fig. D.4. On the curve $\mathcal{C}$ given by the Cartesian equation $f = f(u)$, let $\mathsf{M} = (u, f)$ be a running point, $\mathsf{A} = (0,0)$ and $\mathsf{B} = (1,0)$. Then, w is the slope of AM, while z is that of BM. Conditions (D.62) amount to saying that the tangent to $\mathcal{C}$ at M lies outside the angular sector $(\overrightarrow{\mathsf{MA}}, \overrightarrow{\mathsf{MB}})$.

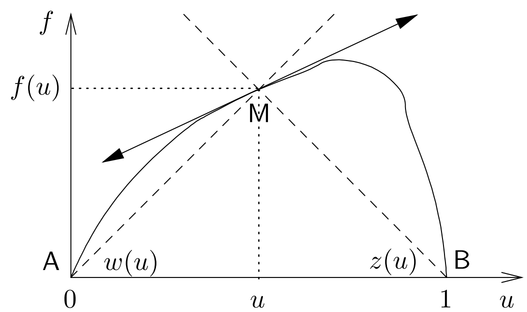

FIG. D.4 – Geometrical interpretation of the Whitham conditions.

We now express (D.62) in terms of w and z alone, viewed as functions of u. This step allows us to connect eligible functions with convexe and concave functions.

Theorem 5 *Conditions (D.62) are satisfied if and only if w and z are strictly decreasing at u if $f(u) > 0$, strictly increasing at u if $f(u) < 0$, critical at u if $f(u) = 0$.*

PROOF- A straightforward computation shows that

$$w' = \frac{uf' - f}{u^2} \qquad \text{and} \qquad z' = -\frac{(1-u)f' + f}{(1-u)^2}. \tag{D.65}$$

Suppose $f > 0$. After (D.3), we have $z < 0 < w$, because $u \in]0,1[$. Thus, conditions (D.62) reduce to

$$z = -\frac{f}{1-u} < f' < \frac{f}{u} = w. \tag{D.66}$$

It becomes clear that

$$-\left[(1-u)f' + f\right] < 0 \qquad \text{and} \qquad uf' - f < 0, \tag{D.67}$$

whence $z' < 0$ and $w' < 0$. We proceed similarly in the case $f < 0$ for which $w < 0 < z$. If $f = 0$, necessarily $w = z = 0$ because $u \notin \{0,1\}$. By (D.62), we have $f' = 0$, from which we deduce that $w' = z' = 0$ via (D.65). To establish the converse, we follow the same steps in reverse order. □

D.2.2.4 Flux monotonicity

The Whitham conditions were earlier derived at the continuous level from the Chapman-Enskog analysis. We are going to show how these are also linked to the monotonicity of the numerical flux at the discrete level.

Consider the first-order explicit scheme

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}[H(u_i^n, u_{i+1}^n) - H(u_{i-1}^n, u_i^n)]. \tag{D.68}$$

We recall [32] that the numerical flux H is said to be monotonous if it is increasing with respect to its first argument and decreasing with respect to its second argument. In terms of derivatives, if we specify $H(u_L, u_R)$ for the arguments, monotonicity amounts to

$$H_{u_L} \geq 0 \qquad \text{and} \qquad H_{u_R} \leq 0. \tag{D.69}$$

We wish to know about the monotonicity of the Born-Infeld numerical flux, namely, $H^{\mathrm{BI}}(u_L, u_R) = F^\star$, where $F^\star$ was defined in section D.2.1.4.

Theorem 6 *The Born-Infeld numerical flux is montonous if and only if the Whitham conditions (D.62) are satisfied.*

Proof- It is easy to see that the only case to deal with is the Case 1 of Theorem 1 with $\widetilde{W} \neq 0$ and $\widetilde{Z} \neq 0$ because, for all other cases, the numerical flux is 0.

Since $\widetilde{W} \neq 0$ and $\widetilde{Z} \neq 0$, none of the L and R states, represented by $(W_L, Z_L) = (w(u_L), z(u_L))$ and $(W_R, Z_R) = (w(u_R), z(u_R))$, can have a nil component. In $\boxed{\text{1a}}$, $W_{L,R} < 0 < Z_{L,R}$ and $F_{L,R} < 0$, which yields $\mathbf{v}^\star = (W_L, Z_R)$. It follows that

$$H^{\mathrm{BI}}(u_L, u_R) = \frac{w(u_L)z(u_R)}{z(u_R) - w(u_L)}, \tag{D.70}$$

and consequently

$$H^{\mathrm{BI}}_{u_L} = w'(u_L)\left[\frac{z(u_R)}{z(u_R) - w(u_L)}\right]^2 \tag{D.71}$$

$$H^{\mathrm{BI}}_{u_R} = -z'(u_R)\left[\frac{w(u_L)}{z(u_R) - w(u_L)}\right]^2. \tag{D.72}$$

The monotonicity conditions (D.69) take place if and only if $w' > 0$ and $z' > 0$. Moreover, since $F < 0$, Theorem 5 leads to the conclusion that the Whitham conditions (D.62) hold. In $\boxed{\text{1b}}$, we have $F_{L,R} > 0$ and

$$H^{\mathrm{BI}}(u_L, u_R) = \frac{w(u_R)z(u_L)}{z(u_L) - w(u_R)}. \tag{D.73}$$

The rest of the discussion works alike. □

Remark 1 *As a matter of fact, it is fair to mention that the equivalence between monotonicity of the numerical flux and the Whitham conditions is also valid for the Xin-Jin relaxation scheme (D.6), the flux of which is*

$$H^{\mathrm{XJ}}(u_L, u_R) = \tfrac{1}{2}[f(u_L) + f(u_R)] + \tfrac{a}{2}[u_L - u_R]. \tag{D.74}$$

We must add that, for a matter of simplicity, we have written here simplified results : more general theorems are given in [5]. Indeed, it can be shown that the Whitham condition appears not only for the particular case $f(u) = u(1-u)g(u)$. Similarly, the equivalence between the Whitham condition and the monotonicity of the scheme can be generalised. However, the results shown here are sufficient to understand the case of the two-phase flow model, which we are going to detail.

D.3 Numerical results

Computer simulations are performed in order to compare Born-Infeld to Xin-Jin. The runs are dedicated to Riemann problems, for which analytical solutions can be found. The simulations are carried out for 4 types of *positive* flux functions, the characteristics of which are summarized in Table D.3. The `Yes` entry in the last two columns means that the property at issue is satisfied over the interval $]0,1[$.

Name	Slip velocity g	w,z decreasing	f concave
Constant g	4	`Yes`	`Yes`
Partial Whitham	$4\,u\,[1-u]$	`No`	`No`

TAB. D.1 – Flux functions for numerical tests

In the Xin-Jin scheme, the parameter a is computed, at each edge, by

$$a(u_L, u_R) = \max\left(|f'(u_L)|, |f'(u_R)|\right). \tag{D.75}$$

In this case, it is known that the Xin-Jin scheme is the Local-Lax-Friedrichs scheme. The numerical simulations are performed with 200 cells.

D.3.1 Constant g

Since $f = 4u(1-u)$ is positive and concave, the Whitham conditions are met (see figures D.5).

Figure D.6 displays the profile of the solution u at $t = 1$, for the initial data $u_L = 0.25$, $u_R = 0.5$. The sharpness of the shock for Born-Infeld is comparable to that of Xin-Jin. In the figure D.7, we analyze the convergence of the scheme toward the exact solution when the number of cells is increased. In this case, we see that the numerical dissipation of the Born-Infeld scheme is between those of the Lax-Friedrichs and Xin-Jin. We have seen this result for all the physical flux $f(u)$ that we have experienced, but we haven't been able to prove it.

If we swap the L and R states, i.e., if now $u_L = 0.5$, $u_R = 0.25$, the solution is a rarefaction, as shown in Fig. D.8. The two schemes have about the same quality.

D.3.2 Partial Whitham

The trouble with the function $f(u) = u^2(1-u)^2$ is that the Whitham condition is not satisfied over the whole interval $[0,1]$.

If the data are pushed toward monophasic states, e.g., if $u_L = 1.$, $u_R = 0.$, we observe a bad result (Fig. D.10). The result is explained by the fact that the left and right states lie in the region where the Whitham condition is not satisfied. Furthermore, the numerical flux is $H = 0$ for the Born-Infeld scheme. The case $u_L = 0.99$, $u_R = 0.01$ seems to be much better (D.11). In fact, it is not : the numerical solution does not converge toward the exact one, as the number of cells increases.

In the figure D.12, we see that in this case, the Born-Infeld scheme is worse than Lax-Friedrichs' scheme!

D.4 Relaxation scheme for the drift-flux model

In this section, we show how to construct an explicit relaxation scheme based on the Born-Infeld system for a two-phase flow model. We start by presenting the drift-flux model which is closed by a

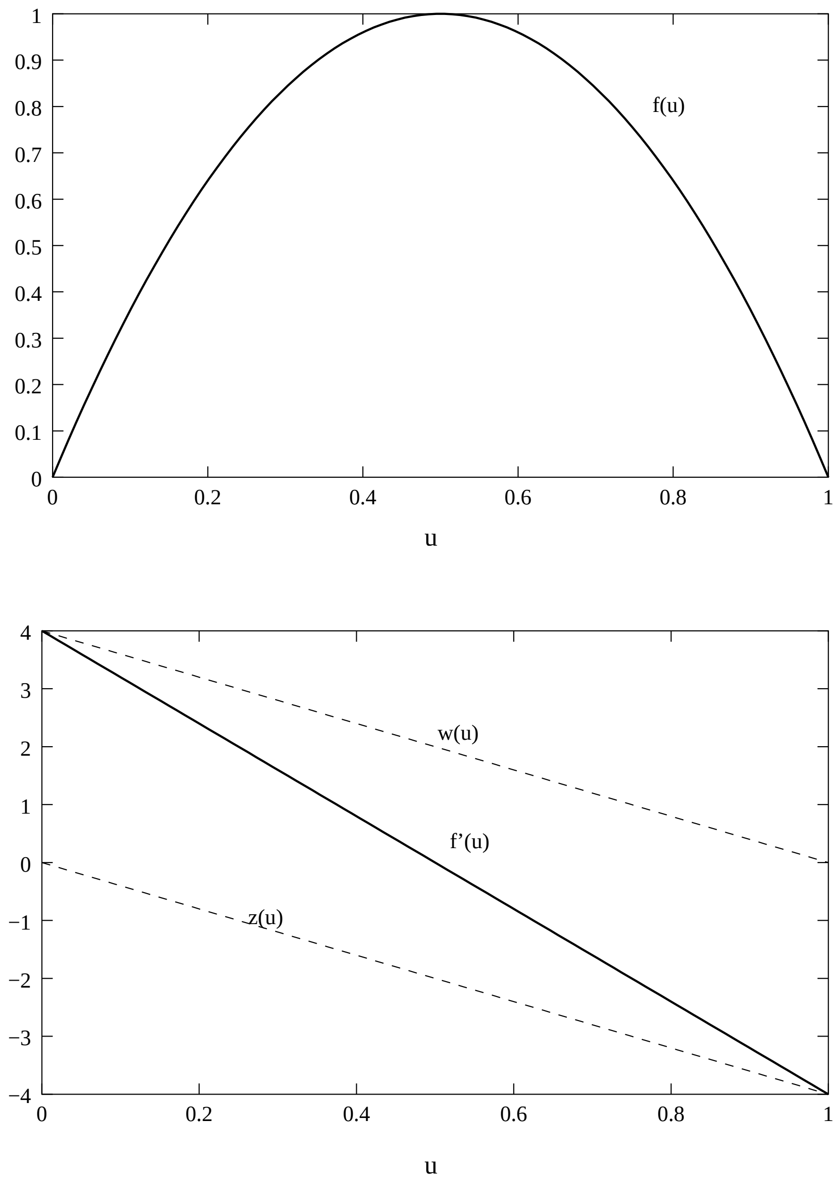

Fig. D.5 – Constant g

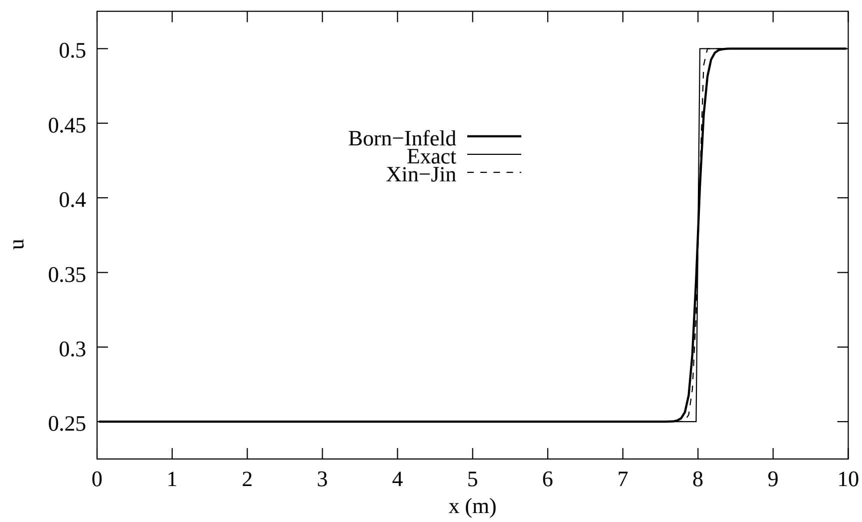

FIG. D.6 – Constant g, $u_L = 0.25$, $u_R = 0.5$

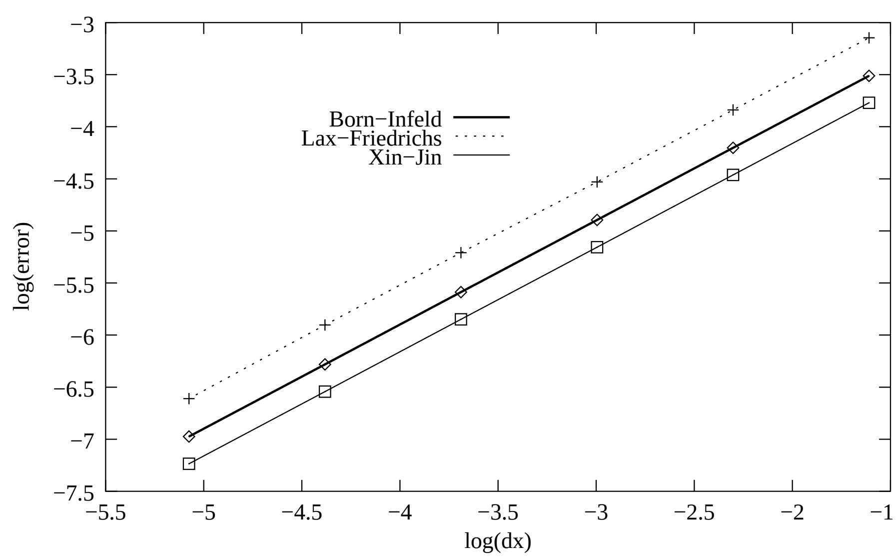

FIG. D.7 – Constant g, $u_L = 0.25$, $u_R = 0.5$

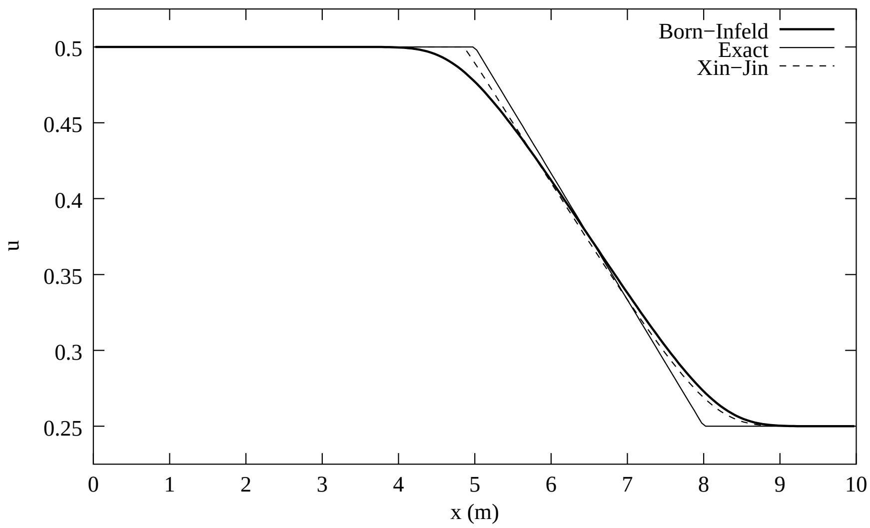

FIG. D.8 – Constant g. $u_L = 0.5$, $u_R = 0.25$

thermodynamic and a hydrodynamic law. Then we design a relaxation scheme in which the 2 first equations are relaxed by introducing a new relaxation variable and the last equation (the gas mass transport) is relaxed thanks to the Born-Infeld system. We will see that classical hydrodynamic laws are satisfying a Whitham condition. This scheme will be presented in his order 1 in space and time version for a matter of simplicity.

D.4.1 The drift-flux model

Let us consider phase space :

$$\Omega = \left\{ \mathbf{u} = (\rho,\ \rho v,\ \rho Y) \in \mathbb{R}^3;\ \rho > 0,\ v \in \mathbb{R},\ Y \in [0,1] \right\},$$

where ρ is the density of the mixture, v is the velocity of the mixture and Y is the gas mass fraction. We are interested in the simulation of two-phase flows in petroleum pipelines[54]. The system, made up of three conservation laws, is a drift-flux type model and is closed by two thermodynamic and hydrodynamic models. We will consider a perfect gas and a uncompressible liquid with the state equations

$$\rho_{\mathrm{G}} = p/a_{\mathrm{G}}^2, \quad \rho_{\mathrm{L}} = \rho_{\mathrm{L}}^0, \tag{D.76}$$

where ρ_{L}^0 is the liquid density and a_{G} is the sound speed in the gas. Then the pressure law is

$$p(\rho, \rho Y) = a_{\mathrm{G}}^2 \frac{\rho Y}{1 - \dfrac{\rho}{\rho_{\mathrm{L}}^0}\left(1 - \dfrac{\rho Y}{\rho}\right)}. \tag{D.77}$$

We consider a general algebraic hydrodynamic law of the type

$$\varphi(\mathbf{u}) = v_{\mathrm{L}} - v_{\mathrm{G}}, \quad \forall\ \mathbf{w} \in (\mathbb{R}^+)^2 \times \mathbb{R}, \tag{D.78}$$

where v_{G} (resp. v_{L}) is the gas velocity (resp. the liquid velocity). We consider the following two-phase system in conservation form :

$$\left\{ \begin{array}{lcl} \partial_t(\rho) & + \ \partial_x(\rho v) & = \ 0, \\ \partial_t(\rho v) & + \ \partial_x(\rho v^2 + P(\mathbf{u})) & = \ S(\mathbf{u}), \\ \partial_t(\rho Y) & + \ \partial_x(\rho Y v - \sigma(\mathbf{u})) & = \ 0. \end{array} \right. \tag{D.79}$$

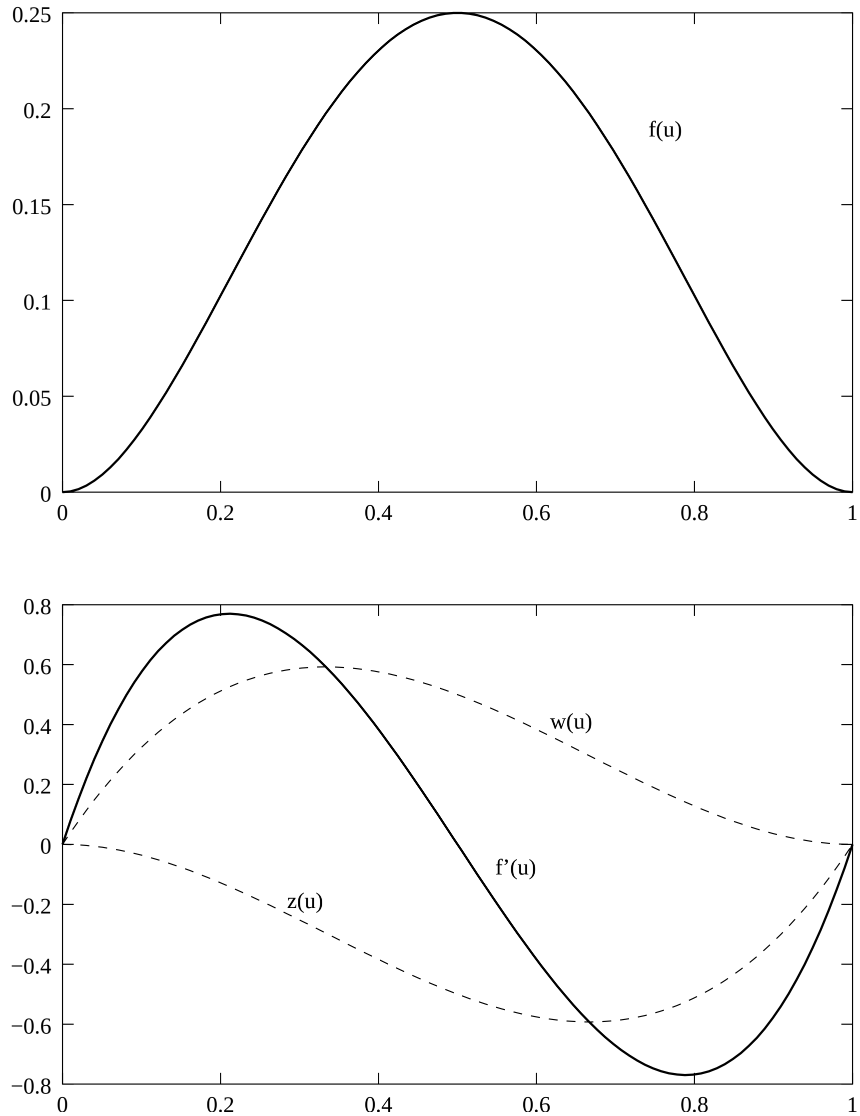

FIG. D.9 – Partial Whitham

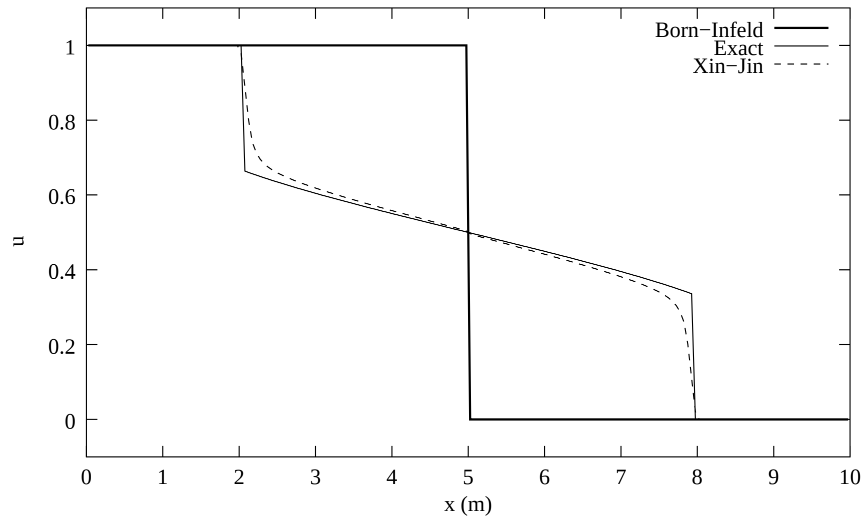

FIG. D.10 – Partial Whitham. $u_L = 1$, $u_R = 0$

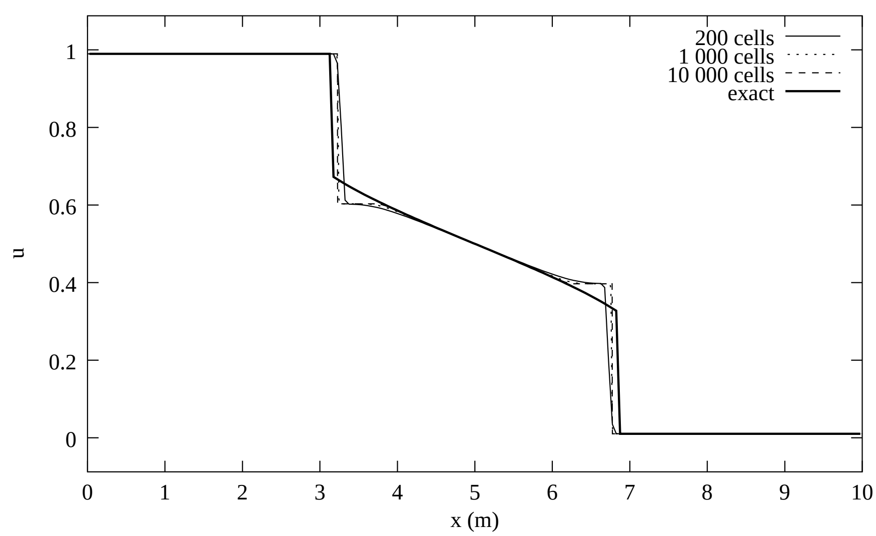

FIG. D.11 – Partial Whitham. $u_L = 0.99$, $u_R = 0.01$

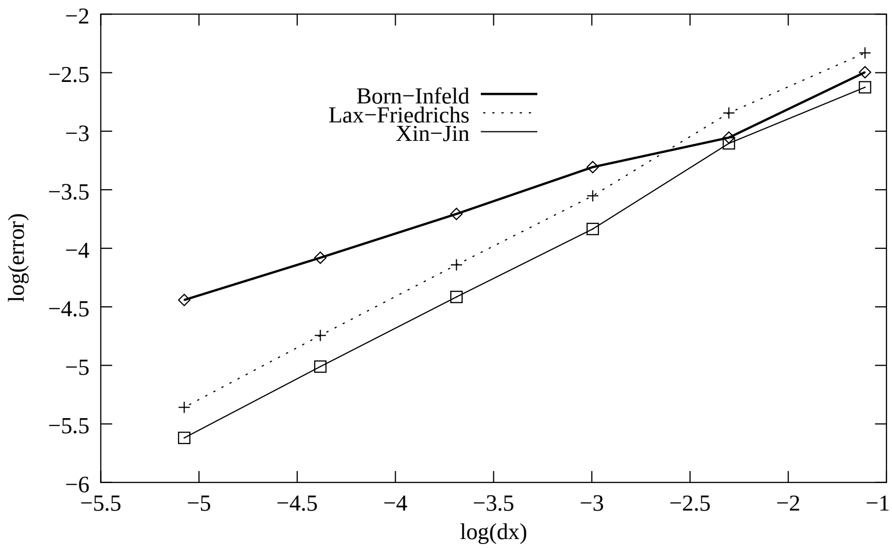

FIG. D.12 – Partial Whitham. $u_L = 0.99$, $u_R = 0.01$

The nonlinear function σ and P are defined by

$$\sigma(\mathbf{u}) = \rho Y(1-Y)\varphi(\mathbf{u}) \qquad \text{and} \qquad P(\mathbf{u}) = p(\mathbf{u}) + \rho Y(1-Y)\varphi(\mathbf{u})^2.$$

We see that the previous function $g(u)$ is now replaced by the slip law $\varphi(\mathbf{u})$.

There is no analytical expression for the physical flux of the considered system, except for very simple hydrodynamic laws. Therefore the eigenvalues of the system (D.79) are not known in full generality. However, in most common situations, i.e., for usual values of $\mathbf{u}$, the system is hyperbolic and has three real eigenvalues $\lambda_1 < \lambda_2 < \lambda_3$ with $\lambda_1 < 0$ and $\lambda_3 > 0$.

D.4.2 Basic ideas

We want to relax all the *genuine* non-linearities of the equilibrium system (D.79) which will appear by means of a Lagrangian change of coordinates. Let us set $\tau = 1/\rho$, and define the Lagrangian mass coordinate y by $dy = \rho dx - \rho v dt$. It turns out that expressing the main nonlinearities of the problem in terms of the variables (τ, v, Y) instead of the conservative one leads to simpler calculations. For the sake of simplicity and with some abuse in the notations, nonlinear functions (like the pressure or the hydrodynamic law) will be given the same notation when expressed in both types of variables. The system (D.79) then rewrites :

$$\left\{ \begin{array}{lll} \partial_t \tau - \partial_y v & = & 0, \\ \partial_t v + \partial_y P(\mathbf{u}) & = & S(\mathbf{u}), \\ \partial_t Y - \partial_y \sigma(\mathbf{u}) & = & 0. \end{array} \right. \tag{D.80}$$

Let us introduce the variables

$$W = -\rho(1-Y)\varphi(\mathbf{u}), \quad Z = \rho Y \varphi(\mathbf{u}). \tag{D.81}$$

We introduce one new state variable Π which is intended to coincide with $P(\mathbf{u})$ in the limit of the relaxation parameter λ. We decide to consider the relaxation system

$$\begin{cases} \partial_t \tau & - \quad \partial_y v & = 0, \\ \partial_t v & + \quad \partial_y \Pi & = 0, \\ \partial_t \Pi & + a^2 \; \partial_y v & = \lambda(P - \Pi), \\ \partial_t W & + Z \; \partial_y W & = \lambda(1-Y)(\varphi - \Phi), \\ \partial_t Z & + W \; \partial_y Z & = -\lambda Y(\varphi - \Phi). \end{cases} \tag{D.82}$$

In the Eulerian frame, the relaxation system is

$$\begin{cases} \partial_t \rho & + \partial_x(\rho v) & = 0, \\ \partial_t(\rho v) & + \partial_x(\rho v^2 + \Pi) & = 0, \\ \partial_t(\rho\Pi) & + \partial_x(\rho\Pi v + a^2 v) & = \lambda\rho(P - \Pi), \\ \partial_t W & + (v + Z\tau)\, \partial_x W & = \lambda\rho(1-Y)(\varphi - \Phi), \\ \partial_t Z & + (v + W\tau)\, \partial_x Z & = -\lambda\rho Y(\varphi - \Phi). \end{cases} \tag{D.83}$$

This system can equivalently be written in the form

$$\begin{cases} \partial_t \rho & + \partial_x(\rho v) & = 0, \\ \partial_t(\rho v) & + \partial_x(\rho v^2 + \Pi) & = 0, \\ \partial_t(\rho\Pi) & + \partial_x(\rho\Pi v + a^2 v) & = \lambda\rho(P - \Pi), \\ \partial_t Y & + \partial_x\{Y(1-Y)\Phi\} & = 0, \\ \partial_t\{(1-2Y)\Phi\} & - \partial_x\{Y(1-Y)\Phi^2\} & = \lambda(1-2Y)[\varphi(\mathbf{u}) - \Phi], \end{cases} \tag{D.84}$$

where Φ is the relaxation variable associated to the function $\varphi(\mathbf{u})$. The above relaxation system will be given hereafter the convenient abstract form :

$$\partial_t \mathbf{v} + \partial_x \mathcal{G}(\mathbf{v}) = \lambda \mathcal{R}(\mathbf{v}) + \mathcal{S}(\mathbf{v}), \quad t > 0, \; x \in \mathbb{R}; \tag{D.85}$$

where the variable $\mathbf{v}$ and the flux function $\mathcal{G}$ receive clear definitions. Here, the relaxation variable associated with the function $\sigma(\mathbf{u})$ is $\Sigma = Y(1-Y)\Phi$.

The important point in this relaxation strategy is that the Born-Infeld system is introduced only for the evolution equation of the gas mass fraction. For the evolution equations on the density ρ and the total momentum ρv, the technique consists in relaxing the nonlinear function $P(\mathbf{u})$, which is the strategy that we used in [6, 7].

In the following, we suppose that

$$a > \max(|W|, |Z|).$$

D.4.3 Whitham condition

We have already noticed (in [7]) that, on one side, the evolution equations on the density ρ and the total momentum ρv (which form a system close to the p-system) and, on the other side, the evolution equation on the gas mass fraction Y, are not totally coupled. In particular, we have hilighted that the stability conditions resulting from the Chapman-Enskog analysis could be obtained while considering only the partial derivatives of P with respect to τ and v and the partial derivatives of σ with respect to Y. If the system was really coupled, we should have consider the partial derivatives with respect to $\mathbf{u}$ but numerical experiments show that the numerical scheme is rough enough with this uncoupled approach. These remarks suggest to consider the Whitham-like condition

$$\min(W, Z) \leq -\sigma_Y(\mathbf{u}) \leq \max(W, Z), \tag{D.86}$$

associated with the relaxation coefficient

$$a = -P_\tau + P_v^2. \tag{D.87}$$

Obviously, for the no-slip law $\varphi(\mathbf{u}) = 0$, the condition (D.86) is satisfied. Therefore, we consider the equivalent condition

$$-\frac{1}{Y} \leq \frac{\varphi_Y(\mathbf{u})}{\varphi(\mathbf{u})} \leq \frac{1}{1-Y}, \quad \varphi \neq 0. \tag{D.88}$$

It is now necessary to verify that the condition (D.88) is satisfied by the classical slip laws. The answer is that there are several classical slip laws (like Zuber-Findlay) which, under classical restriction, are satisfying this condition. Moreover, all the interval $[-\frac{1}{Y}, \frac{1}{1-Y}]$ is used ! This is very surprising because the Whitham condition has not been designed in this goal ... From another point of view, the slip laws have not been designed in order to satisfy this condition.

Let us now detail the results. We supppose that the liquid phase is uncompressible and the gas is perfect. Let us notice that $\rho > 0$ because the void never appears in petroleum pipelines. From another point of view, we restrict ourselves to $\rho < \rho_{\mathrm{L}}^0$ because the liquid is not compressible (if $\rho = \rho_{\mathrm{L}}^0$, the speed of sound is infinite and the system is not hyperbolic anymore). That is why, in the following, the density lie in the interval $]0, \rho_{\mathrm{L}}^0[$.

Let us define the auxiliary function κ by

$$\kappa(\rho, Y) = (1-Y)\left(\frac{\rho}{\rho_{\mathrm{L}}^0} - 1\right) \tag{D.89}$$

The partial derivative of κ with respect to Y is

$$\kappa_Y(\rho, Y) = -\left(\frac{\rho}{\rho_{\mathrm{L}}^0} - 1\right), \quad \forall \rho \in]0, \rho_{\mathrm{L}}^0[,\ Y \in]0,1[, \tag{D.90}$$

and the function κ satisfies the properties

$$\kappa = -(1-Y)\kappa_Y, \quad -1 < \kappa < 0, \quad \forall \rho \in]0, \rho_{\mathrm{L}}^0[,\ Y \in]0,1[. \tag{D.91}$$

Theorem 7 *Let us suppose that the slip law is Zuber-Findlay's law*

$$\varphi(\mathbf{u}) = -\frac{v\,(1-C_0) - C_1}{C_0\,\kappa(\rho, Y) + (1-Y)}, \quad \forall \rho \in]0, \rho_{\mathrm{L}}^0[,\ Y \in]0,1[, \tag{D.92}$$

with $C_0 > 1$ and $C_1 \in \mathbb{R}$ two constants. Then, the slip law $\varphi(\mathbf{u})$ satisfies the Whitham condition (D.88).

Remark 2 *The Zuber-Findlay law is tradionnaly written in the form $v_{\mathrm{G}} = C_0(R_{\mathrm{G}} v_{\mathrm{G}} + R_{\mathrm{L}} v_{\mathrm{L}}) + C_1$ with R_{G} (resp.R_{L}) the surface gas fraction of the gas (resp. the liquid). The closure laws $R_{\mathrm{G}} + R_{\mathrm{L}} = 1$ and $\rho Y = \rho_{\mathrm{G}}(p)$ lead to the equality (D.92).*

Preuve of the theorem(7). One easily see that the partial derivative of φ with respect to Y satisfies

$$\varphi_Y = -\frac{C_0\,\kappa_Y - 1}{C_0\,\kappa + (1-Y)}\varphi, \tag{D.93}$$

which leads, after injection of (D.91), to the equality

$$\frac{\varphi_Y}{\varphi} = \frac{1}{1-Y}. \tag{D.94}$$

This is the maximum authorized by the inequalities (D.88) and concludes the proof.□

Similar results can be obtained for other slip laws.

- The law
$$\varphi(\mathbf{u}) = -\frac{\mu\, v + \nu}{1 + \mu\, \kappa(\rho, Y)}, \quad \forall \rho \in]0, \rho_{\mathrm{L}}^0[,\ Y \in]0, 1[, \tag{D.95}$$
with $0 < \mu < 1$ and $\nu \in \mathbb{R}$ is an extension of the Zuber-Findlay law. It can be shown that
$$-\frac{1}{Y} \leq \frac{\varphi_Y}{\varphi} \leq 0. \tag{D.96}$$
The inequalities (D.88) are then satisfied.
- If the flow is emulsion (that is to say that there is small bubbles of gas in the liquid), the law is
$$\varphi(\mathbf{u}) = -\frac{\delta\, \rho_{\mathrm{L}}^0}{\rho(1 - Y)}, \quad \forall \rho \in]0, \rho_{\mathrm{L}}^0[,\ Y \in]0, 1[, \tag{D.97}$$
with $\delta \in \mathbb{R}$. One can show that
$$\frac{\varphi_Y}{\varphi} = \frac{1}{1 - Y}, \tag{D.98}$$
and, again, the inequalities (D.88) are then satisfied.

More details on these computations can be found in [5].

Now that we have verified that the slip laws are satisfying the Whitham condition, we can design the relaxation scheme.

D.4.4 Relaxation scheme

The pipeline is made of I cells, denoted $(M_i)_{i=1,I}$. Let x_i be the center of the cell and Δx its length. We also denote $x_{i+1/2} = (x_i + x_{i+1})/2$ the interface between two cells, $M_i = [x_{i-1/2},\ x_{i+1/2}]$, $x_{1/2} = 0$ inlet boundary interface and $x_{I+1/2} = L$ the outlet boundary interface. Let $\Delta t^n = t^{n+1} - t^n$ be the time step. Let $\mathbf{u}_i^n \approx \mathbf{u}(x_i,\ t^n)$ be the discrete unknown. The numerical scheme is based on the following splitting method.

1. Relaxation. We take $\lambda = \infty$ and solve the ODE system $\partial_t \mathbf{v} = \lambda \mathcal{R}(\mathbf{v})$ by projecting the variables on the equilibrium variety.
2. Evolution. We take $\lambda = 0$ and solve the system $\partial_t \mathbf{v} + \partial_x \mathcal{G}(\mathbf{v}) = \mathcal{S}(\mathbf{v})$ on one iteration in order to go to the time $t = t^{n+1}$.

At the beginning of the time step, we compute $\mathbf{v}_i^n \approx \mathbf{v}(x_i,\ t^n)$ at equilibrium, i.e.
$$\begin{gathered}\Pi_i^n := P(\mathbf{u}_i^n), \quad \Sigma_i^n := \sigma(\mathbf{u}_i^n), \\ W_i^n = -\rho_i^n (1 - Y_i^n) \varphi(\mathbf{u}_i^n), \quad Z_i^n = \rho_i^n Y_i^n \varphi(\mathbf{u}_i^n).\end{gathered} \tag{D.99}$$
We then consider the system $\partial_t \mathbf{v} + \partial_x \mathcal{G}(\mathbf{v}) = \mathcal{S}(\mathbf{v})$ that we approximate by the following finite volume scheme :
$$\frac{\mathbf{v}_i^{n+1} - \mathbf{v}_i^n}{\Delta t^n} + \frac{\mathbf{H}_{i+1/2}^{n+1/2} - \mathbf{H}_{i-1/2}^{n+1/2}}{\Delta x} = \mathcal{S}(\mathbf{v}_i^n) \tag{D.100}$$
with $\mathbf{H}_{i+1/2}^{n+1/2}$ the numerical flux based on a Godunov scheme. We denote $\mathbf{v_L} = \mathbf{v}_i^n$ and $\mathbf{v_R} = \mathbf{v}_{i+1}^n$ the left and right states of the Riemann Problem and $\mathbf{v}^\star(\mathbf{v_L},\ \mathbf{v_R})$ the value of the solution of this R.P. at $x = x_{i+1/2}$. The Godunov numerical flux is
$$\mathbf{H}_{i+1/2}^{n+1/2} = \mathbf{H}^{Godunov}(\mathbf{v_L},\ \mathbf{v_R}) = \mathcal{G}\left(\mathbf{v}^\star(\mathbf{v_L},\ \mathbf{v_R})\right). \tag{D.101}$$

The explicit computations for $\mathbf{v}^\star$ are given in proposition (D.4.1) at the end of this section. The proposition will not be proven here, since it is done in [6] but is just shown for consistency.

In the following proposition, we denote $\overline{\varphi} = (\varphi_\mathbf{L} + \varphi_\mathbf{R})/2$ the average and $\langle\varphi\rangle = (\varphi_\mathbf{L} - \varphi_\mathbf{R})/2$ the mid-jump of the the two states $\mathbf{v_L}$ and $\mathbf{v_R}$ whatever the quantity φ. For all $v \in \mathbb{R}$, one notes $v^+ = \max(v,0)$ and $v^- = \min(v,0)$.

Proposition D.4.1 *We consider the Riemann problem for the relaxation system (D.84) with $\lambda = 0$ and no source terms and the initial data*

$$\mathbf{v}(x,\ 0) = \begin{cases} \mathbf{v_L} & \text{if } x < 0, \\ \mathbf{v_R} & \text{if } x > 0. \end{cases} \tag{D.102}$$

The Riemann solution is made of six constant states separated by five contact discontinuities

$$\mathbf{v}(x,t) = \begin{cases} \mathbf{v_L} = \mathbf{v}_0 & \text{if } \frac{x}{t} < \mu_1, \\ \mathbf{v}_j & \text{if } \mu_j < \frac{x}{t} < \mu_{j+1}, \quad 1 \le j \le 4 \\ \mathbf{v_R} = \mathbf{v}_5 & \text{if } \frac{x}{t} > \mu_5, \end{cases} \tag{D.103}$$

Let us set

$$\begin{aligned} \tau_\mathbf{L}^\star &= \tau_\mathbf{L} - \tfrac{\langle v\rangle}{a} + \tfrac{\langle \Pi\rangle}{a^2}, \quad \tau_\mathbf{R}^\star = \tau_\mathbf{R} - \frac{\langle v\rangle}{a} - \frac{\langle \Pi\rangle}{a^2}, \\ v^\star &= \overline{v} + \tfrac{\langle \Pi\rangle}{a}, \qquad\qquad \Pi^\star = \overline{\Pi} + a\langle v\rangle. \end{aligned} \tag{D.104}$$

Assume that the parameter a is chosen large enough so that both $\tau_\mathbf{L}^\star$ and $\tau_\mathbf{R}^\star$ in (D.104) are positive.

Let us set $s_\mathbf{L} = \min(W_\mathbf{L}, Z_\mathbf{L})$ and $s_\mathbf{R} = \min(W_\mathbf{R}, Z_\mathbf{R})$. If $a > \max(|W|,|Z|)$, the eigenvalues are increasingly ordered as follows

$$\underbrace{v_\mathbf{L} - a\tau_\mathbf{L}}_{\mu_1(\mathbf{v_L})} \le v^\star + s_\mathbf{L}\tau_\mathbf{L}^\star \le v^\star \le v^\star + s_\mathbf{R}\tau_\mathbf{R}^\star \le \underbrace{v_\mathbf{R} + a\tau_\mathbf{R}}_{\mu_5(\mathbf{v_R})}. \tag{D.105}$$

1. $\boxed{\text{Case 1}}$ *If $\varphi_\mathbf{L}\varphi_\mathbf{R} \ge 0$, the intermediary states are*

$$\mathbf{v}_1 = \begin{pmatrix} \rho_\mathbf{L}^\star \\ \rho_\mathbf{L}^\star v^\star \\ \rho_\mathbf{L}^\star \Pi^\star \\ W_\mathbf{L} \\ Z_\mathbf{L} \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} \rho_\mathbf{L}^\star \\ \rho_\mathbf{L}^\star v^\star \\ \rho_\mathbf{L}^\star \Pi^\star \\ \widetilde{W} \\ \widetilde{Z} \end{pmatrix}, \tag{D.106}$$

$$\mathbf{v}_3 = \begin{pmatrix} \rho_\mathbf{R}^\star \\ \rho_\mathbf{R}^\star v^\star \\ \rho_\mathbf{R}^\star \Pi^\star \\ \widetilde{W} \\ \widetilde{Z} \end{pmatrix}, \ \mathbf{v}_4 = \begin{pmatrix} \rho_\mathbf{R}^\star \\ \rho_\mathbf{R}^\star v^\star \\ \rho_\mathbf{R}^\star \Pi^\star \\ W_\mathbf{R} \\ Z_\mathbf{R} \end{pmatrix}, \tag{D.107}$$

with $\left(\widetilde{W}, \widetilde{Z}\right) = \left(W_\mathbf{L}^- + W_\mathbf{R}^+,\ Z_\mathbf{L}^- + Z_\mathbf{R}^+\right)$.

2. $\boxed{\text{Case 2}}$ *If $\varphi_\mathbf{L}\varphi_\mathbf{R} \le 0$, the intermediary states are*

$$\mathbf{v}_1 = \begin{pmatrix} \rho_\mathbf{L}^\star \\ \rho_\mathbf{L}^\star v^\star \\ \rho_\mathbf{L}^\star \Pi^\star \\ W_\mathbf{L} \\ Z_\mathbf{L} \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} \rho_\mathbf{L}^\star \\ \rho_\mathbf{L}^\star v^\star \\ \rho_\mathbf{L}^\star \Pi^\star \\ W_\mathbf{L}^- \\ Z_\mathbf{L}^- \end{pmatrix}, \tag{D.108}$$

$$\mathbf{v}_3 = \begin{pmatrix} \rho^\star_{\mathbf{R}} \\ \rho^\star_{\mathbf{R}} v^\star \\ \rho^\star_{\mathbf{R}} \Pi^\star \\ W^+_{\mathbf{R}} \\ Z^+_{\mathbf{R}} \end{pmatrix}, \ \mathbf{v}_4 = \begin{pmatrix} \rho^\star_{\mathbf{R}} \\ \rho^\star_{\mathbf{R}} v^\star \\ \rho^\star_{\mathbf{R}} \Pi^\star \\ W_{\mathbf{R}} \\ Z_{\mathbf{R}} \end{pmatrix}. \tag{D.109}$$

D.4.5 Relaxation coefficient

In the numerical scheme, in order to reduce the numerical dissipation, we consider one local relaxation coefficient on each interface

$$a_{i+1/2} = \sqrt{\max\left(A(\mathbf{u}_i^n),\, A(\mathbf{u}_{i+1}^n)\right)} \tag{D.110}$$

where the function A is defined by $A(\mathbf{u}) = -P_\tau(\mathbf{u}) + P_v(\mathbf{u})^2$.

D.5 More numerical results

In this section, we compare the approximate solutions computed with the Born-Infeld relaxation scheme with the relaxation scheme introduced in [6]. The experiments only involve Riemann problems for different types of slip law. These experiments are summarized in Table D.2.

Number	Slip law φ	Exact solution
1	$\varphi = 0$	1-rarefaction
2	Zuber-Findlay	1-shock, 2-contact discontinuity, 3-shock
3	Emulsion	2-contact discontinuity

Tab. D.2 – Flux functions for numerical tests

All the experiments are done by considering a pipe's length of 100 meters and a cell size of 0.5 (m).

D.5.1 Experiment 1

Consider the left (L) and right (R) states

$$\begin{pmatrix} \rho \\ Y \\ v \end{pmatrix}_L = \begin{pmatrix} 500 \\ 0.2 \\ 34.423 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \rho \\ Y \\ v \end{pmatrix}_R = \begin{pmatrix} 400 \\ 0.2 \\ 50 \end{pmatrix},$$

with the slip law $\varphi = 0$. These have been tailored so that the solution to the Riemann problem is a pure 1-rarefaction. With $a_{\mathrm{G}} = 100$ m/s, the speeds of propagation of the fronts are $(\lambda_1)_{\mathbf{L}} = -40.12$ m/s and $(\lambda_1)_{\mathbf{R}} = -15.77$ m/s. Snapshots in Fig. D.13 correspond to time $T = 0.8$ s.

We see that the two schemes exhibit approximate solutions which, first, are correct ones, and, second, are roughly identical. This behavior suggest that the numerical dissipation introduced by the two schemes is the same. This is natural, since for $\varphi = 0$, the two schemes are based on similar relaxation system. This conclusion is confirmed by the two next experiments.

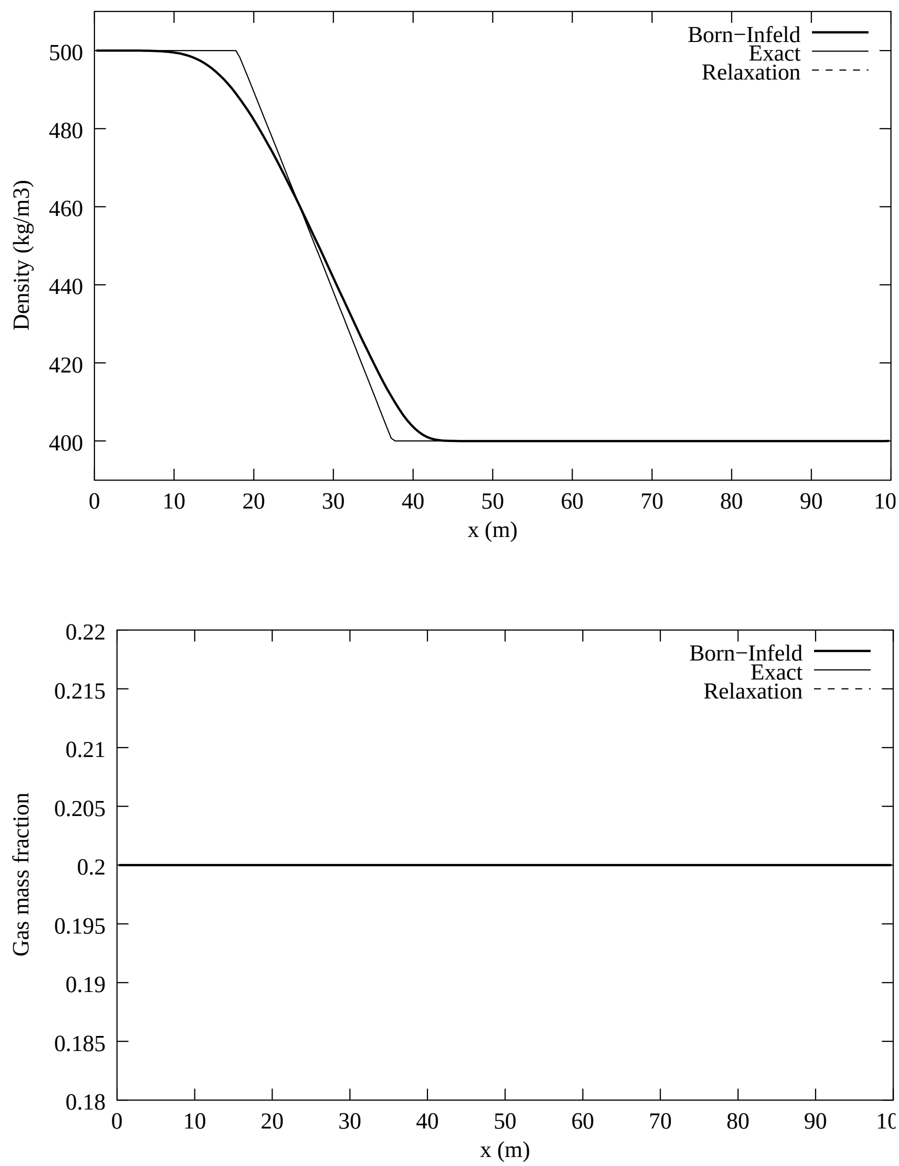

Fig. D.13 – Experiment 1

D.5.2 Experiment 2

Consider the left (L) and right (R) states

$$\begin{pmatrix} \rho \\ Y \\ v \end{pmatrix}_L = \begin{pmatrix} 453.19 \\ 0.70543\,.\,10^{-2} \\ 24.807 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \rho \\ Y \\ v \end{pmatrix}_R = \begin{pmatrix} 454.91 \\ 0.10804\,.\,10^{-1} \\ 1.7460 \end{pmatrix}.$$

with the classical Zuber-Findlay slip law $\varphi = -\frac{v\,(1-C_0)-C_1}{C_0\,\kappa(\rho,Y)+(1-Y)}$, where $C_0 = 1.07$ and $C_1 = 0.21620$. These have been tailored so that the solution to the Riemann problem is composed of a 1-shock, a 2-contact discontinuity and a 3-shock. Here, we consider $a_{\mathrm{G}} = 300$ m/s. The result is showed in figure D.14, which corresponds to time $T = 0.5$ s.

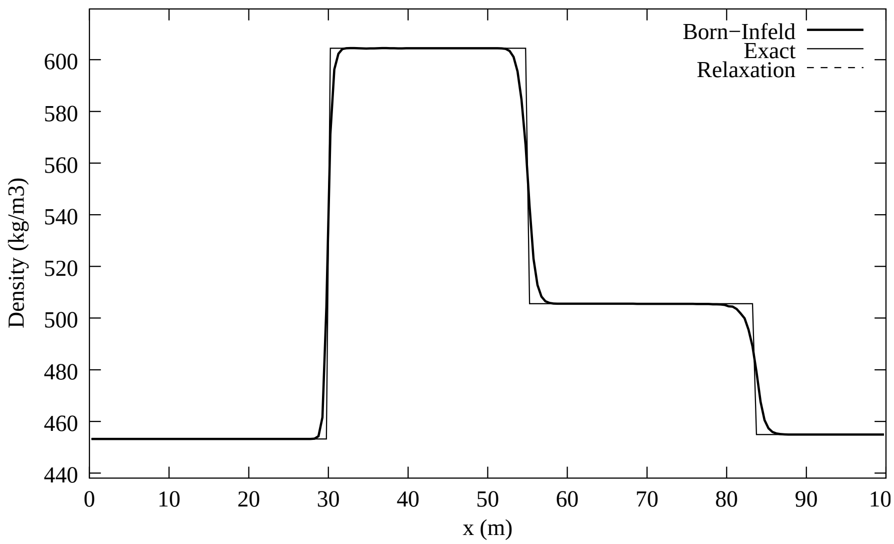

FIG. D.14 – Experiment 2

Again, the two schemes behave the same way.

In order to really distinguish the two schemes, we test our scheme with a slip law different from 0 and we consider the emulsion slip law.

D.5.3 Experiment 3

Consider the left (L) and right (R) states

$$\begin{pmatrix} \rho \\ Y \\ v \end{pmatrix}_L = \begin{pmatrix} 901.11 \\ 1.2330\,.\,10^{-3} \\ 0.95027 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \rho \\ Y \\ v \end{pmatrix}_R = \begin{pmatrix} 208.88 \\ 4.2552\,.\,10^{-2} \\ 0.78548 \end{pmatrix}.$$

with the emulsion slip law $\varphi = -\frac{\delta\,\rho_{\mathrm{L}}^0}{\rho(1-Y)}$, where $\delta = 4.4809$. These have been tailored so that the solution to the Riemann problem is composed of a 2-contact discontinuity. With $a_{\mathrm{G}} = 300$ m/s, this discontinuity propagates with speed 1. (m/s). The result is showed in figure D.14, which corresponds to time $T = 20.$ s.

This experience, whith both a non-zero slip law and a contact discontinuity with small propagating speed, allow, in general, to distinguish the schemes : the numerical dissipation plays an important role. Here, again, the two relaxation schemes behave very similarly.

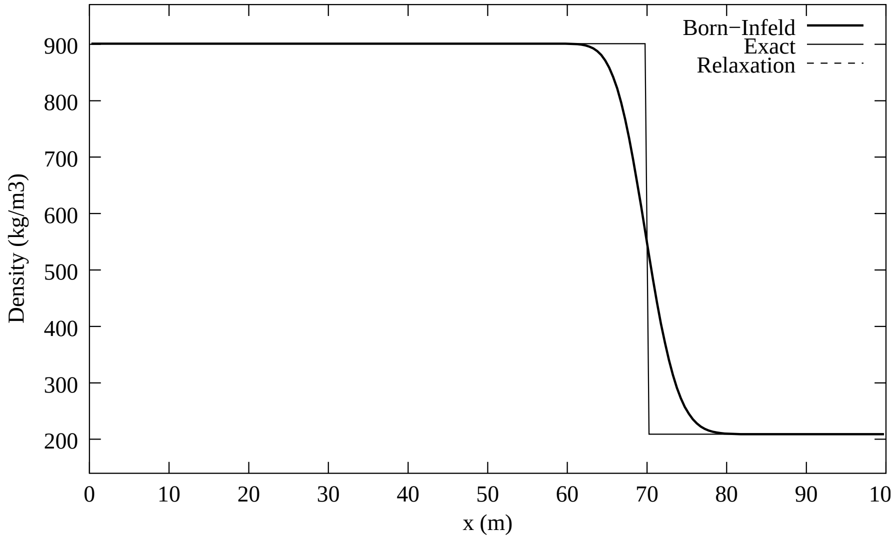

FIG. D.15 – Experiment 3

D.6 Conclusion

There seems to be a very close connection between scalar conservation laws of the two-phase type (D.1) and the Born-Infeld equations (D.20). The newly proposed relaxation method, which takes advantage of this connection, is much more "natural" than the standard Xin-Jin method, to the extent that :

- there is no need for an extra parameter a to be artificially tuned in order for stability and monotonicity to be ensured;
- the eigenvalues of the relaxation system coincide with the physical phase velocities defined in the original equation.

The Born-Infeld relaxation method is all the more attractive that, in terms of quality, it does compete well with the Xin-Jin method in which the least diffusion amount is set.

Unfortunately, the very fact that no free parameter is involved may sometimes turn out to be a drawback. When the flux does not belong to a narrow class of eligible functions which automatically fulfill the Whitham conditions, there is no "screwdriver" at our disposal to force diffusion, and thereby to ensure stability and monotonicity.

This property has been highlighted with numerical experiments involving Riemann problems which show that

- if the Whitham condition is satisfied for all the states between u_L and u_R, the numerical scheme behave well,
- otherwise, the resulting numerical solution does not satisfy any basic properties : maximum principle, entropy condition, etc ...

Finally, the numerical experiments show that, when the Whitham condition is satisfied, the numerical dissipation is bounded from above and below by those of Lax-Friedrichs and Xin-Jin schemes.

This numerical scheme can be applied to two-phase flows in pipelines by considering the drift-flux model. It is surprising to see that the classical slip laws are verifying a Whitham condition. Numerical experiments show that this relaxation scheme is behaving well, compared to other relaxation schemes.