A topographically modified KdV approximation

In this chapter we discuss the validity of the previously derived uncoupled KdV approximation on a large time scale for different bottom topographies. We demonstrate its validity for less restrictive bottoms, but provide two examples of simple bottoms for which the approximation diverges. A new approximation that takes the bottom into account is finally derived.

5.1 Discussion on the validity of the approximation

Starting from the previous theorem, it is worth wondering if this one holds for less restrictive initial data and bottoms, i.e. without any condition of a sufficient decay rate at infinity. In this view, we focus in a more general way on the last three terms of U_1 by supposing that (u_0, n_0) is bounded in $L^\infty([0,t]; H^\sigma(\mathbb{R}))^2$, which is propagated by the KdV equation on (U_0, N_0) (see [38]). Using the classical Cauchy-Schwarz inequality on the first two terms and the proposition 3.2 of [42] on the last term, we can write the following controls for all $t \in [0, \frac{T_0}{\varepsilon}]$, $s \geq 2$ and $\sigma \geq s+5$:

$$\Big|\partial_x U_0(T, . - t)\int_0^t N_0(T, . - t + 2s)ds\Big|_{H^s(\mathbb{R})} \leq C_1\sqrt{t}\ ,$$

$$\Big|\partial_x U_0(T, . - t)\int_0^t b(. - t + s)ds\Big|_{H^s(\mathbb{R})} \leq C_2\, |b|_{L^2(\mathbb{R})}\, \sqrt{t}\ ,$$

$$\Big|\int_0^t \partial_x b(. - t + s) N_0(T, . - t + 2s)ds\Big|_{H^s(\mathbb{R})} \leq C_3\, |\partial_x b|_{H^s(\mathbb{R})}\, \sqrt{t}\ ,$$

where the constants C_1, C_2, C_3 depend exclusively on $|(U_0, N_0)|_{L^\infty([0,t];H^\sigma(\mathbb{R}))^2}$.
These preliminary estimates are at the heart of the proof of the following theorem.

Theorem 5.1.1. *Let $s \geq 2$, $\sigma \geq s+5$, $(v_0, \eta_0) \in H^\sigma(\mathbb{R})^2$, $b \in H^{s+4}(\mathbb{R})$ and $(v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon)_{0\leq\varepsilon\leq\varepsilon_0}$ be a family of solutions of (Σ) with initial data (v_0, η_0). We define $(u_0, n_0) = (v_0 + \eta_0, v_0 - \eta_0)$. Then the solution (U_0, N_0) of the system (Σ_{KdV}) with initial data (u_0, n_0) is bounded in $L^\infty([0, T_0]; H^\sigma(\mathbb{R}))$. Moreover, we have the following error estimate for all $t \in [0, \frac{T_0}{\varepsilon}]$:*

$$\left|(v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon) - (v_{KdV}^\varepsilon, \eta_{KdV}^\varepsilon)\right|_{L^\infty([0,t];H^s(\mathbb{R}))} \leq C\varepsilon\sqrt{t}(1+\varepsilon t)\,,$$

where $(v_{KdV}^\varepsilon, \eta_{KdV}^\varepsilon)$ are as defined in (4.1.7).

Proof. Using the three previous inequalities, one obtains :

$$|(U_1, N_1)|_{L^\infty([0,T_0]\times[0,t];H^{s+3}(\mathbb{R}))} \leq C\sqrt{t}\,.$$

where $C = C(|b|_{H^{s+4}(\mathbb{R})}, (U_0, N_0)_{L^\infty([0,t];H^\sigma(\mathbb{R}))^2})$. The final result follows from Corollary 1.1.6. □

Remark 5.1.2. *The difference between this theorem and Theorem 4.2.3 lies in the assumption made on the bottom topography b. Here, we just need to suppose $b \in H^{s+4}(\mathbb{R})$ whereas we supposed $b \in H^{s+4,1}(\mathbb{R})$ in the first theorem. The following function b defined as*

$$b(x) = \frac{1}{(1+x^2)^{27/4}}$$

is an example of bottom which is in $H^6(\mathbb{R})$ but not in $H^{6,1}(\mathbb{R})$.

This theorem proves that the approximation is less precise on a large time scale if we weaken the assumptions on the initial data and bottom, that is to say if we remove the assumption of a sufficient decay rate at infinity. And yet, it is worth pointing out that the regularity imposed on b in this theorem excludes many physical cases of interest. We focus from now on two simple examples of bottoms which do not fall into the scope of Theorem 5.1.1 : a regular step, and a slowly varying sinusoidal bottom. Our goal is to emphasize the fact that the approximation $(v_{KdV}^\varepsilon, \eta_{KdV}^\varepsilon)$ diverges from the exact solution $(v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon)$ in these two simple cases. A topographically modified KdV approximation which is still valid for such topographies is derived at the end of this chapter.
In order to simplify the analysis, we only consider the approximation corresponding to $a_1 = 1/6, a_2 = 0, a_4 = 0$ which is obtained for $\theta = \sqrt{2/3}, \lambda = 1, \mu = 1$, and the case of a wave propagating to the right. This last condition is realized by taking $n_0 = 0$, which implies that $N_0 = N = 0$.

5.1.1 The case of a step

We consider here a bottom whose shape corresponds to a regular step. The interest of such an example is that in this case, $b \notin L^2(\mathbb{R})$.
The bottom is defined as follows :

$$b(x) = \begin{cases} 0\ , \quad \forall x \leq 0\ , \\ \frac{A}{2}\left(1+\sin\left(\frac{\pi}{l}(x-\frac{l}{2})\right)\right)\ , \quad \forall x \in [0,l]\ , \\ A\ , \quad \forall x \geq l\ . \end{cases} \tag{5.1.1}$$

For a right going wave, the system (Σ_{KdV}) is reduced to the simple KdV equation :

$$\partial_T U_0 + \frac{3}{8}\partial_x U_0^2 + \frac{1}{6}\partial_x^3 U_0 = 0\ ,$$

and we chose the initial condition u_0 such that the solution of this equation is a positive soliton which propagates to the right.
We write the explicit expression of the corrector U_1 when $N_0 = 0$:

$$U_1(t,x) = \frac{1}{4}U_0(x-t)(b(x)-b(x-t)) + \frac{1}{2}\partial_x U_0(x-t)\int_0^t b(x-t+s)ds\ .$$

In this expression, the only possibly secularly growing term is $\partial_x U_0(T,x-t)\int_0^t b(x-t+s)ds$.
The time evolution in amplitude of this term is obviously led by the evolution of $\int_0^t b(x-t+s)$ for all $x \in \mathbb{R}$. When the bottom is a step as defined in (5.1.1), this integral essentially grows linearly in time. We now prove that because of this, $|U_1|_{L^\infty([0,T_0]\times[0,t];H^{s+3}(\mathbb{R}))}$ grows linearly in time. Let $s \geq 2$ and $\sigma \geq s+5$. Starting from the expression of U_1, we get for all $t \in [0, \frac{T_0}{\varepsilon}]$ the following estimates :

$$\begin{aligned} |U_1(T,t,\cdot)|_{H^{s+3}(\mathbb{R})} &= \Big|\frac{1}{4}U_0(T,\cdot-t)\Big(b(\cdot)-b(\cdot-t)\Big) \\ &\quad +\frac{1}{2}\partial_x U_0(T,\cdot-t)\int_0^t b(\cdot-t+s)ds\Big|_{H^{s+3}(\mathbb{R})}, \\ &\geq \Big|\frac{1}{2}\partial_x U_0(T,\cdot-t)\int_0^t b(\cdot-t+s)ds\Big|_{H^{s+3}(\mathbb{R})} - C\ , \end{aligned}$$

with $C = \Big|\frac{1}{4}U_0(T,\cdot - t)(b(\cdot) - b(\cdot - t))\Big|_{H^{s+3}} \leq \frac{1}{8}|b|_{L^\infty}|U_0|_{L^\infty([0,t];H^{s+3})} \equiv C_0$,

$$\begin{aligned}
|U_1(T,t,\cdot)|_{H^{s+3}(\mathbb{R})} &\geq \frac{1}{2}|\partial_x U_0(T,\cdot - t)\int_0^t b(\cdot - t + s)ds|_{L^2(\mathbb{R})} - C_0 \ , \\
&= \frac{1}{2}\sqrt{\int_0^\infty |\partial_x U_0(T,x-t)|^2 \Big|\int_0^t b(x-t+s)ds\Big|^2 dx} - C_0 \ , \\
&\quad \text{since} \int_0^t b(x-t+s)ds = 0 \ , \ \forall x \leq 0 \ , \\
&\geq \frac{1}{2}\sqrt{\int_{l+t}^\infty |\partial_x U_0(T,x-t)|^2 \Big|\int_{x-t}^x b(s)ds\Big|^2 dx} - C_0 \ , \\
&\geq \frac{1}{2}At\sqrt{\int_{l+t}^\infty |\partial_x U_0(T,x-t)|^2 dx} - C_0 \ , \\
&\quad \text{since} \int_{x-t}^x b(s)ds = At \ , \ \forall x \geq l+t \ , \\
&\geq \frac{1}{2}A\,t\sqrt{\int_l^\infty |\partial_x U_0(T,x)|^2 dx} - C_0 \ ,
\end{aligned}$$

which implies that

$$|U_1|_{L^\infty([0,T_0]\times[0,t];H^{s+3}(\mathbb{R}))} \geq C_1 t - C_0 \ , \tag{5.1.2}$$

where the last constant C_1 only depends on $|\partial_x U_0|_{L^2(\mathbb{R})}$.
This linear growth of $|U_1|_{L^\infty([0,T_0]\times[0,t];H^{s+3}(\mathbb{R}))}$ is sharp since it follows from the explicit expression of U_1 that this growth is at most linear. If follows therefore from Proposition 1.1.3 that there exists a constant C_2 such that for all $t \in [0, \frac{T}{\varepsilon}]$:

$$\Big|(v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon) - (v_{app}^\varepsilon, \eta_{app}^\varepsilon)\Big|_{L^\infty([0,t];H^s(\mathbb{R}))} \leq C_2(1+t)\varepsilon^2 t \ . \tag{5.1.3}$$

Furthermore, we recall that

$$(v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon) - (v_{KdV}^\varepsilon, \eta_{KdV}^\varepsilon) = (v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon) - (v_{app}^\varepsilon, \eta_{app}^\varepsilon) + \varepsilon\left(\frac{U_1 + N_1}{2}, \frac{U_1 - N_1}{2}\right) \ . \tag{5.1.4}$$

Using this relation, (5.1.2) and (5.1.3), we get that there exists a constant C_3 such that $\forall t \in [0, \frac{T_0}{\varepsilon}]$,

$$\Big|(v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon) - (v_{KdV}^\varepsilon, \eta_{KdV}^\varepsilon)\Big|_{L^\infty([0,t];H^s(\mathbb{R}))} \geq C_3(1+t)\varepsilon - C_2(1+t)\varepsilon^2 t \ .$$

We finally deduce that there exists two constants C and C' such that $\forall t \in [0, \frac{T_0}{\varepsilon}]$,

$$\Big|(v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon) - (v_{KdV}^\varepsilon, \eta_{KdV}^\varepsilon)\Big|_{L^\infty([0,t];H^s(\mathbb{R}))} \geq C\varepsilon t(C' - \varepsilon t) \ .$$

This proves that in this study case, the error is of order $O(1)$ on times of order $O(1/\varepsilon)$, and the usual KdV approximation is not valid for such a topography.

5.1.2 The case of a sinusoidal bottom

We consider here a bottom defined as follows :

$$b(x) = A\sin(\varepsilon x)\ ,\ \ \forall x \in \mathbb{R}\ . \tag{5.1.5}$$

We mention that such a type of periodic bottom varying on a slow spatial scale has been studied in [21] by Craig-Guyenne-Nicholls-Sulem, with the difference that the authors authorized the bottom to vary also on a small spatial scale.
Again, the amplitude of the term $\partial_x U_0(T, x-t)\int_0^t b(x-t+s)ds$ evolves in time according to $\int_0^t b(x-t+s)ds$. Let us have a look at this quantity for all $x \in \mathbb{R}$ and $t \geq 0$:

$$\begin{aligned}
\int_0^t b(x-t+s)ds &= \int_{x-t}^x b(s)ds \\
&= A\int_{x-t}^x \sin(\varepsilon x)ds \\
&= -\frac{A}{\varepsilon}\left[\cos(\varepsilon x) - \cos(\varepsilon(x-t))\right] \\
&= \frac{2A}{\varepsilon}\sin\left(\varepsilon(x-\frac{t}{2})\right)\sin\left(\frac{\varepsilon t}{2}\right)
\end{aligned}$$

We can see that the amplitude of this term is of order $O(1/\varepsilon)$. We now demonstrate that it is also the case for the corrector U_1 :

$$\begin{aligned}
|U_1(T,t,\cdot)|_{H^{s+3}(\mathbb{R})} &\geq \left|\frac{1}{2}\partial_x U_0(T,\cdot-t)\int_0^t b(\cdot-t+s)ds\right|_{H^{s+3}(\mathbb{R})} - C_0\ , \\
&\geq \frac{1}{2}\left|\partial_x U_0(T,\cdot-t)\int_0^t b(\cdot-t+s)ds\right|_{L^2(\mathbb{R})} - C_0\ , \\
&= \frac{1}{2}\sqrt{\int_{-\infty}^{\infty}|\partial_x U_0(T,x-t)|^2\left|\int_0^t b(x-t+s)ds\right|^2 dx} - C_0\ , \\
&= \frac{A}{\varepsilon}\sqrt{\int_{-\infty}^{\infty}|\partial_x U_0(T,x-t)|^2\sin^2\left(\varepsilon(x-\frac{t}{2})\right)\sin^2\left(\frac{\varepsilon t}{2}\right)dx} - C_0\ , \\
&= \frac{A}{\varepsilon}|\sin^2\left(\frac{\varepsilon t}{2}\right)|\sqrt{\int_{-\infty}^{\infty}|\partial_x U_0(T,x-t)|^2\sin^2\left(\varepsilon(x-\frac{t}{2})\right)dx} - C_0\ .
\end{aligned} \tag{5.1.6}$$

At this point, we remark that

$$0 \leq \int_{-\infty}^{\infty}|\partial_x U_0(T,x-t)|^2\sin^2\left(\varepsilon(x-\frac{t}{2})\right)dx \leq \int_{-\infty}^{\infty}|\partial_x U_0(T,x-t)|^2\,dx\ ,$$

and thus that

$$0 \leq \frac{\int_{-\infty}^{\infty} |\partial_x U_0(T, x-t)|^2 \sin^2\left(\varepsilon(x-\frac{t}{2})\right) dx}{\int_{-\infty}^{\infty} |\partial_x U_0(T, x-t)|^2 dx} \leq 1 \ .$$

We hence deduce that for all $t \geq 0$ there exists $\alpha(t) \in \mathbb{R}$ such that

$$\int_{-\infty}^{\infty} |\partial_x U_0(T, x-t)|^2 \sin^2\left(\varepsilon(x-\frac{t}{2})\right) dx = \sin^2(\alpha(t)) \int_{-\infty}^{\infty} |\partial_x U_0(T, x-t)|^2 dx \ .$$

Pluging this one into (5.1.6) leads to

$$|U_1(T, t, \cdot)|_{H^{s+3}(\mathbb{R})} \geq \frac{A}{\varepsilon} \left| \sin^2\left(\frac{\varepsilon t}{2}\right) \sin^2(\alpha(t)) \right| \left| \partial_x U_0(T, \cdot) \right|_{L^2(\mathbb{R})} - C_0 \ ,$$

which finally implies that there exists a constant C_1 such that

$$|U_1|_{L^\infty([0,T_0]\times[0,t];H^{s+3}(\mathbb{R}))} \geq \frac{C_1}{\varepsilon} - C_0 \ .$$

The estimate (5.1.3) holds again since the time growth of U_1 is still at most linear. Consequently, using the last estimate,(5.1.3) and the relation (5.1.4) leads to the existence of C_3 and C_4 such that

$$\left| (v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon) - (v_{KdV}^\varepsilon, \eta_{KdV}^\varepsilon) \right|_{L^\infty([0,t];H^s(\mathbb{R}))} \geq C_3 - \varepsilon C_4 - C_2(1+t)\varepsilon^2 t \ .$$

We finally deduce that there exists three constants C,C' and C'' such that $\forall t \in [0, \frac{T_0}{\varepsilon}]$,

$$\left| (v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon) - (v_{KdV}^\varepsilon, \eta_{KdV}^\varepsilon) \right|_{L^\infty([0,t];H^s(\mathbb{R}))} \geq C - C'\varepsilon^2 t^2 - C''\varepsilon(1+\varepsilon t) \ ,$$

which proves that the uncoupled KdV approximation diverges on a large time scale in this case too.

5.2 A topographically modified approximation

Both examples clearly show the invalidity of the approximation on a large time scale if we consider general bottoms topographies b which do not have specific decay properties at infinity. Therefore, we obviously need to modify the usual KdV approximation to be able to handle general bathymetries.

The explicit expression of U_1 has shown that the two terms that may exhibit a secular growth are $\frac{1}{2}\partial_x U_0(T, x-t) \int_0^t b(x-t+s)ds$ and $\frac{1}{4}\int_0^t \partial_x b(x-t+s) N_0(T, x-t+2s)ds$. As far as N_1 is concerned, the same possibly problematic terms are $\frac{-1}{2}\partial_x N_0(T, x+t) \int_0^t b(x+t-s)ds$ and $\frac{-1}{4}\int_0^t \partial_x b(x+t-s) U_0(T, x+t-2s)ds$. The idea is as follows : rather than treating these terms as correcting terms, we can include them with the leading order one terms U_0 and N_0 in the approximation.

This idea leads us to propose the following topographically modified KdV approximation which is an alternative version of $(\mathcal{M})$:

$$(\mathcal{M}_b)\left\{\begin{array}{rcl} v_{KdV}^{\varepsilon,b} &=& \dfrac{U_0+N_0}{2}+\dfrac{\varepsilon}{4}\Big[\partial_x U_0(T,x-t)\displaystyle\int_0^t b(x-t+s)ds \\ && -\partial_x N_0(T,x+t)\displaystyle\int_0^t b(x+t-s)ds \\ && +\dfrac{1}{2}\displaystyle\int_0^t \partial_x b(x-t+s)N_0(T,x-t+2s)ds \\ && -\dfrac{1}{2}\displaystyle\int_0^t \partial_x b(x+t-s)U_0(T,x+t-2s)ds \\ && +\dfrac{1}{2}U_0(T,x-t)\left(b(x)-b(x-t)\right) \\ && +\dfrac{1}{2}N_0(T,x+t)\left(b(x+t)-b(x)\right)\Big] \;, \\ \eta_{KdV}^{\varepsilon,b} &=& \dfrac{U_0-N_0}{2}+\dfrac{\varepsilon}{4}\Big[\partial_x U_0(T,x-t)\displaystyle\int_0^t b(x-t+s)ds \\ && -\partial_x N_0(T,x+t)\displaystyle\int_0^t b(x+t-s)ds \\ && +\dfrac{1}{2}\displaystyle\int_0^t \partial_x b(x-t+s)N_0(T,x-t+2s)ds \\ && -\dfrac{1}{2}\displaystyle\int_0^t \partial_x b(x+t-s)U_0(T,x+t-2s)ds \\ && +\dfrac{1}{2}U_0(T,x-t)\left(b(x)-b(x-t)\right) \\ && +\dfrac{1}{2}N_0(T,x+t)\left(b(x+t)-b(x)\right)\Big] \;. \end{array}\right. \tag{5.2.1}$$

where U_0 and N_0 are still solutions of the system $(\Sigma_{KdV}^{\varepsilon})$.

Remark 5.2.1. *We have here also included the terms* $U_0(T,x-t)\left(b(x)-b(x-t)\right)$ *and* $N_0(T,x+t)\left(b(x+t)-b(x)\right)$ *even if these terms remain bounded indepently of* ε *for all time. The reason of this choice is that we are interested in their physical meaning. Indeed, we further see - in Part III - that they are responsible for the reproduction of the phenomenon of shoaling. We hence decided to include these terms in the approximation.*

The main advantage of this modification relies in the following remark : now that the bottom terms have been included with the leading order terms in the approximation, we can easily see that the correcting terms U_1 and N_1 solve a different equation. Indeed, the equation on U_1 becomes :

$$(\Sigma_{corr}^{\,b})\left\{\begin{array}{l} (\partial_t+\partial_x)U_1 = -\dfrac{1}{8}\partial_x N_0^2-\dfrac{1}{4}\partial_x(U_0N_0)+\dfrac{a_2-a_4}{2}\,\partial_x^3 N_0 \;, \\ (\partial_t-\partial_x)N_1 = -\dfrac{1}{8}\partial_x U_0^2-\dfrac{1}{4}\partial_x(U_0N_0)-\dfrac{a_2-a_4}{2}\,\partial_x^3 U_0 \;. \end{array}\right. \tag{5.2.2}$$

It is clear here that all the possibly secularly growing terms of the correctors (U_1, N_1) have been removed. We can now state our final theorem.

Theorem 5.2.2. *Let $s \geq 2$, $\sigma \geq s+5$, $(v_0, \eta_0) \in H^\sigma(\mathbb{R})^2$, $b \in W^{1,\infty}(\mathbb{R})$ and $(v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon)_{0\leq\varepsilon\leq\varepsilon_0}$ be a family of solutions of (Σ) with initial data (v_0, η_0). We define $(u_0, n_0) = (v_0 + \eta_0, v_0 - \eta_0)$. Then the solution (U_0, N_0) of the system (Σ_{KdV}) with initial data (u_0, n_0) is bounded in $L^\infty([0, T_0]; H^\sigma(\mathbb{R}))$. Moreover, we have for all $t \in [0, \frac{T_0}{\varepsilon}]$:*

$$\left|(v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon) - (v_{KdV}^{\varepsilon,b}, \eta_{KdV}^{\varepsilon,b})\right|_{L^\infty([0,t];H^s(\mathbb{R}))} \leq C\varepsilon\sqrt{t}(1 + \varepsilon t)\,,$$

where $(v_{KdV}^{\varepsilon,b}, \eta_{KdV}^{\varepsilon,b})$ is as defined in (5.2.1).

Proof. The proof is straightforward using the previous remark and adapting the prooves of Theorems 4.2.3 and 5.1.1. ☐

Remark 5.2.3. *This modified version is quite interesting numerically since the topographical terms are computed explicitly from the solution of the KdV equations. We thus expect the numerical simulation of this model to be faster than the one of the symmetric Boussinesq model (Σ). This point is checked in Part III.*

In the periodic framework, we saw that the usual approximation is not valid on a large time scale because of the linear growth in time of the term $\partial_x U_0(T, x - t) \int_0^t N_0(T, x - t + 2s)ds$ in U_1, unless we specify a zero mass assumption on the initial data u_0 and n_0. Once more, we can propose a valid approximation just by including this term in the order one terms of the ansatz. We conclude this chapter with the proposition of a new approximation that remains valid in the periodic framework :

$$(\mathcal{M}_b^{per}) \begin{cases} v_{KdV}^{\varepsilon,b\,per} = v_{KdV}^{\varepsilon,b} - \dfrac{\varepsilon}{8}\Big[\partial_x U_0(T, x-t)\displaystyle\int_0^t N_0(T, x-t+2s)ds \\ \qquad\qquad +\partial_x N_0(T, x+t)\displaystyle\int_0^t U_0(T, x+t-2s)ds\Big]\quad, \\ \eta_{KdV}^{\varepsilon,b\,per} = \eta_{KdV}^{\varepsilon,b} - \dfrac{\varepsilon}{8}\Big[\partial_x U_0(T, x-t)\displaystyle\int_0^t N_0(T, x-t+2s)ds \\ \qquad\qquad -\partial_x N_0(T, x+t)\displaystyle\int_0^t U_0(T, x+t-2s)ds\Big]\quad. \end{cases} \tag{5.2.3}$$

Concerning this approximation, the previous theorem remains true and we can even state an improved version :

Theorem 5.2.4. *Let* $s \geq 2$*,* $\sigma \geq s+5$*,* $(v_0, \eta_0) \in H^\sigma(\mathbb{T})^2$*,* $b \in W^{1,\infty}(\mathbb{T})$ *and* $(v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon)_{0\leq\varepsilon\leq\varepsilon_0}$ *be a family of solutions of* (Σ) *with initial data* (v_0, η_0)*. We define* $(u_0, n_0) = (v_0 + \eta_0, v_0 - \eta_0)$*. Then the solution* (U_0, N_0) *of the system* (Σ_{KdV}) *with initial data* (u_0, n_0) *is bounded in* $L^\infty([0, T_0]; H^\sigma(\mathbb{T}))$*. Besides, we have for all* $t \in [0, \frac{T_0}{\varepsilon}]$ *:*

$$\left|(v_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon) - (v_{KdV}^{\varepsilon, b\,per}, \eta_{KdV}^{\varepsilon, b\,per})\right|_{L^\infty([0,t];H^s(\mathbb{T}))} \leq C\varepsilon(1 + \varepsilon t)\,,$$

This theorem remains true in the nonperiodic framework, which means that we have a better precision with this model than with the model $(\mathcal{M}_b)$.

PARTIE III

Simulations numériques