A characterization of NST models and their associated polynomials

Sommaire

2.1 Résumé en français

Dans ce chapitre, on a proposé une caractérisation par fonction variance des modèles NST multivariés sur $\mathbb{R}^k$ décrits par Boubacar Maïnassara & Kokonendji (2014). Ces modèles sont composés d'une variable stable-Tweedie positive fixée et des variables gaussiennes indépendantes conditionnées par la première, de mêmes paramètres de dispersion égale à la variable fixée. Étant donné un vecteur aléatoire $(X_1, \ldots, X_k)^\top \in \mathbb{R}^k$, les modèles NST multivariés sont générés par la mesures σ-finies $\boldsymbol{\nu}_{p,t}$ on $\mathbb{R}^k$ (avec $p \geq 1$ et $t > 0$) données par :

$$\boldsymbol{\nu}_{p,t}(d\mathbf{x}) = \mu_{p,t}(dx_1) \prod_{j=2}^{k} \mu_{0,x_1}(dx_j),$$

où X_1 est une variable stable-Tweedie positive univariée de distribution $\mu_{p,t}$ en Definition 1.5.1 et $X_2, \ldots, X_k$ sachant $X_1 = x_1$ sont $k-1$ variables réelles gaussiennes indépendantes générées par μ_{0,x_1} (de moyenne et de variance x_1) avec $(p-1)(1-\alpha) = 1$. La fonction cumulante $\mathbf{K}_{\nu_{p,t}}(\boldsymbol{\theta}) = \log \int_{\mathbb{R}^k} \exp(\langle \boldsymbol{\theta}, \mathbf{x} \rangle) \boldsymbol{\nu}_{p,t}(d\mathbf{x})$ est explicitement

$$\mathbf{K}_{\nu_{p,t}}(\boldsymbol{\theta}) = tK_{\mu_{p,1}}\left(\theta_1 + \frac{1}{2}\sum_{j=2}^{k}\theta_j^2\right),$$

pour tout $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)^\top$ dans le domaine canonique

$$\boldsymbol{\Theta}(\boldsymbol{\nu}_{p,t}) = \left\{\boldsymbol{\theta} \in \mathbb{R}^k; \left(\theta_1 + \frac{1}{2}\sum_{j=2}^{k}\theta_j^2\right) \in \Theta\left(\mu_{p,1}\right)\right\},$$

où $K_{\mu_{p,1}} =: K_{\mu_p}$ est la fonction cumulante des modèles stables-Tweedie de relation (1.7).

Ces modèles ont pour fonctions variances

$$\mathbf{V}_{\mathbf{G}_{p,t}}(\mathbf{m}) = t^{1-p} m_1^{p-2} \cdot \mathbf{m}\mathbf{m}^\top + \mathbf{Diag}_k(0, m_1, \ldots, m_1), \quad \forall \mathbf{m} \in \mathbf{M}_{\mathbf{G}_{p,t}},$$

avec t la puissance de convolution de mesure associée, p le paramètre de puissance variance et $\mathbf{M}_{\mathbf{G}_{p,t}} =]0, \infty[\times \mathbb{R}^{k-1}$ le domaine des moyennes $\mathbf{m}$. A travers la décomposition matricielle de $\mathbf{V}_{\mathbf{G}_{p,t}}(\mathbf{m})$ via le complément de Schur, on a su déterminer la matrice inverse de $\mathbf{V}_{\mathbf{G}_{p,t}}(\mathbf{m})$, et arriver à caractériser analytiquement les *NST* par leurs fonctions variances en distinguant les cas limites $p = 1$ et $p = 2$. Les lois *NST* étant indéfiniment divisible pour $p \geq 1$, la nature des polynômes associés à ces modèles a été décrite à l'aide des propriétés de la quasi orthogonalité, des systèmes de Lévy-Sheffer et des relations de récurrences correspondantes.

2.2 Introduction

An important problem in statistical analysis is how to choose an adequate family of distributions or statistical model to describe the study. For this purpose, the characterization theorems can be useful because, under general reasonable suppositions related to the nature of the experiment, they allow us to reduce the possible set of distributions that can be used. One of these reasonable assumptions is that the normal stable Tweedie (*NST*) models (Boubacar Maïnassara & Kokonendji, 2014) present particular forms of variance functions based on the first component of mean vector and a probability measure which is not easy to handle. So, a characterization by variance functions or by associated polynomials is required for the analysis related to this model. Recall that variance function plays a significant role in the classification of natural exponential families (*NEF*). Thus, the NEFs can be characterized by variance functions obtained by successive differentiations of the Laplace transform of a positive measure. Also, the variance functions are convenient to identify a family that is, for example, a Laplace transform to identify a probability distribution.

For an accurate presentation of this work, let us introduce some notations. Let $k \in \{2, 3, \ldots\}$, we denote by $(\mathbf{e}_i)_{i=1,\ldots,k}$ an orthonormal basis of $\mathbb{R}^k$ and by $\mathbf{I}_k = \mathbf{Diag}_k(1, \ldots, 1)$ the $k \times k$ unit matrix. For two vectors $\mathbf{a} = (a_1, \ldots, a_k)^\top \in \mathbb{R}^k$ and $\mathbf{b} = (b_1, \ldots, b_k)^\top \in \mathbb{R}^k$, we use the notations $\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a}^\top \mathbf{b}$ and $\mathbf{a} \otimes \mathbf{b} = \mathbf{a}\mathbf{b}^\top$ to denote the scalar $\sum_{j=1}^k a_j b_j$ and the $k \times k$ matrix $\left(a_i b_j\right)_{i,j=1,\ldots,k}$ respectively, and finally $\mathcal{S}\left(\mathbb{R}^k\right)$ the set of symmetric matrices on $\mathbb{R}^k$. About exponential families, variance function, generalized variance function, we are referring to Chapter 1.

Concerning notations of associated polynomials, we also recall that for all $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$ and for all $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$, we write $\mathbf{x}^\mathbf{n} = x_1^{n_1} \cdots x_k^{n_k}$, $\|\mathbf{x}\|_+ = \max\left(-\sum x_i^-, \sum x_i^+\right)$, $\mathbf{n}! = n_1! \cdots n_k!$ and $|\mathbf{n}| = n_1 + \cdots + n_k$. A polynomial in $\mathbf{x} \in \mathbb{R}^k$ of the $|\mathbf{n}|$th degree can be written as follows :

$$Q_\mathbf{n}(\mathbf{x}) = \sum_{|\mathbf{q}| \leq |\mathbf{n}|} \alpha_\mathbf{q} \mathbf{x}^\mathbf{q}, \quad \mathbf{n}, \mathbf{q} \in \mathbb{N}^k,$$

where $\alpha_\mathbf{q} \neq 0$ when $|\mathbf{q}| = |\mathbf{n}|$. Let us now introduce the power of convolution $t > 0$ of any $\boldsymbol{\mu} \in \mathcal{M}(\mathbb{R}^k)$ such that $\boldsymbol{\mu}^{*t} = \boldsymbol{\mu}_t$ generates the NEF $\mathbf{F}_t = \mathbf{F}(\boldsymbol{\mu}_t)$ on $\mathbb{R}^k$ with mean $\mathbf{m}_t \in \mathrm{M}_{\mathbf{F}_t}$. Note that, for $t > 0$ and then $\boldsymbol{\mu}$ is infinitely divisible, we can associate $\mathbf{F}_t$ to a Lévy process $(\boldsymbol{X}_t)_{t>0}$; see, e.g., Sato (1999). For all $\mathbf{n} \in \mathbb{N}^k$ and for all $\mathbf{A} \in \mathbf{GL}\left(\mathbb{R}^k\right)$, following Schoutens & Teugels (1998) we define a family of associated polynomials (linked to the so-called Lévy-Scheffer systems) in $\mathbf{x} \in \mathbb{R}^k$ by

$$\mathrm{P}_{\mathbf{A},\mathbf{n},t}(\mathbf{x}) = f_{\mu_t}^{(\mathbf{n})}(\mathbf{x}, \mathbf{m}_t)\left(\mathbf{A}\mathbf{e}_1, \ldots, \mathbf{A}\mathbf{e}_k\right) / f_{\mu_t}(\mathbf{x}, \mathbf{m}_t), \tag{2.1}$$

where $f_{\mu_t}^{(\mathbf{n})}(\mathbf{x}, \mathbf{m})(\mathbf{A}\mathbf{e}_1, \ldots, \mathbf{A}\mathbf{e}_k)$ is the $|\mathbf{n}|$th derivative of $\mathbf{m} \mapsto f_{\mu_t}(\mathbf{x}, \mathbf{m})$ in $|\mathbf{n}|$th directions $\mathbf{A}\mathbf{e}_1$ (n_1 times), ... , $\mathbf{A}\mathbf{e}_k$ (n_k times) with

$$f_{\mu_t}(\mathbf{x}, \mathbf{m}) = \exp\{\langle \psi_{\mu_t}(\mathbf{m}), \mathbf{x} \rangle - K_{\mu_t}(\psi_{\mu_t}(\mathbf{m}))\}, \quad \forall \mathbf{m} \in \mathrm{M}_{F_t},$$

and the relation $\psi_{\mu_t}(\mathbf{m}) = \psi_\mu(\mathbf{m}/t)$. When $\mathbf{A} = \mathbf{I}$, the expression (2.1) corresponds to $\mathrm{P}_{\mathbf{n},t}(\mathbf{x}) = f_{\mu_t}^{(\mathbf{n})}(\mathbf{x}, \mathbf{m}_t)/f_{\mu_t}(\mathbf{x}, \mathbf{m}_t)$. In particular, $f_{\mu_t}(\cdot, \mathbf{m}_t) := 1$ and $\mathrm{P}_{\mathbf{A},\mathbf{n},t}$ is a polynomial in $\mathbf{x}$ of degree $|\mathbf{n}|$ and the sequence $(\mathrm{P}_{\mathbf{A},\mathbf{n},t=1})_{\mathbf{n}\in\mathbb{N}^k}$ forms a basis of $\mathbb{R}[x_1, \ldots, x_k]$.

In this paper we first characterize the *NST* models through their variance functions and then describe some of their associated polynomials by quasi-othogonality properties. For this, Section 2.3 recalls the description of all k-variate *NST* models which are composed by a fixed univariate stable Tweedie variable having a positive value domain, and the remaining random variables given the fixed one are real independent Gaussian variables with the same variance equal to the fixed component. Section 2.4 is devoted to the result of the *NST* characterization by variance functions, for which the proof is given in the appendix in Section 2.6. In Section 2.5 we examine associated polynomials with these models.

2.3 *NST* models and déterminant of the Hessian cumulant function

According to Boubacar Maïnassara & Kokonendji (2014) and given a random vector $(X_1, \dots, X_k)^\top \in \mathbb{R}^k$, the multivariate *NST* models are generated by σ-finite measures $\boldsymbol{\nu}_{p,t}$ on $\mathbb{R}^k$ (with $p \geq 1$ and $t > 0$) as follows :

$$\boldsymbol{\nu}_{p,t}(d\mathbf{x}) = \mu_{p,t}(dx_1) \prod_{j=2}^{k} \mu_{0,x_1}(dx_j), \tag{2.2}$$

where X_1 is a positive univariate stable-Tweedie distribution $\mu_{p,t}$ of Definition 1.5.1 and $X_2, \dots, X_k$ given $X_1 = x_1$ are $k-1$ real independent Gaussian variables generated by μ_{0,x_1} (with mean 0 and variance x_1) with $(p-1)(1-\alpha) = 1$. The cumulant function $\mathbf{K}_{\nu_{p,t}}(\boldsymbol{\theta}) = \log \int_{\mathbb{R}^k} \exp(\langle \boldsymbol{\theta}, \mathbf{x} \rangle) \boldsymbol{\nu}_{p,t}(d\mathbf{x})$ is explicitly

$$\mathbf{K}_{\nu_{p,t}}(\boldsymbol{\theta}) = t K_{\mu_{p,1}} \left(\theta_1 + \frac{1}{2} \sum_{j=2}^{k} \theta_j^2 \right), \tag{2.3}$$

for all $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k)^\top$ in the canonical domain

$$\boldsymbol{\Theta}(\boldsymbol{\nu}_{p,t}) = \left\{ \boldsymbol{\theta} \in \mathbb{R}^k ; \left(\theta_1 + \frac{1}{2} \sum_{j=2}^{k} \theta_j^2 \right) \in \Theta\left(\mu_{p,1}\right) \right\} \tag{2.4}$$

where $K_{\mu_{p,1}} =: K_{\mu_p}$ is the cumulant function of stables-Tweedie models of relation (1.7).

The fowolling results stat some news importants properties of cumulant of (2.3) which we will use in Chapitre 4.

Theorem 2.3.1 *For $t = 1$, the determinant of Hessian cumulant fonction (2.3) is continuous for all $p \in (1, \infty)$ and satisfies :*

(*i*) $\det \mathbf{K}''_{\nu_p}(\boldsymbol{\theta}) = \left[\mathbf{K}_{\nu_p}(\boldsymbol{\theta})\right]^k$ *pour* $p = 1$.

(*ii*) $\det \mathbf{K}''_{\nu_p}(\boldsymbol{\theta}) = \left[(2-p)\mathbf{K}_{\nu_p}(\boldsymbol{\theta})\right]^{(p+k-1)/(2-p)}$ *pour* $1 < p < 2$.

(*iii*) $\det \mathbf{K}''_{\nu_p}(\boldsymbol{\theta}) = \exp\left[(k+1)\mathbf{K}_{\nu_p}(\boldsymbol{\theta})\right]$ *pour* $p = 2$.

(*iv*) $\det \mathbf{K}''_{\nu_p}(\boldsymbol{\theta}) = \left[1/\left(-(p-2)\mathbf{K}_{\nu_p}(\boldsymbol{\theta})\right)\right]^{(p+k-1)/(p-2)}$ *pour* $p > 2$.

The following lemma is required to prove Theorem 2.3.1.

Lemma 2.3.1 *Let* $\mathbf{L}$ *be a positive definite* $k \times k$ *matrix* $\mathcal{S}\left(\mathbb{R}^k\right)$ *having the form*

$$\mathbf{L} = \begin{pmatrix} \lambda & \mathbf{a}^\top \\ \mathbf{a} & \mathbf{A} \end{pmatrix},$$

with $\lambda \in \mathbb{R} \setminus \{0\}$, $\mathbf{a} = (a_1, \ldots, a_{k-1})^\top \in \mathbb{R}^{k-1}$ *and* $\mathbf{A}$ *is a* $(k-1) \times (k-1)$ *symetric matrix. Then, the determinant of matrix* $\mathbf{L}$ *is given by*

$$\det \mathbf{L} = \lambda \det(\mathbf{A} - \lambda^{-1} \mathbf{a} \otimes \mathbf{a}).$$

Proof : Since $\lambda \neq 0$, one can use Schur complement $\mathbf{A} - \lambda^{-1}\mathbf{a} \otimes \mathbf{a}$ of λ to obtain

$$\mathbf{L} = \begin{pmatrix} 1 & 0_{1\times(k-1)} \\ \lambda^{-1}\mathbf{a} & \mathbf{I}_{k-1} \end{pmatrix} \begin{pmatrix} \lambda & 0_{1\times(k-1)} \\ 0_{(k-1)\times 1} & \mathbf{A} - \lambda^{-1}\mathbf{a} \otimes \mathbf{a} \end{pmatrix} \begin{pmatrix} 1 & \lambda^{-1}\mathbf{a}^\top \\ 0_{(k-1)\times 1} & \mathbf{I}_{k-1} \end{pmatrix},$$

where $\mathbf{I}_{k-1}$ is $(k-1)\times(k-1)$ unit matrix, $0_{1\times(k-1)}$ is $1\times(k-1)$ matrix and $0_{(k-1)\times 1}$ is $(k-1)\times 1$ matrix une matrice a $k-1$ colonnes et une ligne. Then,

$$\begin{aligned} \det \mathbf{L} &= \det \begin{pmatrix} 1 & 0_{1\times(k-1)} \\ \lambda^{-1}\mathbf{a} & \mathbf{I}_{k-1} \end{pmatrix} \det \begin{pmatrix} \lambda & 0_{1\times(k-1)} \\ 0_{(k-1)\times 1} & \mathbf{A} - \lambda^{-1}\mathbf{a} \otimes \mathbf{a} \end{pmatrix} \det \begin{pmatrix} 1 & \lambda^{-1}\mathbf{a}^\top \\ 0_{(k-1)\times 1} & \mathbf{I}_{k-1} \end{pmatrix} \\ &= (\det \mathbf{I}_{k-1}) \left(\lambda \det(\mathbf{A} - \lambda^{-1}\mathbf{a} \otimes \mathbf{a})\right) (\det \mathbf{I}_{k-1}) \\ &= \lambda \det(\mathbf{A} - \lambda^{-1}\mathbf{a} \otimes \mathbf{a}), \end{aligned}$$

and the Lemma 2.3.1 is proved. □

Proof of Theorem 2.3.1 : Setting $g(\boldsymbol{\theta}) = \theta_1 + \frac{1}{2}\sum_{j=2}^{k} \theta_j^2$ and $t = 1$, the relation (2.3) is write

$$\mathbf{K}_{\nu_p}(\boldsymbol{\theta}) = K_{\mu_p}(g(\boldsymbol{\theta})) \tag{2.5}$$

with $p \geqslant 1$. Then,

$$\mathbf{K}''_{\nu_p}(\boldsymbol{\theta}) = K''_{\mu_p}(g(\boldsymbol{\theta})) \times g'(\boldsymbol{\theta}) \otimes g'(\boldsymbol{\theta}) + K'_{\mu_p}(g(\boldsymbol{\theta})) \times g''(\boldsymbol{\theta}),$$

where $g'(\boldsymbol{\theta}) = (1, \theta_2, \ldots, \theta_k)^\top$ et $g''(\boldsymbol{\theta}) = \mathbf{Diag}_k(0, 1, \ldots, 1)$. Therefore

$$\begin{aligned} \mathbf{K}''_{\nu_p}(\boldsymbol{\theta}) &= K''_{\mu_p}(g(\boldsymbol{\theta}))\,(1, \theta_2, \ldots, \theta_k) \otimes (1, \theta_2, \ldots, \theta_k) \\ &\quad + K'_{\mu_p}(g(\boldsymbol{\theta}))\,\mathbf{Diag}_k(0, 1, \ldots, 1). \end{aligned}$$

Using Lemma 2.3.1 with $\lambda = K''_{\mu_p}(g(\boldsymbol{\theta}))$, $\mathbf{a} = K''_{\mu_p}(g(\boldsymbol{\theta}))\,(\theta_2, \ldots, \theta_k)^\top$ and $\mathbf{A} = K''_{\mu_p}(g(\boldsymbol{\theta}))\,(\theta_2, \ldots, \theta_k) \otimes (\theta_2, \ldots, \theta_k) + K'_{\mu_p}(g(\boldsymbol{\theta}))\,\mathbf{I}_{k-1}$. One deduces $\mathbf{A} - \lambda^{-1}\mathbf{a} \otimes \mathbf{a} =$

$K'_{\mu_p}(g(\boldsymbol{\theta}))\,\mathbf{I}_{k-1}$ and

$$\begin{aligned}\det \mathbf{K}''_{\nu_p}(\boldsymbol{\theta}) &= K''_{\mu_p}(g(\boldsymbol{\theta}))\det\left[K'_{\mu_p}(g(\boldsymbol{\theta}))\,\mathbf{I}_{k-1}\right]\\ &= K''_{\mu_p}(g(\boldsymbol{\theta}))\left[K'_{\mu_p}(g(\boldsymbol{\theta}))\right]^{k-1}\det\mathbf{I}_{k-1}\\ &= K''_{\mu_p}(g(\boldsymbol{\theta}))\left[K'_{\mu_p}(g(\boldsymbol{\theta}))\right]^{k-1}\\ &= K''_{\mu_p}\left(\theta_1+\frac{1}{2}\sum_{j=2}^{k}\theta_j^2\right)\left[K'_{\mu_p}\left(\theta_1+\frac{1}{2}\sum_{j=2}^{k}\theta_j^2\right)\right]^{k-1}, \qquad (2.6)\end{aligned}$$

which we use to verify the four relationships listed in the above proposal based on (2.42) and the expressions of K_{μ_p} of Définition 1.5.1 according to the values of p.

$\boxed{(i)}$ For $p=1$, $K_{\mu_p}(\theta_0)=\exp(\theta_0)$, then $K'_{\mu_p}(\theta_0)=\exp(\theta_0)$ and $K''_p(\theta_0)=\exp(\theta_0)$ therefore (2.6) is write

$$\begin{aligned}\det \mathbf{K}''_{\nu_p}(\boldsymbol{\theta}) &= K_{\mu_p}\left(\theta_1+\frac{1}{2}\sum_{j=2}^{k}\theta_j^2\right)\left[K_{\mu_p}\left(\theta_1+\frac{1}{2}\sum_{j=2}^{k}\theta_j^2\right)\right]^{k-1}\\ &= \left[K_{\mu_p}\left(\theta_1+\frac{1}{2}\sum_{j=2}^{k}\theta_j^2\right)\right]^{k}\\ &= \left[\mathbf{K}_{\nu_p}(\boldsymbol{\theta})\right]^{k},\end{aligned}$$

with $\mathbf{K}_{\nu_p}(\boldsymbol{\theta})>0$ pour $p=1$.

$\boxed{(ii)}$ For $1<p<2$, $K_{\mu_p}(\theta_0)=[1/(2-p)]\,[-(p-1)\theta_0]^{1+1/(1-p)}$ with $\Theta_{\mu_p}=(-\infty,0)$. Then the first and second order derivatives gives, respectively $K'_{\mu_p}(\theta_0)=[-(p-1)\theta_0]^{1/(1-p)}=\left[(2-p)K_{\mu_p}(\theta_0)\right]^{1/(2-p)}$ and $K''_{\mu_p}(\theta_0)=[-(p-1)\theta_0]^{p/(1-p)}=\left[(2-p)K_{\mu_p}(\theta_0)\right]^{p/(2-p)}$. The expression of (2.6) becomes

$$\begin{aligned}\det \mathbf{K}''_{\nu_p}(\boldsymbol{\theta}) &= \left[(2-p)K_{\mu_p}\left(\theta_1+\frac{1}{2}\sum_{j=2}^{k}\theta_j^2\right)\right]^{p/(2-p)}\left[(2-p)K_{\mu_p}\left(\theta_1+\frac{1}{2}\sum_{j=2}^{k}\theta_j^2\right)\right]^{(k-1)/(2-p)}\\ &= \left[(2-p)K_{\mu_p}\left(\theta_1+\frac{1}{2}\sum_{j=2}^{k}\theta_j^2\right)\right]^{(p+k-1)/(2-p)}\\ &= \left[(2-p)\mathbf{K}_{\nu_p}(\boldsymbol{\theta})\right]^{(p+k-1)/(2-p)},\end{aligned}$$

with $\mathbf{K}_{\nu_p}(\boldsymbol{\theta})>0$ pour $1<p<2$.

$\boxed{(iii)}$ For $p=2$, $K_{\mu_p}(\theta_0)=-\log(-\theta_0)$ avec $\Theta_{\mu_p}=(-\infty,0)$. Then $K'_{\mu_p}(\theta_0)=-1/\theta_0=\exp\left[K_{\mu_p}(\theta_0)\right]$ et $K''_{\mu_p}(\theta_0)=1/\theta_0^2=\exp\left[2K_{\mu_p}(\theta_0)\right]$ and then the relation

(2.6) is write,

$$\begin{aligned}\det \mathbf{K}''_{\nu_p}(\boldsymbol{\theta}) &= \exp\left[2K_{\mu_p}\left(\theta_1 + \frac{1}{2}\sum_{j=2}^{k}\theta_j^2\right)\right]\left[\exp\left\{K_{\mu_p}\left(\theta_1 + \frac{1}{2}\sum_{j=2}^{k}\theta_j^2\right)\right\}\right]^{k-1} \\ &= \exp\left[(k+1)K_{\mu_p}\left(\theta_1 + \frac{1}{2}\sum_{j=2}^{k}\theta_j^2\right)\right] \\ &= \exp\left[(k+1)\mathbf{K}_{\nu_p}(\boldsymbol{\theta})\right].\end{aligned}$$

(*iv*) For $p > 2$, $K_{\mu_p}(\theta_0) = [-1/(p-2)]\,[-(p-1)\theta_0]^{1+1/(1-p)}$ avec $\Theta_{\mu_p} = (-\infty, 0)$. Then the first and second order derivatives gives, respectively

$$K'_{\mu_p}(\theta_0) = [1/(-(p-1)\theta_0)]^{1/(p-1)} = \left[1/\left(-(p-2)K_{\mu_p}(\theta_0)\right)\right]^{1/(p-2)},$$

and

$$K''_{\mu_p}(\theta_0) = [1/(-(p-1)\theta_0)]^{p/(p-1)} = \left[1/\left(-(p-2)K_{\mu_p}(\theta_0)\right)\right]^{p/(p-2)},$$

that one replaces in the expression (2.6) to get

$$\begin{aligned}\det \mathbf{K}''_{\nu_p}(\boldsymbol{\theta}) &= \left[1/\left(-(p-2)K_{\mu_p}\left(\theta_1 + \frac{1}{2}\sum_{j=2}^{k}\theta_j^2\right)\right)\right]^{(p+k-1)/(p-2)} \\ &= \left[1/\left(-(p-2)\mathbf{K}_{\nu_p}(\boldsymbol{\theta})\right)\right]^{(p+k-1)/(p-2)},\end{aligned}$$

with $\mathbf{K}_{\nu_p}(\boldsymbol{\theta}) < 0$ for $p > 2$.

For $p \in (1,2) \cup (2,\infty)2$, the function $p \mapsto \mathbf{K}_{\nu_p}$ is continuous. We will check the continuity at $p = 2$. Then, For $p \neq 2$,

$$\begin{aligned}
\lim_{p\to 2^-}\left[\det \mathbf{K}''_{\nu_p}(\boldsymbol{\theta})\right] &= \lim_{p\to 2^-}\left[(2-p)\mathbf{K}_{\nu_p}(\boldsymbol{\theta})\right]^{(p+k-1)/(2-p)}\\
&= \lim_{p\to 2^-}\left[(2-p)K_{\mu_p}\left(\theta_1+\frac{1}{2}\sum_{j=2}^{k}\theta_j^2\right)\right]^{(p+k-1)/(2-p)}\\
&= \lim_{p\to 2^-}\left[-(p-1)\left(\theta_1+\frac{1}{2}\sum_{j=2}^{k}\theta_j^2\right)\right]^{(p+k-1)/(1-p)}\\
&= \left[-\left(\theta_1+\frac{1}{2}\sum_{j=2}^{k}\theta_j^2\right)\right]^{-(k+1)}\\
&= \exp\left[-(k+1)\log\left(-\theta_1-\frac{1}{2}\sum_{j=2}^{k}\theta_j^2\right)\right]\\
&= \exp\left[(k+1)K_{\mu_{p=2}}\left(-\theta_1-\frac{1}{2}\sum_{j=2}^{k}\theta_j^2\right)\right]\\
&= \exp\left[(k+1)\mathbf{K}_{\nu_{p=2}}(\boldsymbol{\theta})\right]\\
&= \det \mathbf{K}''_{\nu_{p=2}}(\boldsymbol{\theta}),
\end{aligned}$$

from where the function $p \mapsto \det \mathbf{K}''_{\nu_p}(\boldsymbol{\theta})$ is continuous from the left at $p = 2$.

$$\begin{aligned}
\lim_{p\to 2^+}\left[\det \mathbf{K}''_{\nu_p}(\boldsymbol{\theta})\right] &= \lim_{p\to 2^+}\left[1/\left(-(p-2)\mathbf{K}_{\nu_p}(\boldsymbol{\theta})\right)\right]^{(p+k-1)/(p-2)}\\
&= \lim_{p\to 2^+}\left[1/\left(-(p-2)K_{\mu_p}\left(\theta_1+\frac{1}{2}\sum_{j=2}^{k}\theta_j^2\right)\right)\right]^{(p+k-1)/(p-2)}\\
&= \lim_{p\to 2^+}\left[1/\left(-(p-1)\left(\theta_1+\frac{1}{2}\sum_{j=2}^{k}\theta_j^2\right)\right)\right]^{(p+k-1)/(p-1)}\\
&= \left[-\left(\theta_1+\frac{1}{2}\sum_{j=2}^{k}\theta_j^2\right)\right]^{-(k+1)}\\
&= \exp\left[-(k+1)\log\left(-\theta_1-\frac{1}{2}\sum_{j=2}^{k}\theta_j^2\right)\right]\\
&= \exp\left[(k+1)K_{\mu_{p=2}}\left(-\theta_1-\frac{1}{2}\sum_{j=2}^{k}\theta_j^2\right)\right]\\
&= \exp\left[(k+1)\mathbf{K}_{\nu_{p=2}}(\boldsymbol{\theta})\right]\\
&= \det \mathbf{K}''_{\nu_{p=2}}(\boldsymbol{\theta}),
\end{aligned}$$

from where the function $p \mapsto \det \mathbf{K}''_{\nu_p}(\boldsymbol{\theta})$ is continuous from the right at $p = 2$.
□

For fixed $p \geq 1$ and $t > 0$, the multivariate NEF generated by $\boldsymbol{\nu}_{p,t}$ of (3.2) is the set $\mathbf{G}_{p,t} = \left\{\mathbf{P}(\boldsymbol{\theta}; p, t)\,;\ \boldsymbol{\theta} \in \boldsymbol{\Theta}(\boldsymbol{\nu}_{p,t})\right\}$ of probability distributions

$$\mathbf{P}(\boldsymbol{\theta}; p, t)(d\mathbf{x}) = \exp\left[\langle \boldsymbol{\theta}, \mathbf{x}\rangle - \mathbf{K}_{\boldsymbol{\nu}_{p,t}}(\boldsymbol{\theta})\right] \boldsymbol{\nu}_{p,t}(d\mathbf{x}).$$

Therefore, the variance functions of $\mathbf{G}_{p,t} = \mathbf{G}(\boldsymbol{\nu}_{p,t})$ generated by $\boldsymbol{\nu}_{p,t}$ is

$$\mathbf{V}_{\mathbf{G}_{p,t}}(\mathbf{m}) = t^{1-p} \cdot m_1^{p-2} \cdot \mathbf{m} \otimes \mathbf{m} + \mathbf{Diag}_k(0, m_1, \ldots, m_1), \quad \forall \mathbf{m} \in \mathbf{M}_{\mathbf{G}_{p,t}} \tag{2.7}$$

and the generalized variance functions

$$\det \mathbf{V}_{\mathbf{G}_{p,t}}(\mathbf{m}) = t^{1-p} \cdot m_1^{p+k-1}, \quad \forall \mathbf{m} \in \mathbf{M}_{\mathbf{G}_{p,t}} = (0, \infty) \times \mathbb{R}^{k-1}. \tag{2.8}$$

with $\mathbf{M}_{\mathbf{G}_{p,t}} = (0, \infty) \times \mathbb{R}^{k-1}$. Here is the covariance matrix structure of the trivariate normal Poisson, that is $k = 3$ and $p = 1$:

$$\mathbf{V}_{\mathbf{G}_1}(\mathbf{m}) = \begin{pmatrix} m_1 & m_2 & m_3 \\ m_2 & m_1 + m_1^{-1} m_2^2 & m_1^{-1} m_2 m_3 \\ m_3 & m_1^{-1} m_2 m_3 & m_1 + m_1^{-1} m_3^2 \end{pmatrix}.$$

Tables 2.3.1 summarizes the k-variate *NST* models with power variance parameter $p \geq 1$, power generalized variance parameter $q := p + k - 1$ and support of distribution $\mathbf{S}_p$.

Table 2.3.1 – Summary of k-variate *NST* models with power variance parameter $p \geq 1$, power variance generalized q, mean domain $\mathbf{M}_{\mathbf{G}_1} = (0, \infty) \times \mathbb{R}^k$ and the support of distribution $\mathbf{S}_p$.

Type(s)	p	$q = p + k - 1$	$\mathbf{S}_p$
Normal Poisson	$p = 1$	$q = k$	$\mathbb{N} \times \mathbb{R}^{k-1}$
Normal Poisson composé	$1 < p < 2$	$k < q < k + 1$	$[0, \infty) \times \mathbb{R}^{k-1}$
Normal noncentral gamma	$p = 3/2$	$q = k + 1/2$	$[0, \infty) \times \mathbb{R}^{k-1}$
Normal gamma	$p = 2$	$q = k + 1$	$(0, \infty) \times \mathbb{R}^{k-1}$
Normal positive stable	$p_2 > 2$	$q > k + 1$	$(0, \infty) \times \mathbb{R}^{k-1}$
Normal Inverse Gaussienne	$p = 3$	$q = k + 2$	$(0, \infty) \times \mathbb{R}^{k-1}$

2.4 Characterization of *NST* by variance functions

We now come to the main result. Namely, we want to prove that, given a variance function defined in (2.7) with fixed $p \geq 1$, the corresponding NEF is one of the *NST* models generated by $\boldsymbol{\nu}_{p,t}$ of (3.2). The result of characterization with $t = 1$ can be stated as below and its proof is given in the appendix of Section 2.6.

Theorem 2.4.1 *Let $k \in \{2, 3, \ldots\}$, $p \geq 1$ and $\mathbf{M}_p = (0, \infty) \times \mathbb{R}^{k-1}$. Then the unit variance function $\mathbf{V}_p : \mathbf{M}_p \to \mathcal{S}\left(\mathbb{R}^k\right)$ defined by*

$$\mathbf{V}_p(\mathbf{m}) = m_1^{p-2} \cdot \mathbf{m} \otimes \mathbf{m} + \mathbf{Diag}_k(0, m_1, \ldots, m_1), \tag{2.9}$$

for all $\mathbf{m} = (m_1, \ldots, m_k)^\top \in \mathbf{M}_p$, characterizes up to affinity the NST model generated by $\boldsymbol{\nu}_{p,t}$ of (3.2) with $t = 1$.

Since all *NST* models $\mathbf{G}_{p,t} = \mathbf{G}(\nu_{\alpha,t})$ are infinitely divisible (Sato, 1999), the following result highlights their modified Lévy measures $\rho(\nu_{\alpha,t})$ of the normal gamma type for $p > 1$ and degenerated for $p = 1$. See Kokonendji & Khoudar (2006) for some univariate situations $k = 1$.

Theorem 2.4.2 *(Boubacar Maïnassara & Kokonendji, 2014). Let $\nu_{\alpha(p),t}$ be a generating measure (3.2) of an NST family for given $p = p(\alpha) \geq 1$ and $t > 0$. Denote $\eta = 1 + k/(p-1) = \eta(p,k) > 1$ the modified Lévy measure parameter. Then*

$$\rho(\nu_{\alpha(p),t}) = \begin{cases} t^k (p-1)^{-\eta(p,k)} \cdot \nu_{0,\eta(p,k)} & \text{for} \quad p > 1 \\ t^k \cdot (\delta_{\mathbf{e}_1} \prod_{j=2}^{k} \mu_{2,1})^{*k} & \text{for} \quad p = 1. \end{cases}$$

Corollary 2.4.1 *(Boubacar Maïnassara & Kokonendji, 2014). Let $\mathbf{G}'_p = \mathbf{G}(\rho(\nu_{\alpha(p),t}))$ under assumptions of Theorem 2.4.2. Then*

$$\mathbf{V}_{\mathbf{G}'_p}(\overline{\mathbf{m}}) = \begin{cases} [\eta(p,k)]^{-1} \cdot \overline{\mathbf{m}} \otimes \overline{\mathbf{m}} + \langle \mathbf{e}_1, \overline{\mathbf{m}} \rangle \cdot \mathbf{Diag}_k(0, 1, \ldots, 1) & \text{for} \quad p > 1 \\ k^{-1} \cdot \mathbf{Diag}_k(0, 1, \ldots, 1) & \text{for} \quad p = 1, \end{cases}$$

with $\overline{\mathbf{m}} \in (0, \infty) \times \mathbb{R}^{k-1}$.

The following lemma is also required for the proof of Theorem 2.4.1.

Lemma 2.4.2 *Let $\mathbf{L}$ be a positive definite $k \times k$ matrix in $\mathcal{S}\left(\mathbb{R}^k\right)$ having the form*

$$\mathbf{L} = \begin{pmatrix} \lambda & \mathbf{a}^\top \\ \mathbf{a} & \mathbf{A} \end{pmatrix},$$

with $\lambda \in \mathbb{R} \setminus \{0\}$, $\mathbf{a} \in \mathbb{R}^{k-1}$ and $\mathbf{A}$ a $(k-1) \times (k-1)$ symmetric matrix. Then, the inverse matrix of $\mathbf{L}$ is given by

$$\mathbf{L}^{-1} = \begin{pmatrix} \lambda^{-1} + \lambda^{-2}\mathbf{a}^\top \mathbf{S}^{-1}\mathbf{a} & -\lambda^{-1}\mathbf{a}^\top \mathbf{S}^{-1} \\ -\lambda^{-1}\mathbf{a}\mathbf{S}^{-1} & \mathbf{S}^{-1} \end{pmatrix},$$

with $\mathbf{S} = \mathbf{A} - \lambda^{-1}\mathbf{a} \otimes \mathbf{a}$.

Proof. It is easy to check that $\mathbf{L}\mathbf{L}^{-1} = \mathbf{I}_k$. Which provides the result.□

2.5 Polynomials associated with *NST* models

Throughout this section, the *NST* laws are associated to the Lévy processes $(\mathbf{X}_t)_{t>0}$ because each distribution $\boldsymbol{\nu}_{p,t}$ defined in (3.2) is infinitely divisible for given $p \geq 1$; see, e.g., Sato (1999). As NEF (from μ_t) we associate to *NST* models (from $\boldsymbol{\nu}_{p,t}$) the polynomials in (2.1) denoted here by $\mathrm{P}_{\mathbf{A},\mathbf{n};p,t}$ for a given power variance parameter $p \geq 1$.

2.5.1 Some quasi-orthogonal polynomials and *NST* models

Let us recall here the definition of orthogonality and some of their modifications.

Définition 2.5.1 *A family* $(\mathrm{P}_{\mathbf{n}})_{\mathbf{n}\in\mathbb{N}^k}$ *of polynomials on* $\mathbb{R}^k$ *is said to be :*

(i) $\boldsymbol{\mu}$*-orthogonal (resp.* $\boldsymbol{\mu}$*-pseudo-orthogonal) if for all* $\mathbf{n}$ *and* $\mathbf{q}$ *in* $\mathbb{N}^k$*, one has* $\int_{\mathbb{R}^k} \mathrm{P}_{\mathbf{n}}(\mathbf{x})\mathrm{P}_{\mathbf{q}}(\mathbf{x})\boldsymbol{\mu}(d\mathbf{x}) = 0$ *when* $\mathbf{n} \neq \mathbf{q}$ *(resp.* $|\mathbf{n}| \neq |\mathbf{q}|$*).*

(ii) $\boldsymbol{\mu}$*-transorthogonal (resp.* $\boldsymbol{\mu}$*-2-pseudo-orthogonal) if for all* $\mathbf{n}$ *and* $\mathbf{q}$ *in* $\mathbb{N}^k$*, one has* $\int_{\mathbb{R}^k} \mathrm{P}_{\mathbf{n}}(\mathbf{x})\mathrm{P}_{\mathbf{q}}(\mathbf{x})\boldsymbol{\mu}(d\mathbf{x}) = 0$ *when* $\|\mathbf{n}-\mathbf{q}\|_+ \geq \inf(|\mathbf{n}|;|\mathbf{q}|)$ *(resp.* $|\mathbf{n}| \geq 2|\mathbf{q}|$*).*

(iii) d-orthogonal $(d \geq 1)$ *if and only if the d terms recurrence relations are written*

$$x_i\mathrm{P}_{\mathbf{n}}(\mathbf{x}) = \sum_{j=1}^{k}\sum_{0\leq|\mathbf{q}|\leq d} \alpha_{i,\mathbf{n}+e_j-\mathbf{q}}(\mathbf{n})\mathrm{P}_{\mathbf{n}+e_j-q}(\mathbf{x}),$$

for all $\mathbf{x} = (x_1,\ldots,x_k)^\top \in \mathbb{R}^k$.

The following proposition summarizes some known results on the associated polynomials of *NST* models.

Proposition 2.5.1 *Let* $\mathrm{P}_{\mathbf{A},\mathbf{n};p,t}(\mathbf{x})$ *be polynomials of (2.1) associated to NST models for* $p > 1$*,* $t > 0$ *and* $\mathbf{A} \in \mathbf{GL}(\mathbb{R}^k)$ *with generating measure* $\boldsymbol{\nu}_{p,t}$*. Then, we have the three following assertions :*

(i) $\mathrm{P}_{\mathbf{A},\mathbf{n};p,t}(\mathbf{x})$ *are* $\boldsymbol{\nu}_{p,t}$*-orthogonal or* $\boldsymbol{\nu}_{p,t}$*-pseudo-orthogonal if and only if* $p = 2$ *;*

(ii) $\mathrm{P}_{\mathbf{A},\mathbf{n};p,t}(\mathbf{x})$ *are* $\boldsymbol{\nu}_{p,t}$*-transorthogonal or* $\boldsymbol{\nu}_{p,t}$*-pseudo-orthogonal if and only if* $p = 3$ *;*

(iii) $\mathrm{P}_{\mathbf{A},\mathbf{n};p,t}(\mathbf{x})$ *are p-orthogonal if and only if* $p \in \{2,3,4,\ldots\}$*.*

Proof. *(i)* Since the corresponding *NST* model with $p = 2$ is the gamma-Gaussian type which belongs to simple quadratic NEFs, we therefore refer to Pommeret (1996).

(ii) For the normal inverse Gaussian with $p = 3$ and belonging to simple cubic NEFs, the detailed proofs are given in Hassairi & Zarai (2006) for the $\boldsymbol{\nu}_{p,t}$-pseudo-orthogonality and in Kokonendji & Zarai (2007) for the $\boldsymbol{\nu}_{p,t}$-transorthogonality.

(iii) See Kokonendji & Pommeret (2005). □

As for $p \in [1, \infty) \setminus \{2, 3, \ldots\}$ the characterizations by modifications of orthogonality property are not available. In particular, the normal Poisson polynomials with $p = 1$ need more investigations. Beside this, the Poisson-Gaussian polynomials are orthogonals (see Pommeret, 1996) because they are in the simple quadratic families of Casalis (1996). See Nisa et al. (2015) and Kokonendji & Nisa (2016) for short discussion and differences between normal Poisson and Poisson-Gaussian families.

2.5.2 The *NST* Lévy–Sheffer systems

The concept is defined as follows.

Définition 2.5.2 *A polynomial set* $\left\{Q_{\mathbf{n},t}(\mathbf{x})\,;\mathbf{n} \in \mathbb{N}^k, t \geq 0\right\}$ *is called a Lévy-Sheffer system (see also Schoutens & Teugels, 1998) if there exits a neighborhood of* $\mathbf{m} = 0$ *denoted by* $\mathbf{B}$ *such that*

$$\sum_{\mathbf{n}\in\mathbf{N}^k} \frac{\mathbf{m}^{\mathbf{n}}}{\mathbf{n}!} Q_{\mathbf{n},t}(\mathbf{x}) = \exp\left\{\langle a(\mathbf{m}), \mathbf{x}\rangle - tK_{\mu}\left(a(\mathbf{m})\right)\right\}, \quad \forall (t, \mathbf{m}) \in [0, \infty) \times \mathbf{B},$$

where $a : \mathbf{B} \to \mathbb{R}^k$ *is an analytic function with* $a(0) = 0$ *and* $\boldsymbol{\mu}$ *is an infinitely divisible distribution on* $\mathbb{R}^k$.

It is clear that a Lévy-Sheffer system is connected to a Lévy process $(\mathbf{X}_t)_{t>0}$ with associated distributions $(\boldsymbol{\mu}_t)_{t>0}$; see also Kokonendji (2005b) for further details. Let us recall also that if $(\mathbf{X}_t)_{t>0}$ is a Lévy process, then the basic link between the polynomials and the corresponding Lévy process is given by the following martingale equality for each $\mathbf{n} \in \mathbb{N}^k$ and for $0 < s \leq t$,

$$\mathbb{E}\left[Q_{\mathbf{n},t}(\mathbf{X}_t) \,|\, \mathbf{X}_s\right] = Q_{\mathbf{n},s}(\mathbf{X}_s). \tag{2.10}$$

In the framework of *NST* models, we are looking for the corresponding *NST* Lévy-Sheffer system and a particular case of characterization by pseudo-orthogonality.

Proposition 2.5.2 *Let* $\mathrm{P}_{t\mathbf{A},\mathbf{n};p,t}(\mathbf{x})$ *be polynomials of (2.1) associated to NST models for* $p \geq 1$, $t > 0$ *and* $\mathbf{A} \in \mathbf{GL}(\mathbb{R}^k)$ *with generating measure* $\boldsymbol{\nu}_{p,t}$ *and* $\boldsymbol{\nu}_p := \boldsymbol{\nu}_{p,1}$. *Then, we have the two following assertions :*

(i) $(\mathrm{P}_{t\mathbf{A},\mathbf{n};p,t})_{\mathbf{n}\in\mathbb{N}^k,t>0}$ *forms a Lévy-Sheffer systems and for all* $t \geq s$,

$$\mathbb{E}\left\{\exp\langle\psi_{\boldsymbol{\nu}_p}(\mathbf{A}\mathbf{m} + \mathbf{m}_1), \mathbf{X}_t - \mathbf{X}_s\rangle \,|\, \mathbf{X}_s\right\} = \exp\left\{(t - s)\mathbf{K}_{\boldsymbol{\nu}_p}\left(\psi_{\boldsymbol{\nu}_p}(\mathbf{A}\mathbf{m} + \mathbf{m}_1)\right)\right\}, \tag{2.11}$$

for all $\mathbf{m}$ *in a neighborhood of* $\mathbf{m}_t$ *;*

(ii) $(\mathrm{P}_{t\mathbf{A},\mathbf{n};p,t})_{\mathbf{n}\in\mathbb{N}^k,t\geq 0}$ *is pseudo-orthogonal if and only if* $p = 2$.

Proof. (i) By Taylor expansion, one successively has :

$$\begin{aligned} f_{\nu_p}(\mathbf{x}, t\mathbf{A}\mathbf{m}+\mathbf{m}_t) &= \sum_{\mathbf{n}\in\mathbb{N}^k} \frac{(t\mathbf{A}\mathbf{m}+\mathbf{m}_t-\mathbf{m}_t)^{\mathbf{n}}}{\mathbf{n}!} f^{(\mathbf{n})}_{\nu_p}(\mathbf{x},\mathbf{m}_t) \\ &= \sum_{\mathbf{n}\in\mathbb{N}^k} \frac{\mathbf{m}^{\mathbf{n}}}{\mathbf{n}!} f^{(\mathbf{n})}_{\nu_{p,t}}(\mathbf{x},\mathbf{m}_t)\,(t\mathbf{A}\mathbf{e}_1,\ldots,t\mathbf{A}\mathbf{e}_k) \\ &= \sum_{\mathbf{n}\in\mathbb{N}^k} \frac{\mathbf{m}^{\mathbf{n}}}{\mathbf{n}!} \mathrm{P}_{t\mathbf{A},\mathbf{n};p,t}(\mathbf{x}), \end{aligned}$$

and then,

$$\begin{aligned} \sum_{\mathbf{n}\in\mathbb{N}^k} \frac{\mathbf{m}^{\mathbf{n}}}{\mathbf{n}!} \mathrm{P}_{t\mathbf{A},\mathbf{n};p,t}(\mathbf{x}) &= \exp\left\{\langle \psi_{\nu_p}(t\mathbf{A}\mathbf{m}+\mathbf{m}_t),\mathbf{x}\rangle - \mathbf{K}_{\nu_p}\left(\psi_{\nu_p}(t\mathbf{A}\mathbf{m}+\mathbf{m}_t)\right)\right\} \\ &= \exp\left\{\langle \psi_{\nu_p}(\mathbf{A}\mathbf{m}+\mathbf{m}_1),\mathbf{x}\rangle - t\mathbf{K}_{\nu_p}\left(\psi_{\nu_p}(\mathbf{A}\mathbf{m}+\mathbf{m}_1)\right)\right\}, \end{aligned}$$

with $\mathbf{m}_t = t\mathbf{m}_1$. Setting $a(\mathbf{m}) = \psi_{\nu_p}(\mathbf{A}\mathbf{m}+\mathbf{m}_1)$ and since $\boldsymbol{\nu}_p$ is an infinitely divisible, from this we deduce the desired result.

To prove (2.11) we apply (2.10) to the $\mathrm{P}_{t\mathbf{A},\mathbf{n};p,t}(\mathbf{x})$ and we get

$$\mathbb{E}\left(\mathrm{P}_{t\mathbf{A},\mathbf{n};p,t}(\mathbf{X}_t)\,|\,\mathbf{X}_s\right) = \mathrm{P}_{s\mathbf{A},\mathbf{n};p,s}(\mathbf{X}_s). \tag{2.12}$$

We now combine (2.12) with the form of the generating function. On the left hand side we find

$$\begin{aligned} \sum_{\mathbf{n}\in\mathbb{N}^k} \mathbb{E}\{\mathrm{P}_{t\mathbf{A},\mathbf{n};p,t}(\mathbf{X}_t)\,|\,\mathbf{X}_s\}\frac{\mathbf{m}^{\mathbf{n}}}{\mathbf{n}!} &= \mathbb{E}\left\{\sum_{\mathbf{n}\in\mathbb{N}^k} \frac{\mathbf{m}^{\mathbf{n}}}{\mathbf{n}!}\{\mathrm{P}_{t\mathbf{A},\mathbf{n};p,t}(\mathbf{X}_t)\}\,|\,\mathbf{X}_s\right\} \\ &= \mathbb{E}\left\{\exp\left\{\langle \psi_{\nu_p}(\mathbf{A}\mathbf{m}+\mathbf{m}_1),\mathbf{X}_t\rangle - t\mathbf{K}_{\nu_p}\left(\psi_{\nu_p}(\mathbf{A}\mathbf{m}+\mathbf{m}_1)\right)\right\}\,|\,\mathbf{X}_s\right\} \\ &= \exp\left\{-t\mathbf{K}_{\nu_p}\left(\psi_{\nu_p}(\mathbf{A}\mathbf{m}+\mathbf{m}_1)\right)\right\}\mathbb{E}\left\{\exp\left\{\langle \psi_{\nu_p}(\mathbf{A}\mathbf{m}+\mathbf{m}_1),\mathbf{X}_t\rangle\right\}\,|\,\mathbf{X}_s\right\}. \end{aligned}$$

On the right, we have

$$\begin{aligned} \sum_{\mathbf{n}\in\mathbb{N}^k} \frac{\mathbf{m}^{\mathbf{n}}}{\mathbf{n}!} \mathrm{P}_{s\mathbf{A},\mathbf{n};p,s}(\mathbf{X}_s) &= f_{\nu_p}(\mathbf{X}_s, s\mathbf{A}\mathbf{m}+\mathbf{m}_s) \\ &= \exp\left\{\langle \psi_{\nu_p}(\mathbf{A}\mathbf{m}+\mathbf{m}_1),\mathbf{X}_s\rangle - s\mathbf{K}_{\nu_p}\left(\psi_{\nu_p}(\mathbf{A}\mathbf{m}+\mathbf{m}_1)\right)\right\} \\ &= \exp\left\{-s\mathbf{K}_{\nu_p}\left(\psi_{\nu}(\mathbf{A}\mathbf{m}+\mathbf{m}_1)\right)\right\}\exp\left\{\langle \psi_{\nu_p}(\mathbf{A}\mathbf{m}+\mathbf{m}_1),\mathbf{X}_s\rangle\right\}. \end{aligned}$$

Combination of both previous expressions leads to the relationship

$$\mathbb{E}\left\{\exp\left\{\langle \psi_{\nu_p}(\mathbf{A}\mathbf{m}+\mathbf{m}_1),\mathbf{X}_t-\mathbf{X}_s\rangle\right\}\,|\,\mathbf{X}_s\right\} = \exp\left\{t\mathbf{K}_{\nu_p}\left(\psi_{\nu_p}(\mathbf{A}\mathbf{m}+\mathbf{m}_1)\right) - s\mathbf{K}_{\nu_p}\left(\psi_{\nu_p}(\mathbf{A}\mathbf{m}+\mathbf{m}_1)\right)\right\},$$

which is equivalent to relation (2.11) ; hence, Part (i) is proven.

(ii) For $p = 2$ the corresponding *NST* model is gamma-Gaussian and belongs to quadratic NEF case. We therefore refer to Pommeret (2000). □

2.5.3 Recurrence relation of polynomials associated with *NST*

For simplicity we here omit the power variance parameter $p \geq 1$ related to any *NST* model and their corresponding polynomials and generating measure ; because the following result holds for all $p \geq 1$.

Theorem 2.5.1 *A sequence of polynomials* $(\mathrm{P}_{\mathbf{n}})_{\mathbf{n}\in\mathbb{N}^k}$ *is associated with the NST models if* $(\mathrm{P}_{\mathbf{n}})_{\mathbf{n}\in\mathbb{N}^k}$ *satisfies the following recurrence relationship*

$$\begin{aligned} x_1\mathrm{P}_{\mathbf{n}}(\mathbf{x}) &= n_1\mathrm{P}_{\mathbf{n}}(\mathbf{x}) + \|\mathbf{n}-(p-1)e_1\|_+ A_{\mathbf{n}}^{(p-1)e_1}\mathrm{P}_{\mathbf{n}-(p-1)e_1}(\mathbf{x}) + A_{\mathbf{n}}^{e_1}\mathrm{P}_{\mathbf{n}-e_1}(\mathbf{x}) \\ x_i\mathrm{P}_{\mathbf{n}}(\mathbf{x}) &= A_{\mathbf{n}}^{e_i}\mathrm{P}_{\mathbf{n}-e_i}(\mathbf{x}), \quad \forall i = 2, \dots, k, \end{aligned}$$

where $A_{\mathbf{n}}^{\mathbf{q}} = \mathbf{n}!/(\mathbf{n}-\mathbf{q})!$, *with convention* $A_{\mathbf{n}}^{\mathbf{q}} = 0$ *for* $\mathbf{n}-\mathbf{q} \notin \mathbb{N}^k$.

Proof. By Taylor expansion, for any $\mathbf{m} \in \mathrm{M} := (0,\infty)\times\mathbb{R}^{k-1}$ and for all $\mathbf{x} \in \mathbb{R}^k$ we have

$$f_\nu(\mathbf{x},\mathbf{m}) = \sum_{\mathbf{n}\in\mathbb{N}^k} \frac{\mathbf{m}^{\mathbf{n}}}{\mathbf{n}!}\mathrm{P}_{\mathbf{n}}(\mathbf{x}), \tag{2.13}$$

where $f_\nu(\cdot,\mathbf{m})$ is the associated probability density ; e.g. from Theorem 2.4.1. Then (2.13) can write

$$\exp\{\langle\psi_\nu(\mathbf{m}),\mathbf{x}\rangle - \mathbf{K}_\nu(\psi_\nu(\mathbf{m}))\} = \sum_{\mathbf{n}\in\mathbb{N}^k} \frac{\mathbf{m}^{\mathbf{n}}}{\mathbf{n}!}\mathrm{P}_{\mathbf{n}}(\mathbf{x}),$$

equivalent to

$$\exp\{\langle\psi_\nu(\mathbf{m}),\mathbf{x}\rangle\} = \sum_{\mathbf{n}\in\mathbb{N}^k} \frac{\mathbf{m}^{\mathbf{n}}}{\mathbf{n}!}\mathrm{P}_{\mathbf{n}}(\mathbf{x})\exp\{\mathbf{K}_\nu(\psi_\nu(\mathbf{m}))\},$$

that is

$$\exp\langle\boldsymbol{\theta},\mathbf{x}\rangle = \sum_{\mathbf{n}\in\mathbb{N}^k} \frac{(\mathbf{K}'_\nu(\boldsymbol{\theta}))^{\mathbf{n}}}{\mathbf{n}!}\mathrm{P}_{\mathbf{n}}(\mathbf{x})\exp\{\mathbf{K}_\nu(\boldsymbol{\theta})\}. \tag{2.14}$$

Differentiating with respect to θ_i one obtains :

$$x_i\exp\langle\boldsymbol{\theta},\mathbf{x}\rangle = \sum_{\mathbf{n}\in\mathbb{N}^k} \frac{\mathrm{P}_{\mathbf{n}}(\mathbf{x})}{\mathbf{n}!}\left\{\sum_{j=1}^{k} n_j\mathbf{m}^{\mathbf{n}-e_j}\frac{\partial^2\mathbf{K}_\nu(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j} + \mathbf{m}^{\mathbf{n}}\frac{\partial\mathbf{K}_\nu(\boldsymbol{\theta})}{\partial\theta_i}\right\}\exp\{\mathbf{K}_\nu(\boldsymbol{\theta})\}. \tag{2.15}$$

From (2.14) and (2.15) we successively get

$$
\begin{aligned}
\sum_{\mathbf{n}\in\mathbb{N}^k}\frac{\mathbf{m}^{\mathbf{n}}}{\mathbf{n}!}x_i\mathrm{P}_{\mathbf{n}}(\mathbf{x}) &= \sum_{\mathbf{n}\in\mathbb{N}^k}\frac{\mathrm{P}_{\mathbf{n}}(\mathbf{x})}{\mathbf{n}!}\left\{\sum_{j=1}^{k} n_j\mathbf{m}^{\mathbf{n}-e_j}\mathbf{V}_{ij}(\mathbf{m}) + \mathbf{m}^{\mathbf{n}}m_i\right\}\\
&= \sum_{\mathbf{n}\in\mathbb{N}^k}\frac{\mathrm{P}_{\mathbf{n}}(\mathbf{x})}{\mathbf{n}!}\sum_{j=1}^{i-1} n_j\mathbf{m}^{\mathbf{n}-e_j}m_1^{p-2}m_im_j\\
&\quad+\sum_{\mathbf{n}\in\mathbb{N}^k}\frac{\mathrm{P}_{\mathbf{n}}(\mathbf{x})}{\mathbf{n}!}\left\{\sum_{j=i+1}^{k} n_j\mathbf{m}^{\mathbf{n}-e_j}m_1^{p-2}m_im_j + n_i\mathbf{m}^{\mathbf{n}-e_i}\left(m_1^{p-2}m_i^2+m_1\right)+\mathbf{m}^{\mathbf{n}}m_i\right\}\\
&= \sum_{\mathbf{n}\in\mathbb{N}^k}\frac{\mathrm{P}_{\mathbf{n}}(\mathbf{x})}{\mathbf{n}!}\sum_{j=1}^{k} n_j\mathbf{m}^{\mathbf{n}-e_j}m_1^{p-2}m_im_j\\
&\quad+\sum_{\mathbf{n}\in\mathbb{N}^k}\frac{\mathrm{P}_{\mathbf{n}}(\mathbf{x})}{\mathbf{n}!}\left[-n_i\mathbf{m}^{\mathbf{n}-e_i}m_1^{p-2}m_i^2 + n_i\mathbf{m}^{\mathbf{n}-e_i}\left(m_1^{p-2}m_i^2+m_1\right)+\mathbf{m}^{\mathbf{n}}m_i\right]\\
&= \sum_{\mathbf{n}\in\mathbb{N}^k}\frac{\mathrm{P}_{\mathbf{n}}(\mathbf{x})}{\mathbf{n}!}\left\{\sum_{j=1}^{k} n_j\mathbf{m}^{\mathbf{n}-e_j}m_1^{p-2}m_im_j + n_im_1\mathbf{m}^{\mathbf{n}-e_i}+\mathbf{m}^{\mathbf{n}}m_i\right\}\\
&= \sum_{\mathbf{n}\in\mathbb{N}^k}\frac{\mathrm{P}_{\mathbf{n}}(\mathbf{x})}{\mathbf{n}!}\left\{|\mathbf{n}|\mathbf{m}^{\mathbf{n}+e_i+(p-2)e_1} + n_i\mathbf{m}^{\mathbf{n}-e_i+e_1}+\mathbf{m}^{\mathbf{n}+\mathbf{e_i}}\right\},
\end{aligned}
$$

and identifying the coefficients of $\mathbf{m}^{\mathbf{n}}$, it follows that :

$$
x_i\frac{\mathrm{P}_{\mathbf{n}}(\mathbf{x})}{\mathbf{n}!} = \frac{\|\mathbf{n}-e_i-(p-2)e_1\|_+}{(\mathbf{n}-e_i-(p-2)e_1)!}\mathrm{P}_{\mathbf{n}-e_i-(p-2)e_1}(\mathbf{x}) + \frac{n_i+1-\delta_{i1}}{(\mathbf{n}+e_i-e_1)!}\mathrm{P}_{\mathbf{n}+e_i-e_1}(\mathbf{x}) + \frac{1}{(\mathbf{n}-e_i)!}\mathrm{P}_{\mathbf{n}-e_i}(\mathbf{x}),
$$

equivalent to

$$
x_i\mathrm{P}_{\mathbf{n}}(\mathbf{x}) = \|\mathbf{n}-(e_i+(p-2)e_1)\|_+A_{\mathbf{n}}^{e_i+(p-2)e_1}\mathrm{P}_{\mathbf{n}-(e_i+(p-2))e_1}(\mathbf{x}) + (n_i+1-\delta_{i1})A_{\mathbf{n}}^{e_1-e_i}\mathrm{P}_{\mathbf{n}-(e_1-e_i)}(\mathbf{x}) + A_{\mathbf{n}}^{e_i}\mathrm{P}_{\mathbf{n}-e_i}(\mathbf{x}).
$$

Hence, for $i=1$ we get

$$
x_1\mathrm{P}_{\mathbf{n}}(\mathbf{x}) = n_1\mathrm{P}_{\mathbf{n}}(\mathbf{x}) + A_{\mathbf{n}}^{e_1}\mathrm{P}_{\mathbf{n}-e_1}(\mathbf{x}) + \|\mathbf{n}-(p-1)e_1\|_+A_{\mathbf{n}}^{(p-1)e_1}\mathrm{P}_{\mathbf{n}-(p-1)e_1}(\mathbf{x}),
$$

and for $i \in \{2,3,\dots,k\}$, $A_{\mathbf{n}}^{e_i+(p-2)e_1} = A_{\mathbf{n}}^{e_1-e_i} = 0$, then $x_i\mathrm{P}_{\mathbf{n}}(\mathbf{x}) = A_{\mathbf{n}}^{e_i}\mathrm{P}_{\mathbf{n}-e_i}(\mathbf{x})$. The proof of Theorem 2.5.1 is complete. □

The following tables summarize the classical polynomials associated with *NEF* on $\mathbb{R}^k$.

TABLE 2.5.1 – Orthogonal polynomials on $\mathbb{R}$ for quadratic NEF (Morris, 1982)

Type	$\mathbf{V_F}(m)$	$\mathbf{M_F}$	$Q_n(x)$	Orthog. polynomials
Gaussian	1	$\mathbb{R}$	$(\frac{1}{\sqrt{2}})\mathrm{H}_n(\sqrt{2}x)$	**Hermite**
Poisson	m	$(0,\infty)$	$(-1)^n\mathrm{C}_n^1(x)$	**Charlier**
Gamma	m^2	$(0,\infty)$	$(-1)^n\mathrm{L}_n^0(x+1)$	**Laguerre**
Binomial	$m(1-m)$	$(0,1)$	$n!(\frac{1}{4})^n\mathrm{K}_n^{\frac{1}{2},1}(x)$	**Krawchouk**
Negative-binomial	$m(1+m)$	$(0,\infty)$	$2^{-\frac{3n}{2}}\mathrm{M}_n^{\frac{1}{2},1}(x-1)$	**Meixner**
Hyperbolic	m^2+1	$\mathbb{R}$	$n!\mathrm{P}_n^1(x)$	**Pollaczek**

TABLE 2.5.2 – 2-Orthogonal polynomials on $\mathbb{R}$ for cubic *NEF* (Hassairi & Zarai, 2004)

Type $\mathbf{V}_F(m), m>0$	$Q_n(x)$ (Induction relations $n \geq 2$), $Q_0(x)=1$
Inverse Gaussian m^3	$Q_1(x)=(x-1), \quad Q_2(x)=x^2-6x+3$ $Q_{n+1}(x)=(x-3n-1)Q_n(x)-n(3n-2)Q_{n-1}(x)-A_n^3Q_{n-2}(x)$
Ressel-Kendall $m^2(m+1)$	$Q_1(x)=\frac{1}{2}(x-1), \quad Q_2(x)=\frac{1}{4}(x^2-7x+4)$ $Q_{n+1}(x)=\frac{1}{2}[(x-5n-1)Q_n(x)-n(4n-3)Q_{n-1}(x)-A_n^3Q_{n-2}(x)]$
Abel (GP) $m(m+1)^2$	$Q_1(x)=\frac{1}{4}(x-1), \quad Q_2(x)=\frac{1}{16}(x^2-15x+8)$ $Q_{n+1}(x)=\frac{1}{4}[(x-8n-1)Q_n(x)-n(5n-4)Q_{n-1}(x)-A_n^3Q_{n-2}(x)]$
Takács (GNB) $m(m+1)(2m+1)$	$Q_1(x)=\frac{1}{6}(x-1), \quad Q_2(x)=\frac{1}{36}(x^2-15x+8)$ $Q_{n+1}(x)=\frac{1}{6}[(x-13n-1)Q_n(x)-n(9n-8)Q_{n-1}(x)-2A_n^3Q_{n-2}(x)]$
Strict arcsine $m(m^2+1)$	$Q_1(x)=\frac{1}{2}(x-1), \quad Q_2(x)=\frac{1}{4}(x^2-5x+3)$ $Q_{n+1}(x)=\frac{1}{2}[(x-4n-1)Q_n(x)-n(3n-2)Q_{n-1}(x)-A_n^3Q_{n-2}(x)]$
Large arcsine (GSA) $m(2m^2+2m+1)$	$Q_1(x)=\frac{1}{9}(x-1), \quad Q_2(x)=\frac{1}{81}(x^2-13x+3)$ $Q_{n+1}(x)=\frac{1}{9}[(x-11n-1)Q_n(x)-n(8n-7)Q_{n-1}(x)-2A_n^3Q_{n-2}(x)]$

TABLE 2.5.3 – The twelve G_0-orbits of the real cubic NEF distributed in four G-orbits (Hassairi, 1992)

G-orbit	Quadratic (Morris, 1982)		Cubic (Letac & Mora, 1990)	
1st	Gaussian		Inverse Gaussian	
	1		m^3	
2nd	Poisson	Gamma	Abel	Ressel-Kendall
	m	m^2	$m(m+1)^2$	$m^2(m+1)$
3rd	Binomial	Negative binomial	Takács	
	$m(1-m)$	$m(m+1)$	$m(m+1)(2m+1)$	
4th	Hyperbolic		Large arcsine	Strict arcsine
	m^2+1		$m(2m^2+2m+1)$	$m(m^2+1)$

TABLE 2.5.4 – (d)-(Pseudo) (Orthogonal) polynomials and univariate stable Tweedie model

Types	$V_F(m) = m^p$	$Q_n(x)$
Extreme stable	$p < 0$	?
Normal	$p = 0$	Orthogonal (Hermite)
[Do not exist]	$0 < p < 1$	
Poisson	$p = 1$	Orthogonal (Charlier)
Compound Poisson	$1 < p < 2$	?
Gamma	$p = 2$	Orthogonal (Laguerrre)
Positive stable	$p > 2$	?
Inverse Gaussian	$p = 3$	2-Orthogonal (Hassairi & Zarai, 2004 ; Kokonendji, 2005b)
Positive stable	$p = 2d - 1$, $d \in \{2, 3, \cdots\}$	d-Pseudo-orthogonal (Kokonendji, 2005a) d-Orthogonal/Sheffer syst. (kokonendji, 2005b)

TABLE 2.5.5 – Orthogonal polynomials on $\mathbb{R}^k$ of the Casalis class by Pommeret (1996) with $\mathbf{V}_F(\mathbf{m}) = a\mathbf{m} \otimes \mathbf{m} + \mathbf{B}(\mathbf{m}) + \mathbf{C}$

Types	a	$\mathbf{B}(\mathbf{m}) + \mathbf{C}$	$Q_{\mathbf{n}}(\mathbf{x})$ (Orthogonal)
Poisson-G $_{j=0,1,\cdots,k}$	0	$\mathbf{Diag}_k(m_1, \cdots, m_j, 1, \cdots, 1)$	Charlier & Hermite
Gaussian (PG_0)	0	Σ	Hermite
Poisson (PG_k)	0	$\mathbf{Diag}_k(m_1, \cdots, m_k)$	Charlier
NM-gamma-G $_{j=0,1,\cdots,k}$	1	$\mathbf{Diag}_k(m_1, \cdots, m_j, 0, m_{j+1}, \cdots, m_{j+1})$	Laguerre & Charlier
Gamma-Gaussian ($NMgG_0$)	1	$\mathbf{Diag}_k(0, m_1, \cdots, m_1)$	Hermite & Laguerre
Neg.Multinomial ($NMgG_k$)	1	$\mathbf{Diag}_k(m_1, \cdots, m_k)$	Meixner
Multinomial	-1	$\mathbf{Diag}_k(m_1, \cdots, m_k)$	Laguerre & Meixner
Hyperbolic	1	$\mathbf{Diag}_k(m_1, \cdots, m_{k-1}, \sum_{i=1}^{k-1} m_i + 1)$	Meixner & Pollaczek

TABLE 2.5.6 – (d)-(Pseudo) (Orthogonality) polynomials and multivariate *NST* model with $\mathbf{V}_F(\mathbf{m}) = m_1^{p-2} \cdot \mathbf{m} \otimes \mathbf{m} + \mathbf{Diag}_k(0, m_1, \cdots, m_1)$

Types	$p = p(\alpha)$	$Q_{\mathbf{n}}(\mathbf{x})$
Normal Poisson	$p = 1$	?
Normal Compound Poisson	$1 < p < 2$	?
Normal Gamma	$p = 2$	Orthogonal (Pommeret, 1996) Pseudo-Orthogonal (Pommeret, 2000)
Normal Positive stable	$p > 2$	?
Normal Inverse Gaussian	$p = 3$	Transorthogonal / 2-Pseudo-Orthogonal (Hassairi & Zarai, 2006 ; Kokonendji & Zarai, 2007)
Normal Positive stable	$p \in \{3, 4, \cdots\}$	p-Orthogonal (Kokonendji & Pommeret, 2005) & ??

2.6 Proof of Theorem 2.4.1

We mainly show that, given the variance function (3.7) with fixed $p \geq 1$ then, up to affinity, the corresponding cumulant function is that introduced in (2.3) with $t = 1$. For problems of existence, we first proving the limits cases $(p = 1; p = 2)$ then the cas $p \in (1, 2) \cup (2, \infty)$.

Let $\mathbf{F}_p = \mathbf{F}(\boldsymbol{\nu}_p)$ be a NEF satisfies (3.7), $p \geq$ and for $\mathbf{m} = (m_1, \ldots, m_k)^\top \in \mathbf{M}_{\mathbf{F}_p} = (0, \infty) \times \mathbb{R}^{k-1} =: \mathbf{M}$ (does not depending on $p > 1$). To apply Lemma 2.4.2 with $\mathbf{L} = [\mathbf{V}_p(\mathbf{m})]$ of (3.7), one sets : $\lambda = m_1^p$, $\mathbf{a} = m_1^{p-1}(m_2, ..., m_k)^\top$ and $\mathbf{A} = m_1^{p-2}(m_2, ..., m_k) \otimes (m_2, ..., m_k) + m_1\mathbf{I}_{k-1} = \lambda^{-1}\mathbf{a} \otimes \mathbf{a} + m_1\mathbf{I}_{k-1}$. Then $\mathbf{S} = \mathbf{A} - \lambda^{-1}\mathbf{a} \otimes \mathbf{a} = m_1\mathbf{I}_{k-1}$ and, since $m_1 > 0$, we successively have : $\mathbf{S}^{-1} = (1/m_1)\mathbf{I}_{k-1}$, $\lambda^{-1} + \lambda^{-2}\mathbf{a}^\top\mathbf{S}^{-1}\mathbf{a} = (1/m_1^p) + (1/m_1^3)\sum_{j=2}^k m_j^2$, $-\lambda^{-1}\mathbf{a}^\top\mathbf{S}^{-1} = -(1/m_1^2)\mathbf{a}^\top$ and $-\lambda^{-1}\mathbf{a}\mathbf{S}^{-1} = -(1/m_1^2)\mathbf{a}$. Thus, the inverse of variance-covariance matrix of (3.7) can be written as

$$\left[\mathbf{V}_p(\mathbf{m})\right]^{-1} = \begin{pmatrix} \dfrac{1}{m_1^p} + \dfrac{1}{m_1^3}\displaystyle\sum_{j=2}^k m_j^2 & \dfrac{-1}{m_1^2}\mathbf{a}^\top \\ \dfrac{-1}{m_1^2}\mathbf{a} & \dfrac{1}{m_1}\mathbf{I}_{k-1} \end{pmatrix}. \tag{2.16}$$

Since $\mathbf{m} = \mathbf{K}'_{\nu_p}(\boldsymbol{\theta})$ and $\mathbf{V}_p(\mathbf{m}) = \mathbf{K}''_{\nu_p}(\boldsymbol{\theta})$, then by writing $\boldsymbol{\theta}$ in terms of $\mathbf{m}$ one obtains

$$\mathbf{V}_p(\mathbf{m}) = [\boldsymbol{\theta}'(\mathbf{m})]^{-1}$$

which implies

$$\boldsymbol{\theta}(\mathbf{m}) = \int [\mathbf{V}_p(\mathbf{m})]^{-1} d\mathbf{m}.$$

For $\boldsymbol{\theta} \in \boldsymbol{\Theta} := \boldsymbol{\theta}(\mathbf{M})$ with $\mathbf{M} = (0, \infty) \times \mathbb{R}^{k-1}$, there exist an analytic function $\varphi : \mathbb{R}^k \to \mathbb{R}$ such that, for all $\mathbf{m} \in \mathbf{M}$

$$\boldsymbol{\theta}'(\mathbf{m}) = \left(\frac{\partial^2\varphi(\mathbf{m})}{\partial m_i \partial m_j}\right)_{i,j=1,2,\ldots,k}. \tag{2.17}$$

Case $p = 1$ **:** For $p = 1$, the relation (2.16) can write, for all $\mathbf{m} = (m_1, \ldots, m_k)^T \in \mathbf{M} := (0, \infty) \times \mathbb{R}^{k-1}$

$$[\mathbf{V}_1(\mathbf{m})]^{-1} = \begin{pmatrix} \dfrac{1}{m_1} + \dfrac{1}{m_1^3}\displaystyle\sum_{j=2}^k m_j^2 & -\dfrac{1}{m_1^2}\mathbf{a}^T \\ -\dfrac{1}{m_1^2}\mathbf{a} & \dfrac{1}{m_1}I_{k-1} \end{pmatrix}. \tag{2.18}$$

Using (2.17) and (2.18) for getting the first information on stable-Tweedie component, we have

$$\frac{\partial^2\varphi}{\partial m_1^2} = \frac{1}{m_1} + \frac{1}{m_1^3}\sum_{j=2}^k m_j^2. \tag{2.19}$$

Integrating (2.19) with respect to m_1 where $m_1 > 0$, we have

$$\frac{\partial \varphi}{\partial m_1} = \log m_1 - \frac{1}{2m_1^2} \sum_{j=2}^{k} m_j^2 + g(m_2, \ldots, m_k), \tag{2.20}$$

where $g : \mathbb{R}^{k-1} \longrightarrow \mathbb{R}$ is an analytic function to be determined. Derivative of (2.20) with respect to m_j, for $j = 2, \ldots, k$ gives

$$\frac{\partial^2 \varphi}{\partial m_j \partial m_1} = -\frac{m_j}{m_1^2} + \frac{\partial g}{\partial m_j}. \tag{2.21}$$

Expression (2.21) is equal to the $(1, j)$th element of $[\mathbf{V}_1(\mathbf{m})]^{-1}$ in (2.18), so by identification we get

$$-\frac{m_j}{m_1^2} = -\frac{m_j}{m_1^2} + \frac{\partial g}{\partial m_j},$$

and we deduce that $\partial g / \partial m_j = 0$ implies $g(m_2, \ldots, m_k) = \alpha_1$ (a real constant). And thus, (2.20) becomes

$$\frac{\partial \varphi}{\partial m_1} = \log m_1 - \frac{1}{2m_1^2} \sum_{j=2}^{k} m_j^2 + \alpha_1. \tag{2.22}$$

Integrating (2.22) with respect to m_1 we have

$$\varphi(\mathbf{m}) = m_1 \log m_1 - m_1 + \frac{1}{2m_1} \sum_{j=2}^{k} m_j^2 + \alpha_1 m_1 + h(m_2, \ldots, m_k), \tag{2.23}$$

where $h : \mathbb{R}^{k-1} \longrightarrow \mathbb{R}$ an analytic function to be determined. From now on, complete informations of the model (i.e. normal components) begin to show itself. The two first derivatives of (2.23) with respect to m_j give, respectively,

$$\frac{\partial \varphi}{\partial m_j} = \frac{m_j}{m_1} + \frac{\partial h}{\partial m_j}, \quad \forall j = 2, \ldots, k \tag{2.24}$$

and,

$$\frac{\partial^2 \varphi}{\partial m_j^2} = \frac{1}{m_1} + \frac{\partial^2 h}{\partial m_j^2}, \quad \forall j = 2, \ldots, k. \tag{2.25}$$

Expression (2.25) is equal to the diagonal (j, j) of $[\mathbf{V}_1(\mathbf{m})]^{-1}$ in (2.18) for all $j \in \{2, \ldots k\}$, hence we have

$$\frac{1}{m_1} = \frac{1}{m_1} + \frac{\partial^2 h}{\partial m_j^2}.$$

Consequently $\partial^2 h / \partial m_j^2 = 0$ and $\partial h / \partial m_j = \alpha_j$ (a real constant) for all $j \in \{2, \ldots, k\}$. Then, equation (2.24) becomes

$$\frac{\partial \varphi}{\partial m_j} = \frac{m_j}{m_1} + \alpha_j, \quad j = 2, \ldots, k. \tag{2.26}$$

Using equations (2.22) and (2.26) we get

$$\boldsymbol{\theta}(\mathbf{m}) = \left(\log m_1 - \frac{1}{2m_1^2}\sum_{j=2}^{k} m_j^2, \frac{m_2}{m_1}, \dots, \frac{m_k}{m_1}\right)^T + (\alpha_1, \alpha_2, \dots, \alpha_k)^T,$$

or

$$\boldsymbol{\theta}(\mathbf{m}) = \begin{cases} \theta_1(\mathbf{m}) &= \log m_1 - \frac{1}{2}\sum_{j=2}^{k} \frac{m_j^2}{m_1^2} + \alpha_1 \\ \theta_j(\mathbf{m}) &= \frac{m_j}{m_1} + \alpha_j, \qquad\qquad j = 2, \dots, k. \end{cases} \tag{2.27}$$

From (2.27), each θ_j belongs to $\mathbb{R}$ for $j \in \{1, 2, \dots, k\}$ because $m_1 > 0$ and $\theta_j \in \mathbb{R}$ for $j \in \{2, \dots, k\}$. Thus, one has $\boldsymbol{\theta}(\mathbf{M}) \subseteq \mathbb{R}^k$ and also

$$m_1(\boldsymbol{\theta}) = \exp\left[(\theta_1 - \alpha_1) + \frac{1}{2}\sum_{j=2}^{k}(\theta_j - \alpha_j)^2\right] \tag{2.28}$$

$$m_j(\boldsymbol{\theta}) = (\theta_j - \alpha_j)\exp\left[(\theta_1 - \alpha_1) + \frac{1}{2}\sum_{j=2}^{k}(\theta_j - \alpha_j)^2\right], j = 2, \dots, k. \tag{2.29}$$

Since $\mathbf{M} = (0,1) \times \mathbb{R}^{k-1}$, we deduce that $\boldsymbol{\Theta}(\boldsymbol{\nu}) = \mathbb{R}^k$. Also $\mathbf{m} = \dfrac{\partial \mathbf{K}_\nu(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$, then using (2.28) as follow :

$$\begin{aligned} \mathbf{K}_\nu(\boldsymbol{\theta}) &= \int \frac{\mathbf{K}_\nu(\boldsymbol{\theta})}{\partial \theta_1} d\theta_1 \\ &= \exp\left[(\theta_1 - \alpha_1) + \frac{1}{2}\sum_{j=2}^{k}(\theta_j - \alpha_j)^2\right] + f(\theta_2, \dots, \theta_k), \end{aligned} \tag{2.30}$$

where $f : \mathbb{R}^{k-1} \longrightarrow \mathbb{R}$ is an analytic function to be determined. Taking derivative of (2.30) with respect to θ_j for each $j = 2, \dots, k$, we obtain

$$\frac{\partial \mathbf{K}_\nu(\boldsymbol{\theta})}{\partial \theta_j} = (\theta_j - \alpha_j)\exp\left[(\theta_1 - \alpha_1) + \frac{1}{2}\sum_{j=2}^{k}(\theta_j - \alpha_j)^2\right] + \frac{\partial f}{\partial \theta_j},$$

which is equal to (2.29) ; then one gets $\partial f / \partial \theta_j = 0$ for all $j \in \{2, \dots, k\}$ implying $f(\theta_2, \dots, \theta_k) = b$ (a real constant). Finally, for all $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k)^T \in \boldsymbol{\Theta}(\boldsymbol{\nu})$, (2.30) becomes

$$\mathbf{K}_\nu(\boldsymbol{\theta}) = \exp\left[(\theta_1 - \alpha_1) + \frac{1}{2}\sum_{j=2}^{k}(\theta_j - \alpha_j)^2\right] + b.$$

By Proposition 1.4.1 one can see that, up to affinity, this $\mathbf{K}_{\nu_p}$ is a *NST* cumulant function as given in (2.3) with $t = 1$ on its corresponding canonical domain in (3.4) for $p = 1$. The proof of Theorem 2.4.1 is therefore completed by use of the analytical property of $\mathbf{K}_{\nu_p}$. □

Case $p = 2$: Casalis (1996) dealt this case in the context of $2d + 4$ quadratic NEFs families and corresponds to gamma-Gaussian family. For $p = 2$, the inverse matrix $\left[\mathbf{V}_p(\mathbf{m})\right]^{-1}$ in (2.16) can write

$$[\mathbf{V}_2\,(\mathbf{m})]^{-1} = \begin{pmatrix} \dfrac{1}{m_1^2} + \dfrac{1}{m_1^3}\displaystyle\sum_{j=2}^{k} m_j^2 & -\dfrac{1}{m_1^2}\mathbf{a}^T \\ -\dfrac{1}{m_1^2}\mathbf{a} & \dfrac{1}{m_1} I_{k-1} \end{pmatrix}, \tag{2.31}$$

for all $\mathbf{m} = (m_1, \ldots, m_k)^T$ in $\mathbf{M} := (0, \infty) \times \mathbb{R}^{k-1}$. Using (2.17) and (2.31) for getting the first information on stable-Tweedie component, we have

$$\frac{\partial^2 \varphi}{\partial m_1^2} = \frac{1}{m_1^2} + \frac{1}{m_1^3}\sum_{j=2}^{k} m_j^2,$$

and then

$$\frac{\partial \varphi}{\partial m_1} = -\frac{1}{m_1} - \frac{1}{2m_1^2}\sum_{j=2}^{k} m_j^2 + g(m_2, \ldots, m_k), \tag{2.32}$$

where $g : \mathbb{R}^{k-1} \longrightarrow \mathbb{R}$ is an analytic function to be determined. Derivative of (2.32) with respect to θ_j gives

$$\frac{\partial^2 \varphi}{\partial m_j \partial m_1} = -\frac{m_j}{m_1^2} + \frac{\partial g}{\partial m_j}, j = 2, \ldots, k. \tag{2.33}$$

Expression (2.33) is equal to $(1, j)$th element of $[\mathbf{V}_2(\mathbf{m})]^{-1}$ in (2.31), that is

$$-\frac{m_j}{m_1^2} + \frac{\partial g}{\partial m_j} = -\frac{m_j}{m_1^2},$$

therefore $\dfrac{\partial g}{\partial m_j} = 0$ for each $j = 2, \ldots, k$ indeed $g(m_2, \ldots, m_k) = \alpha_1$ (a real constant). The relation (2.32) becomes,

$$\frac{\partial \varphi}{\partial m_1} = -\frac{1}{m_1} - \frac{1}{2m_1^2}\sum_{j=2}^{k} m_j^2 + \alpha_1, \tag{2.34}$$

and its primitives can be written as

$$\varphi(\mathbf{m}) = -\log m_1 + \frac{1}{2m_1}\sum_{j=2}^{k} m_j^2 + \alpha_1 m_1 + h(m_2, \ldots, m_k), \tag{2.35}$$

where $h : \mathbb{R}^{k-1} \longrightarrow \mathbb{R}$ is an analytic function to be determined. From now on, complete information of the model (i.e. normal components) begin to show itself. The two first derivatives of (2.35) with respect to m_j give, respectively,

$$\frac{\partial \varphi}{\partial m_j} = \frac{m_j}{m_1} + \frac{\partial h}{\partial m_j}, \qquad \forall j \in \{2, \ldots, k\} \tag{2.36}$$

and

$$\frac{\partial^2 \varphi}{\partial m_j^2} = \frac{1}{m_1} + \frac{\partial^2 h}{\partial m_j^2}, \qquad \forall j \in \{2, \ldots, k\}. \tag{2.37}$$

Expression (2.37) is equal to diagonal (j, j)th element of $[\mathbf{V}_2(\mathbf{m})]^{-1}$ in (2.31) for $j \in \{2, \ldots k\}$, so we deduce

$$\frac{1}{m_1} = \frac{1}{m_1} + \frac{\partial^2 h}{\partial m_j^2}.$$

Consequently $\partial^2 h/\partial m_j^2 = 0$ and $\partial h/\partial m_j = \alpha_j$ (a real constant) for all $j = 2, \ldots, k$. Then, equation (2.36) becomes

$$\frac{\partial \varphi}{\partial m_j} = \frac{m_j}{m_1} + \alpha_j, \quad \forall j \in \{2, \ldots, k\}, \tag{2.38}$$

using equations (2.34) and (2.38) one obtains

$$\boldsymbol{\theta}(\mathbf{m}) = \begin{cases} \theta_1(\mathbf{m}) & = & -\dfrac{1}{m_1} - \dfrac{1}{2m_1^2} \sum_{j=2}^k m_j^2 + \alpha_1 \\ \theta_j(\mathbf{m}) & = & \dfrac{m_j}{m_1} + \alpha_j, \quad j = 2, \ldots, k. \end{cases} \tag{2.39}$$

From (2.39), each θ_j belongs to $\mathbb{R}$ because $m_1 > 0$ and $m_j \in \mathbb{R}$ for $j \in \{2, \ldots, k\}$. And also

$$m_1 = \frac{-1}{(\theta_1 - \alpha_1) + \dfrac{1}{2} \sum_{j=2}^k (\theta_j - \alpha_j)^2}, \tag{2.40}$$

$$m_j = \frac{-(\theta_j - \alpha_j)}{(\theta_1 - \alpha_1) + \dfrac{1}{2} \sum_{j=2}^k (\theta_j - \alpha_j)^2}, \; j = 2, \ldots, k, \tag{2.41}$$

with $\theta_j(\mathbf{m}) - \alpha_j \neq 0$ for all $j \in \{1 \ldots, k\}$. Since $m_1 > 0$, then in (2.40), $(\theta_1 - \alpha_1) + \frac{1}{2} \sum_{j=2}^k (\theta_j - \alpha_j)^2 < 0$ and thus $\boldsymbol{\Theta}(\boldsymbol{\nu}) = \left\{\boldsymbol{\theta} \in \mathbb{R}^k ; (\theta_1 - \alpha_1) + \frac{1}{2} \sum_{j=2}^k (\theta_j - \alpha_j)^2 < 0\right\}$. Also, $\mathbf{m} = \dfrac{\partial \mathbf{K}_\nu(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$, then using (2.40) one can obtain $\mathbf{K}_\nu(\boldsymbol{\theta})$ as follow :

$$\begin{aligned} \mathbf{K}_\nu(\boldsymbol{\theta}) &= \int \frac{\mathbf{K}_\nu(\boldsymbol{\theta})}{\partial \theta_1} d\theta_1 \\ &= -\log\Big[-(\theta_1 - \alpha_1) - \frac{1}{2} \sum_{j=2}^k (\theta_j - \alpha_j)^2 \Big] + f(\theta_2, \ldots, \theta_k), \end{aligned} \tag{2.42}$$

where $f : \mathbb{R}^{k-1} \longrightarrow \mathbb{R}$ is an analytic function to be determined. Again, derivative of (2.42) with respect to θ_j produces

$$\frac{\mathbf{K}_{\nu}(\boldsymbol{\theta})}{\partial \theta_j} = \frac{-(\theta_1 - \alpha_1)}{(\theta_1 - \alpha_1) + \frac{1}{2}\sum_{j=2}^{k}(\theta_j - \alpha_j)^2} + \frac{\partial f}{\partial \theta_j}, \tag{2.43}$$

which is equal to (2.41) ; then one gets, $\partial f / \partial \theta_j = 0$ for all $j \in \{2, \ldots, k\}$ implying $f(\theta_2, \ldots, \theta_k) = c$ (a real constant). Finally, it ensues from it that we have

$$\mathbf{K}_{\nu}(\boldsymbol{\theta}) = -\log\Big[-(\theta_1 - \alpha_1) - \frac{1}{2}\sum_{j=2}^{k}(\theta_j - \alpha_j)^2\Big] + c, \quad \forall \boldsymbol{\theta} \in \boldsymbol{\Theta}(\boldsymbol{\nu}).$$

By Proposition 1.4.1 one can see that, up to affinity, this $\mathbf{K}_{\nu_p}$ is a *NST* cumulant function as given in (2.3) with $t = 1$ on its corresponding canonical domain in (3.4) for $p = 2$. □

Case $p \in (1, 2) \cup (2, \infty)$: Using (2.17) into (2.16) for getting the first informations on the stable Tweedie component, we have

$$\frac{\partial^2 \varphi(\mathbf{m})}{\partial m_1^2} = \frac{1}{m_1^p} + \frac{1}{m_1^3}\sum_{j=2}^{k} m_j^2,$$

and then

$$\frac{\partial \varphi(\mathbf{m})}{\partial m_1} = \frac{-1}{(p-1)m_1^{p-1}} - \frac{1}{2m_1^2}\sum_{j=2}^{k} m_j^2 + g(m_2, \ldots, m_k), \tag{2.44}$$

where $g : \mathbb{R}^{k-1} \to \mathbb{R}$ is a analytic function to be determined. Note that since $m_1 > 0$ and $p \in (1, 2) \cup (2, \infty)$ then (2.44) is well defined. Derivative of (2.44) with respect to m_j gives

$$\frac{\partial^2 \varphi(\mathbf{m})}{\partial m_j \partial m_1} = -\frac{m_j}{m_1^2} + \frac{\partial g(m_2, \ldots, m_k)}{\partial m_j}, \quad \forall j \in \{2, \ldots, k\}. \tag{2.45}$$

Expression (2.45) is equal to the $(1, j)$th element of $[\mathbf{V}_p(\mathbf{m})]^{-1}$ in (2.16), that is

$$\frac{-m_j}{m_1^2} + \frac{\partial g(m_2, \ldots, m_k)}{\partial m_j} = \frac{-m_j}{m_1^2}, \quad \forall j \in \{2, \ldots, k\};$$

therefore, $\partial g(m_2, \ldots, m_k)/\partial m_j = 0$, for all $j \in \{2, \ldots, k\}$ implies $g(m_2, \ldots, m_k) = a_1$ (a real constant). Thus (2.44) becomes

$$\frac{\partial \varphi(\mathbf{m})}{\partial m_1} = \frac{-1}{(p-1)m_1^{p-1}} - \frac{1}{2m_1^2}\sum_{j=2}^{k} m_j^2 + a_1, \tag{2.46}$$

and, by integration with respect to m_1, one gets

$$\varphi(\mathbf{m}) = \frac{-1}{(p-1)(p-2)m_1^{p-2}} - \frac{1}{2m_1}\sum_{j=2}^{k} m_j^2 + a_1 m_1 + h(m_2,\ldots,m_k), \qquad (2.47)$$

where $h : \mathbb{R}^{k-1} \to \mathbb{R}$ is an analytic function to be determined. From now on, complete informations of the model (i.e. normal components) begin to show itself. The two first derivatives of (2.47) with respect to m_j give, respectively,

$$\frac{\partial\varphi(\mathbf{m})}{\partial m_j} = \frac{m_j}{m_1} + \frac{\partial h(m_2,\ldots,m_k)}{\partial m_j}, \quad \forall j \in \{2,\ldots,k\} \qquad (2.48)$$

and

$$\frac{\partial\varphi^2(\mathbf{m})}{\partial m_j^2} = \frac{1}{m_1} + \frac{\partial^2 h(m_2,\ldots,m_k)}{\partial m_j^2}, \quad \forall j \in \{2,\ldots,k\}. \qquad (2.49)$$

Expression (2.49) is equal to the diagonal (j,j)th element of $[\mathbf{V}_p(\mathbf{m})]^{-1}$ in (2.16) for all $j \in \{2,\ldots,k\}$, hence we have

$$\frac{1}{m_1} + \frac{\partial^2 h(m_2,\ldots,m_k)}{\partial m_j^2} = \frac{1}{m_1}.$$

Consequently, $\partial^2 h(m_2,\ldots,m_k)/\partial m_j^2 = 0$ and $\partial h(m_2,\ldots,m_k)/\partial m_j = a_j$ (a real constant) for all $j \in \{2,\ldots,k\}$. Then equation (2.48) becomes

$$\frac{\partial\varphi(\mathbf{m})}{\partial m_j} = \frac{m_j}{m_1} + a_j, \quad \forall j = 2,\ldots,k. \qquad (2.50)$$

Using equations (2.46) and (2.50) one obtains

$$\boldsymbol{\theta}(\mathbf{m}) = \left\{ \begin{array}{lcll} \theta_1 & = & \dfrac{-1}{(p-1)m_1^{p-1}} - \dfrac{1}{2m_1^2}\sum_{j=2}^{k} m_j^2 + a_1 & \\ \theta_j & = & \dfrac{m_j}{m_1} + a_j, & j = 2,\ldots,k. \end{array} \right. \qquad (2.51)$$

From (2.51) and since $m_1 > 0$ and $m_j \in \mathbb{R}$ for all $p \in (1,2) \cup (2,\infty)$, one has : $(\theta_1 - a_1) < 0$ which can be extended to 0 when $p > 2$ (see also (2.54) below) and $(\theta_j - a_j) \in \mathbb{R}$ for $j \in \{2,\ldots,k\}$. Hence, up to affinity (see Proposition 1.4.1) the set Θ introduced for (2.17) is similar to the one of (3.4). Also, the system (2.51) leads to its converse as

$$m_1(\boldsymbol{\theta}) = \left(-(p-1)\left[(\theta_1 - a_1) + \frac{1}{2}\sum_{j=2}^{k}(\theta_j - a_j)^2\right]\right)^{-1/(p-1)} \qquad (2.52)$$

and

$$m_j(\boldsymbol{\theta}) = (\theta_j - a_j)\left(-(p-1)\left[(\theta_1 - a_1) + \frac{1}{2}\sum_{j=2}^{k}(\theta_j - a_j)^2\right]\right)^{-1/(p-1)}, \; j = 2,\ldots,k, \qquad (2.53)$$

with $\theta_j - a_j \neq 0$ for all $j \in \{2, \ldots, k\}$. Since $\mathbf{m} = \partial \mathbf{K}_{\nu_p}(\boldsymbol{\theta})/\partial \boldsymbol{\theta}$ then using (2.52) one can write $\mathbf{K}_{\nu_p}(\boldsymbol{\theta})$ as follow :

$$\begin{aligned} \mathbf{K}_{\nu_p}(\boldsymbol{\theta}) &= \int \frac{\partial \mathbf{K}'_{\nu_p}(\boldsymbol{\theta})}{\partial \theta_1} d\theta_1 \\ &= \frac{-1}{p-2}\left(-(p-1)\Big[(\theta_1 - a_1) + \frac{1}{2}\sum_{j=2}^{k}(\theta_j - a_j)^2\Big]\right)^{1-1/(p-1)} \\ &\quad + f(\theta_2, \ldots, \theta_k), \end{aligned} \tag{2.54}$$

where $f : \mathbb{R}^{k-1} \to \mathbb{R}$ is an analytic function to be determined. Again, derivative of (2.54) with respect to θ_j gives

$$\frac{\partial \mathbf{K}_{\nu_p}(\boldsymbol{\theta})}{\partial \theta_j} = (\theta_j - a_j)\left(-(p-1)\Big[(\theta_1 - a_1) + \frac{1}{2}\sum_{j=2}^{k}(\theta_j - a_j)^2\Big]\right)^{-1/(p-1)} + \frac{\partial f(\theta_2, \ldots, \theta_k)}{\partial \theta_j}$$

which is equal to (2.53) ; then, one gets $\partial f(\theta_2, \ldots, \theta_k)/\partial \theta_j = 0$ for all $j \in \{2, \ldots, k\}$ implying $f(\theta_j, \ldots, \theta_k) = b$ (a real constant). Finally, it ensues from it that we have

$$\mathbf{K}_{\nu_p}(\boldsymbol{\theta}) = \frac{1}{2-p}\left(-(p-1)\Big[(\theta_1 - a_1) + \frac{1}{2}\sum_{j=2}^{k}(\theta_j - a_j)^2\Big]\right)^{1+1/(1-p)} + b.$$

By Proposition 1.4.1 one can see that, up to affinity, this $\mathbf{K}_{\nu_p}$ is a *NST* cumulant function as given in (2.3) with $t = 1$ on its corresponding canonical domain in (3.4) for all $p \in (1,2) \cup (2, \infty)$. The proof of Theorem 2.4.1 is therefore completed by use of the analytical property of $\mathbf{K}_{\nu_p}$. □