FONCTIONNELLES DE COÛT DE GRANDS ARBRES ALÉATOIRES (UNIFORMES ET SIMPLEMENT GÉNÉRÉS)

Version non modifiée de l'article [54]
Cost functionals for large (uniform and simply generated) random trees
paru dans *Electronic Journal of Probability.* Des notes de bas de page ont été ajoutées pour corriger quelques inexactitudes et préciser certaines conventions.

Abstract. Additive tree functionals allow to represent the cost of many divide-and-conquer algorithms. We give an invariance principle for such tree functionals for the Catalan model (random tree uniformly distributed among the full binary ordered trees with given number of nodes) and for simply generated trees (including random tree uniformly distributed among the ordered trees with given number of nodes). In the Catalan model, this relies on the natural embedding of binary trees into the Brownian excursion and then on elementary L^2 computations. We recover results first given by Fill and Kapur (2004) and then by Fill and Janson (2009). In the simply generated case, we use convergence of conditioned Galton-Watson trees towards stable Lévy trees, which provides less precise results but leads us to conjecture a different phase transition value between "global" and "local" regimes. We also recover results first given by Janson (2003 and 2016) in the Brownian case and give a generalization to the stable case.

3.1 Introduction

Ordered trees have many applications in various fields such as computer science for data structures or in biology for genealogical or phylogenetic trees of extant species. Related to those applications, the study of large trees has attracted some attention. In this paper, we shall consider asymptotics in the global regime for general additive functionals of large trees corresponding to the Catalan model and some simply generated trees. Such additive functionals give indexes of trees which are used in computer science, physics or biology to summarize some properties of trees.

For instance, the total path length $P(\mathbf{t})$ of a tree $\mathbf{t}$, see (3.1) and (3.2) for a precise definition, which sums the distances to the root of all nodes, in the context of binary search trees, counts the number of key comparisons needed by Hoare's sorting algorithm Quicksort to sort a list of randomly permuted items, see Rösler [172]. Its convergence towards the Airy distribution was first established by Takács [183], see also Aldous [9, 10] and Janson [113] for binary trees under the Catalan model, Régnier [166], Rösler [172] for binary search trees under the random permutation model (RPM) and Fill and Kapur [90, 91] for m-ary search trees.

The Wiener index $W(\mathbf{t})$ of a tree $\mathbf{t}$, see again definitions (3.1) and (3.2) for a precise definition, which sums the distances between all pairs of nodes of $\mathbf{t}$, was introduced by the chemist Wiener [193] in 1947. It was initially defined as the number of bonds between all pairs of atoms in an acyclic molecule. It also plays an important role in physicochemical properties of chemical structures (boiling points, heat of formation, crystal defects, ...), see Dobrynin, Entringera and Gutman [66] or Trinajstic [185], chapter 10. Its asymptotics has been studied by Janson [113] for binary trees under the Catalan model, Neininger [152] for binary search trees under the RPM and recursive trees and Janson [113] for simply generated trees.

The study of additive functionals associated with monomials, that is $f(x) = x^{\beta-1}$ in (3.1) or equivalently $b_n = n^\beta$ in (3.4), with $\beta > 0$, is interesting because many usual additive functionals can be expressed in terms of those elementary functionals. Moreover, a phase transition in the limiting behavior appears when β varies, see Fill and Kapur [89], Fill and Janson [88] for uniform binary trees, Neininger [151] for binary search trees under RPM and Fill and Kapur for m-ary trees [90, 91].

Additive functionals also appears naturally for the study of phylogenetic trees (rooted binary trees with n labeled leaves corresponding to species and $n-1$ internal vertices). When the number of species in studies of phylogenies grows, it can be interesting to look at the shapes of these trees through indices. Among these indices, we can cite the Sackin index $S(\mathbf{t})$ of a tree $\mathbf{t}$, see definition (3.7), introduced in 1972 by Sackin [175] and also studied in computer science for binary search trees (named as external path length), see Régnier [166] and Rösler [172]. Blum, François and Janson [27] studied its asymptotics. We can also consider the Colless index $C(\mathbf{t})$ of a tree $\mathbf{t}$, see definition (3.6), introduced by Colless [50] in 1982. Its asymptotics have also been studied by by Blum, François and Janson [27]. The cophenetic index $\mathrm{Co}(\mathbf{t})$ of a tree $\mathbf{t}$ was introduced in 2013 by Mir, Rosseló and Rotger [146] and Cardona, Mir and Rosseló [42] who studied its limiting behavior.

We stress that additive functionals in the local regime, such as the total size, the number of leaves, the number of protected nodes, the number of sub-trees or the shape functional (take $f(x) = \log(x)/x$ in (3.1) or $b_n = \log(n)$ in (3.4)) are not covered by our results. See Wagner [191], Holmgren et Janson [108], Janson [115] and Ralaivaosaona and Wagner [165] for asymptotic results in the local regime.

3.1.1 A finite measure indexed by a tree

Let $\mathbb{T}$ denote the set of all rooted finite ordered trees. For $\mathbf{t} \in \mathbb{T}$, let $|\mathbf{t}|$ be the the number of nodes of $\mathbf{t}$; for a node $v \in \mathbf{t}$, let $\mathbf{t}_v$ denote the sub-tree of $\mathbf{t}$ above v (see (3.11) in Section 3.2.1 for a precise definition). We consider the following unnormalized non-negative finite measure $\mathcal{A}_\mathbf{t}$:

$$\mathcal{A}_\mathbf{t}(f) = \sum_{v\in\mathbf{t}} |\mathbf{t}_v| f\left(\frac{|\mathbf{t}_v|}{|\mathbf{t}|}\right), \tag{3.1}$$

where f is a measurable real-valued function defined on $[0, 1]$. We are interested in the asymptotic distribution of $\mathcal{A}_\mathbf{t}(f)$ when $\mathbf{t}$ belongs to a certain class of trees and $|\mathbf{t}|$ goes to infinity. We shall consider two classes of trees: the binary trees (and more precisely the Catalan model) and some simply generated trees.

We give some examples related to the measure $\mathcal{A}_\mathbf{t}$ which are commonly used in the analysis of trees. In what follows, for a tree $\mathbf{t} \in \mathbb{T}$, we denote by $\emptyset$ its root and by d the usual graph distance on $\mathbf{t}$. For $v, w \in \mathbf{t}$, we say that w is an ancestor of v and write $w \preccurlyeq v$ if $d(\emptyset, v) = d(\emptyset, w) + d(w, v)$. For $u, v \in \mathbf{t}$, we denote by $u \wedge v$, the most recent common ancestor of u and v: $u \wedge v$ is the only element of $\mathbf{t}$ such that: $w \preccurlyeq u$ and $w \preccurlyeq v$ implies $w \preccurlyeq u \wedge v$.

- The **total path length** of $\mathbf{t}$ is defined by $P(\mathbf{t}) = \sum_{w\in\mathbf{t}} d(\emptyset, w)$. As $d(\emptyset, w) = \sum_{v\in\mathbf{t}} \mathbf{1}_{\{v\preccurlyeq w\}} -$

1, we get: $P(\mathbf{t}) = \sum_{v\in\mathbf{t}}\sum_{w\in\mathbf{t}} \mathbf{1}_{\{v\preccurlyeq w\}} - |\mathbf{t}| = \mathcal{A}_\mathbf{t}(1) - |\mathbf{t}|$ [1].

- The **shape functional** of $\mathbf{t}$ is defined by $\sum_{w\in\mathbf{t}} \log(|\mathbf{t}_w|)$. We also have the equality $\sum_{w\in\mathbf{t}} \log(|\mathbf{t}_w|) = |\mathbf{t}|^{-1}\mathcal{A}_\mathbf{t}(\log(x)/x) + |\mathbf{t}|\log(|\mathbf{t}|)$. (The function $\log(x)/x$ will not be covered by the main results of this paper.)

- The **Wiener index** of $\mathbf{t}$ is defined by $W(\mathbf{t}) = \sum_{u,w\in\mathbf{t}} d(u,w)$. Since

$$d(u,w) = \sum_{v\in\mathbf{t}} (\mathbf{1}_{\{v\preccurlyeq u\}} + \mathbf{1}_{\{v\preccurlyeq w\}} - 2\mathbf{1}_{\{v\preccurlyeq u,\, v\preccurlyeq w\}}),$$

we deduce that $W(\mathbf{t}) = 2|\mathbf{t}|\,(\mathcal{A}_\mathbf{t}(1) - \mathcal{A}_\mathbf{t}(x))$.

In a nutshell, for $\mathbf{t}\in\mathbb{T}$, we have:

$$\Big(P(\mathbf{t}),\, W(\mathbf{t})\Big) = \Big(\mathcal{A}_\mathbf{t}(1) - |\mathbf{t}|,\, 2|\mathbf{t}|(\mathcal{A}_\mathbf{t}(1) - \mathcal{A}_\mathbf{t}(x))\Big). \tag{3.2}$$

The measure $\mathcal{A}_\mathbf{t}$ is also related to other additive functionals in the particular case of binary trees, see Section 3.1.2.

3.1.2 Additive functionals and toll functions for binary trees

Additive functionals on binary trees allow to represent the cost of algorithms such as "divide and conquer", see Fill and Kapur [89]. For $\mathbf{t}\in\mathbb{T}$ a full binary tree, we shall denote by 1 (resp. 2) the left (resp. right) child of the root. Thus $\mathbf{t}_1$ (resp. $\mathbf{t}_2$) will be the left (resp. right) sub-tree of the root of $\mathbf{t}$. A functional F on binary trees is called an additive functional if it satisfies the following recurrence relation:

$$F(\mathbf{t}) = F(\mathbf{t}_1) + F(\mathbf{t}_2) + b_{|\mathbf{t}|}, \tag{3.3}$$

for all trees $\mathbf{t}$ such that $|\mathbf{t}|\geq 3$ and with $F(\{\emptyset\}) = b_1$. The given sequence $(b_n, n\in\mathbb{N}^*)$ is called the toll function. Notice that:

$$F(\mathbf{t}) = \sum_{v\in\mathbf{t}} b_{|\mathbf{t}_v|}. \tag{3.4}$$

In the particular case where the toll function is a power function, that is $b_n = n^\beta$ for $n\in\mathbb{N}^*$ and some $\beta > 0$, we get $F(\mathbf{t}) = |\mathbf{t}|^{-\beta+1}\mathcal{A}_\mathbf{t}(x^{\beta-1})$. In such cases, the asymptotic study of the measure $\mathcal{A}_\mathbf{t}$ will provide the asymptotic of the additive functionals.

We say that $v\in\mathbf{t}$ is a leaf if $|\mathbf{t}_v| = 1$. We denote by $\mathcal{L}(\mathbf{t})$ the set of leaves of $\mathbf{t}$ and, when $|\mathbf{t}|>1$, by $\mathbf{t}^* = \mathbf{t}\setminus\mathcal{L}(\mathbf{t})$ the tree $\mathbf{t}$ without its leaves. We stress that the additive functional considered in [89] is exactly

$$\tilde{F}(\mathbf{t}) = F(\mathbf{t}^*) = \sum_{v\in\mathbf{t}^*} b_{|\mathbf{t}^*_v|}. \tag{3.5}$$

However the asymptotics will be the same as the one for F when the toll function is a power function, see Remark 3.9. We complete the examples of the previous section for binary trees.

- The **Sackin index** (or external path length) of a tree $\mathbf{t}$, used to study the balance of the tree, is similar to the total path length of $\mathbf{t}$ when one considers only the leaves: $S(\mathbf{t}) = \sum_{w\in\mathcal{L}(\mathbf{t})} d(\emptyset, w)$. Using that for a full binary tree we have $|\mathbf{t}| = 2|\mathcal{L}(\mathbf{t})| - 1$, we deduce that $2S(\mathbf{t}) = \sum_{v\in\mathbf{t}} |\mathbf{t}_v| - 1 = \mathcal{A}_\mathbf{t}(1) - 1$.

[1] By convention, for $a\in\mathbb{R}$, we denote the function $x\mapsto x^a\mathbf{1}_{(0,1]}(x)$ defined on $[0,1]$ by x^a.

- The **Colless index** of a binary tree $\mathbf{t}$ is defined as $C(\mathbf{t}) = \sum_{v\in\mathbf{t}^*} |L_v - R_v|$, where $L_v = |\mathcal{L}(\mathbf{t}_{v1})|$ (resp. $R_v = |\mathcal{L}(\mathbf{t}_{v2})|$) is the number of leaves of the left (resp. right) sub-tree above v. Since $\mathbf{t}$ is a full binary tree, we get $2L_v - 2R_v = |\mathbf{t}_{v1}| - |\mathbf{t}_{v2}|$ and $|\mathbf{t}_{v1}| + |\mathbf{t}_{v2}| = |\mathbf{t}_v| - 1$. We obtain that $2C(\mathbf{t}) = \sum_{v\in\mathbf{t}} |\mathbf{t}_v| - |\mathbf{t}| - 2\chi(\mathbf{t})$ [2], with

$$\chi(\mathbf{t}) = \sum_{v\in\mathbf{t}^*} \min(|\mathbf{t}_{v1}|, |\mathbf{t}_{v2}|). \tag{3.6}$$

  That is $2C(\mathbf{t}) = \mathcal{A}_{\mathbf{t}}(1) - |\mathbf{t}| - 2\chi(\mathbf{t})$.

- The **cophenetic index** of a tree $\mathbf{t}$ (which is used in [146] to study the balance of the tree) is defined by $\mathrm{Co}(\mathbf{t}) = \sum_{u,w\in\mathcal{L}(\mathbf{t}),\, u\neq w} d(\emptyset, u\wedge w)$. Using again that $\mathbf{t}$ is a full binary tree, we get $4\mathrm{Co}(\mathbf{t}) = 4\sum_{v\in\mathbf{t}} |\mathcal{L}(\mathbf{t}_v)|(|\mathcal{L}(\mathbf{t}_v)| - 1) - 4|\mathcal{L}(\mathbf{t})|(|\mathcal{L}(\mathbf{t})| - 1) = \sum_{v\in\mathbf{t}} |\mathbf{t}_v|^2 - |\mathbf{t}|^2 - |\mathbf{t}| + 1$. That is $4\mathrm{Co}(\mathbf{t}) = |\mathbf{t}|\mathcal{A}_{\mathbf{t}}(x) - |\mathbf{t}|^2 - |\mathbf{t}| + 1$.

In a nutshell, for $\mathbf{t}\in\mathbb{T}$ full binary, we have:

$$\Big(2S(\mathbf{t}), 2C(\mathbf{t}), 4\mathrm{Co}(\mathbf{t})\Big) = \Big(\mathcal{A}_{\mathbf{t}}(1) - 1,\, \mathcal{A}_{\mathbf{t}}(1) - |\mathbf{t}| - 2\chi(\mathbf{t}),\, |\mathbf{t}|\mathcal{A}_{\mathbf{t}}(x) - |\mathbf{t}|^2 - |\mathbf{t}| + 1\Big). \tag{3.7}$$

3.1.3 Main results on the asymptotics of additive functionals in the Catalan model

We consider the Catalan model: let T_n be a random tree uniformly distributed among the set of full binary ordered trees with n internal nodes (and thus $n+1$ leaves), which has cardinality $C_n = (2n)!/[(n!^2)(n+1)]$. We have:

$$\boxed{|\mathrm{T}_n| = 2n+1}.$$

Recall that T_n is a (full binary) Galton-Watson tree (also known as simply generated tree) conditioned on having n internal nodes (see Janson [114], Example 10.3). It is well known, see Takàcs [183], Aldous [9, 10] and Janson [113], that $|\mathrm{T}_n|^{-3/2}P(\mathrm{T}_n)$ converges in distribution, as n goes to infinity, towards $2\int_0^1 B_s\, ds$, where $B = (B_s, s\in[0,1])$ is the normalized positive Brownian excursion. This result, see Corollary 3.16, can be seen as a consequence of the convergence in distribution of T_n (in fact the contour process) properly scaled towards the Brownian continuum tree whose contour process is B, see [9] and Duquesne [69], or Duquesne and Le Gall [70] in the setting of Brownian excursion. For a combinatorial approach, which can be extended to other families of trees, see also Fill and Kapur [90, 91] or Fill, Flajolet and Kapur [86].

In [89], the authors considered the toll functions $b_n = n^\beta$ with $\beta > 0$ and they proved that with a suitable scaling the corresponding additive functional $F_\beta(\mathrm{T}_n) = |\mathrm{T}_n|^{-\beta+1}\mathcal{A}_{\mathrm{T}_n}(x^{\beta-1})$ converge in distribution to a limit, say Y_β. The distribution of Y_β is characterized by its moments. (In [85, 89], the authors considered also the toll function $b_n = \log(n)$.) See also Janson and Chassaing [116] for asymptotics of the Wiener index, which is a consequence of the joint convergence in distribution of $(\mathcal{A}_{\mathrm{T}_n}(1), \mathcal{A}_{\mathrm{T}_n}(x))$ with a suitable scaling and Blum, François and Janson [27] for the convergence of the Sackin and Colless indexes. We give a natural representation of the family $(\mathrm{T}_n, n\in\mathbb{N}^*)$ such that we have an a.s. convergence of the additive functional instead of a convergence in distribution (see Section 3.2.4). In Theorem 3.4 (take $\alpha = 2$), we prove that, in the Catalan model, the random measure $|\mathrm{T}_n|^{-3/2}\mathcal{A}_{\mathrm{T}_n}$ converges weakly a.s., as n goes to infinity, to a random measure $2\Phi_B$, built on the Brownian normalized excursion B, see (3.16) with $h = B$. Using the notation $\mathrm{T}_{n,v} = (\mathrm{T}_n)_v$ for $v\in\mathrm{T}_n$, this proves in particular the following a.s. convergence. See also the fluctuations for this a.s. convergence, in Proposition 3.10.

[2]By convention, $\sum_k a_k + b = (\sum_k a_k) + b$.

Theorem. *We have that a.s. for all real-valued continuous function f defined on $[0,1]$:*

$$|\mathrm{T}_n|^{-3/2} \sum_{v\in\mathrm{T}_n} |\mathrm{T}_{n,v}|\, f\left(\frac{|\mathrm{T}_{n,v}|}{|\mathrm{T}_n|}\right) \xrightarrow[n\to+\infty]{a.s.} 2\Phi_B(f). \tag{3.8}$$

Notice that Theorem 3.4 is more general as the convergences hold jointly for all measurable real-valued functions f defined on $[0,1]$ such that f is continuous on $(0,1]$ and $\sup_{x\in(0,1]} x^a|f(x)|$ is finite for some $a < 1/2$. Notice this covers the case of toll functions $b_n = n^\beta$ with $\beta > 1/2$ in [89] which corresponds to the so called "global" regime. The limit $2\Phi_B(x^{\beta-1})$ gives a representation of Y_β for $\beta > 1/2$, which, thanks to Corollary 3.6, corresponds when $\beta \geq 1$ to the one announced in Fill and Janson [88]. In particular, we have the following representation for Φ_B on monomials, see Lemma 3.5:

Corollary. *We have, for all $\beta > 1$:*

$$\Phi_B(x^{\beta-1}) = \frac{1}{2}\,\beta(\beta-1)\int_{[0,1]^2} |t-s|^{\beta-2}\, m_B(s,t)\, ds\, dt,$$

where $m_B(s,t) = \inf_{u\in[s\wedge t, s\vee t]} B(u)$.

In the "local" regime, that is $\beta \in (0,1/2]$, according to Corollary 3.6 and Lemma 3.5, the convergence (3.8) is not relevant as $\Phi_B(x^{\beta-1}) = +\infty$ a.s.; see [89] for the relevant normalization.

The proof of Theorem 3.4 relies on the natural embedding of T_n into the Brownian excursion, see [10] and Le Gall [134], so that the convergence in distribution of the random measure $|\mathrm{T}_n|^{-3/2}\mathcal{A}_{\mathrm{T}_n}$ or of the additive functionals F_β (which holds simultaneously for all $\beta > 1/2$) is then an a.s. convergence. In Remark 3.8, we provide, as a direct consequence of Theorem 3.4, the joint convergence of the total length path, the Wiener, Sackin, Colless and cophenetic indexes defined in Sections 3.1.1 and 3.1.2.

Remark 3.1. The method presented in this section based on the embedding of T_n into a Brownian excursion can not be extended directly to other models of trees such as binary search trees, recursive trees or simply generated trees.

Concerning binary search trees (or random permutation model or Yule trees), see [166] and [172] for the convergence of the external path length (which corresponds in our setting to the Sackin index), [151] for toll function $b_n = n^\beta$, [152] for the Wiener index (and [113] for simply generated trees), [27] (and [93] for other trees) for the Sackin and Colless indexes, and [85] for the shape function.

Concerning recursive trees, see [141, 65] for the convergence of the total path length and [152] for the Wiener index. In the setting of recursive trees, then (3.3) is a stochastic fixed point equation, which can be analyzed using the approach of [173].

Remark 3.2. One can replace the toll function $b_{|\mathbf{t}|}$ in (3.3) by a function of the tree, say $\mathbf{b}(\mathbf{t})$. For example, if one consider $\mathbf{b}(\mathbf{t}) = \mathbf{1}_{\{\mathbf{t}=\mathbf{t}_0\}}$, with $\mathbf{t}_0$ a given tree, then the corresponding additive functional gives the number of occurrence of the motif $\mathbf{t}_0$. The case of "local" toll function $\mathbf{b}$ (with finite support or fast decreasing rate) has been considered in the study of fringe trees, see [7], [60, 92] for binary search trees, and [115] for simply generated trees and [108] for binary search trees and recursive trees.

The terms "local" and "global" are used to stress the phase transition of the limit laws from normal to non-normal. If the toll function is small then the contribution $b_{|\mathbf{t}|}$ from each sub-tree $\mathbf{t}$ is small so that the limit law is normal. But, if the toll function is large, then the main contribution comes from a few sub-trees of large size so that the limit law is non-normal. See [111] for the study of the phase transition on asymptotics of additive functionals with toll functions $b_n = n^\beta$ on binary search trees between the "local" regime (corresponding to $\beta \leq 1/2$) and the "global" regime ($\beta > 1/2$). The same phase transition is observed for the Catalan model, see [89]. Our main result, see Theorem 3.4, concerns specifically the "global" regime.

3.1.4 Main results on the asymptotics of additive functionals for simply generated trees

We consider a weight sequence $\mathfrak{p} = (\mathfrak{p}(k), k \in \mathbb{N})$ on $\mathbb{R}_+$ with generating function $g_{\mathfrak{p}}$. We assume that $g_{\mathfrak{p}}$ has a positive radius of convergence, $g_{\mathfrak{p}}(0) > 0$, $g''_{\mathfrak{p}} \neq 0$ and $\mathfrak{p}$ is generic, that is there exists a positive root to the equation $g_{\mathfrak{p}}(q) = q g'_{\mathfrak{p}}(q)$. A simply generated tree of size $p \in \mathbb{N}^*$ with weight function $\mathfrak{p}$ is a random tree $\tau^{(p)}$ such that the probability of $\tau^{(p)}$ to be equal to $\mathbf{t}$, with $|\mathbf{t}| = p$, is proportional to $\prod_{v\in\mathbf{t}} \mathfrak{p}(k_v(\mathbf{t}))$, where $k_v(\mathbf{t})$ is the number of children of the node v in $\mathbf{t}$. According to Section 3.2.5, since $g_{\mathfrak{p}}$ is generic, without loss of generality we can assume that $\mathfrak{p}$ is a critical probability ($g_{\mathfrak{p}}(1) = g'_{\mathfrak{p}}(1) = 1$), so that $\tau^{(p)}$ is distributed as a Galton-Watson (GW) tree τ with offspring distribution $\mathfrak{p}$ conditioned to $|\tau| = p$. Global convergence of scaled GW trees τ to Lévy trees has been studied in Le Gall and Le Jan [136] and in [70] using the convergence of contour process.

Assume $\mathfrak{p}$ belongs to the domain of attraction of a stable distribution of Laplace exponent $\psi(\lambda) = \kappa\lambda^\gamma$ with $\gamma \in (1, 2]$ and $\kappa > 0$. Then, the convergence of $\tau^{(p)}$ properly scaled to the normalized Lévy trees holds according to [69]. This result is recalled in Section 3.4.2. We recall that the normalized Lévy tree is a real tree coded by the normalized positive excursion of the height function $H = (H(s), s \in [0, 1])$.

We have the following convergence in distribution, see Corollary 3.15 for a precise statement.

Theorem. *There exists a sequence $(a_p, p \in \mathbb{N}^*$ s.t. $\mathbb{P}(|\tau| = p) > 0)$ such that we have the following convergence in distribution:*

$$\frac{a_p}{p^2} \sum_{v\in\tau^{(p)}} |\tau_v^{(p)}|\, f\left(\frac{|\tau_v^{(p)}|}{p}\right) \xrightarrow[p\to+\infty]{(d)} \Phi_H(f), \tag{3.9}$$

simultaneously for all real-valued continuous function f defined on $[0, 1]$ [3].

The convergence (3.9) has to be understood along the infinite sub-sequence of p such that $\mathbb{P}(|\tau| = p) > 0$. The proof relies on the fact that one can approximate $\mathcal{A}_{\mathbf{t}}(x^k)$, for $k \in \mathbb{N}^*$, by an elementary continuous functional of the contour process of $\mathbf{t}$, see Section 3.4.4. Then, we use the convergence of the contour process of $\tau^{(p)}$ to the contour process of H to conclude. We also provide the first moment of $\Phi_H(x^{\beta-1})$, see Lemma 3.17 and conjecture that $\beta = 1/\gamma$ corresponds to the phase transition between the "global" and "local" regime in this setting.

Remark 3.3. We make the following comments.

- Assume that $\mathfrak{p}$ has finite variance, say σ^2. Then one can take $a_p = \sqrt{p}$ and H is equal to $(2/\sigma)B$ which corresponds to $\psi(\lambda) = \sigma^2\lambda^2/2$. By scaling, or using that the limit in Theorem 3.4 does not depend on α, we deduce that $\Phi_{cB} = c\Phi_B$. We can then rewrite (3.9) as:

$$p^{-3/2} \sum_{v\in\tau^{(p)}} |\tau_v^{(p)}|\, f\left(\frac{|\tau_v^{(p)}|}{p}\right) \xrightarrow[p\to+\infty]{\text{(d)}} \frac{2}{\sigma}\Phi_B(f), \tag{3.10}$$

  where the convergence holds simultaneously for all real-valued continuous function f defined on $[0, 1]$ and along the infinite sub-sequence of p such that $\mathbb{P}(|\tau| = p) > 0$.

- If one consider the binary offspring distribution $\mathfrak{p}$ such that $\mathfrak{p}(2) + \mathfrak{p}(0) = 1$ (recall that $1 > \mathfrak{p}(0) > 0$ by assumption), one gets that $\tau^{(2n+1)}$ is uniformly distributed among the full binary trees with n internal nodes (and $n+1$ leaves), that is $\tau^{(2n+1)}$ is distributed as T_n, see the Catalan model studied in Section 3.1.3. Take $\mathfrak{p}(0) = 1/2$ to get the critical

[3] The right wording is in terms of the convergence of measures given in Corollary 3.15.

case, and notice that $\sigma = 1$ in (3.10). The convergence (3.10), with $p = 2n + 1$, is then a weaker version of (3.8) (convergence in distribution instead of a.s. convergence, and continuous functions on $[0, 1]$ instead of continuous functions on $(0, 1]$ with possible blow up at $0+$).

- If one consider the (shifted) geometric distribution: $\mathfrak{p}(k) = q(1 - q)^k$ for $k \in \mathbb{N}$ with $q \in (0, 1)$, one gets that $\tau^{(p)}$ is uniformly distributed among the rooted ordered trees with p nodes. Take $q = 1/2$ to get the critical case, and notice that $\sigma = 2$ in (3.10).

- If one consider the Poisson offspring distribution: $\mathfrak{p}(k) = \frac{\lambda^k}{k!} e^{-\lambda}$ for $k \in \mathbb{N}$ with $\lambda \in \mathbb{R}^+$, one gets that $\tau^{(p)}$ is uniformly distributed among the labeled unordered rooted trees with p nodes (also known as Cayley trees). Take $\lambda = 1$ to get the critical case, and notice that $\sigma = 1$ in (3.10). In particular, we recover the result of Zohoorian Azad [196] for the additive functional associated to the toll function $b_n = n^2$ for $n \in \mathbb{N}^*$ (apply Corollary 3.16 with $\beta = 2$).

3.1.5 Organization of the paper

Section 3.2 is devoted to the definition of the main objects used in this paper (ordered rooted discrete trees using Neveu's formalism, real trees defined by a contour function, Brownian tree whose contour function is a Brownian normalized excursion, the embedding of the discrete binary trees from the Catalan model into the Brownian tree, and simply generated random trees). We present our main result about the Catalan model in Section 3.3 on the convergence (3.8), see Theorem 3.4 and Corollary 3.6. (The proofs are given in Sections 3.5 and 3.6.) The corresponding fluctuations are stated in Proposition 3.10. (The proof is given in Section 3.7.) Section 3.4 is devoted to the main results concerning the convergence of $\mathcal{A}_\tau$ when τ is a simply generated tree, see Corollaries 3.15 and 3.16. Some technical results are gathered in Section 3.8.

3.2 Notations

Let I be an interval of $\mathbb{R}$ with positive Lebesgue measure. We denote by $\mathcal{B}(I)$ the set of real-valued measurable functions defined on I. We denote by $\mathcal{C}(I)$ (resp. $\mathcal{C}_+(I)$) the set of real-valued (resp. non-negative) continuous functions defined on I. For $f \in \mathcal{B}(I)$ we denote by $\| f \|_\infty$ the supremum norm and by $\| f \|_{\text{esssup}}$ the essential supremum of $|f|$ over I. The two supremums coincide when f is continuous.

3.2.1 Ordered rooted discrete trees

We recall Neveu's formalism [153] for ordered rooted discrete trees, which we shall simply call trees. We set $\mathcal{U} = \bigcup_{n \geq 0} (\mathbb{N}^*)^n$ the set of finite sequences of positive integers with the convention $(\mathbb{N}^*)^0 = \{\emptyset\}$. For $n \geq 0$ and $u \in (\mathbb{N}^*)^n \subset \mathcal{U}$, we set $|u| = n$ the length of u. Let $u, v \in \mathcal{U}$. We denote by uv the concatenation of the two sequences, with the convention that $uv = u$ if $v = \emptyset$ and $uv = v$ if $u = \emptyset$. We say that v is an ancestor of u (in a large sense) and write $v \preccurlyeq u$ if there exists $w \in \mathcal{U}$ such that $u = vw$. If $v \preccurlyeq u$ and $v \neq u$, then we shall write $v \prec u$. The set of ancestors of u is the set $\bar{\mathrm{A}}_u = \{v \in \mathcal{U}; v \preccurlyeq u\}$. The most recent common ancestor of a subset $\mathbf{s}$ of $\mathcal{U}$, denoted by $\mathfrak{m}(\mathbf{s})$, is the unique element v of $\bigcap_{u \in \mathbf{s}} \bar{\mathrm{A}}_u$ with maximal length. We consider the lexicographic order on $\mathcal{U}$: for $u, v \in \mathcal{U}$, we set $v < u$ either if $v \prec u$ or if $v = wjv'$ and $u = wiu'$ with $w = \mathfrak{m}(\{v, u\})$, $u, u' \in \mathcal{U}$ and $j < i$ for some $i, j \in \mathbb{N}^*$.

A tree $\mathbf{t}$ is a subset of $\mathcal{U}$ that satisfies:

- $\emptyset \in \mathbf{t}$,
- If $u \in \mathbf{t}$, then $\bar{\mathrm{A}}_u \subset \mathbf{t}$.
- For every $u \in \mathbf{t}$, there exists $k_u(\mathbf{t}) \in \mathbb{N}$ such that, for every $i \in \mathbb{N}^*$, $ui \in \mathbf{t}$ if and only if $1 \leq i \leq k_u(\mathbf{t})$.

Let $u \in \mathbf{t}$. The integer $k_u(\mathbf{t})$ represents the number of offsprings of the node u. The node u is called a leaf (resp. internal node) if $k_u(\mathbf{t}) = 0$ (resp. $k_u(\mathbf{t}) > 0$). The node $\emptyset$ is called the root of $\mathbf{t}$. We define the sub-tree $\mathbf{t}_u \in \mathbb{T}$ of $\mathbf{t}$ "above" u as:

$$\mathbf{t}_u = \{v \in \mathcal{U},\ uv \in \mathbf{t}\}. \tag{3.11}$$

We denote by $|\mathbf{t}| = \mathrm{Card}\,(\mathbf{t})$ the number of nodes of $\mathbf{t}$ and we say that $\mathbf{t}$ is finite if $|\mathbf{t}| < +\infty$. Let $d_{\mathbf{t}}$ denote the usual graph distance on $\mathbf{t}$. In particular, we have $d_{\mathbf{t}}(\emptyset, u) = |u|$ for $u \in \mathbf{t}$. When the context is clear, we shall write d for $d_{\mathbf{t}}$.

We denote by $\mathbb{T}$ the set of finite trees and by $\mathbb{T}^{(p)} = \{\mathbf{t} \in \mathbb{T},\ |\mathbf{t}| = p\}$ the set of trees with p nodes, for $p \in \mathbb{N}^*$. Let us recall that, for a tree $\mathbf{t} \in \mathbb{T}$, we have

$$\sum_{u \in \mathbf{t}} k_u(\mathbf{t}) = |\mathbf{t}| - 1. \tag{3.12}$$

3.2.2 Real trees

We recall the definition of a real tree, see [78]. A real tree is a metric space $(\mathcal{T}, d)$ which satisfies the following two properties for every $x, y \in \mathcal{T}$:

(i) There exists a unique isometric map $f_{x,y}$ from $[0, d(x,y)]$ into $\mathcal{T}$ such that $f_{x,y}(0) = x$ and $f_{x,y}(d(x,y)) = y$.

(ii) If ϕ is a continuous injective map from $[0,1]$ into $\mathcal{T}$ such that $\phi(0) = x$ and $\phi(1) = y$, then we have $\phi([0,1]) = f_{x,y}([0, d(x,y)])$.

Equivalently, a metric space $(\mathcal{T}, d)$ is a real tree if and only if $\mathcal{T}$ is connected and d satisfies the four point condition:

$$d(s,t) + d(x,y) \leq \max(d(s,x) + d(t,y), d(s,y) + d(t,x)) \quad \text{for all} \quad s,t,x,y \in \mathcal{T}.$$

A rooted real tree is a real tree $(\mathcal{T}, d)$ with a distinguished element $\emptyset$ called the root. For $x, y \in \mathcal{T}$, we denote by $[\![x, y]\!]$ the range of the map $f_{x,y}$ described above. Let $x, y \in \mathcal{T}$. We denote by $x \wedge y$ their most recent common ancestor which is the only $z \in \mathcal{T}$ such that $[\![\emptyset, z]\!] = [\![\emptyset, x]\!] \bigcap [\![\emptyset, y]\!]$. The out-degree $d_x(\mathcal{T})$ of x is the number of connected components of $\mathcal{T} \backslash \{x\}$ which do not contain the root. We say x is a leaf (resp. branching point) if $d_x(\mathcal{T}) = 0$ (resp. $d_x(\mathcal{T}) \geq 2$). We say $\mathcal{T}$ is binary if $d_x(\mathcal{T}) \in \{0,1,2\}$ for all $x \in \mathcal{T}$.

For $h \in \mathcal{C}_+([0,1])$, we define its minimum over the interval with bounds $s, t \in [0,1]$:

$$m_h(s,t) = \inf_{u \in [s \wedge t, s \vee t]} h(u). \tag{3.13}$$

We shall also use the length of the excursion of h above level r straddling s defined by:

$$\sigma_{r,s}(h) = \int_0^1 dt\, \mathbf{1}_{\{m_h(s,t) \geq r\}}. \tag{3.14}$$

For $\beta > 0$, we set:

$$Z_\beta^h = \int_0^1 ds \int_0^{h(s)} dr\ \sigma_{r,s}(h)^{\beta - 1}. \tag{3.15}$$

Let $h \in \mathcal{C}_+([0,1])$ be such that $m_h(0,1) = 0$. For every $x, y \in [0,1]$, we set $d_h(x,y) = h(x) + h(y) - 2m_h(x,y)$. It is easy to check that d_h is symmetric and satisfies the triangle inequality. The relation $\sim_h$ defined on $[0,1]^2$ by $x \sim_h y \Leftrightarrow d_h(x,y) = 0$ is an equivalence relation. Let $\mathcal{T}_h = [0,1]/\sim_h$ be the corresponding quotient space. The function d_h on $[0,1]^2$ induces a function on $\mathcal{T}_h^2$, which we still denoted by d_h, and which is a distance on $\mathcal{T}_h$. It is not difficult to check that $(\mathcal{T}_h, d_h)$ is then a compact real tree. We denote by $\mathbf{p}_h$ the canonical projection from $[0,1]$ into $\mathcal{T}_h$. Thus, the metric space $(\mathcal{T}_h, d_h)$ can be viewed as a rooted real tree by setting $\emptyset = \mathbf{p}_h(0)$. The image of the Lebesgue measure on $[0,1]$ by $\mathbf{p}_h$ is a measure μ_h on $\mathcal{T}_h$.

For $\mathbf{t} \in \mathbb{T}$, we define the unnormalized measure $\mathcal{A}_\mathbf{t}$ on $[0,1]$ by:

$$\mathcal{A}_\mathbf{t}(f) = \sum_{v \in \mathbf{t}} |\mathbf{t}_v| f\left(\frac{|\mathbf{t}_v|}{|\mathbf{t}|}\right), \quad f \in \mathcal{C}([0,1]).$$

For $h \in \mathcal{C}_+([0,1])$, we also consider the measure Φ_h on $[0,1]$ defined by:

$$\Phi_h(f) = \int_0^1 ds \int_0^{h(s)} dr\ f(\sigma_{r,s}(h)), \quad f \in \mathcal{B}([0,1]). \tag{3.16}$$

We endow the space of non-negative finite measures on $[0,1]$ with the topology of weak convergence.

3.2.3 The Brownian continuum random tree $\mathcal{T}$

Let $B = (B_t, 0 \leq t \leq 1)$ be a positive normalized Brownian excursion. Informally, B is just a linear standard Brownian path started from the origin and conditioned to stay positive on $(0,1)$ and to come back to 0 at time 1. For $\alpha > 0$, let $e = \sqrt{2/\alpha}\, B$ and let $\mathcal{T}_e$ denote the associated real tree called Brownian continuum random tree. (We recall the associated branching mechanism is $\psi(\lambda) = \alpha\lambda^2$.) The continuum random tree introduced in [8] corresponds to $\alpha = 1/2$ and the Brownian tree associated to the normalized Brownian excursion corresponds to $\alpha = 2$. We shall keep the parameter α so that the two previous cases are easy to read on the results. See [135] for properties of the Brownian continuum random tree. In particular $\mu_e(dx)$-a.s. x is a leaf and a.s. $\mathcal{T}_e$ is binary.

We shall forget to stress the dependence in e in the notations, when there is no ambiguity, so that for example we simply write $\mathcal{T}$, μ, $\sigma_{r,s}$ and Z_β for respectively $\mathcal{T}_e$, μ_e, $\sigma_{r,s}(e)$ which is defined in (3.14) and Z_β^e which is defined in (3.15). For $r \geq 0$ and $s \in [0,1]$, we also have:

$$\sigma_{r,s} = \mu(x \in \mathcal{T},\, d(\emptyset, x \wedge \mathbf{p}(s)) \geq r)),$$

which is the mass of the sub-tree of $\mathcal{T}$ containing $\mathbf{p}(s)$ and at distance r from the root.

3.2.4 The discrete binary tree from the Brownian tree

A marked tree $\tilde{\mathbf{t}} = (\mathbf{t}, (h_v, v \in \mathbf{t}))$ is a tree $\mathbf{t} \in \mathbb{T}$ with a label on each node. The label $h_v \in (0, +\infty)$ will be interpreted as the length of the branch from below v. (Notice, there is a branch below the root.) We define the concatenation of two marked trees $\tilde{\mathbf{t}}^{(i)} = (\mathbf{t}^{(i)}, (h_v^{(i)}, v \in \mathbf{t}^{(i)}))$ with $i \in \{1,2\}$ and $r > 0$ as $\tilde{\mathbf{t}} = [\tilde{\mathbf{t}}^{(1)}, \tilde{\mathbf{t}}^{(2)}; r]$ with $\mathbf{t} = \{\emptyset\} \bigcup_{i=1}^2 \{v = iu,\, u \in \mathbf{t}^{(i)}\}$ and for $v \in \mathbf{t}$, we have $h_v = r$ if $v = \emptyset$ and $h_v = h_u^{(i)}$ if $v = iu$ with $u \in \mathbf{t}^{(i)}$ and $i \in \{1,2\}$.

Let $g \in \mathcal{C}_+([0,1])$ be such that $\mathcal{T}_g$ is binary. Let $n \in \mathbb{N}$ and $0 < t_1 < \cdots < t_{n+1} < 1$ such that $(\mathbf{p}_g(t_k), 1 \leq k \leq n+1)$ are $n+1$ distinct leaves. Set $G_n = (g; t_1, \ldots, t_{n+1})$. We denote by $\mathcal{T}_g(G_n) = \bigcup_{k=1}^{n+1} [\![\emptyset, \mathbf{p}_g(t_k)]\!]$ the random real tree spanned by the $n+1$ leaves $\mathbf{p}_g(t_1), \ldots, \mathbf{p}_g(t_{n+1})$ with root $\emptyset$. We define recursively the associated marked tree $\tilde{\mathbf{t}}(G_n) =$

$(\mathbf{t}(G_n), (h_{n,v}(G_n), v \in \mathbf{t}(G_n)))$, where intuitively $\mathbf{t}(G_n)$ is similar to $\mathcal{T}_g(G_n)$ but with the branch lengths equal to 1 and no branch below the root, and $h_{n,v}(G_n)$ is the length of the branch in $\mathcal{T}_g(G_n)$ below the node corresponding to $v \in \mathbf{t}(G_n)$. More precisely, for $n = 0$, we set $\mathbf{t}(G_0) = \{\emptyset\}$ and $h_{0,\emptyset}(G_0) = g(t_1)$. Let $n \geq 1$. Since $\mathcal{T}_g$ is binary and $(\mathbf{p}_g(t_k), 1 \leq k \leq n+1)$ are $n+1$ distinct leaves, there exists a unique $s \in (t_1, t_{n+1})$ and a unique $\ell \in \{1, \ldots, n\}$ such that $g(s) = m_g(t_1, t_{n+1})$ and $t_\ell < s < t_{\ell+1}$. We define $g_1(t) = (g(t) - g(s))\mathbf{1}_{[t_1,s]}(t)$ and $g_2(t) = (g(t) - g(s))\mathbf{1}_{[s,t_{n+1}]}(t)$. Notice that $\mathcal{T}_{g_i}$ is binary and $(\mathbf{p}_g(t_k), 1 \leq k \leq \ell)$ (resp. $(\mathbf{p}_g(t_k), \ell+1 \leq k \leq n+1)$) are ℓ (resp. $n - \ell + 1$) distinct leaves of $\mathcal{T}_{g_1}$ (resp. $\mathcal{T}_{g_2}$). Set $G'_{\ell-1} = (g_1; t_1, \ldots, t_\ell)$ and $G''_{n-\ell} = (g_2; t_{\ell+1}, \ldots, t_{n+1})$ and define $\tilde{\mathbf{t}}(G_n)$ as the concatenation $[\mathbf{t}(G'_{\ell-1}), \mathbf{t}(G''_{n-\ell}); g(s)]$.

Let e be the Brownian excursion defined in Section 3.2.3. Let $(U_n, n \in \mathbb{N}^*)$ be a sequence of independent random variables uniform on $[0,1]$, independent of e. In particular $(\mathbf{p}(U_n), n \in \mathbb{N}^*)$ are a.s. distinct leaves of $\mathcal{T}$. Let $(U_{1,n}, \ldots, U_{n+1,n})$ be the a.s. increasing reordering of $(U_1, \ldots, U_{n+1})$ and set $G_n = (e; (U_{1,n}, \ldots, U_{n+1,n}))$. We write $\mathcal{T}_{[n]} = \mathcal{T}(G_n)$ the random real tree spanned by the $n+1$ leaves $\mathbf{p}(U_1), \ldots, \mathbf{p}(U_{n+1})$ and the root and $\tilde{\mathrm{T}}_n = (\mathrm{T}_n; (h_{n,v}, v \in \mathrm{T}_n)) = \tilde{\mathbf{t}}(G_n)$ the associated marked tree. According to Pitman [163], Theorem 7.9 or Aldous [10], the tree $\mathcal{T}_{[n]}$ can also be obtained by stick-breaking procedure or Poisson line-breaking construction. For $1 \leq k \leq n+1$, we denote by $u(U_k)$ the leaf in T_n corresponding to the leaf $\mathbf{p}(U_k)$ in $\mathcal{T}_{[n]}$. See Figure (3.1) for an example with $n = 4$. It is well known that T_n is uniform among the discrete full binary ordered trees with n internal nodes.

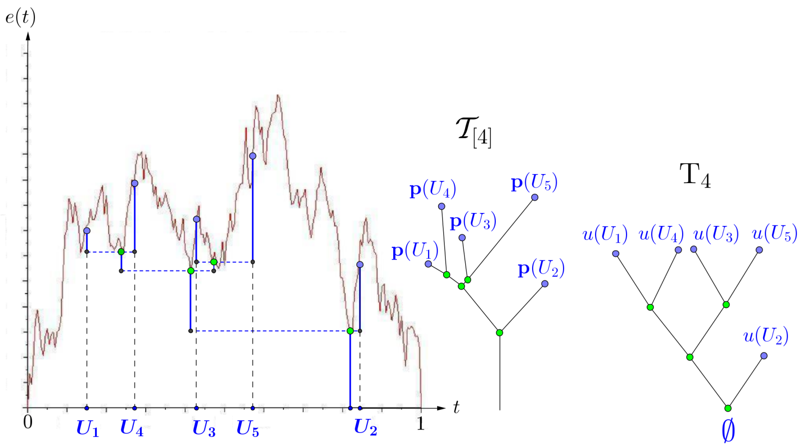


Figure 3.1 – The Brownian excursion, $\mathcal{T}_{[n]}$ and T_n (for $n = 4$).

3.2.5 Simply generated random tree

The presentation of simply generated trees is common in combinatorics. The tools involved in our proofs use Galton-Watson trees. For these reasons, we recall the link between simply generated trees and Galton-Watson trees (see also the survey of Janson [114] for more details). We consider a weight sequence $\mathfrak{p} = (\mathfrak{p}(k), k \in \mathbb{N})$ of non-negative real numbers such that $\sum_{k\in\mathbb{N}} \mathfrak{p}(k) > \mathfrak{p}(1) + \mathfrak{p}(0)$ and $\mathfrak{p}(0) > 0$. For $\mathbf{t} \in \mathbb{T}$, we define its weight as:

$$w(\mathbf{t}) = \prod_{v\in\mathbf{t}} \mathfrak{p}(k_v(\mathbf{t})).$$

We set $w(\mathbb{T}^{(p)}) = \sum_{\mathbf{t}\in\mathbb{T}^{(p)}} w(\mathbf{t})$. For $p \in \mathbb{N}^*$ such that $w(\mathbb{T}^{(p)}) > 0$, a simply generated tree taking values in $\mathbb{T}^{(p)}$ with weight sequence $\mathfrak{p}$ is a $\mathbb{T}^{(p)}$-random variable $\tau^{(p)}$ whose distribution is characterized by, for all $\mathbf{t} \in \mathbb{T}^{(p)}$:

$$\mathbb{P}(\tau^{(p)} = \mathbf{t}) = \frac{w(\mathbf{t})}{w(\mathbb{T}^{(p)})}.$$

Let $g_\mathfrak{p}$ be the generating function of $\mathfrak{p}$: $g_\mathfrak{p}(\theta) = \sum_{k\in\mathbb{N}} \theta^k \mathfrak{p}(k)$ for $\theta > 0$. From now on, we assume there exists $\theta > 0$ such that $g_\mathfrak{p}(\theta)$ is finite. For $q > 0$ such that $g_\mathfrak{p}(q) < +\infty$, let $\mathfrak{p}_q$ be the probability distribution with generating function $\theta \mapsto g_\mathfrak{p}(q\theta)/g_\mathfrak{p}(q)$. According to [121] see also [3], the distribution of the GW tree τ with offspring distribution $\mathfrak{p}_q$ conditioned on $\{|\tau| = p\}$ is the distribution of $\tau^{(p)}$ and thus does not depend on q. It is easy to check there exists at most one positive root, say $q_\mathfrak{p}$, of the equation $g_\mathfrak{p}(q) = q g'_\mathfrak{p}(q)$. We say that $\mathfrak{p}$ is generic (for the total progeny) if such root $q_\mathfrak{p}$ exists and non-generic otherwise. In particular, all weight sequences such that there exists $q > 0$ with $g_\mathfrak{p}(q)$ finite and $g_\mathfrak{p}(q) < q g'_\mathfrak{p}(q)$ (that is $\mathfrak{p}_q$ is a super-critical offspring distribution), are generic.

From now on, we shall assume that $\mathfrak{p}$ is generic. Without loss of generality, by replacing $\mathfrak{p}$ by the probability distribution with generating function $\theta \mapsto g_\mathfrak{p}(q_\mathfrak{p}\theta)/g_\mathfrak{p}(q_\mathfrak{p})$, we will assume that $\mathfrak{p}$ is a critical probability distribution, that is:

$$\sum_{k\in\mathbb{N}} \mathfrak{p}(k) = \sum_{k\in\mathbb{N}} k\mathfrak{p}(k) = 1.$$

We recall that $\tau^{(p)}$ is distributed as a critical GW tree τ with offspring distribution $\mathfrak{p}$ conditioned on $\{|\tau| = p\}$, as for all finite tree $\mathbf{t}$, $\mathbb{P}(\tau = \mathbf{t}) = w(\mathbf{t})$.

Local limits for critical GW trees conditioned on having a large total progeny go back to [121] for the generic case (infinite spine case) and [114] for the non-generic case (condensation case), see also [3, 4] and reference therein for more general conditionings. Scaling limits or global limits for GW tree conditioned on having a large total progeny have been studied in [70] for forests (that is collection of GW trees) and in [69, 130] for critical GW tree in the domain of attraction of Lévy trees, see also [129] for more general conditioning of GW trees and [131] for non-generic cases.

3.3 Catalan model

Let $\alpha > 0$ and recall $e = \sqrt{2/\alpha}\, B$, where $B = (B_t, t \in [0,1])$ denotes the normalized Brownian excursion. We also recall that the discrete binary tree T_n, defined in Section 3.2.4 from the Brownian tree $\mathcal{T}_e$, is uniformly distributed among the full ordered rooted binary trees with n internal nodes. In particular, we have $|\mathrm{T}_n| = 2n + 1$. For $n \in \mathbb{N}^*$, we define the weighted random measure A_n on $[0,1]$ defined by $A_n = |\mathrm{T}_n|^{-3/2} \mathcal{A}_{\mathrm{T}_n}$, that is for $f \in \mathcal{B}([0,1])$:

$$A_n(f) = |\mathrm{T}_n|^{-3/2} \sum_{v\in\mathrm{T}_n} |\mathrm{T}_{n,v}|\, f\left(\frac{|\mathrm{T}_{n,v}|}{|\mathrm{T}_n|}\right), \tag{3.17}$$

where $\mathrm{T}_{n,v} = (\mathrm{T}_n)_v$ is the sub-tree of T_n "above" v. Notice that $A_n(\{0\}) = 0$. The next result is proved in Section 3.6.

Theorem 3.4. *We have that a.s. for all $f \in \mathcal{B}([0,1])$, continuous on $(0,1]$ and such that $\lim_{x\to 0+} x^a f(x) = 0$ for some $a \in [0,1/2)$:*

$$A_n(f) \xrightarrow[n\to+\infty]{} \sqrt{2\alpha}\, \Phi_e(f).$$

We deduce from this Theorem that $(A_n, n \in \mathbb{N}^*)$ converges a.s. for the weak topology towards $\sqrt{2\alpha}\, \Phi_e$.

By convention, for $a \in \mathbb{R}$, we denote the function $x \mapsto x^a \mathbf{1}_{(0,1]}(x)$ defined on $[0,1]$ by x^a. We consider the random variable $Z_\beta = \Phi_e(x^{\beta-1})$, see definition (3.16). The behavior of this random variable and its first moment are given in the following Lemma, whose short proof is given in Remark 3.18.

Lemma 3.5. *We have that a.s. for all* $1/2 \geq \beta > 0$, $Z_\beta = +\infty$. *We have that a.s. for all* $\beta > 1/2$, Z_β *is finite and*

$$\mathbb{E}\left[Z_\beta\right] = \frac{1}{2\sqrt{\alpha}} \, \frac{\Gamma\left(\beta - \frac{1}{2}\right)}{\Gamma(\beta)}. \tag{3.18}$$

We also have the representation formulas $Z_1 = \int_0^1 e(s)\, ds$ *and for* $\beta > 1$:

$$Z_\beta = \frac{1}{2}\,\beta(\beta-1) \int_{[0,1]^2} |t-s|^{\beta-2}\, m(s,t)\, ds\, dt. \tag{3.19}$$

We get the following convergence.

Corollary 3.6. *We have that a.s. for all* $\beta > 0$,

$$\lim_{n\to+\infty} |\mathrm{T}_n|^{-(\beta+\frac{1}{2})} \sum_{v\in \mathrm{T}_n} |\mathrm{T}_{n,v}|^\beta = \sqrt{2\alpha}\, Z_\beta.$$

Proof. Notice that $|\mathrm{T}_n|^{-(\beta+\frac{1}{2})} \sum_{v\in\mathrm{T}_n} |\mathrm{T}_{n,v}|^\beta = A_n(x^{\beta-1})$. For $\beta > 1/2$, the Corollary is then a direct consequence of Theorem 3.4 with $f = x^{\beta-1}$. We now consider the case $1/2 \geq \beta > 0$. Let $c > 0$. Using Theorem 3.4, we have that a.s.:

$$\liminf_{n\to+\infty} A_n(x^{\beta-1}) \geq \lim_{n\to+\infty} A_n(c \wedge x^{\beta-1}) = \sqrt{2\alpha}\, \Phi_e(c\wedge x^{\beta-1}).$$

Letting c goes to infinity, and using that, by Lemma 3.5, $\Phi_e(x^{\beta-1}) = Z_\beta = +\infty$ a.s., we get that a.s. $\liminf_{n\to+\infty} A_n(x^{\beta-1}) \geq \sqrt{2\alpha}\, Z_\beta = +\infty$. Then use a monotonicity argument in β to deduce the results holds a.s. for all $\beta \in (0, 1/2]$. □

Remark 3.7. All the moments of Z_β, for $\beta > 1/2$, are given in [89] (see Proposition 3.5 therein), thanks to the identification provided by Corollary 3.6. The representation formula (3.19) for Z_β is motivated by the formulation of our Corollary 3.6 given in [89] and [88].

Remark 3.8. Corollary 3.6 gives directly that $(|\mathrm{T}_n|^{-3/2} \sum_{v\in\mathrm{T}_n} |\mathrm{T}_{n,v}|, |\mathrm{T}_n|^{-5/2} \sum_{v\in\mathrm{T}_n} |\mathrm{T}_{n,v}|^2)$ is asymptotically distributed as $\sqrt{2\alpha}\,(Z_1, Z_2)$. Recall $\chi(\mathbf{t})$ defined in (3.6). According to Lemma 3 of [27] or [93], there exists a finite constant K such that, for all $n \geq 3$, we have $\mathbb{E}[\min(|\mathrm{T}_{n,1}|, |\mathrm{T}_{n,2}|)] \leq K|\mathrm{T}_n|^{1/2}$. Since conditionally on $\{v \in \mathrm{T}_n\}$ and $|\mathrm{T}_{n,v}|$, we have that $\mathrm{T}_{n,v}$ is uniformly distributed on the trees with $|\mathrm{T}_{n,v}|$ nodes, we deduce that $\mathbb{E}[\chi(\mathrm{T}_n)] \leq K\mathbb{E}[\mathcal{A}_{\mathrm{T}_n}(\sqrt{x})]$. According to Theorem 3.8 in [89], we have $\mathbb{E}[\mathcal{A}_{\mathrm{T}_n}(\sqrt{x})] = O(n\log(n))$ and thus $\mathbb{E}[\chi(\mathrm{T}_n)] = O(n\log(n))$. Noticing that $\chi(\mathrm{T}_n)$ is non-decreasing in n, using Borel-Cantelli lemma and arguments on convergence determining of measures [4](see proof of Theorem 3.4 in Section 3.6 for a detailed proof in the same spirit), we deduce that a.s. $\lim_{n\to+\infty} |\mathrm{T}_n|^{-3/2}\chi(\mathrm{T}_n) = 0$. Then, we can directly recover the joint asymptotic distribution of the total length path, the Wiener, Sackin, Colless and cophenetic indexes defined by (3.2) in Section 3.1.1 and (3.7) in Section 3.1.2 for the Catalan model. More precisely, we have:

$$\left(\frac{P(\mathrm{T}_n)}{|\mathrm{T}_n|^{3/2}}, \frac{W(\mathrm{T}_n)}{|\mathrm{T}_n|^{5/2}}, \frac{S(\mathrm{T}_n)}{|\mathrm{T}_n|^{3/2}}, \frac{C(\mathrm{T}_n)}{|\mathrm{T}_n|^{3/2}}, \frac{\mathrm{Co}(\mathrm{T}_n)}{|\mathrm{T}_n|^{5/2}}\right) \xrightarrow[n\to\infty]{\text{a.s.}} \sqrt{2\alpha}\left(Z_1, 2(Z_1 - Z_2), \frac{Z_1}{2}, \frac{Z_1}{2}, \frac{Z_2}{4}\right).$$

Remark 3.9. We complete Corollary 3.6 by considering the additive functionals $\tilde{F}$, see definition (3.5) used in [89], instead F defined by (3.4). For $\mathbf{t} \in \mathbb{T}$ and $|\mathbf{t}| > 1$, recall $\mathbf{t}^* = \mathbf{t} \setminus \mathcal{L}(\mathbf{t})$ is the tree $\mathbf{t}$ without its leaves. We have that a.s. for all $\beta > 0$,

$$\lim_{n\to+\infty} |\mathrm{T}_n^*|^{-(\beta+\frac{1}{2})} \sum_{v\in\mathrm{T}_n^*} |\mathrm{T}_{n,v}^*|^\beta = 2\sqrt{\alpha}\, Z_\beta. \tag{3.20}$$

[4]i.e. using the characterization of random measures.

This result differs from Corollary 3.6 as $\sqrt{2}$ is replaced by 2. To prove (3.20), first notice that for a full binary tree $|\mathbf{t}^*| = |\mathbf{t}| - |\mathcal{L}(\mathbf{t})| = (|\mathbf{t}| - 1)/2$ so that:

$$|\mathrm{T}_n^*|^{-(\beta+\frac{1}{2})} \sum_{v\in\mathrm{T}_n^*} |\mathrm{T}_{n,v}^*|^\beta = \sqrt{2}\,(|\mathrm{T}_n| - 1)^{-(\beta+\frac{1}{2})} \sum_{v\in\mathrm{T}_n} (|\mathrm{T}_{n,v}| - 1)^\beta.$$

Let $x_+ = \max(x, 0)$ denote the positive part of $x \in \mathbb{R}$. We have $x^\beta \geq (x-1)^\beta \geq x^\beta - c_\beta x^{(\beta-1)_+}$ for all $x \geq 1$ with $c_\beta = 1$ if $0 < \beta \leq 1$ and $c_\beta = \beta$ if $\beta \geq 1$. Then use Corollary 3.6 (two times) to deduce that a.s. for all $\beta > 0$:

$$\lim_{n\to+\infty} |\mathrm{T}_n^*|^{-(\beta+\frac{1}{2})} \sum_{v\in\mathrm{T}_n^*} |\mathrm{T}_{n,v}^*|^\beta = \sqrt{2} \lim_{n\to+\infty} |\mathrm{T}_n|^{-(\beta+\frac{1}{2})} \sum_{v\in\mathrm{T}_n} |\mathrm{T}_{n,v}|^\beta = 2\sqrt{\alpha}\, Z_\beta.$$

The next proposition, whose proof is given in Section 3.7, gives the fluctuations corresponding to the invariance principles of Theorem 3.4. Notice the speed of convergence in the invariance principle is of order $|\mathrm{T}_n|^{-1/4}$.

Proposition 3.10. *Let $f \in \mathcal{C}([0,1])$ be locally Lipschitz continuous on $(0,1]$ with $\| x^a f' \|_{esssup}$ finite for some $a \in (0,1)$. We have the following convergence in distribution:*

$$\left(|\mathrm{T}_n|^{1/4}(A_n - \sqrt{2\alpha}\,\Phi_e)(f),\, A_n\right) \xrightarrow[n\to\infty]{(d)} \left((2\alpha)^{1/4}\sqrt{\Phi_e(xf^2)}\; G,\, \sqrt{2\alpha}\; \Phi_e\right),$$

where G is a standard (centered reduced) Gaussian random variable independent of the excursion e.

Notice the fluctuations for the a.s. convergence towards Z_β with $\beta \geq 1$, given in Corollary 3.6, have an asymptotic variance (up to a multiplicative constant) given by $Z_{2\beta}$.

Remark 3.11. The contribution to the fluctuations is given by the error of approximation of $A_{n,1}(f)$ by $A_{n,2}(f)$, see notations from the proof of Theorem 3.4. This corresponds to the fluctuations coming from the approximation of the branch lengths $(h_{n,v}, v \in \mathrm{T}_n)$ by their mean, which relies on the explicit representation on their joint distribution given in Lemma 3.23. In particular, there is no other contribution to the fluctuations from the approximation of the continuum tree $\mathcal{T}$ by the sub-tree $\mathcal{T}_{[n]}$.

3.4 Simply generated trees model

The main result of this section is Corollary 3.15 in Section 3.4.3. The Sections 3.4.1 and 3.4.2 present the contour process of discrete trees and its convergence towards the contour process of a continuous random tree.

We keep notations from Section 3.2.5 on simply generated random tree. We assume the weight sequence $\mathfrak{p} = (\mathfrak{p}(k), k \in \mathbb{N})$ of non-negative real numbers such that $\sum_{k\in\mathbb{N}} \mathfrak{p}(k) > \mathfrak{p}(1) + \mathfrak{p}(0)$ and $\mathfrak{p}(0) > 0$ is generic. As stated in Section 3.2.5, without loss of generality, we will assume that $\mathfrak{p}$ is a critical probability distribution, that is:

$$\sum_{k\in\mathbb{N}} \mathfrak{p}(k) = \sum_{k\in\mathbb{N}} k\mathfrak{p}(k) = 1.$$

3.4.1 Contour process

Let $\mathbf{t} \in \mathbb{T}$ be a finite tree. The contour process $C^{\mathbf{t}} = (C^{\mathbf{t}}(s), s \in [0, 2|\mathbf{t}|])$ is defined as the distance to the root of a particle visiting continuously each edge of $\mathbf{t}$ at speed one (where all edges are of length 1) according to the lexicographic order of the nodes. More precisely, we set $\emptyset = u(0) < u(1) < \ldots < u(|\mathbf{t}| - 1)$ the nodes of $\mathbf{t}$ ranked in the lexicographic order. By convention, we set $u(|\mathbf{t}|) = \emptyset$.

We set $\ell_0 = 0$, $\ell_{|\mathbf{t}|+1} = 2$ and for $k \in \{1, \ldots, |\mathbf{t}|\}$, $\ell_k = d(u(k-1), u(k))$. We set $L_k = \sum_{i=0}^{k} \ell_i$ for $k \in \{0, \ldots, |\mathbf{t}|+1\}$, and $L'_k = L_k + d(u(k), \mathfrak{m}(u(k), u(k+1)))$ for $k \in \{0, \ldots, |\mathbf{t}|-1\}$. (Notice that $L'_k = L_k$ if and only if $u(k) \prec u(k+1)$.) We have $L_{|\mathbf{t}|} = 2|\mathbf{t}|-2$ and $L_{|\mathbf{t}|+1} = 2|\mathbf{t}|$. We define for $k \in \{0, \ldots, |\mathbf{t}|-1\}$:

- for $s \in [L_k, L'_k)$, the particle goes down from $u(k)$ to $\mathfrak{m}(u(k), u(k+1))$: $C^{\mathbf{t}}(s) = |u(k)| - (s - L_k)$;
- for $s \in [L'_k, L_{k+1})$, the particle goes up from $\mathfrak{m}(u(k), u(k+1))$ to $u(k+1)$: $C^{\mathbf{t}}(s) = |\mathfrak{m}(u(k), u(k+1))| + (s - L'_k)$,

and $C^{\mathbf{t}}(s) = 0$ for $s \in [2|\mathbf{t}|-2, 2|\mathbf{t}|]$. Notice that $C^{\mathbf{t}}$ is continuous.

For $u \in \mathbf{t}$, we define $\mathcal{I}_u$ the time interval during which the particle explores the edge attached below u. More precisely for $k \in \{1, \ldots, |\mathbf{t}|-1\}$, we set:

$$\mathcal{I}_{u(k)} = [L_k - 1, L_k) \bigcup [L''_k, L''_k + 1),$$

where $L''_k = \inf\{s \geq L_k,\, C^{\mathbf{t}}(s) < |u(k)|\}$ and $\mathcal{I}_\emptyset = [2|\mathbf{t}|-2, 2|\mathbf{t}|]$. The sets $(\mathcal{I}_u, u \in \mathbf{t})$ are disjoints 2 by 2 with $\bigcup_{u \in \mathbf{t}} \mathcal{I}_u = [0, 2|\mathbf{t}|]$. For $u \in \mathbf{t}$, we have that the Lebesgue measure of $\mathcal{I}_u$ is 2 and

$$C^{\mathbf{t}}(s) \leq d(\emptyset, u) \leq C^{\mathbf{t}}(s) + 1 \quad \text{for all } s \in \mathcal{I}_u. \tag{3.21}$$

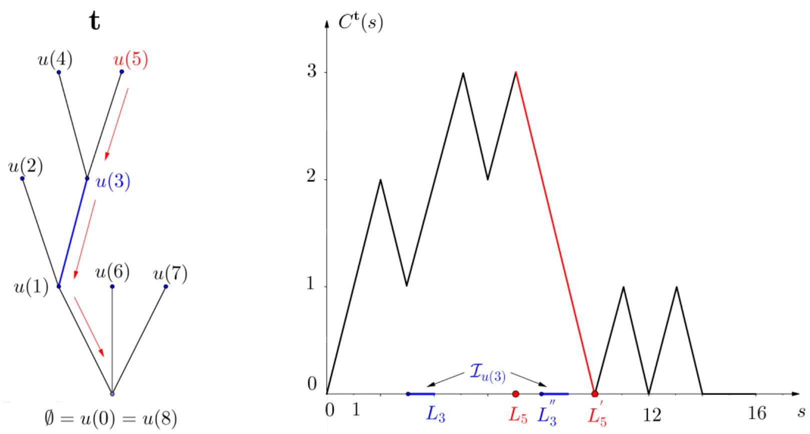


Figure 3.2 – A tree $\mathbf{t}$ with 8 nodes and its contour process $C^{\mathbf{t}}$: for $s \in [L_5, L'_5) = [7, 10)$, the particle goes down from $u(5)$ to $\mathfrak{m}(u(5), u(6)) = \emptyset$; $\mathcal{I}_{u(3)} = [L_3 - 1, L_3) \cup [L''_3, L''_3 + 1) = [4, 5) \cup [8, 9)$ is the time interval during which the particle explores the edge attached below $u(3)$.

3.4.2 Convergence of contour processes

We assume that $\mathfrak{p}$ is a probability distribution on $\mathbb{N}$ such that $1 > \mathfrak{p}(1) + \mathfrak{p}(0) \geq \mathfrak{p}(0) > 0$ and which is critical (that is $\sum_{k \in \mathbb{N}} k\mathfrak{p}(k) = 1$). We also assume that $\mathfrak{p}$ is in the domain of attraction of a stable distribution of Laplace exponent $\psi(\lambda) = \kappa \lambda^\gamma$ with $\gamma \in (1, 2]$ and $\kappa > 0$, and renormalizing sequence $(a_p, p \in \mathbb{N}^*)$ of positive reals: if $(U_k, k \in \mathbb{N}^*)$ are independent random variables with the same distribution $\mathfrak{p}$, and $W_p = \sum_{k=1}^{p} U_k - p$, then W_p/a_p converges in distribution, as p goes to infinity, towards a random variable X with Laplace exponent $-\psi$ (that is $\mathbb{E}[\mathrm{e}^{-\lambda X}] = \mathrm{e}^{\psi(\lambda)}$ for $\lambda \geq 0$). Notice this convergence implies that:

$$\lim_{p \to +\infty} \frac{a_p}{p} = 0. \tag{3.22}$$

Remark 3.12. If $\mathfrak{p}$ has finite variance, say σ^2, then one can take $a_p = \sqrt{p}$ and X is then a centered Gaussian random variable with variance σ^2, so that $\psi(\lambda) = \sigma^2\lambda^2/2$.

The main theorem in Duquesne [69] on the functional convergence in distribution of the contour process stated when $\mathfrak{p}$ is aperiodic, can easily be extended to the case $\mathfrak{p}$ periodic. (Indeed the lack of periodicity hypothesis is mainly used in Lemma 4.5 in [69] which is based on Gnedenko local limit theorem. Since the latter holds *a fortiori* for lattice distributions in the domain of attraction of stable law, it allows to extend the result to such periodic distribution, as soon as one uses sub-sequences on which the conditional probabilities are well defined.) It will be stated in this more general version, see Theorem 3.13 below. Since the contour process is continuous as well as its limit, the convergence in distribution holds on the space $\mathcal{C}([0,1])$ of real continuous functions endowed with the supremum norm.

Theorem 3.13. *Let $\mathfrak{p}$ be a critical probability distribution on $\mathbb{N}$, with $1 > \mathfrak{p}(1) + \mathfrak{p}(0) \geq \mathfrak{p}(0) > 0$, which belongs to the domain of attraction of a stable distribution of Laplace exponent $\psi(\lambda) = \kappa\lambda^\gamma$ with $\gamma \in (1,2]$ and $\kappa > 0$, and renormalizing sequence $(a_p, p \in \mathbb{N}^*)$. Let τ be a GW tree with offspring distribution $\mathfrak{p}$, and $\tau^{(p)}$ be distributed as τ conditionally on $\{|\tau| = p\}$. There exists a random non-negative continuous process $H = (H_s, s \in [0,1])$, such that the following convergence on the space $\mathcal{C}([0,1])$ holds in distribution:*

$$\frac{a_p}{p}\left(C^{\tau^{(p)}}(2ps), s \in [0,1]\right) \xrightarrow[p\to+\infty]{(d)} H,$$

where the convergence is taken along the infinite sub-sequence of p such that $\mathbb{P}(|\tau| = p) > 0$.

The process H, see [69] for a construction of H, is the so called normalized excursion for the height process, introduced in [136], of a Lévy tree with branching mechanism ψ.

Remark 3.14. If $\psi(\lambda) = \alpha\lambda^2$, for some $\alpha > 0$, then H is distributed as $\sqrt{2/\alpha}\, B$, where B is the positive Brownian excursion, see [70].

3.4.3 Main result

The next result is a direct consequence of [69] on the convergence of the contour process of random discrete tree, see Theorem 3.13 given in Section 3.4.2. We keep notations and definitions of Sections 3.4.1 and 3.4.2 below, with H the normalized excursion of the height function associated to the branching mechanism ψ. The proof of the next corollary is given in Section 3.4.4.

Corollary 3.15. *Let $\mathfrak{p}$ be a critical probability distribution on $\mathbb{N}$, with $1 > \mathfrak{p}(1) + \mathfrak{p}(0) \geq \mathfrak{p}(0) > 0$, which belongs to the domain of attraction of a stable distribution of Laplace exponent $\psi(\lambda) = \kappa\lambda^\gamma$ with $\gamma \in (1,2]$ and $\kappa > 0$, and renormalizing sequence $(a_p, p \in \mathbb{N}^*)$. Let τ be a GW tree with offspring distribution $\mathfrak{p}$, and $\tau^{(p)}$ be distributed as τ conditionally on $\{|\tau| = p\}$. We have the following convergence in distribution:*

$$\frac{a_p}{p^2}\mathcal{A}_{\tau^{(p)}} \xrightarrow[p\to+\infty]{(d)} \Phi_H,$$

where we endow the space of non-negative measures with the topology of the weak convergence and where the convergence is taken along the infinite sub-sequence of p such that $\mathbb{P}(|\tau| = p) > 0$.

We set for $\beta > 0$ and $\mathbf{t} \in \mathbb{T}$:

$$Z^*_\beta(\mathbf{t}) = \sum_{v\in\mathbf{t}} |\mathbf{t}_v|^\beta.$$

Using the Skorohod representation theorem, we deduce the following result.

Corollary 3.16. *Assume hypothesis of Corollary 3.15 hold and let Z^H_β be given by (3.15) for $\beta \geq 1$. There exist continuous functions defined on $[1, \infty)$, Θ_p distributed as $\left(\frac{a_p}{p^{\beta+1}} Z^*_\beta(\tau^{(p)}), \, \beta \geq 1\right)$ and Θ distributed as $\left(Z^H_\beta, \, \beta \geq 1\right)$ such that*

$$\Theta_p \xrightarrow[p \to +\infty]{(p.s.)} \Theta$$

for the simple convergence of functions and where the convergence is taken along the infinite sub-sequence of p such that $\mathbb{P}(|\tau| = p) > 0$.

The technical proof of the first part of the next Lemma is given in Section 3.8.6. The second part, which is the representation formula, is a direct consequence of the deterministic Lemma 3.39 in Section 3.8.5 (with $\beta = a + 1$).

Lemma 3.17. *Assume the height function H is associated to the Laplace exponent $\psi(\lambda) = \kappa\lambda^\gamma$ with $\gamma \in (1, 2]$ and $\kappa > 0$. We have that a.s. for all $1/\gamma \geq \beta > 0$, $Z^H_\beta = +\infty$, that a.s. for all $\beta > 1/\gamma$, Z^H_β is finite and*

$$\mathbb{E}\left[Z^H_\beta\right] = \frac{1}{\gamma\kappa^{1/\gamma}} \frac{\Gamma\left(\beta - \frac{1}{\gamma}\right)}{\Gamma\left(\beta + 1 - \frac{2}{\gamma}\right)}. \tag{3.23}$$

We also have the representation formulas $Z^H_1 = \int_0^1 H(s)\, ds$ and, for $\beta > 1$, $Z^H_\beta = \frac{1}{2}\beta(\beta - 1) \int_{[0,1]^2} |t - s|^{\beta - 2}\, m_H(s, t)\, ds\, dt$.

Remark 3.18. Lemma 3.5 given in Section 3.3 is a consequence of Lemma 3.17 applied with $H = e$, $\gamma = 2$ and $\kappa = \alpha$.

Remark 3.19. For $\beta \in (0, 1/\gamma]$, we deduce from Corollary 3.15 and Lemma 3.17, using the same arguments as in the proof of Corollary 3.6, the convergence in distribution of the sequence $\left(\frac{a_p}{p^{\beta+1}} Z^*_\beta(\tau^{(p)}), \, p \in \mathbb{N}^* \text{ s.t. } \mathbb{P}(|\tau| = p) > 0\right)$ towards infinity. So the normalization is not relevant to get a proper limit, suggesting we have a "local" regime. The convergence in distribution of this sequence for $\beta \in (1/\gamma, 1)$ towards Z^H_β (which is a.s. finite) is an open question, but we conjecture it holds. This conjecture and Corollary 3.16 would then give that for simply generated trees, under the hypothesis of Corollary 3.15, there is a phase transition at $\beta = 1/\gamma$ between a "global" regime ($\beta > 1/\gamma$) and a "local" regime ($\beta \leq 1/\gamma$).

Remark 3.20. If $\mathfrak{p}$ has finite variance, say σ^2, then one can take $a_p = \sqrt{p}$ in Corollaries 3.15 and 3.16 and H is equal to $(2/\sigma)B$ which corresponds to $\psi(\lambda) = \sigma^2\lambda^2/2$, see Remarks 3.12 and 3.14. By scaling, or using that the limit in Theorem 3.4 does not depend on α, we deduce that in this case $\Phi_H = \frac{2}{\sigma}\Phi_B$ and $Z^H_\beta = \frac{2}{\sigma} Z^B_\beta$ in Corollaries 3.15 and 3.16.

3.4.4 Proof of Corollary 3.15

Elementary functionals of finite trees

Let $\mathbf{t} \in \mathbb{T}$ be a finite tree and $k \in \mathbb{N}^*$. For $\mathbf{u} = (u_1, \ldots, u_k) \in \mathbf{t}^k$, we define $\mathfrak{m}(\mathbf{u}) = \mathfrak{m}(\{u_1, \ldots, u_k\})$ the most recent common ancestor of $u_1, \ldots, u_k$. We consider the following elementary functional of a tree, defined for $\mathbf{t} \in \mathbb{T}$:

$$D_k(\mathbf{t}) = \sum_{\mathbf{u} \in \mathbf{t}^k} d(\emptyset, \mathfrak{m}(\mathbf{u})). \tag{3.24}$$

We have:

$$\sum_{v\in\mathbf{t}} |\mathbf{t}_v|^k = D_k(\mathbf{t}) + |\mathbf{t}|^k, \tag{3.25}$$

which we obtain from the following equalities

$$\sum_{v\in\mathbf{t}} |\mathbf{t}_v|^k = \sum_{v\in\mathbf{t}} \sum_{\mathbf{u}\in\mathbf{t}^k} \mathbf{1}_{\{v\preccurlyeq\mathfrak{m}(\mathbf{u})\}} = \sum_{\mathbf{u}\in\mathbf{t}^k} \sum_{v\in\mathbf{t}} \mathbf{1}_{\{v\preccurlyeq\mathfrak{m}(\mathbf{u})\}} = \sum_{\mathbf{u}\in\mathbf{t}^k} (d(\emptyset,\mathfrak{m}(\mathbf{u}))+1).$$

For $x=(x_1,\dots,x_k)\in\mathbb{R}^k$, denote by $(x_{(1)},\dots,x_{(k)})$ its order statistics which are uniquely defined by $x_{(1)}\leq\cdots\leq x_{(n)}$ and $\sum_{i=1}^k \delta_{x_i} = \sum_{i=1}^k \delta_{x_{(i)}}$, with δ_z the Dirac mass at z. Recall the notation $m_h(s,t)$, see (3.13), for the minimum of h over the interval with bounds s and t. We set:

$$\mathcal{D}_k(\mathbf{t}) = \int_{[0,|\mathbf{t}|]^k} m_{C^{\mathbf{t}}}(2x_{(1)}, 2x_{(k)})\, dx, \tag{3.26}$$

with the conventions that if $k=1$, then $\mathcal{D}_1(\mathbf{t}) = \int_{[0,|\mathbf{t}|]} C^{\mathbf{t}}(2x)\,dx$.

We have the following lemma.

Lemma 3.21. *We have for* $\mathbf{t}\in\mathbb{T}$ *and* $k\in\mathbb{N}^*$*:*

$$0 \leq D_k(\mathbf{t}) - \mathcal{D}_k(\mathbf{t}) \leq |\mathbf{t}|^k. \tag{3.27}$$

Proof. For $\mathbf{u}=(u_1,\dots,u_k)\in\mathbf{t}^k$, we have the following generalization of (3.21): for $x=(x_1,\dots,x_k)\in\prod_{i=1}^k \mathcal{I}_{u_i}$,

$$m_{C^{\mathbf{t}}}(x_{(1)},x_{(k)}) \leq d(\emptyset,\mathfrak{m}(\mathbf{u})) \leq m_{C^{\mathbf{t}}}(x_{(1)},x_{(k)}) + 1.$$

(Notice that $m_{C^{\mathbf{t}}}(x_{(1)},x_{(k)}) = d(\emptyset,\mathfrak{m}(\mathbf{u}))$ as soon as $\mathfrak{m}(\mathbf{u})\prec u_i$ for all $i\in\{1,\dots,k\}$.) We deduce that:

$$0 \leq 2^k d(\emptyset,\mathfrak{m}(\mathbf{u})) - \int_{\prod_{i=1}^k \mathcal{I}_{u_i}} m_{C^{\mathbf{t}}}(x_{(1)},x_{(k)})\,dx \leq 2^k.$$

By summing over $\mathbf{u}\in\mathbf{t}^k$, we get:

$$0 \leq 2^k D_k(\mathbf{t}) - \int_{[0,2|\mathbf{t}|]^k} m_{C^{\mathbf{t}}}(x_{(1)},x_{(k)})\,dx \leq 2^k|\mathbf{t}|^k.$$

Use the change of variable $2y=x$ to get (3.27). □

Convergence of additive functionals

We now give the main result of this Section.

Corollary 3.22. *Under the hypothesis and notations of Theorem 3.13, we have the following convergences in distribution for all* $k\in\mathbb{N}^*$*:*

$$\lim_{p\to+\infty} \frac{a_p}{p^{k+1}} \sum_{v\in\tau^{(p)}} |\tau_v^{(p)}|^k \overset{(d)}{=} \lim_{p\to+\infty} \frac{a_p}{p^{k+1}} \sum_{\mathbf{u}\in(\tau^{(p)})^k} d(\emptyset,\mathfrak{m}(\mathbf{u})) \overset{(d)}{=} \int_0^1 ds \int_0^{H(s)} dr\, \sigma_{r,s}(H)^{k-1},$$

where $\sigma_{r,s}(H)$ *is the length of the excursion of the height process* H *above* r *straddling* s *defined in (3.14) and where the convergence is taken along the infinite sub-sequence of* p *such that* $\mathbb{P}(|\tau|=p)>0$.

Proof. Recall notation $m_h(s,t)$ and $\sigma_{r,s}(h)$ given in (3.13) and (3.14). We shall take limits along the infinite sub-sequence of p such that $\mathbb{P}(|\tau| = p) > 0$.

Recall definitions (3.24) of $D_k(\mathbf{t})$ and (3.26) of $\mathcal{D}_k(\mathbf{t})$. Thanks to Lemma 3.21 and (3.22) which implies that $(p^{-(k+1)}a_p(D_k(\tau^{(p)}) - \mathcal{D}_k(\tau^{(p)})), p \in \mathbb{N}^*)$ converges in probability towards 0 and to (3.25), we see the proof of the corollary is complete as soon as we obtain that for all $k \in \mathbb{N}^*$:

$$\lim_{p\to+\infty} \frac{a_p}{p^{k+1}} \mathcal{D}_k(\tau^{(p)}) \overset{(d)}{=} \int_0^1 ds \int_0^{H(s)} dr\, \sigma_{r,s}(H)^{k-1}. \tag{3.28}$$

We deduce from Theorem 3.13 the following convergence in law:

$$\frac{a_p}{p^2}\mathcal{D}_1(\tau^{(p)}) = \frac{a_p}{p^2} \int_{[0,p]} C^{\tau^{(p)}}(2x)\, dx = \int_{[0,1]} \frac{a_p}{p} C^{\tau^{(p)}}(2ps)\, ds \xrightarrow[p\to+\infty]{(d)} \int_{[0,1]} ds\, H(s).$$

This gives (3.28) for $k = 1$. We have that for $k \geq 2$ and $\mathbf{t} \in \mathbb{T}$:

$$\begin{aligned} 2\mathcal{D}_k(\mathbf{t}) &= 2\int_{[0,|\mathbf{t}|]^k} m_{C^{\mathbf{t}}}(2x_{(1)}, 2x_{(k)})\, dx \\ &= k(k-1) \int_{[0,|\mathbf{t}|]^2} dx_1 dx_2\, |x_2 - x_1|^{k-2}\, m_{C^{\mathbf{t}}(2\bullet)}(x_1, x_2) \\ &= k(k-1)|\mathbf{t}|^k \int_{[0,1]^2} dx_1 dx_2\, |x_2 - x_1|^{k-2}\, m_{C^{\mathbf{t}}(2|\mathbf{t}|\bullet)}(x_1, x_2), \end{aligned}$$

where we used (3.26) in the first equality, we choose $x_{(1)}$ and $x_{(k)}$ among $x_1, \ldots, x_k$ for the second one and we used the change of variable x_i to $|\mathbf{t}|x_i$ for the last one. We deduce from Theorem 3.13 the following convergence in law for all $k \in \mathbb{N}^*$ such that $k \geq 2$:

$$\frac{a_p}{p^{k+1}}\mathcal{D}_k(\tau^{(p)}) \xrightarrow[p\to+\infty]{(d)} \frac{k(k-1)}{2} \int_{[0,1]^2} ds ds'\, |s' - s|^{k-2}\, m_H(s, s').$$

Then use (3.57) from Lemma 3.39 to get (3.28). This ends the proof. □

Conclusion

We deduce from the proof of Corollary 3.22, using the Skorohod representation theorem, that all the convergences in distribution of Corollary 3.22 hold simultaneously for all $k \in \mathbb{N}^*$. We thus get that $\lim_{n\to+\infty} \frac{a_p}{p^2}\mathcal{A}_{\tau^{(p)}}(x^{k-1}) \overset{(d)}{=} \Phi_H(x^{k-1})$, simultaneously for all $k \in \mathbb{N}^*$. Since on $[0,1]$, the convergence of moments implies the weak convergence of finite measures, we deduce that the random measure $\frac{a_p}{p^2}\mathcal{A}_{\tau^{(p)}}$ converges in distribution towards Φ_H for the topology of weak convergence of finite measures on $[0,1]$.

3.5 Preliminary Lemmas

Recall $\mathcal{T}$ is the real tree coded by the excursion e, see Section 3.2.3 and $\mathcal{T}_{[n]}$ is the (smallest) sub-tree of $\mathcal{T}_e$ containing $n+1$ leaves picked uniformly at random and the root, see Section 3.2.4. Recall $(\mathrm{T}_n, (h_{n,v}, v \in \mathrm{T}_n))$ denote the corresponding marked tree. Intuitively, for $v \in \mathrm{T}_n$, $h_{n,v}$ is the length of the branch below the branching point with label v in $\mathcal{T}_{[n]}$ (when keeping the order on the leaves). We recall, see [10], [163] (Theorem 7.9) or [70], that the density of $(h_{n,v}, v \in \mathrm{T}_n)$ is, conditionally on T_n, given by:

$$f_n((h_{n,v}, v \in \mathrm{T}_n)) = 2\frac{(2n)!}{n!}\, \alpha^{n+1}\, L_n\, \mathrm{e}^{-\alpha L_n^2} \prod_{v\in\mathrm{T}_n} \mathbf{1}_{\{h_{n,v}>0\}}, \tag{3.29}$$

where $L_n = \sum_{v \in \mathrm{T}_n} h_{n,v}$ denotes the total length of $\mathcal{T}_{[n]}$. Notice that the edge-lengths have an exchangeable distribution and are independent of the shape tree T_n. Furthermore, elementary computations give that $(h_{n,v}, v \in \mathrm{T}_n)$, with $v \in \mathrm{T}_n$ ranked in the lexicographic order, has, conditionally on T_n and L_n, the same distribution as $(L_n\Delta_1, \dots, L_n\Delta_{2n+1})$, where $\Delta_1, \dots, \Delta_{2n+1}$ represents the lengths of the $2n+1$ intervals obtained by cutting $[0,1]$ at $2n$ independent uniform random variables on $[0,1]$ and independent of L_n. We thus deduce the following elementary Lemma.

Lemma 3.23. *Conditionally on $\mathrm{T}_n = \mathbf{t}$, the random vector $(h_{n,v}, v \in \mathbf{t})$ has the same distribution as $(L_n E_v / S_{\mathbf{t}}, v \in \mathbf{t})$, where $(E_u, u \in \mathcal{U})$ are independent exponential random variables with mean 1, independent of T_n and L_n, and $S_{\mathbf{t}} = \sum_{v \in \mathbf{t}} E_v$.*

According to [2], we have that a.s. $\lim_{n\to+\infty} L_n/\sqrt{n} = 1/\sqrt{\alpha}$. We then deduce from Lemma 3.23 that $(2n+1)\sqrt{\alpha}\, h_{n,\emptyset}/\sqrt{n}$ converges in distribution towards $E_\emptyset$ as n goes to infinity. Intuitively, we get that $2\sqrt{\alpha n}\, \mathbb{E}[h_{n,v}]$ is of order 1, for $v \in \mathrm{T}_n$. Recall the random measure A_n is defined in (3.17). We introduce the random measure:

$$A_{1,n} = 2\sqrt{\alpha n}\, \mathbb{E}[h_{n,\emptyset}] A_n.$$

Lemma 3.24. *Let $a \in [0, 1/2)$. There exists a finite constant C such that for all $f \in \mathcal{B}([0,1])$ and $n \in \mathbb{N}^*$, we have:*

$$\mathbb{E}\left[|A_n(f) - A_{1,n}(f)|\right] \le C \|x^a f\|_\infty\, n^{-1}.$$

Proof. Let $a \in [0, 1/2)$ and $f \in \mathcal{B}([0,1])$. Using (3.55) in the Appendix, we deduce that for all $n \in \mathbb{N}^*$, we have $\left|1 - 2\sqrt{\alpha n}\, \mathbb{E}[h_{n,\emptyset}]\right| \le 1/2n$. Using (3.44) in Lemma 3.34, we deduce that:

$$\mathbb{E}[|A_n(f) - A_{1,n}(f)|] \le \frac{1}{2n} \mathbb{E}[|A_n(f)|] \le \frac{C_{1,1-a}}{2n} \|x^a f\|_\infty.$$

□

Intuitively, $h_{n,v}$ is of the same order of its expectation. Since the random variables $(h_{n,v}, v \in \mathrm{T}_n)$ are exchangeable, we deduce that $h_{n,v}$ is of the same order as $\mathbb{E}[h_{n,\emptyset}]$. Based on this intuition, we define the random measure $A_{2,n}$ as follows. For $f \in \mathcal{B}([0,1])$, we set:

$$A_{2,n}(f) = 2\sqrt{\alpha n}\, |\mathrm{T}_n|^{-3/2} \sum_{v \in \mathrm{T}_n} |\mathrm{T}_{n,v}| f\left(\frac{|\mathrm{T}_{n,v}|}{|\mathrm{T}_n|}\right) h_{n,v}.$$

Lemma 3.25. *Let $a \in [0, 1/2)$. There exists a finite constant C such that for all $f \in \mathcal{B}([0,1])$ and $n \in \mathbb{N}^*$, we have:*

$$\mathbb{E}[|A_{1,n}(f) - A_{2,n}(f)|] \le C \|x^a f\|_\infty\, n^{-1/4}.$$

Proof. Let $a \in [0, 1/2)$ and $f \in \mathcal{B}([0,1])$. For $v \in \mathrm{T}_n$, we set $Y_{n,v} = \sqrt{n}(\mathbb{E}[h_{n,v}] - h_{n,v})$ and

$$K_n = \frac{1}{2\sqrt{\alpha}}(A_{1,n}(f) - A_{2,n}(f)) = |\mathrm{T}_n|^{-3/2} \sum_{v \in \mathrm{T}_n} |\mathrm{T}_{n,v}| f\left(\frac{|\mathrm{T}_{n,v}|}{|\mathrm{T}_n|}\right) Y_{n,v}.$$

Using that $(h_{n,v}, v \in \mathrm{T}_n)$ is exchangeable, elementary computations give:

$$\mathbb{E}\left[K_n^2 | \mathrm{T}_n\right] \le |\mathrm{T}_n|^{-1/2} A_n(xf^2) \mathbb{E}[Y_{n,\emptyset}^2] + A_n(|f|)^2 |\mathbb{E}[Y_{n,\emptyset} Y_{n,1}]|.$$

Then using (3.44) and (3.45) in Lemma 3.34 and (3.56) in Lemma 3.38, we get:

$$\mathbb{E}[K_n^2] = \mathbb{E}\left[\mathbb{E}[K_n^2 | \mathrm{T}_n]\right] \le \frac{C_{1,1}}{2\alpha\sqrt{2n+1}} \|x^{1/2} f\|_\infty^2 + \frac{C_{2,1-a}^2}{8\alpha n} \|x^a f\|_\infty^2 \le \frac{c}{\sqrt{n}} \|x^a f\|_\infty^2,$$

for some finite constant c which does not depend on n and f. □

Let $\mathcal{L}_{n,v} = \{u \in \mathrm{T}_n;\, v \preccurlyeq u,\, k_u(\mathrm{T}_n) = 0\}$ be the set of leaves of T_n with ancestor v, and $|\mathcal{L}_{n,v}|$ be its cardinality. Notice the number of leaves of $\mathrm{T}_{n,v}$ is exactly $|\mathcal{L}_{n,v}|$. We now approximate the multiplying factor $|\mathrm{T}_{n,v}|$ in $A_{2,n}$ by twice the number of leaves in $\mathrm{T}_{n,v}$ as $2|\mathcal{L}_{n,v}| = |\mathrm{T}_{n,v}| + 1$. For this reason, we set for $f \in \mathcal{B}([0,1])$:

$$A_{3,n}(f) = 4\sqrt{\alpha n}\; |\mathrm{T}_n|^{-3/2} \sum_{v\in \mathrm{T}_n} |\mathcal{L}_{n,v}|\, f\left(\frac{|\mathrm{T}_{n,v}|}{|\mathrm{T}_n|}\right) h_{n,v}.$$

Lemma 3.26. *Let $a \in [0,1/2)$. For all $f \in \mathcal{B}([0,1])$ and $n \in \mathbb{N}^*$, we have:*

$$\mathbb{E}[|A_{2,n}(f) - A_{3,n}(f)|] \leq \|x^a f\|_\infty\, n^{a-\frac{1}{2}}.$$

Proof. Let $a \in [0,1/2)$ and $f \in \mathcal{B}([0,1])$. Since $2|\mathcal{L}_{n,v}| = |\mathrm{T}_{n,v}| + 1$, we get that:

$$|A_{2,n}(f) - A_{3,n}(f)| \leq 2\sqrt{\alpha n}\, |\mathrm{T}_n|^{-3/2} \sum_{v\in \mathrm{T}_n} |f| \left(\frac{|\mathrm{T}_{n,v}|}{|\mathrm{T}_n|}\right) h_{n,v}.$$

As $|\mathrm{T}_{n,v}| \geq 1$ and $a \geq 0$, we get that $|f|\left(\frac{|\mathrm{T}_{n,v}|}{|\mathrm{T}_n|}\right) \leq \|\, x^a f\,\|_\infty\, |\mathrm{T}_n|^a$. We deduce that:

$$|A_{2,n}(f) - A_{3,n}(f)| \leq 2\sqrt{\alpha n}\, L_n |\mathrm{T}_n|^{a-\frac{3}{2}}\, \|\, x^a f\,\|_\infty\,.$$

According to (3.54), we have $2\sqrt{\alpha n}\,\mathbb{E}[L_n] \leq |\mathrm{T}_n|$. We deduce that $\mathbb{E}[|A_{2,n}(f) - A_{3,n}(f)|] \leq |\mathrm{T}_n|^{a-\frac{1}{2}}\, \|\, x^a f\,\|_\infty$. □

We define $\mathcal{N}_{n,r,U_k}$ as the number of leaves of the sub-tree $\mathcal{T}_{[n]}$ which are distinct from $\mathbf{p}(U_k)$ and such that their most recent common ancestor with $\mathbf{p}(U_k)$ is at distance further than r from the root. More precisely, using the definition (3.13) of m, we have:

$$\mathcal{N}_{n,r,U_k} + 1 = \mathrm{Card}\,\{i \in \{1,\dots,n+1\},\, m(U_i,U_k) \geq r\}.$$

In particular, we deduce from the construction of $\mathcal{T}_{[n]}$ and T_n that for $1 \leq k \leq n+1$:

$$\sum_{v\preccurlyeq u(U_k)} f\left(\frac{|\mathrm{T}_{n,v}|}{|\mathrm{T}_n|}\right) h_{n,v} = \int_0^{e(U_k)} dr\, f\left(\frac{2\mathcal{N}_{n,r,U_k}+1}{2n+1}\right), \tag{3.30}$$

where $u(U_k)$ is the leaf in T_n corresponding to the leaf $\mathbf{p}(U_k)$ in $\mathcal{T}_{[n]}$.

Recall that, for $v \in \mathrm{T}_n$, $\mathcal{L}_{n,v}$ denotes the set of leaves of T_n with ancestor v and $\mathcal{L}(\mathrm{T}_n) = \mathcal{L}_{n,\emptyset}$ denotes the set of leaves of T_n. We deduce that:

$$\begin{aligned} A_{3,n}(f) &= 4\sqrt{\alpha n}\; |\mathrm{T}_n|^{-3/2} \sum_{v\in \mathrm{T}_n} |\mathcal{L}_{n,v}|\, f\left(\frac{|\mathrm{T}_{n,v}|}{|\mathrm{T}_n|}\right) h_{n,v} \\ &= 4\sqrt{\alpha n}\; |\mathrm{T}_n|^{-3/2} \sum_{u\in \mathcal{L}(\mathrm{T}_n)} \sum_{v\preccurlyeq u} f\left(\frac{|\mathrm{T}_{n,v}|}{|\mathrm{T}_n|}\right) h_{n,v} \\ &= 4\sqrt{\alpha n}\; |\mathrm{T}_n|^{-3/2} \sum_{k=1}^{n+1} \int_0^{e(U_k)} dr\, f\left(\frac{2\mathcal{N}_{n,r,U_k}+1}{2n+1}\right), \end{aligned}$$

where we used (3.30) for the last equality. Notice that by construction, conditionally on e and U_k, the random variable $\mathcal{N}_{n,r,U_k}$ is binomial with parameter (n, σ_{r,U_k}). For this reason, we consider the following approximation of $A_{3,n}(f)$. For $f \in \mathcal{B}([0,1])$ non-negative, we set:

$$A_{4,n}(f) = 4\sqrt{\alpha n}\; |\mathrm{T}_n|^{-3/2} \sum_{k=1}^{n+1} \int_0^{e(U_k)} dr\, f(\sigma_{r,U_k}).$$

Lemma 3.27. *We have the following properties.*

(i) For $a \in (0,1)$, there exists a finite constant $C(a)$ such that if $f \in \mathcal{B}([0,1])$ is locally Lipschitz continuous on $(0,1]$, we have for all $n \in \mathbb{N}^*$:

$$\mathbb{E}[|A_{3,n}(f) - A_{4,n}(f)|] \leq C(a) \, \| x^a f' \|_{esssup} \, n^{-1/2}.$$

(ii) If $a \in (-1/2, 0]$, there exists a finite constant $C(a)$ such that we have for all $n \in \mathbb{N}^*$:

$$\mathbb{E}[|A_{3,n}(x^a) - A_{4,n}(x^a)|] \leq C(a) \, n^{-(2a+1)/8}.$$

Remark 3.28. We can extend (i) of Lemma 3.27 to get that for a uniformly Hölder continuous function f with exponent $\lambda > 1/2$, we have $\mathbb{E}[|A_{3,n}(f) - A_{4,n}(f)|] = O(n^{-\lambda/2})$. This allows the extension of Proposition 3.10 to such functions.

Proof. For $s \in [0,1]$, let $\mathcal{N}_{n,r,s}$ be a random variable which is, conditionally on e, binomial with parameter $(n, \sigma_{r,s})$. Notice, this is consistent with the definition of $\mathcal{N}_{n,r,U_k}$. Hence we get, for $f \in \mathcal{B}([0,1])$,

$$\begin{aligned} \mathbb{E}\left[|A_{3,n}(f) - A_{4,n}(f)|\right] &\leq 4\sqrt{\alpha n} |\mathrm{T}_n|^{-\frac{3}{2}} \sum_{k=1}^{n+1} \mathbb{E}\left[\int_0^{e(U_k)} \left| f\left(\frac{2\mathcal{N}_{n,r,U_k}+1}{2n+1}\right) - f(\sigma_{r,U_k}) \right| dr \right] \\ &\leq 4\sqrt{\alpha} \int_0^1 ds \, \mathbb{E}\left[\int_0^{e(s)} dr \, \mathbb{E}\left[\left| f\left(\frac{2\mathcal{N}_{n,r,s}+1}{2n+1}\right) - f(\sigma_{r,s}) \right| \, \Big| \, e\right] dr\right]. \end{aligned} \tag{3.31}$$

We first prove property (i). Let $a \in (0,1)$ and $f \in \mathcal{B}([0,1])$ be locally Lipschitz continuous on $(0,1]$. Using (ii) of Lemma 3.35, we have that for $s \in (0,1)$ and $r \in (0, e(s))$,

$$\mathbb{E}\left[\left| f\left(\frac{2\mathcal{N}_{n,r,s}+1}{2n+1}\right) - f(\sigma_{r,s}) \right| \, \Big| \, e\right] \leq \frac{\| x^a f' \|_{\text{esssup}}}{1-a} \left(\sigma_{r,s}^{-\frac{a}{2}} + \sigma_{r,s}^{\frac{1}{2}-a}\right) n^{-1/2}. \tag{3.32}$$

We recall that $Z_\beta = \int_0^1 ds \int_0^{e(s)} dr \, \sigma_{r,s}^{\beta-1}$ for $\beta > 0$. Thus, we have $\mathbb{E}\left[Z_{\frac{3}{2}-a}\right] \leq \mathbb{E}\left[Z_{1-\frac{a}{2}}\right]$; the last term being finite thanks to Lemma 3.5. We deduce from (3.31) and (3.32) that

$$\mathbb{E}\left[|A_{3,n}(f) - A_{4,n}(f)|\right] \leq 8\sqrt{\alpha} \frac{\| x^a f' \|_{\text{esssup}}}{1-a} \mathbb{E}\left[Z_{1-\frac{a}{2}}\right] n^{-1/2}.$$

This achieves the proof of property (i).

We now prove property (ii). We consider $a \in (-1/2, 0)$ and $f(x) = x^a$, as the case $a = 0$ is obvious. Let $\gamma > 0$. We write:

$$\int_0^1 ds \, \mathbb{E}\left[\int_0^{e(s)} dr \, \mathbb{E}\left[\left| \left(\frac{2\mathcal{N}_{n,r,s}+1}{2n+1}\right)^a - \sigma_{r,s}^a \right| \, \Big| \, e\right]\right] = \kappa_1 + \kappa_2 + \kappa_3,$$

with $\kappa_i = \int_0^1 ds \, \mathbb{E}\left[\int_0^{e(s)} dr \, \mathbb{E}\left[\mathbf{1}_{D_i} \left| \left(\frac{2\mathcal{N}_{n,r,s}+1}{2n+1}\right)^a - \sigma_{r,s}^a \right| \, \Big| \, e\right]\right]$ and:

$$D_1 = \left\{\sigma_{r,s} > 2n^{-\gamma}, \frac{2\mathcal{N}_{n,r,s}+1}{2n+1} > n^{-\gamma}\right\}, \quad D_2 = \left\{\sigma_{r,s} > 2n^{-\gamma}, \frac{2\mathcal{N}_{n,r,s}+1}{2n+1} \leq n^{-\gamma}\right\},$$

and $D_3 = (D_1 \bigcup D_2)^c$. We start considering κ_1. Notice that, thanks to (3.49) with $b = 1 + a$, we have $|x^a - y^a| \leq x^a y^{-1} |x - y| \leq n^{\gamma(1-a)} |x - y|$ if $x, y \in [n^{-\gamma}, +\infty)$. Using this inequality with $x = \frac{2\mathcal{N}_{n,r,s}+1}{2n+1}$ and $y = \sigma_{r,s}$, we obtain:

$$\kappa_1 \leq n^{\gamma(1-a)} \int_0^1 ds \, \mathbb{E}\left[\int_0^{e(s)} \mathbb{E}\left[\left|\frac{2\mathcal{N}_{n,r,s}+1}{2n+1} - \sigma_{r,s}\right| \,\Big|\, e\right] dr\right]. \tag{3.33}$$

Moreover, if X is a binomial random variable with parameter (n, p), then we have:

$$\mathbb{E}\left[\left|\frac{2X+1}{2n+1} - p\right|\right]^2 \leq \mathbb{E}\left[\left(\frac{2X+1}{2n+1} - p\right)^2\right] \leq \frac{1}{2n+1} \leq \frac{1}{n}.$$

With $X = \mathcal{N}_{n,r,s}$ and $p = \sigma_{r,s}$, we get that $\mathbb{E}\left[\left|\frac{2\mathcal{N}_{n,r,s}+1}{2n+1} - \sigma_{r,s}\right| \,\big|\, e\right] \leq \frac{1}{\sqrt{n}}$. We deduce from (3.33) that:

$$\kappa_1 \leq \mathbb{E}\left[\int_0^1 e(s)\, ds\right] n^{\gamma(1-a)-\frac{1}{2}}. \tag{3.34}$$

We give an upper bound of κ_2. We first recall Hoeffding's inequality: if X is a binomial random variable with parameter (n, p), and $t > 0$, then we have $\mathbb{P}(np - X > nt) \leq \exp(-2nt^2)$. Using that $\{p - \frac{2X+1}{2n+1} > n^{-\gamma}\} \subset \{np - X > n^{1-\gamma}\}$, we deduce that:

$$\mathbb{P}\left(p - \frac{2X+1}{2n+1} > n^{-\gamma}\right) \leq \mathbb{P}\left(np - X > n^{1-\gamma}\right) \leq \exp\left(-2n^{1-2\gamma}\right). \tag{3.35}$$

Notice that on D_2, we have $0 \leq \left(\frac{2\mathcal{N}_{n,r,s}+1}{2n+1}\right)^a - \sigma_{r,s}^a \leq \left(\frac{2\mathcal{N}_{n,r,s}+1}{2n+1}\right)^a \leq (2n+1)^{-\gamma a}$ as well as $\sigma_{r,s} - \frac{2\mathcal{N}_{n,r,s}+1}{2n+1} > n^{-\gamma}$. Hence, we obtain:

$$\begin{aligned}\kappa_2 &\leq (2n+1)^{-\gamma a} \int_0^1 ds \, \mathbb{E}\left[\int_0^{e(s)} \mathbb{P}\left(\sigma_{r,s} - \frac{2\mathcal{N}_{n,r,s}+1}{2n+1} > n^{-\gamma} \Big| e\right) dr\right] \\ &\leq \mathbb{E}[Z_1]\,(2n+1)^{-\gamma a}\, \mathrm{e}^{-2n^{1-2\gamma}}.\end{aligned} \tag{3.36}$$

Finally, we consider κ_3. Let $\eta \in (0, a + 1/2)$. We have:

$$\begin{aligned}&\mathbb{E}\left[\int_0^{e(s)} \mathbf{1}_{\{\sigma_{r,s} \leq 2n^{-\gamma}\}} \mathbb{E}\left[\left|\left(\frac{2\mathcal{N}_{n,r,s}+1}{2n+1}\right)^a - \sigma_{r,s}^a\right| \,\Big|\, e\right] dr\right] \\ &\qquad \leq \mathbb{E}\left[\int_0^{e(s)} \mathbf{1}_{\{\sigma_{r,s} \leq 2n^{-\gamma}\}} \mathbb{E}\left[\left(\frac{2\mathcal{N}_{n,r,s}+1}{2n+1}\right)^a + \sigma_{r,s}^a \,\Big|\, e\right] dr\right] \\ &\qquad \leq 3\mathbb{E}\left[\int_0^{e(s)} \mathbf{1}_{\{\sigma_{r,s} \leq 2n^{-\gamma}\}} \sigma_{r,s}^a \, dr\right] \\ &\qquad \leq 3 \cdot 2^\eta n^{-\gamma\eta}\, \mathbb{E}\left[\int_0^{e(s)} dr\, \sigma_{r,s}^{a-\eta}\right],\end{aligned}$$

where we used (i) of Lemma 3.35 for the second inequality. Recall that $D_3 = \{\sigma_{r,s} \leq 2n^{-\gamma}\}$. We deduce that:

$$\kappa_3 \leq \int_0^1 ds\, 3 \cdot 2^\eta n^{-\gamma\eta}\, \mathbb{E}\left[\int_0^{e(s)} dr\, \sigma_{r,s}^{a-\eta}\right] = 3 \cdot 2^\eta n^{-\gamma\eta}\, \mathbb{E}[Z_{a-\eta+1}]. \tag{3.37}$$

Choose $\gamma = 1/3$ and $\eta = 3(2a+1)/8$. Thanks to Lemma 3.5, we get that $\mathbb{E}\left[\int_0^1 e(s)\, ds\right] = \mathbb{E}[Z_1]$ is finite and that $\mathbb{E}[Z_{a-\eta+1}]$ is also finite since $a - \eta + 1 > 1/2$ as $a > -1/2$. Therefore, we deduce from (3.31) and then (3.34), (3.36) and (3.37) that there exists a finite constant $C(a)$ such that we have for all $n \in \mathbb{N}^*$:

$$\mathbb{E}[|A_{3,n}(x^a) - A_{4,n}(x^a)|] \le C(a)\, n^{-(2a+1)/8}.$$

□

Lemma 3.29. *For all $f \in \mathcal{B}([0,1])$ such that $f \ge 0$ and $\|x^a f\|_\infty < +\infty$ for some $a \in [0, 1/2)$, we have:*

$$A_{4,n}(f) \xrightarrow[n\to+\infty]{a.s.} \sqrt{2\alpha}\ \Phi_e(f).$$

Proof. Let $f \in \mathcal{B}([0,1])$ such that $f \ge 0$ and $\|x^a f\|_\infty < +\infty$ for some $a \in [0, 1/2)$. Let U be uniform on $[0,1]$ and independent of e. Recall $Z_\beta = \int_0^1 ds \int_0^1 dr\, \sigma_{r,s}^{\beta-1}$ defined in (3.15). Notice that:

$$\mathbb{E}\Big[\int_0^{e(U)} dr\, f(\sigma_{r,U}) \Big| e\Big] \le \|x^a f\|_\infty Z_{1-a}.$$

Since $1-a > 1/2$, we deduce from Lemma 3.5 that a.s. $Z_{1-a} < +\infty$. Then, use the strong law of large numbers (conditionally on e) to deduce that $A_{4,n}(f)$ converges a.s. towards $\sqrt{2\alpha}\ \Phi_e(f)$ as n goes to infinity. □

3.6 Proof of Theorem 3.4

Let $a > -1/2$. According to Lemmas 3.24, 3.25, 3.26 and 3.27 (use (i) for $a > 0$ and (ii) for $a \in (-1/2, 0]$), there exists $\varepsilon > 0$ and a finite constant c such that for all $n \in \mathbb{N}^*$, we have $\mathbb{E}[|A_n(x^a) - A_{4,n}(x^a)|] \le cn^{-\varepsilon}$. Since according to Lemma 3.29, we have a.s. that $\lim_{n\to+\infty} A_{4,n}(x^a) = \sqrt{2\alpha}\ \Phi_e(x^a)$, we deduce from Borel-Cantelli lemma that, with $\varphi(n) = \lceil n^{2/\varepsilon} \rceil$, we have a.s. $\lim_{n\to+\infty} A_{\varphi(n)}(x^a) = \sqrt{2\alpha}\ \Phi_e(x^a)$.

For $n' \ge n \ge 1$, we have $\mathcal{T}_{[n]} \subset \mathcal{T}_{[n']}$. Unfortunately, by the construction of T_n, we don't have in general that $v \in \mathrm{T}_n$ implies that $v \in \mathrm{T}_{n'}$, see the example of Figure 3.3.

However, it is still true, as $1 + a > 0$, that:

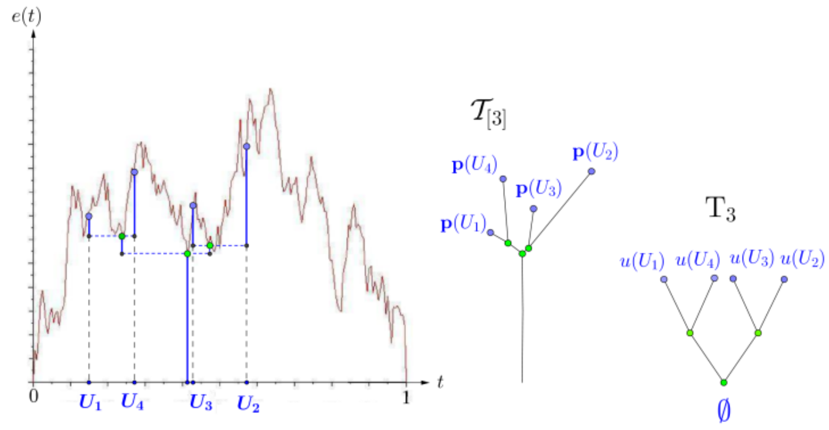
$$\sum_{v\in \mathrm{T}_n} |\mathrm{T}_{n,v}|^{1+a} \le \sum_{v'\in \mathrm{T}_{n'}} |\mathrm{T}_{n',v'}|^{1+a}. \tag{3.38}$$

Let $n \in \mathbb{N}^*$. There exists a unique $n' \in \mathbb{N}^*$ such that $\varphi(n') \le n < \varphi(n'+1)$. We obtain from (3.38) that:

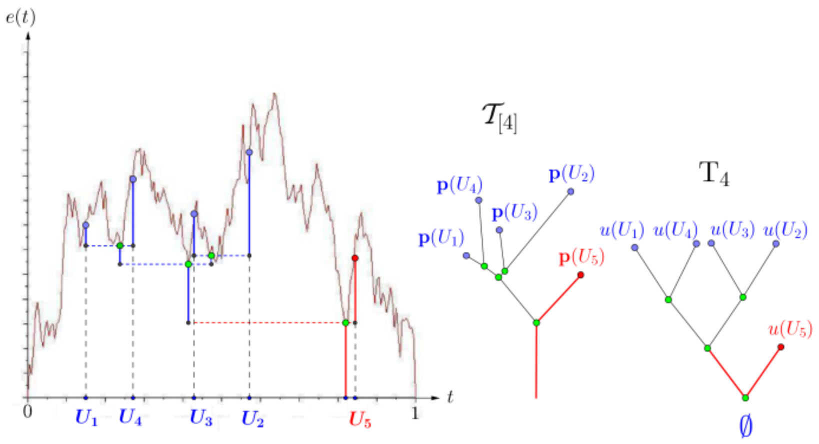
$$\left(\frac{2\varphi(n')+1}{2\varphi(n'+1)+1}\right)^{a+\frac{3}{2}} A_{\varphi(n')}(x^a) \le A_n(x^a) \le \left(\frac{2\varphi(n'+1)+1}{2\varphi(n')+1}\right)^{a+\frac{3}{2}} A_{\varphi(n'+1)}(x^a).$$

As $\lim_{n'\to+\infty} \varphi(n')/\varphi(n'+1) = 1$, we deduce that a.s. $\lim_{n\to+\infty} A_n(x^a) = \sqrt{2\alpha}\ \Phi_e(x^a)$.

In particular, for all $a \in (-1/2, 0]$, a.s. for all $k \in \mathbb{N}$, we have $\lim_{n\to+\infty} A_n(x^{a+k}) = \sqrt{2\alpha}\ \Phi_e(x^{a+k})$. Since on $[0,1]$, the convergence of moments implies the weak convergence of measure, we deduce that a.s. the random measure $A_n(x^a \bullet)$ converges weakly towards $\sqrt{2\alpha}\ \Phi_e(x^a \bullet)$. By taking a dense subset of a in $(-1/2, 0]$ and using monotonicity, we deduce that a.s. for all $a \in (-1/2, 0]$ the random measure $A_n(x^a \bullet)$ converges weakly towards $\sqrt{2\alpha}\ \Phi_e(x^a \bullet)$. This ends the proof of Theorem 3.4.



(a) $n = 4$



(b) $n' = 5$

Figure 3.3 – $\mathcal{T}_{[3]} \subset \mathcal{T}_{[4]}$ but $\mathrm{T}_3 \not\subset \mathrm{T}_4$

3.7 Proof of Proposition 3.10

3.7.1 A preliminary convergence in distribution

Let $(E_v, v \in \mathcal{U})$ be independent exponential random variables with mean 1 and independent of e. Let $f \in \mathcal{C}([0,1])$. We set for $v \in \mathrm{T}_n$:

$$X_{n,v} = |\mathrm{T}_n|^{-5/4} |\mathrm{T}_{n,v}| f\left(\frac{|\mathrm{T}_{n,v}|}{|\mathrm{T}_n|}\right) \quad \text{and} \quad Z_n(f) = \sum_{v \in \mathrm{T}_n} X_{n,v}\,(E_v - 1). \tag{3.39}$$

We have the following lemma.

Lemma 3.30. *Let $f \in \mathcal{C}([0,1])$ be locally Lipschitz continuous on $(0,1]$ such that $\| x^a f' \|_{esssup}$ is finite for some $a \in (0,1)$. We have the following convergence in distribution:*

$$(Z_n(f), A_n) \xrightarrow[n\to+\infty]{(d)} \left((2\alpha)^{1/4} \sqrt{\Phi_e(xf^2)}\; G, \sqrt{2\alpha}\;\Phi_e \right), \tag{3.40}$$

where G is a standard Gaussian random variable independent of e.

Proof. Let $f \in \mathcal{C}([0,1])$. We first assume that f is non-negative. We compute the Laplace transform of $Z_n(f)$ conditionally on T_n. Let $\lambda > 0$. Elementary computations give:

$$\mathbb{E}\left[\mathrm{e}^{-\lambda Z_n(f)} \,|\mathrm{T}_n \right] = \mathrm{e}^{\lambda \sum_{v\in \mathrm{T}_n} X_{n,v}}\, \mathbb{E}\left[\mathrm{e}^{-\lambda \sum_{v\in\mathrm{T}_n} X_{n,v}\, E_v} \,|\mathrm{T}_n \right] = \mathrm{e}^{\sum_{v\in\mathrm{T}_n} (\lambda X_{n,v} - \log(1+\lambda X_{n,v}))}\,.$$

For $x \geq 0$, we have $\frac{x^2}{2} - \frac{x^3}{3} \leq x - \log(1+x) \leq \frac{x^2}{2}$. Thanks to Theorem 3.4, we have:

$$\sum_{v \in \mathrm{T}_n} X_{n,v}^2 = A_n(xf^2) \xrightarrow[n\to+\infty]{\text{a.s.}} \sqrt{2\alpha}\, \Phi_e(xf^2)$$

and

$$\sum_{v \in \mathrm{T}_n} X_{n,v}^3 = |\mathrm{T}_n|^{-1/4} A_n(x^2 f^3) \xrightarrow[n\to+\infty]{\text{a.s.}} 0.$$

We deduce that a.s. $\lim_{n\to+\infty} \mathbb{E}\left[\mathrm{e}^{-\lambda Z_n(f)} \,|\mathrm{T}_n\right] = \exp\left(\lambda^2 \sqrt{2\alpha}\, \Phi_e(xf^2)/2\right)$. Let $K > 0$, and consider the event $B_K = \bigcap_{n\in\mathbb{N}} \{A_n(xf^2) \leq K\}$. Since on B_K, the term $\mathbb{E}\left[\mathrm{e}^{-\lambda Z_n(f)} \,|\mathrm{T}_n\right]$ is bounded by $\exp(\lambda^2 K/2)$, we deduce from dominated convergence that for any continuous bounded function g on the set of finite measure on $[0,1]$ (endowed with the topology of weak convergence), we have:

$$\begin{aligned}
\lim_{n\to+\infty} \mathbb{E}\left[\mathrm{e}^{-\lambda Z_n(f)}\, g(A_n) \mathbf{1}_{B_K}\right] &= \lim_{n\to+\infty} \mathbb{E}\left[\mathbb{E}\left[\mathrm{e}^{-\lambda Z_n(f)} \,|\mathrm{T}_n\right] g(A_n) \mathbf{1}_{B_K}\right] \\
&= \mathbb{E}\left[\mathrm{e}^{\lambda^2 \sqrt{2\alpha}\, \Phi_e(xf^2)/2}\, g(\sqrt{2\alpha}\, \Phi_e) \mathbf{1}_{B_K}\right] \\
&= \mathbb{E}\left[\mathrm{e}^{-\lambda (2\alpha)^{1/4} \sqrt{\Phi_e(xf^2)}\, G}\, g(\sqrt{2\alpha}\, \Phi_e) \mathbf{1}_{B_K}\right],
\end{aligned}$$

where G is a standard Gaussian random variable independent of e. We deduce that the convergence in distribution (3.40) holds conditionally on B_K. Since $A_n(xf^2)$ is finite for every n and converges a.s. to a finite limit, we get that for any $\varepsilon > 0$, there exists K_ε finite such that $\mathbb{P}(B_{K_\varepsilon}) \geq 1 - \varepsilon$. Then use Lemma 3.31 below to conclude that (3.40) holds for f non-negative.

In the general case, we set $f_+ = \max(0, f)$ and $f_- = \max(0, -f)$ so that $f = f_+ - f_-$. Notice that f_+ and f_- are non-negative and continuous. We have proved that (3.40) holds with f replaced by $\lambda_+ f_+ + \lambda_- f_-$ for any $\lambda_+ \geq 0$ and $\lambda_- \geq 0$. Since $f_+ f_- = 0$, this implies the following convergence in distribution:

$$(Z_n(f_+), Z_n(f_-), A_n) \xrightarrow[n\to+\infty]{(d)} \left((2\alpha)^{1/4} \sqrt{\Phi_e(xf_+^2)}\, G_+, (2\alpha)^{1/4} \sqrt{\Phi_e(xf_-^2)}\, G_-, \sqrt{2\alpha}\, \Phi_e\right),$$

where G_+ and G_- are independent standard Gaussian random variables independent of e. Then, using again that $f_+ f_- = 0$, we obtain that, conditionally on e, $\sqrt{\Phi_e(xf_+^2)}\, G_+ - \sqrt{\Phi_e(xf_-^2)}\, G_-$ is distributed as $\sqrt{\Phi_e(xf^2)}\, G$, where G is a standard Gaussian random variable independent of e. We deduce that (3.40) holds. This ends the proof. □

Lemma 3.31. *Let $(\Gamma_\varepsilon, \varepsilon > 0)$ be a sequence of events such that $\lim_{\varepsilon\to 0} \mathbb{P}(\Gamma_\varepsilon) = 1$. Let $(W_n, n \in \mathbb{N})$ and W be random variables taking values in a Polish space $\mathcal{M}$. Assume that for all $\varepsilon > 0$, conditionally on Γ_ε, the sequence $(W_n, n \in \mathbb{N})$ converges in distribution towards W. Then $(W_n, n \in \mathbb{N})$ converges in distribution towards W.*

Proof. Let g be a real-valued bounded continuous function defined on $\mathcal{M}$. It is enough to prove that $\lim_{n\to+\infty} |\mathbb{E}[g(W_n)] - \mathbb{E}[g(W)]| = 0$. By hypothesis, we have that for all $\varepsilon > 0$:

$$\lim_{n\to+\infty} \mathbb{E}[g(W_n)|\Gamma_\varepsilon] = \mathbb{E}[g(W)|\Gamma_\varepsilon].$$

We get:

$$|\mathbb{E}[g(W_n)] - \mathbb{E}[g(W)]| \leq |\mathbb{E}[g(W_n)|\Gamma_\varepsilon] - \mathbb{E}[g(W)|\Gamma_\varepsilon]| \mathbb{P}(\Gamma_\varepsilon) + 2\, \|g\|_\infty\, \mathbb{P}(\Gamma_\varepsilon^c)$$

We deduce that $\limsup_{n\to+\infty} |\mathbb{E}[g(W_n)] - \mathbb{E}[g(W)]| \leq 2\, \|g\|_\infty\, \mathbb{P}(\Gamma_\varepsilon^c)$. Since $\lim_{\varepsilon\to 0} \mathbb{P}(\Gamma_\varepsilon^c) = 0$, we deduce that $\lim_{n\to+\infty} |\mathbb{E}[g(W_n)] - \mathbb{E}[g(W)]| = 0$. This ends the proof. □

3.7.2 Proof of Proposition 3.10

We deduce Proposition 3.10 directly from Lemmas 3.32 and 3.33 below.
Using notations from Section 3.5, we set:

$$\Delta_n = \frac{1}{2\sqrt{\alpha}} |\mathrm{T}_n|^{1/4}(A_{1,n} - A_{2,n}).$$

Lemma 3.32. *Let $f \in \mathcal{C}([0,1])$ be locally Lipschitz continuous on $(0,1]$ such that $\| x^a f' \|_{esssup}$ is finite for some $a \in (0,1)$. We have the following convergence in probability:*

$$|\mathrm{T}_n|^{1/4}(A_n - \sqrt{2\alpha}\, \Phi_e)(f) - 2\sqrt{\alpha}\, \Delta_n(f) \xrightarrow[n\to+\infty]{\mathbb{P}} 0.$$

Proof. We keep notations from Section 3.5. We have:

$$\left| |\mathrm{T}_n|^{1/4}(A_n - \sqrt{2\alpha}\, \Phi_e)(f) - 2\sqrt{\alpha}\, \Delta_n(f) \right| \leq \Delta_{1,n} + \Delta_{3,n} + \Delta_{4,n} + \Delta_{5,n},$$

where

$$\Delta_{1,n} = |\mathrm{T}_n|^{1/4}|A_n(f) - A_{1,n}(f)|, \quad \Delta_{3,n} = |\mathrm{T}_n|^{1/4}|A_{2,n}(f) - A_{3,n}(f)|,$$
$$\Delta_{4,n} = |\mathrm{T}_n|^{1/4}|A_{3,n}(f) - A_{4,n}(f)|, \quad \Delta_{5,n} = |\mathrm{T}_n|^{1/4}|A_{4,n}(f) - \sqrt{2\alpha}\, \Phi_e(f)|.$$

Using Lemmas 3.24, 3.26 and 3.27 part (i), we deduce the following convergence in probability:

$$\Delta_{1,n} \xrightarrow[n\to+\infty]{\mathbb{P}} 0, \quad \Delta_{3,n} \xrightarrow[n\to+\infty]{\mathbb{P}} 0 \quad \text{and} \quad \Delta_{4,n} \xrightarrow[n\to+\infty]{\mathbb{P}} 0.$$

We study the convergence of $\Delta_{5,n}$. We set:

$$I_n = \frac{1}{n+1} \sum_{k=1}^{n+1} \int_0^{e(U_k)} dr\, f(\sigma_{r,U_k}) - \int_0^1 ds \int_0^{e(s)} dr\, f(\sigma_{r,s}).$$

By conditioning with respect to e, we deduce that:

$$\mathbb{E}[I_n^2] \leq \frac{1}{n+1} \mathbb{E}\left[\left(\int_0^{e(U_1)} dr\, f(\sigma_{r,U_1}) \right)^2 \right] \leq \frac{\| f \|_\infty^2}{n+1} \mathbb{E}\Big[\int_0^1 ds\, e(s)^2 \Big]. \tag{3.41}$$

Using the definition of $A_{4,n}(f)$, we get $\Delta_{5,n} \leq \Delta_{6,n} + \sqrt{2\alpha}\, \Delta_{7,n}$ with

$$\Delta_{6,n} = |\mathrm{T}_n|^{1/4} \left| 1 - \frac{|\mathrm{T}_n|^{3/2}}{2(n+1)\sqrt{2n}} \right| A_{4,n}(|f|) \quad \text{and} \quad \Delta_{7,n} = |\mathrm{T}_n|^{1/4} |I_n|.$$

From the a.s. convergence of $A_{4,n}(|f|)$ towards a finite limit, see Lemma 3.29, we deduce that a.s. $\lim_{n\to+\infty} \Delta_{6,n} = 0$. Since $\mathbb{E}\Big[\int_0^1 ds\, e(s)^2 \Big]$ is finite, see [169], we deduce from (3.41) that $\lim_{n\to+\infty} \mathbb{E}[\Delta_{7,n}^2] = 0$. We obtain that:

$$\Delta_{5,n} \xrightarrow[n\to+\infty]{\mathbb{P}} 0.$$

Then, we collect all the convergences together to get the result. □

Now, we study the convergence in distribution of $\Delta_n(f)$.

Lemma 3.33. *Let $f \in \mathcal{C}([0,1])$ be locally Lipschitz continuous on $(0,1]$ such that $\| x^a f' \|_{esssup}$ is finite for some $a \in (0,1)$. We have the following convergence in distribution:*

$$(2\sqrt{\alpha}\,\Delta_n(f), A_n) \xrightarrow[n\to+\infty]{(d)} \left((2\alpha)^{1/4}\sqrt{\Phi_e(xf^2)}\; G, \sqrt{2\alpha}\;\Phi_e\right), \tag{3.42}$$

where G is a standard Gaussian random variable independent of e.

Proof. According to Lemma 3.23, we get that $(\Delta_n(f), A_n)$ is distributed as $(\Delta'_n(f), A_n)$ where:

$$\Delta'_n(f) = |\mathrm{T}_n|^{-5/4} \sum_{v\in\mathrm{T}_n} |\mathrm{T}_{n,v}| f\left(\frac{|\mathrm{T}_{n,v}|}{|\mathrm{T}_n|}\right) Y'_{n,v}, \quad \text{with} \quad Y'_{n,v} = \sqrt{n}\left(\mathbb{E}\left[\frac{L'_n E_v}{S_{\mathrm{T}_n}}\right] - \frac{L'_n E_v}{S_{\mathrm{T}_n}}\right),$$

and $S_{\mathbf{t}} = \sum_{v\in\mathbf{t}} E_v$ for $\mathbf{t} \in \mathbb{T}$, with L'_n a random variable distributed as L_n, and thus with density given by (3.52), independent of T_n and $(E_u, u \in \mathcal{U})$ independent exponential random variables with mean 1, independent of L'_n and T_n. So it is enough to prove (3.42) with Δ_n replaced by Δ'_n.

Recall the definition (3.39) of $Z_n(f)$. Since L'_n is independent of $(E_u, u \in \mathcal{U})$ and T_n, we get:

$$\Delta'_n(f) = \frac{\sqrt{n}}{\sqrt{|\mathrm{T}_n|}}(\kappa_{1,n} + \kappa_{2,n})A_n(f) - \sqrt{n}\,\frac{L'_n}{S_{\mathrm{T}_n}} Z_n(f)$$

with

$$\kappa_{1,n} = |\mathrm{T}_n|^{3/4}\,(\mathbb{E}[L'_n] - L'_n)\mathbb{E}\left[\frac{E_\emptyset}{S_{\mathrm{T}_n}}\right] \quad \text{and} \quad \kappa_{2,n} = |\mathrm{T}_n|^{3/4}\,L'_n\left(\mathbb{E}\left[\frac{E_\emptyset}{S_{\mathrm{T}_n}}\right] - \frac{1}{S_{\mathrm{T}_n}}\right).$$

Thanks to Corollary 3.37 with $\alpha = \gamma = 1$ and $\beta = 0$, we have that:

$$\mathbb{E}[E_\emptyset/S_{\mathrm{T}_n}] = \Gamma(2n+1)/\Gamma(2n+2) = 1/|\mathrm{T}_n|.$$

Using (3.54), we get:

$$\mathbb{E}[|\kappa_{1,n}|] \leq |\mathrm{T}_n|^{3/4}\sqrt{\mathrm{Var}\;(L'_n)}\;\frac{\Gamma(2n+1)}{\Gamma(2n+2)} \leq \frac{1}{\sqrt{\alpha}}\;\frac{1}{(2n+1)^{1/4}}\,\cdot$$

We deduce that $\lim_{n\to+\infty}\kappa_{1,n} = 0$ in probability. Using (3.53) and Corollary 3.37 (three times), we get:

$$\begin{aligned}\mathbb{E}[\kappa_{2,n}^2] &= |\mathrm{T}_n|^{3/2}\frac{n+1}{\alpha}\left(\frac{\Gamma(2n+1)^2}{\Gamma(2n+2)^2} + \frac{\Gamma(2n-1)}{\Gamma(2n+1)} - 2\frac{\Gamma(2n+1)}{\Gamma(2n+2)}\frac{\Gamma(2n)}{\Gamma(2n+1)}\right)\\ &= \frac{|\mathrm{T}_n|^{3/2}\;(n+1)(2n+3)}{\alpha 2n(2n+1)^2(2n-1)}.\end{aligned}$$

We deduce that $\lim_{n\to+\infty}\kappa_{2,n} = 0$ in probability.

We deduce from the law of large numbers that $\lim_{n\to+\infty} S_{\mathrm{T}_n}/|\mathrm{T}_n| = 1$ in probability. According to [2], we have that a.s. $\lim_{n\to+\infty} L_n/\sqrt{n} = 1/\sqrt{\alpha}$. This implies the following convergence in probability $\lim_{n\to+\infty} L'_n/\sqrt{n} = 1/\sqrt{\alpha}$. We obtain that:

$$\sqrt{n}\,\frac{L'_n}{S_{\mathrm{T}_n}} \xrightarrow[n\to+\infty]{\mathbb{P}} \frac{1}{2\sqrt{\alpha}}\,\cdot$$

We deduce that $(2\sqrt{\alpha}\,\Delta'_n(f), A_n)$ has the same limit in distribution as $(-Z_n(f), A_n)$ as n goes to infinity. Then use Lemma 3.30 to get that (3.42) holds with Δ_n replaced by Δ'_n. This ends the proof of the Lemma. □

3.8 Appendix

3.8.1 Upper bounds for moments of the cost functional

According to [89], for $\beta > \frac{1}{2}$ and $k \in \mathbb{N}^*$, there exists a finite constant $C_{k,\beta}$ such that for all $n \in \mathbb{N}^*$,

$$\mathbb{E}\left[\left(\sum_{v\in \mathrm{T}_n} |\mathrm{T}_{n,v}|^\beta\right)^k\right] \leq C_{k,\beta}\, |\mathrm{T}_n|^{k(\beta+\frac{1}{2})}. \tag{3.43}$$

(Notice that (3.43) is stated in [89] with $\mathrm{T}^*_{n,v} = \mathrm{T}_{n,v}\backslash\mathcal{L}(\mathrm{T}_{n,v})$ instead of $\mathrm{T}_{n,v}$; but using that $|\mathrm{T}_{n,v}| = 2|\mathrm{T}^*_{n,v}| + 1$ it is elementary to get (3.43).)

The following lemma, which plays a key role in the proofs of Lemmas 3.24 and 3.25, is a direct consequence of these upper bounds.

Lemma 3.34. *For all $a \in [0, 1/2)$ and $f \in \mathcal{B}([0,1])$, we have for $k \in \mathbb{N}^*$:*

$$\mathbb{E}\left[|A_n(f)|^k\right] \leq C_{k,1-a} \|x^a f\|^k_\infty, \tag{3.44}$$

$$\mathbb{E}\left[A_n(xf^2)\right] \leq C_{1,2-2a} \|x^a f\|^2_\infty. \tag{3.45}$$

Proof. Let $k \in \mathbb{N}^*$. Using (3.43), we have:

$$\mathbb{E}\left[|A_n(f)|^k\right] \leq |\mathrm{T}_n|^{-\frac{3}{2}k} \|x^a f\|^k_\infty\, \mathbb{E}\left[\left(\sum_{v\in\mathrm{T}_n} \frac{|\mathrm{T}_{n,v}|^{1-a}}{|\mathrm{T}_n|^{-a}}\right)^k\right] \leq C_{k,1-a}\|x^a f\|^k_\infty,$$

which gives (3.44). Moreover, we also have:

$$\mathbb{E}\left[A_n(xf^2)\right] \leq |\mathrm{T}_n|^{-\frac{3}{2}} \|x^a f\|^2_\infty\, \mathbb{E}\left[\sum_{v\in\mathrm{T}_n} \frac{|\mathrm{T}_{n,v}|^{2-2a}}{|\mathrm{T}_n|^{1-2a}}\right] \leq C_{1,2-2a}\|x^a f\|^2_\infty$$

and we get (3.45). □

3.8.2 A lemma for binomial random variables

We give a lemma used for the proof of Lemma 3.27.

Lemma 3.35. *Let X be a binomial random variable with parameter $(n,p) \in \mathbb{N}^* \times (0,1)$.*

(i) For $a \in (0,1]$, we have

$$\mathbb{E}\left[(2X+1)^{-a}\right] \leq \left(1 \wedge \frac{1}{p(n+1)}\right)^a.$$

(ii) Let $f \in \mathcal{C}((0,1])$ be locally Lipschitz continuous and $b \in (0,1)$. Then we have:

$$\mathbb{E}\left[\left|f\left(\frac{2X+1}{2n+1}\right) - f(p)\right|\right] \leq \frac{\|x^b f'\|_{esssup}}{1-b}\left(p^{-\frac{b}{2}} + p^{\frac{1}{2}-b}\right) n^{-1/2}.$$

Proof. We prove (i). Let $a \in (0,1]$. Let X be a binomial random variable with parameter (n,p). An elementary computation gives that:

$$\mathbb{E}\left[\frac{1}{1+X}\right] = \frac{1-(1-p)^{n+1}}{p(n+1)}. \tag{3.46}$$

Using Jensen inequality and (3.46), we get

$$\mathbb{E}\left[\left(\frac{1}{2X+1}\right)^a\right] \le \mathbb{E}\left[\frac{1}{2X+1}\right]^a \le \mathbb{E}\left[\frac{1}{1+X}\right]^a \le \left(1 \wedge \frac{1}{p(n+1)}\right)^a.$$

We prove (ii). Let $b \in (0,1)$. We have $\left|f\left(\frac{2X+1}{2n+1}\right) - f(p)\right| \le \|x^b f'\|_{\text{esssup}} \left|\int_p^{\frac{2X+1}{2n+1}} x^{-b} dx\right|$ and thus

$$\left|f\left(\frac{2X+1}{2n+1}\right) - f(p)\right| \le \frac{\|x^b f'\|_{\text{esssup}}}{1-b} \left|\left(\frac{2X+1}{2n+1}\right)^{1-b} - p^{1-b}\right|. \tag{3.47}$$

We decompose the right-hand side term into two parts:

$$\left|\left(\frac{2X+1}{2n+1}\right)^{1-b} - p^{1-b}\right| \le \left|p^{1-b} - \left(\frac{X}{n}\right)^{1-b}\right| + \left|\left(\frac{2X+1}{2n+1}\right)^{1-b} - \left(\frac{X}{n}\right)^{1-b}\right|. \tag{3.48}$$

We shall use the following key inequality (consider first the case $x \ge y$ and then the case $x < y$): for all $x, y > 0$ and $0 < b < 1$, we have:

$$|x^{1-b} - y^{1-b}| \le x^{-b}|x - y|. \tag{3.49}$$

For the first term of the right hand side of (3.48), using (3.49), we have $\left|p^{1-b} - \left(\frac{X}{n}\right)^{1-b}\right| \le p^{-b}\left|p - \frac{X}{n}\right|$. Hence, we get:

$$\mathbb{E}\left[\left|p^{1-b} - \left(\frac{X}{n}\right)^{1-b}\right|\right] \le p^{-b}\sqrt{\text{Var }(X/n)} \le p^{\frac{1}{2}-b} n^{-1/2}. \tag{3.50}$$

For the second term of the right hand side of (3.48), using (3.49) again, we get:

$$\left|\left(\frac{2X+1}{2n+1}\right)^{1-b} - \left(\frac{X}{n}\right)^{1-b}\right| \le \left(\frac{2X+1}{2n+1}\right)^{-b} \left|\frac{2X+1}{2n+1} - \frac{X}{n}\right| \le \frac{(2n+1)^{b-1}}{(2X+1)^b}.$$

This gives, using (i) and $|1 \wedge (1/x)|^b \le x^{-b/2}$ for $x > 0$, that:

$$\mathbb{E}\left[\left|\left(\frac{2X+1}{2n+1}\right)^{1-b} - \left(\frac{X}{n}\right)^{1-b}\right|\right] \le (2n+1)^{b-1} p^{-\frac{b}{2}} (n+1)^{-\frac{b}{2}} \le p^{-\frac{b}{2}} n^{-1/2}. \tag{3.51}$$

Using (3.47), (3.48), (3.50) and (3.51), we get the expected result. □

3.8.3 Some results on the Gamma function

We give here some results on the moments of Gamma random variables.

Lemma 3.36. *Let $k, \ell, n \in (0, +\infty)$ and $\alpha, \beta, \gamma \in [0, +\infty)$ such that $k + \ell + n + \alpha + \beta > \gamma$. Let $\Gamma_k, \Gamma_\ell, \Gamma_n$ be three independent Gamma random variables with respective parameter $(k, 1)$, $(\ell, 1)$ and $(n, 1)$. Then we have:*

$$\mathbb{E}\left[\frac{\Gamma_k^\alpha \Gamma_\ell^\beta}{(\Gamma_k + \Gamma_\ell + \Gamma_n)^\gamma}\right] = \frac{\Gamma(k+\alpha)}{\Gamma(k)} \frac{\Gamma(\ell+\beta)}{\Gamma(\ell)} \frac{\Gamma(k+\ell+n+\alpha+\beta-\gamma)}{\Gamma(k+\ell+n+\alpha+\beta)}.$$

Proof. Elementary computations give that for all non negative function $f \in \mathcal{B}([0,\infty))$,

$$\mathbb{E}\left[\Gamma_k^\alpha f(\Gamma_k)\right] = \mathbb{E}\left[\Gamma_k^\alpha\right] \mathbb{E}\left[f(\Gamma_{k+\alpha})\right] = \frac{\Gamma(k+\alpha)}{\Gamma(k)} \mathbb{E}\left[f(\Gamma_{k+\alpha})\right].$$

We deduce that:

$$\begin{aligned}\mathbb{E}\left[\frac{\Gamma_k^\alpha \Gamma_\ell^\beta}{(\Gamma_k+\Gamma_\ell+\Gamma_n)^\gamma}\right] &= \mathbb{E}\left[\mathbb{E}\left[\frac{\Gamma_k^\alpha \Gamma_\ell^\beta}{(\Gamma_k+\Gamma_\ell+\Gamma_n)^\gamma}\middle|\Gamma_\ell,\Gamma_n\right]\right]\\ &= \mathbb{E}\left[\Gamma_k^\alpha\right]\mathbb{E}\left[\frac{\Gamma_\ell^\beta}{\left(\Gamma_\ell+\tilde\Gamma_{k+n+\alpha}\right)^\gamma}\right]\\ &= \mathbb{E}\left[\Gamma_k^\alpha\right]\mathbb{E}\left[\Gamma_\ell^\beta\right]\mathbb{E}\left[\frac{1}{\Gamma^\gamma_{k+\ell+n+\alpha+\beta}}\right]\\ &= \frac{\Gamma(k+\alpha)}{\Gamma(k)}\frac{\Gamma(\ell+\beta)}{\Gamma(\ell)}\frac{\Gamma(k+\ell+n+\alpha+\beta-\gamma)}{\Gamma(k+\ell+n+\alpha+\beta)},\end{aligned}$$

where $\tilde\Gamma_{k+n+\alpha}$ is a Gamma random variable with parameter $(k+n+\alpha,1)$ independent of Γ_ℓ, and $\Gamma_{k+\ell+n+\alpha+\beta}$ is a Gamma random variable with parameter $(k+\ell+n+\alpha+\beta,1)$. □

We directly deduce the following result.

Corollary 3.37. *Let $m\geq 2$. Let $(E_i, 1\leq i\leq m)$ be independent exponential random variables with parameter 1 and $S_m=\sum_{i=1}^m E_i$. Then for all $\alpha,\beta,\gamma\in[0,+\infty)$ such that $m+\alpha+\beta>\gamma$, we have*

$$\mathbb{E}\left[\frac{E_1^\alpha E_2^\beta}{S_m^\gamma}\right]=\Gamma(1+\alpha)\Gamma(1+\beta)\frac{\Gamma(m+\alpha+\beta-\gamma)}{\Gamma(m+\alpha+\beta)}.$$

3.8.4 Elementary computations on the branch length of $\mathcal{T}_{[n]}$

We keep notations from Section 3.5. Recall that the density of $(h_{n,v}, v\in \mathrm{T}_n)$ is, conditionally on T_n, given by (3.29). Recall $L_n=\sum_{v\in\mathrm{T}_n} h_{n,v}$ denotes the total length of $\mathcal{T}_{[n]}$. From Aldous [10], Pitman [163] (see Theorem 7.9) or Duquesne and Le Gall [71] or by standard computations, we get that the density of L_n, conditionally on T_n, is given by:

$$f_{L_n}(x)=2\frac{\alpha^{n+1}}{n!}x^{2n+1}\,\mathrm{e}^{-\alpha x^2}\,\mathbf{1}_{\{x>0\}}. \tag{3.52}$$

In particular, the random variable L_n is independent of T_n. The first two moments of L_n are given by

$$\mathbb{E}[L_n]=\frac{1}{\sqrt{\alpha}}\frac{\Gamma(n+\frac32)}{\Gamma(n+1)}=\frac{n+1}{\sqrt{\alpha}}\frac{\Gamma(n+\frac32)}{\Gamma(n+2)}\quad\text{and}\quad \mathbb{E}[L_n^2]=\frac{n+1}{\alpha}. \tag{3.53}$$

According to [98], we have that $(n+1)^{s-1}\leq\Gamma(n+s)/\Gamma(n+1)\leq n^{s-1}$ for $n\in\mathbb{N}^*$ and $s\in[0,1]$. Hence, we obtain:

$$\frac{1}{\sqrt{\alpha}}\frac{n+1}{\sqrt{n+2}}\leq\mathbb{E}[L_n]\leq\frac{\sqrt{n+1}}{\sqrt{\alpha}}\quad\text{and}\quad \mathrm{Var}\,(L_n)\leq\frac{1}{\alpha}. \tag{3.54}$$

Using that $L_n=\sum_{v\in\mathrm{T}_n}h_{n,v}$ and that, conditionally on T_n, the random variables $(h_{n,v}, v\in\mathrm{T}_n)$ are exchangeable, we deduce that $\mathbb{E}[h_{n,\emptyset}]=\mathbb{E}[L_n]/(2n+1)$ and thus:

$$\frac{1}{2\sqrt{\alpha(n+1)}}\leq\mathbb{E}[h_{n,\emptyset}]=\frac{1}{2\sqrt{\alpha}}\frac{\Gamma(n+\frac12)}{\Gamma(n+1)}\leq\frac{1}{2\sqrt{\alpha n}}. \tag{3.55}$$

We finish by a result on the covariance of the branch lengths, used in Lemma 3.25. We define $Y_{n,v}=\sqrt{n}(\mathbb{E}[h_{n,v}]-h_{n,v})$ for $v\in\mathrm{T}_n$. Notice that $(Y_{n,v}, v\in\mathrm{T}_n)$ has an exchangeable distribution conditionally on T_n.

Lemma 3.38. *Let $n \in \mathbb{N}^*$. We have:*

$$\left|\mathbb{E}[Y_{n,\emptyset} Y_{n,1}]\right| \leq \frac{1}{8\alpha n} \quad \text{and} \quad \mathbb{E}[Y_{n,\emptyset}^2] \leq \frac{1}{2\alpha}. \tag{3.56}$$

Proof. Using Lemma 3.23 and its notations, and Corollary 3.37 and (3.53), we have, with $\mathbf{t} \in \mathbb{T}$ full binary such that $|\mathbf{t}| = 2n+1$:

$$\mathbb{E}[h_{n,\emptyset} h_{n,1}] = \mathbb{E}\left[L_n^2\right] \mathbb{E}\left[\frac{E_\emptyset E_1}{S_{\mathbf{t}}^2}\right] = \frac{n+1}{\alpha} \frac{1}{2(n+1)(2n+1)} = \frac{1}{2\alpha} \frac{1}{2n+1},$$

and

$$\mathbb{E}[h_{n,\emptyset}^2] = \mathbb{E}\left[L_n^2\right] \mathbb{E}\left[\frac{E_\emptyset^2}{S_{\mathbf{t}}^2}\right] = \frac{n+1}{\alpha} \frac{1}{(n+1)(2n+1)} = \frac{1}{\alpha} \frac{1}{2n+1}.$$

The lemma is then a consequence of these equalities and (3.55). □

3.8.5 A deterministic representation formula

Lemma 3.39. *Let $h \in \mathcal{C}_+([0,1])$. We have that for all $a > 0$:*

$$2 \int_0^1 ds \int_0^{h(s)} dr\, \sigma_{r,s}(h)^a = a(a+1) \int_{[0,1]^2} |s'-s|^{a-1}\, m_h(s,s')\, ds ds'. \tag{3.57}$$

Proof. In this proof only, we shall write $m(s,t)$ and $\sigma_{r,s}$ respectively for $m_h(s,t)$ and $\sigma_{r,s}(h)$. Recall that $\sigma_{r,s} = \int_0^1 dt\, \mathbf{1}_{\{m(s,t) \geq r\}}$. We deduce that $\int_0^1 dt\, m(s,t) = \int_0^{h(s)} dr\, \sigma_{r,s}$ for every $s \in [0,1]$. Hence, the result is obvious for $a = 1$.

If $g \in \mathcal{B}([0,1])$ is a non negative function such that $x^2 g \in \mathcal{C}^2([0,1])$ or if $g = x^{a-1}$ for $a > 0$, we set:

$$I(g) = \int_0^1 ds \int_0^{h(s)} dr\ \sigma_{r,s}\, g(\sigma_{r,s}) \quad \text{and} \quad J(g) = \int_{0<s<t<1} ds\, dt\, [x^2 g]''(t-s)\, m(s,t).$$

We then have to prove that $J(g) = I(g)$ for $g = x^{a-1}$ for all $a > 0$. First of all, remark that if $f, g \in \mathcal{B}([0,1])$ are non negative functions such that $x^2 f, x^2 g \in \mathcal{C}^2([0,1]$, we have:

$$|I(g) - I(f)| \leq \int_0^1 ds \int_0^{h(s)} dr\ \sigma_{r,s}\, |g(\sigma_{s,r}) - f(\sigma_{s,r})| \leq \|g - f\|_\infty \,\|h\|_\infty \tag{3.58}$$

and

$$|J(g) - J(f)| \leq \|(x^2 g)'' - (x^2 f)''\|_\infty \int_{0<s<t<1} ds\, dt m(s,t) \leq \|(x^2 g)'' - (x^2 f)''\|_\infty \,\|h\|_\infty. \tag{3.59}$$

The proof of $J(g) = I(g)$ when $g = x^{a-1}$ is divided in 3 steps. First of all, we prove the result when $a \in \mathbb{N}^*$, which gives the equality when g is polynomial. Then we get the case when $g \in \mathcal{C}^2([0,1])$ by Bernstein's approximation. This gives the case $a \geq 3$. Finally, we give the result for $a \in (0,3) \backslash \{1,2\}$.

1st step

Let $g = x^{a-1}$ with $a \in \mathbb{N}^*$. We have:

$$\begin{aligned}
I(x^{a-1}) &= \int_0^1 ds \int_0^{h(s)} dr \left(\int_0^1 dt\, \mathbf{1}_{\{m(s,t) \geq r\}} \right)^a \\
&= \int_{[0,1]^{a+1}} ds\, ds_1 \ldots ds_a \left(\int_0^{h(s_1)} dr\, \mathbf{1}_{\{m(s_1,s_2)\geq r\}} \ldots \mathbf{1}_{\{m(s_1,s_{a+1})\geq r\}} \right) \\
&= \int_{[0,1]^{a+1}} ds\, ds_1 \ldots ds_a \left(\min(m(s,s_1), \ldots, m(s,s_a)\right) \\
&= \int_{[0,1]^{a+1}} ds_1 \ldots ds_{a+1} \left(\min(m(s_1,s_2), \ldots, m(s_1,s_{a+1})\right) \\
&= \int_{[0,1]^{a+1}} ds_1 \ldots ds_{a+1} \left(m\left(\min_{1\leq i\leq a+1} s_i, \max_{1\leq i\leq a+1} s_i \right)\right),
\end{aligned}$$

where we used that $\bigcup_{i=2}^{a+1}[s_1, s_i] = [\min_{1\leq i\leq a+1} s_i, \max_{1\leq i\leq a+1} s_i]$ for the last equality. By choosing $s = \min_{1\leq i\leq a+1} s_i$ and $t = \max_{1\leq i\leq a+1} s_i$, we have:

$$\begin{aligned}
\int_{[0,1]^{a+1}} ds_1 \ldots ds_{a+1} &\left(m\left(\min_{1\leq i\leq a+1} s_i, \max_{1\leq i\leq a+1} s_i \right)\right) \\
&= a(a+1) \int_{0<s<t<1} ds\, dt \int_{[s,t]^{a-1}} ds_1 \ldots ds_{a-1}\, m(s,t) \\
&= a(a+1) \int_{0<s<t<1} m(s,t)(t-s)^{a-1}\, ds\, dt.
\end{aligned} \tag{3.60}$$

This gives $I(x^{a-1}) = J(x^{a-1})$.

2nd step

Let $g \in \mathcal{C}^2([0,1])$ be a non negative function. For $n \in \mathbb{N}$, we define the associated Bernstein polynomial $B_n(g)$ by:

$$B_n(g)(x) = \sum_{k=0}^n \binom{n}{k} g(k/n)\, x^k (1-x)^{n-k}, \quad x \in [0,1].$$

It is well known (see for instance, Theorem 6.3.2 in [52]) that for every $k \in \mathbb{N}$ and for every $f \in \mathcal{C}^k([0,1])$, $\lim_{n\to\infty} \|f^{(k)} - B_n^{(k)}(f)\|_\infty = 0$. Using that $\|(x^2 B_n(g))'' - (x^2 g)''\|_\infty \leq 2\|B_n(g) - g\|_\infty + 4\|B_n'(g) - g'\|_\infty + \|B_n''(g) - g''\|_\infty$, we deduce from (3.58) and (3.59) that $J(g) = I(g)$.

3rd step

Let $g = x^{a-1}$ with $a \in (0,3)\backslash\{1,2\}$. We approximate g by functions in $\mathcal{C}^2([0,1])$. For $\delta \in (0,1)$, we define:

$$g_\delta(x) = \begin{cases} P_\delta(x) & \text{if } 0 \leq x \leq \delta, \\ g(x) & \text{if } \delta \leq x \leq 1, \end{cases}$$

where P_δ is the polynomial with degree 2 such that $P_\delta(\delta) = g(\delta) = \delta^{a-1}$, $P_\delta'(\delta) = g'(\delta) = (a-1)\delta^{a-2}$ and $P_\delta''(\delta) = g''(\delta) = (a-1)(a-2)\delta^{a-3}$.

We shall prove that $\lim_{\delta\to 0} I(g_\delta) = I(g)$. We have:

$$g_\delta''(x) = \begin{cases} g''(\delta) & \text{if } 0 \leq x \leq \delta, \\ g''(x) & \text{if } \delta \leq x \leq 1. \end{cases}$$

- Assume $a \in (0,1)$. Let $h = g_\gamma - g_\delta$ with $\delta, \gamma \in (0,1)$ such that $\delta < \gamma$. It is easy to check that $h'' \leq 0$ on $[0,1]$. Since $h'(1) = h(1) = 0$ by construction, we deduce that $h \leq 0$ on $[0,1]$. Hence, when δ tends to 0, the sequence $(g_\delta, 0 < \delta < 1)$ is non decreasing and converges on $(0,1]$ towards g. By monotone convergence theorem, we get $\lim_{\delta\to 0} I(g_\delta) = I(g)$.

- Assume $a \in (1,3)$. Notice that $(g_\delta, 0 < \delta < 1)$ is uniformly bounded by a constant. Hence, by dominated convergence theorem, we obtain that $\lim_{\delta\to 0} I(g_\delta) = I(g)$.

We now prove that $\lim_{\delta\to 0} J(g_\delta) = J(g)$. Remark that if $x \in (\delta, 1]$, $(x^2 g_\delta(x))'' = (x^2 g(x))''$, and that there exists a constant $C(a)$, which does not depend on δ, such that for all $x \in (0,\delta]$, we have $|(x^2 g_\delta(x))''| \leq C(a)\delta^{a-1}$. We get that:

$$\begin{aligned}
|J(g_\delta) - J(g)| &= \left| \int_{0<s<t<1} m(s,t) \left((x^2 g_\delta(x))'' - (x^2 g(x))'' \right)_{x=t-s} dsdt \right| \\
&\leq \| h \|_\infty \int_0^\delta \left(\left| (x^2 g_\delta(x))'' - (x^2 g(x))'' \right| \right)_{x=r} dr \\
&\leq \| h \|_\infty \int_0^\delta \left(a(a+1) r^{a-1} + C(a)\delta^{a-1} \right) dr.
\end{aligned}$$

We deduce that $\lim_{\delta\to 0} J(g_\delta) = J(g)$. Thanks to the 2nd step, we have $J(g_\delta) = I(g_\delta)$ for all $\delta \in (0,1)$. Letting δ goes down to 0, we deduce that $J(g) = I(g)$. □

3.8.6 Proof of the first part of Lemma 3.17 (finiteness of Z_β^H and (3.23))

We use the setting of [70] on Lévy trees. Let H be the height function of a stable Lévy tree with branching mechanism $\psi(\lambda) = \kappa\lambda^\gamma$, with $\gamma \in (1,2]$ and $\kappa > 0$.

Let $\mathbb{N}$ be the excursion measure of the height process and set $\sigma = \inf\{s > 0, H(s) = 0\}$ for the duration of the excursion so that: $\mathbb{N}[1 - e^{-\lambda\sigma}] = \psi^{-1}(\lambda)$ for all $\lambda > 0$. According to Chapter VII in [21], ψ^{-1} is the Laplace exponent of a subordinator whose Lévy measure is denoted by π_*. Thus, the distribution of σ under $\mathbb{N}$ is π_* given by:

$$\pi_*(da) = \frac{1}{\gamma\kappa^{1/\gamma}\Gamma((\gamma-1)/\gamma)} \frac{da}{a^{1+\frac{1}{\gamma}}}.$$

Let $\mathbb{N}^{(a)}[\bullet] = \mathbb{N}[\bullet|\sigma = a]$ be the distribution of the excursion of the height process with duration a, so that:

$$\mathbb{N}[\bullet] = \int_0^\infty \pi_*(da)\, \mathbb{N}^{(a)}[\bullet].$$

In particular, we shall prove the result of Lemma 3.17 under $\mathbb{N}^{(1)}$. In this proof only, we shall write m for m_H defined by (3.13). We extend the definitions (3.14) and (3.15) as follows:

$$\sigma_{r,s} = \int_0^\sigma dt\, \mathbf{1}_{\{\min(s,t)\geq r\}} \quad \text{and} \quad Z_\beta^H = \int_0^\sigma ds \int_0^{H(s)} dr\, \sigma_{r,s}^{\beta-1} \quad \text{for } \beta > 0.$$

The integral in ds/σ in Z_β^H corresponds to taking a leaf at random in the Lévy tree. Using Bismut's decomposition of the Lévy tree, see Theorem 4.5 in [71] or Theorem 2.1 in [1], we get that, since $\psi'(0) = 0$, then under $\mathbb{N}[\sigma\bullet]$, the height $H(U)$, with U uniformly distributed over $[0,\sigma]$, is "distributed" as $\mathcal{H}$ with Lebesgue "distribution" on $(0,+\infty)$. It also implies that under $\mathbb{N}[\sigma\bullet]$, the random variable $\big(H(U), (\sigma_{H(U)-r,U}, r \in [0, H(U)])\big)$ is "distributed" as $(\mathcal{H}, (S_t, t \in [0,\mathcal{H}]))$, where $S = (S_t, t \geq 0)$ is a subordinator, with Laplace exponent say ϕ, independent of $\mathcal{H}$.

We prove (3.23) and get as a direct consequence using monotonicity, that $\mathbb{N}^{(1)}$-a.s., for all $\beta > 1/\gamma$, Z_β^H is finite. Using that:

$$(\psi^{-1})'(\lambda) = \mathbb{N}\left[\sigma\,\mathrm{e}^{-\lambda\sigma}\right] = \mathbb{N}\left[\sigma\,\mathrm{e}^{-\lambda\sigma_{0,U}}\right] = \mathbb{E}\left[\mathrm{e}^{-\lambda S_{\mathcal{H}}}\right] = \int_0^\infty dt\,\mathbb{E}\left[\mathrm{e}^{-\lambda S_t}\right] = \frac{1}{\phi(\lambda)}, \quad (3.61)$$

we deduce that:

$$\phi(\lambda) = \frac{1}{(\psi^{-1})'(\lambda)} = \gamma\kappa^{1/\gamma}\lambda^{(\gamma-1)/\gamma}. \quad (3.62)$$

Notice in particular that S_t is distributed as $t^{\gamma/(\gamma-1)}S_1$. We shall need later in the proof the following computation:

$$\mathbb{E}\left[S_1^{-(\gamma-1)/\gamma}\right] = \frac{1}{\Gamma\left(\frac{\gamma-1}{\gamma}\right)}\int_0^\infty dt\,t^{-1/\gamma}\mathbb{E}\left[\mathrm{e}^{-tS_1}\right] = \frac{1}{\kappa^{1/\gamma}(\gamma-1)\Gamma\left(\frac{\gamma-1}{\gamma}\right)}\cdot \quad (3.63)$$

We set $\Lambda(\lambda) = \mathbb{N}\left[Z_\beta^H\,\mathrm{e}^{-\lambda\sigma}\right]$ for $\lambda > 0$. Using Bismut's decomposition again, we get:

$$\begin{aligned}\Lambda(\lambda) = \mathbb{N}\left[\sigma\int_0^{H(U)} dr\,\sigma_{H(U)-r,U}^{\beta-1}\,\mathrm{e}^{-\lambda\sigma_{0,U}}\right] &= \mathbb{E}\left[\int_0^{\mathcal{H}} dr\,S_r^{\beta-1}\,\mathrm{e}^{-\lambda S_{\mathcal{H}}}\right]\\ &= \mathbb{E}\left[\int_0^\infty dt\int_0^t dr\,S_r^{\beta-1}\,\mathrm{e}^{-\lambda S_t}\right]\\ &= \mathbb{E}\left[\int_0^\infty dt\int_0^\infty dr\,S_r^{\beta-1}\,\mathrm{e}^{-\lambda S_{t+r}}\right].\end{aligned}$$

We have:

$$\begin{aligned}\Lambda(\lambda) &= \mathbb{E}\left[\int_0^\infty dt\,\mathrm{e}^{-\lambda S_t}\right]\mathbb{E}\left[\int_0^\infty dr\,S_r^{\beta-1}\,\mathrm{e}^{-\lambda S_r}\right]\\ &= \frac{1}{\phi(\lambda)}\,\mathbb{E}\left[S_1^{\beta-1}\int_0^\infty dr\,r^{(\beta-1)\gamma/(\gamma-1)}\,\mathrm{e}^{-\lambda r^{\gamma/(\gamma-1)}S_1}\right]\\ &= \frac{1}{\phi(\lambda)}\,\mathbb{E}\left[S_1^{-(\gamma-1)/\gamma}\right]\lambda^{-\beta+(1/\gamma)}\,\frac{\gamma-1}{\gamma}\int_0^\infty du\,u^{\beta-1-(1/\gamma)}\,\mathrm{e}^{-u},\end{aligned}$$

where we used that S has stationary independent increments for the first equality, (3.61) and that S_r is distributed as $r^{\gamma/(\gamma-1)}S_1$ for the second, and the change of variable $u = \lambda S_1 r^{\gamma/(\gamma-1)}$ for the last. Then use (3.63) and (3.62) to deduce that:

$$\Lambda(\lambda) = \frac{\Gamma\left(\beta-\frac{1}{\gamma}\right)}{\gamma^2\kappa^{2/\gamma}\Gamma\left(\frac{\gamma-1}{\gamma}\right)}\,\lambda^{-1-\beta+\frac{2}{\gamma}}. \quad (3.64)$$

On the other hand, we set $G(a) = \mathbb{N}^{(a)}[Z_\beta^H]$ so that:

$$\Lambda(\lambda) = \int_0^\infty \pi_*(da)\,G(a)\,\mathrm{e}^{-\lambda a}.$$

We deduce from the scaling property of the height function that, under $\mathbb{N}^{(a)}$, the random variable $\Big((H(s), s\in[0,a]), (\sigma_{r,s}; r\in[0,H(s)], s\in[0,a])\Big)$ is distributed as the random variable $\Big((a^{(\gamma-1)/\gamma}H(s/a),\ s\in[0,a]), (a\sigma_{r,s/a}; r\in[0,a^{(\gamma-1)/\gamma}H(s/a)], s\in[0,a])\Big)$ under

$\mathbb{N}^{(1)}$. This implies that Z_β^H is under $\mathbb{N}^{(a)}$ distributed as $a^{\beta+1-1/\gamma} Z_\beta^H$ under $\mathbb{N}^{(1)}$. This gives $G(a) = a^{\beta+1-1/\gamma} G(1)$. We deduce that:

$$\Lambda(\lambda) = G(1) \int_0^\infty \pi_*(da)\, a^{\beta+1-\frac{1}{\gamma}}\, \mathrm{e}^{-\lambda a} = G(1) \frac{\Gamma\left(\beta+1-\frac{2}{\gamma}\right)}{\gamma \kappa^{1/\gamma} \Gamma\left(\frac{\gamma-1}{\gamma}\right)} \lambda^{-\beta-1+\frac{2}{\gamma}}.$$

Then use (3.64) to get that for all $\beta > 0$:

$$\mathbb{N}^{(1)}[Z_\beta^H] = G(1) = \frac{1}{\gamma \kappa^{1/\gamma}} \frac{\Gamma\left(\beta - \frac{1}{\gamma}\right)}{\Gamma\left(\beta+1-\frac{2}{\gamma}\right)}.$$

This gives (3.23) and that $\mathbb{N}^{(1)}$-a.s., for all $\beta > 1/\gamma$, Z_β^H is finite.

We prove now that $\mathbb{N}^{(1)}$-a.s., for all $\beta \in (0, 1/\gamma]$, Z_β^H is infinite. Let $\beta \in (0, 1/\gamma]$. Let U be uniform on $[0, \sigma]$ under $\mathbb{N}$. According to the first part of the proof, we deduce from the Bismut's decomposition that $\int_0^{H(U)} dr\ \sigma_{r,U}^{\beta-1}$ is, under $\mathbb{N}[\sigma \bullet | H(U) = t]$, distributed as $\int_0^t dr\, S_r^{\beta-1}$. Thanks to [21] see Theorem 11 in chapter III and since S is a stable subordinator with index $(\gamma-1)/\gamma$, we have that $\limsup_{r\to 0+} S_r/h(r) > 0$ a.s. for $h(r) = r^{\gamma/(\gamma-1)} \log(|\log(r)|)^{-1/(\gamma-1)}$. As $\beta \in (0, 1/\gamma]$, we have $\int_0 dr\, h(r)^{\beta-1} = +\infty$. This implies that a.s. $\int_0 dr\, S_r^{\beta-1} = +\infty$. We deduce that $\mathbb{N}$-a.e. ds-a.e. on $[0, \sigma]$, $\int_0^{H(s)} dr\ \sigma_{r,s}^{\beta-1} = +\infty$. This gives that $\mathbb{N}$-a.e. $Z_\beta^H = +\infty$. Then use the scaling to deduce that $\mathbb{N}^{(1)}$-a.s. $Z_\beta^H = +\infty$.

Acknowledgments. We would like to thank the Referees and the Associate Editor for their useful comments which helped to improve the paper. We also thank T. Duquesne for discussions on the height process of Lévy trees.