INTENSITY ESTIMATION PERTURBED POINT PROCESSES

Abstract

This article proposes new kernel estimators of the intensity function of spatial point processes taking into account position errors. The asymptotic properties of these estimators are derived. A simulation study compares their results to the results of the classical kernel estimator and shows that the edge-corrected deconvoluting kernel estimator is the most appropriate.

6.1. Introduction

In the theory of kernel density estimation, many authors have considered the problem of estimating the density from noisy observations. Indeed, one may consider that each measurement reflects the true value polluted by the addition of a stochastic error. This problem is usually handled by a deconvolution method, either when the distribution of the errors is known (Stefanski & Carroll, 1990) or unknown (Diggle & Hall, 1993).

When dealing with a spatial point pattern, a systematic exploratory tool is the intensity function, which is the equivalent of the trend for geostatistical data. Some authors (Ogata & Katsura, 1986) propose a parametric estimation of the intensity function but nonparametric estimators are more frequent. The most common nonparametric estimators of the intensity function are derived from the multivariate density estimation theory : mainly kernel and nearest-neighbour estimators (Cressie, 1993). Recently, a new approach based on a hierarchical Bayesian model and the Voronoi tessellation has also been proposed (Heikkinen & Arjas, 1998 and Byers & Raftery, 2002).

All of these methods use locational data of the events, which are often difficult to collect and subsequently whose measurements are subject to errors. Lund & Rudemo (2000) try to make inference on such point processes observed with noise while Bar-Hen & *al.* (2005) study the influence of measurement errors on descriptive statistics used for testing the complete spatial randomness. In this paper, we propose a new kernel estimator of the intensity function which takes into account the location errors by a deconvolution method. For simplicity, we develop it in the case of bidimensional point processes.

Each kernel method is subject to a crucial choice, which is the bandwidth selection, much more important than the kernel choice itself (Silverman, 1986). This choice has been extensively discussed in the literature and original procedures have been proposed either for the deconvolution kernel density estimation problem (Delaigle & Gijbels, 2004) or for the kernel intensity estimation problem (Xu & *al.*, 2003).

Section 2 is an introduction to the perturbed point processes. We then define the new estimator and discuss its properties in Section 3. We present an asymptotic study in Section 4 and an adaptation of an existing bandwidth selection procedure to this specific problem in Section 5. The usefulness of the estimator is assessed by its application to simulated data in Section 6.

6.2. Perturbed point processes

Consider a Poisson point process $\mathbf{Y}$ in $\mathbb{R}^2$ with intensity function $\lambda_Y()$.

We only observe the point pattern $Z = \{z_1, \cdots, z_N\}$ in the domain $D \subset \mathbb{R}^2$ according to the model :

$$z_i = y_i + \epsilon_i, \tag{1}$$

where $(y_i : i = 1, \cdots, N)$ are events issued from the process $\mathbf{Y}$ and $(\epsilon_i : i = 1, \cdots, n)$ are i.i.d. with known isometric density function $g(.)$ and represent the location errors. We will also assume that the errors ϵ_i are independant from the true locations y_i.

This additive error model is very common in statistics, for example in the regression framework (Carroll, Maca & Ruppert, 1999). It has been used in the point process framework by Bar-Hen & *al.* (2005). As in their paper, we denote by $\mathbf{Y}$ the unperturbed (true) point process and by $\mathbf{Z}$ the perturbed (observed) point process.

Our goal is to estimate the intensity function $\lambda_Y(s)$ for every point $s \in D$.

6.3. The deconvoluting kernel intensity estimators

Denote $\lambda_Z(.)$ the intensity function of the perturbed process $\mathbf{Z}$.

Based on the observations Z, the edge-corrected kernel estimator for $\lambda_Z(.)$ is (Diggle, 1985) :

$$\forall s \in \mathbb{R}^2, \hat{\lambda}_{Z,h}(s) = \begin{cases} \frac{\sum_{j=1}^{n} \frac{1}{h^2} K\left(\frac{s-z_j}{h}\right)}{\int_D \frac{1}{h^2} K\left(\frac{s-u}{h}\right)\nu(du)} & \text{if } \int_D \frac{1}{h^2} K\left(\frac{s-u}{h}\right)\nu(du) \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $K(.)$ is a kernel function and ν represents the Lebesgue measure.

From now on, we will assume that

$$\int_{\mathbb{R}^2} ||u|||K(u)|\nu(du) < \infty. \tag{2}$$

The denominator $p_h(s) = \int_D \frac{1}{h^2} K\left(\frac{s-u}{h}\right)\nu(du)$ ensures that this estimator is asymptotically unbiased when $h \to 0$ and its practical usefulness has been shown (Zheng & *al.*, 2004). Denote $G_h = \{s \in \mathbb{R}^2 : p_h(s) \neq 0\}$.

The bidimensional Fourier transform of $g(.)$ is

$$\mathcal{F}(g)(t) = \int_{\mathbb{R}^2} e^{-it'z} g(z)\nu(dz)$$

where $z = (z_{(1)}z_{(2)})', t = (t_{(1)}t_{(2)})'$ and $t'z = t_{(1)}z_{(1)} + t_{(2)}z_{(2)}$.

Assume
$$\forall t \in \mathbb{R}^2, |\mathcal{F}(g)(t)| > 0. \tag{3}$$

In the density estimation framework, Stefanski & Carroll (1990) introduced a deconvoluting estimator adapted to noisy observations. Without taking into account the limited domain constraint, an estimator of $\lambda_Y(s)$ inspired by this is

$$\begin{aligned}\lambda^*_{Y,h}(s) &= \sum_{j=1}^{n} \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{is't} \Big\{ \int_{\mathbb{R}^2} e^{-it'z} \frac{1}{h^2} K\big(\frac{z-z_j}{h}\big) \nu(dz) / \mathcal{F}(g)(t) \Big\} \nu(dt) \\ &= \sum_{j=1}^{n} \frac{1}{h^2} K^*_h\Big(\frac{s-z_j}{h}\Big),\end{aligned}$$

where $K^*_h(t) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{it'y} \mathcal{F}(K)(y)/\mathcal{F}(g)(y/h) dy$.

Here, the choice of a band-limited kernel K, combined to (3), ensures that the inverse Fourier transform can be applied.

Now, from (1) we get $\lambda_Z(.) = \lambda_Y(.) * g(.) \Rightarrow \mathcal{F}(\lambda_Z)(.) = \mathcal{F}(\lambda_Y)(.)\ \mathcal{F}(g)(.)$ and a natural estimator of $\lambda_Y(s)$ is then

$$\begin{aligned}\hat{\lambda}_{Y,h}(s) &= \mathcal{F}^{-1}\big(\mathcal{F}(\hat{\lambda}_{Z,h})(t)/\mathcal{F}(g)(t)\big)(s) \\ &= \sum_{j=1}^{n} \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{is't} \Big\{ \int_{G_h} \frac{e^{-it'z} \frac{1}{h^2} K\big(\frac{z-z_j}{h}\big)}{p_h(z)} \nu(dz)/\mathcal{F}(g)(t) \Big\} \nu(dt).\end{aligned}$$

Unfortunately, due to the presence of the edge-correction term, it is not clear to find a condition concerning the kernel K ensuring that the inverse Fourier transform can be applied. This is a main difference with the estimator $\lambda^*_{Y,h}$ previously introduced as it prevents its practical use.

A way of adapting the estimator $\lambda^*_{Y,h}$ to the limited domain context is to define
$$\lambda^{**}_{Y,h}(s) = \begin{cases} \frac{\lambda^*_{Y,h}(s)}{p^*_h(s)} & \text{if } p^*_h(s) \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $p^*_h(s)$ is an edge-correction term judiciously chosen, which will be discussed in the next section. We will denote $G'_h = \{s \in \mathbb{R}^2 : p^*_h(s) \neq 0\}$.

Whatever estimator we may choose, we can remark that, if the support of the density function g is $\mathbb{R}^2$, the estimation of λ_Y in each point of D requires the estimation of λ_Z in each point of $\mathbb{R}^2$. Here λ_Z may be highly underestimated in $\bar{D}$ as events in this domain are not observed. We believe that this makes the problem difficult and that the quality of any estimator will be affected by this drawback.

6.4. The asymptotic framework

In kernel density estimation in $\mathbb{R}^d$, the asymptotic framework is usually the following : the sample size n tends to infinity and the bandwidth h tends to 0 s.t. $nh^d \to \infty$, allowing the estimated density in every point to depend on an expected number of observations tending to infinity. In the point process theory, one often assumes that the expectation of the number of observed events N will tend to infinity with the size of the domain D : this is described as the increasing-domain asymptotics. However, in this case, with a given intensity, letting the bandwidth h tend to 0 implies that the estimated intensity in every point will depend on an expected number of events tending to 0.

The solution adopted by Diggle & Marron (1988) is to set up an increasing-intensity asymptotic framework. Denote $m_Y = \int_{\mathbb{R}^2} \lambda_Y(s)\nu(ds)$. Letting m_Y tend to infinity, Cucala & Thomas-Agnan (2006) obtain a consistent kernel estimator of $\frac{\lambda_Y(s)}{m_Y}$ in the error-free unbounded-domain case (no measurement error, unbounded domain). We decide to adopt the same scheme here so we will study the asymptotic behaviour of $\hat{\lambda}^0_{Y,h}(s) = \frac{\hat{\lambda}_{Y,h}(s)}{N}\mathbb{1}[N \neq 0]$ and $\lambda^{**0}_{Y,h}(s) = \frac{\lambda^{**}_{Y,h}(s)}{N}\mathbb{1}[N \neq 0]$ when m_Y tends to infinity.

Following the idea of Lahiri & *al.* (1999), it is also possible to set up a mixed asymptotic framework in which both the intensity and the observation domain increase to infinity, the first at a faster rate than the second.

Finally, let us mention two other alternative asymptotic frameworks. The first relies on replacing the increasing-domain asymptotic framework by an increasing-time asymptotic framework, as defined by Ellis (1991) : the length of the observation time T tends to infinity s.t. $Th^d \to \infty$ and the intensity is assumed to be constant in time. On the other hand, Kutoyants (1998) considers several realizations of the process on a finite domain and lets this number of realizations tend to infinity.

6.4.1. Preliminary results. — Denote $m_Z = \int_D \lambda_Z(s)\nu(ds)$ and $\lambda_Z^0(s) = \frac{\lambda_Z(s)}{m_Z}$, $\lambda_Y^0(s) = \frac{\lambda_Y(s)}{m_Z}$. It has been shown (Cucala & Thomas-Agnan, 2006) that, if we denote the random variable $X = \frac{1}{N}\sum_{i=1}^N f(Z_i)\mathbb{1}[N \neq 0]$, where $f()$ is any given measurable function,

$$\mathbb{E}X = (1 - e^{-m_Z}) \int_D f(s)\lambda_Z^0(s)\nu(ds) \tag{4}$$

and

$$\mathrm{Var}X = \int_D f^2(s)\lambda_Z^0(s)\nu(ds)A(m_Z) - \Big(\int_D f(s)\lambda_Z^0(s)\nu(ds)\Big)^2 \big(A(m_Z) - e^{-m_Z} + e^{-2m_Z}\big) \tag{5}$$

where $A(m_Z) = e^{-m_Z}\sum_{k=1}^{\infty} \frac{m_Z^k}{k k!} = \mathbb{E}\Big[\frac{1}{N}\mathbb{1}[N \neq 0]\Big]$.

6.4.2. Asymptotic bias of the estimator $\hat{\lambda}_{Y,h}$. — Even if the Fourier transform leading to this estimator is not ensured to be finite, we would like to know if a suitable choice of the kernel K can lead to an unbiased estimator.

From (4), we have $\mathbb{E}\hat{\lambda}_{Y,h}^0(s) = \dfrac{1 - e^{-m_Z}}{m_Z(2\pi)^2} \times$

$$\int_{\mathbb{R}^2} e^{is't}\Big\{\int_{G_h} \frac{e^{-it'z}}{p_h(z)} \int_D \frac{1}{h^2} K\Big(\frac{z-x}{h}\Big)\lambda_Z(x)\nu(dx)\nu(dz)/\mathcal{F}(g)(t)\Big\}\nu(dt)$$

which finally leads to (see Appendix)

$$\frac{m_Z(2\pi)^2}{1-e^{-m_Z}}\,\mathbb{E}(\hat{\lambda}^0_{Y,h}(s))$$
$$=\int_{\mathbb{R}^2} e^{-is't}\Big\{\frac{\int_{\mathbb{R}^2}\int_{G_h} e^{-it'(z-\epsilon)}\lambda_Y(z-\epsilon)\nu(dz)e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}{\int_{\mathbb{R}^2} e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}\Big\}\nu(dt)$$
$$-h\int_{\mathbb{R}^2} e^{-is't}\Big\{\frac{\int_{\mathbb{R}^2}\int_{G_h}\frac{\int_{B_{z,h}}u_{(1)}K(u)\nu(du)}{\int_{B_{z,h}}K(u)\nu(du)}e^{-it'(z-\epsilon)}\frac{\partial\lambda_Y}{\partial s_{(1)}}(z-\epsilon)\nu(dz)e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}{\int_{\mathbb{R}^2} e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}\Big\}\nu(dt)$$
$$-h\int_{\mathbb{R}^2} e^{-is't}\Big\{\frac{\int_{\mathbb{R}^2}\int_{G_h}\frac{\int_{B_{z,h}}u_{(2)}K(u)\nu(du)}{\int_{B_{z,h}}K(u)\nu(du)}e^{-it'(z-\epsilon)}\frac{\partial\lambda_Y}{\partial s_{(2)}}(z-\epsilon)\nu(dz)e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}{\int_{\mathbb{R}^2} e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}\Big\}\nu(dt)$$
$$+\frac{h^2}{2}\int_{\mathbb{R}^2} e^{-is't}\Big\{\frac{\int_{\mathbb{R}^2}\int_{G_h}\frac{\int_{B_{z,h}}u^2_{(1)}K(u)\nu(du)}{\int_{B_{z,h}}K(u)\nu(du)}e^{-it'(z-\epsilon)}\frac{\partial^2\lambda_Y}{\partial {s_{(1)}}^2}(z-\epsilon)\nu(dz)e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}{\int_{\mathbb{R}^2} e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}\Big\}\nu(dt)$$
$$+h^2\int_{\mathbb{R}^2} e^{-is't}\Big\{\frac{\int_{\mathbb{R}^2}\int_{G_h}\frac{\int_{B_{z,h}}u_{(1)}u_{(2)}K(u)\nu(du)}{\int_{B_{z,h}}K(u)\nu(du)}e^{-it'(z-\epsilon)}\frac{\partial^2\lambda_Y}{\partial s_{(1)}\partial s_{(2)}}(z-\epsilon)\nu(dz)e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}{\int_{\mathbb{R}^2} e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}\Big\}\nu(dt)$$
$$+\frac{h^2}{2}\int_{\mathbb{R}^2} e^{-is't}\Big\{\frac{\int_{\mathbb{R}^2}\int_{G_h}\frac{\int_{B_{z,h}}u^2_{(2)}K(u)\nu(du)}{\int_{B_{z,h}}K(u)\nu(du)}e^{-it'(z-\epsilon)}\frac{\partial^2\lambda_Y}{\partial {s_{(2)}}^2}(z-\epsilon)\nu(dz)e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}{\int_{\mathbb{R}^2} e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}\Big\}\nu(dt)+O(h^3).$$

Let $m_Z\to\infty$ and $h\to 0$.

First case : If the kernel K has a compact support, then as $h\to 0$, $G_h\to D$ in a monotone way and $\forall z\in D, B_{z,h}\to\mathbb{R}^2$ in a monotone way. Thus the expectation is asymptotically equal to

$$\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{-is't}\Big\{\frac{\int_{\mathbb{R}^2}\int_{D} e^{-it'(z-\epsilon)}\lambda^0_Y(z-\epsilon)\nu(dz)e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}{\int_{\mathbb{R}^2} e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}\Big\}\nu(dt)+O(h^2).$$

Second case : If the kernel K does not have a compact support, then $G_h=\mathbb{R}^2$ and $\forall z\in D, B_{z,h}\to\mathbb{R}^2$ in a monotone way. But $\forall z\in\bar{D}, B_{z,h}$ has no limit. Thus the bias is asymptotically equal to

$$\frac{1}{(2\pi)^2}\Bigg\{-h\int_{\mathbb{R}^2} e^{-is't}\Big\{\frac{\int_{\mathbb{R}^2}\int_{\bar{D}}\frac{\int_{B_{z,h}}u_{(1)}K(u)\nu(du)}{\int_{B_{z,h}}K(u)\nu(du)}e^{-it'(z-\epsilon)}\frac{\partial\lambda_Y}{\partial s_{(1)}}(z-\epsilon)\nu(dz)e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}{\int_{\mathbb{R}^2}e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}\Big\}\nu(dt)$$

$$-h\int_{\mathbb{R}^2} e^{-is't}\Big\{\frac{\int_{\mathbb{R}^2}\int_{\bar{D}}\frac{\int_{B_{z,h}}u_{(2)}K(u)\nu(du)}{\int_{B_{z,h}}K(u)\nu(du)}e^{-it'(z-\epsilon)}\frac{\partial\lambda_Y}{\partial s_{(2)}}(z-\epsilon)\nu(dz)e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}{\int_{\mathbb{R}^2}e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}\Big\}\nu(dt)$$

$$+\frac{h^2}{2}\int_{\mathbb{R}^2} e^{-is't}\Big\{\frac{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{\int_{B_{z,h}}u_{(1)}^2K(u)\nu(du)}{\int_{B_{z,h}}K(u)\nu(du)}e^{-it'(z-\epsilon)}\frac{\partial^2\lambda_Y}{\partial {s_{(1)}}^2}(z-\epsilon)\nu(dz)e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}{\int_{\mathbb{R}^2}e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}\Big\}\nu(dt)$$

$$+h^2\int_{\mathbb{R}^2} e^{-is't}\Big\{\frac{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{\int_{B_{z,h}}u_{(1)}u_{(2)}K(u)\nu(du)}{\int_{B_{z,h}}K(u)\nu(du)}e^{-it'(z-\epsilon)}\frac{\partial^2\lambda_Y}{\partial s_{(1)}\partial s_{(2)}}(z-\epsilon)\nu(dz)e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}{\int_{\mathbb{R}^2}e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}\Big\}\nu(dt)$$

$$+\frac{h^2}{2}\int_{\mathbb{R}^2} e^{-is't}\Big\{\frac{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{\int_{B_{z,h}}u_{(2)}^2K(u)\nu(du)}{\int_{B_{z,h}}K(u)\nu(du)}e^{-it'(z-\epsilon)}\frac{\partial^2\lambda_Y}{\partial {s_{(2)}}^2}(z-\epsilon)\nu(dz)e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}{\int_{\mathbb{R}^2}e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}\Big\}\nu(dt)+O(h^3)\Bigg\}.$$

But now the terms depending on h, as $h\frac{\int_{B_{z,h}}u_{(1)}K(u)\nu(du)}{\int_{B_{z,h}}K(u)\nu(du)}$, are asymptotically equivalent to constants. Indeed, for example,

$$h\frac{\int_{B_{z,h}}u_{(1)}K(u)\nu(du)}{\int_{B_{z,h}}K(u)\nu(du)}=\frac{\int_{B_{z,1}}u_{(1)}K(u/h)\nu(du)}{\int_{B_{z,1}}K(u/h)\nu(du)}\in[\min_{B_{z,1}}u_{(1)};\max_{B_{z,1}}u_{(1)}].$$

So it seems that, whatever kernel one chooses, it is not possible that the estimator $\hat{\lambda}^0_{Y,h}(s)$ is asymptotically unbiased.

6.4.3. Asymptotic bias of the estimator $\lambda^{**}_{Y,h}$. — In this paragraph, we demonstrate the difficulty of finding an edge correction factor p^*_h so that the estimator $\lambda^{**}_{Y,h}$ is unbiased. But we propose a judicious choice of this edge correction factor leading to a tractable estimator and to asymptotic unbiasedness in the case of no measurement error and in the case of constant intensity.

We have, $\forall s\in G'_h$,

$$\mathbb{E}\lambda^{**0}_{Y,h}(s)=\frac{1-e^{-m_Z}}{m_Z(2\pi)^2p^*_h(s)}\int_{\mathbb{R}^2}e^{is't}\Big\{\int_{\mathbb{R}^2}e^{-it'z}\int_D\frac{1}{h^2}K\Big(\frac{z-x}{h}\Big)\lambda_Z(x)\nu(dx)\nu(dz)/\mathcal{F}(g)(t)\Big\}\nu(dt).$$

And the asymptotic expectation is, when K is a band-limited kernel

$$
\begin{aligned}
&\frac{(2\pi)^{-2}}{p_h^*(s)}\Bigg\{\int_{\mathbb{R}^2} e^{-is't}\Big\{\frac{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{B_{z,h}} K(u)\nu(du)e^{-it'(z-\epsilon)}\lambda_Y^0(z-\epsilon)\nu(dz)e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}{\int_{\mathbb{R}^2} e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}\Big\}\nu(dt)\\
&-h\int_{\mathbb{R}^2} e^{-is't}\Big\{\frac{\int_{\mathbb{R}^2}\int_{\bar{D}}\int_{B_{z,h}} u_{(1)}K(u)\nu(du)e^{-it'(z-\epsilon)}\frac{\partial\lambda_Y^0}{\partial s_{(1)}}(z-\epsilon)\nu(dz)e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}{\int_{\mathbb{R}^2} e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}\Big\}\nu(dt)\\
&-h\int_{\mathbb{R}^2} e^{-is't}\Big\{\frac{\int_{\mathbb{R}^2}\int_{\bar{D}}\int_{B_{z,h}} u_{(2)}K(u)\nu(du)e^{-it'(z-\epsilon)}\frac{\partial\lambda_Y^0}{\partial s_{(2)}}(z-\epsilon)\nu(dz)e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}{\int_{\mathbb{R}^2} e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}\Big\}\nu(dt)\\
&+\frac{h^2}{2}\int_{\mathbb{R}^2} e^{-is't}\Big\{\frac{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{B_{z,h}} u_{(1)}^2K(u)\nu(du)e^{-it'(z-\epsilon)}\frac{\partial^2\lambda_Y^0}{\partial {s_{(1)}}^2}(z-\epsilon)\nu(dz)e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}{\int_{\mathbb{R}^2} e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}\Big\}\nu(dt)\\
&+h^2\int_{\mathbb{R}^2} e^{-is't}\Big\{\frac{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{B_{z,h}} u_{(1)}u_{(2)}K(u)\nu(du)e^{-it'z}\frac{\partial^2\lambda_Y^0}{\partial s_{(1)}\partial s_{(2)}}(z-\epsilon)\nu(dz)g(\epsilon)\nu(d\epsilon)}{\int_{\mathbb{R}^2} e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}\Big\}\nu(dt)\\
&+\frac{h^2}{2}\int_{\mathbb{R}^2} e^{-is't}\Big\{\frac{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{B_{z,h}} u_{(2)}^2K(u)\nu(du)e^{-it'z}\frac{\partial^2\lambda_Y^0}{\partial {s_{(2)}}^2}(z-\epsilon)\nu(dz)g(\epsilon)\nu(d\epsilon)}{\int_{\mathbb{R}^2} e^{-it'\epsilon}g(\epsilon)\nu(d\epsilon)}\Big\}\nu(dt)+O(h)^3\Bigg\}.
\end{aligned}
$$

In this expression, the terms depending on h such as $h\int_{B_{z,h}} u_{(1)}K(u)\nu(du)$ are asymptotically negligible. Indeed, for example, from (2),

$$\Big|h\int_{B_{z,h}} u_{(1)}K(u)\nu(du)\Big| < h\int_{\mathbb{R}^2} ||u|||K(u)|\nu(du) \to 0.$$

We realize that the ideal edge-correction term $p_h^*(s)$, leading to asymptotic unbiasedness, should be $\mathcal{F}^{-1}\Big(\frac{\mathcal{F}\big((\lambda_Y^0*g)\times p_h\big)(t)}{\mathcal{F}(g)(t)}\Big)(s)/\lambda_Y(s)$ which is of course unknown.

If we use this formula for a constant intensity, i.e. $\forall s \in \mathbb{R}^2, \lambda_Y^0(s) = 1$, we obtain

$$\begin{aligned}
p_h^*(s) &= \mathcal{F}^{-1}\Big(\frac{\mathcal{F}(p_h)(t)}{\mathcal{F}(g)(t)}\Big)(s) \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{is't}\frac{\int_{\mathbb{R}^2} e^{-it'z}\frac{1}{h^2}\int_D K\big(\frac{z-u}{h}\big)\nu(du)\nu(dz)}{\int_{\mathbb{R}^2} e^{-it'z}g(z)\nu(dz)}\nu(dt) \\
&= \int_D \frac{1}{h^2}\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{is't}\frac{\int_{\mathbb{R}^2} e^{-it'z}K\big(\frac{z-u}{h}\big)\nu(dz)}{\int_{\mathbb{R}^2} e^{-it'z}g(z)\nu(dz)}\nu(dt)\nu(du) \\
&= \int_D \frac{1}{h^2}K_h^*\Big(\frac{s-u}{h}\Big)\nu(du).
\end{aligned}$$

This edge-correction term is finite and $p_h^*(s)$ reduces to $p_h(s)$ when g reduces to a Dirac function (no measurement error).

Finally, we conclude that it seems that no consistent estimator is available for this complex problem. That is why we choose to focus on

$$\lambda_{Y,h}^{**}(s) = \frac{\sum_{j=1}^n \frac{1}{h^2}K_h^*\Big(\frac{s-z_j}{h}\Big)}{\int_D \frac{1}{h^2}K_h^*\Big(\frac{s-u}{h}\Big)\nu(du)}, \forall s \in G_h',$$

where K_h^* is the so-called deconvoluting kernel introduced by Stefanski & Carroll (1990).

Indeed, this estimator is much more tractable than $\hat{\lambda}_{Y,h}(s)$ as it uses the Fourier transform of the kernel K which is explicit, and the use of a band-limited kernel K ensures its existence. Moreover it reduces to Diggle's estimator when there is no measurement error.

6.4.4. Asymptotic variance of the estimator $\lambda^{**}_{Y,h}$. — The first integral from expression (5) is, $\forall s \in G'_h$,

$$
\begin{aligned}
B &= \int_D \left\{ \frac{1}{(2\pi)^2 p^*_h(s)} \int_{\mathbb{R}^2} e^{is't} \Big\{ \int_{\mathbb{R}^2} e^{-it'z} \frac{1}{h^2} K\big(\frac{z-x}{h}\big) \nu(dz)/\mathcal{F}(g)(t) \Big\} \nu(dt) \right\}^2 \lambda^0_Z(x)\nu(dx) \\
&= \frac{1}{(2\pi)^4 p^{*2}_h(s)} \int_D \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{is't} e^{is'v} \Big\{ \int_{\mathbb{R}^2} e^{-it'z} \frac{1}{h^2} K\Big(\frac{z-x}{h}\Big) \nu(dz)/\mathcal{F}(g)(t) \Big\} \\
&\quad \Big\{ \int_{\mathbb{R}^2} e^{-iv'w} \frac{1}{h^2} K\Big(\frac{w-x}{h}\Big) \nu(dw)/\mathcal{F}(g)(v) \Big\} \nu(dt)\nu(dv)\lambda^0_Z(x)\nu(dx) \\
&= \frac{1}{(2\pi)^4 p^{*2}_h(s) m_Z} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{e^{is't} e^{is'v}}{\mathcal{F}(g)(t)\mathcal{F}(g)(v)} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-it'z} e^{-iv'w} \\
&\quad \Big\{ \int_D \frac{1}{h^4} K\Big(\frac{z-x}{h}\Big) K\Big(\frac{w-x}{h}\Big) \lambda_Z(x)\nu(dx) \Big\} \nu(dw)\nu(dz)\nu(dt)\nu(dv).
\end{aligned}
$$

Now

$$
\int_D K\big(\frac{z-x}{h}\big) K\big(\frac{w-x}{h}\big) \frac{\lambda_Z(x)}{h^4} \nu(dx) = \int_{\mathbb{R}^2} \int_D K\big(\frac{z-x}{h}\big) K\big(\frac{w-x}{h}\big) \frac{\lambda_Y(x-\epsilon)}{h^4} \nu(dx) g(\epsilon)\nu(d\epsilon)
$$

and

$$
\begin{aligned}
& \int_D \frac{1}{h^4} K\big(\frac{z-x}{h}\big) K\big(\frac{w-x}{h}\big) \lambda_Y(x-\epsilon)\nu(dx) \\
= \ & \frac{1}{h^2} \int_{B_{z,h}} K(u) K\big(u - \frac{z-w}{h}\big) \lambda_Y(z-uh-\epsilon)\nu(du) \\
= \ & \frac{\lambda_Y(z-\epsilon)}{h^2} \int_{B_{z,h}} K(u) K\big(u - \frac{z-w}{h}\big) \nu(du) - \frac{1}{h} \frac{\partial \lambda_Y}{\partial s_{(1)}}(z-\epsilon) \int_{B_{z,h}} u_{(1)} K(u) \\
& K\big(u - \frac{z-w}{h}\big) \nu(du) - \frac{1}{h} \frac{\partial \lambda_Y}{\partial s_{(2)}}(z-\epsilon) \int_{B_{z,h}} u_{(2)} K(u) K\big(u - \frac{z-w}{h}\big) \nu(du) + O(1) \\
\sim_{h\to 0} \ & \frac{\lambda_Y(z-\epsilon)}{h^2} \int_{B_{z,h}} K^2(u) \mathbb{1}(z=w)\nu(du) - \frac{1}{h} \frac{\partial \lambda_Y}{\partial s_{(1)}}(z-\epsilon) \\
& \int_{B_{z,h}} u_{(1)} K^2(u) \mathbb{1}(z=w)\nu(du) - \frac{1}{h} \frac{\partial \lambda_Y}{\partial s_{(2)}}(z-\epsilon) \int_{B_{z,h}} u_{(2)} K^2(u) \mathbb{1}(z=w)\nu(du)
\end{aligned}
$$

So we get

$$B \sim_{h\to 0} \frac{1}{h^2(2\pi)^4 p_h^{*2}(s)} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{e^{is't}e^{is'v}}{\mathcal{F}(g)(t)\mathcal{F}(g)(v)} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-it'(z-\epsilon)}e^{-iv'(z-\epsilon)}\lambda_Y^0(z-\epsilon)$$
$$\int_{B_{z,h}} K^2(u)\nu(du)\nu(dz)e^{-it'\epsilon}e^{-iv'\epsilon}g(\epsilon)\nu(d\epsilon)\nu(dt)\nu(dv).$$

Moreover we have $\int_{B_{z,h}} K^2(u)\nu(du) \underset{h\to 0}{\longrightarrow} \begin{cases} \int_{\mathbb{R}^2} K^2(u)\nu(du) & \text{if } z \in D, \\ 0 & \text{otherwise.} \end{cases}$

Consequently,

$$B \sim_{h\to 0} \frac{\int_{\mathbb{R}^2} K^2(u)\nu(du)}{h^2(2\pi)^4 p_h^{*2}(s)} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{e^{is't}e^{is'v}}{\mathcal{F}(g)(t)\mathcal{F}(g)(v)} \int_{\mathbb{R}^2}\int_{D} e^{-it'(z-\epsilon)}e^{-iv'(z-\epsilon)}\lambda_Y^0(z-\epsilon)$$
$$\nu(dz)e^{-it'\epsilon}e^{-iv'\epsilon}g(\epsilon)\nu(d\epsilon)\nu(dt)\nu(dv).$$

Now, the second integral from expression (5) is

$$C = \left\{ \int_D \frac{1}{(2\pi)^2 p_h^*(s)} \int_{\mathbb{R}^2} e^{is't}\Big\{ \int_{\mathbb{R}^2} e^{-it'z}\frac{1}{h^2}K\big(\frac{z-x}{h}\big)\nu(dz)/\mathcal{F}(g)(t)\Big\}\nu(dt)\lambda_Z^0(x)\nu(dx)\right\}^2$$
$$= O(p_h^*(s)^{-2}).$$

So the asymptotic variance of $\lambda_{Y,h}^{**0}(s)$ is the product of $\dfrac{A(m_Z)}{h^2}\dfrac{\int_{\mathbb{R}^2} K^2(u)\nu(du)}{(2\pi)^4 p_h^{*2}(s)}$ by

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{e^{is't}e^{is'v}}{\mathcal{F}(g)(t)\mathcal{F}(g)(v)} \int_{\mathbb{R}^2}\int_{D} e^{-it'(z-\epsilon)}e^{-iv'(z-\epsilon)}\lambda_Y^0(z-\epsilon)\nu(dz)e^{-it'\epsilon}e^{-iv'\epsilon}g(\epsilon)\nu(d\epsilon)\nu(dt)\nu(dv).$$

6.4.5. The link with the classical deconvolution estimator. — As mentioned before, we could also let the observation domain D tend to $\mathbb{R}^2$ and thus use an asymptotic framework similar to Lahiri & *al* (1999) 's mixed framework or Fuentes (2002) 's "shrinking asymptotics" framework.

For example, let $D = \gamma D_0$ with $\gamma \to \infty$ and $\frac{m_Z}{\gamma} \to \infty$ and consider a product kernel $K(s) = K_0(s^{(1)})K_0(s^{(2)}), \forall s = (s^{(1)}, s^{(2)})' \in \mathbb{R}^2$. In that case, the results of 4.3 and 4.4 would be exactly the two-dimensional equivalent of those obtained by the estimator introduced by Stefanski & Carroll (1990) in the unbounded-domain case. By "equivalent" we mean that the asymptotic bias is $\frac{h^2}{2}\int_{\mathbb{R}} x^2 K_0(x)dx\nabla^2\lambda_Y^0(s)$ and the asymptotic variance is $\frac{\mathbb{E}(1/N)}{h^2}K_h^*(t)^2\lambda_Y^0(s)$, the only difference being the factor $\frac{1}{n}$ in the asymptotic variance term replaced by $A(m_Z) = \mathbb{E}\Big(\frac{1}{N}\Big)$.

6.5. The bandwidth selection procedure

Let us look for the bandwidth h minimizing the mean integrated square error (MISE)

$$MISE(h) = \mathbb{E}\int_D \{\lambda_{Y,h}^{**0}(s) - \lambda_Y^0(s)\}^2\nu(ds).$$

Due to their complexity and their dependance on the domain D, we will not use the expressions of the asymptotic bias and variance to set up a bandwidth selection procedure. Instead, we will rely on the procedures described by Delaigle & Gijbels (2004) in the unbounded-domain framework and adapt them to the bidimensional case.

The expression of the asymptotic MISE obtained by Stefanski & Carroll (1990) comes directly from the asymptotic bias and variance expressed before and Parseval's identity. We get for dimension 2

$$AMISE(h) = \frac{\mathbb{E}(1/N)}{(2\pi h)^2}\int_{\mathbb{R}^2}\frac{\mathcal{F}(K)(t)^2}{|\mathcal{F}(g)(t/h)|^2}\nu(dt) + \frac{h^4}{4}\alpha^2\int_{\mathbb{R}^2}\big(\nabla^2\lambda_Y^0(s)\big)^2\nu(ds),$$

where $\alpha = \int_{\mathbb{R}} x^2K_0(x)dx$.

In order to minimize this expression, we need to estimate the term

$\int_{\mathbb{R}^2}\big(\nabla^2\lambda_Y^0(s)\big)^2\nu(ds)$. We propose to use a normal-reference rule. Suppose λ_Y^0 is the density of a Gaussian distribution with variance matrix

$\Sigma = \begin{pmatrix} \sigma_{Y,1}^2 & \rho_Y \sigma_{Y,1}\sigma_{Y,2} \\ \rho_Y \sigma_{Y,1}\sigma_{Y,2} & \sigma_{Y,2}^2 \end{pmatrix}$, then we have $\int_{\mathbb{R}^2} \left(\nabla^2 \lambda_Y^0(s)\right)^2 \nu(ds)$

$$\begin{aligned}
= & \quad \frac{(\sigma_{Y,1}^2+\sigma_{Y,2}^2)^2}{4\pi\sigma_{Y,1}^5\sigma_{Y,2}^5(1-\rho_Y^2)^{5/2}} - \frac{\sigma_{Y,1}^2+\sigma_{Y,2}^2}{4\pi\sigma_{Y,1}^3\sigma_{Y,2}^5(1-\rho_Y^2)^{3/2}} + \frac{3}{16\pi\sigma_{Y,1}\sigma_{Y,2}^5\sqrt{1-\rho_Y^2}} \\
- & \quad \frac{\sqrt{1-\rho_Y^2}\left(\sigma_{Y,1}^4\rho_Y^2+\sigma_{Y,1}^2\sigma_{Y,2}^2(1+\rho_Y^2)+\sigma_{Y,2}^4\right)}{4\pi\sigma_{Y,1}^5\sigma_{Y,2}^5(1-\rho_Y^2)^3} + \frac{(3\rho_Y^2\sigma_{Y,1}^2+\sigma_{Y,2}^2)\sqrt{1-\rho_Y^2}}{8\pi\sigma_{Y,1}^3\sigma_{Y,2}^5(1-\rho_Y^2)^2} \\
+ & \quad \frac{3\sqrt{1-\rho_Y^2}(\rho_Y^2\sigma_{Y,1}^2+\sigma_{Y,2}^2)^2}{16\pi\sigma_{Y,1}^5\sigma_{Y,2}^5(1-\rho_Y^2)^3} \\
= & \quad H(\sigma_{Y,1},\sigma_{Y,2},\rho_Y).
\end{aligned}$$

Denote $\sigma_{Z,1}^2 = Var(z^{(1)})$, $\sigma_{Z,2}^2 = Var(z^{(2)})$ and $\rho_Z = \frac{Cov(z^{(1)},z^{(2)})}{\sqrt{Var(z^{(1)})}\sqrt{Var(z^{(2)})}}$, where z is distributed according to λ_Z^0.

Denote $\sigma_{\epsilon,1}^2 = Var(\epsilon^{(1)})$, $\sigma_{\epsilon,2}^2 = Var(\epsilon^{(2)})$ and $\rho_\epsilon = \frac{Cov(\epsilon^{(1)},\epsilon^{(2)})}{\sqrt{Var(\epsilon^{(1)})}\sqrt{Var(\epsilon^{(2)})}}$, where ϵ is distributed according to g.

Now we get $\sigma_{Z,1}^2 = \sigma_{Y,1}^2 + \sigma_{\epsilon,1}^2$, $\sigma_{Z,2}^2 = \sigma_{Y,2}^2 + \sigma_{\epsilon,2}^2$ and $\rho_Z = \frac{\rho_Y\sigma_{Y,1}\sigma_{Y,2}+\rho_\epsilon\sigma_{\epsilon,1}\sigma_{\epsilon,2}}{\sigma_{Z,1}\sigma_{Z,2}}$.

$\sigma_{Z,1}^2$ can be estimated by $\hat{\sigma}_{Z,1}^2 = \frac{1}{n}\sum_{i=1}^n (z_i^{(1)} - \bar{z^{(1)}})^2$, where $\bar{z^{(1)}} = \frac{1}{n}\sum_{i=1}^n z_i^{(1)}$.

$\sigma_{Z,2}^2$ can be estimated by $\hat{\sigma}_{Z,2}^2 = \frac{1}{n}\sum_{i=1}^n (z_i^{(2)} - \bar{z^{(2)}})^2$, where $\bar{z^{(2)}} = \frac{1}{n}\sum_{i=1}^n z_i^{(2)}$.

ρ_Z can be estimated by $\hat{\rho}_Z = \frac{\sum_{i=1}^n (z_i^{(1)}-\bar{z^{(1)}})(z_i^{(2)}-\bar{z^{(2)}})}{\sqrt{\sum_{i=1}^n (z_i^{(1)}-\bar{z^{(1)}})^2}\sqrt{\sum_{i=1}^n (z_i^{(2)}-\bar{z^{(2)}})^2}}$.

And finally an estimator of $\int_{\mathbb{R}^2} \left(\nabla^2 \lambda_Y^0(s)\right)^2 \nu(ds)$ is

$$H\Big(\hat{\sigma}_{Z,1}^2 - \sigma_{\epsilon,1}^2, \hat{\sigma}_{Z,2}^2 - \sigma_{\epsilon,2}^2, \frac{\hat{\rho}_Z\hat{\sigma}_{Z,1}\hat{\sigma}_{Z,2} - \rho_\epsilon\sigma_{\epsilon,1}\sigma_{\epsilon,2}}{\sqrt{(\hat{\sigma}_{Z,1}^2 - \sigma_{\epsilon,1}^2)(\hat{\sigma}_{Z,2}^2 - \sigma_{\epsilon,2})^2}}\Big).$$

On the other hand, $\mathbb{E}(1/N)$ will be estimated by $1/n$.

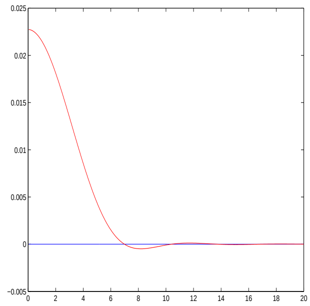

FIGURE 1. Profile of the kernel K_0

6.6. Computation of the estimator

6.6.1. A band-limited kernel. — As already said, the choice of the kernel is of secondary importance for the quality of our estimator. Here, for practical purpose, we choose a bidimensional kernel whose Fourier transform has compact support. The chosen kernel is a product kernel $K(x, y) = K_0(x)K_0(y)$, where

$$K_0(t) = \frac{48}{\pi}\frac{t^3\cos(t) - 6t^2\sin(t) + 15\sin(t) - 15t\cos(t)}{t^7}$$

is a one-dimensional band-limited kernel also used by Delaigle & Gijbels (2004). Figure 1 gives its profile.

We notice that it is very similar to the triangular kernel. It can lead to negative values for $\lambda_Z(s)$ but a nonnegative kernel may also lead to negative values for $\lambda_Y(s)$ due to the deconvolution method.

6.6.2. The Fourier transforms. — The Fourier transform of the chosen kernel is

$$\mathcal{F}(K)(t) = (1 - t_1^2)^3(1 - t_2^2)^3 \mathbb{1}_{[-1,1]^2}(t).$$

The Fourier transform of the density function of the errors g can usually be calculated analytically. For example, if the locational errors are normally distributed with mean $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and variance matrix $\Sigma = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}$, then we have $\mathcal{F}(g)(t) = e^{-\frac{\sigma^2}{2}|t|^2}$. If the marginal locational errors are independent Laplace random variables with mean 0 and variance σ^2, we have $\mathcal{F}(g)(t) = \frac{1}{1+\sigma^2 t_1^2}\frac{1}{1+\sigma^2 t_2^2}$.

As in Stefanski & Carroll (1990), the inverse Fourier transforms are evaluated by a numerical Simpson procedure, slower but more accurate than the FFT procedure.

6.7. A simulation study

An inhomogeneous Poisson process is simulated in $[0,1]^2$ enlarged by a guard area with intensity

$$\lambda_Y(s) = C\big[1 + 0.7\cos\big(2\pi(||s|| - 0.5)\big)\big],$$

where C is a constant chosen such that the expected number of events in $[0,1]^2$ is 100. This is done by an acceptation-rejection method (Gentle, 2002).

The location errors $\{\epsilon_i,\ i = 1, \cdots, n\}$ are then simulated and added to the simulated locations :

$$z_i = y_i + \epsilon_i.$$

Only the observations z_i in $[0,1]^2$ will be used to estimate the intensity.

From the simulated sample, we compute the estimates $\hat{\lambda}_{Z,h_{opt}}$, λ^*_{Y,h^*} and λ^{**}_{Y,h^*}, where h_{opt} is the bandwidth obtained by the classical cross-validation procedure (Silverman, 1986) and h^* is the bandwidth obtained via the procedure described in section 5.

Denote $ISE = \int_{[0,1]^2} \big(\hat{\lambda}_{Z,h_{opt}} - \lambda_Y(s)\big)^2 \nu(ds)$,

$ISE^* = \int_{[0,1]^2} \big(\lambda^*_{Y,h^*}(s) - \lambda_Y(s)\big)^2 \nu(ds)$, $ISE^{**} = \int_{[0,1]^2} \big(\lambda^{**}_{Y,h^*}(s) - \lambda_Y(s)\big)^2 \nu(ds)$.

This procedure is repeated m times and we compute the empirical quartiles of ISE, ISE^* and ISE^{**}. Tables 1, 2 and 3 give the results when ϵ follows a Gaussian distribution with mean $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and variance matrix $\Sigma = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}$, and the number m of realizations is equal to 10.

Tables 4, 5 and 6 give the results when ϵ follows a Laplace distribution with same mean and variance matrix, and the number m of realizations is equal to 10.

TABLE 1. Gaussian error, σ=0.02

	ISE	ISE^*	ISE^{**}
1st quartile ($*10^3$)	1.0600	1.6745	0.9038
median ($*10^3$)	1.3939	1.9613	1.0279
3rd quartile ($*10^3$)	1.5899	2.2432	1.3158

TABLE 2. Gaussian error, σ=0.05

	ISE	ISE^*	ISE^{**}
1st quartile ($*10^3$)	0.8185	1.4153	0.6655
median ($*10^3$)	1.2474	1.7199	0.9298
3rd quartile ($*10^3$)	1.5281	1.8908	1.2138

TABLE 3. Gaussian error, σ=0.1

	ISE	ISE^*	ISE^{**}
1st quartile ($*10^3$)	0.7669	1.2194	0.7223
median ($*10^3$)	0.8854	1.4123	0.8733
3rd quartile ($*10^3$)	1.4305	1.6451	1.2544

TABLE 4. Laplace error, σ=0.02

	ISE	ISE^*	ISE^{**}
1st quartile ($*10^3$)	1.0444	1.4676	0.8274
median ($*10^3$)	1.4129	1.7275	1.0025
3rd quartile ($*10^3$)	2.1357	1.9753	1.2334

TABLE 5. Laplace error, σ=0.05

	ISE	ISE^*	ISE^{**}
1st quartile ($*10^3$)	0.7869	1.1814	0.7689
median ($*10^3$)	1.4859	1.4223	1.1308
3rd quartile ($*10^3$)	2.0375	1.5114	1.4210

In each case, the estimator λ^{**}_{Y,h^*} gives the best results. The results of the estimator λ^*_{Y,h^*} are not better, or even worse, than the ones obtained by the classical Diggle estimator $\hat{\lambda}_{Z,h_{opt}}$, suggesting that deconvolution and edge-correction should both be considered when dealing with perturbed locations in a bounded domain.

TABLE 6. Laplace error, σ=0.1

	ISE	ISE^*	ISE^{**}
1st quartile ($*10^3$)	1.4211	1.2435	1.2350
median ($*10^3$)	1.7803	1.7003	1.4141
3rd quartile ($*10^3$)	2.1798	1.9612	1.6842

To get a better understanding of the use of the deconvolution kernel estimator, Figure 3 shows the contours of the true intensity and of the mean values of the three estimators when ϵ follows a Gaussian distribution with $\sigma = 0.05$.

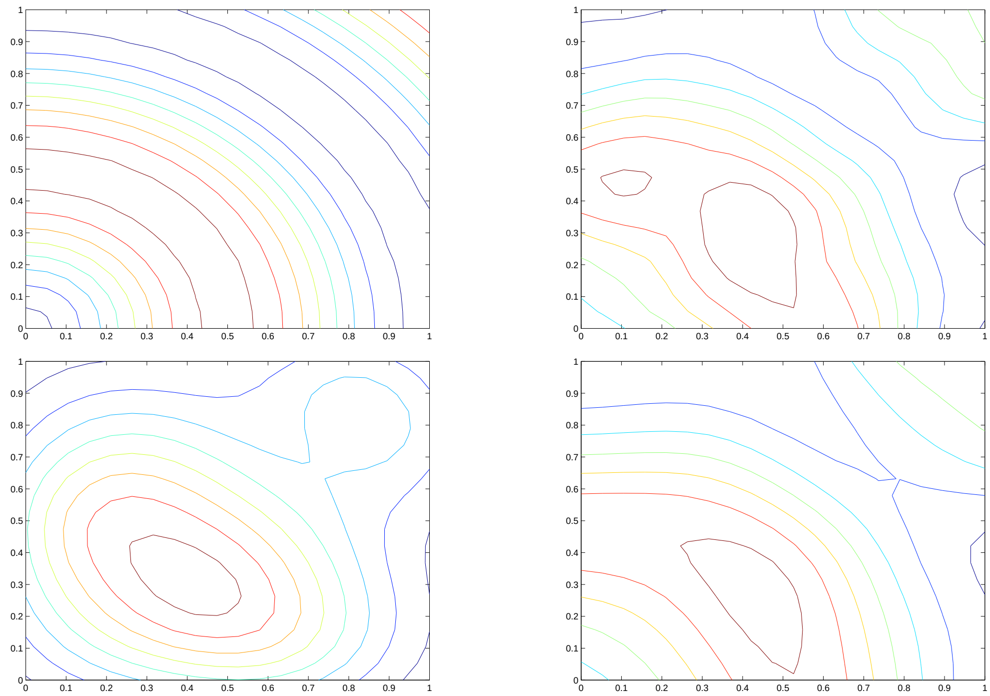

Figure 3 : Up-left figure : Contours of λ_Y. Up-right figure : Contours of $\hat{\lambda}_{Z,h_{opt}}$. Down-left figure : Contours of λ^*_{Y,h^*}. Down-right figure : Contours of λ^{**}_{Y,h^*}

It appears that the values taken by λ^*_{Y,h^*} close to the boundary of the square are too low, due to the absence of edge-correction. At the same time, the deconvolution technique used to get λ^{**}_{Y,h^*} leads to a better recognition of the peaks and troughs than the classical estimator $\hat{\lambda}_{Z,h_{opt}}$.

Finally, we consider how to handle uniform locational errors. Indeed, in this case, condition (3) is not satisfied and there is no appropriate deconvoluting intensity estimator. A solution can be to use the equivalent estimator for another error distribution. To illustrate this, Table 7 shows the results obtained when using the convoluting kernel estimator adapted to Laplace (index L) or Gaussian (index G) errors to uniform errors. The simulation procedure remains the same.

TABLE 7. uniform error, σ=0.05

	ISE	ISE_L^*	ISE_L^{**}	ISE_G^*	ISE_G^{**}
1st quartile ($*10^3$)	0.6939	1.3107	0.6804	1.3125	0.6823
median ($*10^3$)	1.0755	1.6191	1.0158	1.6181	1.0167
3rd quartile ($*10^3$)	1.1079	1.7944	1.1942	1.7955	1.2008

It appears that, even when the error distribution is misspecified, the deconvoluting kernel estimator remains useful. This goes along with the results of Hesse (1999) in the deconvoluting kernel density estimation framework asserting that the important point to specify is the error variance more than the error distribution.

6.8. An application to real data

In this section we illustrate our method on the spatial distributions of trees observed at Paracou site, which are data provided by the Forest department of CIRAD (Gourlet-Fleury & *al.*, 2004). This experimental station is located in the coastal part of French Guyana. It is composed of 14 experimental permanent sample plots of 6.25 ha each and one of 16 ha. In 1984, on each plot, all trees of diameter at breast height greater than 10 cm were localized by cartesian coordinates and botanically identified, when possible. The station is used for various ecological studies.

The trees were located in the following way : each plot was squared (12.5m $\times$ 12.5m) with ropes placed at the edge of the plot with decametre and compass. The coordinates of a tree were then measured with respect to the nearest origin (of the system of ropes axis) with decametre and compass (to keep the orthogonality). It can be noted that GPS is not well working around the equator and is not at all precise under canopy. Thus the trees were approximately localized independently of each other, with the same error that is a sum of the metrology error, a bad estimation of the center of a tree whose trunk could be deformed (that is not circular) in tropical context, plus various entry errors (on the field, the coordinates were called out by the measurer to someone else who recorded the values). Finally, the localization errors are suspected to follow approximately a gaussian distribution with standard deviation equal to 4m.

Figure 4 presents the results obtained when applying both the classical Diggle estimator (on the left) and the deconvoluting kernel estimator (on the right) to one of the data sets from Paracou, representing the spatial distribution of a tree species called Dicorynia. The estimated standard deviation of the location errors is quite important here so that the strong aggregation exhibited by Diggle estimator becomes less obvious when applying the deconvolution estimator. This could also come from the different bandwidth selection procedure adapted to each estimator : a larger bandwidth leads to a smoother estimation.

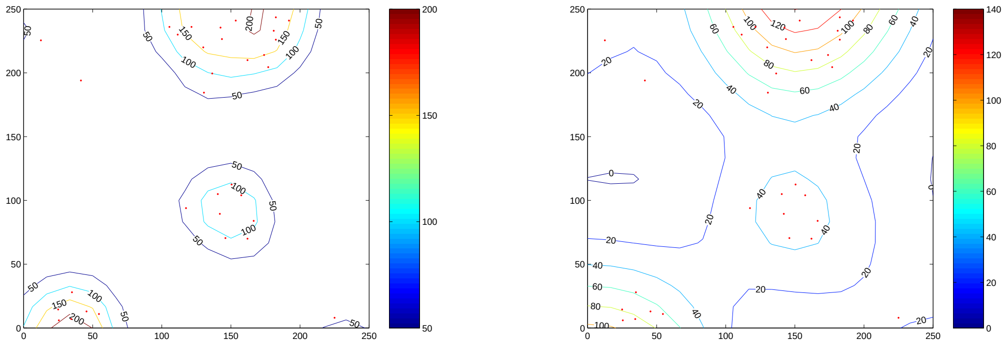

Figure 4 : Left figure : Contours of $\hat{\lambda}_{Z,h_{opt}}$. Right figure : Contours of λ^{**}_{Y,h^*}

Appendix

Denote

$$J = \int_D \frac{1}{h^2} K\big(\frac{z-x}{h}\big)\lambda_Z(x)\nu(dx) = \int_{\mathbb{R}^2}\int_D \frac{1}{h^2} K\big(\frac{z-x}{h}\big)\lambda_Y(x-\epsilon)\nu(dx)g(\epsilon)\nu(d\epsilon)$$
$$= \int_{\mathbb{R}^2}\int_{B_{z,h}} K(u)\lambda_Y(z-uh-\epsilon)\nu(du)g(\epsilon)\nu(d\epsilon),$$

where $B_{z,h} = \{\frac{z-x}{h} : x \in D\}$, as illustrated in Figure 2.

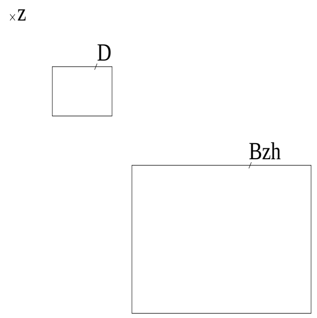

FIGURE 2. Illustration of the different sets

Then $\lambda_Y(z-\epsilon-uh) = \lambda_Y(z-\epsilon) - h\big(u_{(1)}\frac{\partial\lambda_Y}{\partial s_{(1)}}(z-\epsilon) + u_{(2)}\frac{\partial\lambda_Y}{\partial s_{(2)}}(z-\epsilon)\big) + \frac{h^2}{2}\big(u_{(1)}^2\frac{\partial^2\lambda_Y}{\partial {s_{(1)}}^2}(z-\epsilon) + u_{(1)}u_{(2)}\frac{\partial^2\lambda_Y}{\partial s_{(1)}\partial s_{(2)}}(z-\epsilon)\big) + u_{(2)}^2\frac{\partial^2\lambda_Y}{\partial {s_{(2)}}^2}(z-\epsilon)\big) + O(h^3)$.

So

$$\begin{aligned}
J = \int_{\mathbb{R}^2} \Big\{ & \lambda_Y(z-\epsilon)\int_{B_{z,h}} K(u)\nu(du) - h\frac{\partial\lambda_Y}{\partial s_{(1)}}(z-\epsilon)\int_{B_{z,h}} u_{(1)}K(u)\nu(du) \\
& -h\frac{\partial\lambda_Y}{\partial s_{(2)}}(z-\epsilon)\int_{B_{z,h}} u_{(2)}K(u)\nu(du) + \frac{h^2}{2}\frac{\partial^2\lambda_Y}{\partial {s_{(1)}}^2}(z-\epsilon)\int_{B_{z,h}} u_{(1)}^2K(u)\nu(du) \\
& +h^2\frac{\partial^2\lambda_Y}{\partial s_{(1)}\partial s_{(2)}}(z-\epsilon)\int_{B_{z,h}} u_{(1)}u_{(2)}K(u)\nu(du) \\
& +\frac{h^2}{2}\frac{\partial^2\lambda_Y}{\partial {s_{(2)}}^2}(z-\epsilon)\int_{B_{z,h}} u_{(2)}^2K(u)\nu(du) + O(h^3)\Big\} g(\epsilon)\nu(d\epsilon).
\end{aligned}$$

And

$$
\begin{aligned}
I &= \int_{G_h} \frac{e^{-it'z}}{p_h(z)} \int_D \frac{1}{h^2} K\big(\frac{z-x}{h}\big) \lambda_Z(x)\nu(dx)\nu(dz) \\
&= \int_{G_h} \frac{e^{-it'z}}{p_h(z)} \Bigg\{ \int_{\mathbb{R}^2} \Big\{ \lambda_Y(z-\epsilon) \int_{B_{z,h}} K(u)\nu(du) - h \frac{\partial \lambda_Y}{\partial s_{(1)}}(z-\epsilon) \int_{B_{z,h}} u_{(1)} K(u)\nu(du) \\
&- h \frac{\partial \lambda_Y}{\partial s_{(2)}}(z-\epsilon) \int_{B_{z,h}} u_{(2)} K(u)\nu(du) + \frac{h^2}{2} \frac{\partial^2 \lambda_Y}{\partial {s_{(1)}}^2}(z-\epsilon) \int_{B_{z,h}} u_{(1)}^2 K(u)\nu(du) \\
&+ h^2 \frac{\partial^2 \lambda_Y}{\partial s_{(1)} \partial s_{(2)}}(z-\epsilon) \int_{B_{z,h}} u_{(1)} u_{(2)} K(u)\nu(du) + \frac{h^2}{2} \frac{\partial^2 \lambda_Y}{\partial {s_{(2)}}^2}(z-\epsilon) \int_{B_{z,h}} u_{(2)}^2 K(u)\nu(du) \\
&+ O(h^3) \Big\} g(\epsilon)\nu(d\epsilon) \Bigg\} \nu(dz) \\
&= \int_{G_h} \frac{e^{-it'z}}{\int_{B_{z,h}} K(u)\nu(du)} \Bigg\{ \int_{\mathbb{R}^2} \Big\{ \lambda_Y(z-\epsilon) \int_{B_{z,h}} K(u)\nu(du) \\
&- h \frac{\partial \lambda_Y}{\partial s_{(1)}}(z-\epsilon) \int_{B_{z,h}} u_{(1)} K(u)\nu(du) - h \frac{\partial \lambda_Y}{\partial s_{(2)}}(z-\epsilon) \int_{B_{z,h}} u_{(2)} K(u)\nu(du) \\
&+ \frac{h^2}{2} \frac{\partial^2 \lambda_Y}{\partial {s_{(1)}}^2}(z-\epsilon) \int_{B_{z,h}} u_{(1)}^2 K(u)\nu(du) + h^2 \frac{\partial^2 \lambda_Y}{\partial s_{(1)} \partial s_{(2)}}(z-\epsilon) \int_{B_{z,h}} u_{(1)} u_{(2)} K(u)\nu(du) \\
&+ \frac{h^2}{2} \frac{\partial^2 \lambda_Y}{\partial {s_{(2)}}^2}(z-\epsilon) \int_{B_{z,h}} u_{(2)}^2 K(u)\nu(du) + O(h^3) \Big\} g(\epsilon)\nu(d\epsilon) \Bigg\} \nu(dz) \\
&= \int_{\mathbb{R}^2} \int_{G_h} e^{-it'(z-\epsilon)} \lambda_Y(z-\epsilon)\nu(dz) e^{-it'\epsilon} g(\epsilon)\nu(d\epsilon) \\
&- \int_{\mathbb{R}^2} \int_{G_h} \frac{h \int_{B_{z,h}} u_{(1)} K(u)\nu(du)}{\int_{B_{z,h}} K(u)\nu(du)} e^{-it'(z-\epsilon)} \frac{\partial \lambda_Y}{\partial s_{(1)}}(z-\epsilon)\nu(dz) e^{-it'\epsilon} g(\epsilon)\nu(d\epsilon) \\
&- \int_{\mathbb{R}^2} \int_{G_h} \frac{h \int_{B_{z,h}} u_{(2)} K(u)\nu(du)}{\int_{B_{z,h}} K(u)\nu(du)} e^{-it'(z-\epsilon)} \frac{\partial \lambda_Y}{\partial s_{(2)}}(z-\epsilon)\nu(dz) e^{-it'\epsilon} g(\epsilon)\nu(d\epsilon) \\
&+ \int_{\mathbb{R}^2} \int_{G_h} \frac{h^2 \int_{B_{z,h}} u_{(1)}^2 K(u)\nu(du)}{2 \int_{B_{z,h}} K(u)\nu(du)} e^{-it'(z-\epsilon)} \frac{\partial^2 \lambda_Y}{\partial {s_{(1)}}^2}(z-\epsilon)\nu(dz) e^{-it'\epsilon} g(\epsilon)\nu(d\epsilon) \\
&+ \int_{\mathbb{R}^2} \int_{G_h} \frac{h^2 \int_{B_{z,h}} u_{(1)} u_{(2)} K(u)\nu(du)}{\int_{B_{z,h}} K(u)\nu(du)} e^{-it'(z-\epsilon)} \frac{\partial^2 \lambda_Y}{\partial s_{(1)} \partial s_{(1)}}(z-\epsilon)\nu(dz) e^{-it'\epsilon} g(\epsilon)\nu(d\epsilon) \\
&+ \int_{\mathbb{R}^2} \int_{G_h} \frac{h^2 \int_{B_{z,h}} u_{(2)}^2 K(u)\nu(du)}{2 \int_{B_{z,h}} K(u)\nu(du)} e^{-it'(z-\epsilon)} \frac{\partial^2 \lambda_Y}{\partial {s_{(2)}}^2}(z-\epsilon)\nu(dz) e^{-it'\epsilon} g(\epsilon)\nu(d\epsilon) + O(h^3).
\end{aligned}
$$