# Singular Lagrangian

## 2.1 Introduction

To study the dynamic of a system that described by Lagrangian, we need to calculate the Euler-Lagrange equations that lead us to get finally the motion equations, where all the accelerations are expected to be expressed in functions of positions and velocities as a standard model for treatment. On the other hand, if we do not reach this expection, it is obvious that we are dealing with the opposite case where our Lagrangian seemed to be *singular*. The dilemma is in this last type of systems which is characterized by constraints presence submitted on the initial data and assumed generally to be independent of time. Besides that the Lagrangian type may be predicted from the constraints, there exists a definitive way to determine its quality from the determinant of what is known as *the Hessian matrix* . The singular Lagrangian expected to be treated in exception way that made physicists to search for methods to deal with it.

The aim of this first chapter is to give an introduction to singular Lagrangian which is the main motivation that leads us to expose two effective ways to treat its systems as we will show in the next chapters, depending on simple and illustrative examples. However, this can not be approached directly without going through important concepts in analytical mechanics seemed to be related to what is known as the Lagrangian and the Hamltonian formalism that are descibed respectively in configuration and phase spaces.

## 2.2 Lagrangian formalism

To describ a dynamic system, we give the Lagrangian $L(q_i, \dot{q}_i)$ with $N$ number of freedom degrees where $q_i$ and $\dot{q}_i$ represent coordinates and velocities respectively, while $(i = 1, ..., n)$. The action $S$ between two points $t_1$ and $t_2$ is given by the expression

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i)\, dt. \tag{2.1}$$

Most of the basic equations in physics can be deduced from what we call least action principle which stipulates that the action $S$ must be stationary, and its small variation $\delta S$ tends towards zero between two close moments $t_1$ and $t_2$ verifynig conditions that $\delta q(t_1) = \delta q(t_2) = 0$ .Indeed , the variation of the action is then written :

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} \delta L(q_i, \dot{q}_i)\, dt \\ &= \int_{t_1}^{t_2} \sum_i \left( \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt, \end{aligned}$$

where we'll integrate by using

$$\delta \dot{q}_i = \delta \frac{dq_i}{dt} = \frac{d}{dt} \delta q_i \text{ and } \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i = \left( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right) - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \right),$$

to get

$$\delta S = \sum_i \frac{\partial L}{\partial \dot{q}_i} \delta q_i \bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \sum_i \left( \frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i dt,$$

taking into account the conditions at the boundary that we have already mentioned above, we arrive to

$$\delta S = \int_{t_1}^{t_2} \sum_i \left( \frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i dt,$$

this variation must be null regardless of $\delta q_i$ value, this is only possible if

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0, \quad i = 1, ..., n, \tag{2.2}$$

this equations called Euler-Lagrange equation can be written by $p_i$ as follows

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{2.3}$$

$$\dot{p}_i = \frac{\partial L}{\partial q_i}, \tag{2.4}$$

where $p_i$ defined in (2.3) called conjugate momenta, while (2.4) is the veritable motion equation according to the sense of Newton and Lagrange.

## 2.3 Hamiltonian formalism

Starting from the Lagrangian and using the transformation of Legendre, we can construct the Hamiltonian wich is a new description much effective in symmetric systems than lagrangian formalism. It depends on moving from the configuration space with $n$ dimensions to the phase one with $2n$ dimensions, by remplacing the $n$ generalized velocities $\dot{q}_i$ according to the momenta $p_i$ defined in (2.3 ), where $i = 1, ..., n$. Thus, the Hamiltonian experssion is given as follows

$$H(q_i, p_i) = p_i \dot{q}_i - L(q_i, \dot{q}_i). \tag{2.5}$$

The action principle (2.1) gives

$$\begin{aligned} S &= \int_{t_1}^{t_2} L dt \\ &= \int_{t_1}^{t_2} (p_i \dot{q}_i - H(q_i, p_i))\, dt. \end{aligned} \tag{2.6}$$

The principle of least action stipulates that ($\delta S = 0$) between two times $t_1$ and $t_2$ as follows

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} \delta (p_i \dot{q}_i - H(q_i, p_i))\, dt \quad = \int_{t_1}^{t_2} (\delta p_i \dot{q}_i + p_i \delta \dot{q}_i - \delta H(q_i, p_i))\, dt \\ &= \int_{t_1}^{t_2} \left( \delta p_i \dot{q}_i + p_i \delta \dot{q}_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i \right) dt \\ &= \int_{t_1}^{t_2} \left( \delta p_i \dot{q}_i + \frac{d}{dt}(p_i \delta q_i) - \dot{p}_i \delta q_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i \right) dt, \end{aligned}$$

that can be written

$$\delta S = (p_i \delta q_i)|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left( \left( \dot{q}_i - \frac{\partial H}{\partial p_i} \right) \delta p_i - \left( \dot{p}_i + \frac{\partial H}{\partial q_i} \right) \delta q_i \right) dt.$$

Starting from that $\delta q(t_1) = \delta q(t_2) = 0$, the first term is null. Moreover, the variations $\delta p_i$ and $\delta q_i$are independents. So to have $\delta S = 0$ we must offer that

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad i = 1, ..., n \tag{2.7}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad i = 1, ..., n, \tag{2.8}$$

which are called Hamilton's equations. These equations are principally equivalents with Euler-Lagrange equations (2.2).

## 2.4 General form of Poisson brackets

Defining the ordinary form of the Poisson bracket that depends on the two functions $f\,(q_i, p_i)$ and $g\;(q_i, p_i)$ as follows

$$\{f, g\} = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right), \tag{2.9}$$

where Poisson bracket verify the next proprieties

$$\begin{aligned} \{f, g\} &= -\{g, f\} && \text{(Antisymmetry)} \\ \{f + h, g\} &= \{f, g\} + \{f, h\} && \text{(Linearity)} \\ \{fh, g\} &= f\,\{h, g\} + \{f, g\}\,h && \text{(Leibniz's identity)} \\ \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} &= 0 && \text{(Jacobi's identity)} \end{aligned} \quad .$$

We can express Hamilton's equations as follows

$$\dot{q}_i = \{q_i, H\}, \quad i = 1, ..., n \tag{2.10}$$

$$\dot{p}_i = \{p_i, H\}, \quad i = 1, ..., n. \tag{2.11}$$

We can rewrite the formula of Poisson bracket as more general and practical form that will be used later in the next chapters

$$\{f, g\}_{GPB} = \sum_{ij} J_{ij} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \xi_j}, \qquad i,\; j = 1, 2, ....., 2n \tag{2.12}$$

where $J_{ij} = \left\{\xi_i, \xi_j\right\}$ is an antisymmetric matrix element called *structure matrix*.So, the motion equation is written as

$$\dot{f} = \{f, H\}_{GPB}$$

For our phase space, the dynamic variables are given by

$$\left(\xi_1,\; \xi_2, ..., \xi_n,\; \xi_{n+1}, ..., \xi_{2n}\right) = \left(q_1,\; q_2, ...,\; q_n,\; p_1, ..., p_n\right).$$

For the dynamic variable $\xi_i$, we have this relation

$$\{\xi_i, f\}_{GPB} = \sum_j J_{ij} \frac{\partial f}{\partial \xi_j} \tag{2.13}$$

## 2.5 Singular Lagrangian

The determination of Lagrangian quality depends on the determinant of *the Hessian matrix*, that can be constructed from the differential derivative of momenta with respect to velocities, where $p_i = p_i\left(q_i, \dot{q}_i\right)$ defined by (2.3) in a system with $N$ number of freedom degrees according to the Lagrangian $L\left(q_i, \dot{q}_i\right)$, $i = 1, ..., n$ , as follows

$$dp_i = \sum_j \frac{\partial p_i}{\partial q_j} dq_j + \sum_j \frac{\partial p_i}{\partial \dot{q}_j} d\dot{q}_j, \tag{2.14}$$

and

$$\frac{dp_i}{dt} = \sum_j \frac{\partial p_i}{\partial q_j} \dot{q}_j + \sum_j \frac{\partial p_i}{\partial \dot{q}_j} \ddot{q}_j, \tag{2.15}$$

remplacing the relation (2.3) in (2.15) we obtian

$$\frac{dp_i}{dt} = \sum_j \frac{\partial L^2}{\partial q_j \partial \dot{q} i} \dot{q}_j + \sum_j \frac{\partial L^2}{\partial \dot{q}_j \partial \dot{q}_i} \ddot{q}_j, \tag{2.16}$$

we use now the equation (2.4), we get the equality

$$\sum_j \frac{\partial L^2}{\partial q_j \partial \dot{q}_j} \dot{q}_j + \sum_j \frac{\partial L^2}{\partial \dot{q}_j \partial \dot{q}_i} \ddot{q}_j - \frac{\partial L}{\partial q_i} = 0,$$

or else

$$\sum_j W_{ij}\left(q, \dot{q}\right) \ddot{q}_j = \frac{\partial L}{\partial q_i} - \sum_j \frac{\partial L^2}{\partial q_j \partial \dot{q}_i} \dot{q}_j, \tag{2.17}$$

where $W$ is the Hessian matrix defined by the next elements

$$W_{ij} = \frac{\partial L^2}{\partial \dot{q}_j \partial \dot{q}_i} = \frac{\partial p_i}{\partial \dot{q}_j}, \tag{2.18}$$

If $\det W \neq 0$, the marix $W$ is invertible, it means that we can express all the $\ddot{q}_i$ as functions of $\dot{q}_i$ and $q_i$. This signifies that a unique solution of (E-L) equations exists, and we are dealing with non-singular Lagrangian. Contrariwise, if $\det W = 0$, the matrix $W$ is not invertible, and the Lagrangian is seemed to be *singular.*

As we know, to pass from the Lagrangian formulation to the Hamiltonian one, it must be that all the velocities $\dot{q}_i$ expressed by functions of $q_i$ and $p_i$ as follows :

$$\dot{q}_i = f\left(q_i, p_i\right), \tag{2.19}$$

while the Hamiltonian (2.5) can be constructed by the Legendre transformation as

$$H = \sum_i p_i f\left(q_i, p_i\right) - L\left(q_i, f\left(q_i, p_i\right)\right). \tag{2.20}$$

It is clear that the procedure of having the Hamiltonian (2.20) is based particularly on the possibility of solving $p_i = \partial L/\partial \dot{q}_i$. This requires that the Jacobian matrix $\partial p_i/\partial \dot{q}_j$ is invertible, and it leads to

$$\frac{\partial p_i}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i} = W_{ij}. \tag{2.21}$$

Thus, in the case of a singular Lagrangian, it is impossible to pass to the Hamiltonian formulation in a standard way. We will illustrate this point with the following example

Considering the Lagrangian with two degrees of freedom [6] as follows

$$L = \frac{1}{2}\left(\dot{x} - y\right)^2, \tag{2.22}$$

The Hessian matrix $W$ correspondent is

$$W = \begin{pmatrix} \frac{\partial^2 L}{\partial \dot{x} \partial \dot{x}} & \frac{\partial^2 L}{\partial \dot{x} \partial \dot{y}} \\ \frac{\partial^2 L}{\partial \dot{y} \partial \dot{x}} & \frac{\partial^2 L}{\partial \dot{y} \partial \dot{y}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.23}$$

This Lagrangian is singular since that $\det W = 0$. The conjugate momenta are

$$p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} - y \text{ and } p_y = \frac{\partial L}{\partial \dot{y}} = 0. \tag{2.24}$$

which define the momenta that are insoluble with respect to $\dot{y}$, as what it was expected for a singular Lagrangian.

# Chapter 3

# Dirac's method for systems with constraints

## 3.1 Introduction

Hamiltonian of constrained systems represents an important class of physical systems described by singular Lagrangians. In this case, our conjugate momenta will not all be invertible with respect to velocities as already mentioned in the previous chapter.The Hamiltonian can be always formulated by the Legendre transformation, but in singular systems, it must be corrected so that it contains the constraints in question multiplied by what is called *Dirac's multipliers*. As a result,the canonical Hamiltonian equations changed automatically to be equivalent with Euler-Lagrange equations.

Dirac was the first who succeeded in treating singular systems by standard and consistent manner [1]. In Dirac's formalism, the inherent constraints would be generated and called primary constraints. Due to the consistency conditions, these primary constraints may generate new constraints, called secondary constraints.This iterative way of calculating the different constraints in the Dirac formalism is called the Dirac-Bergmann algorithm that ends when we determine Dirac's multipliers. The Poisson brackets must be replaced by another brackets called Dirac brackets which are more adequate in the presence of constraints.

Thus, the aim of this chapter is to expose this algorithm step by step till we will end with Dirac brackets determination that may lead us to correct quantizations of constrained systems.

## 3.2 Primary constraints and the new Hamiltonian formalism

In a system that desccribed by a singular Lagrangian in wich $\det W = 0$, and the conjugate momenta are defined by (2.3), may not all be invertible to velocities.We can't work directly by standard way to get the Hamiltonian equations as we did above. Therefore, we use Dirac's method to fix the problem starting on constructing contraints as follows:

the momenta are not all independent, but there are rather some relations of the type $\phi_m(q,p) = 0$ called primary constraints, that was obtained automatically from the canonical definition of momenta $p_i = \partial L/\partial\dot{q}_i$, $i = 1, ...., n$. where $M$ is the constraints number

$$\phi_m(q,p) = 0,\ m = 1, .., M \quad \text{where } q = (q,p) \quad \text{and} \quad M = \dim(W) - rank\,(W). \tag{3.1}$$

In line to the primary constraints existance, our system must be descibed by new total Hamiltonian $H_T$ or new Lagrangian $\tilde{L}$ depend on them besides to the older canonical form of $H_c$ or $L$ respectively, where $\lambda_m$ is the Dirac's multipliers, and the total Hamiltonian expression is given by

$$H_T(p,q) = H_c(p,q) + \lambda_m\phi_m(p,q), \tag{3.2}$$

it can be expressed also by the transformation of Legendre in the opposite direction, and allows to extract the new Lagrangian as follows

$$H_T(p,q) = p_i\dot{q}_i - \tilde{L} \ \text{ leads to } \tilde{L} = p_i\dot{q}_i - H_T(p,q) = p_i\dot{q}_i - H_c(p,q) - \lambda_m\phi_m(p,q). \tag{3.3}$$

The principle of least action stipulates that ($\delta S = 0$) between two times $t_1$ and $t_2$ giving

$$\delta S = \delta\int_{t_i}^{t_f}\tilde{L}dt = \delta\left[\int_{t_i}^{t_f} p_i\dot{q}_i - H_c(p,q) - \lambda_m\phi_m(p,q)\,dt\right] = \int_{t_i}^{t_f}\left[\delta\left(p_i\dot{q}_i - H_c\right) - \delta\left(\lambda_m\phi_m\right)\right]dt,$$

leads to

$$\delta S = \int_{t_i}^{t_f}\left[\left(\dot{q}_i - \frac{\partial H_c}{\partial q_i} - \lambda_m\frac{\partial \phi_m}{\partial p_i}\right)\delta p_i + \left(-\dot{p}_i - \frac{\partial H_c}{\partial q_i} - \lambda_m\frac{\partial \phi_m}{\partial q_i}\right)\delta q_i - \delta\lambda_m\phi_m\right]dt, \tag{3.4}$$

Since $\phi_m(q,p) = 0$ and $\delta S \to 0$, moreover, $\forall\ \delta p_i$ , $\delta q_i$ and $\delta\lambda_m$ that are independents, we get finally the new Hamiltonian equations

$$\dot{q}_i = \frac{\partial H_c}{\partial q_i} + \lambda_m \frac{\partial \phi_m}{\partial p_i}, \quad i = 1, .., n \tag{3.5}$$

$$\dot{p}_i = -\frac{\partial H_c}{\partial q_i} - \lambda_m \frac{\partial \phi_m}{\partial q_i}, \quad i = 1, .., n \tag{3.6}$$

$$\phi_m = 0, \qquad m = 1, .., M. \tag{3.7}$$

To have the Poisson bracktes form of these equations, we constuct the general formula of the differential equation with respect to time of the function $F = F(q, p)$ using the usual mathematical relation

$$\dot{F} = \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial p_i}\dot{p}_i, \tag{3.8}$$

using (3.5), (3.6) and (3.7) we have

$$\dot{F} = \frac{\partial F}{\partial q_i}\frac{\partial H_c}{\partial q_i} - \frac{\partial F}{\partial p_i}\frac{\partial H_c}{\partial q_i} + \left(\frac{\partial F}{\partial q_i}\frac{\partial \phi_m}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial \phi_m}{\partial q_i}\right)\lambda_m \quad ; \quad \phi_m = 0,$$

where $\dot{F}$ may take the Poisson bracket form as follows

$$\dot{F} = \{F, H_c\} + \lambda_m \{F, \phi_m\} \quad ; \quad \phi_m = 0. \tag{3.9}$$

According to Dirac, it is necessary to calculate the Poisson brackets before using the constraints $\phi_m = 0$. It is therefore convenient to rewrite the previous equation in this form

$$\dot{F} = (\{F, H_c\} + \lambda_m \{F, \phi_m\})|_{\phi_m=0} \tag{3.10}$$

or

$$\dot{F} = \{F, H_T\}|_{\phi_m=0}. \tag{3.11}$$

**Exemple**

Considering the Lagrangian from [6]

$$L = \frac{1}{2}\dot{x}^2 + x\dot{y} + f(x, y).$$

Calculating the (E-L) equations

$$\dot{y} + \tfrac{\partial f}{\partial x} - \ddot{x} = 0, \qquad \tfrac{\partial f}{\partial y} - \dot{x} = 0, \tag{3.12}$$

and the conjugate momenta

$$p_x = \tfrac{\partial L}{\partial \dot{x}} = \dot{x}, \quad p_y = \tfrac{\partial L}{\partial \dot{y}} = x,$$

we have the primary constraint $\phi_1 = p_y - x = 0$. Forming the canonical Hamiltonian

$$H_c = \dot{x}p_x + \dot{y}p_y - L = \tfrac{1}{2}p_x^2 - f(x,y)$$

If we try to calculate Hamilton's equations from $H_c$, we will obtain equations which are not equivalent to the equations of (E-L). Indeed, we will obtain the equations

$$\begin{cases} \dot{x} = p_x \\ \dot{p}_x = \frac{\partial f}{\partial x}. \end{cases}, \quad \begin{cases} \dot{y} = 0 \\ \dot{p}_y = \frac{\partial f}{\partial y}. \end{cases} \tag{3.13}$$

Therefore, we must hamiltonize $H_c$ i.e Finding $H_T$ for which the corresponding hamiltonian equations will be equivalent to the E-L one.Writing $H_T$ as follows

$$H_T = H_c + \lambda_1\phi_1 = \frac{1}{2}p_x^2 - f(x,y) + \lambda_1 (p_y - x)$$

Thus, the Hamiltonian equations lead to

$$\begin{cases} \dot{x} = p_x \\ \dot{p}_x = \frac{\partial f}{\partial x} + \lambda_1 \end{cases}, \quad \begin{cases} \dot{y} = \lambda_1 \\ \dot{p}_y = \frac{\partial f}{\partial y} \end{cases}, \text{ and } \; p_y - x = 0 \tag{3.14}$$

## 3.3 Weak and strong equality

Dirac introduced the notion of the weak equality under that sign (" $\approx$ ") replacing the constraints condition given by $\phi_m = 0$, where the system was described by $(= ()|_{\phi_m=0})$ to express the dynamic only in the sub space of constraints, otherwise the notion of strong equality ("=") is vailable in all the space. Thus, the evolution equations may be written as follows

$$\dot{F} = \{F, H_T\}|_{\phi_m=0} \tag{3.15}$$

$$\dot{F} \approx \{F, H_T\} \approx \{F, H_c\} + \lambda_m \{F, \phi_m\}, \tag{3.16}$$

Therefore we can write the Hamiltonian equations in the form of Poisson brackets as well

$$\dot{q}_i \approx \{q_i, H_T\}, \quad \dot{p}_i \approx \{p_i, H_T\}. \tag{3.17}$$

## 3.4 Secondary constraints and Dirac-Bergmann algorithm

The primary constraints must be preserved over time during an evoltion, we can write

$$\tfrac{d\phi_{m'}}{dt} = \dot{\phi}_{m'} \approx 0, \quad m' = 1, ...M, \tag{3.18}$$

but according to (3.16), we'll have

$$\dot{\phi}_{m'} = \{\phi_{m'}, H_T\} \approx 0 \Leftrightarrow \{\phi_{m'}, H_c\} + \lambda_m \{\phi_{m'}, \phi_m\} \approx 0, \qquad m', m = 1, ...M. \tag{3.19}$$

That are called *consistency conditions (the CCs )*, where they are related to primary constraints here specifically.The system (3.19) is a system of non-homogeneous algebraic equations, which will help us to verify the Dirac' multipliers $\lambda_m$. In reality, the study of this system will lead us to one of the following three situations :

1) The CCs determine the Dirac's multipliers either all (all equations give values of $\lambda_m$ with $m = 1, ...M$) or some (in addition to some equations which are identically true such that $0 \approx 0$). In this case, the iteration stops.

2) The CCs do not determine multipliers and gives at least one incorrect equation such as for example ( $1 = 0$). In this case, there is certainly an anomaly, so it is useless to go further before modifying the Lagrangian itself, and restarting again the steps.

3) The CCs do not determine the multipliers directly, and give new different relations between $p_i$ and the $q_i$ described by the formula $\varphi_k(q,p) \approx 0$ , $k = 1, ..., K$, that expresses a new restarting called *secondary constraints* can have also CCs according to (3.16) and need to be treated to give cases as the both that we have already mentioned besides to this one itself. The iteration stops in the end, where we may determine mutipliers.

The logical analysis above was formulated in a sequential consistent manner with restricted iteration may be stopped or continued according to the existing situation that ends by the determination of multipliers as a goal. This process is known as *The Dirac-Bergmann algorithm.*

## 3.5 Constraints classification

Considering $\{\phi_j \approx 0\}$ with $j = 1, ..., J = M + K$ that describs all the constraints (secondary and primary), where $M$ is the number of primary constraints, and $K$ the one of secondary constraints. According to Dirac we say that the function $F(q, p)$ is first class if its Poisson bracket with each of the constraints (primary or secondary) that are included under the previous relation, is null on the surface of constraints, i.e $\left\{F,\phi_j\right\}\approx 0$.Otherwise, we say that the function $F(q,p)$ is second class, if $\left\{F,\phi_j\right\}\not\approx 0$ (at least for one $j$).

## 3.6 Dirac brackets

We will assume that all the constraints of our system (primary and secondary) are secondary class. We notice that $\phi_m$, $m=1,...,M$ the primary constraints, while $\phi_k$ , $k=1,...,K$ secondry constraints. Writing the CCs of the set of constraints, we get

$$\left\{\phi_j,H_c\right\}+\lambda_m\left\{\phi_j,\phi_m\right\}\approx 0,\quad m=1,...,M\ \text{ et }\ j=1,...,J=K+M \tag{3.20}$$

where

$$H_T=H_c+\lambda_m\phi_m,\quad m=1,...,M.$$

Rewriting( 3.20) in matrix form as follows

$$\underbrace{\begin{pmatrix}\{\phi_1,\phi_1\} & .... & \{\phi_1,\phi_M\}\\ \vdots & \ddots & \vdots\\ \{\phi_J,\phi_1\} & .... & \{\phi_J,\phi_M\}\end{pmatrix}}_{=\Omega}\underbrace{\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_M\end{pmatrix}}_{=\boldsymbol{\lambda}}\approx\underbrace{\begin{pmatrix}-\{\phi_1,H_c\}\\ \vdots\\ -\{\phi_J,H_c\}\end{pmatrix}}_{=\boldsymbol{\eta}}, \tag{3.21}$$

Or else

$$\Omega\lambda\approx\eta, \tag{3.22}$$

where $\Omega$ is a matrix of $K$ lines and $M$ columns. Forming now the square matrix $\Delta$ defined by

$$\Delta_{\alpha,\alpha'}=\{\phi_\alpha,\phi_{\alpha'}\}\quad,\ \alpha,\alpha'=1,...,J\ \text{ where } J=M+K, \tag{3.23}$$

this matrix is antisymmetric and contains the matrix $\Omega$ as a block; explicitly

$$\Delta=\begin{pmatrix}\{\phi_1,\phi_1\} & .... & \{\phi_1,\phi_M\} & \{\phi_1,\phi_{M+1}\} & .... & \{\phi_1,\phi_J\}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \{\phi_J,\phi_1\} & .... & \{\phi_J,\phi_M\} & \{\phi_J,\phi_{M+1}\} & .... & \{\phi_J,\phi_J\}\end{pmatrix}$$
$$=\begin{pmatrix}\underbrace{\begin{matrix}0 & .... & \{\phi_1,\phi_M\}\\ \vdots & \ddots & \vdots\\ \{\phi_J,\phi_1\} & .... & \{\phi_J,\phi_M\}\end{matrix}}_{=\Omega} & \underbrace{\begin{matrix}\{\phi_1,\phi_{M+1}\} & .... & \{\phi_1,\phi_J\}\\ \vdots & \ddots & \vdots\\ \{\phi_J,\phi_{M+1}\} & .... & 0\end{matrix}}_{=\omega}\end{pmatrix},$$

Where $\omega$ is a matrix with $J$ lines and $J-M$ columns. Dirac has shown that $\det(\Delta) \neq 0$ (for the demonstration, see [1]), moreover the matrix $\Delta$ must be of even dimension, because the determinant of an odd antisymmetric matrix must be null . Considering now the column vector $\theta$ at $J$ components

$$\theta = \left( \begin{array}{cccccc} \lambda_1 & .... & \lambda_M & \underbrace{0 \quad .... \quad 0}_{J-M} \end{array} \right)^t , \tag{3.24}$$

or otherwise written

$$\boldsymbol{\theta} = \left( \begin{array}{c} \boldsymbol{\lambda} \\ \mathbf{0} \end{array} \right). \tag{3.25}$$

Calculating the product $\Delta\theta$ by block as follows

$$\Delta\boldsymbol{\theta} = (\Omega\omega) \left( \begin{array}{c} \boldsymbol{\lambda} \\ \mathbf{0} \end{array} \right) = \Omega\boldsymbol{\lambda}, \tag{3.26}$$

then by comparing between( 3.22)and( 3.26), we get

$$\Delta\boldsymbol{\theta} \approx \boldsymbol{\eta}, \tag{3.27}$$

since $\Delta$ is invertible,we can obtain

$$\boldsymbol{\theta} \approx \Delta^{-1}\boldsymbol{\eta},$$

or else

$$\boldsymbol{\theta}_\alpha \approx \Delta^{-1}_{\alpha,\alpha'}\eta_{\alpha'} \quad , \quad \alpha, \alpha' = 1, ..., J,$$

but as $\theta = (\boldsymbol{\lambda}, \mathbf{0})^t$, we deduce that

$$\theta_m = \lambda_m \approx \Delta^{-1}_{m,\alpha'}\eta_{\alpha'} \quad , \quad m = 1, ..., M \quad \text{and } \alpha' = 1, ..., J \tag{3.28}$$

$$\theta_\alpha = 0 \approx \Delta^{-1}_{\alpha,\alpha'}\eta_{\alpha'} \quad , \quad \alpha = M+1, ..., J \text{ and } \alpha' = 1, ..., J. \tag{3.29}$$

Since the matrix elements $\Delta$ are the brackets $\Delta_{\alpha,\alpha'} = \{\phi_\alpha, \phi_{\alpha'}\}$ , $\alpha, \alpha' = 1, ..., J$, the elements of the inverse matrix $\Delta^{-1}$ will be noted by $\Delta^{-1}_{\alpha,\alpha'} = \{\phi_\alpha, \phi_{\alpha'}\}^{-1}$ , $\alpha, \alpha' = 1, ..., J$. According to the equations (3.28),(3.29)and (3.21), we write

$$\lambda_m \approx -\{\phi_m, \phi_{\alpha'}\}^{-1}\{\phi_{\alpha'}, H_c\} \ , \ m = 1, ..., M \quad \text{and } \alpha' = 1, ..., J \tag{3.30}$$

$$0 \approx \{\phi_\alpha, \phi_{\alpha'}\}^{-1}\{\phi_{\alpha'}, H_c\} \quad , \quad \alpha = M+1, ..., J \text{ and } \alpha' = 1, ..., J. \tag{3.31}$$

Recalling the evolution equation of the function $F(q,p)$ that was given by (3.16) as follows

$$\dot{F} \approx \{F, H_c\} + \lambda_m \{F, \phi_m\},$$

taking into account (3.30), we'll have

$$\begin{aligned} \dot{F} &\approx \{F, H_c\} - \{F, \phi_m\} \{\phi_m, \phi_{\alpha'}\}^{-1} \{\phi_{\alpha'}, H_c\} \\ \text{with } m &= 1, ..., M \text{ and } \alpha' = 1, ..., J, \end{aligned} \tag{3.32}$$

but according to (3.31), we have $\{\phi_\alpha, \phi_{\alpha'}\}^{-1} \{\phi_{\alpha'}, H_c\} \approx 0$, with $\alpha = M+1, ..., J$, that allows to generalize (3.32) without any problem as follows

$$\dot{F} \approx \{F, H_c\} - \{F, \phi_\alpha\} \{\phi_\alpha, \phi_{\alpha'}\}^{-1} \{\phi_{\alpha'}, H_c\}, \text{ with } \quad \alpha,\ \alpha' = 1, ..., J, \tag{3.33}$$

Dirac defined (3.33) as brackts that take his name

$$\{F, H_c\}_D = \{F, H_c\} - \{F, \phi_\alpha\} \{\phi_\alpha, \phi_{\alpha'}\}^{-1} \{\phi_{\alpha'}, H_c\}, \tag{3.34}$$

while the reduced form is given by

$$\dot{F} \approx \{F, H_c\}_D. \tag{3.35}$$

The generalization of Dirac bracket to the case of two functions $f$ and $g$ in phase space is

$$\boxed{\{f, g\}_D = \{f, g\} - \{f, \phi_\alpha\} \{\phi_\alpha, \phi_{\alpha'}\}^{-1} \{\phi_{\alpha'}, g\}}. \tag{3.36}$$

The consistency conditions $\{\phi_\alpha, H_T\} \approx 0$ allows to write

$$\{F, H_T\}_D = \{F, H_T\} - \{F, \phi_\alpha\} \{\phi_\alpha, \phi_{\alpha'}\}^{-1} \underbrace{\{\phi_{\alpha'}, H_T\}}_{\approx 0},$$

we obtain the equality

$$\{F, H_T\}_D \approx \{F, H_T\} \approx \dot{F}.$$

In the special case where $F = q$ or $F = p$, we obtain the Hamiltonian equations

$$\dot{q} \approx \{q, H_T\}_D \tag{3.37}$$

$$\dot{p} \approx \{p, H_T\}_D \tag{3.38}$$

Dirac brackets have properties similar to those of Poisson brackets, besides to another two properties given by

$$\{f, \phi_\alpha\}_D = 0 \qquad (\phi_\alpha \text{ second class constraint}) \text{ and } \{f, G\}_D \approx \{f, G\} \qquad (G \text{ first class function}), \tag{3.39}$$

where $f$ depend on $q$ and $p$. For the demonstration of (3.39), we can have look to [6].

The evolution equation of a quantity $F(q,p)$ is given as a function of these new brackets as

$$\dot{F} \approx \{F, H_c\}_D \,. \tag{3.40}$$

Dirac brackets have a simple interpretation, it bears the information of constrained systems inside itselfs. Otherwise, we can say that the Dirac's method takes the information on the constraint starting from the Lagrangian to give it in the end to the canonical brackets of himself.

# Chapter 4

# Faddeev and jackiw method for systems with constraints

## 4.1 Introduction

In order to search for new much simpler methods to deal with constrainted systems, Faddeev-Jackiw proposed an alternative treatment seems technically different and does not have the same Dirac's conjecture, thus it has evoked much attention [3]. Noting that the original Faddeev-Jackiw method was addressed to unconstrained systems, while Barcelos-Neto and Wotzasek had been proposed an extension called symplectic algorithm to deal with constraints systems [9, 10], that we are dealing with it in this thesis.

The Faddeev-Jackiw (F-J) formalism pursues a classical geometric treatment based on the sympletic structure of the phase space and it is only applied to first order Lagrangians, linear with respect to velocities [3]. This method is rised basically on Lagrangian formalism and the matrix form of Euler-Lagrange equations as a main source of studying, without missing an important passage in converting the Lagrangian to linear one with respect to velocities and conjugate momenta using the Legendre transformation. The matrix form of (E-L) equations lead us to introduce the *(F-J) matrix* that gives us two cases can be treated according to its determinant as we will see later.

Thus, the objective of this chapter is to treat the (F-J) matrix cases with a symplectic algorithm step by step till we will end with an invertible matrix represent the basic geometric structure called generalized Poisson brackets and coincide with Dirac's brackets, that will be the bridge to the commutators of the quantized theory, as we have already mentioned in the previous chapter, while our real aim is to make a clear comparaison later between those methods in that crossing road.

## 4.2 Lagrangian linearization

As we have already evoked in the preceding chapter, we will not be able to express for a singular systems all velocities ( the $\dot{q}_i$) according to the coordinates ( the $q_i$), and the conjugate momenta (the $p_i$) using the relations $p_i = \partial L / \partial \dot{q}_i$, $i = 1, ...., n$. As we know in this case the Hessian matrix $W$ is not invertible. Considering $R = rank\,(W)$, this means that it is possible to reverse the equations $p_i = \partial L / \partial \dot{q}_i$ only with respect to $R$ generalized velocities $\dot{q}_a$ with $a = 1, ..., R$, writing them as functions of the other velocities, generalized coordinates and conjugate momenta as follows : $\dot{q}_a = f_a\left(q_i, p_b, \dot{q}_s\right)$ , $a, b = 1, ..., R$ , $i = 1, .., n$ , $s = R + 1, ..., n$

Since $s = n - R$, we make appear $s$ relations noted as :

$$\phi_s = p_s - g_s\left(q_i, p_b\right), \; b = 1, ..., \; R, \; s = R + 1, ..., n, \quad i = 1, .., n, \tag{4.1}$$

the $s$ relations express constraints that come automatically from the system.

The associated Hamiltonian $H$ to the Lagrangian $L\left(q_i, \dot{q}_i\right)$ takes the form

$$\begin{aligned} H &= p_i \dot{q}_i - L \\ &= p_a \dot{q}_a + p_s \dot{q}_s - L \\ &= p_a f_a\left(q_i, p_b, \dot{q}_s\right) + g_s\left(q_i, p_b\right) \dot{q}_s - L. \end{aligned} \tag{4.2}$$

The $H$ does not depend on generalized velocities despite their apparent presence.We can prove that fact by deriving( 4.2) with respect to $\dot{q}_c$, while it appears directly in illustrative example since $H = H\left(q_i, p_i\right)$.

Very often, the Lagrangian is nonlinear with respect to velocities. Linearization consists in passing from this Lagrangian $L(q_i, \dot{q}_i)$ to a canonical Hamiltonian $H\left(q_i, p_i\right)$, to then return to have directly a linear Lagrangian $L(q_i, \dot{q}_i, p_i)$.The main controller in this process is the Legendre transformation in the both directions. In a specific way, we define the inverse of Legendre transformation as follows

$$L = p_i \dot{q}_i - H,$$

as well as the constraints (4.1), we have

$$L\left(q_i, \dot{q}_i, p_a\right) = p_a \dot{q}_a + g_s\left(q_i, p_a\right) \dot{q}_s - H\left(q_i, p_a\right). \tag{4.3}$$

The Faddeev and Jackiw method consists in treating the $q_i$ and $p_a$ to be independents for the Lagrangian that had been constructed as we will see in the next example

**Example**

To explain this point well, considering the following nonlinear Lagrangian [5]

$$L = \frac{1}{2}(y\dot{x} + x\dot{y})^2 - xy. \tag{4.4}$$

The conjugate monenta are

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = y(y\ \dot{x} + \dot{y}\ x) \\ p_y &= \frac{\partial L}{\partial \dot{y}} = x(x\ \dot{y} + y\ \dot{x}) \\ (y\ \dot{x} + \dot{y}\ x) &= \frac{p_x}{y} = \frac{p_y}{x} \quad (\text{constraint})\,. \end{aligned}$$

We can deduce one constraint $p_y = \frac{x}{y}\ p_x$.Using this constraint the Hamiltonian gets the expression

$$\begin{aligned} H &= p_x\dot{x} + p_y\dot{y} - L \\ &= p_x\dot{x} + p_y\dot{y} - \frac{1}{2}(y\dot{x} + x\dot{y})^2 + xy \\ &= p_x\left(\dot{x} + \frac{x}{y}\dot{y}\right)\frac{y^2}{y^2} - \frac{1}{2}(\frac{p_x}{y})^2 + xy \\ &= \frac{p_x^2}{y^2} - \frac{1}{2}(\frac{p_x}{y})^2 + xy \\ &= \frac{1}{2}(\frac{p_x}{y})^2 + xy, \end{aligned}$$

$H$ doesn't depend on velocities clearly. Now the linear Lagrangian is

$$\begin{aligned} L &= p_i\dot{q}_i - H \\ &= p_x\dot{x} + p_y\dot{y} - \frac{1}{2}(\frac{p_x}{y})^2 - xy \\ &= p_x\dot{x} + \frac{xp_x}{y}\dot{y} - \frac{1}{2}(\frac{p_x}{y})^2 - xy \end{aligned}$$

The independent variables are then $x, y$ and $p_x$, while the momentum $p_y$ depends on the other variables through the mentioned constraint above $p_y = \frac{x}{y}\ p_x$. We will see later that the (E-L) equations apply on the independent variables of any system according to the constraints.