Faddeev and jackiw method for systems with constraints

4.1 Introduction

In order to search for new much simpler methods to deal with constrainted systems, Faddeev-Jackiw proposed an alternative treatment seems technically different and does not have the same Dirac's conjecture, thus it has evoked much attention [3]. Noting that the original Faddeev-Jackiw method was addressed to unconstrained systems, while Barcelos-Neto and Wotzasek had been proposed an extension called symplectic algorithm to deal with constraints systems [9, 10], that we are dealing with it in this thesis.

The Faddeev-Jackiw (F-J) formalism pursues a classical geometric treatment based on the sympletic structure of the phase space and it is only applied to first order Lagrangians, linear with respect to velocities [3]. This method is rised basically on Lagrangian formalism and the matrix form of Euler-Lagrange equations as a main source of studying, without missing an important passage in converting the Lagrangian to linear one with respect to velocities and conjugate momenta using the Legendre transformation. The matrix form of (E-L) equations lead us to introduce the *(F-J) matrix* that gives us two cases can be treated according to its determinant as we will see later.

Thus, the objective of this chapter is to treat the (F-J) matrix cases with a symplectic algorithm step by step till we will end with an invertible matrix represent the basic geometric structure called generalized Poisson brackets and coincide with Dirac's brackets, that will be the bridge to the commutators of the quantized theory, as we have already mentioned in the previous chapter, while our real aim is to make a clear comparaison later between those methods in that crossing road.

4.2 Lagrangian linearization

As we have already evoked in the preceding chapter, we will not be able to express for a singular systems all velocities (the $\dot{q}_i$) according to the coordinates (the q_i), and the conjugate momenta (the p_i) using the relations $p_i = \partial L/\partial\dot{q}_i$, $i = 1, ..., n$. As we know in this case the Hessian matrix W is not invertible. Considering $R = rank\,(W)$, this means that it is possible to reverse the equations $p_i = \partial L/\partial\dot{q}_i$ only with respect to R generalized velocities $\dot{q}_a$ with $a = 1, ..., R$, writing them as functions of the other velocities, generalized coordinates and conjugate momenta as follows : $\dot{q}_a = f_a\,(q_i, p_b, \dot{q}_s)$, $a, b = 1, ..., R$, $i = 1, .., n$, $s = R + 1, ..., n$

Since $s = n - R$, we make appear s relations noted as :

$$\phi_s = p_s - g_s\,(q_i, p_b)\,,\ b = 1, ...,\ R,\ s = R + 1, ..., n,\ \ i = 1, .., n, \tag{4.1}$$

the s relations express constraints that come automatically from the system.

The associated Hamiltonian H to the Lagrangian $L\,(q_i, \dot{q}_i)$ takes the form

$$\begin{aligned} H &= p_i\dot{q}_i - L \\ &= p_a\dot{q}_a + p_s\dot{q}_s - L \\ &= p_a f_a\,(q_i, p_b, \dot{q}_s) + g_s\,(q_i, p_b)\,\dot{q}_s - L. \end{aligned} \tag{4.2}$$

The H does not depend on generalized velocities despite their apparent presence.We can prove that fact by deriving(4.2) with respect to $\dot{q}_c$, while it appears directly in illustrative example since $H = H\,(q_i, p_i)$.

Very often, the Lagrangian is nonlinear with respect to velocities. Linearization consists in passing from this Lagrangian $L(q_i, \dot{q}_i)$ to a canonical Hamiltonian $H\,(q_i, p_i)$, to then return to have directly a linear Lagrangian $L(q_i, \dot{q}_i, p_i)$.The main controller in this process is the Legendre transformation in the both directions. In a specific way, we define the inverse of Legendre transformation as follows

$$L = p_i\dot{q}_i - H,$$

as well as the constraints (4.1), we have

$$L\,(q_i, \dot{q}_i, p_a) = p_a\dot{q}_a + g_s\,(q_i, p_a)\,\dot{q}_s - H\,(q_i, p_a)\,. \tag{4.3}$$

The Faddeev and Jackiw method consists in treating the q_i and p_a to be independents for the Lagrangian that had been constructed as we will see in the next example

Example

To explain this point well, considering the following nonlinear Lagrangian [5]

$$L = \frac{1}{2}(y\dot{x} + x\dot{y})^2 - xy. \tag{4.4}$$

The conjugate monenta are

$$\begin{aligned}
p_x &= \frac{\partial L}{\partial \dot{x}} = y(y\ \dot{x} + \dot{y}\ x) \\
p_y &= \frac{\partial L}{\partial \dot{y}} = x(x\ \dot{y} + y\ \dot{x}) \\
(y\ \dot{x} + \dot{y}\ x) &= \frac{p_x}{y} = \frac{p_y}{x} \quad (\text{constraint})\,.
\end{aligned}$$

We can deduce one constraint $p_y = \frac{x}{y}\ p_x$.Using this constraint the Hamiltonian gets the expression

$$\begin{aligned}
H &= p_x\dot{x} + p_y\dot{y} - L \\
&= p_x\dot{x} + p_y\dot{y} - \frac{1}{2}(y\dot{x} + x\dot{y})^2 + xy \\
&= p_x\left(\dot{x} + \frac{x}{y}\dot{y}\right)\frac{y^2}{y^2} - \frac{1}{2}(\frac{p_x}{y})^2 + xy \\
&= \frac{p_x^2}{y^2} - \frac{1}{2}(\frac{p_x}{y})^2 + xy \\
&= \frac{1}{2}(\frac{p_x}{y})^2 + xy,
\end{aligned}$$

H doesn't depend on velocities clearly. Now the linear Lagrangian is

$$\begin{aligned}
L &= p_i\dot{q}_i - H \\
&= p_x\dot{x} + p_y\dot{y} - \frac{1}{2}(\frac{p_x}{y})^2 - xy \\
&= p_x\dot{x} + \frac{xp_x}{y}\dot{y} - \frac{1}{2}(\frac{p_x}{y})^2 - xy
\end{aligned}$$

The independent variables are then x, y and p_x, while the momentum p_y depends on the other variables through the mentioned constraint above $p_y = \frac{x}{y}\ p_x$. We will see later that the (E-L) equations apply on the independent variables of any system according to the constraints.

4.3 Faddeev and Jackiw approach

Faddeev-Jackiw method is based on two main maneuvers

i) The linearization of the Lagrangian with respect to the generalized velocities.

ii) The inversion of the Faddeev-Jackiw matrix obtained using the (E-L) equations.

This method allows to derive the set of Dirac brackets in one fell swoop without needing to calculate any Poisson brackets separately .

The idea is to treat the independent variables (the q_i , $i = 1, .., n$ and the p_a , $a = 1, ...,$ R), on an equal footing by introducing new variables $\xi_i = q_i$, $i = 1, .., n$ and $\xi_{n+a} = p_a$ with $a = 1, ..., R$, in such a way that the Lagrangian (4.3) is written

$$L = A_J \, \dot{\xi}_J - H \quad , \quad J = 1, ..., n + R, \tag{4.5}$$

so that

$$\begin{aligned} A_a &= p_a \ , \ \ a = 1, ..., \ R \\ A_s &= g_s\left(q_i, p_a\right) \ , \ s = R+1, ..., n \\ A_{n+a} &= 0. \end{aligned}$$

We write the Euler-Lagrange equations relating to the dynamic variables $(\xi_J, \dot{\xi}_J)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_J}\right) - \frac{\partial L}{\partial \xi_J} = 0. \tag{4.6}$$

We have

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_J}\right) &= \frac{d}{dt} A_J = \frac{\partial A_J}{\partial \xi_I}\frac{d\xi_I}{dt} = \frac{\partial A_J}{\partial \xi_I} \, \dot{\xi}_I \\ \frac{\partial L}{\partial \xi_J} &= \frac{\partial A_I}{\partial \xi_J} \, \dot{\xi}_I - \frac{\partial H}{\partial \xi_J}, \end{aligned}$$

thus, (4.6) gives

$$\left(\frac{\partial A_I}{\partial \xi_J} - \frac{\partial A_J}{\partial \xi_I}\right) \dot{\xi}_I = \frac{\partial H}{\partial \xi_J}, \tag{4.7}$$

or else

$$f_{IJ} \, \dot{\xi}_I = \frac{\partial H}{\partial \xi_J} \quad , \quad I, J = 1, ..., n + R, \tag{4.8}$$

where

$$f_{IJ} = \frac{\partial A_I}{\partial \xi_J} - \frac{\partial A_J}{\partial \xi_I}, \tag{4.9}$$

is the element of Faddeev-Jackiw matrix f .This matrix is antisymmetric since $f_{IJ} = -f_{JI}$. Thus, two cases arise

i) if the matrix f is invertible, we can deduce from (4.8) the expression

$$\dot{\xi}_I = f_{IJ}^{-1} \frac{\partial H}{\partial \xi_J}. \tag{4.10}$$

On the other hand, Hamilton's equations must be on the form

$$\dot{\xi}_I = \{\xi_I, H\}, \tag{4.11}$$

recalling the general form of the Poisson bracket given by the equation (2.13)

$$\{\xi_I, H\} = \{\xi_I, \xi_J\} \frac{\partial H}{\partial \xi_J}, \tag{4.12}$$

it leads that

$$f_{IJ}^{-1} = \{\xi_I, \xi_J\}, \tag{4.13}$$

The bracket$\{\xi_I, \xi_J\}$ Are nothing but just the Dirac brackets obtained by Faddeev-Jackiw approach.

Exemple

Considering the nonlinear Lagrangian from [10], while we choosed $m = 1$

$$L = \frac{1}{2}\dot{q}^2 - V(q).$$

the conjugate momentum

$$p = \frac{\partial L}{\partial \dot{q}} = \dot{q}.$$

The canonical Hamiltonian

$$\begin{aligned} H &= p\dot{q} - L \\ &= p\dot{q} - \frac{1}{2}\dot{q}^2 + V(q) \\ &= \frac{1}{2}p^2 + V(q). \end{aligned}$$

Thus, the linear Lagrangian will be

$$\begin{aligned} L &= p\dot{q} - H \\ &= p\dot{q} - \frac{1}{2}p^2 - V(q). \end{aligned}$$

The independent variables q and p .The (E-L) equations are

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}=0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}}\right)-\frac{\partial L}{\partial p}=0 \end{cases} \Rightarrow \begin{cases} \dot{p}+\frac{\partial V}{\partial q}=0 \\ p-\dot{q}=0 \end{cases}, \tag{4.14}$$

the matrix form of (4.14) is given by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix}=\begin{pmatrix} \frac{\partial V}{\partial q} \\ p \end{pmatrix}, \tag{4.15}$$

where f is

$$f=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

f is invertible, thus its inverse is

$$f^{-1}=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}=\begin{pmatrix} \{q,q\} & \{q,p\} \\ -\{q,p\} & \{p,p\} \end{pmatrix}$$

As we have no constraints, we get the from f^{-1} directly, the canonical Poisson brackets

$$\{q,q\}=0,\ \ \{q,p\}=1,\quad \{p,p\}=0$$

ii) If f is not invertible, we may deal with two sub cases :

a- there exists supplementary conditions.

Since f is not invertible, it means that $rank(f)<n+R$, then this matrix admits $n+R$ $-rank(f)$ independents zero *mode* υ^m , $m=1,...,n+R$ $-rank(f)$. These modes are the line vectors verifying the relation

$$\upsilon^m f=0, \tag{4.16}$$

or explicitly

$$\upsilon_I^m f_{IJ}=0. \tag{4.17}$$

Multipling the equation (4.8) in the left side by υ_I^m will principally give a rise to the constraints

$$\phi_m=\upsilon_I^m\frac{\partial H}{\partial \xi_J}=0,\quad m=1,...,n+R-rank(f), \tag{4.18}$$

These constraints ϕ_m are relations between ξ_J that must be conserved with respect to time. We can write their derivation as follows

$$\dot{\phi}_m=\frac{d\phi_m}{dt}=\frac{\partial \phi_m}{\partial \xi_J}\dot{\xi}_J=0.$$

Proceeding this path, we must add to the Lagrangian (4.5) terms of the form $\left(\lambda_m \frac{\partial \phi_m}{\partial \xi_J} \dot{\xi}_J\right)$, or in the form $\left(\dot{\lambda}_m \phi_m\right)$. We obtain a new linear Lagrangian according to $\dot{\xi}_J$ and $\dot{\lambda}_m$ having the expression

$$L = A_J \dot{\xi}_J + \dot{\lambda}_m \phi_m - H. \tag{4.19}$$

The $\lambda_m$are treated as new independent variables . Thus, (E-L) equations in this case will be

$$\xi_I \rightarrow \left(\frac{\partial A_I}{\partial \xi_J} - \frac{\partial A_J}{\partial \xi_I}\right) \dot{\xi}_I + \frac{\partial \phi_m}{\partial \xi_J} \dot{\lambda}_m = \frac{\partial H}{\partial \xi_J} \tag{4.20}$$

$$\lambda_m \rightarrow \frac{d\phi_m}{dt} = \frac{\partial \phi_m}{\partial \xi_J} \dot{\xi}_J = 0 \quad \text{(conservation of } \phi_m \text{ with respect to time.)} \quad , \tag{4.21}$$

in matrix form, the equations will be

$$\underbrace{\begin{pmatrix} \frac{\partial A_I}{\partial \xi_J} - \frac{\partial A_J}{\partial \xi_I} & \frac{\partial \phi_m}{\partial \xi_J} \\ \frac{\partial \phi_m}{\partial \xi_J} & 0 \end{pmatrix}}_{\text{the matrix } f} \begin{pmatrix} \dot{\xi}_I \\ \dot{\lambda}_m \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial \xi_J} \\ 0 \end{pmatrix}$$

This new matrix f is an antisymmetric square matrix of dimension $n+R+(n+R-rank(f)) = 2\,(n+R) - rank(f)$.

Exemple

Considering the linear Lagrangian

$$L = \frac{1}{2}\dot{x}^2 - ax\dot{y}, \qquad a = cte \neq 0. \tag{4.22}$$

The conjugate momenta

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = \dot{x} \\ p_y &= \frac{\partial L}{\partial \dot{y}} = -ax, \end{aligned}$$

where the primary constraint is $p_y + ax = 0$. The canonical Hamiltonian is

$$\begin{aligned} H &= p_x\dot{x} + p_y\dot{y} - L \\ &= p_x\dot{x} + p_y\dot{y} - \left(\frac{1}{2}\dot{x}^2 - ax\dot{y}\right) \\ &= p_x^2 + p_y\dot{y} - \frac{1}{2}p_x^2 + ax\dot{y} \\ &= \frac{1}{2}p_x^2 + (p_y + ax)\,\dot{y} \end{aligned} \tag{4.23}$$

Since $p_y + ax = 0$, the Hamiltonian becomes

$$H = \frac{1}{2}p_x^2.$$

Thus, the linear Lagrangien is

$$\begin{aligned} L &= p_x\dot{x} + p_y\dot{y} - H \\ &= p_x\dot{x} + p_y\dot{y} - \frac{1}{2}p_x^2 \\ &= p_x\dot{x} - ax\dot{y} - \frac{1}{2}p_x^2. \end{aligned} \tag{4.24}$$

The independent variables here are x, y et p_x. The corresponding (E-L) equations

$$\begin{aligned} \dot{p}_x + a\dot{y} &= 0 \\ -a\dot{x} &= 0 \\ -\dot{x} + p_x &= 0 \end{aligned}$$

The matrix form of the system is given by

$$\underbrace{\begin{pmatrix} 0 & a & 1 \\ -a & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}}_{f^{(0)}} \underbrace{\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{p}_x \end{pmatrix}}_{\dot{\xi}} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ p_x \end{pmatrix}}_{\partial H/\partial \xi}, \tag{4.25}$$

$f^{(0)}$ is singular $rank\left(f^{(0)}\right) = 2$. Thus, this matrix admits one zero mode ; $n+R-rank(f^{(0)}) = 2+1-2 = 1$ that is given as follows (check the annex)

$$\upsilon = \begin{pmatrix} 0 & -\frac{1}{a} & 1 \end{pmatrix}. \tag{4.26}$$

Multiplying (4.25) by (4.26) on the left side, we get the next supplementary constraint

$$p_x = 0,$$

that must be preserved with respect to time, therefore we may add the term $\dot{\lambda}p_x$ to the linear Lagrangian

$$L = p_x\dot{x} - ax\dot{y} - \frac{1}{2}p_x^2 + \dot{\lambda}p_x$$

The independent variables now are x, y , p_x and λ. The corresponding (E-L) equations

$$\begin{aligned} \dot{p}_x + a\dot{y} &= 0 \\ -a\dot{x} &= 0 \\ -\dot{x} + p_x - \dot{\lambda} &= 0 \\ \dot{p}_x &= 0, \end{aligned}$$

or else

$$\underbrace{\begin{pmatrix} 0 & a & 1 & 0 \\ -a & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{f^{(1)}} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{p}_x \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ p_x \\ 0 \end{pmatrix}, \tag{4.27}$$

$f^{(1)}$ is invertible, where the inverse is

$$\left(f^{(1)}\right)^{-1} = \begin{pmatrix} 0 & -\frac{1}{a} & 0 & 0 \\ \frac{1}{a} & 0 & 0 & -\frac{1}{a} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{a} & -1 & 0 \end{pmatrix} \tag{4.28}$$

The generalized Poisson brackets (that are identical to Dirac's one) between the dynamic variables according to the Lagrangian of the begining are

$$\begin{aligned} \{x, x\} &= \{y, y\} = \{p_x, p_x\} = 0 \\ \{x, y\} &= -\frac{1}{a}, \quad \{x, p_x\} = 0, \ \ \{y, p_x\} = 0. \end{aligned}$$

b-There exists no supplementary constraints, but only identities of the type $(0 = 0)$ produced by multiplying the equation (4.8) in the left side by υ_I^m. This is due to the presence of gauge symmetry that we lead us to add term coincide with the Lagrangian, where we fix the gauge in certain conditions.

To make it clear, we recall the previous example mentioned in (4.4)

$$L = \frac{1}{2}(y\dot{x} + x\dot{y})^2 - xy, \tag{4.29}$$

where its linear form was as follows

$$L = p_x\dot{x} + \frac{xp_x}{y}\dot{y} - \frac{1}{2}(\frac{p_x}{y})^2 - xy.$$

The independent variables are then x, y and p_x, while the momentum p_y depends on the other variables through the mentioned constraint above $p_y = \frac{x}{y}\, p_x$. the (E-L) equations apply on the independent variables as follows

$$\begin{aligned}
\dot{y}\frac{p_x}{y} - \dot{p}_x &= y \\
-\dot{x}\frac{p_x}{y} + \dot{p}_x\frac{x}{y} &= -\frac{p_x^2}{y^3} + x \\
\dot{x} + \dot{y}\frac{x}{y} &= \frac{p_x}{y^2},
\end{aligned}$$

their matrix form is given by

$$\underbrace{\begin{pmatrix} 0 & \frac{p_x}{y} & -1 \\ -\frac{p_x}{y} & 0 & \frac{-x}{y} \\ 1 & \frac{x}{y} & 0 \end{pmatrix}}_{f^{(0)}} \underbrace{\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{p}_x \end{pmatrix}}_{\dot{\xi}} = \underbrace{\begin{pmatrix} y \\ -\frac{p_x^2}{y^3} + x \\ \frac{p_x}{y^2} \end{pmatrix}}_{\partial H/\partial \xi}, \tag{4.30}$$

$f^{(0)}$ is singular of $rank\left(f^{(0)}\right) = 2$. Thus the matrix admits one zero mode; $n+R - rank(f^{(0)}) = 2 + 1 - 2 = 1$ is given as follows

$$\upsilon = \begin{pmatrix} -\frac{x}{p_x} & \frac{y}{p_x} & 1 \end{pmatrix} \tag{4.31}$$

Multiplying (4.30) by (4.31) in the left side, we get only identities of the type $(0 = 0)$, so there is no generated constraint in this case, and the matrix keeps singular to express that we are dealing exactely with the presence of gauge symmetry. We choose the gauge condition $y = 1$ by adding the term $\dot{w}(y - 1)$ to the Lagrangian as follows

$$L = p_x\dot{x} + \frac{xp_x}{y}\dot{y} - \frac{1}{2}(\frac{p_x}{y})^2 - xy + \dot{w}(y - 1).$$

The independant variables now x, y , p_x and w.The corresponding (E-L) equations

$$\begin{aligned}
\frac{p_x}{y}\dot{y} - \dot{p}_x &= y \\
-\frac{p_x}{y}\dot{x} - \frac{x}{y}\dot{p}_x + \dot{w} &= -\frac{p_x^2}{y^3} + x \\
\dot{x} + \frac{x}{y}\dot{y} &= \frac{p_x}{y^2} \\
-\dot{y} &= 0,
\end{aligned}$$

using the matrix form we get

$$\underbrace{\begin{pmatrix} 0 & \frac{p_x}{y} & -1 & 0 \\ -\frac{p_x}{y} & 0 & -\frac{x}{y} & 1 \\ 1 & \frac{x}{y} & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}}_{f^{(1)}} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{p}_x \\ \dot{w} \end{pmatrix} = \begin{pmatrix} y \\ -\frac{p_x^2}{y^3} + x \\ \frac{p_x}{y^2} \\ 0 \end{pmatrix}, \tag{4.32}$$

$f^{(1)}$ is invertible, and it inverse is given by

$$f^{(1)^{-1}} = \begin{pmatrix} 0 & 0 & 1 & \frac{x}{y} \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & -\frac{1}{y}p_x \\ -\frac{x}{y} & 1 & \frac{1}{y}p_x & 0 \end{pmatrix} = \begin{pmatrix} \{x,x\} & \{x,y\} & \{x,p_x\} & \{x,w\} \\ \{y,x\} & \{y,y\} & \{y,p_x\} & \{y,w\} \\ \{p_x,x\} & \{p_x,y\} & \{p_x,p_x\} & \{p_x,w\} \\ \{w,x\} & \{w,y\} & \{w,p_x\} & \{w,w\} \end{pmatrix}, \tag{4.33}$$

where then we can extract the following brackets

$$\{x, p_x\} = 1 \ , \quad \{y, p_x\} = 0, \text{ and } \{x, y\} = 0.$$

At this level, we can summarize the existence of three cases that characterize Faddeev and Jackiw method as follows

i) f is invertible and the brackets are obtained using f^{-1}as a matrix elements, and the algorithm ends here.

ii) f is not invertible and there is no generated constraints, this is a sign of gauge symmetry presence. In this case ,the supplementary conditions $\zeta_n(\xi) = 0$ are necessary in order to fix the gauge and have an invertible matrix f . We add terms to the Lagrangian (4.5) as $\dot{\omega}_n \zeta_n(\xi)$ where ω_n represent multipliers. Then we have to write the E-L equations with respect to these variables ξ_I , λ_m and ω_n. Algorithm ends when we find f^{-1}.

iii) f is not invertible and the zero modes give new constraints. We must then add them to the Lagrangian (4.19) with a different lagrangian multipliers, and restart the zero procedure.

Chapter 5

Special applications

There is no doubt that the comparaison study between Dirac's method and (F-J) approach in introducing correct brackets supposed to be the bridge to the quantize theory for constrained systems highlights effectively under the shadow of illustrative applications more than giving analysis to the general principles. In order that, we will show two applications of particle moving on circle and other one moving on ellipse. These two applications will be studied by those methods mentioned above for giving remarks later.

5.1 Applications treated by Dirac's method

5.1.1 Particale moving on a circle

Considering here a particle of mass m moving on a circle of radius $(r = a)$. We will calculate the Dirac brackets for this system. Thus, the corresponding Lagrangian is written

$$L\left(x, \dot{x}, y, \dot{y}, \mu\right) = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) - \mu \left(x^2 + y^2 - a^2\right) \tag{5.1}$$

where the quantity μ is treated here as an independent dynamic variable that called Lagrangian multiplier. The corresponding conjugate momenta are

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$
$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$$
$$p_\mu = \frac{\partial L}{\partial \dot{\mu}} = 0$$

The Hessian matrix W corresponding is

$$W = \begin{pmatrix} \frac{\partial^2 L}{\partial \dot{x} \partial \dot{x}} & \frac{\partial^2 L}{\partial \dot{x} \partial \dot{y}} & \frac{\partial^2 L}{\partial \dot{x} \partial \dot{\mu}} \\ \frac{\partial^2 L}{\partial \dot{y} \partial \dot{x}} & \frac{\partial^2 L}{\partial \dot{y} \partial \dot{y}} & \frac{\partial^2 L}{\partial \dot{y} \partial \dot{\mu}} \\ \frac{\partial^2 L}{\partial \dot{\mu} \partial \dot{x}} & \frac{\partial^2 L}{\partial \dot{\mu} \partial \dot{y}} & \frac{\partial^2 L}{\partial \dot{\mu} \partial \dot{\mu}} \end{pmatrix} = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.2}$$

$\det(W) = 0$, therefore Lagrangian (5.1) is singular. We pose the relation $p_\mu \approx 0$ as a primary constraint

$$\phi_1 = p_\mu \approx 0. \tag{5.3}$$

The constraint $\phi_1$is our only primary constraint, then we construct the canonical Hamiltonian

$$\begin{aligned} H_c &= p_x \dot{x} + p_y \dot{y} + p_\mu \dot{\mu} - L \\ &= \frac{1}{2m}\left(p_x^2 + p_y^2\right) + p_\mu \dot{\mu} + \mu\left(x^2 + y^2 - a^2\right), \end{aligned}$$

then the total Hamiltonian

$$H_T = H_c + \lambda_1 \phi_1, \tag{5.4}$$

where λ_1 is Dirac's multiplier. Explicitly H_T is

$$H_T = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \mu\left(x^2 + y^2 - a^2\right) + \gamma p_\mu, \tag{5.5}$$

where $\lambda_1 + \dot{\mu} = \gamma$. The consistency condition for ϕ_1 is

$$\dot{\phi}_1 = \{\phi_1, H_T\} \approx 0 \Rightarrow \{p_\mu, H_T\} \approx 0 \Rightarrow -\left(x^2 + y^2 - a^2\right) \approx 0, \tag{5.6}$$

which is a new constraint, that the Lagrange multiplier had already implicitly imposed. So, we write

$$\phi_2 = a^2 - x^2 - y^2 \approx 0 \tag{5.7}$$

Likewise, the consistency condition for ϕ_2 gives

$$\dot{\phi}_2 = \{\phi_2, H_T\} \approx 0 \Rightarrow \left\{a^2 - x^2 - y^2, \frac{1}{2m}\left(p_x^2 + p_y^2\right)\right\} \approx 0 \Rightarrow -\frac{2}{m}\left(x p_x + y p_y\right) \approx 0,$$

which also a new constraint that we note by

$$\phi_3 = -\frac{2}{m}\left(x p_x + y p_y\right) \approx 0, \tag{5.8}$$

we do the same for ϕ_3, we obtain

$$\begin{aligned}\dot{\phi}_3 &= \{\phi_3, H_T\} \approx 0 \Rightarrow \left\{-\frac{2}{m}(xp_x + yp_y), \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \mu\left(x^2 + y^2\right)\right\} \approx 0 \\ &\Rightarrow \frac{4\mu}{m}\left(x^2 + y^2\right) - \frac{2}{m^2}\left(p_x^2 + p_y^2\right) \approx 0,\end{aligned}$$

which is also a new constraint, can be defined as

$$\phi_4 = \frac{4\mu}{m}\left(x^2 + y^2\right) - \frac{2}{m^2}\left(p_x^2 + p_y^2\right) \approx 0. \tag{5.9}$$

Finally, if we apply again the consistency condition for ϕ_4, we get expression for γ (or else for λ_1)

$$\gamma = -\frac{4\mu}{m\left(x^2 + y^2\right)}(xp_x + yp_y). \tag{5.10}$$

The algorithm ends.

We have four constraints, ϕ_1, ϕ_2, ϕ_3 and ϕ_4 and the Poisson brackets between these constraints can be written in the form of an antisymmetric matrix Δ whose elements are noted $\Delta_{ij} = \left\{\phi_i, \phi_j\right\}$. This matrix is known as the constraints matrix. Explicitly

$$\Delta = \begin{pmatrix} 0 & \Delta_{12} & \Delta_{13} & \Delta_{14} \\ -\Delta_{12} & 0 & \Delta_{23} & \Delta_{24} \\ -\Delta_{13} & -\Delta_{14} & 0 & \Delta_{34} \\ -\Delta_{14} & -\Delta_{24} & -\Delta_{34} & 0 \end{pmatrix}, \tag{5.11}$$

where

$$\begin{aligned}\Delta_{12} &= \{\phi_1, \phi_2\} = \left\{p_\mu, a^2 - x^2 - y^2\right\} = 0 \\ \Delta_{13} &= \{\phi_1, \phi_3\} = \left\{p_\mu, -\frac{2}{m}(xp_x + yp_y)\right\} = 0 \\ \Delta_{14} &= \{\phi_1, \phi_4\} = -\frac{4}{m}\left(x^2 + y^2\right) = -\frac{4a^2}{m} \\ \Delta_{23} &= \{\phi_2, \phi_3\} = \frac{4}{m}\left(x^2 + y^2\right) = \frac{4a^2}{m} \\ \Delta_{24} &= \{\phi_2, \phi_4\} = \frac{8}{m^2}(xp_x + yp_y) = 0 \\ \Delta_{34} &= \{\phi_3, \phi_4\} = \frac{16\mu}{m^2}\left(x^2 + y^2\right) + \frac{8}{m^3}\left(p_x^2 + p_y^2\right) = \frac{32a^2}{m^2}\mu,\end{aligned}$$

where we used the constraints as strong equalities after the computation of these brackets .

Therefore the constraints matrix is going to be

$$\Delta = \begin{pmatrix} 0 & 0 & 0 & -\frac{4a^2}{m} \\ 0 & 0 & \frac{4a^2}{m} & 0 \\ 0 & -\frac{4a^2}{m} & 0 & \frac{32a^2}{m^2}\mu \\ \frac{4a^2}{m} & 0 & -\frac{32a^2}{m^2}\mu & 0 \end{pmatrix} \tag{5.12}$$

the inverse is

$$\Delta^{-1} = \begin{pmatrix} 0 & \frac{2\mu}{a^2} & 0 & \frac{m}{4a^2} \\ -\frac{2\mu}{a^2} & 0 & -\frac{m}{4a^2} & 0 \\ 0 & \frac{m}{4a^2} & 0 & 0 \\ -\frac{m}{4a^2} & 0 & 0 & 0 \end{pmatrix} \tag{5.13}$$

Calculating now the Dirac's brackets of the dynamic variables using the formula (3.36) to know that

$$\{f, g\}_D = \{f, g\} - \sum_{i,j=1}^{4} \{f, \phi_i\} \Delta_{ij}^{-1} \{\phi_j, g\}. \tag{5.14}$$

For example, we calculate the bracket $\{\mu, p_\mu\}_D$

$$\begin{aligned} \{\mu, p_\mu\}_D &= 1 - \sum_{i,j=1}^{4} \{\mu, \phi_i\} \Delta_{ij}^{-1} \{\phi_j, p_\mu\} \\ &= 1 - \{\mu, \phi_1\} \Delta_{14}^{-1} \{\phi_4, p_\mu\} \\ &= 1 - \{\mu, p_\mu\} \frac{m}{4a^2} \left\{ \frac{4\mu}{m} \left(x^2 + y^2\right) \ , \ p_\mu \right\} \\ &= 1 - \frac{m}{4a^2} \{\mu, p_\mu\} \frac{4}{m} \left(x^2 + y^2\right) \\ &= 1 - \frac{m}{4a^2} \frac{4}{m} \left(x^2 + y^2\right) \quad , \quad x^2 + y^2 = a^2 \\ &= 0. \end{aligned}$$

Likewise we can obtain the bracket

$$\begin{aligned} \{x, p_x\}_D &= 1 - \sum_{i,j=1}^{4} \{x, \phi_i\} \Delta_{ij}^{-1} \{\phi_j, p_x\} \\ &= 1 - \sum_{j=1}^{4} \{x, \phi_3\} \Delta_{3j}^{-1} \{\phi_j, p_x\} - \sum_{j=1}^{4} \{x, \phi_4\} \Delta_{4j}^{-1} \{\phi_j, p_x\} \\ &= 1 - \{x, \phi_3\} \Delta_{32}^{-1} \{\phi_2, p_x\} \\ &= 1 - \left\{ x, -\frac{2}{m} \left(xp_x + yp_y\right) \right\} \frac{m}{4a^2} \left\{a^2 - x^2 - y^2, p_x\right\} \\ &= 1 - \frac{x^2}{a^2}. \end{aligned}$$

We can equaly verify that we have the Dirac's brackets as follows

$$\begin{aligned} \{y, p_y\}_D &= 1 - \frac{y^2}{a^2}, \quad \{x, p_y\}_D = -\frac{xy}{a^2}, \quad \{y, p_x\}_D = -\frac{xy}{a^2}, \\ \{x, y\}_D &= 0, \quad \{p_x, p_y\}_D = -\frac{1}{a^2}(xp_y - yp_x) = -\frac{1}{a^2}L_Z, \end{aligned} \tag{5.15}$$

where L_Z is the angular momentum for the component Z.

5.1.2 Particale moving on ellipse

Considering here a particle of mass m moving on a ellipse that was centered at the origin with width $2a$ and height $2b$. We will calculate the Dirac brackets for this system.

Thus, the corresponding Lagrangian is written

$$L(x, \dot{x}, y, \dot{y}, \mu) = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - \mu\left(b^2x^2 + a^2y^2 - a^2b^2\right), \tag{5.16}$$

where the quantity μ is treated here as an independent dynamic variable.

The corresponding conjugate momenta are

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = m\dot{x} \\ p_y &= \frac{\partial L}{\partial \dot{y}} = m\dot{y} \\ p_\mu &= \frac{\partial L}{\partial \dot{\mu}} = 0 \end{aligned}$$

The Hessian matrix W is

$$W = \begin{pmatrix} \frac{\partial^2 L}{\partial \dot{x}\partial \dot{x}} & \frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}} & \frac{\partial^2 L}{\partial \dot{x}\partial \dot{\mu}} \\ \frac{\partial^2 L}{\partial \dot{y}\partial \dot{x}} & \frac{\partial^2 L}{\partial \dot{y}\partial \dot{y}} & \frac{\partial^2 L}{\partial \dot{y}\partial \dot{\mu}} \\ \frac{\partial^2 L}{\partial \dot{\mu}\partial \dot{x}} & \frac{\partial^2 L}{\partial \dot{\mu}\partial \dot{y}} & \frac{\partial^2 L}{\partial \dot{\mu}\partial \dot{\mu}} \end{pmatrix} = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.17}$$

$\det(W) = 0$, therefore Lagrangian (5.16) is singular. We pose the relation $p_\mu \approx 0$ as a primary constraint.i.e

$$\phi_1 = p_\mu \approx 0 \tag{5.18}$$

The constraint $\phi_1$is our only primary constraint. then we construct the canonical Hamiltonian

$$\begin{aligned} H_c &= p_x\dot{x} + p_y\dot{y} + p_\mu\dot{\mu} - L \\ &= \frac{1}{2m}\left(p_x^2 + p_y^2\right) + p_\mu\dot{\mu} + \mu\left(b^2x^2 + a^2y^2 - a^2b^2\right), \end{aligned}$$

while, the total Hamiltonian

$$H_T = H_c + \lambda_1 \phi_1, \tag{5.19}$$

where λ_1 is Dirac's multiplier. Explicitly H_T is

$$H_T = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \mu\left(b^2x^2 + a^2y^2 - a^2b^2\right) + \gamma p_\mu, \tag{5.20}$$

where $\lambda_1 + \dot{\mu} = \gamma$. The consistency condition for ϕ_1 is

$$\dot{\phi}_1 = \{\phi_1, H_T\} \approx 0 \Rightarrow \{p_\mu, H_T\} \approx 0 \Rightarrow -\left(b^2x^2 + a^2y^2 - a^2b^2\right) \approx 0, \tag{5.21}$$

which is a new constraint, that the Lagrangian multiplier had already implicitly imposed. So we write

$$\phi_2 = a^2b^2 - b^2x^2 - a^2y^2 \approx 0, \tag{5.22}$$

Likewise, the consistency condition for ϕ_2 gives

$$\dot{\phi}_2 = \{\phi_2, H_T\} \approx 0 \Rightarrow \left\{a^2b^2 - b^2x^2 - a^2y^2, \frac{1}{2m}\left(p_x^2 + p_y^2\right)\right\} \approx 0 \Rightarrow -\frac{2}{m}\left(b^2xp_x + a^2yp_y\right) \approx 0,$$

which also is a new constraint that we note by

$$\phi_3 = -\frac{2}{m}\left(b^2xp_x + a^2yp_y\right) \approx 0, \tag{5.23}$$

We do the same for ϕ_3, we obtain

$$\begin{aligned} \dot{\phi}_3 &= \{\phi_3, H_T\} \approx 0 \Rightarrow \left\{-\frac{2}{m}\left(b^2xp_x + a^2yp_y\right), \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \mu\left(b^2x^2 + a^2y^2 - a^2b^2\right)\right\} \approx 0 \\ &\Rightarrow \frac{4\mu}{m}\left(b^4x^2 + a^4y^2\right) - \frac{2}{m^2}\left(b^2p_x^2 + a^2p_y^2\right) \approx 0, \end{aligned}$$

which is also a new constraint, can be defined as

$$\phi_4 = \frac{4\mu}{m}\left(b^4x^2 + a^4y^2\right) - \frac{2}{m^2}\left(b^2p_x^2 + a^2p_y^2\right) \approx 0, \tag{5.24}$$

Finally, if we apply again the consistency condition for ϕ_4, we get expression for γ (or else for λ_1)

$$\gamma = -\frac{4\mu}{m\left(b^4x^2 + a^4y^2\right)}\left(b^4xp_x + a^4yp_y\right). \tag{5.25}$$

The algorithm ends.

We have four constraints, ϕ_1, ϕ_2, ϕ_3 and ϕ_4 and the Poisson brackets between these constraints can be written in the form of an antisymmetric matrix Δ whose elements are noted $\Delta_{ij} = \left\{\phi_i, \phi_j\right\}$. This matrix is known as the constraints matrix. Explicitly

$$\Delta = \begin{pmatrix} 0 & \Delta_{12} & \Delta_{13} & \Delta_{14} \\ -\Delta_{12} & 0 & \Delta_{23} & \Delta_{24} \\ -\Delta_{13} & -\Delta_{14} & 0 & \Delta_{34} \\ -\Delta_{14} & -\Delta_{24} & -\Delta_{34} & 0 \end{pmatrix}, \tag{5.26}$$

where

$$\begin{aligned} \Delta_{12} &= \{\phi_1, \phi_2\} = \left\{p_\mu, a^2b^2 - b^2x^2 - a^2y^2\right\} = 0 \\ \Delta_{13} &= \{\phi_1, \phi_3\} = \left\{p_\mu, -\frac{2}{m}\left(b^2xp_x + a^2yp_y\right)\right\} = 0 \\ \Delta_{14} &= \{\phi_1, \phi_4\} = -\frac{4}{m}\left(b^4x^2 + a^4y^2\right) \\ \Delta_{23} &= \{\phi_2, \phi_3\} = \frac{4}{m}\left(b^4x^2 + a^4y^2\right) \\ \Delta_{24} &= \{\phi_2, \phi_4\} = \frac{8}{m^2}\left(b^4xp_x + a^4yp_y\right) \\ \Delta_{34} &= \{\phi_3, \phi_4\} = \frac{16\mu}{m^2}\left(b^6x^2 + a^6y^2\right) + \frac{8}{m^3}\left(b^4p_x^2 + a^4p_y^2\right). \end{aligned}$$

Where we have used the constraints as strong equalities after the computation of these brackets . Therefore the constraints matrix is

$$\Delta = \begin{pmatrix} 0 & 0 & 0 & -\frac{4\left(b^4x^2+a^4y^2\right)}{m} \\ 0 & 0 & \frac{4\left(b^4x^2+a^4y^2\right)}{m} & \frac{8\left(b^4xp_x+a^4yp_y\right)}{m^2} \\ 0 & -\frac{4\left(b^4x^2+a^4y^2\right)}{m} & 0 & \frac{16\mu\left(b^6x^2+a^6y^2\right)}{m^2} + \frac{8\left(b^4p_x^2+a^4p_y^2\right)}{m^3} \\ \frac{4\left(b^4x^2+a^4y^2\right)}{m} & -\frac{8\left(b^4xp_x+a^4yp_y\right)}{m^2} & -\frac{16\mu\left(b^6x^2+a^6y^2\right)}{m^2} - \frac{8\left(b^4p_x^2+a^4p_y^2\right)}{m^3} & 0 \end{pmatrix}.$$

The inverse is

$$\Delta^{-1} = \begin{pmatrix} 0 & \frac{\mu\left(b^6x^2+a^6y^2\right)}{(b^4x^2+a^4y^2)^2} + \frac{\left(b^4p_x^2+a^4p_y^2\right)}{2m(b^4x^2+a^4y^2)^2} & -\frac{\left(b^4xp_x+a^4yp_y\right)}{2(b^4x^2+a^4y^2)^2} & \frac{m}{4(b^4x^2+a^4y^2)} \\ -\frac{\mu\left(b^6x^2+a^6y^2\right)}{(b^4x^2+a^4y^2)^2} - \frac{\left(b^4p_x^2+a^4p_y^2\right)}{2m(b^4x^2+a^4y^2)^2} & 0 & -\frac{m}{4(b^4x^2+a^4y^2)} & 0 \\ \frac{\left(b^4xp_x+a^4yp_y\right)}{2(b^4x^2+a^4y^2)^2} & \frac{m}{4(b^4x^2+a^4y^2)} & 0 & 0 \\ -\frac{m}{4(b^4x^2+a^4y^2)} & 0 & 0 & 0 \end{pmatrix}.$$

Calculating now the Dirac's brackets of the dynamic variables in the same frequency of circle application, for example, we calculate the bracket $\{\mu, p_\mu\}_D$

$$\begin{aligned}
\{\mu, p_\mu\}_D &= 1 - \sum_{i,j=1}^{4} \{\mu, \phi_i\} \Delta_{ij}^{-1} \{\phi_j, p_\mu\} \\
&= 1 - \{\mu, \phi_1\} \Delta_{14}^{-1} \{\phi_4, p_\mu\} \\
&= 1 - \{\mu, p_\mu\} \frac{m}{4\left(b^4x^2 + a^4y^2\right)} \left\{ \frac{4\mu}{m} \left(b^4x^2 + a^4y^2\right) \ , \ p_\mu \right\} \\
&= 1 - \frac{m}{4\left(b^4x^2 + a^4y^2\right)} \{\mu, p_\mu\} \frac{4}{m} \left(b^4x^2 + a^4y^2\right) \\
&= 1 - \frac{m}{4\left(b^4x^2 + a^4y^2\right)} \frac{4}{m} \left(b^4x^2 + a^4y^2\right) \\
&= 0.
\end{aligned}$$

Likewise we can obtain the bracket

$$\begin{aligned}
\{x, p_x\}_D &= 1 - \sum_{i,j=1}^{4} \{x, \phi_i\} \Delta_{ij}^{-1} \{\phi_j, p_x\} \\
&= 1 - \sum_{j=1}^{4} \{x, \phi_3\} \Delta_{3j}^{-1} \{\phi_j, p_x\} - \sum_{j=1}^{4} \{x, \phi_4\} \Delta_{4j}^{-1} \{\phi_j, p_x\} \\
&= 1 - \{x, \phi_3\} \Delta_{32}^{-1} \{\phi_2, p_x\} \\
&= 1 - \left\{ x, -\frac{2}{m} \left(b^2xp_x + a^2yp_y\right) \right\} \frac{m}{4\left(b^4x^2 + a^4y^2\right)} \left\{a^2b^2 - b^2x^2 - a^2y^2, p_x\right\} \\
&= 1 - \frac{b^4x^2}{\left(b^4x^2 + a^4y^2\right)}.
\end{aligned}$$

We can equaly verify that we have the Dirac's brackets as follows

$$\begin{aligned}
\{y, p_y\}_D &= 1 - \frac{a^4y^2}{\left(b^4x^2 + a^4y^2\right)} \ , \quad \{x, p_y\}_D = -\frac{a^2b^2xy}{\left(b^4x^2 + a^4y^2\right)}, \quad \{y, p_x\}_D = -\frac{a^2b^2yx}{\left(b^4x^2 + a^4y^2\right)}, \\
\{x, y\}_D &= 0, \qquad \{p_x, p_y\}_D = -\frac{L_Z}{\left(b^4x^2 + a^4y^2\right)}.
\end{aligned} \tag{5.27}$$

5.2 Applications treated by Fadeev and Jackiw method

5.2.1 Particle moving on a circle

The Lagrangian of the system is given by

$$L\left(x, \dot{x}, y, \dot{y}, \mu\right) = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) - \mu \left(x^2 + y^2 - a^2\right). \tag{5.28}$$

The correspondant conjugate momenta are

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \ , \quad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \ , \ p_\mu = \frac{\partial L}{\partial \dot{\mu}} = 0, \tag{5.29}$$

where $p_\mu = 0$ is the primary constraints . The canonical Hamiltonian for the system is

$$\begin{aligned} H &= p_x\dot{x} + p_y\dot{y} + p_\mu\dot{\mu} - L \\ &= p_x\dot{x} + p_y\dot{y} + p_\mu\dot{\mu} - \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \mu\left(x^2 + y^2 - a^2\right) \\ &= \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \mu\left(x^2 + y^2 - a^2\right) \quad , \ p_\mu = 0. \end{aligned} \tag{5.30}$$

Thus, the linear Lagrangian will be

$$\begin{aligned} L &= p_x\dot{x} + p_y\dot{y} - H \\ &= p_x\dot{x} + p_y\dot{y} - \frac{1}{2m}\left(p_x^2 + p_y^2\right) - \mu\left(x^2 + y^2 - a^2\right). \end{aligned} \tag{5.31}$$

We arrive to an important situation that deserves to be given some observations. if we follow directly the algorithm above, we may find as follows in the next step using our independent variables x, y, μ, p_x and p_y. The correspondent (E-L) equations lead us to

$$\begin{aligned} \dot{p}_x + 2\mu x &= 0 \\ \dot{p}_y + 2\mu y &= 0 \\ x^2 + y^2 - a^2 &= 0 \\ -\dot{x} + \frac{p_x}{m} &= 0 \\ -\dot{y} + \frac{p_y}{m} &= 0. \end{aligned}$$

Under the matrix form, we have

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}}_{=f} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\mu} \\ \dot{p}_x \\ \dot{p}_y \end{pmatrix} = \begin{pmatrix} 2x\mu \\ 2y\mu \\ a^2 - x^2 - y^2 \\ \frac{p_x}{m} \\ \frac{p_y}{m} \end{pmatrix}. \tag{5.32}$$

The calculation of the determinant of f leads that it is singular with $rank(f) = 4$. Therefore, this matrix admits one zero mode under this relation $n+R-rank(f) = 3+2-4 = 1$, that is given by

$$\upsilon = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{5.33}$$

Multiplying(5.32) on the left side by (5.33), we may obtain a supplementary constraint

$$\phi = a^2 - x^2 - y^2 = 0, \tag{5.34}$$

which is nothing but expresses the circle equation as it should be. However, we know that this constraint must be introduced in the Lagrangian (5.31) either like $\dot{\alpha}\phi$ or $\alpha\dot{\phi}$, where α is the Lagrangian multiplier. Thus, it is now more practical to replace easily $\mu \to \dot{\alpha}$ in the begining. By doing this, we simply introduce a total derivative to the Lagrangian

$$\mu\phi \to \mu\phi - \frac{d}{dt}(\alpha\phi) = (\mu - \dot{\alpha})\,\phi - \alpha\dot{\phi}. \tag{5.35}$$

Choosing $\mu = \dot{\alpha}$. After this digression, we then write our Lagrangian (5.31) as follows

$$L = p_x\dot{x} + p_y\dot{y} - \frac{1}{2m}\left(p_x^2 + p_y^2\right) - \dot{\alpha}\left(x^2 + y^2 - a^2\right). \tag{5.36}$$

Our independent variables become $(x, y,\ p_x,\ p_y$ and $\alpha)$. The (E-L) equations give

$$\begin{aligned} \dot{p}_x + 2x\dot{\alpha} &= 0 \\ \dot{p}_y + 2y\dot{\alpha} &= 0 \\ -\dot{x} + \frac{p_x}{m} &= 0 \\ -\dot{y} + \frac{p_y}{m} &= 0 \\ -2x\dot{x} - 2y\dot{y} &= 0. \end{aligned}$$

Under the matrix form, we get

$$\underbrace{\begin{pmatrix} 0 & 0 & -1 & 0 & -2x \\ 0 & 0 & 0 & -1 & -2y \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 2x & 2y & 0 & 0 & 0 \end{pmatrix}}_{f^{(0)}} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{p}_x \\ \dot{p}_y \\ \dot{\alpha} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{p_x}{m} \\ \frac{p_y}{m} \\ 0 \end{pmatrix}, \tag{5.37}$$

$f^{(0)}$ is singular and $rank\left(f^{(0)}\right) = 4$, where it has one zero mode given by

$$v = \begin{pmatrix} 0 & 0 & -2x & -2y & 1 \end{pmatrix} \tag{5.38}$$

Multiplying the system (5.37) by that latter (5.38) , we obtain a new supplementary constraint

$$-\frac{2}{m}\left(xp_x + yp_y\right) = 0 \tag{5.39}$$

This constraint must be introduced in the Lagrangian of the starting (5.36). Thus, we write

$$L = p_x\dot{x} + p_y\dot{y} - \frac{1}{2m}\left(p_x^2 + p_y^2\right) - \dot{\alpha}\left(x^2 + y^2 - a^2\right) - 2\frac{\dot{\beta}}{m}\left(xp_x + yp_y\right) \tag{5.40}$$

Our independent variables now are x, y , p_x, p_y, α and β.The corresponding (E-L) equations are

$$\begin{aligned}
\dot{p}_x + 2x\dot{\alpha} + 2\frac{p_x}{m}\dot{\beta} &= 0\\
\dot{p}_y + 2y\dot{\alpha} + 2\frac{p_y}{m}\dot{\beta} &= 0\\
-\dot{x} + \frac{p_x}{m} + 2\frac{x}{m}\dot{\beta} &= 0\\
-\dot{y} + \frac{p_y}{m} + 2\frac{y}{m}\dot{\beta} &= 0\\
-2x\dot{x} - 2y\dot{y} &= 0\\
-\frac{2}{m}\left(\dot{x}p_x + \dot{y}p_y + x\dot{p}_x + y\dot{p}_y\right) &= 0
\end{aligned}$$

Under the matrix form, we get

$$\underbrace{\begin{pmatrix} 0 & 0 & -1 & 0 & -2x & -\frac{2p_x}{m}\\ 0 & 0 & 0 & -1 & -2y & -\frac{2p_y}{m}\\ 1 & 0 & 0 & 0 & 0 & -\frac{2x}{m}\\ 0 & 1 & 0 & 0 & 0 & -\frac{2y}{m}\\ 2x & 2y & 0 & 0 & 0 & 0\\ \frac{2p_x}{m} & \frac{2p_y}{m} & \frac{2x}{m} & \frac{2y}{m} & 0 & 0 \end{pmatrix}}_{f^{(1)}} \begin{pmatrix} \dot{x}\\ \dot{y}\\ \dot{p}_x\\ \dot{p}_y\\ \dot{\alpha}\\ \dot{\beta} \end{pmatrix} = \begin{pmatrix} 0\\ 0\\ \frac{p_x}{m}\\ \frac{p_y}{m}\\ 0\\ 0 \end{pmatrix} \tag{5.41}$$

Noting in the begining that the matrix $f^{(1)}$contains the matrix $f^{(0)}$ as a sub matrix. Moreover, $f^{(1)}$ is invertible and its inverse is

$$\left(f^{(1)}\right)^{-1} = \begin{pmatrix} 0 & 0 & 1-\frac{x^2}{a^2} & -\frac{xy}{a^2} & \frac{x}{2a^2} & 0\\ 0 & 0 & -\frac{xy}{a^2} & 1-\frac{y^2}{a^2} & \frac{y}{2a^2} & 0\\ \frac{x^2}{a^2}-1 & \frac{xy}{a^2} & 0 & -\frac{L_z}{a^2} & -\frac{p_x}{2a^2} & \frac{mx}{2a^2}\\ \frac{xy}{a^2} & \frac{y^2}{a^2}-1 & \frac{L_z}{a^2} & 0 & -\frac{p_y}{2a^2} & \frac{my}{2a^2}\\ -\frac{x}{2a^2} & -\frac{y}{2a^2} & \frac{p_x}{2a^2} & \frac{p_y}{2a^2} & 0 & -\frac{m}{4a^2}\\ 0 & 0 & -\frac{mx}{2a^2} & -\frac{my}{2a^2} & \frac{m}{4a^2} & 0 \end{pmatrix} \tag{5.42}$$

The generalized Poisson brackets of the dynamic variables contained in the symplectic matrix $\left(f^{(1)}\right)^{-1}$ are identical to the Dirac's brackets obtained by his method in the same treatment.

For example, we mention the next brackets

$$\begin{aligned} \{x, p_x\}_{GPB} &= 1 - \frac{x^2}{a^2} = \{x, p_x\}_D \\ \{y, p_y\}_{GPB} &= 1 - \frac{y^2}{a^2} = \{y, p_y\}_D \end{aligned}$$

5.2.2 Particle moving on ellipse

The Lagrangian of the system is given by

$$L(x, \dot{x}, y, \dot{y}, \mu) = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) - \mu \left(bx^2 + a^2 y^2 - a^2 b^2\right). \tag{5.43}$$

The correspondent conjugate momenta are

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}\ , \quad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}\ , \ p_\mu = \frac{\partial L}{\partial \dot{\mu}} = 0 \tag{5.44}$$

where $p_\mu = 0$ is the primary constraints . The canonical Hamiltonian for the system is

$$\begin{aligned} H &= p_x\dot{x} + p_y\dot{y} + p_\mu\dot{\mu} - L \\ &= p_x\dot{x} + p_y\dot{y} + p_\mu\dot{\mu} - \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) + \mu \left(bx^2 + a^2 y^2 - a^2 b^2\right) \\ &= \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \mu \left(bx^2 + a^2 y^2 - a^2 b^2\right) \quad , \ p_\mu = 0. \end{aligned} \tag{5.45}$$

Thus, the linear Lagrangian will be

$$\begin{aligned} L &= p_x\dot{x} + p_y\dot{y} - H \\ &= p_x\dot{x} + p_y\dot{y} - \frac{1}{2m}\left(p_x^2 + p_y^2\right) - \mu \left(bx^2 + a^2 y^2 - a^2 b^2\right). \end{aligned} \tag{5.46}$$

We arrive to an important situation that deserves to be given some observations. if we follow directly the algorithm above, we may find as follows in the next step using our independent variables x, y, μ, p_x and p_y. The correspondent (E-L) equations lead us to

$$\begin{aligned} \dot{p}_x + 2\mu b^2 x &= 0 \\ \dot{p}_y + 2\mu a^2 y &= 0 \\ \left(bx^2 + a^2 y^2 - a^2 b^2\right) &= 0 \\ -\dot{x} + \frac{p_x}{m} &= 0 \\ -\dot{y} + \frac{p_y}{m} &= 0. \end{aligned}$$

Under the matrix form, we have

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}}_{=f} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\mu} \\ \dot{p}_x \\ \dot{p}_y \end{pmatrix} = \begin{pmatrix} 2xb^2\mu \\ 2ya^2\mu \\ a^2b^2 - b^2x^2 - a^2y^2 \\ \frac{p_x}{m} \\ \frac{p_y}{m} \end{pmatrix}. \tag{5.47}$$

The calculation of the determinant of f leads that it is singular with $rank(f) = 4$. Therefore, this matrix admits one zero mode under this relation $n+R-rank(f) = 3+2-4 = 1$, that is given by

$$v = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{5.48}$$

Multiplying(5.47) on the left side by (5.48), we may obtain a supplementary constraint

$$\phi = a^2b^2 - b^2x^2 - a^2y^2 = 0 \tag{5.49}$$

which is nothing but expresses the ellipse equation as it should be. However, we know that this constraint must be introduced in the Lagrangian (5.46). As in circle application we choose $\mu = \dot{\alpha}$, and we write our Lagrangian (5.46) as follows

$$L = p_x\dot{x} + p_y\dot{y} - \frac{1}{2m}\left(p_x^2 + p_y^2\right) - \dot{\alpha}\left(b^2x^2 + a^2y^2 - a^2b^2\right). \tag{5.50}$$

Our independent variables becomes ($x, y,\ p_x$, p_y and α).

$$\begin{aligned} \dot{p}_x + 2b^2x\dot{\alpha} &= 0 \\ \dot{p}_y + 2a^2y\dot{\alpha} &= 0 \\ -\dot{x} + \frac{p_x}{m} &= 0 \\ -\dot{y} + \frac{p_y}{m} &= 0 \\ -2b^2x\dot{x} - 2a^2y\dot{y} &= 0. \end{aligned}$$

Under the matrix form, we get

$$\underbrace{\begin{pmatrix} 0 & 0 & -1 & 0 & -2b^2x \\ 0 & 0 & 0 & -1 & -2a^2y \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 2b^2x & 2a^2y & 0 & 0 & 0 \end{pmatrix}}_{f^{(0)}} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{p}_x \\ \dot{p}_y \\ \dot{\alpha} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{p_x}{m} \\ \frac{p_y}{m} \\ 0 \end{pmatrix}, \tag{5.51}$$

$f^{(0)}$ is singular and $rank\left(f^{(0)}\right)=4$, where it has one zero mode given by

$$\upsilon=\begin{pmatrix} 0 & 0 & -2b^2x & -2a^2y & 1 \end{pmatrix}. \tag{5.52}$$

Multiplying the system (5.51) by that latter (5.52), we obtain a new supplementary constraint

$$-\frac{2}{m}\left(b^2xp_x+a^2yp_y\right)=0. \tag{5.53}$$

This constraint must be introduced in the Lagrangian of the starting (5.50). Thus, we write

$$L=p_x\dot{x}+p_y\dot{y}-\frac{1}{2m}\left(p_x^2+p_y^2\right)-\dot{\alpha}\left(b^2x^2+a^2y^2-a^2\right)-2\frac{\dot{\beta}}{m}\left(b^2xp_x+a^2yp_y\right). \tag{5.54}$$

Our variables now are x, y , p_x, p_y, α and β.The corresponding E-L equations are

$$\begin{aligned}
\dot{p}_x+2b^2x\dot{\alpha}+2b^2\frac{p_x}{m}\dot{\beta} &= 0\\
\dot{p}_y+2a^2y\dot{\alpha}+2a^2\frac{p_y}{m}\dot{\beta} &= 0\\
\dot{x}-2b^2\frac{x}{m}\dot{\beta} &= \frac{p_x}{m}\\
\dot{y}-2a^2\frac{y}{m}\dot{\beta} &= \frac{p_y}{m}\\
-2b^2x\dot{x}-2a^2y\dot{y} &= 0\\
-\frac{2}{m}\left[b^2\left(\dot{x}p_x+x\dot{p}_x\right)+a^2\left(\dot{y}p_y+y\dot{p}_y\right)\right] &= 0
\end{aligned}$$

Under the matrix form, we get

$$\underbrace{\begin{pmatrix}
0 & 0 & -1 & 0 & -2b^2x & -\frac{2b^2p_x}{m}\\
0 & 0 & 0 & -1 & -2a^2y & -\frac{2a^2p_y}{m}\\
1 & 0 & 0 & 0 & 0 & -\frac{2b^2x}{m}\\
0 & 1 & 0 & 0 & 0 & -\frac{2a^2y}{m}\\
2b^2x & 2a^2y & 0 & 0 & 0 & 0\\
\frac{2b^2p_x}{m} & \frac{2a^2p_y}{m} & \frac{2b^2x}{m} & \frac{2a^2y}{m} & 0 & 0
\end{pmatrix}}_{f^{(1)}}
\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{p}_x\\ \dot{p}_y\\ \dot{\alpha}\\ \dot{\beta}\end{pmatrix}
=\begin{pmatrix}0\\ 0\\ \frac{p_x}{m}\\ \frac{p_y}{m}\\ 0\\ 0\end{pmatrix} \tag{5.55}$$

Noting in the begining that the matrix $f^{(1)}$contains the matrix $f^{(0)}$ as a sub matrix. Moreover, $f^{(1)}$is invertible and its inverse is

$$\left(f^{(1)}\right)^{-1}=\begin{pmatrix} 0 & 0 & \frac{a^4y^2}{a^4y^2+b^4x^2} & -\frac{a^2b^2xy}{a^4y^2+b^4x^2} & \frac{b^2x}{2(a^4y^2+b^4x^2)} & 0 \\ 0 & 0 & -\frac{a^2b^2xy}{a^4y^2+b^4x^2} & \frac{b^4x^2}{a^4y^2+b^4x^2} & \frac{a^2y}{2(a^4y^2+b^4x^2)} & 0 \\ -\frac{a^4y^2}{a^4y^2+b^4x^2} & \frac{a^2b^2xy}{a^4y^2+b^4x^2} & 0 & -\frac{a^2b^2L_z}{a^4y^2+b^4x^2} & -\frac{b^2p_x}{2(a^4y^2+b^4x^2)} & \frac{b^2mx}{2(a^4y^2+b^4x^2)} \\ \frac{a^2b^2xy}{a^4y^2+b^4x^2} & -\frac{b^4x^2}{a^4y^2+b^4x^2} & \frac{a^2b^2L_z}{a^4y^2+b^4x^2} & 0 & -\frac{a^2p_y}{2(a^4y^2+b^4x^2)} & \frac{a^2my}{2(a^4y^2+b^4x^2)} \\ -\frac{b^2x}{2(a^4y^2+b^4x^2)} & -\frac{a^2y}{2(a^4y^2+b^4x^2)} & \frac{b^2p_x}{2(a^4y^2+b^4x^2)} & \frac{a^2p_y}{2(a^4y^2+b^4x^2)} & 0 & -\frac{m}{4(a^4y^2+b^4x^2)} \\ 0 & 0 & \frac{-b^2mx}{2(a^4y^2+b^4x^2)} & \frac{-a^2my}{2(a^4y^2+b^4x^2)} & \frac{m}{4(a^4y^2+b^4x^2)} & 0 \end{pmatrix}.$$

The generalized Poisson brackets of the dynamic variables contained in the symplectic matrix $\left(f^{(1)}\right)^{-1}$ are identical to the Dirac's brackets obtained by his method in the same treatment. For example, we mention the next brackets

$$\begin{aligned} \{x,p_x\}_{GPB} &= \frac{a^4y^2}{a^4y^2+b^4x^2}=1-\frac{b^4x^2}{(b^4x^2+a^4y^2)}=\{x,p_x\}_D \\ \{y,p_y\}_{GPB} &= \frac{b^4x^2}{a^4y^2+b^4x^2}=1-\frac{a^4y^2}{(b^4x^2+a^4y^2)}=\{y,p_y\}_D \end{aligned}$$

5.3 Notes and results

It must be noted that we dealt in the two above-mentioned applications with Lagrangians of the first order with two ways that are technically different of Dirac and (F-J).These both methods enabled us to reach the Dirac's brackets which considerd as important entrance to the quantize theory with fully compatible results. There is no doubt that the F-J method was much faster and more economical .We can recognize that efficitively in givinig us those Dirac's brackets in one fell swoop as a matrix elements, while Dirac's conjecture gave us the same result, one by one under many Poisson brackets calculations.

It is clear also that we didn't use much steps and notions such as weak and strong equality, constraint classifications, and there is also reduction in constraints number in F-J method.

We need to mention that it is axiomatic that the brackets obtained in the ellipse application can lead us to the same one obtained for a particle moving on circle in specific condition where $a=b$.Indeed, this is what we may get clearly in our brackets.

Finally we may say that the effective role of Dirac's conjecture can't be denied, but (F-J) method is considered to be more successful and attractive in practice.