وزارة التعليم العالي والبحث العلمي

BADJI MOKHTAR –ANNABA UNIVERSITY

جامعة باجي مختار عنابة

Faculty of Sciences

Department of Mathematics

Year: 2021/2022

THESIS

Presented for the Degree of
DOCTOR in Mathematics

The stability of some porous systems

Option

Applied Mathematics

By

FOUGHALI FOUZIA

Examination Committee Members

First Supervisor:	ZITOUNI Salah	Dr	Univ. of Souk-Ahras
Second Supervisor:	DJEBABLA Abdelhak	Pr.	Univ. of Annaba
President:	KERKER Mohamed Amine	Dr	Univ. of Annaba
Examiner:	BERKANE Ahmed	Pr.	Univ. of Annaba
Examiner:	BOUZETTOUTA Lamine	Dr	Univ. of Skikda
Examiner:	BERHAIL Amel	Dr	Univ. of Guelma

The stability of some porous systems

A Doctoral Thesis
By Fouzia Foughali

Advisors: Dr. Salah Zitouni and
Pr. Abdelhak Djebabla

Dedication

There is no way I can express how much I owe to my family for their love, generous spirit and support through the many years of my education.
I dedicate this research to my tender mother for her never ending- love.
I will be always grateful to my father for his incomparable love and moral support.
To my adorable sisters and brothers, each one by her/his name.
I will be always grateful to my husband for his confidence in me, patience and understanding who helped me to complete this project.
To my children, the reason for my happiness.

Acknowledgement

Firstly, I would thank Allah, the almighty for providing me with patience and will to study and finish my thesis; all great praise for him.
I would also like to express my deepest gratitude to my supervisor, Dr. Salah Zitouni at Souk Ahres University, for the encouragement, guidance assistance provided me during the period of my study. I am very grateful and indebted to him.
I would like to express my sincere gratitude to my research co-advisor Prof. Abdelhak Djebabla at Annaba University, for his helps and kindness. I'am honored to know him.
My thanks go also to Dr. Kerker Mohamed Amine a jury president of this thesis, for his willingness to evaluate my work.
I am very grateful to Pr. Berkane Ahmed, Annaba University, Dr Bouzettouta Lamine, Skikda University and Dr Berhail Amel, Ghelma University, for their acceptance to be members of my thesis committee.
Finally, I also want to use this occasion to thank my family, friends and colleagues for the immense assistance and kindness.

Abstract

In this thesis, we are interested in studying the well posedness and the stability of some linear one-dimensional porous-elastic systems. The first is a porous-thermoelastic system with second sound and a distributed delay term acting on the transverse displacement, where the heat flux of the system is governed by Cattaneo's law. The second is a porous-elastic system with microtemperatures and varying delay term, and the last is a swelling porous thermoelastic soils mixture with second sound, where the thermal conduction is given by the theory of Green and Naghdi called thermoelasticity type III.

Under suitable assumptions, we prove the well-posedness of the systems by using semigroups theory. For the stability of these systems, we use a multipliers technique which is based on the construction of a Lyapunov functional equivalent to energy.

Keywords: Porous system, swelling porous systems, Cattaneo's law, second sound, distributed delay, varying delay, semigroup theory, exponential stability, polynomial stability, Lyapunov functional.

ملخص

في هذه الأطروحة، سندرس وجود ووحدانية الحل واستقرار بعض الأنظمة الخطية المرنة أحادية البعد ذات مسامات. أول نظام هو نظام حراري مرن ذو مسامات مع الصوت الثاني و التأخير الموزع الذي يعمل على الإزاحة العرضية، حيث يخضع التدفق الحراري للنظام لقانون Cattaneo. ثاني نظام، هو عبارة عن نظام مسامي مرن مع تأثير حراري دقيق و تأخير متغير بالنسبة للزمن. و آخر نظام عبارة عن نظام مسامي حراري منتفخ مختلط بالصوت الثاني حيث تم إعطاء التوصيل الحراري من خلال نظرية Green و Naghdi و التي تسمى بالمرونة الحرارية من النوع الثالث.

في ظل شروط مناسبة، سنبرهن وجود ووحدانية الحل بالاعتماد على نظرية شبه الزمر، و استقرار هذه الأنظمة سيتم برهانه باستخدام تقنية المضاعف الذي يقوم على بناء دالة Lyapunov المكافئة للطاقة.

الكلمات المفتاحية : نظام مسامي، نظام مسامي منتفخ، نظام حراري، قانون Cattaneo، الصوت الثاني، تأخر موزع، تأخر متغير، نظرية شبه الزمر، استقرار أسي، استقرار متعدد الحدود، دالة Lyapunov .

Résumé

Dans cette thèse, nous nous intéressons à l'étude de l'existence, de l'unicité de la solution et de la stabilité de certains systèmes poreux-élastiques unidimensionnels linéaires. Le premier est un système poreux thermoélastique avec un second son et un terme de retard distribué agissant sur le déplacement transversal, où le flux thermique du système est régi par la loi de Cattaneo. Le second est un système poreux élastique avec micro-températures et un terme de retard variant, et le dernier est un mélange d'un système poreux thermoélastique gonflé avec deuxième son, où la conduction thermique est donnée par la théorie de Green et Naghdi appelée thermoélasticité de type III.

Sous des hypothèses appropriées, nous prouvons l'existence et l'unicité de la solution par la théorie des Semi-groupes. Pour la stabilité de ses systèmes, nous utilisons une technique des multiplicateurs qui se base sur la construction d'une fonctionnelle de Lyapunov équivalente à l'énergie.

Mots-clés: Système poreux, systèmes poreux gonflés, loi de Cattaneo, deuxième son, retard distribué, retard varié, théorie de semi-groupe, stabilité exponentielle, stabilité polynomiale, fonction de Lyapunov.

Contents

Chapter 1

Introduction

In recent years, elastic materials with voids, which have nice physical properties, are used widely in engineering, such as vehicles, aeroplanes, and large space structures. Due to their extensive applications, the elasticity problems of these kinds of materials have become hot issues, which have attracted the attention of many authors, and numerous stability results have been established (see [8],[19],[56],[57],[63]).

The classical thermoelasticity theory, based on Fourier's law of heat conduction, suffers from the deficiency of admitting thermal signals propagating with infinite speed. To overcome this deficiency many theories were developed, one of which would allow heat to propagate as wave with finite speed. Results concerning existence, nonexistence and stability in this regard have established by many mathematicians.

By the end of last century Green and Naghdi [47, 48] introduced three new types of thermoelastic theories in the aim of replacing the usual entropy production inequality with an entropy balance law. In each of these theories, the heat flux is given by a different constitutive assumption. As a result, three theories were obtained and respectively called thermoelasticity type I, type II and type III. When the theory of type I is linearized we obtain the classical system of thermoelasticity. The systems arising in thermoelasticity of type III are of dissipative nature whereas those of type II thermoelasticity do not sustain energy dissipation.

The theory of porous materials is an important generalization of the classical theory of elasticity for the treatment of porous solids in which the skeletal materials is thermoelastic and the interstices are void of material. This theory deals with materials containing small pores or voids.

An extension of this theory to linear thermoelastic bodies was proposed by Ieşan [27]. In addition, Ieşan [28], [29] added the microtemperature element to this theory.

On the basic of micromorphic continua theory, Grot [34] developed a theory of thermodynamics of elastic material with inner structure whose micro-elements, in addition to micro-deformations, possess micro-temperatures. The importance of materials with microstructure has been demonstrated by huge number of papers appeared in different fields of applications such as petroleum industry, material science, biology and many others.

The basic evolution equations for one-dimensional theories of porous materials with temperature and micro-temperature are given by

$$\begin{cases} \rho u_{tt} = T_x\ , & \rho\eta_t = q_x\ , \\ \\ J\varphi_{tt} = H_x + G\ , & \rho E_t = P_x + q - Q\ , \end{cases}$$

where T is the stress tensor, H is the equilibrated stress vector, G is the equilibrated body force, q is the heat flux, P is the first heat flux moment, Q is the mean heat flux, E is the first moment of energy, and η the entropy. The variables u and φ are, respectively, the displacement of the solid elastic material and the volume fraction. The constitutive equations are

$$\begin{cases} T = \mu u_x + b\varphi - \beta\theta, & q = k\theta_x + k_1 w, \\ \\ H = \delta\varphi_x - dw, & P = -k_2 w_x, \\ \\ G = -bu_x - \xi\varphi + m\theta - \tau\varphi_t, & Q = k_3 w + k_4\theta_x, \\ \\ \rho\eta = \beta u_x + c\theta + m\varphi, & \rho E = -\alpha w - d\varphi_x, \end{cases}$$

where ρ, J, μ, α, β, δ, ξ, b, d, m, τ, c, k, k_1, k_2, k_3 and k_4 are the constitutive coefficients whose physical meaning is well known, θ and w are the temperature and microtemperature, respectively.

Introducing the delay term makes the problem different from those considered in the literatures. Delay effect arises in many applications depending not only on the present state but also on some past occurrences. It may turn a well-behaved system into a wild one. The presence of delay may be a source of instability. For example, it was showed in [[3]-[6],[40],[51],[70]] that an arbitrarily small decay may destabilize a system, which is uniformly asymptotically stable in the absence of delay unless additional conditions or control terms have been used.

1.1 Delay differential equations

It is generally know that many systems in science and engineering can be described by models that include past effects. These systems, where the rate of change in a state is not only determined by the present states but also by the past states, are described by delay differential equations (DDEs). In other words, DDes are differential equations in which the derivatives of some unknown functions at present time depend on the values of the functions at previous times.

A simple delay differential equation for $x(t) \in \mathbb{R}^n$ takes the form

$$\frac{d}{dt}x(t) = f(t, x_t),$$

where $x_t = \{x(\tau) : \tau \leq t\}$ represents the trajectory of the solution in the past.

The functional operator f takes a time input and continuous function x_t and generates a real number $\frac{d}{dt}x(t)$ as its output.

Examples of such equation include:

(1) discrete/ constant delay $\frac{d}{dt}x(t) = f(t, x(t - \tau)),$

(2) time-varying delay $\frac{d}{dt}x(t) = f(t, x(t - \tau(t))),$

(3) distributed delay $\frac{d}{dt}x(t) = f\left(t, \int_0^\tau \mu(s)x(t - s)ds\right).$

(see Tijani [62]).

1.2 Stabilization of evolution problems

Problems of global existence and stability in time of Partial Differential Equations are subject, recently, of many works. In this thesis we are interested in the study of the global existence and the stabilization of some evolution equations. The purpose of the stabilization is to attenuate the vibrations by feedback, thus consists in guaranteeing the decrease of energy of the solutions to 0 in a more or less fast way by a mechanism of dissipation.

More precisely, the problem of stabilization consists in determining the asymptotic behavior of the energy by $E(t)$, to study its limits in order to determine if this limit is null or not and if this limit is null, to give an estimate of the decay rate of the energy to zero.

This problem has been studied by many authors for various systems. They are several type of stabilization,

(1) Strong stabilization:

$$E(t) \longrightarrow 0, \text{ as } t \longrightarrow \infty.$$

(2) Logarithmic stabilization:

$$E(t) \leq c(\log t)^{-\delta}, \forall t > 0, (c, \delta > 0).$$

(3) Polynomial stabilization:

$$E(t) \leq ct^{-\delta}, \forall t > 0, (c, \delta > 0).$$

(4) Uniform stabilization:

$$E(t) \leq ce^{-\delta t}, \forall t > 0, (c, \delta > 0).$$

The subject of this thesis is study the well-posedness of a linear one-dimensional porous-elastic system by using the theory of semi-groups to establish the existence and uniqueness of the solutions. For the stability results, we used the multiplier method based on the construction of a Lyapunov function.

1.3 Methodology

In this thesis, to ensure the well-posed of our problems, we use the theory of semi-groups to establish the existence and uniqueness of the solutions. In semigroups theory, the Hille-Yosida theorem is a powerful and fundamental tool relating the energy dissipation properties of an unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset H \longrightarrow H$ to the existence, uniqueness and regularity of the solutions of a stationary differential equation (Cauchy problem)

$$\begin{cases} \Phi'(t) = \mathcal{A}(t)\Phi(t), \; t > 0 \\ \\ \Phi(0) = \Phi_0. \end{cases}$$

For the stability results, we use the multiplier method based on the construction of a Lyapunov function $\pounds$ equivalent to the energy E of the solution. We denote by $\pounds \sim E$ the equivalence

$$c_1 E(t) \leq \pounds(t) \leq c_2 E(t), \; \forall t > 0, \tag{1.1}$$

for two positive constants $c_1$and c_2. To establish exponential stability, it suffices to show that

$$\pounds'(t) \leq -c\pounds(t), \; \forall t > 0, \tag{1.2}$$

for some $c > 0$. A simple integration of (1.2) over $[0, t]$ with (1.1) leads to the desired result of exponential stability.

It is worth noting that Lyapunov theorems are only sufficient conditions for the stability and the difficulty here is to find the adequate Lyapunov function.

1.4 The main results of this thesis

This thesis contains five chapters.

Chapter 3. In this chapter, we consider the thermoelastic system of porous type with a linear frictional damping and an internal distributed delay acting on the transverse displacement, where the heat flux is given by Cattaneo's law. The system is written as:

$$\begin{cases} \rho u_{tt} = \mu_1 u_{xx} + b\varphi_x - \mu_0 u_t - \int_{\tau_1}^{\tau_2} \mu(s)\, u_t(x, t-s)\, ds, & \text{in } (0,1)\times(0,+\infty), \\ J\varphi_{tt} = \alpha\varphi_{xx} - bu_x - \xi\varphi + \beta\theta_x, & \text{in } (0,1)\times(0,+\infty), \\ c\theta_t = -q_x + \beta\varphi_{tx} - \delta\theta, & \text{in } (0,1)\times(0,+\infty), \\ \tau_0 q_t + q + k\theta_x = 0, & \text{in } (0,1)\times(0,+\infty). \end{cases} \tag{1.3}$$

Under suitable assumptions on the weight of distributed delay, we first prove the well-posedness of the system by using the semigroup theory. Also, we establish the exponential stability of the solution by introducing a suitable Lyapunov functional. It was published in an international journal:

F. Foughali, S. Zitouni, H. E. Khouchemane, A. Djebabla; Well-posedness and exponential decay for a porous-thermoelastic system with second sound and distributed delay. Mathematics in Engineering, Science and Aerospace (MESA). Vol. 11, No. 4, 2020: 1003-1020.

Chapter 4. In this chapter, we are concerned with the one-dimensional porous-elastic system with microtemparatures and a time-varying delay, the system is written as

$$\begin{cases} \rho_1 u_{tt} = \mu u_{xx} + b\varphi_x - \gamma_1 u_t - \gamma_2 u_t(x, t-\tau(t)), & \text{in } (0,1)\times(0,+\infty), \\ J\varphi_{tt} = \delta\varphi_{xx} - bu_x - \xi\varphi - dw_x, & \text{in } (0,1)\times(0,+\infty), \\ \alpha w_t = \beta w_{xx} - d\varphi_{tx} - kw, & \text{in } (0,1)\times(0,+\infty). \end{cases} \tag{1.4}$$

The aim of this chapter is that under suitable assumptions on the weight of the damping and the weight of the delay term, we prove the well-posedness of the system by using the semigroup method. We then investigate the asymptotic behavior of the system through the perturbed energy method. Also, by using the multiplier method, we prove that the energy of system decays exponentially in the case of equal wave speeds and decays polynomially in the case of nonequal wave speeds. Under the case of nonequal wave speeds, we also investigate the lack of exponential stability of the system.

Chapter 5. This chapter is devoted to the study of swelling porous thermoelastic soils with second sound, where the heat conduction is given by Cattaneo's law, which has the form

$$
\begin{cases}
\rho u_{tt} = a_1 u_{xx} + a_2 \varphi_{xx} \ , & \text{in } (0,1)\times(0,+\infty), \\
J\varphi_{tt} = a_3 \varphi_{xx} + a_2 u_{xx} + \beta\theta_x \ , & \text{in } (0,1)\times(0,+\infty), \\
\alpha\theta_t = -q_x + \beta\varphi_{tx} - \gamma\theta, & \text{in } (0,1)\times(0,+\infty), \\
\tau q_t = -q - k\theta_x \ , & \text{in } (0,1)\times(0,+\infty).
\end{cases}
\tag{1.5}
$$

The aim of this chapter is that , we study the existence and the uniqueness of the solution using the semigroup theory. Also, we show that the energy associated with the system is dissipative and we establish the exponential stability of the solution by introducing a suitable Lyapunov functional.

Chapter 2

Preliminary

In this preliminary we shall introduce and state some necessary notations needed in the proof of our results, and some the basic results which concerning the well-posed of our problems, the semi-groupe theory and Layponov functionals and other theorems. The knowledge of all these notations and results are important for our study, see, e.g., ([1]), ([54]), ([13]) and ([66]).

2.1 Some functional analysis concepts

Let Ω be an open subset of $\mathbb{R}^n$, $n \in \mathbb{N}$ supplied with the Lebesgue measure dx.

2.1.1 Hilbert space

Definition 2.1 A Hilbert space H is a vectorial space supplied with inner product $\langle u, v\rangle$, such that $\|u\| = \langle u, u\rangle^{\frac{1}{2}}$ is the norm which let H complete.

2.1.2 $L^P(\Omega)$ space

Definition 2.2 Let $1 \leq p < \infty$, and let Ω be an open domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Define the standard lebesgue space $L^P(\Omega)$, by

$$L^P(\Omega) = \left\{ u : \Omega \to \mathbb{R} : \ f \text{ is measurable and } \int_\Omega |u|^p \, dx < \infty \right\}.$$

The functional $\|.\|_{L^P}$ defined by

$$\|u\|_{L^P} = \left[\int_\Omega |u|^p \, dx \right]^{\frac{1}{p}}$$

is a norm on $L^P(\Omega)$.

Definition 2.3 For $p = \infty$, we have

$$L^{\infty}(\Omega) = \left\{ \begin{array}{c} u : \Omega \to \mathbb{R} : \; u \text{ is measurable and there exists a constant } C \text{ such that} \\ |u| \leq C \text{ a.e in } \Omega. \end{array} \right\}.$$

We denote

$$\|u\|_{\infty} = \inf \{C, \;\; |u| \leq C \text{ a.e in } \Omega\}.$$

Remark 2.1 For $p = 2$, $L^2(\Omega)$ equipped with the scalar product

$$\langle u, v \rangle = \int_{\Omega} u(x)v(x)dx$$

is a Hilbert space. Then

$$\|u\|^2_{L^2(\Omega)} = \langle u, u \rangle .$$

2.1.3 Sobolev space $W^{m,\,p}(\Omega)$

Definition 2.4 (Sobolev Space) For any positive integer m and $1 \leq p \leq \infty$, the $W^{m,\,p}(\Omega)$ is the space defined by

$$W^{m,\,p}(\Omega) \equiv \{u \in L^p(\Omega) : D^{\alpha}u \in L^p(\Omega) \text{ for } \; 0 \leq |\alpha| \leq m\},$$

where $D^{\alpha}u$ is the weak (or distributional) partial derivative, and

$$W_0^{m,\,p}(\Omega) \equiv \text{ the closure of } C_0^{\infty}(\Omega) \text{ in the space } W^{m,\,p}(\Omega).$$

Clearly $W^{0,\,p}(\Omega) = L^p(\Omega)$, and if $1 \leq p < \infty$, $W_0^{0,\,p}(\Omega) = L^p(\Omega)$ because $C_0^{\infty}(\Omega)$ is dense in $L^p(\Omega)$.

Definition 2.5 (The Sobolev Norms) We define a norm $\|.\|_{W^{m,\,p}(\Omega)}$, where m is a positive integer and $1 \leq p \leq \infty$, as follows:

$$\begin{aligned} \|u\|_{W^{m,\,p}(\Omega)} &= \left(\sum_{0 \leq |\alpha| \leq m} \|D^{\alpha}u\|^p_{L^p(\Omega)} \right)^{1/p} \quad \text{if } \; 1 \leq p < \infty \, , \\ \|u\|_{W^{m,\,\infty}(\Omega)} &= \max_{0 \leq |\alpha| \leq m} \|D^{\alpha}u\|_{\infty} . \end{aligned}$$

Definition 2.6 For $p = 2$, we denote

$$H^m(\Omega) = W^{m,2}(\Omega), \quad \text{and} \quad H_0^m(\Omega) = W_0^{m,2}(\Omega).$$

Theorem 2.1 *Let $u \in W^{1,\,p}(I)$, then $u \in W_0^{1,\,p}(\Omega)$ if and only if $u = 0$ on $\partial\Omega$.*

2.2 Existence and uniqueness theorem

The existence and uniqueness of a solution to weak formulation of the problem can be proved by using the Lax-Milgram's Lemma. This states that the weak formulation admits a unique solution.

Lemma 2.1 ***(Lax-Milgram's Lemma)*** *Let $a(.,.)$ be a bilinear form on a Hilbert space H equipped with norm $\|.\|_H$ and the following properties:*

1) $a(.,.)$ is continuous, that is

$$\exists\gamma_1 > 0 \text{ such that } |a(w,v)| \le \gamma_1 \|w\|_H \|v\|_H \ , \forall w,\ v \in H,$$

2) $a(.,.)$ coercive (or H-elliptic), that is

$$\exists\alpha > 0 \text{ such that } |a(v,v)| \ge \alpha \|v\|_H^2 \ , \forall v \in H,$$

3) L is a linear mapping on H (thus L is continuous), that is

$$\exists\gamma_2 > 0 \text{ such that } |L(w)| \le \gamma_2 \|w\|_H \ , \forall w \in H.$$

Then there exists a unique $u \in H$ such that

$$a(w,u) = L(w),\ \forall w \in H.$$

Definition 2.7 An unbounded linear operator $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ is said to be monotone if it satisfies

$$(\mathcal{A}u, u) \ge 0,\ \forall u \in D(\mathcal{A}).$$

It is called maximal monotone if, in addition

$$R(I + \mathcal{A}) = H \ \text{ i.e.}$$

$$\forall f \in H, \exists u \in D(\mathcal{A}) \text{ such that } u + \mathcal{A}u = f,$$

where $R(I + \mathcal{A})$ is the range of $(I + \mathcal{A})$.

Proposition 2.1 *Let $\mathcal{A}$ be a maximal monotone operator. Then $D(\mathcal{A})$ is dense in H.*

Theorem 2.2 ***(Hille-Yosida)*** *Let $\mathcal{A}$ be a maximal monotone operator. Then, given any $u_0 \in D(\mathcal{A})$ there exists a unique function*

$$u \in C([0,\infty), D(\mathcal{A}) \cap C^1([0,\infty), H)$$

satisfying

$$\begin{cases} \dfrac{du}{dt} + \mathcal{A}u = 0 \\ u(0) = u_0. \end{cases}$$

Moreover,

$$|u(t)| \le |u_0| \quad \text{ and } \quad \left|\frac{du}{dt}(t)\right| = |\mathcal{A}u(t)| \le |\mathcal{A}u_0| \ , \forall t \ge 0.$$

2.3 Semigroups of bounded linear operators

In this chapter we will present some definitions, some results on C_0-semigroups, including some theorems on exponential stability.

2.3.1 Some definitions

Definition 2.8 Let H be a real or complex Hilbert space equipped with the inner product $(,)$ and the induced norm $\|.\|$. Let $\mathcal{A}$ be a densely defined linear operator on H, i.e., $\mathcal{A} : D(\mathcal{A}) \subseteq H \to H$. We say that is dissipative if for any $x \in D(\mathcal{A})$,

$$\mathrm{Re}\,(\mathcal{A}x, x) \leq 0.$$

Definition 2.9 A family $S(t)$ $(0 \leq t > \infty)$ of bounded linear operators in a Hilbert space H is called a strongly continuous semigroup (in short, a C_0-semigroups) if
(i) $S\,(0) = Id_x$,
(ii) $S\,(t_1 + t_2) = S\,(t_1)\,S\,(t_2)\,, \forall t_1, t_2 \geq 0$,
(iii) For each $x \in H$, $S\,(t)\,x$ is continuous in t on $[0, \infty)\,.$

For such a semigroup $S\,(t)$, we define an operator $\mathcal{A}$ with domain $D(\mathcal{A})$ consisting of points x such that the limit

$$\mathcal{A}x = \lim_{h \longrightarrow 0} \frac{S\,(h)\,x - x}{h}, \ \ x \in D(\mathcal{A})$$

exists. Then $\mathcal{A}$ is called the infinitesimal generator of the semigroup $S\,(t)\,.$ Given an operator $\mathcal{A}$, if $\mathcal{A}$ coincides with the infinitesimal generator of $S(t)$, then we say that it generates a strongly continuous semigroup $S(t)$, $t \geq 0$. Sometimes we also denote $S(t)$ by $e^{\mathcal{A}t}$.

Definition 2.10 $\left\{e^{\mathcal{A}t}\right\}_{t\geq 0}$ is said to be exponentially stable if there exists positive constants α and $M \geq 0$ such that

$$\left\|e^{\mathcal{A}t}\right\| \leq Me^{-\alpha t}\ , \forall t \geq 0.$$

If $\alpha = 0$, the semigroup $(S(t))_{t\geq 0}$ is called uniformly bounded and if moreover $M = 1$, then it is called a C_0-semigroup of contractions.

2.3.2 C_0-semigroup generated by dissipative operator

Suppose that the linear operator $\mathcal{A}$ generates a C_0-semigroup $e^{\mathcal{A}t}$ on a Hilbert space H. Then we have (see Pazy [54]):

Theorem 2.3 ***(Hille-Yosida)*** *A linear (unbounded) operator $\mathcal{A}$ is the infinitesimal generator of a C_0- semigroup of contraction $S(t)$, $t \geq 0$, if and only if*
(i) $\mathcal{A}$ is closed and $\overline{D(\mathcal{A})} = H$,
(ii) the resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$ contains $\mathbb{R}^+$ and for every $\lambda > 0$,

$$\left\|(\lambda I - \mathcal{A})^{-1}\right\| \leq \frac{1}{\lambda}.$$

Theorem 2.4 ***(Lumer-Phillips)*** *Let $\mathcal{A}$ be a linear operator with dense domain $D(\mathcal{A})$ in a Hilbert space H. If $\mathcal{A}$ is dissipative and there is $\lambda_0 > 0$ such that the range, $R(\lambda_0 I - \mathcal{A})$, of $\lambda_0 I - \mathcal{A}$ is H, then $\mathcal{A}$ is the infinitesimal generator of a C_0- semigroup of contractions on H.*

As a collorary of the above theorem, the following result will be frequently used in this thesis:

Theorem 2.5 *Let $\mathcal{A}$ be a linear operator with dense domain $D(\mathcal{A})$ in a Hilbert space H. If $\mathcal{A}$ is dissipative and $0 \in \rho(\mathcal{A})$, the resolvent set of $\mathcal{A}$, then $\mathcal{A}$ is the infinitesimal generator of a C_0- semigroup of contractions on H.*

2.3.3 Exponential stability

By collect some result in the literature concerning the necessary and sufficient conditions for a C_0- semigroup being exponentially stable. The result was obtained by Gearhart and Huang [25], independently (see also Prüss [55]).

Theorem 2.6 *Let $S(t) = e^{\mathcal{A}t}$ be a C_0- semigroup of contractions on Hilbert space. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\lambda,\ \lambda \in \mathbb{R}\} \equiv i\mathbb{R}$$

and

$$\overline{\lim_{|\lambda|\to\infty}} \left\|(i\lambda I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(H)} < \infty$$

hold.

We use the above theorem to prove the lack of exponential stability.

2.4 Some useful inequalities

Our study based on some important inequalities, These inequalities is very useful in applied mathematics.

Theorem 2.7 ***(Hölder's Inequality)*** *Let* $1 \leq p,\ q \leq \infty$ *such that* $\dfrac{1}{p} + \dfrac{1}{q} = 1$, *assume that* $f \in L^p(\Omega)$ *and* $g \in L^q(\Omega)$ *then,* $fg \in L^1(\Omega)$ *and*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q .$$

If $p = q = 2$ then we obtain the **Cauchy-Schwarz** inequality:

$$\int_\Omega |fg|\, dx \leq \left(\int_\Omega |f|^2\, dx\right)^{\frac{1}{2}} \left(\int_\Omega |g|^2\, dx\right)^{\frac{1}{2}}.$$

Lemma 2.2 ***(Poincaré's inequality)*** *Suppose* I *is a bounded interval. Then there exists a constant* C *(depending on* $|I| > \infty$*) such that*

$$\|u\|_{w^{1,p}(I)} \leq C \left\|u^{'}\right\|_{L^p(I)} \text{ , for all } u \in w_0^{1,p}(I).$$

Lemma 2.3 ***(Inequality of Poincaré-Friedrich's type)*** *Let* u *is a function satisfies the following conditions:* $u \in C^1(\overline{\Omega})$ *where* Ω *is a domain in* $\mathbb{R}^n$ *and* $u\backslash_{\partial\Omega} = 0$, *then*

$$\int_\Omega |u|^2\, dx \leq c \int_\Omega |\nabla u|^2\, dx,$$

where c *is a constant depends only on the domain is* Ω.

Lemma 2.4 ***(Young's inequality)*** *For all* $a,\ b \in \mathbb{R}^+$ *, we have*

$$ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon},$$

where $\varepsilon > 0$.

Chapter 3

Well-posedness and exponential decay for a porous-thermoelastic system with second sound and a distributed delay term

3.1 Introduction

In this chapter we are concerned with the thermoelastic system of porous type with a linear frictional damping and an internal distributed delay acting on the transverse displacement, where the heat flux is given by Cattaneo's law. The system is written as:

$$\begin{cases} \rho u_{tt} = \mu_1 u_{xx} + b\varphi_x - \mu_0 u_t - \int_{\tau_1}^{\tau_2} \mu(s)\, u_t(x, t-s)\, ds, & \text{in } (0,1)\times(0,+\infty), \\ J\varphi_{tt} = \alpha\varphi_{xx} - bu_x - \xi\varphi + \beta\theta_x, & \text{in } (0,1)\times(0,+\infty), \\ c\theta_t = -q_x + \beta\varphi_{tx} - \delta\theta, & \text{in } (0,1)\times(0,+\infty), \\ \tau_0 q_t + q + k\theta_x = 0, & \text{in } (0,1)\times(0,+\infty), \end{cases} \tag{3.1}$$

with the following initial and boundary conditions

$$\begin{cases} u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in (0,1), \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x), & x \in (0,1), \\ \theta(x,0) = \theta_0(x),\ \ q(x,0) = q_0(x), & x \in (0,1), \\ u(0,t) = \varphi_x(0,t) = \theta(0,t) = 0, & t \in (0,+\infty), \\ u(1,t) = \varphi_x(1,t) = \theta(1,t) = q(1,t) = 0, & t \in (0,+\infty), \\ u_t(x,-t) = f_0(x,-t), & (x,t) \in (0,1)\times(0,\tau_2), \end{cases} \tag{3.2}$$

where u is the transversal displacement, φ is the volume fraction difference, θ is the temperature difference, q is the heat flux and the coefficients. The parameter ρ is the mass density and J equals to the product of the equilibrated inertia by the mass density. The coefficients b, μ_0, μ_1, α, β, ξ, τ_0, k are positive constant coefficients. The parameters with b, μ_1, ξ satisfying $\mu_1\xi > b^2$ and τ_1, τ_2 are two real numbers where $0 \leq \tau_1 < \tau_2$, and $\mu : [\tau_1, \tau_2] \longrightarrow \mathbb{R}$ is a bounded function verify the following assumption

$$\mu_0 \geq \int_{\tau_1}^{\tau_2} |\mu(s)|\, ds. \tag{3.3}$$

The initial data $(u_0, u_1, \varphi_0, \varphi_1, \theta_0, q_0, f_0)$ are assumed to belong to a suitable functional space.

We see that it is better to start our literature review with the pioneer work of Goodman and Cowin [17], where they introduced the concept of a continuum theory of granular materials with interstitial voids into the theory of elastic solids with voids. The importance of such materials often arise in many practical problems, for instance, in petroleum industry, soil mechanics, engineering, power technology, biology, material science. We refer the reader to Cowin and Nunziato [18, 19] and the references therein for more details. The system (3.1)-(3.2) arises in the theory of linear elastic materials, which governs the mechanical deformations in elastic structures, where the heat flux is given by Cattaneo's law. Many results in this contests can be obtained, and numerous stability have been established [21, 41, **?**]. For the porous thermoelectricity systems coupled with the heat equation by Cattaneo's law, Messaoudi and Fareh [44] considered the following system

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\phi_x - \gamma u_t, & \text{in } (0,1)\times(0,+\infty), \\ J\phi_{tt} = \alpha\phi_{xx} - bu_x - \xi\phi + \beta\theta_x, & \text{in } (0,1)\times(0,+\infty), \\ c\theta_t = -q_x + \beta\phi_{tx} - \delta\theta, & \text{in } (0,1)\times(0,+\infty), \\ \tau_0 q_t + q + k\theta_x = 0\ , & \text{in } (0,1)\times(0,+\infty), \end{cases} \tag{3.4}$$

they established an exponential stability result by using the spectral theory.

On the other hand, the systems with delay term have attracted extensive attention due to the evolution tendency depends not only on the current state but also on a certain or some past occurrence (see [5]-[68]). An arbitrarily small delay may be the source of instability, see [26, 53, 58]. In [40] Wenjun Liu and Miaomiao Chen, considered the following porous thermoelastic system with second sound and time-varying delay term

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\phi_x - \gamma_1 u_t - \gamma_2 u_t(x, t-\tau(t)), & (x,t)\in(0,1)\times(0,\infty), \\ J\phi_{tt} = \alpha\phi_{xx} - bu_x - \xi\phi + \beta\theta_x, & (x,t)\in(0,1)\times(0,\infty), \\ c\theta_t = -q_x + \beta\phi_{tx} - \delta\theta, & (x,t)\in(0,1)\times(0,\infty), \\ \tau_0 q_t + q + k\theta_x = 0, & (x,t)\in(0,1)\times(0,\infty). \end{cases} \tag{3.5}$$

The authors established the global existence and uniqueness of the system (3.5) by using the semigroup theory and variable norm technique of Kato and proved that the system is exponentially stable under a certain condition on the weight of the delay term.

Introducing a distributed delay term makes our problem different from those considered so far in the literatures, importance of this term appears in many works and this is due to the fact on it's influence on the asymptotic behavior of the solution for the different types of PDEs problems for this we refer the readers to [4]-[67].

Recently, Khochemane and Bouzettouta [38] considered a one-dimensional porous-elastic system with distributed delay

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, & \text{in } (0,1)\times(0,\infty), \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \mu_1\phi_t + \int_{\tau_1}^{\tau_2} \mu_2(s)\,\phi_t(t-s)\,ds = 0, & \text{in } (0,1)\times(0,\infty), \end{cases} \tag{3.6}$$

and studied the well-posedness of the system by using the semigroup theory and they showed that the dissipation given by this complementary control stabilizes exponentially the system for the case of equal speeds of wave propagation.

Motivate and inspired by above works we consider the porous-thermoelastic system (3.1)-(3.2), and prove the existence and uniqueness of the solution. By construct some Lyapunov functionals, we obtain the exponential decay result under the assumption (3.3). Our work extends the stability results in [44, 40, 38] to porous systems with second sound and distributed delay acting on the displacement equation.

The rest of this chapter is organized as follows. In Section 2, we prove the well-posedness result of the system by using the semigroup theory. In Section 3, we establish an exponential stability result of the energy.

3.2 Preliminaries

We introduce as in [50] the new variable

$$z(x,\rho,t,s) = u_t\,(x, t-\rho s)\,, \quad x\in(0,1)\,,\ \rho\in(0,1)\,,\ s\in(\tau_1,\tau_2)\,,\ t>0.$$

Then, we have

$$sz_t(x,\rho,t,s) + z_\rho(x,\rho,t,s) = 0, \quad x\in(0,1)\,,\ \rho\in(0,1)\,,\ s\in(\tau_1,\tau_2)\,,\ t>0. \tag{3.7}$$

Therefore, problem (3.1) takes the form

$$\begin{cases} \rho u_{tt} = \mu_1 u_{xx} + b\varphi_x - \mu_0 u_t - \int_{\tau_1}^{\tau_2} \mu(s)\,z(x,1,t,s)\,ds, & \text{in } (0,1)\times(0,+\infty), \\ J\varphi_{tt} = \alpha\varphi_{xx} - bu_x - \xi\varphi + \beta\theta_x, & \text{in } (0,1)\times(0,+\infty), \\ c\theta_t = -q_x + \beta\varphi_{tx} - \delta\theta, & \text{in } (0,1)\times(0,+\infty), \\ \tau_0 q_t + q + k\theta_x = 0, & \text{in } (0,1)\times(0,+\infty), \end{cases} \tag{3.8}$$

with the following initial and boundary conditions

$$\begin{cases} u(x,0)=u_0(x)\,,\ u_t(x,0)=u_1(x)\,, & x\in(0,1),\\ \varphi(x,0)=\varphi_0(x)\,,\ \varphi_t(x,0)=\varphi_1(x)\,, & x\in(0,1),\\ \theta(x,0)=\theta_0(x)\,,\ q(x,0)=q_0(x)\,, & x\in(0,1),\\ u(0,t)=\varphi_x(0,t)=\theta(0,t)=0,\ , & t\in(0,\infty),\\ u(1,t)=\varphi_x(1,t)=\theta(1,t)=q(1,t)=0, & t\in(0,\infty),\\ z(x,0,t,s)=u_t(x,t)\,, & (x,t,s)\in(0,1)\times(0,\infty)\times(\tau_1,\tau_2)\,,\\ z(x,\rho,0,s)=f_0(x,\rho,s)\,, & (x,\rho,s)\in(0,1)\times(0,1)\times(\tau_1,\tau_2)\,. \end{cases} \tag{3.9}$$

By using $(3.8)_2$, $(3.8)_4$ and the boundary conditions, we conclude that

$$\frac{d^2}{dt^2}\int_0^1\varphi(x,t)\,dx+\frac{\xi}{J}\int_0^1\varphi(x,t)\,dx=0 \text{ and } \frac{d}{dt}\int_0^1 q(x,t)dx+\frac{1}{\tau_0}\int_0^1 q(x,t)dx=0. \tag{3.10}$$

So, By solving (3.10) and using the initial data of φ and q, we obtain

$$\int_0^1\varphi(x,t)\,dx=\int_0^1\varphi_0(x,t)\,dx\cos\sqrt{\frac{\xi}{J}}t+\sqrt{\frac{J}{\xi}}\left(\int_0^1\varphi_1(x)\,dx\right)\sin\sqrt{\frac{\xi}{J}}t$$

and

$$\int_0^1 q(x,t)dx=\left(\int_0^1 q_0(x,t)dx\right)\exp(-\frac{1}{\tau_0}t).$$

Consequently, if we let

$$\begin{aligned}\bar\varphi(x,t) &= \varphi(x,t)-\left(\int_0^1\varphi_0(x)\,dx\right)\cos\sqrt{\frac{\xi}{J}}t-\sqrt{\frac{J}{\xi}}\left(\int_0^1\varphi_1(x)\,dx\right)\sin\sqrt{\frac{\xi}{J}}t,\\ \bar q(x,t) &= q(x,t)-\left(\int_0^1 q_0(x,t)dx\right)\exp(-\frac{1}{\tau_0}t).\end{aligned}$$

Then it follows that

$$\int_0^1\bar\varphi(x,t)\,dx=0 \quad\text{and}\quad \int_0^1\bar q(x,t)dx=0\ , \quad \forall t\geq 0.$$

Therefore, the use of Poincare's inequality is applicable for $\bar\varphi$ and $\bar q$ is justified. in addition, simple substitution shows that $(u,\bar\varphi,\theta,\bar q,z)$ satisfies system(3.8) with initial data for $\bar\varphi$ and $\bar q$ but write φ and q given as

$$\begin{aligned}\bar\varphi_0(x,t) &= \varphi_0(x,t)-\int_0^1\varphi_0(x)\,dx\ \ ,\quad \bar\varphi_1(x,t)=\varphi_1(x,t)-\int_0^1\varphi_1(x)\,dx,\\ \bar q_0(x,t) &= q_0(x,t)-\int_0^1 q_0(x)\,dx,\end{aligned}$$

instead of φ_0, φ_1, for φ and q_0 for q, respectively. Henceforth, we work with $\bar\varphi$ and $\bar q$ instead of φ and q but write φ and q for simplicity of notation.

Throughout this chapter, c_p is used to denote the Poincaré-type constant.

3.2. Preliminaries

3.3 Well-posedness of the problem

In this section, we give a brief idea about the existence and uniqueness of solutions for (3.1)-(3.2) using the semigroup theory [54].

We set $v = u_t$, $\phi = \varphi_t$ and let

$$U = (u, u_t, \varphi, \varphi_t, q, \theta, z)^T ,$$

then

$$\partial_t U = (u_t, v_t, \varphi_t, \phi_t, q_t, \theta_t, z_t)^T .$$

Therefore, problem (3.8)-(3.9) can be rewritten as

$$\begin{cases} \partial_t U = \mathcal{A}U, \\ U(0) = U_0 = (u_0, u_1, \varphi_0, \varphi_1, q_0, \theta_0, f_0)^T, \end{cases} \tag{3.11}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ u_t \\ \varphi \\ \varphi_t \\ q \\ \theta \\ z \end{pmatrix} = \begin{pmatrix} u_t \\ \frac{\mu_1}{\rho}u_{xx} + \frac{b}{\rho}\varphi_x - \frac{\mu_0}{\rho}u_t - \frac{1}{\rho}\int_{\tau_1}^{\tau_2}\mu(s)\, z(x,1,t,s)\, ds \\ \varphi_t \\ \frac{\alpha}{J}\varphi_{xx} - \frac{b}{J}u_x - \frac{\xi}{J}\varphi + \frac{\beta}{J}\theta_x \\ -\frac{1}{\tau_0}q - \frac{k}{\tau_0}\theta_x \\ -\frac{1}{c}q_x + \frac{\beta}{c}\varphi_{tx} - \frac{\delta}{c}\theta \\ -s^{-1}z_\rho \end{pmatrix}. \tag{3.12}$$

We define the energy space as

$$\begin{aligned} \mathcal{H} \ : \ &= H_0^1(0,1) \times L^2(0,1) \times H_*^1(0,1) \times L^2(0,1) \times L^2(0,1) \\ &\times L^2(0,1) \times L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)), \end{aligned}$$

where

$$H_*^1(0,1) := \left\{\phi \in H^1(0,1) : \phi_x(0) = \phi_x(1) = 0\right\},$$

be the Hilbert space equipped with the inner product

$$\begin{aligned}\left\langle U,\widetilde{U}\right\rangle_{\mathcal{H}} &= \rho\int_0^1 u_t\widetilde{u}_t dx + J\int_0^1 \varphi_t\widetilde{\varphi}_t dx + c\int_0^1 \theta\widetilde{\theta} dx + \mu_1\int_0^1 u_x\widetilde{u}_x dx \\ &+\xi\int_0^1 \varphi\widetilde{\varphi} dx + \alpha\int_0^1 \varphi_x\widetilde{\varphi}_x dx + \frac{\tau_0}{k}\int_0^1 q\widetilde{q} dx + b\int_0^1 (u_x\widetilde{\varphi} + \widetilde{u}_x\varphi) dx \\ &+\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu(s)\right|\int_0^1 z(x,\rho,s)\,\widetilde{z}(x,\rho,s)\,dpdsdx.\end{aligned}$$

The domain of $\mathcal{A}$ is

$$D(\mathcal{A}) = \left\{\begin{array}{l} U\in \left(H^2(0,1)\cap H_0^1(0,1)\right)\times H_0^1(0,1)\times\left(H^2(0,1)\cap H_*^1(0,1)\right) \\ \\ \times H_*^1(0,1)\times H^1(0,1)\times H_0^1(0,1)\times L^2\left((0,1)\times(0,1)\times(\tau_1,\tau_2)\right), \\ \\ u_t(x,t) = z(x,0,t,s) \text{ in } (0,1) \end{array}\right\}.$$

Clearly, $D(\mathcal{A})$ is dense in $\mathcal{H}$.
Using semigroup arguments, we can obtain a the following well-posedness result.

Theorem 3.1 *Suppose that $\int_{\tau_1}^{\tau_2}|\mu(s)|\,ds \le \mu_0$. For all $U_0\in\mathcal{H}$, problem (3.11) possesses then a unique solution $U\in C(\mathbb{R}^+;\mathcal{H})$.*

Moreover, if $U_0\in D(\mathcal{A})$, the solution satisfies

$$U\in C\left(\mathbb{R}^+; D(\mathcal{A})\cap C^1\left(\mathbb{R}^+;\mathcal{H}\right)\right).$$

Proof. We use the semigroup approach. So, we prove that $\mathcal{A}$ is a maximal monotone operator. First, we prove that the operator $\mathcal{A}$ is dissipative.

For any $U=(u,u_t,\varphi,\varphi_t,q,\theta,z)^T\in D(\mathcal{A})$, by using the inner product and integrating by parts

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = \left\langle \begin{pmatrix} u_t \\ \frac{\mu_1}{\rho}u_{xx} + \frac{b}{\rho}\varphi_x - \frac{\mu_0}{\rho}u_t - \frac{1}{\rho}\int_{\tau_1}^{\tau_2}\mu(s)\, z(x,1,t,s)\, ds \\ \varphi_t \\ \frac{\alpha}{J}\varphi_{xx} - \frac{b}{J}u_x - \frac{\xi}{J}\varphi + \frac{\beta}{J}\theta_x \\ -\frac{1}{\tau_0}q - \frac{k}{\tau_0}\theta_x \\ -\frac{1}{c}q_x + \frac{\beta}{c}\varphi_{tx} - \frac{\delta}{c}\theta \\ -s^{-1}z_\rho \end{pmatrix}, \begin{pmatrix} u \\ u_t \\ \varphi \\ \varphi_t \\ q \\ \theta \\ z \end{pmatrix} \right\rangle.$$

Then

$$\begin{aligned} \langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= -\mu_0 \int_0^1 u_t^2 dx - \delta \int_0^1 \theta^2 dx - \frac{1}{k}\int_0^1 q^2 dx - \int_0^1 \int_{\tau_1}^{\tau_2} \mu(s)\, z(x,1,t,s)\, u_t ds dx \\ &\quad - \int_0^1 \int_{\tau_1}^{\tau_2} |\mu(s)| \int_0^1 z_\rho(x,\rho,t,s)\, z(x,\rho,t,s)\, d\rho ds dx. \end{aligned}$$

Integrating by parts in ρ, we have

$$\begin{aligned} &\int_0^1 \int_{\tau_1}^{\tau_2} |\mu(s)| \int_0^1 z_\rho(x,\rho,t,s)\, z(x,\rho,t,s)\, d\rho ds dx \\ &= \frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} |\mu(s)| \left[z^2(x,1,t,s) - z^2(x,0,t,s)\right] ds dx. \end{aligned}$$

We can imply that

$$\begin{aligned} \langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= -\mu_0 \int_0^1 u_t^2 dx - \delta \int_0^1 \theta^2 dx - \frac{1}{k}\int_0^1 q^2 dx - \int_0^1 \int_{\tau_1}^{\tau_2} \mu(s)\, z(x,1,t,s)\, u_t ds dx \\ &\quad - \frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,t,s)\, ds dx + \frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} |\mu(s)|\, u_t^2 ds dx. \end{aligned}$$

Now, using Young's inequality, we can estimate

$$\begin{aligned} -\int_0^1 \int_{\tau_1}^{\tau_2} \mu(s)\, z(x,1,t,s)\, u_t ds dx &\leq \frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu(s)|\, ds \int_0^1 u_t^2 dx \\ &\quad + \frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,t,s)\, ds dx. \end{aligned}$$

Therefore, from the assumption (3.3) we have

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \leq -\left(\mu_0 - \int_{\tau_1}^{\tau_2} |\mu(s)|\, ds\right) \int_0^1 u_t^2 dx - \delta \int_0^1 \theta^2 dx - \frac{1}{k}\int_0^1 q^2 dx \leq 0.$$

Consequently, $\mathcal{A}$ is a dissipative operator.

Next, we prove the operator $\mathcal{A}$ is maximal. It is sufficient to show that the operator $(I_d - \mathcal{A})$ is surjective. Indeed, given $G = (g_1, g_2, g_3, g_4, g_5, g_6, g_7)^T \in \mathcal{H}$, we prove that there exists a unique $U = (u, u_t, \varphi, \varphi_t, q, \theta, z)^T \in D(\mathcal{A})$ such that

$$(I_d - \mathcal{A})\, U = G. \tag{3.13}$$

That is

$$\begin{cases} u - v = g_1, \\ \int_{\tau_1}^{\tau_2} \mu(s)\, u_t(x, t-s)\, ds - \mu_1 u_{xx} - b\varphi_x + (\rho + \mu_0)v = \rho g_2, \\ \varphi - \phi = g_3, \\ \phi - \alpha\varphi_{xx} + bu_x + \xi\varphi - \beta\theta_x = Jg_4, \\ (1+\tau_0)q + k\theta_x = \tau_0 g_5, \\ q_x - \beta\varphi_{tx} + (1+\delta)\theta = cg_6, \\ sz + z_\rho = sg_7. \end{cases} \tag{3.14}$$

From (3.14)$_1$,(3.14)$_3$ and (3.14)$_5$ we have

$$\begin{cases} v = u - g_1, \\ \phi = \varphi - g_3, \\ \theta_x = -\dfrac{(\tau_0+1)}{k} q + \dfrac{\tau_0}{k} g_5, \\ \theta = -\dfrac{(\tau_0+1)}{k} \int_0^x q(y)dy + \dfrac{\tau_0}{k}\int_0^x g_5(y)dy. \end{cases} \tag{3.15}$$

Inserting (3.15) into $(3.14)_2$, $(3.14)_4$ and $(3.14)_6$, we get

$$\begin{cases} -\mu_1 u_{xx} - b\varphi_x + \mu_2 u = h_1 \in L^2(0,1), \\ \\ -\alpha\varphi_{xx} + bu_x + (1+\xi)\varphi + \beta\dfrac{(\tau_0+1)}{k} q = h_2 \in L^2(0,1), \\ \\ q_x - \beta\varphi_{tx} - (1+\delta)\dfrac{(\tau_0+1)}{k}\int_0^x q(y)dy = h_3 \in L^2(0,1), \\ \\ z + s^{-1}z_\rho = g_7 \in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)), \end{cases} \tag{3.16}$$

where

$$\begin{aligned} \mu_2 &= (\rho+\mu_0) + \int_{\tau_1}^{\tau_2} \mu(s)\, e^s ds, \\ h_1 &= \rho g_2 - \mu_2 g_1 - \int_{\tau_1}^{\tau_2} s e^s \mu(s) \int_0^1 g_7(x,\tau,s)\, e^{-s\tau} d\tau ds, \\ h_2 &= g_3 - Jg_4 + \beta\frac{\tau_0}{k} g_5, \\ h_3 &= -(1+\delta)\frac{\tau_0}{k}\int_0^x g_5(y)dy + cg_6, \end{aligned}$$

and, by (3.14) we can find as

$$z(x,0,t,s) = u_t(x,t) \;\; = v(x,t) \text{ for } x \in (0,1), t \in (0,1), s \in (\tau_1,\tau_2), \tag{3.17}$$

and from (3.14), we have

$$z(x,\rho,t,s) - s^{-1}z_\rho(x,\rho,t,s) = g_7(x,\rho,s) \;\text{ for } x \in (0,1),\; \rho \in (0,1),\; s \in (\tau_1,\tau_2). \tag{3.18}$$

Then, by (3.17) and (3.18), we obtain

$$z(x,\rho,t,s) = (g_1 - u)\, e^{s\rho} - s e^{s\rho}\int_0^\rho g_7(x,\tau,s)\, e^{-s\tau} d\tau.$$

So, from (3.14) on $(0,1)\times(0,1)\times(\tau_1,\tau_2)$,

$$z(x,\rho,t,s) = v e^{s\rho} - s e^{s\rho}\int_0^\rho g_7(x,\tau,s)\, e^{-s\tau} d\tau, \tag{3.19}$$

and in particular,

$$z(x,1,t,s) = v e^s - z_0(x,s),$$

with

$$z_0 \in L^2((0,1)\times(\tau_1,\tau_2))$$

defined by

$$z_0(x,s) = -se^{s}\int_0^1 g_7(x,\tau,s)\,e^{-s\tau}d\tau.$$

Multiplying the third equations of system (3.16)$_1$,(3.16)$_2$ and (3.16)$_3$ by $\widetilde{u}$, $\widetilde{\varphi}$ and $(-\int_0^x \widetilde{q}(y)dy)$ respectively, and integrating over $(0,1)$, we arrive at

$$\begin{cases} -\mu_1\int_0^1 u_{xx}\tilde{u}dx - b\int_0^1 \varphi_x\tilde{u}dx + \mu_2\int_0^1 u\tilde{u}dx = \int_0^1 h_1\tilde{u}dx, \\ -\alpha\int_0^1 \varphi_{xx}\tilde{\varphi}dx + b\int_0^1 u_x\tilde{\varphi}dx + (1+\xi)\int_0^1 \varphi\tilde{\varphi}dx + \beta\dfrac{(\tau_0+1)}{k}\int_0^1 q\tilde{\varphi}dx = -\int_0^1 h_2\tilde{\varphi}dx, \\ -\int_0^1 q_x\int_0^x \tilde{q}(y)dydx + \beta\int_0^1 \varphi_x\int_0^x \tilde{q}(y)dydx + (1+\delta)\dfrac{(\tau_0+1)}{k}\int_0^1(\int_0^x q(y)dy\int_0^x \tilde{q}(y)dy)dx \\ = -\int_0^1 h_3\int_0^x \tilde{q}(y)dydx. \end{cases} \tag{3.20}$$

Consequently, problem (3.20) is equivalent to the problem

$$a\left((u,\varphi,q),(\widetilde{u},\widetilde{\varphi},\widetilde{q})\right) = F\left(\widetilde{u},\widetilde{\varphi},\widetilde{q}\right), \tag{3.21}$$

where

$$a:\left[H^2(0,1)\cap H_0^1(0,1)\times H^2(0,1)\cap H_*^1(0,1)\times H^1(0,1)\right]^2 \longrightarrow \mathbb{R}$$

is the bilinear form given by

$$\begin{aligned} a\left((u,\varphi,q),(\widetilde{u},\widetilde{\varphi},\widetilde{q})\right) = {} & \mu_1\int_0^1 u_x\tilde{u}_xdx + b\int_0^1 \varphi\tilde{u}_xdx + \mu_2\int_0^1 u\tilde{u}dx + \alpha\int_0^1 \varphi_x\tilde{\varphi}_xdx \\ & + b\int_0^1 u_x\tilde{\varphi}dx + (1+\xi)\int_0^1 \varphi\tilde{\varphi}dx + \beta\frac{(\tau_0+1)}{k}\int_0^1 q\tilde{\varphi}dx \\ & + \int_0^1 q\tilde{q}dx - \beta\int_0^1 \varphi\tilde{q}dx \\ & + (1+\delta)\frac{(\tau_0+1)^2}{k^2}\int_0^1(\int_0^x q(y)dy\int_0^x \tilde{q}(y)dy)dx, \end{aligned}$$

and

$$F:\left[H^2(0,1)\cap H_0^1(0,1)\times H^2(0,1)\cap H_*^1(0,1)\times H^1(0,1)\right] \longrightarrow \mathbb{R}$$

is the linear form defined by

$$F\left(\widetilde{u},\widetilde{\varphi},\widetilde{q}\right)^T = \int_0^1 h_1\tilde{u}dx - \int_0^1 h_2\tilde{\varphi}dx - \int_0^1 h_3\int_0^x \tilde{q}(y)dydx.$$

3.3. Well-posedness of the problem

New, for $V = H^2(0,1) \cap H_0^1(0,1) \times H^2(0,1) \cap H_*^1(0,1) \times H^1(0,1)$ equipped with the norm

$$\|(u,\varphi,q)\|_V^2 = \left\|(u_x + \frac{b}{\mu_1}\varphi)\right\|_2^2 + \|u\|_2^2 + \|\varphi_x\|_2^2 + \|q\|_2^2 .$$

One easily to see that $a(.,.)$ and $F(.)$ are bounded. Furthermore, using integration by parts, we obtain

$$\begin{aligned}
a((u,\varphi,q),(u,\varphi,q)) &= \mu_1 \int_0^1 u_x^2 dx + \mu_2 \int_0^1 u^2 dx + 2b \int_0^1 u_x \tilde{\varphi} dx + \alpha \int_0^1 \varphi_x^2 dx \\
&\quad + (1+\xi) \int_0^1 \varphi^2 dx + \beta \frac{(\tau_0+1)}{k} \int_0^1 q\varphi dx + \int_0^1 q^2 dx \\
&\quad + \beta \int_0^1 \varphi q dx + (1+\delta) \frac{(\tau_0+1)}{k} \int_0^1 (\int_0^x q(y)dy)^2 dx \\
&= \mu_1 (u_x + \frac{b}{\mu_1}\varphi)^2 + (\xi - \frac{b^2}{\mu_1})\varphi^2 + \mu_2 \int_0^1 u^2 dx + \alpha \int_0^1 \varphi_x^2 dx \\
&\quad + \int_0^1 \varphi^2 dx + \frac{(\tau_0 - 1)}{k} \int_0^1 q^2 dx \\
&\quad + (1+\delta) \frac{(\tau_0+1)^2}{k^2} \int_0^1 (\int_0^x q(y)dy)^2 dx \\
&\geqslant \check{c} \|(u,\varphi,q)\|_V^2 ,
\end{aligned}$$

for some $\check{c} > 0$, for all $\tau_0 \geq 1$, $\delta \in]0,1]$ and $\mu_2 \geq 0$, thus a is coercive.

Consequently, by the Lax-Milgram theorem, we deduce that problem (3.21) admits a unique solution $(u,\varphi,q) \in H^2(0,1) \cap H_0^1(0,1) \times H^2(0,1) \cap H_*^1(0,1) \times H^1(0,1)$ for all $(\widetilde{u},\widetilde{\varphi},\widetilde{q}) \in H^2(0,1) \cap H_0^1(0,1) \times H^2(0,1) \cap H_*^1(0,1) \times H^1(0,1)$.

Substituting u, φ and q in (3.15), we obtain

$$\begin{cases} v \in H^2(0,1) \cap H_0^1(0,1), \\ \\ \phi \in H^2(0,1) \cap H_*^1(0,1), \\ \\ \theta \in H_0^1(0,1). \end{cases}$$

Inserting v in (3.19) and bearing in mind $(3.16)_4$, we obtain

$$z,\ z_p \in L^2((0,1) \times (0,1) \times (\tau_1,\tau_2)).$$

Now, if $(\tilde{u},\tilde{q}) \equiv (0,0) \in (H^2(0,1) \cap H_0^1(0,1)) \times H^1(0,1)$, then $(3.20)_2$ reduces to

$$\alpha \int_0^1 \varphi_x \tilde{\varphi}_x dx + b \int_0^1 u_x \tilde{\varphi} dx + (1+\xi) \int_0^1 \varphi \tilde{\varphi} dx + \beta \frac{(\tau_0+1)}{k} \int_0^1 q \tilde{\varphi} dx = -\int_0^1 h_2 \tilde{\varphi} dx,$$

$$\forall \tilde{\varphi} \in H^2(0,1) \cap H_*^1(0,1), \tag{3.22}$$

which implies

$$\alpha\varphi_{xx} = bu_x + (1+\xi)\varphi + \beta\frac{(\tau_0+1)}{k}q + h_2 \in L^2(0,1). \tag{3.23}$$

Equation (3.22) is also true for any $\Phi \in C^1([0,1])$, $\Phi_x(0) = \Phi_x(1) = 0$ which is in $[H^2(0,1) \cap H^1_*(0,1)]$.

Hence, we have

$$\alpha\int_0^1 \varphi_x\Phi_x dx + \int_0^1 (bu_x + (1+\xi)\varphi + \beta\frac{(\tau_0+1)}{k}q + h_2)\Phi dx = 0,$$

for any $\Phi \in C^1([0,1])$, $\Phi_x(0) = \Phi_x(1) = 0$.

Thus, using integration by parts and bearing in mind (3.23), we get

$$\varphi_x(1)\Phi(1) - \varphi_x(0)\Phi(0) = 0, \forall\Phi \in C^1([0,1])\ , \Phi_x(0) = \Phi_x(1) = 0,$$

therefore, $\varphi_x(1) = \varphi_x(0) = 0$. Consequently, we obtain

$$\varphi \in H^2(0,1) \cap H^1_*(0,1).$$

Similarly, we obtain

$$\mu_1 u_{xx} = -b\varphi_x + \mu_2 u + h_1 \in L^2(0,1),$$

$$q_x = \beta\varphi_{tx} - (1+\delta)\frac{(\tau_0+1)}{k}\int_0^x q(y)dy + h_3 \in L^2(0,1),$$

thus, we have

$$u \in H^2(0,1) \cap H^1_0(0,1), \quad q \in H^1(0,1).$$

Finally, the application of the regularity theory for the linear elliptic equations guarantees the existence of unique $U \in D(\mathcal{A})$ such that (3.13) is satisfied. Hence, the operator $(I_d - \mathcal{A})$ is surjective. Therefore, $\mathcal{A}$ is a maximal monotone operator, by Hille-Yousida theorem (see [54, 13]) we have the well-posedness result stated in the theorem 3.1. ■

3.4 Exponential stability of solution

In this section, we state and prove the stability result for the energy of the system (3.8)-(3.9). For the regular solution of the system (3.8)-(3.9),we define the energy functional $E(t)$ as

$$\begin{aligned} E(t) := {}& \frac{1}{2}\textstyle\int_0^1 \left[\rho u_t^2 + J\varphi_t^2 + c\theta^2 + \frac{\tau_0}{k}q^2 + \alpha\varphi_x^2 + \mu_1 u_x^2 + \xi\varphi^2 + 2bu_x\varphi\right] dx \\ &+ \frac{1}{2}\textstyle\int_0^1\int_{\tau_1}^{\tau_2}\int_0^1 s\left|\mu(s)\right| z^2(x,\rho,t,s)\, d\rho ds dx. \end{aligned} \tag{3.24}$$

Remark 3.1 Note that $E(t)$ is stirctly positive. In fact, by considering

$$\mu_1 u_x^2 + 2bu_x\varphi + \xi\varphi^2 = \mu_1(u_x + \frac{b}{\mu_1}\varphi)^2 + (\xi - \frac{b^2}{\mu_1})\varphi^2,$$

and using the fact $\mu_1\xi > b^2$, we get

$$\mu_1 u_x^2 + 2bu_x\varphi + \xi\varphi^2 > 0.$$

Consequently, it follows that $E(t) > 0$.

The stability result reads as follows.

Theorem 3.2 *Suppose that $\int_{\tau_1}^{\tau_2} |\mu(s)| ds \leq \mu_0$. Then, the classical solution of (3.8)-(3.9) satisfies, for two positive constants c_0 and α_1, the following estimate:*

$$E(t) \leq c_0 e^{-\alpha_1 t}, \quad t \geq 0. \tag{3.25}$$

In order to prove this result, we need the following lemmas.

Lemma 3.1 *Let (u, φ, θ, q) be the solution of (3.8)-(3.9) and assume (3.3) holds. Then the energy functional, defined by (3.24) satisfies*

$$\frac{d}{dt}E(t) \leq -(\mu_0 - \int_{\tau_1}^{\tau_2} |\mu(s)| ds) \int_0^1 u_t^2 dx - \delta \int_0^1 \theta^2 dx - \frac{1}{k}\int_0^1 q^2 dx \leq 0, \forall t \geq 0. \tag{3.26}$$

Proof. Multiplying the first equation in (3.8) by u_t, the second by φ_t , the third by θ and the fourth by $\frac{q}{k}$, integrating over $(0,1)$ with respect to x, we obtain

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\int_0^1 \left[\rho u_t^2 + J\varphi_t^2 + c\theta^2 + \frac{\tau_0}{k}q^2 + \alpha\varphi_x^2 + \mu_1 u_x^2 + \xi\varphi^2\right] dx + b\int_0^1 u_x\varphi dx\right] \\ = &-\int_0^1 (\mu_0 u_t^2 + \delta\theta^2 + \frac{1}{k}q^2) dx - \int_0^1 \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,t,s) u_t ds dx. \end{aligned} \tag{3.27}$$

On the other hand, multiplying (3.7) by $|\mu(s)| z(x,\rho,t,s)$ and integrating over $(0,1) \times (0,1) \times (\tau_1, \tau_2)$ with respect to ρ, x and s, we obtain

$$\begin{aligned} &\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu(s)| z(x,p,t,s) z_t(x,\rho,t,s) ds d\rho dx \\ &+ \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu(s)| z(x,\rho,t,s) z_\rho(x,\rho,t,s) ds d\rho dx = 0, \end{aligned}$$

which gives

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu(s)| z^2(x,\rho,t,s) ds d\rho dx = -\frac{1}{2}\frac{d}{d\rho}\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu(s)| z^2(x,\rho,s,t) ds d\rho dx.$$

Thus, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu(s)\right| z^2(x,\rho,t,s)\,dsd\rho dx = -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu(s)\right| z^2(x,1,t,s)\,dsdx + \frac{1}{2}\int_{\tau_1}^{\tau_2}\left|\mu(s)\right| ds\int_0^1 u_t^2 dx. \quad (3.28)$$

Summing up (3.27)-(3.28), we arrive at

$$\frac{d}{dt}E(t) = -\int_0^1(\mu_0 u_t^2+\delta\theta^2+\frac{1}{k}q^2)dx - \int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\,z(x,1,t,s)\,u_t dsdx - \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu(s)\right| z^2(x,1,t,s)\,dsdx + \frac{1}{2}\int_{\tau_1}^{\tau_2}\left|\mu(s)\right| ds\int_0^1 u_t^2 dx. \quad (3.29)$$

Using integration by parts and Young's inequality, we have

$$-\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\,z(x,1,t,s)\,u_t dsdx \le \frac{1}{2}\int_{\tau_1}^{\tau_2}\left|\mu(s)\right| ds\int_0^1 u_t^2 dx + \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu(s)\right| z^2(x,1,t,s)\,dsdx. \quad (3.30)$$

Simple substitution of (3.30) into (3.29) and using (3.3) give (3.26), which concludes the proof. ■

Now, we are going to construct a Lyapunov functional equivalent to the energy. For this, we will prove several lemmas with the purpose of creating negative counterparts of the terms that appear in the energy.

Lemma 3.2 *Let (u,φ,θ,q) be the solution of (3.8)-(3.9). Then the functional*

$$K_1(t) := \rho\int_0^1 u u_t dx + J\int_0^1 \varphi\varphi_t dx + \frac{\mu_0}{2}\int_0^1 u^2 dx \quad (3.31)$$

satisfies, for any $\xi_1>0$, the estimate

$$K_1'(t) \le -\frac{\ell}{2}\int_0^1 u_x^2 dx + \rho\int_0^1 u_t^2 dx + J\int_0^1\varphi_t^2 dx - \xi_1\int_0^1\varphi^2 dx + \frac{\alpha}{2}\int_0^1\varphi_x^2 dx + \frac{\beta^2}{2\alpha}\int_0^1\theta^2 dx + \hat{c}_0\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu(s)\right| z^2(x,1,t,s)\,dsdx, \quad (3.32)$$

where $\hat{c}_0 = \dfrac{\mu_0}{2\ell c_p}$, $\xi_1 = \xi - \dfrac{b^2}{\mu_1}$ and $\ell = \mu_1 - \dfrac{b^2}{\xi}$.

Proof. By differentiating $K_1(t)$ with respect to t, using the first and the second equation of (3.8), and integrating by parts, we obtain

$$\begin{aligned} K_1'(t) &= \rho\int_0^1 u_t^2dx+\int_0^1 u(\mu_1 u_{xx}+b\varphi_x-\mu_0 u_t-\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,t,s)ds)dx+J\int_0^1\varphi_t^2dx \\ &+\int_0^1\varphi(\alpha\varphi_{xx}-bu_x-\xi\varphi+\beta\theta_x)dx+\frac{\mu_0}{2}\int_0^1 2uu_tdx. \\ &= \rho\int_0^1 u_t^2dx-\mu_1\int_0^1 u_x^2dx-2b\int_0^1 u_x\varphi dx-\int_0^1 u\int_{\tau_1}^{\tau_2}\mu(s)u_t(x,t-s)dsdx \\ &+J\int_0^1\varphi_t^2dx-\alpha\int_0^1\varphi_x^2dx-\xi\int_0^1\varphi^2dx-\beta\int_0^1\varphi_x\theta dx. \end{aligned}$$

By using Young's inequalities, we obtain

$$\begin{aligned} -\beta\int_0^1\varphi_x\theta dx &= -\int_0^1\varphi_x(\beta\theta)dx=-\int_0^1\sqrt{\alpha}\varphi_x\left(\frac{\beta}{\sqrt{\alpha}}\theta\right)dx \\ -\beta\int_0^1\varphi_x\theta dx &\le \frac{\alpha}{2}\int_0^1\varphi_x^2dx+\frac{\beta^2}{2\alpha}\int_0^1\theta^2dx, \end{aligned}$$

on the other hand we have

$$-\mu_1\int_0^1 u_x^2dx-2b\int_0^1 u_x\varphi dx-\xi\int_0^1\varphi^2dx\le-\frac{1}{2}\left(\mu_1-\frac{b^2}{\xi}\right)\int_0^1 u_x^2dx-\frac{1}{2}\left(\xi-\frac{b^2}{\mu_1}\right)\int_0^1\varphi^2dx.$$

By using Young's Poincaré's and Cauchy Schwartz inequalities , we obtain

$$\begin{aligned} &\int_0^1 u\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,t,s)dsdx \\ \le\ &\frac{\ell}{2}\int_0^1 u_x^2(x,t)dx+\frac{1}{2\ell c_p}\int_0^1\left(\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,t,s)ds\right)^2dx \\ \le\ &\frac{\ell}{2}\int_0^1 u_x^2(x,t)dx+\frac{1}{2\ell c_p}\int_{\tau_1}^{\tau_2}|\mu(s)|ds\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)z^2(x,1,t,s)dsdx \\ \le\ &\frac{\ell}{2}\int_0^1 u_x^2(x,t)dx+\frac{\mu_0}{2\ell c_p}\int_0^1\int_{\tau_1}^{\tau_2}|\mu(s)|z^2(x,1,t,s)dsdx. \end{aligned}$$

Then, (3.32) is established. ■

Lemma 3.3 *Let (u,φ,θ,q) be the solution of (3.8)-(3.9). Then the functional*

$$K_2(t):=-cJ\int_0^1\varphi_t(\int_0^x\theta(y,t)dy)dx \tag{3.33}$$

satisfies, for any $\varepsilon>0$, the estimate

$$\begin{aligned} K_2'(t) \le\ &-\frac{J\beta}{2}\int_0^1\varphi_t^2dx+\frac{J}{\beta}\int_0^1 q^2dx+\varepsilon\int_0^1\varphi_x^2dx \\ &+c_p\varepsilon\int_0^1 u_x^2dx+\varepsilon\int_0^1\varphi^2dx \\ &+C(\varepsilon)\int_0^1\theta^2dx, \end{aligned}\tag{3.34}$$

where $C(\varepsilon) = c\beta + \frac{c^2\alpha^2}{4\varepsilon} + \frac{J\delta^2}{\beta} + \frac{c^2\xi^2}{4\varepsilon} + \frac{c^2b^2}{4\varepsilon}$.

Proof. By differentiating $K_2(t)$ with respect to t, then exploiting the second and the third equation in (3.8), and integrating by parts, we obtain

$$\begin{aligned} K_2'(t) &= -J\int_0^1 \varphi_t(\int_0^x -q_x + \beta\varphi_{tx} - \delta\theta)dx \\ &\quad -c\int_0^1 ((\alpha\varphi_{xx} - bu_x - \xi\varphi + \beta\theta_x)\int_0^x \theta(y,t)dy)dx \\ &= -J\beta\int_0^1 \varphi_t^2 dx + c\beta\int_0^1 \theta^2 dx + J\int_0^1 \varphi_t q dx + c\alpha\int_0^1 \varphi_x\theta dx \\ &\quad +J\delta\int_0^1 \varphi_t(\int_0^x \theta(y,t)dy)dx - bc\int_0^1 u\theta dx + c\xi\int_0^1 \varphi(\int_0^x \theta(y,t)dy)dx. \end{aligned}$$

By using Young's, Chauchy-Schwartz and Poincaré inequalities, we obtain for any $\varepsilon > 0$,

$$\begin{aligned} J\int_0^1 \varphi_t q dx &\leq \frac{J\beta}{4}\int_0^1 \varphi_t^2 dx + \frac{J}{\beta}\int_0^1 q^2 dx, \\ c\alpha\int_0^1 \varphi_x\theta dx &\leq \varepsilon\int_0^1 \varphi_x^2 dx + \frac{c^2\alpha^2}{4\varepsilon}\int_0^1 \theta^2 dx, \\ J\delta\int_0^1 \varphi_t(\int_0^x \theta(y,t)dy)dx &\leq \frac{J\beta}{4}\int_0^1 \varphi_t^2 dx + \frac{J\delta^2}{\beta}\int_0^1 \theta^2 dx, \\ c\xi\int_0^1 \varphi(\int_0^x \theta(y,t)dy)dx &\leq \varepsilon\int_0^1 \varphi^2 dx + \frac{c^2\xi^2}{4\varepsilon}\int_0^1 \theta^2 dx, \\ -bc\int_0^1 u\theta dx &\leq c_p\varepsilon\int_0^1 u_x^2 dx + \frac{c^2b^2}{4\varepsilon}\int_0^1 \theta^2 dx. \end{aligned}$$

Combining all the above inequalities, we obtain (3.34). ■

Lemma 3.4 *Let* (u, φ, θ, q) *be the solution of (3.8)-(3.9) and (3.7). Then the functional*

$$K_3(t) := \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}|\mu(s)|\, z^2(x,\rho,t,s)\, dsd\rho dx \tag{3.35}$$

satisfies, for some positive constant m_1, *the following estimate*

$$\begin{aligned} K_3'(t) &\leq -m_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu(s)|\, z^2(x,\rho,t,s)\, dsd\rho dx \\ &\quad -m_1\int_0^1\int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,t,s)\, dsdx + \mu_0\int_0^1 u_t^2 dx. \end{aligned} \tag{3.36}$$

Proof. By differentiating $K_3(t)$ with respect to t, and using the equation (3.7), we obtain,

$$\begin{aligned} K_3'(t) &= -2\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} e^{-s\rho}\left|\mu(s)\right| z(x,\rho,t,s)\, z_\rho(x,\rho,t,s)\, dsd\rho dx \\ &= -\frac{d}{d\rho}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} e^{-s\rho}\left|\mu(s)\right| z^2(x,\rho,t,s)\, dsd\rho dx \\ &\quad -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}\left|\mu(s)\right| z^2(x,\rho,t,s)\, dsd\rho dx \\ &= -\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu(s)\right|\left[e^{-s}z^2(x,1,t,s)-z^2(x,0,t,s)\right] dsd\rho dx \\ &\quad -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}\left|\mu(s)\right| z^2(x,\rho,t,s)\, dsd\rho dx. \end{aligned}$$

Using the fact that $z(x,0,t,s)=u_t$ and $e^{-s}\le e^{-s\rho}\le 1$, for all $0<\rho<1$, we obtain

$$\begin{aligned} K_3'(t) &\le -\int_0^1\int_{\tau_1}^{\tau_2} e^{-s}\left|\mu(s)\right| z^2(x,1,t,s)\, dsd\rho dx+\int_{\tau_1}^{\tau_2}\left|\mu(s)\right| ds\int_0^1 u_t^2 dx \\ &\quad -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} se^{-s}\left|\mu(s)\right| z^2(x,\rho,t,s)\, dsd\rho dx. \end{aligned}$$

Because $-e^{-s}$ is an increasing function, we have $-e^{-s}\le -e^{-\tau_2}$, for all $s\in[\tau_1,\tau_2]$. Finally, setting $m_1=e^{-\tau_2}$ and recalling (3.3), we obtain (3.36). ■

Next, we define a Lyapunov function L and show that it is equivalent to the energy functional E.

Lemma 3.5 *For N sufficiently large, the functional defined by*

$$L(t):=NE(t)+K_1(t)+N_1K_2(t)+N_2K_3(t), \tag{3.37}$$

where N, N_1, N_2 are positive constants to be chosen appropriately later, satisfies

$$c_1E(t)\le L(t)\le c_2E(t),\ \forall t\ge 0, \tag{3.38}$$

for two positive constants c_1 and c_2.

Proof. Let

$$\pounds(t):=\left|L(t)-NE(t)\right|=K_1(t)+N_1K_2(t)+N_2K_3(t),$$

then

$$\begin{aligned} \left|\pounds(t)\right| &\le \rho\int_0^1\left|uu_t\right| dx+J\int_0^1\left|\varphi\varphi_t\right| dx+\frac{\mu_0}{2}\int_0^1\left|u^2\right| dx+N_1cJ\int_0^1\left|\varphi_t\int_0^x\theta(y,t)dy\right| dx \\ &\quad +N_2\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\left|e^{-s\rho}\mu(s)\right| z^2(x,\rho,t,s)\, dsd\rho dx. \end{aligned}$$

Exploiting Young's, Cauchy-schwartz inequalities, we obtain for all $\varepsilon > 0$

$$\begin{aligned} \int_0^1 u u_t dx &\leq \frac{\varepsilon}{2}\int_0^1 u_t^2 dx + \frac{1}{2\varepsilon}\int_0^1 u^2 dx, \\ \int_0^1 \varphi\varphi_t dx &\leq \frac{\varepsilon}{2}\int_0^1 \varphi_t^2 dx + \frac{1}{2\varepsilon}\int_0^1 \varphi^2 dx, \\ \int_0^1 \varphi_t \int_0^x \theta(y,t)dydx &\leq \frac{\beta}{4\delta}\int_0^1 \varphi_t^2 dx + \frac{\delta}{\beta}\int_0^1 \theta^2 dx. \end{aligned}$$

By (3.24)) and the fact that $|e^{-s\rho}| \leq 1$ for all $\rho \in [0,1]$, we obtain

$$\begin{aligned} |\pounds(t)| &\leq \frac{\varepsilon\rho}{2}\int_0^1 u_t^2 dx + (\frac{\mu_0}{2} + \frac{1}{2\varepsilon})\int_0^1 u^2 dx + (\frac{\varepsilon J}{2} + \frac{N_1 cJ\beta}{4\delta})\int_0^1 \varphi_t^2 dx + \frac{J}{2\varepsilon}\int_0^1 \varphi^2 dx \\ &+ \frac{N_1 cJ\delta}{\beta}\int_0^1 \theta^2 dx + N_2\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu\left(s\right)\right| z^2\left(x,\rho,t,s\right) dsd\rho dx \\ &\leq C\int_0^1 (u_t^2 + u_x^2 + \varphi_t^2 + \varphi_x^2 + u^2 + \theta^2 + \varphi^2)dx \\ &+ N_2\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu\left(s\right)\right| z^2\left(x,\rho,t,s\right) dsd\rho dx \\ &\leq ME(t), \ M \geq 0, \end{aligned}$$

where $C > 0$.

Consequently, $|L(t) - NE(t)| \leq ME(t)$ which yields

$$(N-M)E(t) \leq L(t) \leq (N+M)E(t).$$

Choosing N such that $(N - M) \geq 0$. ■

Proof. (Of Theorem (3.2))

By differentiating (3.37) and recalling (3.32), (3.34), (3.36) and (3.26) we arrive at

$$\begin{aligned} L'(t) &\leq -\left[Nm_1 - N_2\mu_0 - \rho\right]\int_0^1 u_t^2 dx - \left[\frac{J\beta N_1}{2} - J\right]\int_0^1 \varphi_t^2 dx \\ &- \left[N\delta - N_1 C\left(\varepsilon\right) - \frac{\beta^2}{2\alpha}\right]\int_0^1 \theta^2 dx \\ &- \left[\frac{N}{k} - \frac{N_1 J}{\beta}\right]\int_0^1 q^2 dx - \left[\frac{\alpha}{2} - N_1\varepsilon\right]\int_0^1 \varphi_x^2 dx \\ &- \left[\frac{\ell}{2} - c_p\varepsilon N_1\right]\int_0^1 u_x^2 dx - \left[\xi_1 - N_1\varepsilon\right]\int_0^1 \varphi^2 dx \\ &- \left(N_2 - \hat{c}_0\right)\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu\left(s\right)\right| z^2\left(x,1,t,s\right) dsdx \\ &- N_2\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu\left(s\right)\right| z^2\left(x,\rho,t,s\right) dsd\rho dx. \end{aligned} \tag{3.39}$$

At this point, we need to choose our constants very carefully. First, we choose N_1 and N_2 large enough such that

$$N_1 > \frac{2J}{J\beta}, \quad N_2 > \hat{c}_0 \ ,$$

then, we pick ε small enough such that

$$\varepsilon < \min\left\{\frac{\alpha}{2N_1}, \frac{\ell}{2c_p N_1}, \frac{\xi_1}{N_1}\right\}.$$

Finally, we choose N large enough, so that

$$Nm_1 - N_2\mu_0 - \rho > 0 \text{ and } N\delta - N_1 C(\varepsilon) - \frac{\beta^2}{2\alpha} > 0.$$

Therefore, we deduce that there exist a positive constant α_0 such that (3.39) becomes

$$L'(t) \leq -\alpha_0 E(t), \tag{3.40}$$

and, further, for some c_1, $c_2 > 0$, we have

$$c_1 E(t) \leq L(t) \leq c_2 E(t), \ \forall t \geq 0. \tag{3.41}$$

A Combining (3.40) and the right-hand side of (3.41), we conclude that

$$L'(t) \leq -\alpha_1 L(t), \ \forall t \geq 0, \tag{3.42}$$

where $\alpha_1 = \dfrac{\alpha_0}{c_2}$.

A simple integration of (3.42) over $(0, t)$ leads to

$$L(t) \leq L(0) e^{-\alpha_1 t}, \quad \forall t \geq 0. \tag{3.43}$$

Finally, by combining (3.41) and (3.43) we obtain (3.25). ■

Chapter 4

Well-posedness and general decay for a porous-elastic system with microtemperatures and a time-varying delay term

4.1 Introduction

In this chapter, we are concerned with the one-dimensional porous-elastic system with micro-temperatures and a time-varying delay, the system is written as

$$\begin{cases} \rho_1 u_{tt} = \mu u_{xx} + b\varphi_x - \gamma_1 u_t - \gamma_2 u_t\left(x, t - \tau(t)\right), \\ \\ J\varphi_{tt} = \delta\varphi_{xx} - bu_x - \xi\varphi - dw_x, \\ \\ \alpha w_t = \beta w_{xx} - d\varphi_{tx} - kw, \end{cases} \tag{4.1}$$

where $(x, t) \in (0, 1) \times (0, +\infty)$,with the initial datum and boundary conditions

$$\begin{cases} u\left(x, 0\right) = u_0\left(x\right), \; u_t\left(x, 0\right) = u_1\left(x\right), & x \in (0, 1), \\ \varphi\left(x, 0\right) = \varphi_0\left(x\right), \; \varphi_t\left(x, 0\right) = \varphi_1\left(x\right), & x \in (0, 1), \\ w\left(x, 0\right) = w_0\left(x\right), & x \in (0, 1), \\ u\left(0, t\right) = \varphi_x\left(0, t\right) = w\left(0, t\right) = 0, & t \in (0, +\infty), \\ u\left(1, t\right) = \varphi_x\left(1, t\right) = w\left(1, t\right) = 0, & t \in (0, +\infty), \\ u_t(x, t - \tau(0)) = f_0(x, t - \tau(0)), & (x, t) \in (0, 1) \times (0, \tau(0)), \end{cases} \tag{4.2}$$

where u the transversal displacement, φ is the volume fraction difference, w is the microtemperature difference and the coefficients, ρ_1, b μ, γ_1, γ_2, J, α, β, ξ, d, δ and k are positive constant coefficients where

$$\frac{\mu}{\rho_1} - \frac{\delta}{J} = \chi, \tag{4.3}$$

and

$$\mu\xi > b^2. \tag{4.4}$$

The initial data $(u_0, u_1, \varphi_0, \varphi_1, w_0, f_0)$ are assumed to belong to a suitable functional space.

System (4.1)-(4.2) arises in the theory of linear elastic materials with voids, the study of this problems had stimulated the interest of many researchers due to the extensive practical applications of such materials in different fields of human endeavors most importantly, in petroleum industry, foundation engineering, biology, material science and many others.

To construct the system (4.1), we consider the following three basic evolution equations of the one-dimensional porous materials with micro-temperatures theory

$$\begin{cases} \rho_1 u_{tt} = T_x - R \ , \\ \\ J\varphi_{tt} = H_x + G \ , \\ \\ \rho_1 E_t = P_x + q - Q \ , \end{cases}$$

where T is the stress tensor, H is the equilibrated stress vector, G is the equilibrated body force, q is the heat flux, P is the first heat flux moment, Q is the mean heat flux, and E is the first moment of energy with the following constitutive equations:

$$\begin{cases} T = \mu u_x + b\varphi \ , \quad R = \gamma_1 u_t + \gamma_2 u_t\,(x, t - \tau(t)) \ , \\ \\ H = \delta\varphi_x - dw \ , \quad G = -bu_x - \xi\varphi \ , \\ \\ \rho_1 E = -\alpha w - d\varphi_x \ , \ P = -\beta w_x \ , \quad q = k_1 w \ , \ Q = k_2 w \ , \end{cases}$$

where $k = k_1 - k_2 > 0$.

We assume as in [51], that there exist positive constants τ_1, τ_2, such that

$$0 < \tau_1 \le \tau(t) \le \tau_2, \quad \forall t > 0. \tag{4.5}$$

Moreover, we assume that the speed of the delay satisfies

$$\tau'(t) \le d_1 \le 1, \quad \forall t > 0 \tag{4.6}$$

and

$$\tau \in W^{2,\infty}([0,T]), \quad \forall T > 0, \tag{4.7}$$

where d_1 is a positive constant, and that $\gamma_{1,}$ γ_2 satisfy

$$|\gamma_2| < \sqrt{1-d_1}\gamma_{1.} \tag{4.8}$$

Several results concerning the exponential or the polynomial decay of solutions for the thermoelastic systems were obtained [15, 16, 30, 32, 43, 44, 60]. A sample model describing the one-dimensional porous-elasticity with micro-temperatures, which was developed in [7], is given by the following system:

$$\begin{cases} \rho_1 u_{tt} - \mu u_{xx} - b\varphi_x = 0 \text{ in } (0,1)\times(0,+\infty), \\ \\ J\varphi_{tt} - \delta\varphi_{xx} + bu_x + \xi\varphi + dw_x = 0 \text{ in } (0,1)\times(0,+\infty), \\ \\ \alpha w_t - \beta w_{xx} + d\varphi_{tx} + kw = 0 \quad \text{in } (0,1)\times(0,+\infty). \end{cases}$$

Under suitable conditions, the authors used the semi-groupe method to prove that the system is exponentially stable if and only if $\chi = 0$.

When $\chi \neq 0$, they proved that the system is stable polynomially decaying at a rate in the form $\dfrac{1}{\sqrt{t}}$, which is proved to be optimal.

Time delays so often arise in many physical, chemical, thermal and economical phenomena (see [40, 52, 65, 70]). The presence of delay may be a source of instability. In recent years, the control of PDEs with time-varying delay effects has become an active area of research. For example, Zitouni and Ardjouni [68] studied the transmission system with varying delay in $\mathbb{R}$ of the form:

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t) + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau(t)) = 0 \text{ in } \Omega\times(0,+\infty), \\ \\ v_{tt}(x,t) - bv_{xx}(x,t) = 0 \text{ in } (l_1,l_2)\times(0,+\infty), \\ \\ u(0,t) = u(l_3,t) = 0, \ t>0, \\ \\ u(l_i,t) = v(l_i,t), \quad au(l_i,t) = bv(l_i,t), \ i=1,2, \\ \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)), \ x\in\Omega, \\ \\ (u_t(x,0), v_t(x,0)) = (u_1(x), v_1(x)), \quad x\in\,]l_1,l_2[\,, \end{cases} \tag{4.9}$$

where $0 < l_1 < l_2 < l_3$, $\Omega =]l0, l_1[\cup]l_2, l_3[$, a, b, μ_1, μ_2 are positive constants, and they used the semigroup theory to prove the well-posedness and the uniqueness of solution. Also they showed the exponential stability by introducing an appropriate Lyapunov functional.

On the other hand, in [69], Zitouni and Ardjouni considered a linear damped wave equation with interior delays where two feedback terms have a delay of the form:

$$u_{tt}(x,t) - \triangle u(x,t) + a_0 u_t(x,t) + a_1 u_t(x,t-\tau_1(t)) + a_2 u_t(x,t-\tau_2(t)) = 0$$

$$\begin{aligned} & & \text{in } \Omega \times (0,+\infty), \\ u(x,t) &=& 0 \ \text{ on } \ \Gamma \times (0,+\infty), \\ u(x,0) &=& u_0(x), \ \ u_t(x,0) == u_1(x) \ \text{ in } \Omega, \\ u_t(x,t) &=& g_0(x,t) \ \text{ in } \Omega \times (-\max(\tau_1(0),\tau_2(0)),0), \end{aligned}$$

where $\tau_1(t) > 0$ and $\tau_2(t) > 0$ are the time-varying delays, a_0, a_1 and a_2 are real numbers with $a_0 > 0$, and the initial datum $(u_0,\ u_1,\ g_0)$ belongs to a suitable space. By using semigroup arguments, they proved the well-posednes and uniqueness of the solution for the initial-boundary value problem and they showed the exponential stability of solution by introducing suitable Lyapunov functionals.

The asymptotic behavior of the solution of porous-elastic system with time varying delay effects has been studied by many researchers. For example, in [12], Borges Filho and Santos considered the following one-dimensional equations of an homogeneous and isotropic porous-elastic solid with interior time-dependent delay term feedbacks:

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\varphi_x = 0 & \text{in } \ (0,1)\times(0,+\infty), \\ \\ J\varphi_{tt} - \delta\varphi_{xx} + bu_x + \xi\varphi + +\mu_1\varphi_t + \mu_2\varphi_t(x,t-\tau(t)) = 0 & \text{in } \ (0,1)\times(0,+\infty). \end{cases}$$

They proved that the system is well-posed under some hypothesis adopted by using the variable norm technique of T. Kato. And they also showed that the system is exponentially stable via a suitable Lyapunov functional under suitable conditions.

In [24], Hao and Wang studied the viscoelastic porous-thermoelastic system of the type III with boundary time-varying delay of the form:

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x+\psi)_x + \theta_x = 0, & \text{in } (0,1)\times(0,+\infty), \\ \rho_2\psi_{tt} - \alpha\psi_{xx} + k(\varphi_x+\psi) - \theta + \int_0^t g(t-s)\psi_{xx}(x,s)ds = 0, & \text{in } (0,1)\times(0,+\infty), \\ \rho_3\theta_{tt} - k\theta_{xx} - \delta\theta_{xtt} + \beta\psi_{tt} = 0, & \text{in } (0,1)\times(0,+\infty), \\ (\varphi(x,0),\psi(x,0),\theta(x,0)) = (\varphi_0(x),\psi_0(x),\theta_0(x)), & x\in(0,1), \\ (\varphi_t(x,0),\psi_t(x,0),\theta_t(x,0)) = (\varphi_1(x),\psi_1(x),\theta_1(x)), & x\in(0,1), \\ \varphi(0,t) = \psi(0,t) = \theta(0,t) = \psi(1,t) = \theta(1,t) = 0, & t\in(0,+\infty), \\ \varphi_x(1,t) = -k_1\varphi_t(1,t) - k_2\varphi(1,t-\tau(t)), & t\in(0,+\infty), \\ \varphi_t(1,t-\tau(0)) = f^0(1,t-\tau(0)), & t\in(0,\tau(0)). \end{cases}$$

They established the exponential decay result of the system in which the damping is strong enough to stabilize the thermoelastic system in the presence of time delay.

In this work, we considered the porous- elastic system (4.1)-(4.2) with a time-varying delay term. we proved the well-posedness and uniqueness of the solution by using the variable norm technique of Kato. By introducing an appropriate Lyapunov functional, we proved the exponential decay for the case of equal speeds of propagation. Furthermore, when $\dfrac{\mu}{\rho_1} \neq \dfrac{\delta}{J}$, we obtain the lack of exponential stability by using Gearhart- Herbst-Prüss-Huang theorem. For this case, by introducing the second-order energy, we proved the polynomial decay result.

This chapter is organized as follows. In Section 2, we present some assumptions and prove the well-posedness of problem (4.1)-(4.2). In Section 3, we use the energy method to prove the exponential stability result under the condition $\chi = 0$ and (4.4). In Section 4, we show that the system is not exponentially stable if $\chi \neq 0$. Finally, Section 5 is devoted to the statement and proof of the polynomial stability.

Throughout this chapter, C_p is used to denote the Poincaré-type constant and c a generic positive constant. We use the standard Lebesgue space $L^2(0,1)$ and the Sobolev space $H_0^1(0,1)$ with their usual scalar products and norms.

Meanwhile, from the second equation in (4.1) and the boundary conditions, we obtain

$$\frac{d^2}{dt^2}\int_0^1 \varphi(x,t)dx + \frac{\xi}{J}\int_0^1 \varphi(x,t)dx = 0. \tag{4.10}$$

By solving Eq. (4.10) and using the initial data of φ, we obtain

$$\int_0^1 \varphi(x,t)dx = \left(\int_0^1 \varphi_0(x,t)dx\right)\cos\left(\sqrt{\frac{\xi}{J}}t\right) + \sqrt{\frac{J}{\xi}}\left(\int_0^1 \varphi_1(x,t)dx\right)\sin\left(\sqrt{\frac{\xi}{J}}t\right),$$

consequently, if we set

$$\begin{aligned}\tilde{\varphi}(x,t)dx &= \varphi(x,t)dx - \left(\int_0^1 \varphi_0(x,t)dx\right)\cos\left(\sqrt{\frac{\xi}{J}}t\right)\\ &\quad -\sqrt{\frac{J}{\xi}}\left(\int_0^1 \varphi_1(x,t)dx\right)\sin\left(\sqrt{\frac{\xi}{J}}t\right),\end{aligned}$$

we obtain

$$\int_0^1 \tilde{\varphi}(x,t)dx = 0,\ \forall t \geq 0.$$

Hence, the use of Poincaré's inequality for $\tilde{\varphi}$ is justified. In addition, $(u,\tilde{\varphi},w)$ satisfies system Eqs. (4.1) with initial data of $\tilde{\varphi}$ given by

$$\tilde{\varphi}_0(x) = \varphi_0(x) - \int_0^1 \tilde{\varphi}_0(x)dx \text{ and } \tilde{\varphi}_1(x) = \varphi_1(x) - \int_0^1 \tilde{\varphi}_1(x)dx.$$

In what follows in this chapter, we will work with $\tilde{\varphi}$ but write φ for simplicity of notation.

4.1. Introduction

4.2 Well-posedness

In this section, we prove the existence and uniqueness of solutions for (4.1)-(4.2) using semigroup theory. As in [39], let us introduce the following new variable

$$z(x,p,t) = u_t(x, t - \tau(t)p), \quad (x,p,t) \in (0,1) \times (0,1) \times (0,+\infty), \tag{4.11}$$

which satisfies

$$\tau(t)z_t(x,p,t) + (1 - \tau'(t)p)z_p(x,p,t) = 0 \tag{4.12}$$

for $(x,p,t) \in (0,1) \times (0,1) \times (0,+\infty)$.

Therefore, Problem (4.1) is equivalent to

$$\begin{cases} \rho_1 u_{tt} = \mu u_{xx} + b\varphi_x - \gamma_1 u_t - \gamma_2 u_t\left(x, t - \tau(t)\right), & (x,t) \in (0,1) \times (0,\infty), \\ J\varphi_{tt} = \delta\varphi_{xx} - bu_x - \xi\varphi - dw_x, & (x,t) \in (0,1) \times (0,\infty), \\ \alpha w_t = \beta w_{xx} - d\varphi_{tx} - kw, & (x,t) \in (0,1) \times (0,\infty), \\ \tau(t)z_t(x,p,t) + (1 - \tau'(t)p)z_p(x,p,t) = 0, & (x,p,t) \in (0,1) \times (0,1) \times (0,\infty), \end{cases} \tag{4.13}$$

with the initial data and boundary conditions

$$\begin{cases} u\left(x,0\right) = u_0\left(x\right), \ u_t\left(x,0\right) = u_1\left(x\right), & x \in (0,1), \\ \varphi\left(x,0\right) = \varphi_0\left(x\right), \ \varphi_t\left(x,0\right) = \varphi_1\left(x\right), & x \in (0,1), \\ w\left(x,0\right) = w_0\left(x\right), & x \in (0,1), \\ u\left(0,t\right) = \varphi_x\left(0,t\right) = w\left(0,t\right) = 0, & t \in (0,+\infty), \\ u\left(1,t\right) = \varphi_x\left(1,t\right) = w\left(1,t\right) = 0, & t \in (0,+\infty), \\ z(x,0,t) = u_t(x,t) & (x,t) \in (0,1) \times (0,\infty), \\ z(x,p,0) = f_0(x, -p\tau(0)), & (x,t) \in (0,1) \times (0,1). \end{cases} \tag{4.14}$$

Now, we set $v = u_t$, $\psi = \varphi_t$ and let $V = (u, v, \varphi, \psi, w, z)^T$, then (4.13)-(4.14) can be written as

$$\begin{cases} V_t(t) = \mathcal{A}(t)V(t), & t > 0, \\ V(0) = (u_0, u_1, \varphi_0, \varphi_1, w_0, f_0(x, -p\tau(0)))^T, \end{cases} \tag{4.15}$$

where the time-varying operator $\mathcal{A}(t)$ is defined by

$$\mathcal{A}(t)\begin{pmatrix} u \\ v \\ \varphi \\ \psi \\ w \\ z \end{pmatrix} = \begin{pmatrix} v \\ \frac{\mu}{\rho_1}u_{xx} + \frac{b}{\rho_1}\varphi_x - \frac{\gamma_1}{\rho_1}u_t - \frac{\gamma_2}{\rho_1}z\left(x,1,t\right) \\ \psi \\ \frac{\delta}{J}\varphi_{xx} - \frac{b}{J}u_x - \frac{\xi}{J}\varphi - \frac{d}{J}w_x \\ \frac{\beta}{\alpha}w_{xx} - \frac{d}{\alpha}\varphi_{tx} - \frac{k}{\alpha}w \\ -(1-\tau'(t)p)z_p/\tau(t) \end{pmatrix},$$

with domain

$$D(\mathcal{A}(t)) = \left\{ \begin{array}{l} (u,v,\varphi,\psi,w,z)^T \in (H^2(0,1)\cap H_0^1(0,1)) \times H_0^1(0,1) \\ \times(H_*^2(0,1)\cap H_*^1(0,1)) \times H_*^1(0,1) \times (H^2(0,1)\cap H_0^1(0,1)) \\ \times L^2((0,1)\times H^1(0,1)),\ z(.,0)=v(.) \text{ in } (0,1) \end{array} \right\}, \tag{4.16}$$

where

$$\begin{aligned} L_*^2(0,1) &= \left\{\Psi \in L^2(0,1) : \int_0^1 \Psi(x)dx = 0\right\}, \quad H_*^1(0,1) = H^1(0,1)\cap L_*^2(0,1), \\ H_*^2(0,1) &= \left\{\Psi \in H^2(0,1) :\ \Psi_x(0) = \Psi_x(1) = 0\right\}. \end{aligned}$$

We define the Hilbert space

$$\mathbf{H} = \left\{H_0^1(0,1)\times L^2(0,1)\times H_*^1(0,1)\times L_*^2(0,1)\times L^2(0,1)\times L^2((0,1)\times(0,1))\right\}$$

endowed with the inner product

$$\begin{aligned} \left\langle V,\tilde{V}\right\rangle_H &= \rho_1\int_0^1 u_t\tilde{u}_t dx + J\int_0^1 \varphi_t\tilde{\varphi}_t dx + \alpha\int_0^1 w\tilde{w}dx + \mu\int_0^1 u_x\tilde{u}_x dx \\ &+\delta\int_0^1 \varphi_x\tilde{\varphi}_x dx + \xi\int_0^1 \varphi\tilde{\varphi}dx + b\int_0^1 (u_x\tilde{\varphi} + \varphi\tilde{u}_x)dx \\ &+\int_0^1\int_0^1 z(x,p)\tilde{z}(x,p)dxdp \end{aligned}$$

for any $V = (u,v,\varphi,\psi,w,z)^T$, $\tilde{V} = (\tilde{u},\tilde{v},\tilde{\varphi},\tilde{\psi},\tilde{w},\tilde{z})^T \in H$.

Our well-posedness result was obtained in [37]:

Theorem 4.1 *Let (4.5)-(4.7) be satisfied and assume that (4.8) holds. Then for any $V_0 \in D(\mathcal{A}(0))$, there exist a unique solution V of problem (4.13)-(4.14) satisfying*

$$V \in C([0,\infty), D(\mathcal{A}(0)) \cap C^1([0,\infty), H).$$

Proof of Theorem 4.1. The proof of the global existence and uniqueness of (4.13)-(4.14) is given by:

Theorem 4.2 *[37] Assume that*
(i) $D(\mathcal{A}(0))$ is dense subset of H;
(ii) $D(\mathcal{A}(t)) = D(\mathcal{A}(0))$, $\forall t > 0$;
(iii) For all $t \in [0, t]$, $\mathcal{A}(t)$ generates a strongly continuous semi-group on H and the family $\mathcal{A} = \{\mathcal{A}(t);\ t \in [0, T]\}$ is stable with stability constants ς and m independent of t, i.e., the semi-group $S_t(s)_{s\geq 0}$ generated by $\mathcal{A}(t)$ satisfied

$$\|S_t(s)(u)\|_H \leq \varsigma e^{ms} \|u\|_H \ ,\ \ \forall u \in H\ ,\ \ s \geq 0;$$

(iv) $\partial_t \mathcal{A}(t) \in L^\infty_*([0, T], B(D(\mathcal{A}(0)), H))$, where $L^\infty_*([0, T], B(D(\mathcal{A}(0)), H)$ is the space of equivalent class of essentially bounded, strongly measurable functions from $[0, T]$ into the set $B(D(\mathcal{A}(0)), H)$ of bounded operators from $D(\mathcal{A}(0))$ into H.
Then problem (4.15) has a unique solution

$$V \in C([0, T),\ D(\mathcal{A}(0)) \cap C^1([0, T),\ H)),$$

for any initial data in $D(\mathcal{A}(0))$.

■
Proof of theorem 4.2. To prove Theorem 4.2, we will follow the method used in [39, 40, 51] with the necessary modification imposed by the nature of our problem.
(i) First, we show that $D(\mathcal{A}(0))$ is dense in H. Let $F = (f_1, f_2, f_3, f_4, f_5, f_6) \in H$ be orthogonal to all elements of $D(\mathcal{A}(0))$ with respect to the inner product $\langle , \rangle_H$:

$$\begin{aligned} 0 &= \langle V, F\rangle_H \\ &= \int_0^1 \{\rho_1 v f_2 + J\psi f_4 + \alpha w f_5 + \mu u_x f_{1x} + \delta\varphi_x f_{3x} + \xi\varphi f_3 + b(u_x f_3 + f_{1x}\varphi)\}\, dx \\ &\quad + \int_0^1\int_0^1 z(x,p) f_6(x,p) dx dp \end{aligned} \tag{4.17}$$

for all $V = (u, v, \varphi, \psi, w, z)^T \in D(\mathcal{A}(0))$. Our goal is to prove that

$$f_i = 0,\ \ i = 1, ..., 6.$$

Let us first take $z \in D((0, 1) \times (0, 1))$ and $u = v = \varphi = \psi = w = 0$, so the vector $V = (0, 0, 0, 0, 0, z)^T \in D(\mathcal{A}(0))$, and therefore, from (4.17), we deduce that

$$\int_0^1\int_0^1 z(x,p) f_6(x,p) dx dp = 0.$$

Since $D((0,1)\times(0,1))$ is dense in $L^2((0,1)\times(0,1))$, it follows from then that $f_6=0$. Then, let $v\in D(0,1)$, then $V=(0,v,0,0,0,0)^T\in D(\mathcal{A}(0))$, which implies from (4.17) that

$$\int_0^1 vf_2dx=0,$$

so as above, $f_2=0$.

Similarly, we have $f_5=f_4=0$.

Next, Let $V=(u,0,0,0,0,0)^T$, then we obtain from (4.17) that

$$\int_0^1 u_xf_{1x}dx=0.$$

It is obvious that $(u,0,0,0,0,0)^T\in D(\mathcal{A}(0))$ if and only if $u\in H^2(0,1)\cap H_0^1(0,1)$ and since $H^2(0,1)\cap H_0^1(0,1)$ is dense in $H_0^1(0,1)$ with respect to the inner produce

$$\langle g,h\rangle_{H_0^1}=\int_0^1 g_xh_xdx,$$

we get $f_1=0$. By the same ideas as above, we can also show that $f_3=0$.

(ii) With our choice, $D(\mathcal{A}(t))$ is independent of t, consequently

$$D(\mathcal{A}(t))=D(\mathcal{A}(0)),\ \ \forall t>0.$$

(iii) Now, we show that the operator $\mathcal{A}(t)$ generates a C_0-semigroup in H for a fixed t. We define the time-dependent inner product on H

$$\begin{aligned}\left\langle V,\tilde{V}\right\rangle_t &= \int_0^1\{\rho_1v\tilde{v}+J\psi\tilde{\psi}+\alpha w\tilde{w}+\mu u_x\tilde{u}_x+\delta\varphi_x\tilde{\varphi}_x+\xi\varphi\tilde{\varphi}\\ &\quad+b(u_x\tilde{\varphi}+\tilde{u}_x\varphi)\}dx+\eta\tau(t)\int_0^1\int_0^1 z(x,p)\tilde{z}(x,p)dxdp \end{aligned}\tag{4.18}$$

where η satisfies

$$\frac{|\gamma_2|}{\sqrt{1-d_1}}<\eta<2\gamma_1-\frac{|\gamma_2|}{\sqrt{1-d_1}}\tag{4.19}$$

thanks to hypothesis (4.8).

Let us set

$$h(t)=\frac{(\tau'(t)^2+1)^{\frac{1}{2}}}{2\tau(t)}.$$

In this step, we prove the dissipativity of the operator $\tilde{A}(t)=\mathcal{A}(t)-h(t)I$.

for a fixed t and $V=(u,v,\varphi,\psi,w,z)^T\in D(\mathcal{A}(t))$, we have

$$\begin{aligned}\langle\mathcal{A}(t)V,V\rangle_t &= -\gamma_1\int_0^1 v^2dx-k\int_0^1 w^2dx-\gamma_2\int_0^1 z(x,1)v(x)dx\\ &\quad-\eta\int_0^1\int_0^1(1-\tau'(t)p)z(x,p)z_p(x,p)dxdp.\end{aligned}\tag{4.20}$$

Observe that

$$\int_0^1\int_0^1 (1-\tau'(t)p)z(x,p)z_p(x,p)dxdp = \int_0^1\int_0^1 \frac{1}{2}\frac{\partial}{\partial p}z^2(1-\tau'(t)p)dxdp = \frac{\tau'(t)}{2}\int_0^1\int_0^1 z^2(x,p)dxdp + \frac{1}{2}\int_0^1 \left\{z^2(x,1)(1-\tau'(t)) - z^2(x,0)\right\}dx, \tag{4.21}$$

whereupon

$$\langle \mathcal{A}(t)V,V\rangle_t = -\gamma_1\int_0^1 v^2dx - k\int_0^1 w^2dx - \gamma_2\int_0^1 z(x,1)v(x)dx - \frac{\eta\tau'(t)}{2}\int_0^1\int_0^1 z^2(x,p)dxdp - \frac{\eta}{2}\int_0^1 z^2(x,1)(1-\tau'(t))dx + \frac{\eta}{2}\int_0^1 v^2(x)dx. \tag{4.22}$$

By using Chauchy-Schwarz inequality and (4.6), we get

$$\langle \mathcal{A}(t)V,V\rangle_t \le \left(-\gamma_1 + \frac{|\gamma_2|}{\sqrt{1-d_1}} + \frac{\eta}{2}\right)\int_0^1 v^2(x)dx - k\int_0^1 w^2dx + \left(\frac{|\gamma_2|\sqrt{1-d_1}}{2} - \eta\frac{(1-d_1)}{2}\right)\int_0^1 z^2(x,1)dx + h(t)\langle V,V\rangle_t .$$

Condition (4.19) allows to write

$$-\gamma_1 + \frac{|\gamma_2|}{\sqrt{1-d_1}} + \frac{\eta}{2} \le 0 \quad , \quad \frac{|\gamma_2|\sqrt{1-d_1}}{2} - \eta\frac{(1-d_1)}{2} \le 0 .$$

Consequently, the operator $\mathcal{A}(t)$ is dissipative.
Next, we prove the surjectivity of the operator $(\lambda I - \mathcal{A}(t))$ for fixed $t>0$ and $\lambda>0$. Let $F = (f_1, f_2, f_3, f_4, f_5, f_6)^T \in H$, we seek $V \in D(\mathcal{A})$ satisfying

$$(\lambda I - \mathcal{A})V = F.$$

This gives

$$\begin{cases} \lambda u - v = f_1, \\ \lambda v - \frac{\mu}{\rho_1}\mu u_{xx} - \frac{b}{\rho_1}\varphi_x + \frac{\gamma_1}{\rho_1}v + \frac{\gamma_2}{\rho_1}z(.,1) = f_2, \\ \lambda\varphi - \psi = f_3, \\ \lambda\psi - \frac{\delta}{J}\varphi_{xx} + \frac{b}{J}u_x + \frac{\xi}{J}\varphi + \frac{d}{J}w_x = f_4, \\ \lambda w - \frac{\beta}{\alpha}w_{xx} + \frac{d}{\alpha}\varphi_{tx} + \frac{k}{\alpha}w = f_5, \\ \lambda z + \frac{(1-\tau'(t))}{\tau(t)}z_p = f_6. \end{cases} \tag{4.23}$$

Suppose that we have found u, φ and w. Then, the first and the third equations in (4.23) give

$$\begin{aligned} \lambda u - v &= f_1, \\ \lambda\varphi - \psi &= f_3. \end{aligned} \tag{4.24}$$

Furthermore, by (4.23) we can find z as

$$z(x,0) = v(x), \; x \in (0,1). \tag{4.25}$$

Following the same approach as in [51],

$$z(x,p) = v(x)e^{-\lambda p\tau(t)} + \tau(t)e^{-\lambda p\tau(t)} \int_0^1 f_6(x,\sigma)e^{\lambda\sigma\tau(t)}d\sigma, \;\text{ if }\; \tau'(t) = 0,$$

and

$$z(x,p) = v(x)e^{\vartheta_p(t)} + e^{\vartheta_p(t)} \int_0^1 \frac{f_6(x,\sigma)\tau(t)}{1-\tau'(t)\sigma} e^{-\vartheta_\sigma(t)}d\sigma, \;\text{ if }\; \tau'(t) \neq 0$$

where $\vartheta_p(t) = \lambda\dfrac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t)p)$. Whereupon, from (4.24), we obtain

$$z(x,p) = \begin{cases} \lambda u(x)e^{-\lambda p\tau(t)} - f_1 e^{-\lambda p\tau(t)} \\ +\tau(t)e^{-\lambda p\tau(t)} \int_0^1 f_6(x,\sigma)e^{\lambda\sigma\tau(t)}d\sigma, \text{ if } \tau'(t) = 0, \\ \\ \lambda u(x)e^{\vartheta_p(t)} - f_1 e^{-\vartheta_p(t)} + e^{\vartheta_p(t)} \displaystyle\int_0^1 \frac{f_6(x,\sigma)\tau(t)}{1-\tau'(t)\sigma} e^{-\vartheta_\sigma(t)}d\sigma, \text{ if } \tau'(t) \neq 0. \end{cases} \tag{4.26}$$

Now, we have to find u, φ, w as solution of the equations

$$\begin{cases} \lambda^2 u - \dfrac{\mu}{\rho_1}\mu u_{xx} - \dfrac{b}{\rho_1}\varphi_x + \dfrac{\gamma_1}{\rho_1}\lambda u + \dfrac{\gamma_2}{\rho_1} z(.,1) = f_2 + \lambda f_1 + \dfrac{\gamma_1}{\rho_1}\lambda f_1, \\ \lambda^2\varphi - \dfrac{\delta}{J}\varphi_{xx} + \dfrac{b}{J}u_x + \dfrac{\xi}{J}\varphi + \dfrac{d}{J}w_x = f_4 + \lambda f_3, \\ \lambda w - \dfrac{\beta}{\alpha}w_{xx} + \dfrac{d}{\alpha}\varphi_{tx} + \dfrac{k}{\alpha}w = f_5. \end{cases} \tag{4.27}$$

Solving system (4.27) is equivalent to finding

$$u,\varphi,w \in \left(H^2(0,1)\cap H_0^1(0,1)\right) \times \left(H_*^2(0,1)\cap H_*^1(0,1)\right) \times \left(H^2(0,1)\cap H_0^1(0,1)\right),$$

such that

$$\begin{cases} \displaystyle\int_0^1 \left[\rho_1\lambda^2 u v_1 + \mu u_x v_{1x} + b\varphi v_{1x} + \gamma_1\lambda u v_1 + \gamma_2 z(.,1)\, v_1\right] dx \\ = \displaystyle\int_0^1 \left[\rho_1 f_2 v_1 + \rho_1\lambda f_1 v_1 + \gamma_1\lambda f_1 v_1\right] dx \;,\; v_1 \in H_0^1(0,1), \\ \displaystyle\int_0^1 \left[J\lambda^2\varphi v_2 + \delta\varphi_x v_{2x} + b u_x v_2 + \xi\varphi v_2 - d w v_{2x}\right] dx \\ = \displaystyle\int_0^1 \left[J f_4 v_2 + J\lambda f_3 v_2\right] dx \;,\; v_2 \in H_*^1(0,1), \\ \displaystyle\int_0^1 \left[(\alpha\lambda + k)\, w v_3 + \beta w_x v_{3x} - d\varphi_t v_{3x}\right] dx = \int_0^1 \alpha f_5 v_3 dx \;,\;\; v_3 \in H_0^1(0,1). \end{cases} \tag{4.28}$$

From (4.26), we have

$$z(x,1)=\begin{cases}\lambda u(x)e^{-\lambda\tau(t)}+z_0(x), & \text{if } \tau'(t)=0,\\ \lambda u(x)e^{\vartheta_1(t)}+z_0(x), & \text{if } \tau'(t)\neq 0\end{cases} \tag{4.29}$$

where $x\in(0,1)$ and

$$z_0(x)=\begin{cases}-f_1e^{-\lambda\tau(t)}+\tau(t)e^{-\lambda\tau(t)}\int_0^1 f_6(x,\sigma)e^{\lambda\sigma\tau(t)}d\sigma\ , & \text{if } \tau'(t)=0,\\ \\ -f_1e^{-\vartheta_1(t)}+e^{\vartheta_1(t)}\displaystyle\int_0^1\frac{f_6(x,\sigma)\tau(t)}{1-\tau'(t)\sigma}e^{-\vartheta_\sigma(t)}d\sigma\ , & \text{if } \tau'(t)\neq 0.\end{cases} \tag{4.30}$$

From the above formula, z_0 depends only on f_i, $i=1,6$. Consequently, problem (4.28) is equivalent to the problem

$$\text{Ғ}((u,\varphi,w),(\upsilon_1,\upsilon_2,\upsilon_3))=l(\upsilon_1,\upsilon_2,\upsilon_3) \tag{4.31}$$

where the bilinear form $\text{Ғ}:[H_0^1(0,1)\times H_*^1(0,1)\times L^2(0,1)]^2\rightarrow\mathbb{R}$ and the linear form $l:H_0^1(0,1)\times H_*^1(0,1)\times L^2(0,1)\rightarrow\mathbb{R}$ are defined by

$$\begin{aligned}\text{Ғ}((u,\varphi,w),(\upsilon_1,\upsilon_2,\upsilon_3)) &= \int_0^1\left[\rho_1\lambda^2u\upsilon_1+\mu u_x\upsilon_{1x}+b\varphi\upsilon_{1x}\right]dx\\ &+\int_0^1(\gamma_1+\gamma_2e^{-\lambda\tau(t)})\lambda u\upsilon_1dx\\ &+\int_0^1\left[J\lambda^2\varphi\upsilon_2+\delta\varphi_x\upsilon_{2x}+bu_x\upsilon_2+\xi\varphi\upsilon_2-dw\upsilon_{2x}\right]dx\\ &+\int_0^1\left[(\alpha\lambda+k)w\upsilon_3+\beta w_x\upsilon_{3x}-d\varphi_t\upsilon_{3x}\right]dx,\end{aligned}$$

and

$$\begin{aligned}l(\upsilon_1,\upsilon_2,\upsilon_3) &= \int_0^1\left[\rho_1f_2\upsilon_1+\rho_1\lambda f_1\upsilon_1+\gamma_1\lambda f_1\upsilon_1\right]dx-\int_0^1\gamma_2z_0(x)\upsilon_1dx\\ &+\int_0^1\left[Jf_4\upsilon_2+J\lambda f_3\upsilon_2\right]dx+\int_0^1\alpha f_5\upsilon_3dx\end{aligned} \tag{4.32}$$

if $\tau'(t)=0$, where $z_0(x)$ satisfies the first equation in (4.30).

If $\tau'(t)\neq 0$, we define

$$\begin{aligned}\text{Ғ}((u,\varphi,w),(\upsilon_1,\upsilon_2,\upsilon_3)) &= \int_0^1\left[\rho_1\lambda^2u\upsilon_1+\mu u_x\upsilon_{1x}+b\varphi\upsilon_{1x}\right]dx\\ &+\int_0^1(\gamma_1+\gamma_2e^{\vartheta_1(t)})\lambda u\upsilon_1dx\\ &+\int_0^1\left[J\lambda^2\varphi\upsilon_2+\delta\varphi_x\upsilon_{2x}+bu_x\upsilon_2+\xi\varphi\upsilon_2-dw\upsilon_{2x}\right]dx\\ &+\int_0^1\left[(\alpha\lambda+k)w\upsilon_3+\beta w_x\upsilon_{3x}-d\varphi_t\upsilon_{3x}\right]dx,\end{aligned}$$

and the operator l is defined by the same formula (4.32), where $z_0(x)$ satisfies the second equation in (4.30). It is easy to verify that Ғ is continuous and coercive, and l is continuous. So applying the Lax-Milgram theorem, problem (4.31) admits a unique solution $(u, \varphi, w) \in H_0^1(0,1) \times H_*^1(0,1) \times L^2(0,1)$ for all $(v_1, v_2, v_3) \in H_0^1(0,1) \times H_*^1(0,1) \times L^2(0,1)$. Applying the classical elliptic regularity, it follows from (4.28) that $(u, \varphi, w) \in (H^2(0,1) \cap H_0^1(0,1)) \times (H_*^2(0,1) \cap H_*^1(0,1)) \times (H^2(0,1) \cap H_0^1(0,1))$.

Therefore, the operator $\lambda I - \mathcal{A}(t)$ is surjective for any fixed $t > 0$ and $x > 0$.Since $h(t) > 0$ and

$$\lambda I - \tilde{A}(t) = (\lambda + h(t))\, I - \mathcal{A}(t),$$

we deduce that the operator $\lambda I - \tilde{A}(t)$ is also surjective for any $\lambda > 0$ and $t > 0$.

To complete the proof of (iii), it's suffices to show that

$$\frac{\|\Psi\|_t}{\|\Psi\|_s} \leq e^{\frac{\flat}{2\tau_1}|t-s|}, \quad \forall t,\ s \in [0,T] \tag{4.33}$$

where $\flat$ is a positive constant, $\Psi = (u,\ v,\ \varphi,\ \psi,\ w,\ z)^T$ and $\|.\|_t$ is the norm associated with the inner product (4.18). For $t,\ s \in [0,T]$, we have from (4.18),

$$\begin{aligned} &\|\Psi\|_t^2 - \|\Psi\|_s^2\, e^{\frac{\flat}{\tau_1}|t-s|} \\ = &\left(1 - e^{\frac{\flat}{\tau_1}|t-s|}\right) \int_0^1 \left\{\rho_1 v^2 + J\psi^2 + \alpha w^2 + \mu u_x^2 + \delta\varphi_x^2 + \xi\varphi^2 + 2bu_x\varphi\right\} dx \\ &+\eta \left(\tau(t) - \tau(s) e^{\frac{\flat}{\tau_1}|t-s|}\right) \int_0^1 \int_0^1 z^2(x,p)\, dx\, dp. \end{aligned} \tag{4.34}$$

We notice that $1 - e^{\frac{\flat}{\tau_1}|t-s|} \leq 0$. Now, we will prove that $\tau(t) - \tau(s)e^{\frac{\flat}{\tau_1}|t-s|} \leq 0$ for some $\flat > 0$. To do this, we have

$$\tau(t) = \tau(s) + \tau'(a)(t-s)$$

where $a \in (s,t)$, which implies

$$\frac{\tau(t)}{\tau(s)} \leq 1 + \frac{|\tau'(t)|}{\tau(s)} |t-s|.$$

By (4.6), τ' is bounded on $[0,T]$ and therefore, recalling also (4.5), we deduce that

$$\frac{\tau(t)}{\tau(s)} \leq 1 + \frac{\flat}{\tau_1} |t-s| \leq e^{\frac{\flat}{\tau_1}|t-s|}$$

which proves (4.33) and therefore (iii).

(iv) It is clear that

$$\frac{d}{dt}\mathcal{A}(t)V = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \dfrac{(\tau''(t)\tau(t)p - \tau'(t)\left(\tau'(t)p - 1\right))}{\tau^2(t)} z_p \end{pmatrix},$$

then by using (4.7) and (4.5), (iv) holds exactly as in [51].

Consequently, from above analysis, we deduce that the problem

$$\begin{cases} \tilde{V}_t = \tilde{A}(t)\tilde{V} \\ \tilde{V}(0) = V_0 \end{cases} \tag{4.35}$$

has a unique solution $\tilde{V} \in C([0,+\infty), D(\mathcal{A}(0)))$ and if $V_0 \in D(\mathcal{A}(0))$, then

$$\tilde{V} \in C([0,+\infty), D(\mathcal{A}(0))) \cap C^1([0,+\infty), H).$$

Now, let

$$V(t) = e^{B(t)}\tilde{V}(t)$$

with $B(t) = \int_0^t h(s)ds$, then we have by using (4.35)

$$\begin{aligned} V_t(t) &= h(t)e^{B(t)}\tilde{V}(t) + e^{B(t)}\tilde{V}_t(t) \\ &= h(t)e^{B(t)}\tilde{V}(t) + e^{B(t)}\tilde{A}(t)\tilde{V}(t) \\ &= e^{B(t)}(h(t)\tilde{V}(t) + \tilde{A}(t)\tilde{V}(t)) \\ &= \mathcal{A}(t)e^{B(t)}\tilde{V}(t) \\ &= \mathcal{A}(t)V(t). \end{aligned}$$

Consequently, $V(t)$ is the unique solution of (4.15).

This ends the proof of Theorem 4.2. ■

4.3 Exponential stability

To state our decay result, we introduce the following energy functional:

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^1 (\rho_1 u_t^2 + J\varphi_t^2 + \alpha w^2 + \mu u_x^2 + \delta\varphi_x^2 + \xi\varphi^2 + 2bu_x\varphi)dx \\ &\quad + \frac{\eta}{2}\int_{t-\tau(t)}^t \int_0^1 e^{\lambda(s-t)} u_t^2(x,s)dxds \end{aligned} \tag{4.36}$$

where η and λ are suitable positive constants. We will fix η such that

$$2\gamma_1 - \frac{|\gamma_2|}{\sqrt{1-d_1}} - \eta > 0 \quad \text{and} \quad \eta - \frac{|\gamma_2|}{\sqrt{1-d_1}} > 0 \tag{4.37}$$

and

$$\lambda < \frac{1}{\tau_2}\left|\log \frac{\gamma_2}{\eta\sqrt{1-d_1}}\right|.$$

Remark 4.1 Note that $E(t)$ is stirctly positive. In fact, by considering

$$\mu u_x^2 + \xi\varphi^2 + 2bu_x\varphi = \mu(u_x + \frac{b}{\mu}\varphi)^2 + (\xi - \frac{b^2}{\mu})\varphi^2$$

and using the fact $\mu\xi > b^2$, we get

$$\mu u_x^2 + \xi\varphi^2 + 2bu_x\varphi > 0.$$

Consequently, it follows that $E(t) > 0$.

If the wave speeds are equal, we have the following exponentially stable result.

Theorem 4.3 *Assume that* $\frac{\mu}{\rho_1} = \frac{\delta}{J}$ *hold. Let (4.5)-(4.7) be satisfied and (4.8) holds, then there exist two positive constants* λ_0 *and* ϖ *such that the energy* $E(t)$ *associated with problem (4.1)-(4.2) satisfies*

$$E(t) \leq \lambda_0 e^{-\varpi t}, \quad \forall t \geq 0. \tag{4.38}$$

To prove the Theorem3, we use the following lemmas.

Lemma 4.1 *Assume that (4.8) holds and the hypotheses (4.5)-(4.7) are satisfied. Then the energy* $E(t)$ *is non-increasing, and there exists a positive constant* C_1 *such that for any solution of (4.1)-(4.2), and for any* $t \geq 0$*, we have*

$$\begin{aligned} E'(t) &\leq -C_1\left[\int_0^1 u_t^2(x,t)dx + \int_0^1 u_t^2(x,t-\tau(t))dx\right] - \beta\int_0^1 w_x^2 dx \\ &\quad -k\int_0^1 w^2 dx - \frac{\lambda\eta}{2}\int_{t-\tau(t)}^t \int_0^1 e^{\lambda(s-t)} u_t^2(x,t)dsdx \\ &\leq 0. \end{aligned} \tag{4.39}$$

Proof. *Multiplying the first three equations of (4.13) by* u_t, φ_t, *and* w *respectively, and integrating by parts over* $(0,1)$*, and using the boundary conditions, we obtain*

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_0^1 \rho_1 u_t^2 dx &= -\frac{1}{2}\frac{d}{dt}\int_0^1 \mu u_x^2 + b\int_0^1 \varphi_x u_t dx - \gamma_2 \int_0^1 u_t(x,t)u_t\left(x,t-\tau(t)\right)dx \\
&\quad -\gamma_1 \int_0^1 u_t^2 dx, \\
\frac{1}{2}\frac{d}{dt}\int_0^1 J\varphi_t^2 dx &= -\frac{1}{2}\frac{d}{dt}\int_0^1 \delta\varphi_x^2 + b\int_0^1 \varphi_{xt} u dx - \frac{1}{2}\frac{d}{dt}\int_0^1 \xi\varphi^2 dx + d\int_0^1 w\varphi_{tx} dx, \\
\frac{1}{2}\frac{d}{dt}\int_0^1 \alpha w^2 dx &= -\beta\int_0^1 w_x^2 dx - d\int_0^1 \varphi_{tx} w dx - k\int_0^1 w^2 dx.
\end{aligned}$$

As we have

$$\begin{aligned}
\frac{\eta}{2}\frac{d}{dt}\int_{t-\tau(t)}^t \int_0^1 e^{\lambda(s-t)} u_t^2(x,s)dxds &= \frac{\eta}{2}\int_0^1 u_t^2(x,t)dx \\
&\quad -\frac{\lambda\eta}{2}\int_{t-\tau(t)}^t \int_0^1 e^{\lambda(s-t)}u_t^2(x,s)dxds. \\
&\quad -\frac{\eta}{2}\int_0^1 e^{-\lambda\tau(t)}u_t^2(x,t-\tau(t))(1-\tau'(t))dx.
\end{aligned}$$

By summing them and using (4.5) and (4.6), we obtain

$$\begin{aligned}
\frac{dE(t)}{dt} &\leq -\beta\int_0^1 w_x^2 dx - k\int_0^1 w^2 dx - \gamma_1\int_0^1 u_t^2 dx - \gamma_2\int_0^1 u_t(x,t)u_t\left(x,t-\tau(t)\right)dx \\
&\quad +\frac{\eta}{2}\int_0^1 u_t^2(x,t)dx - \frac{\eta}{2}(1-d_1)e^{-\lambda\tau_2}\int_0^1 u_t^2(x,t-\tau(t))dx \\
&\quad -\frac{\lambda\eta}{2}\int_{t-\tau(t)}^t\int_0^1 e^{\lambda(s-t)}u_t^2(x,s)dxds.
\end{aligned}$$

Thinks to Young's inequality, we obtain

$$\begin{aligned}
-\gamma_2\int_0^1 u_t(x,t)u_t\left(x,t-\tau(t)\right)dx &\leq \frac{|\gamma_2|}{2\sqrt{1-d_1}}\int_0^1 u_t^2(x,t)dx \\
&\quad +\frac{|\gamma_2|\sqrt{1-d_1}}{2}\int_0^1 u_t^2(x,t-\tau(t))dx.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{dE(t)}{dt} &\leq -\beta\int_0^1 w_x^2 dx - k\int_0^1 w^2 dx - \left(\gamma_1 - \frac{|\gamma_2|}{2\sqrt{1-d_1}} - \frac{\eta}{2}\right)\int_0^1 u_t^2(x,t)dx \\
&\quad -\left(\frac{\eta}{2}(1-d_1)e^{-\lambda\tau_2} - \frac{|\gamma_2|\sqrt{1-d_1}}{2}\right)\int_0^1 u_t^2(x,t-\tau(t))dx \\
&\quad -\frac{\lambda\eta}{2}\int_{t-\tau(t)}^t\int_0^1 e^{\lambda(s-t)}u_t^2(x,t)dsdx.
\end{aligned}$$

4.3. Exponential stability

Combining (4.37), (4.39) is established. ■

Now, we will construct a Lyapunov functional L equivalent to E satisfying

$$\frac{dL(t)}{dt} \leq -\kappa L(t), \qquad \forall t \geq 0 \tag{4.40}$$

where κ is a positive constant. This needs several lemmas.

Lemma 4.2 *Let (u, φ, w) be the solution of eqs (4.1)-(4.2). Then the functional*

$$K_1(t) = -\rho_1 \int_0^1 u.u_t dx - \frac{\gamma_1}{2} \int_0^1 u^2 dx \tag{4.41}$$

satisfies

$$K_1'(t) \leq -\rho_1 \int_0^1 u_t^2 dx + 2\mu \int_0^1 u_x^2 dx + c \int_0^1 \varphi^2 dx + c \int_0^1 u_t^2 (x, t - \tau(t))\, dx. \tag{4.42}$$

Proof. *A differentiation of $K_1(t)$ leads to*

$$K_1'(t) = -\rho_1 \int_0^1 u_t^2 dx + \mu \int_0^1 u_x^2 dx + b \int_0^1 u_x \varphi dx + \gamma_2 \int_0^1 u.u_t (x, t - \tau(t))\, dx.$$

Applying Young's and Poincarè's inequalities, we obtain

$$b \int_0^1 u_x \varphi dx \leq \frac{\mu}{2} \int_0^1 u_x^2 dx + \frac{b^2}{2\mu} \int_0^1 \varphi^2 dx,$$

$$\gamma_2 \int_0^1 u.u_t (x, t - \tau(t))\, dx \leq \frac{\mu}{2} \int_0^1 u_x^2 dx + \frac{\gamma_2^2}{2\mu C_p} \int_0^1 u_t^2 (x, t - \tau(t))\, dx.$$

Then

$$K_1'(t) \leq -\rho_1 \int_0^1 u_t^2 dx + 2\mu \int_0^1 u_x^2 dx + \frac{b^2}{2\mu} \int_0^1 \varphi^2 dx + \frac{\gamma_2^2}{2\mu C_p} \int_0^1 u_t^2 (x, t - \tau(t))\, dx.$$

Therefore, (4.42) holds.

Lemma 4.3 *Let (u, φ, w) be the solution of eqs (4.1)-(4.2). Then the functional*

$$K_2(t) = J \int_0^1 \varphi.\varphi_t dx - \frac{b\rho_1}{\mu} \int_0^1 u_t \int_0^x \varphi(y) dy dx \tag{4.43}$$

satisfies

$$\begin{aligned} K_2'(t) \;\; \leq \;\; & -\frac{\delta}{2} \int_0^1 \varphi_x^2 dx - (\mu_1 - c) \int_0^1 \varphi^2 dx + c \int_0^1 \varphi_t^2 dx \\ & + c \int_0^1 w^2 dx + c \int_0^1 u_t^2 dx + c \int_0^1 u_t^2 (x, t - \tau(t))\, dx \end{aligned} \tag{4.44}$$

where $\mu_1 = \xi - \dfrac{b^2}{\mu} > 0$.

■

Proof. *A differentiation of* $K_2(t)$ *leads to*

$$\begin{aligned} K_2'(t) &= J\int_0^1 \varphi_t^2 dx - \delta \int_0^1 \varphi_x^2 dx - \xi \int_0^1 \varphi^2 dx - d\int_0^1 \varphi w_x dx \\ &-\frac{b\rho_1}{\mu}\int_0^1 u_t \int_0^x \varphi_t(y)dydx - \frac{b^2}{\mu}\int_0^1 \varphi_x \int_0^x \varphi(y)dydx \\ &+\frac{\gamma_1 b}{\mu}\int_0^1 u_t \int_0^x \varphi(y)dydx + \frac{\gamma_2 b}{\mu}\int_0^1 u_t\left(x,t-\tau(t)\right)\int_0^x \varphi(y)dydx. \end{aligned}$$

Using integration by parts, we get

$$\begin{aligned} K_2'(t) &= J\int_0^1 \varphi_t^2 dx - \delta \int_0^1 \varphi_x^2 dx - (\xi - \frac{b^2}{\mu})\int_0^1 \varphi^2 dx + d\int_0^1 \varphi_x w dx \\ &-\frac{b\rho_1}{\mu}\int_0^1 u_t \int_0^x \varphi_t(y)dydx + \frac{\gamma_1 b}{\mu}\int_0^1 u_t \int_0^x \varphi(y)dydx \\ &+\frac{\gamma_2 b}{\mu}\int_0^1 u_t\left(x,t-\tau(t)\right)\int_0^x \varphi(y)dydx. \end{aligned}$$

By Young's and Cauchy-Schwarz inequalities, we obtain

$$d\int_0^1 \varphi_x w dx \leq \frac{\delta}{2}\int_0^1 \varphi_x^2 dx + \frac{d^2}{2\delta}\int_0^1 w^2 dx,$$

$$-\frac{b\rho_1}{\mu}\int_0^1 u_t \int_0^x \varphi_t(y)dydx \leq \frac{b}{4}\int_0^1 u_t^2 dx + \frac{b\rho_1^2}{\mu^2}\int_0^1 \varphi_t^2 dx,$$

$$\frac{\gamma_1 b}{\mu}\int_0^1 u_t \int_0^x \varphi(y)dydx \leq \frac{b}{4}\int_0^1 u_t^2 dx + \frac{\gamma_1^2 b}{\mu^2}\int_0^1 \varphi^2 dx,$$

$$\frac{\gamma_2 b}{\mu}\int_0^1 u_t\left(x,t-\tau(t)\right)\int_0^x \varphi(y)dydx \leq \frac{b}{4}\int_0^1 u_t^2\left(x,t-\tau(t)\right)dx + \frac{\gamma_2^2 b}{\mu^2}\int_0^1 \varphi^2 dx.$$

Combining all the above inequalities, we obtain

$$\begin{aligned} K_2'(t) &\leq -\frac{\delta}{2}\int_0^1 \varphi_x^2 dx - [(\xi - \frac{b^2}{\mu}) - \frac{b(\gamma_1^2+\gamma_2^2)}{\mu^2}]\int_0^1 \varphi^2 dx + (J + \frac{b\rho_1^2}{\mu^2})\int_0^1 \varphi_t^2 dx \\ &+\frac{d^2}{2\delta}\int_0^1 w^2 dx + \frac{b}{2}\int_0^1 u_t^2 dx + \frac{b}{4}\int_0^1 u_t^2\left(x,t-\tau(t)\right)dx. \end{aligned}$$

Therefore, (4.44) holds.

Lemma 4.4 *Let* (u,φ,w) *be the solution of eqs (4.1)-(4.2). Then the functional*

$$K_3(t) = -\alpha\int_0^1 w\int_0^x \varphi_t(y)dydx \tag{4.45}$$

satisfies, for any $\varepsilon_1 > 0$*, the estimate*

$$K_3'(t) \leq -\frac{d}{2}\int_0^1 \varphi_t^2 dx + \varepsilon_1\int_0^1 (u_x^2+\varphi_x^2+\varphi^2)dx + c\int_0^1 w_x^2 dx + c(1+\frac{1}{\varepsilon_1})\int_0^1 w^2 dx. \tag{4.46}$$

4.3. Exponential stability

■

Proof. *A differentiation of* $K_3(t)$*, using (4.1) and then integrating by parts, gives*

$$\begin{aligned} K_3'(t) &= \beta\int_0^1 w_x\varphi_t(y)dx - d\int_0^1 \varphi_t^2dx + k\int_0^1 w\int_0^x \varphi_t(y)dydx - \frac{\alpha\delta}{J}\int_0^1 w\varphi_x dx \\ &+\frac{\alpha b}{J}\int_0^1 uwdx + \frac{\alpha\xi}{J}\int_0^1 w\int_0^x \varphi(y)dydx + \frac{\alpha d}{J}\int_0^1 w^2dx. \end{aligned}$$

Using Young's, Cauchy-Schwarz and Poincaré's inequalities, for any $\varepsilon_1 > 0$*, we obtain*

$$\beta\int_0^1 w_x\varphi_t(y)dx \leq \frac{d}{4}\int_0^1 \varphi_t^2dx + \frac{\beta^2}{d}\int_0^1 w_x^2dx,$$

$$k\int_0^1 w\int_0^x \varphi_t(y)dydx \leq \frac{d}{4}\int_0^1 \varphi_t^2dx + \frac{k^2}{d}\int_0^1 w^2dx,$$

$$-\frac{\alpha\delta}{J}\int_0^1 w\varphi_x dx \leq \varepsilon_1\int_0^1 \varphi_x^2dx + \frac{\alpha^2\delta^2}{4J^2\varepsilon_1}\int_0^1 w^2dx,$$

$$\frac{\alpha\xi}{J}\int_0^1 w\int_0^x \varphi(y)dydx \leq \varepsilon_1\int_0^1 \varphi^2dx + \frac{\alpha^2\xi^2}{4J^2\varepsilon_1}\int_0^1 w^2dx,$$

$$\frac{\alpha b}{J}\int_0^1 uwdx \leq \varepsilon_1\int_0^1 u_x^2dx + \frac{\alpha^2 b^2}{4J^2C_p\varepsilon_1}\int_0^1 w^2dx.$$

Combining all the above inequalities, we obtain

$$\begin{aligned} K_3'(t) &\leq -\frac{d}{2}\int_0^1 \varphi_t^2dx + \frac{\beta^2}{d}\int_0^1 w_x^2dx + \varepsilon_1\int_0^1 (u_x^2+\varphi_x^2+\varphi)dx \\ &+\left(\frac{\alpha d}{J} + \frac{k^2}{d} + \frac{1}{\varepsilon_1}\left[\frac{\alpha^2\delta^2}{4J^2\varepsilon_1} + \frac{\alpha^2\xi^2}{4J^2\varepsilon_1} + \frac{\alpha^2 b^2}{4J^2C_p\varepsilon_1}\right]\right)\int_0^1 w^2dx. \end{aligned}$$

Therefore, (4.46) holds.

Lemma 4.5 *Let* (u,φ,w) *be the solution of eqs (4.1)-(4.2). Then the functional*

$$K_4(t) = \frac{\rho_1\delta}{b}\int_0^1 u_t\varphi_x dx + \frac{J\mu}{b}\int_0^1 \varphi_t u_x dx \tag{4.47}$$

satisfies the estimate

$$\begin{aligned} K_4'(t) &\leq -\frac{\mu}{2}\int_0^1 u_x^2dx + c\int_0^1 \varphi_x^2dx + c\int_0^1 w_x^2dx - \frac{J\rho_1}{b}\chi\int_0^1 u_t\varphi_{xt}dx \\ &+c\int_0^1 u_t^2dx + c\int_0^1 u_t^2(x,t-\tau(t))dx. \end{aligned} \tag{4.48}$$

■

Proof. *A differentiation of* $K_4(t)$ *leads to*

$$\begin{aligned} K_4'(t) &= -\mu\int_0^1 u_x^2dx - \frac{\xi\mu}{b}\int_0^1 u_x\varphi dx + \delta\int_0^1 \varphi_x^2dx - \frac{d\mu}{b}\int_0^1 u_xw_xdx \\ &+\frac{\rho_1\delta}{b}\int_0^1 u_t\varphi_{xt}dx + \frac{J\mu}{b}\int_0^1 u_{xt}\varphi_tdx - \frac{\gamma_1\delta}{b}\int_0^1 u_t\varphi_xdx \\ &-\frac{\gamma_2\delta}{b}\int_0^1 \varphi_xu_t(x,t-\tau(t))dx. \end{aligned}$$

By using Young's and Poincaré's inequalities, we obtain

$$-\frac{\xi\mu}{b}\int_0^1 u_x\varphi dx \leq \frac{\mu}{4}\int_0^1 u_x^2dx + \frac{\xi^2\mu}{b^2C_p}\int_0^1 \varphi_x^2dx,$$

$$-\frac{d\mu}{b}\int_0^1 u_xw_xdx \leq \frac{\mu}{4}\int_0^1 u_x^2dx + \frac{d^2\mu}{b^2}\int_0^1 w_x^2dx,$$

$$-\frac{\gamma_1\delta}{b}\int_0^1 u_t\varphi_xdx \leq \frac{\delta}{4}\int_0^1 u_t^2dx + \frac{\gamma_1^2\delta}{b^2}\int_0^1 \varphi_x^2dx,$$

$$-\frac{\gamma_2\delta}{b}\int_0^1 \varphi_xu_t(x,t-\tau(t))dx \leq \frac{\delta}{4}\int_0^1 \varphi_x^2dx + \frac{\gamma_2^2\delta}{b^2}\int_0^1 u_t^2(x,t-\tau(t))dx.$$

Combining all the above inequalities, we obtain

$$\begin{aligned} K_4'(t) &\leq -\frac{\mu}{2}\int_0^1 u_x^2dx + (\frac{\delta}{4} + \frac{\xi^2\mu}{b^2C_p} + \frac{\gamma_1^2\delta}{b^2})\int_0^1 \varphi_x^2dx + \frac{d^2\mu}{b^2}\int_0^1 w_x^2dx \\ &-\frac{J\rho_1}{b}\left(\frac{\mu}{\rho_1} - \frac{\delta}{J}\right)\int_0^1 u_t\varphi_{xt}dx + \frac{\delta}{4}\int_0^1 u_t^2dx + \frac{\gamma_2^2\delta}{b^2}\int_0^1 u_t^2(x,t-\tau(t))dx. \end{aligned}$$

Therefore, (4.48) holds. ■

Lemma 4.6 *Let* (u,φ,w) *be the solution of eqs (4.1)-(4.2), we define the functional*

$$I(t) = \int_0^1\int_{t-\tau(t)}^t e^{s-t}u_t^2(x,s)dsdx. \tag{4.49}$$

Then

$$\frac{dI(t)}{dt} \leq \int_0^1 u_t^2(x,s)dx - (1-d_1)e^{-\tau_1}\int_0^1 u_t^2(x,t-\tau(t))dx - e^{-\tau_1}\int_0^1\int_{t-\tau(t)}^t u_t^2(x,s)dx. \tag{4.50}$$

Next, we define a Lyapunov function L and show that it is equivalent to the energy functional E.

Proof of theorem 4.2. Let us define the Lyapunov functional

$$L(t) = NE(t) + K_1(t) + N_1K_2(t) + N_2K_3(t) + 8K_4(t) + I(t) \tag{4.51}$$

where N, N_1 and N_2 are positive real numbers which will be chosen later.

Taking into account (4.39), (4.42), (4.44), (4.46), (4.48) and (4.50), we arrived at

$$\begin{aligned} L'(t) \leq & -(2\mu - \varepsilon_1 N_2)\int_0^1 u_x^2 dx - (C_1 N + \rho_1 - c(N_1+8) - 1)\int_0^1 u_t^2 dx \\ & -(\frac{\delta}{2}N_1 - \varepsilon_1 N_2 - 8c)\int_0^1 \varphi_x^2 dx - (\frac{d}{2}N_2 - cN_1)\int_0^1 \varphi_t^2 dx \\ & -((\mu_1 - c)N_1 - \varepsilon_1 N_2 - c)\int_0^1 \varphi^2 dx - (\beta N - c(N_2+8))\int_0^1 w_x^2 dx \\ & -\left(kN - c\left(N_1 + N_2(1+\frac{1}{\varepsilon_1})\right)\right)\int_0^1 w^2 dx \\ & -(C_1 N + (1-d_1)e^{-\tau_1} - c(N_1+9))\int_0^1 u_t^2(x,t-\tau(t))dx \\ & -\frac{\lambda\eta N}{2}\int_{t-\tau(t)}^t \int_0^1 e^{\lambda(s-t)} u_t^2(x,t)dsdx - e^{-\tau_1}\int_0^1 \int_{t-\tau(t)}^t u_t^2(x,s)dx. \end{aligned}$$

Now, we choose the constant N_1 large enough such that

$$\alpha 1 = \frac{\delta}{2}N_1 - 8c > 0 \quad \text{and} \quad \alpha_2 = (\mu_1 - c)N_1 - c > 0,$$

then we choose N_2 large enough such that

$$\alpha_3 = \frac{d}{2}N_2 - cN_1 > 0.$$

At this point, we pick ε_1 small enough such that

$$\varepsilon_1 < \min(\frac{2\mu}{N_2},\ \frac{\alpha_1}{N_2},\ \frac{\alpha_2}{N_2}).$$

Consequently, we obtain

$$\alpha_4 = 2\mu - \varepsilon_1 N_2 > 0,\ \ \alpha_5 = \frac{\delta}{2}N_1 - \varepsilon_1 N_2 - 8c > 0,\ \ \alpha_6 = (\mu_1 - c)N_1 - \varepsilon_1 N_2 - c > 0.$$

Finally, we choose N large enough such that

$$\begin{aligned} \alpha_7 &= C_1 N + \rho_1 - c(N_1+8) - 1 > 0,\ \ \alpha_8 = \beta N - c(N_2+8) > 0, \\ \alpha_9 &= kN - c\left(N_1 + N_2(1+\frac{1}{\varepsilon_1})\right) > 0,\ \ \alpha_{10} = C_1 N + (1-d_1)e^{-\tau_1} - c(N_1+9). \end{aligned}$$

So, we arrive at

$$\begin{aligned} L'(t) \leq & -\alpha_4\int_0^1 u_x^2 dx - \alpha_7\int_0^1 u_t^2 dx - \alpha_5\int_0^1 \varphi_x^2 dx - \alpha_3\int_0^1 \varphi_t^2 dx - \alpha_6\int_0^1 \varphi^2 dx \\ & -\alpha_8\int_0^1 w_x^2 dx - \alpha_9\int_0^1 w^2 dx - \alpha_{10}\int_0^1 u_t^2(x,t-\tau(t))dx \\ & -\frac{\lambda\eta N}{2}\int_{t-\tau(t)}^t \int_0^1 e^{\lambda(s-t)} u_t^2(x,t)dsdx - e^{-\tau_1}\int_0^1 \int_{t-\tau(t)}^t u_t^2(x,s)dx. \end{aligned}$$

Then

$$\begin{aligned} L'(t) &\le -\alpha_4\int_0^1 u_x^2dx-\alpha_7\int_0^1 u_t^2dx-\alpha_5\int_0^1\varphi_x^2dx-\alpha_3\int_0^1\varphi_t^2dx-\alpha_6\int_0^1\varphi^2dx \\ &\quad -\alpha_9\int_0^1 w^2dx-\frac{\lambda\eta N}{2}\int_{t-\tau(t)}^t\int_0^1 e^{\lambda(s-t)}u_t^2(x,t)dsdx \qquad (4.52)\\ &\le -\tilde{c}_1\left(\int_0^1(u_t^2+\varphi_t^2+w^2+u_x^2+\varphi_x^2+\varphi^2)dx+\int_{t-\tau(t)}^t\int_0^1 e^{\lambda(s-t)}u_t^2(x,s)dxds\right)\end{aligned}$$

where

$$\tilde{c}_1=\min(\frac{\lambda\eta N}{2},\alpha_i),\quad i=3,...,7,9.$$

On other hand, from Eq (4.36), using Young's inequality and taking $\varepsilon=b$, we obtain

$$\begin{aligned} E(t) &\le \frac{1}{2}\int_0^1(\rho_1u_t^2+J\varphi_t^2+\alpha w^2+(\mu+b)u_x^2+\delta\varphi_x^2+(\xi+b)\varphi^2)dx \\ &\quad +\frac{\eta}{2}\int_{t-\tau(t)}^t\int_0^1 e^{\lambda(s-t)}u_t^2(x,s)dxds.\end{aligned}$$

Then, there exist $\check{C}>0$, such that

$$E(t)\le\check{C}\left(\int_0^1(u_t^2+\varphi_t^2+w^2+u_x^2+\varphi_x^2+\varphi^2)dx+\int_{t-\tau(t)}^t\int_0^1 e^{\lambda(s-t)}u_t^2(x,s)dxds\right),$$

which implies that

$$-\left(\int_0^1(u_t^2+\varphi_t^2+w^2+u_x^2+\varphi_x^2+\varphi^2)dx+\int_{t-\tau(t)}^t\int_0^1 e^{\lambda(s-t)}u_t^2(x,s)dxds\right)\le-\tilde{c}_1E(t). \qquad (4.53)$$

The combination of Eq (4.52) and Eq (4.53) gives

$$L'(t)\le-\breve{k}E(t),\ \forall t\ge0, \qquad (4.54)$$

for $\breve{k}>0$.

On the other hand , we are in position to compare $L(t)$ with $E(t)$, this is given in the following lemma.

Lemma 4.7 *For N sufficiently large, there exist two positive constants a_1 and a_2 depending on N, N_1 and N_2 such that*

$$a_1E(t)\le L(t)\le a_2E(t),\ \forall t\ge0. \qquad (4.55)$$

■

Proof. We consider the functional

$$\pounds(t)=K_1(t)+N_1K_2(t)+N_2K_3(t)+8K_4(t)+I(t)$$

and show that

$$|\pounds\,(t)| \leq \hat{C}_2 E(t),\ \ \hat{C}_2 > 0.$$

From (4.41), (4.43), (4.45), (4.47) and (4.49), we obtain

$$\begin{aligned} |\pounds(t)| &\leq |L\,(t) - NE\,(t)| \\ &\leq \rho_1 \int_0^1 |u.u_t|\,dx + \frac{\gamma_1}{2}\int_0^1 u^2 dx + JN_1 \int_0^1 |\varphi|\,.\,|\varphi_t|\,dx \\ &\quad + \frac{b\rho_1 N_1}{\mu}\int_0^1 |u_t| \int_0^x |\varphi(y)|\,dydx + \alpha N_2 \int_0^1 |w| \int_0^x |\varphi_t(y)|\,dydx \\ &\quad + \frac{8\rho_1\delta}{b}\int_0^1 |u_t|\,|\varphi_x|\,dx + \frac{8J\mu}{b}\int_0^1 |\varphi_t|\,|u_x|\,dx + \int_0^1\int_{t-\tau(t)}^t e^{s-t} u_t^2(x,s)dsdx. \end{aligned}$$

By using Young's and Cauchy-Schwarz inequalities, we get

$$\begin{aligned} JN_1 \int_0^1 |\varphi|\,.\,|\varphi_t|\,dx &\leq \frac{J}{4}\int_0^1 \varphi^2 dx + JN_1^2 \int_0^1 \varphi_t^2 dx \\ \frac{b\rho_1 N_1}{\mu}\int_0^1 |u_t| \int_0^x |\varphi(y)|\,dydx &\leq \frac{b\rho_1 N_1}{4\mu}\int_0^1 u_t^2 dx + \frac{b\rho_1 N_1}{\mu}\int_0^1 \varphi^2 dx \\ \alpha N_2 \int_0^1 |w| \int_0^x |\varphi_t(y)|\,dydx &\leq \frac{\alpha N_2}{4}\int_0^1 w^2 dx + \alpha N_2 \int_0^1 \varphi_t^2 dx \\ \frac{8\rho_1\delta}{b}\int_0^1 |u_t|\,|\varphi_x|\,dx &\leq \frac{2\rho_1\delta}{b}\int_0^1 u_t^2 dx + \frac{8\rho_1\delta}{b}\int_0^1 \varphi_x^2 dx \\ \frac{8J\mu}{b}\int_0^1 |\varphi_t|\,|u_x|\,dx &\leq \frac{2J\mu}{b}\int_0^1 \varphi_t^2 dx + \frac{8J\mu}{b}\int_0^1 u_x^2 dx. \end{aligned}$$

Also, using Young's and Poincaré's inequality gives

$$\rho_1 \int_0^1 |u.u_t|\,dx \leq \frac{C_p}{2}\int_0^1 u_x^2 dx + \frac{\rho_1^2}{2}\int_0^1 u_t^2 dx.$$

Combining all the above inequalities, we obtain

$$\begin{aligned} |\pounds(t)| &\leq \tilde{C}_1 (\int_0^1 u_t^2 dx + \int_0^1 u_x^2 dx + \int_0^1 \varphi^2 dx + \int_0^1 \varphi_t^2 dx + \int_0^1 \varphi_x^2 dx + \int_0^1 w^2 dx) \\ &\quad + \int_0^1 \int_{t-\tau(t)}^t e^{s-t} u_t^2(x,s)dsdx) \end{aligned}$$

where

$$\tilde{C}_1 = \max \left\{ \begin{array}{c} \frac{b\rho_1 N_1}{4\mu} + \frac{2\rho_1\delta}{b} + \frac{\rho_1^2}{2},\ \frac{8J\mu}{b} + \frac{C_p}{2},\ \frac{J}{4} + \frac{b\rho_1 N_1}{\mu}, \\ JN_1^2 + \alpha N_2 + \frac{2J\mu}{b},\ \frac{8\rho_1\delta}{b},\ \frac{\alpha N_2}{4} \end{array} \right\}.$$

On other hand, from Eq (4.5) and(4.37), and using the fact that

$$e^{s-t} \leq 1,\ \text{for all } 0 < s < t,\ \forall t > 0$$

and

$$0 < e^{\lambda(s-t)} < e^{\left[\frac{1}{\tau_2}\left|\log\frac{\gamma_2}{\eta\sqrt{1-d_1}}\right|\right](s-t)} < 1, \text{ for all } 0 < s < t, \ \forall t > 0,$$

then, there exists $\tilde{C}_2 > 0$ such that

$$|\pounds(t)| \leq \tilde{C}_2 E(t).$$

Consequently, we obtain

$$|L(t) - NE(t)| \leq \tilde{C}_2 E(t)$$

that is

$$(N - \tilde{C}_2)E(t) \leq L(t) \leq (N + \tilde{C}_2)E(t). \tag{4.56}$$

Now, by choosing N large enough such that

$$a_1 = (N - \tilde{C}_2) > 0, \ a_2 = (N + \tilde{C}_2) > 0.$$

Then (4.55) holds true.

Now, combining (4.54) and (4.55), we obtain

$$L'(t) \leq -\varpi L(t), \ \forall t \geq 0 \tag{4.57}$$

where $\varpi = \dfrac{\breve{k}}{a_2}$.

A simple integration of Eq (4.57) over $(0, t)$ yields

$$L(t) \leq L(0)e^{-\varpi t} \ , \ \forall t \geq 0. \tag{4.58}$$

The desired result (4.38) follows by using estimates (4.55) and (4.58). ■

4.4 The lack of exponential stability

This section is concerning the lack of exponential stability. Our result is achieved by Gearhart-Herbst-Prüss-Huang theorem to dissipative systems, see Prüss [55] and Huang [25].

Theorem 4.4 *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on Hilbert space H. Then $S(t)$ is exponentially stable if and only if*

$$i\mathbb{R} \equiv \{i\lambda : \lambda \in \mathbb{R}\} \subset \rho(\mathcal{A})$$

and

$$\overline{\lim_{|\lambda|\to\infty}} \left\| (i\lambda I - \mathcal{A})^{-1} \right\|_{\pounds(H)} < \infty$$

hold, where $\rho(\mathcal{A})$ is the resolvent set of the differential operator $\mathcal{A}$.

Next, we state and prove the main result of this section.

Theorem 4.5 *Assume that* $\frac{\mu}{\rho_1} \neq \frac{\delta}{J}$ *hold. Then the semigroup associated to problem (4.1)-(4.2) is not exponentially stable.*

Proof. *We will prove that there exists a sequence of values* λ_n *such that*

$$\left\|(i\lambda I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(H)} \to \infty$$

which is equivalent to prove that there exists $F_n \in H$ *with* $\|F_n\|_H \leq 1$ *and* $V_n \in D(\mathcal{A})$ *such that*

$$\left\|(i\lambda_n I - \mathcal{A})^{-1} F_n\right\|_{\mathcal{L}(H)} = \|V_n\|_H \to \infty$$

where

$$i\lambda_n V_n - \mathcal{A}V_n = F_n \tag{4.59}$$

In other words, we consider the solution of spectral equation (4.59) and show that the corresponding solution V_n *is not bounded when* F_n *is bounded in* H. *Rewrite spectral equation in term of its components, for* $\lambda_n = \lambda$, *we have*

$$\begin{aligned} i\lambda u - v &= f_1 \\ i\lambda v - \frac{\mu}{\rho_1}u_{xx} - \frac{b}{\rho_1}\varphi_x + \frac{\gamma_1}{\rho_1}v + \frac{\gamma_2}{\rho_1}z\,(.,1) &= f_2 \\ i\lambda\varphi - \psi &= f_3 \\ i\lambda\psi - \frac{\delta}{J}\varphi_{xx} + \frac{b}{J}u_x + \frac{\xi}{J}\varphi + \frac{d}{J}w_x &= f_4 \\ i\lambda w - \frac{\beta}{\alpha}w_{xx} + \frac{d}{\alpha}\varphi_{tx} + \frac{k}{\alpha}w &= f_5 \\ \lambda z + \frac{(1-\tau'(t))}{\tau(t)}z_p &= f_6 \end{aligned} \tag{4.60}$$

where $\lambda \in \mathbb{R}$ *and* $F_n = (f_1, f_2\ , f_3, f_4, f_5)^T \in H$. *Taking*

$$f_1 = f_2 = f_3 = f_5 = f_6 = 0 \text{ and } f_4 = \cos n\pi x,$$

then, by using the first and third equation in (4.60), we obtain

$$\begin{aligned} -\lambda^2 u - \frac{\mu}{\rho_1}u_{xx} - \frac{b}{\rho_1}\varphi_x + \frac{\gamma_1}{\rho_1}i\lambda u + \frac{\gamma_2}{\rho_1}z\,(.,1) &= 0 \\ -\lambda^2\varphi - \frac{\delta}{J}\varphi_{xx} + \frac{b}{J}u_x + \frac{\xi}{J}\varphi + \frac{d}{J}w_x &= \cos n\pi x \\ i\lambda w - \frac{\beta}{\alpha}w_{xx} + \frac{d}{\alpha}\varphi_{tx} + \frac{k}{\alpha}w &= 0 \\ i\lambda z + \frac{(1-\tau'(t))}{\tau(t)}z_p &= 0 \end{aligned} \tag{4.61}$$

Taking the boundary conditions into consideration, we can suppose that

$$u = a_1 \sin(n\pi x), \quad \varphi = a_2 \cos(n\pi x), \quad w = a_3 \sin(n\pi x)$$

where a_1, a_2 and a_3 depend on λ and will be determined explicitly in what follows. Therefore, the solution of (4.61) is equivalent to finding a_1, a_2 and a_3, such that

$$\begin{aligned}
\left(\left[-\lambda^2 + \frac{\mu n^2\pi^2}{\rho_1}\right] a_1 + \frac{bn\pi}{\rho_1} a_2 + \frac{\gamma_1}{\rho_1} i\lambda a_1\right) \sin(n\pi x) + \frac{\gamma_2}{\rho_1} z\,(.,1) &= 0 \\
\left[-\lambda^2 + \frac{\delta n^2\pi^2}{J} + \frac{\xi}{J}\right] a_2 + \frac{bn\pi}{J} a_1 + \frac{dn\pi}{J} a_3 &= 1 \\
\left[i\lambda + \frac{\beta n^2\pi^2}{\alpha} + \frac{k}{\alpha}\right] a_3 - \frac{i\lambda dn\pi}{\alpha} a_2 &= 0 \\
i\lambda z + \frac{(1-\tau'(t))}{\tau(t)} z_p &= 0
\end{aligned} \tag{4.62}$$

Furthermore, by (4.25) we can find z as

$$z(x,0) = v(x), \ x \in (0,1). \tag{4.63}$$

Following the same approach as in [51],

$$z(x,p) = v(x) e^{-i\lambda p \tau(t)}, \quad \text{if} \ \ \tau'(t) = 0$$

and

$$z(x,p) = v(x) e^{\vartheta_p(t)}, \quad \text{if} \ \ \tau'(t) \neq 0$$

where $\vartheta_p(t) = i\lambda \dfrac{\tau(t)}{\tau'(t)} \ln(1 - \tau'(t)p)$. Whereupon, we obtain

$$z(x,p) = \begin{cases} i\lambda u(x) e^{-i\lambda p\tau(t)}, & \text{if} \ \ \tau'(t) = 0, \\ i\lambda u(x) e^{\vartheta_p(t)}, & \text{if} \ \ \tau'(t) \neq 0. \end{cases}$$

It follow that

$$z(x,1) = \begin{cases} i\lambda u(x) e^{-i\lambda \tau(t)}, & \text{if} \ \ \tau'(t) = 0, \\ i\lambda u(x) e^{\vartheta_1(t)}, & \text{if} \ \ \tau'(t) \neq 0. \end{cases} \tag{4.64}$$

System (4.62) is equivalent to

$$\begin{aligned}
\left[-\lambda^2 + \frac{\mu n^2\pi^2}{\rho_1} + \frac{\gamma_1}{\rho_1} i\lambda + \frac{\gamma_2}{\rho_1} i\lambda \varrho(t)\right] a_1 + \frac{bn\pi}{\rho_1} a_2 &= 0 \\
\left[-\lambda^2 + \frac{\delta n^2\pi^2}{J} + \frac{\xi}{J}\right] a_2 + \frac{bn\pi}{J} a_1 + \frac{dn\pi}{J} a_3 &= 1 \\
\left[i\lambda + \frac{\beta n^2\pi^2}{\alpha} + \frac{k}{\alpha}\right] a_3 - \frac{i\lambda dn\pi}{\alpha} a_2 &= 0
\end{aligned} \tag{4.65}$$

where

$$\varrho(t) = \begin{cases} e^{-i\lambda\tau(t)}, & if \ \ \tau'(t) = 0, \\ e^{\vartheta_1(t)}, & if \ \ \tau'(t) \neq 0. \end{cases}$$

The above system can be written as

$$\begin{aligned} P_1(\lambda)a_1 + \frac{bn\pi}{\rho_1}a_2 &= 0, \\ \frac{bn\pi}{J}a_1 + P_2(\lambda)a_2 + \frac{dn\pi}{J}a_3 &= 1, \\ -\frac{i\lambda dn\pi}{\alpha}a_2 + P_3(\lambda)a_3 &= 0 \end{aligned} \tag{4.66}$$

where

$$\begin{cases} P_1(\lambda) = -\lambda^2 + \dfrac{\mu n^2\pi^2}{\rho_1} + \dfrac{\gamma_1}{\rho_1}i\lambda + \dfrac{\gamma_2}{\rho_1}i\lambda\varrho(t), \\ P_2(\lambda) = -\lambda^2 + \dfrac{\delta n^2\pi^2}{J} + \dfrac{\xi}{J}, \\ P_3(\lambda) = i\lambda + \dfrac{\beta n^2\pi^2}{\alpha} + \dfrac{k}{\alpha}. \end{cases}$$

From $(4.66)_1$ *and* $(4.66)_3$*, we get*

$$\begin{aligned} a_1 &= -\frac{bn\pi}{\rho_1 P_1}a_2, \\ a_3 &= \frac{i\lambda dn\pi}{\alpha P_3}a_2. \end{aligned}$$

Substituting a_1 *and* a_3 *into* $(4.66)_2$*, we get*

$$a_2 := a_{2_n} = \frac{P_1P_3}{P_1P_2P_3 + \dfrac{i\lambda\,(dn\pi)^2}{\alpha J}P_1 - \dfrac{(bn\pi)^2}{J\rho_1}P_3}$$

Now, we choose λ *such that*

$$P_2(\lambda) = -\lambda^2 + \frac{\delta n^2\pi^2}{J} + \frac{\xi}{J} = \sigma_0 \implies -\lambda^2 = \sigma_0 - \frac{\delta n^2\pi^2}{J} - \frac{\xi}{J}$$

where σ_0 *will be chosen later. Note that*

$$\begin{aligned} P_1P_2P_3 + \frac{i\lambda\,(dn\pi)^2}{\alpha J}P_1 &= P_1\left[P_2P_3 + \frac{i\lambda\,(dn\pi)^2}{\alpha J}\right] \\ &= P_1\left[\sigma_0\left(i\lambda + \frac{\beta n^2\pi^2}{\alpha} + \frac{k}{\alpha}\right) + \frac{i\lambda\,(dn\pi)^2}{\alpha J}\right] \\ &= P_1\left[\frac{n^2\pi^2}{\alpha}\left(\beta\sigma_0 + \frac{i\lambda d^2}{J}\right) + \sigma_0\left(i\lambda + \frac{k}{\alpha}\right)\right]. \end{aligned}$$

So, we take σ_0 *such that*

$$\sigma_0 = -\frac{i\lambda d^2}{J\beta},$$

we have

$$P_1P_2P_3 + \frac{i\lambda\left(dn\pi\right)^2}{\alpha J}P_1 \approx O(n^4).$$

Consequently, we have

$$P_1P_2P_3 - \frac{\left(bn\pi\right)^2}{J\rho_1}P_3 + \frac{i\lambda\left(dn\pi\right)^2}{\alpha J}P_1 \approx O(n^4).$$

Since

$$P_1P_2 \approx O(n^4),$$

since $\chi \neq 0$, *we obtain*

$$a_2 := a_{2n} \approx \frac{J\chi}{\dfrac{\alpha\delta d^2}{\beta^2 J}\chi - \dfrac{b^2}{\rho_1}}$$

for n *large. Finally, for* $\dfrac{\alpha\delta d^2}{\beta^2 J}\chi \neq \dfrac{b^2}{\rho_1}$, *we have*

$$\left\|V_n\right\|_H^2 \geq J\left\|\psi_n\right\|^2 = J\left|\lambda_n\right|^2\left\|\varphi_n\right\|^2 = J\left|\lambda_n\right|^2\left|a_{2n}\right|^2\int_0^1\left|\cos(n\pi x)\right|^2 dx \approx O(n^2).$$

Then

$$\left\|V_n\right\|_H \geq \sqrt{\frac{J}{2}}\left|\lambda_n\right|\left|a_{2n}\right| \approx O(n) \to \infty \quad as \quad n \to \infty.$$

Consequently, applying the theorem 4.4, we conclude that the semigroup $S(t)$ *associated with the system (4.1)-(4.2) does not have exponential decay.* ■

4.5 Polynomial Stability

In this section, we prove that, in case $\chi \neq 0$, the system (4.1)-(4.2) goes to zero polynomially as $\dfrac{1}{\sqrt{t}}$ and moreover, this rate of decay is optimal. For the regular solution of (4.1)-(4.2), we define the second-order energy functionals

$$\begin{aligned} E_2(t) &= \frac{1}{2}\int_0^1\left(\rho_1 u_{tt}^2 + J\varphi_{tt}^2 + \alpha w_t^2 + \mu u_{xt}^2 + \delta\varphi_{xt}^2 + \xi\varphi_t^2 + 2bu_{xt}\varphi_t\right)dx \\ &+\frac{\eta}{2}\int_{t-\tau(t)}^{t}\int_0^1 e^{\lambda(s-t)}u_{tt}^2(x,s)\,dx\,ds. \end{aligned} \tag{4.67}$$

By (4.36) and (4.39), it follows that E_2 satisfies

$$\begin{aligned} E_2'(t) &\leq -C_1\left[\int_0^1 u_{tt}^2(x,t)dx + \int_0^1 u_{tt}^2(x,t-\tau(t))dx\right] - \beta\int_0^1 w_{xt}^2dx \\ &\quad -k\int_0^1 w_t^2dx - \frac{\lambda\eta}{2}\int_{t-\tau(t)}^t\int_0^1 e^{\lambda(s-t)}u_{tt}^2(x,t)dsdx \\ &\leq 0,\ \forall t>0. \end{aligned} \tag{4.68}$$

Before we state and prove the main result of this section, we first establish the following important lemma.

Lemma 4.8 *Let (u,φ,w) be a regular solution of problem (4.1)-(4.2). Then the functional*

$$K_5(t) = -\frac{J\rho_1\beta}{bd}\chi\int_0^1 u_x w_x dx \tag{4.69}$$

satisfies, for any $\varepsilon_2>0$, the estimate

$$K_5'(t) \leq \frac{J\rho_1}{b}\chi\int_0^1 u_t\varphi_{xt}dx + \varepsilon_2\int_0^1(u_x^2+u_t^2)dx + \frac{c}{\varepsilon_2}\int_0^1(w^2+w_t^2+w_{xt}^2)dx. \tag{4.70}$$

Proof. *Taking a derivative of K_5 and using integration by parts, we obtain*

$$K_5'(t) = \frac{J\rho_1\beta}{bd}\chi\int_0^1 u_t w_{xx}dx - \frac{J\rho_1\beta}{bd}\chi\int_0^1 u_x w_{xt}dx. \tag{4.71}$$

From the third equation in (4.1), we have

$$\beta w_{xx} = \alpha w_t + d\varphi_{xt} + kw. \tag{4.72}$$

The combination of (4.71) and (4.72) yields

$$\begin{aligned} K_5'(t) &= \frac{J\rho_1\alpha}{bd}\chi\int_0^1 u_t w_t dx + \frac{J\rho_1}{b}\chi\int_0^1 u_x\varphi_{xt}dx + \frac{J\rho_1 k}{bd}\chi\int_0^1 u_t w dx \\ &\quad -\frac{J\rho_1\beta}{bd}\chi\int_0^1 u_x w_{xt}dx. \end{aligned}$$

Using Young's inequality, we obtain that for any $\varepsilon_2>0$

$$\begin{aligned} \frac{J\rho_1\alpha}{bd}\chi\int_0^1 u_t w_t dx &\leq \frac{\varepsilon_2}{2}\int_0^1 u_t^2dx + \frac{1}{2\varepsilon_2}\left(\frac{J\rho_1\alpha}{bd}\chi\right)^2\int_0^1 w_t^2dx, \\ \frac{J\rho_1 k}{bd}\chi\int_0^1 u_t w dx &\leq \frac{\varepsilon_2}{2}\int_0^1 u_t^2dx + \frac{1}{2\varepsilon_2}\left(\frac{J\rho_1 k}{bd}\chi\right)^2\int_0^1 w^2dx, \\ -\frac{J\rho_1\beta}{bd}\chi\int_0^1 u_x w_{xt}dx &\leq \varepsilon_2\int_0^1 u_x^2dx + \frac{1}{4\varepsilon_2}\left(\frac{J\rho_1\beta}{bd}\chi\right)^2\int_0^1 w_{xt}^2dx. \end{aligned}$$

Combining all the above inequalities, we obtain

$$\begin{aligned} K_5'(t) &\leq \frac{J\rho_1}{b}\chi\int_0^1 u_x\varphi_{xt}dx + \varepsilon_2\int_0^1 (u_x^2+u_t^2)dx + \frac{1}{2\varepsilon_2}\left(\frac{J\rho_1 k}{bd}\chi\right)^2\int_0^1 w^2dx \\ &+\frac{1}{2\varepsilon_2}\left(\frac{J\rho_1\alpha}{bd}\chi\right)^2\int_0^1 w_t^2dx + \frac{1}{4\varepsilon_2}\left(\frac{J\rho_1\beta}{bd}\chi\right)^2\int_0^1 w_{xt}^2dx, \end{aligned}$$

which is the required estimate (4.70). ∎

Now, we are ready to prove the following result:

Theorem 4.6 *Assume that* $\frac{\mu}{\rho_1}-\frac{\delta}{J}\neq 0$ *hold and let* (u,φ,w) *be a regular solution of problem (4.1)-(4.2). Then there exists a positive constant* ϖ_1 *such that the energy functional (4.36) satisfies, for all* $t>0$,

$$E(t)\leq\frac{\varpi_1}{t} \tag{4.73}$$

Proof. *As in Theorem 4.2,* $N, N_1,\ N_2>0$, *define*

$$\tilde{\mathcal{L}}(t) = N(E(t)+E_2(t)) + K_1(t) + N_1K_2(t) + N_2K_3(t) + 8(K_4(t)+K_5(t)) + I(t). \tag{4.74}$$

Remark 4.2 The Lyapunov functional $\tilde{\mathcal{L}}$ defined by Eq. (4.74) is not equivalent to the energy functional E. In other words, Eq. (4.55) no longer holds.

Taking the derivation of Eq. (4.74) and using Eqs.(4.39), (4.42), (4.44), (4.46), (4.48), (4.50), (4.68), and (4.70) with the same choice of ε_1 *as in the proof of Theorem 2, we arrive at*

$$\begin{aligned} \tilde{\mathcal{L}}'(t) &\leq -(\alpha_4-8\varepsilon_2)\int_0^1 u_x^2dx - (\alpha_7-8\varepsilon_2)\int_0^1 u_t^2dx - \alpha_5\int_0^1\varphi_x^2dx - \alpha_3\int_0^1\varphi_t^2dx \\ &-\alpha_6\int_0^1\varphi^2dx - \left(\alpha_9-\frac{8c}{\varepsilon_2}\right)\int_0^1 w^2dx - (kN-\frac{8c}{\varepsilon_2})\int_0^1 w_t^2dx \\ &-(\beta N-\frac{8c}{\varepsilon_2})\int_0^1 w_{xt}^2dx - \frac{\lambda\eta}{2}\int_{t-\tau(t)}^t\int_0^1 e^{\lambda(t-s)}u_t^2(x,t)dsdx \\ &-\frac{\lambda\eta}{2}N\int_{t-\tau(t)}^t\int_0^1 e^{\lambda(s-t)}u_{tt}^2(x,t)dsdx \\ &-C_1N\left[\int_0^1 u_{tt}^2(x,t)dx + \int_0^1 u_{tt}^2(x,t-\tau(t))dx\right]. \\ &\leq -(\alpha_4-8\varepsilon_2)\int_0^1 u_x^2dx - (\alpha_7-8\varepsilon_2)\int_0^1 u_t^2dx - \alpha_5\int_0^1\varphi_x^2dx - \alpha_3\int_0^1\varphi_t^2dx \\ &-\alpha_6\int_0^1\varphi^2dx - \left(kN - c\left(1+\frac{1}{\varepsilon_2}\right)\right)\int_0^1 w^2dx - (kN-\frac{8c}{\varepsilon_2})\int_0^1 w_t^2dx \\ &-(\beta N-\frac{8c}{\varepsilon_2})\int_0^1 w_{xt}^2dx - \frac{\lambda\eta}{2}\int_{t-\tau(t)}^t\int_0^1 e^{\lambda(t-s)}u_t^2(x,t)dsdx. \end{aligned}$$

Now, we pick ε_2 small enough such that

$$\varepsilon_2 < \min(\frac{\alpha_4}{8}, \frac{\alpha_7}{8}).$$

Next, we choose N large enough such that

$$kN - \frac{8c}{\varepsilon_2} > 0, \ kN - c\left(1 + \frac{1}{\varepsilon_2}\right) > 0, \ \beta N - \frac{8c}{\varepsilon_2} > 0.$$

Now, using Eq. (4.36), we get

$$\tilde{\pounds}'(t) \leq -\lambda_0 E(t), \ \forall t > 0 \tag{4.75}$$

where λ_0 is a positive constant. Integrating Eq. (4.75) over $(0, t)$ and using the fact that E is positive and non-increasing, we obtain

$$tE(t) \leq \int_0^t E(s)ds \leq \frac{1}{\lambda_0}\left(\tilde{\pounds}(0) - \tilde{\pounds}(t)\right) \leq \frac{1}{\lambda_0}\tilde{\pounds}(0), \ \forall t > 0.$$

Finally, for $\varpi_1 = \dfrac{1}{\lambda_0}\tilde{\pounds}(0) = \dfrac{E(0) + E_2(0)}{\lambda_0}$, we have

$$E(t) \leq \frac{\varpi_1}{t}, \ \forall t > 0,$$

which completes the proof. ■

Chapter 5

Exponential decay for a swelling porous thermoelastic soils mixture with second sound

5.1 Introduction

In this chapter, we intend to study the stabilization of swelling porous thermoelastic soils with second sound, where the heat conduction is given by Cattaneo's law, The system is written as:

$$\begin{cases} \rho u_{tt} = a_1 u_{xx} + a_2 \varphi_{xx} \ , & \text{in } (0,1) \times (0,+\infty), \\ J\varphi_{tt} = a_3 \varphi_{xx} + a_2 u_{xx} + \beta \theta_x \ , & \text{in } (0,1) \times (0,+\infty), \\ \alpha \theta_t = -q_x + \beta \varphi_{tx} - \gamma \theta, & \text{in } (0,1) \times (0,+\infty), \\ \tau q_t = -q - k\theta_x \ , & \text{in } (0,1) \times (0,+\infty), \end{cases} \tag{5.1}$$

with the following initial and boundary conditions

$$\begin{cases} u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), & x \in (0,1), \\ \varphi(x,0) = \varphi_0(x), \ \varphi_t(x,0) = \varphi_1(x), & x \in (0,1), \\ \theta(x,0) = \theta_0(x), \ \ q(x,0) = q_0(x), & x \in (0,1), \\ u(0,t) = u_x(1,t) = \varphi(0,t) = \varphi_x(1,t) = 0, & t \in (0,+\infty), \\ \theta(0,t) = \theta(1,t) = q(0,t) = 0, & t \in (0,+\infty) \end{cases} \tag{5.2}$$

where u is the transversal displacement of the fluid, φ is the elastic solid material, θ is the temperature difference, q is the heat flux and the coefficients, ρ and J are densities of each constituent, a_1, a_3, α, β, τ, k, γ are positive constant coefficients and $a_2 \neq 0$ is a real number. The parameters with a_1, a_2, a_3 satisfying

$$a_1 a_3 > a_2^2. \tag{5.3}$$

Swelling is porous media theory field of study. This theory considered swelling soils as one of its investigations and priorities area. It sets its attention on every material that suffers from swelling; from a smaller soil's component to a bigger part of plants. This issue has been worked on during the past years and many researchers attempted to discover some systems, such as one-dimensional system to reach stability and suitability in the field. Additionally, many applications in various practical problems such as field of swelling have been applied.

There are several recent articles introducing continuum theories for fluids infiltrating elastic porous media (see [18, 19, 57, 63]). For example, in [63], Wang and Guo considered the linear field equation of swelling porous elastic soils with fluid saturation with the following constitutive equations

$$\begin{cases} \rho_z z_{tt} = a_1 z_{xx} + a_2 u_{xx} - \rho_z \gamma(x) z_t, & \text{in } (0,l) \times (0,+\infty), \\ \rho_u u_{tt} = a_3 z_{xx} + a_2 u_{xx}, & \text{in } (0,l) \times (0,+\infty) \end{cases} \tag{5.4}$$

where z and u represent the displacements of fluid and solid elastic materials respectively. By using the spectral method, they proved that the whole system can be exponentially stabilized by only one internal viscous damping with variable feedback gain imposed in the fluid part. On the other hand, in [57], Quintanilla studied the following system

$$\begin{cases} \rho_z z_{tt} = a_1 z_{xx} + a_2 u_{xx} - \xi(z_t - u_t) + z_{xxt}, & \text{in } (0,l) \times (0,+\infty), \\ \rho_u u_{tt} = a_3 z_{xx} + a_2 u_{xx} + \xi(z_t - u_t), & \text{in } (0,l) \times (0,+\infty). \end{cases} \tag{5.5}$$

Using the energy method, he showed that the system is exponentially stable for $a_2^2 < a_1\xi$.

In the same field of research, Apalara [7] considered a swelling porous elastic system with a single memory term as the only damping source

$$\begin{cases} \rho_z z_{tt} - a_1 z_{xx} - a_2 u_{xx} = 0, & \text{in } (0,1) \times (0,+\infty), \\ \rho_u u_{tt} - a_3 z_{xx} - a_2 u_{xx} + \int_0^t g(t-s) u_{xx}(x,s) ds = 0, & \text{in } (0,1) \times (0,+\infty). \end{cases} \tag{5.6}$$

By using the multiplier method, he established a general decay result irrespective of the wave speeds of system.

On the other hand, the classical theory of heat was under the microscope by so many researchers in the last decades, so that, they overcame its limitation and gain solutions. This idea gave birth to the theory of believing in the possibility of combining heat conduction law and the second sound theory, thus, it emerges that the speed of heat can be finite and the system could be stabilized.

The nonclassical thermoelasticity theories have a major impact these previous years, believing and proving that the speed of heat propagation on physics can be finite by the use of hyperbolic-type and the heat is showed up as a wave phenomenon, as it is called second sound theory. Many results in this contest can be obtained, and numerous stability has been established [22, 31, 40]. For the porous thermoelasticity systems coupled with the heat equation by Cattaneo's law, Messaoudi and Fareh [42] considered the following system

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\phi_x - \gamma u_t, & \text{in } (0,1)\times(0,+\infty), \\ J\phi_{tt} = \alpha\phi_{xx} - bu_x - \xi\phi + \beta\theta_x, & \text{in } (0,1)\times(0,+\infty), \\ c\theta_t = -q_x + \beta\phi_{tx} - \delta\theta, & \text{in } (0,1)\times(0,+\infty), \\ \tau_0 q_t + q + k\theta_x = 0\ , & \text{in } (0,1)\times(0,+\infty), \end{cases} \tag{5.7}$$

and proved an exponential stability result under suitable conditions by using the spectral theory.

In this study, motivated by the above results, we expose the thermoelastic problem with second sound, that shows the possibility of mixing several components (solid, fluid, gas) in the system without breaking down materials and, especially, we are interested in studying one-dimensional system of swelling porous thermoelastic soils mixture with second sound, that shows the whole system can be exponentially stabilized.

After we proved the existence and uniqueness of the solution, we have obtained the exponential decay result under the assumption (5.3) by construct some Lyapunov functionals. Our work extends the stability results from [2, 7, 40, 59] to swelling porous thermoelastic systems with second sound.

The rest of this chapter is organized as follows. In Section 2, we prove the well-posedness by using some results from the semigroup theory. In Section 3, we establish an exponential stability result of the energy.

5.2 Existence and uniqueness of the solutions

In this section, we show the well-posedness of the system (5.1)-(5.2) using the semigroup theory [54].

We set $v = u_t$, $\phi = \varphi_t$ and let

$$U = (u, u_t, \varphi, \varphi_t, \theta, q)^T\ ,$$

then

$$\partial_t U = (u_t, v_t, \varphi_t, \phi_t, \theta_t, q_t)^T\ .$$

Therefore, problem (5.1)-(5.2) can be rewritten as

$$\begin{cases} \partial_t U = \mathcal{A}U,\ t > 0 \\ U(0) = U_0 = (u_0, u_1, \varphi_0, \varphi_1, \theta_0, q_0)^T \end{cases} \tag{5.8}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}=\begin{pmatrix} 0 & Id & 0 & 0 & 0 & 0 \\ \frac{a_1}{\rho}\partial_x^2 & 0 & \frac{a_2}{\rho}\partial_x^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & Id & 0 & 0 \\ \frac{a_2}{J}\partial_x^2 & 0 & \frac{a_3}{J}\partial_x^2 & 0 & \frac{\beta}{J}\partial_x & 0 \\ 0 & 0 & 0 & \frac{\beta}{\alpha}\partial_x & -\frac{\gamma}{\alpha}Id & -\frac{1}{\alpha}\partial_x \\ 0 & 0 & 0 & 0 & -\frac{k}{\tau}\partial_x & -\frac{1}{\tau}Id \end{pmatrix}. \tag{5.9}$$

The domain of $\mathcal{A}$ is

$$D(\mathcal{A})=\left\{U\in\left(H_*^2(0,1)\times H_*^1(0,1)\right)^2\times H_0^1(0,1)\times H_*^1(0,1)\right\}$$

where

$$H_*^1(0,1) \ :\ =\left\{\phi\in H^1(0,1):\phi(0)=0\right\},$$

$$H_*^2(0,1) \ :\ =\left\{\phi\in H^2(0,1):\phi(0)=\phi_x(1)=0\right\}.$$

We consider the following Hilbert space

$$\mathcal{H}:=H_*^1(0,1)\times L^2(0,1)\times H_*^1(0,1)\times L^2(0,1)\times L^2(0,1)\times L^2(0,1).$$

The inner product on $\mathcal{H}$ is

$$\begin{aligned}\left\langle U,\widetilde{U}\right\rangle_{\mathcal{H}} &= \rho\int_0^1 u_t\widetilde{u}_t dx+J\int_0^1\varphi_t\widetilde{\varphi}_t dx+\alpha\int_0^1\theta\widetilde{\theta}dx+a_1\int_0^1 u_x\widetilde{u}_x dx \\ &\quad +a_3\int_0^1\varphi_x\widetilde{\varphi}_x dx+\frac{\tau}{k}\int_0^1 q\widetilde{q}dx+a_2\int_0^1(u_x\widetilde{\varphi}_x+\widetilde{u}_x\varphi_x)dx.\end{aligned}$$

The norm induced by the inner product is

$$\|U\|_{\mathcal{H}}=\int_0^1\left(\rho u_t^2+J\varphi_t^2+a_1u_x^2+2a_2u_x\varphi_x+a_3\varphi_x^2+\alpha\theta^2+\frac{\tau}{k}q^2\right)dx.$$

Clearly, $D(\mathcal{A})$ is dense in $\mathcal{H}$.

It is easy to show that $\mathcal{A}$ is dissipative, for each $U=(u,u_t,\varphi,\varphi_t,\theta,q)^T\in D(\mathcal{A})$, by using the inner product and integration by parts, we have

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= \left\langle \begin{pmatrix} u_t \\ \frac{a_1}{\rho}u_{xx} + \frac{a_2}{\rho}\varphi_{xx} \\ \varphi_t \\ \frac{a_3}{J}\varphi_{xx} + \frac{a_2}{J}u_{xx} + \frac{\beta}{J}\theta_x \\ -\frac{1}{\alpha}q_x + \frac{\beta}{\alpha}\varphi_{tx} - \frac{\gamma}{\alpha}\theta \\ -\frac{1}{\tau}q - \frac{k}{\tau}\theta_x \end{pmatrix}, \begin{pmatrix} u \\ u_t \\ \varphi \\ \varphi_t \\ \theta \\ q \end{pmatrix} \right\rangle \\ &= -\gamma\int_0^1 \theta^2 dx - \frac{1}{k}\int_0^1 q^2 dx \leq 0. \end{aligned} \tag{5.10}$$

Since $\mathcal{A}$ is a dissipative operator. On the other hand, it is easy to show that 0 belongs to the resolvent of $\mathcal{A}$. Consequently, the Lumer-Phillips Theorem implies that the operator $\mathcal{A}$ is the infinitesimal generator of C_0-semigroup of contractions $S(t) = e^{\mathcal{A}t}$ over $\mathcal{H}$ (see [54], Theorem 1.4). From this, we can state the following result:

Theorem 5.1 *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. The system (5.8) is well posed, i.e., for any $U_0 \in \mathcal{H}$, the system (5.8) has a unique weak solution $U(t) = U_0 e^{\mathcal{A}t} \in C(\mathbb{R}^+;\mathcal{H})$. Furthermore, if $U_0 \in D(\mathcal{A})$, $U(t) \in C^1(\mathbb{R}^+; D(\mathcal{A}) \cap C^0(\mathbb{R}^+;\mathcal{H}))$ becomes the classic solution for (5.8).*

5.3 Energy dissipation

In this section, we prove that the energy of the system (5.1)-(5.2) is dissipative over time. The energy functional $E(t)$ is given by

$$E(t) := \frac{1}{2}\int_0^1 \left[\rho u_t^2 + J\varphi_t^2 + a_1 u_x^2 + a_3\varphi_x^2 + 2a_2 u_x\varphi_x + \alpha\theta^2 + \frac{\tau}{k}q^2\right] dx, \tag{5.11}$$

then, consider the following result related to the dissipation of energy.

Lemma 5.1 *Let (u, φ, θ, q) be the solution of (5.1)-(5.2). Then the energy functional, defined by (5.11) satisfies*

$$\frac{d}{dt}E(t) = -\gamma\int_0^1 \theta^2 dx - \frac{1}{k}\int_0^1 q^2 dx \leq 0, \forall t \geq 0. \tag{5.12}$$

Proof. Multiplying the first equation in (5.1) by u_t, the second by φ_t , the third by θ and the fourth by $\frac{q}{k}$ and integrating over $(0,1)$ with respect to x, performing integration by parts and the boundary conditions, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \rho u_t^2 dx = -\frac{1}{2}\frac{d}{dt}\int_0^1 a_1 u_x^2 dx - a_2\int_0^1 \varphi_x u_{tx} dx,$$

$$\frac{1}{2}\frac{d}{dt}\int_0^1 J\varphi_t^2 dx = -\frac{1}{2}\frac{d}{dt}\int_0^1 a_3\varphi_x^2 dx - a_2\int_0^1 u_x\varphi_{tx} dx + \beta\int_0^1 \theta_x\varphi_t dx,$$

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \alpha\theta^2 dx = -\int_0^1 q_x\theta dx - \beta\int_0^1 \varphi_t\theta_x dx - \gamma\int_0^1 \theta^2 dx,$$

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \frac{\tau}{k} q^2 dx = -\frac{1}{k}\int_0^1 q^2 dx + \int_0^1 \theta q_x dx.$$

Summation them leads to

$$\frac{d}{dt}\left[\frac{1}{2}\int_0^1\left[\rho u_t^2 + J\varphi_t^2 + a_1u_x^2 + a_3\varphi_x^2 + \alpha\theta^2 + \frac{\tau}{k}q^2\right]dx + a_2\int_0^1 u_x\varphi_x dx\right]$$
$$= -\int_0^1\left(\gamma\theta^2 + \frac{1}{k}q^2\right)dx. \tag{5.13}$$

■

Therefore, the constants γ and k are positive, this concludes the proof of this lemma.

Remark 5.1 Note that $E(t)$ is stirctly positive. In fact, by considering

$$a_1u_x^2 + a_3\varphi_x^2 + 2a_2u_x\varphi_x = \frac{1}{2}\left[a_1\left(u_x + \frac{a_2}{a_1}\varphi_x\right)^2 + a_3\left(\varphi_x + \frac{a_2}{a_3}u_x\right)^2 + \left(a_1 - \frac{a_2^2}{a_3}\right)u_x^2 + \left(a_3 - \frac{a_2^2}{a_1}\right)\varphi_x^2\right],$$

since $a_1a_3 > a_2^2$, we deduce that

$$a_1u_x^2 + a_3\varphi_x^2 + 2a_2u_x\varphi_x > \frac{1}{2}\left[\left(a_1 - \frac{a_2^2}{a_3}\right)u_x^2 + \left(a_3 - \frac{a_2^2}{a_1}\right)\varphi_x^2\right],$$

we conclude that the energy satisfies

$$E(t) > \frac{1}{2}\int_0^1\left[\rho u_t^2 + J\varphi_t^2 + \tilde{a}_1u_x^2 + \tilde{a}_3\varphi_x^2 + \alpha\theta^2 + \frac{\tau}{k}q^2\right]dx$$

where

$$\tilde{a}_1 = \frac{1}{2}\left(a_1 - \frac{a_2^2}{a_3}\right) > 0\ ,\ \tilde{a}_3 = \frac{1}{2}\left(a_3 - \frac{a_2^2}{a_1}\right) > 0\ .$$

Consequently, it follows that $E(t) > 0$.

5.3. Energy dissipation

5.4 Exponential stability of solution

The stability result reads as follows.

Theorem 5.2 *Suppose that $a_1a_3 > a_2^2$. Then, the classical solution of (5.1)-(5.2) satisfies, for two positive constants c_0 and α_1, the following estimate:*

$$E(t) \le c_0 e^{-\alpha_1 t}, \quad t \ge 0. \tag{5.14}$$

Now, we are going to construct a Lyapunov functional equivalent to the energy. For this, we will prove several lemmas with the purpose of creating negative counterparts of the terms that appear in the energy.

Lemma 5.2 *Let (u, φ, θ, q) be the solution of (5.1)-(5.2). Then the functional*

$$K_1(t) := J\int_0^1 \varphi\varphi_t dx - \frac{a_2}{a_1}\rho\int_0^1 u_t\varphi dx \tag{5.15}$$

satisfies the estimate

$$K_1'(t) \le \left(J + \frac{\rho^2 a_2^2}{2a_0 a_1^2}\right)\int_0^1 \varphi_t^2 dx - \frac{a_0}{2}\int_0^1 \varphi_x^2 dx + \frac{a_0}{2}\int_0^1 u_t^2 dx + \frac{\beta^2}{2a_0}\int_0^1 \theta^2 dx. \tag{5.16}$$

Proof. By differentiating $K_1(t)$ with respect to t, using the first and the second equation of (5.1), and integrating by parts, we obtain

$$\begin{aligned} K_1'(t) &= J\int_0^1 \varphi_t^2 dx + \int_0^1 \varphi(a_3\varphi_{xx} + a_2 u_{xx} + \beta\theta_x)dx - \frac{a_2}{a_1}\rho\int_0^1 \varphi_t u_t dx \\ &\quad -\frac{a_2}{a_1}\int_0^1 \varphi(a_1 u_{xx} + a_2\varphi_{xx})dx \\ &= J\int_0^1 \varphi_t^2 dx - (a_3 - \frac{a_2^2}{a_1})\int_0^1 \varphi_x^2 dx - \frac{a_2}{a_1}\rho\int_0^1 u_t\varphi_t dx - \beta\int_0^1 \varphi_x\theta dx. \end{aligned} \tag{5.17}$$

By using Young's inequalities, we obtain

$$-\beta\int_0^1 \varphi_x\theta dx \le \varepsilon_1\int_0^1 \varphi_x^2 dx + \frac{\beta^2}{4\varepsilon_1}\int_0^1 \theta^2 dx, \tag{5.18}$$

$$-\frac{a_2}{a_1}\rho\int_0^1 u_t\varphi_t dx \le \varepsilon_1\int_0^1 u_t^2 dx + \frac{\rho^2 a_2^2}{4\varepsilon_1 a_1^2}\int_0^1 \varphi_t^2 dx. \tag{5.19}$$

Combining (5.18) and (5.19) , we and up with

$$\begin{aligned} K_1'(t) &\le \left(J + \frac{\rho^2 a_2^2}{4\varepsilon_1 a_1^2}\right)\int_0^1 \varphi_t^2 dx - (a_3 - \frac{a_2^2}{a_1} - \varepsilon_1)\int_0^1 \varphi_x^2 dx + \varepsilon_1\int_0^1 u_t^2 dx \\ &\quad + \frac{\beta^2}{4\varepsilon_1}\int_0^1 \theta^2 dx. \end{aligned} \tag{5.20}$$

For $a_0 = a_3 - \dfrac{a_2^2}{a_1} > 0$ and taking $\varepsilon_1 = \dfrac{a_0}{2}$. Then, (5.16) is established. ■

Lemma 5.3 *Let (u, φ, θ, q) be the solution of (5.1)-(5.2). Then the functional*

$$K_2(t) := -\alpha J \int_0^1 \varphi_t \Big(\int_0^x \theta(y,t)dy \Big) dx \tag{5.21}$$

satisfies the estimate

$$\begin{aligned} K_2'(t) \leq & -\frac{J\beta}{2}\int_0^1 \varphi_t^2 dx + \frac{J}{\beta}\int_0^1 q^2 dx + \Big(\frac{J\gamma^2}{\beta} + (a_3^2 + a_2^2) + \alpha\beta\Big)\int_0^1 \theta^2 dx \\ & + \frac{\alpha^2}{4}\int_0^1 \varphi_x^2 dx + \frac{\alpha^2}{4}\int_0^1 u_x^2 dx. \end{aligned} \tag{5.22}$$

Proof. By differentiating $K_2(t)$ with respect to t, then exploiting the second and the third equation in (5.1), and integrating by parts, we obtain

$$\begin{aligned} K_2'(t) = & -J\int_0^1 \varphi_t \Big(\int_0^x (-q_x + \beta\varphi_{tx} - \gamma\theta)\, dy \Big) dx \\ & -\alpha \int_0^1 ((a_3\varphi_{xx} + a_2 u_{xx} + \beta\theta_x) \int_0^x \theta(y,t)dy) dx \\ = & -J\beta\int_0^1 \varphi_t^2 dx + J\int_0^1 \varphi_t q dx + J\gamma\int_0^1 \varphi_t \Big(\int_0^x \theta(y,t)dy\Big)dx + \alpha a_3 \int_0^1 \varphi_x \theta dx \\ & + \alpha a_2 \int_0^1 u_x\theta dx + \alpha\beta\int_0^1 \theta^2 dx. \end{aligned} \tag{5.23}$$

By using Young's, Chauchy-Schwartz and Poincaré inequalities, we obtain for any $\varepsilon_2 > 0$,

$$\begin{aligned} J\int_0^1 \varphi_t q dx &\leq \frac{J\beta}{4}\int_0^1 \varphi_t^2 dx + \frac{J}{\beta}\int_0^1 q^2 dx, \\ \alpha a_3 \int_0^1 \varphi_x\theta dx &\leq \varepsilon_2 \int_0^1 \varphi_x^2 dx + \frac{\alpha^2 a_3^2}{4\varepsilon_2}\int_0^1 \theta^2 dx, \\ \alpha a_2 \int_0^1 u_x\theta dx &\leq \varepsilon_2 \int_0^1 u_x^2 dx + \frac{\alpha^2 a_2^2}{4\varepsilon_2}\int_0^1 \theta^2 dx. \\ J\gamma\int_0^1 \varphi_t \Big(\int_0^x \theta(y,t)dy\Big)dx &\leq \frac{J\beta}{4}\int_0^1 \varphi_t^2 dx + \frac{J\gamma^2}{\beta}\int_0^1 \theta^2 dx. \end{aligned}$$

Combining all the above inequalities, we obtain

$$\begin{aligned} K_2'(t) \leq & -\frac{J\beta}{2}\int_0^1 \varphi_t^2 dx + \frac{J}{\beta}\int_0^1 q^2 dx + \Big(\frac{J\gamma^2}{\beta} + \frac{\alpha^2(a_3^2 + a_2^2)}{4\varepsilon_2} + \alpha\beta\Big)\int_0^1 \theta^2 dx \\ & + \varepsilon_2 \int_0^1 \varphi_x^2 dx + \varepsilon_2 \int_0^1 u_x^2 dx. \end{aligned} \tag{5.24}$$

By taking $\varepsilon_2 = \dfrac{\alpha^2}{4}$, then (5.22) is established. ■

5.4. Exponential stability of solution

Lemma 5.4 *Let (u, φ, θ, q) be the solution of (5.1)-(5.2) and (5.3). Then the functional*

$$K_3(t) := a_2 \int_0^1 (u\varphi_t - \varphi u_t)\, dx \tag{5.25}$$

satisfies the following estimate

$$\begin{aligned} K_3'(t) \;\; &\leq \;\; -\left(\frac{a_2^2}{2J} - \frac{c_p}{J}\right) \int_0^1 u_x^2 dx + \left(\frac{a_2^2}{\rho} + \frac{J}{2}\left(\frac{a_1}{\rho} - \frac{a_3}{J}\right)^2\right) \int_0^1 \varphi_x^2 dx \\ &\quad + \frac{a_2^2 \beta^2}{4J} \int_0^1 \theta^2 dx. \end{aligned} \tag{5.26}$$

Proof. By differentiating $K_3(t)$ with respect to t, and using assumption (5.3), we obtain,

$$\begin{aligned} K_3'(t) &= a_2 \int_0^1 (u\varphi_{tt} - \varphi u_{tt})\, dx \\ &= a_2 \int_0^1 u(\frac{a_3}{J}\varphi_{xx} + \frac{a_2}{J}u_{xx} + \frac{\beta}{J}\theta)dx - a_2 \int_0^1 \varphi(\frac{a_1}{\rho}u_{xx} + \frac{a_2}{\rho}\varphi_{xx})dx \\ &= a_2\left(\frac{a_1}{\rho} - \frac{a_3}{J}\right) \int_0^1 u_x\varphi_x dx - \frac{a_2^2}{J}\int_0^1 u_x^2 dx + \frac{a_2\beta}{J}\int_0^1 u\theta dx \\ &\quad + \frac{a_2^2}{\rho}\int_0^1 \varphi_x^2 dx. \end{aligned} \tag{5.27}$$

By using Young's inequalities, we have, for $\varepsilon_3 > 0$,

$$a_2\left(\frac{a_1}{\rho} - \frac{a_3}{J}\right) \int_0^1 u_x\varphi_x dx \leq a_2^2 \frac{\varepsilon_3}{2} \int_0^1 u_x^2 dx + \frac{1}{2\varepsilon_3}\left(\frac{a_1}{\rho} - \frac{a_3}{J}\right)^2 \int_0^1 \varphi_x^2 dx. \tag{5.28}$$

Also, using Young's and Poincarè's inequality gives

$$-\frac{a_2\beta}{J}\int_0^1 u\theta dx \leq c_p\varepsilon_3 \int_0^1 u_x^2 dx + \frac{a_2^2\beta^2}{4J^2\varepsilon_3}\int_0^1 \theta^2 dx. \tag{5.29}$$

By substituting (5.28) - (5.29), we have

$$\begin{aligned} K_3'(t) \;\; &\leq \;\; \left(a_2^2\frac{\varepsilon_3}{2} + c_p\varepsilon_3 - \frac{a_2^2}{J}\right) \int_0^1 u_x^2 dx + \left(\frac{a_2^2}{\rho} + \frac{1}{2\varepsilon_3}\left(\frac{a_1}{\rho} - \frac{a_3}{J}\right)^2\right) \int_0^1 \varphi_x^2 dx \\ &\quad + \frac{a_2^2\beta^2}{4J^2\varepsilon_3}\int_0^1 \theta^2 dx. \end{aligned} \tag{5.30}$$

By taking $\varepsilon_3 = \dfrac{1}{J}$, we obtain (5.26). ■

Lemma 5.5 *Let (u, φ, θ, q) be the solution of (5.1)-(5.2). Then the functional*

$$K_4(t) := -\rho \int_0^1 u_t u dx \tag{5.31}$$

satisfies the estimate

$$K_4'(t) \leq -\rho \int_0^1 u_t^2 dx + 2a_1 \int_0^1 u_x^2 dx + \frac{a_3}{4} \int_0^1 \varphi_x^2 dx. \tag{5.32}$$

Proof. By differentiating $K_4(t)$ with respect to t, we obtain

$$K_4'(t) = -\rho \int_0^1 u_t^2 dx + a_1 \int_0^1 u_x^2 dx + a_2 \int_0^1 u_x \varphi_x dx. \tag{5.33}$$

Using Young's and Poincarè's inequality gives, for $\varepsilon_4 > 0$,

$$\begin{aligned} a_2 \int_0^1 u_x \varphi_x dx &\leq \frac{a_2^2}{a_3} \int_0^1 u_x^2 dx + \frac{a_3}{4} \int_0^1 \varphi_x^2 dx \\ &\leq a_1 \int_0^1 u_x^2 dx + \frac{a_3}{4} \int_0^1 \varphi_x^2 dx. \end{aligned} \tag{5.34}$$

By the fact that $a_1 \geq \dfrac{a_2^2}{a_3}$, we end up. ■

Next, we define a Lyapunov function L and show that it is equivalent to the energy functional E.

Lemma 5.6 *For N sufficiently large, the functional defined by*

$$L(t) := NE(t) + N_1 K_1(t) + N_2 K_2(t) + N_3 K_3(t) + K_4(t) \tag{5.35}$$

where N, N_1, N_2 are positive constants to be chosen appropriately later, satisfies

$$c_1 E(t) \leq L(t) \leq c_2 E(t), \ \forall t \geq 0 \tag{5.36}$$

for two positive constants c_1 and c_2.

Proof. Let

$$\pounds(t) := |L(t) - NE(t)| = N_1 K_1(t) + N_2 K_2(t) + N_3 K_3(t) + K_4(t).$$

By Young's, Cauchy-schwartz and Poincaré's inequalities, there exists a positive $\sigma > 0$ such that

$$|\pounds(t)| \leq \sigma E(t) \Leftrightarrow (N - \sigma) E(t) \leq L(t) \leq (N + \sigma) E(t), \ \forall t \geq 0.$$

Therefore, by taking $N > \sigma$, the proof is complete. ■

Proof. (Of Theorem 5.2)

By differentiating (5.35) and recalling (5.12), (5.16), (5.22), (5.26) and (5.32) we arrive at

$$
\begin{aligned}
L'(t) \leq & -\left[N_3\left(\frac{a_2^2}{2J}-\frac{c_p}{J}\right)-\frac{\alpha^2}{4}N_2-2a_1\right]\int_0^1 u_x^2dx-\left(\rho-\frac{a_0}{2}N_1\right)\int_0^1 u_t^2dx \\
& -\left[\frac{J\beta}{2}N_2-N_1\left(J+\frac{\rho^2a_2^2}{2a_0a_1^2}\right)\right]\int_0^1\varphi_t^2dx-\left(\frac{N}{k}-\frac{J}{\beta}N_2\right)\int_0^1 q^2dx \\
& -\left[\frac{a_0}{2}N_1-\frac{\alpha^2}{4}N_2-N_3\left(\frac{a_2^2}{\rho}+\frac{J}{2}\left(\frac{a_1}{\rho}-\frac{a_3}{J}\right)^2\right)-\frac{a_3}{4}\right]\int_0^1\varphi_x^2dx \\
& -\left[\gamma N-\frac{\beta^2}{2a_0}N_1-N_2\left(\frac{J\gamma^2}{\beta}+(a_3^2+a_2^2)+\alpha\beta\right)-\frac{a_2^2\beta^2}{4J}N_3\right]\int_0^1\theta^2dx.
\end{aligned}
\tag{5.37}
$$

At this point, we need to choose our constants very carefully. First, we choose N_1 enough such that

$$\rho-\frac{a_0}{2}N_1>0. \tag{5.38}$$

Once N_1 is fixed, we take N_2 large enough so that

$$\frac{J\beta}{2}N_2-N_1\left(J+\frac{\rho^2a_2^2}{2a_0a_1^2}\right)>0. \tag{5.39}$$

After that, we choose N_3 large enough such that

$$
\begin{cases}
N_3\left(\frac{a_2^2}{2J}-\frac{c_p}{J}\right)-\frac{\alpha^2}{4}N_2-2a_1>0 \\
\text{and} \\
\frac{a_0}{2}N_1-\frac{\alpha^2}{4}N_2-N_3\left(\frac{a_2^2}{\rho}+\frac{J}{2}\left(\frac{a_1}{\rho}-\frac{a_3}{J}\right)^2\right)-\frac{a_3}{4}>0.
\end{cases}
\tag{5.40}
$$

Finally, we choose N large enough so that

$$
\begin{cases}
\frac{N}{k}-\frac{J}{\beta}N_2>0, \\
\text{and} \\
\gamma N-\frac{\beta^2}{2a_0}N_1-N_2\left(\frac{J\gamma^2}{\beta}+(a_3^2+a_2^2)+\alpha\beta\right)-\frac{a_2^2\beta^2}{4J}N_3>0.
\end{cases}
\tag{5.41}
$$

Consequently, there exist a positive constant $\tilde{\alpha}$ such that

$$\frac{d}{dt}L(t)\leq-\tilde{\alpha}\int_0^1\left[u_t^2+\varphi_t^2+u_x^2+\varphi_x^2+\theta^2+q^2\right]dx. \tag{5.42}$$

On the other hand, by Young's inequality, we have

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^1 \left[\rho u_t^2 + J\varphi_t^2 + a_1 u_x^2 + a_3\varphi_x^2 + 2a_2 u_x\varphi_x + \alpha\theta^2 + \frac{\tau}{k}q^2\right] dx \\ &\leq \frac{1}{2}\int_0^1 \left[\rho u_t^2 + J\varphi_t^2 + (a_1+a_2)\, u_x^2 + (a_2+a_3)\,\varphi_x^2 + \alpha\theta^2 + \frac{\tau}{k}q^2\right] dx \\ &\leq \tilde{c}\int_0^1 \left[u_t^2 + \varphi_t^2 + u_x^2 + \varphi_x^2 + \theta^2 + q^2\right] dx \end{aligned} \tag{5.43}$$

where $\tilde{c} = \frac{1}{2}\max\left\{\rho,\ J,\ (a_1+a_2),\ (a_2+a_3),\ \alpha,\ \frac{\tau}{k}\right\}$.

Therefore, we deduce that there exist positive constant α_0 such that

$$L'(t) \leq -\alpha_0 E(t), \tag{5.44}$$

and, further, for some c_1, $c_2 > 0$, we have

$$c_1 E(t) \leq L(t) \leq c_2 E(t),\ \forall t \geq 0. \tag{5.45}$$

A Combining (5.44) and the right-hand side of (5.45), we conclude that

$$L'(t) \leq -\alpha_1 L(t),\ \forall t \geq 0 \tag{5.46}$$

where $\alpha_1 = \frac{\alpha_0}{c_2}$.

A simple integration of (5.46) over $(0,t)$ leads to

$$L(t) \leq L(0)\, e^{-\alpha_1 t}, \quad \forall t \geq 0. \tag{5.47}$$

Finally, by combining (5.45) and (5.47) we obtain (5.14). ∎

Conclusion

In this thesis, we studied the well posedness and the stability of some linear one-dimensional porous-elastic systems. The first is a porous-thermoelastic system with second sound and a distributed delay term acting on the transverse displacement, where the heat flux of the system is governed by Cattaneo's law. The second is a porous-elastic system with microtemperatures and varying delay term, and the last is a swelling porous thermoelastic soils mixture with second sound, where the thermal conduction is given by the theory of Green and Naghdi called thermoelasticity type III.

Under suitable assumptions, we have proved the well-posedness of the systems by using semigroups theory. For the stability of these systems, we used a multipliers technique which is based on the construction of a Lyapunov functional equivalent to energy.

We intend in the future to generalize our results to viscoelasticity problems, in addition it will be interesting to make numerical simulations of the different problems studied in this thesis.

Bibliography

[1] R. A. Adams, John J.F. Fournier; Sobolev spaces, Elsevier 2003.

[2] M. S. Alves, M. V. Ferreira, J. E. Munoz Rivera, O. Vera Villagran; Stability of a thermoelasltic mixture with second sound. Math and Mech of Solids. Phys, 2018: 1-15.

[3] K. Ammari ,S. Nicaise, C. Pignotti; Feedback boundary stabilization of wave equations with interior delay. Systems. Cont. Letters. 2010(59): 623-628.

[4] T. A. Apalara; Well-posedness and exponential stability for a linear damped Timoshenko system with second sound and internal distributed delay. Elect J. Differ. Eqns., 2014; (254): 1-15.

[5] T. A. Apalara, S. A. Messaoudi; An Exponential Stability Result of a Timoshenko System with Thermoelasticity with Second Sound and in the Presence of Delay. Journal Applied Mathematics and Optimization, 2015, 71 (3): 449-472.

[6] T. A. Apalara; Uniform decay in weakly dissipative Timoshenko system with internal distributed delay feedbacks. Acta Math Scient, 2016; 36B(3): 815-830.

[7] T. A. Apalara; On the stability of porous-elastic system with microtemperatures, J. Thermal Stresses, 2018, Doi: 10.1080/01495739.2018.1486688.

[8] T. A. Apalara; General stability result of swelling porous elasltic soils with a viscoelastic damping. Z. Angew. Math. Phys, 2020 (71): 200.

[9] L. Bouzatouta; Existence and Asymptotic behavior for damped Bresse system. A Doctoral Thesis, Annaba univ, 2020.

[10] L. Bouzettouta, S. Zitouni, Kh. Zennir, H. Sissaoui; Well-posedness and decay of solutions to Bresse system with internal distributed delay. Int. J. Appl. Math. Stat., 2017; (56): 153–168.

[11] L. Bouzettouta, A. Djebabla; Exponential stabilization of the full von Kármán beam by a thermal effect and a frictional damping and distributed delay. J. Math. Phys, 2019; 60: 041506.

[12] E. F. Borges, M. L. Santos; On porous-elastic system with a time-varying delay term in the internal feedbacks. J. Appl Math & Mech, 2020, Doi: 10.1002/zamm.2018.00247.

[13] H. Brezis; Functional analysis, Sobolev spaces and partial differential equations. Springer, 2010.

[14] T. Caraballo, J. Real, L. Shaikhet; Method of Lyapunov functionals construction in stability of delay evolution equations. J. Math. Anal. Appl. 2007; 334(2): 1130–1145.

[15] P. S. Casas, R. Quintanilla; Exponential stability in thermoelasticity with microtemperatures. Int J Eng Sci. 2005(45):33-47.

[16] P. S. Casas, R. Quintanilla; Exponential decay in one-dimensional porous-thermoelasticity. Mech Res Commun. 2005;(6):652-659.

[17] M. A. Goodman, S. C. Cowin; A continuum theory for granular materials. Arch Ration Mech Anal. 1972;44(4):249–266.

[18] S. C. Cowin, J. W. Nunziato; Linear elastic materials with voids, J. Elast., 13(2): 125–147 (1983).

[19] S. C. Cowin; The viscoelastic behavior of linear elastic materials with voids, J. Elast., 15(2): 185–191 (1985).

[20] A. C. Eringen; A continuum theory of swelling porous elastic soils. International Journal of Engineering Science, 1994 (32): 1337-1349.

[21] A. Fareh, S. A. Messaoudi; Energy decay for a porous thermoelastic system with thermoelasticity of second sound and with a non-necessary positive definite energy. Appl Math Comput. 2017; 293: 493–507.

[22] A. Fareh, S. A. Messaoudi; Energy decay for a porous thermoelastic system with thermoelasticity of second sound and with a non-necessary positive definite energy. Appl Math Comput. 2017; 293: 493–507.

[23] F. Foughali, S. Zitouni, H. E. Khouchemane, A. Djebabla; Well-posedness and exponential decay for a porous-thermoelastic system with second sound and distributed delay .J. MESA. Vol. 11, No. 4, 2020: 1003-1020.

[24] J. Hao, P. Wang; Exponential decay of solution to the viscoelastic porous-thermoelastic system of type III with boundary time-varying delay. Math Meth in the Appl Scien, 2015, Doi: 10.1002/mma.3813.

[25] F. L. Huang; Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces, Ann. Differential Equations 1, no. 1, 1985: 43-56.

[26] S. Grace; Oscillation criteria for third order nonlinear delay differential equations with damping. Opusc. Math. 2015; 35: 485–497.

[27] D. Ieşan; A theory of thermoelastic matherials with voids. Acta Mech. 60 (1986): 67-89.

[28] D. Ieşan; On the theory of micromorphic elastic solids with microtemperature. J. Thermal stresses 24 (2001): 737-752.

[29] D. Ieşan, and R. Quintanilla; On thermoelastic bodies with inner structure and microtemperatures, J. Math. Anal. Appl. 354 (2009): 12-23

[30] G. Li, Y. Luan, W. Liu; Well-posedness and exponential stability of a thermoelastic Bress system with second sound and delay. Hacettepe J Math & Stat, 2019, 10.15672/HJMS.xx: 1-16.

[31] M. M. A. Gharabli, S. A. Messaoudi ; Well posedness and general decay for a non-linear damped porous thermoelastic system with second sound. Georgian Math J. Forthcoming.

[32] A. E. Green, P. M. Naghdi; Thermoelasticity without energy dissipation. Elasticity, 1993(31): 189–208.

[33] L. Gongwei; Well-posedness and exponential decay of solutions for a transmission problem with distributed delay. 2017; (174): 1-13.

[34] R. Grot; Thermodynamics of continuum with microstructure, Int. J. Eng. Sci. 7 (1969): 801-814.

[35] J. Hao, P. Wang; Exponential decay of solution to the viscoelastic porous thermoelastic system of type III with boundary time-varying delay. Math. Methods Appl. Sci. 2016; 39(13): 3659–3668.

[36] M. Houasni, S. Zitouni, R. Amiar; General decay for a viscoelastic damped Timoshenko system of second sound with distributed delay. Math. Engin. Scien. Aero., 2019; Vol 10, No. 2: 323-340.

[37] T. Kato; Linear and quasi-linear equations of evolution of hyperbolic type, In: Hyperbolicity, pp.125-191, C.I.M.E. Summer Sch., 72, Springer, Heidelberg, 2011.

[38] H. D. Khochemane, L. Bouzettouta, A. Guerouah; Exponential decay and well-posedness for a one dimentional porous-elastic system with distribution delay. Applicable Analysis, 2019; 1703958, DOI: 10.1080/000386811.

[39] M. Kirane, B. Said-Houari, A. M. Naim; Stability result for the Timoshenko system with a time-varying delay term in the internal feedbacks. Applied Analysis, vol. 10, no. 2, pp. 667-686, 2011, DOI: 10.3934/cpaa.2011.10.667.

[40] W. Liu, M. Chen; Well posedness and exponential decay for a porous thermoelastic system with second sound and a time-varying delay term in the internal feedback. Termo-dyn, 2017; doi 10.1007/s00161-017-05566z.

[41] S. A. Messaoudi, A. Fareh; Exponential decay for linear damped porous thermoelastic systems with second sound. Discrete Contin Dyn Syst Ser B. 2015;20(2):599–611.

[42] S. A. Messaoudi, A. Fareh; Exponential decay for linear damped porous thermoelastic systems with second sound. Discrete Contin Dyn. Syst. Ser, 2015; B 20(2): 599-612.

[43] S. A. Messaoudi, A. Fareh; General decay for a porous-thermoelastic system with memory: the case of nonequal speeds. Acta Mathematica Scientia, 2013, 33B(1): 23-40.

[44] S. A. Messaoudi, A. Fareh; Exponential decay for linear damped porous thermoelastic systems with second sound. Discrete Continuous Dyn. Syst. Ser, 2015; B 20(2): 599-612.

[45] M. I. Mustafa; Asymptotic behavior of second sound thermoelasticity with internal time-varying delay. Zeitschrift für angewandte Mathematik und Physik 2013; 64:1353–1362.

[46] M. Mustafa; A uniform stability result for thermoelasticity of type III with boundary distributed delay. J Math. Anal. Appl., 2014; (415): 148-158.

[47] A.E. Green, P.E. Naghdi; A re-examination of the basic postulates of thermomechanics. Proc. Royal Society London A 432 (1991): 174-194.

[48] A.E. Green, P.E. Naghdi; Thermoelasticity without energy dissipassion. J. Elasticity 91(1993): 189-208.

[49] S. Nicaise, C. Pignotti; Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks. SIAM J. Control Optim. (2006; 45(5): 1561-1585.

[50] S. Nicaise, C. Pignotti; Stabilization of the wave equation with boundary or internal distributed delay. Di. Int. Equs, 2008; 21(9-10): 935-958.

[51] S. Nicaise, C. Pignotti, J. Valein; Exponential stability of the wave equation with boundary time-varying delay. Discrete Continuous Dyn.Syst. Ser. 2011 , S 4(3):693-722.

[52] S. Nicaise, C. Pignotti; Interior feedback stabilization of wave equations with time dependent delay. Electron. J. Differ. Equ. 2011(41): 01-20.

[53] Ö. Öztürk, E. Akmon; Nonoscillatory solutions of two dimensional nonlinear delay dynamical systems. Opusc. Math. 2016; 36: 651–669.

[54] A. Pazy; Semigroups of Linear Operator and Applications to Partial Differential Equations. Springer, 1983.

[55] J. Prüss; On the spectrum of C_0-semigroups, Trans. Amer. Math. Soc. no. 2, 1984(284): 847-857.

[56] R. Quintanilla; Slow decay for one-dimensional porous dissipation elasticity. Appl. Math. Lett. 16(4): 487–491 (2003).

[57] R. Quintanilla; Exponential stability for one-dimensional problem of swelling porous elastic soils with fluid saturation. J. of Comp. and App. Math, 2002; (145): 525-533.

[58] R. Racke; Instability of coupled systems with delay. Commun. Pure Appl. Anal. 2012; 11: 1753–1773.

[59] A. J. A; Ramos, D. S. Almeida Jr, M. M. Freitas, A. S. Noé, M. J. Dos Santos; Stabilization of swelling porous elastic soils with fluid saturation and delay time terms. AIP Journal of Mathematical Physics. 2021 (62): 021507.

[60] M. L. Santos, A; D; S. Campelo, D. S. J. Almeida; On the Decay Rates of Porous thermo-elastic systems. J Elast. 2017(127): 79-101.

[61] M. Swann, Y. Chitour, P. Christophe; Stability analysis of dissipative systems subject to nonlinear damping via Lyapunov techniques. 2020, 65(5): 2139-2146.

[62] T. A. A. Apalara; On the stability of some systems of thermoelasticity of type III. A Doctoral Thesis, King Fahd Univ, 2003.

[63] J. M. Wang, B. Z. Guo; On the stability of swelling porous elastic soils with fluid saturation by one internal damping. IMA Journal of Applied Mathematics. 2006 (71): 565-582.

[64] S. T. Wu; Asymptotic behavior for a viscoelastic wave equation with a delay term. Taiwanese J. Math. (2013); 17(3), 765–784.

[65] C. Q. Xu, S. P. Yung, L. K. Li; Stabilization of the wave system with input delay in the boundary control. ESAIM: Control Optim. Calc. Var., 2006(12): 770-785.

[66] S. Zheng, Z. Liu; Semigroups Associated with Dissipative Systems. p.224. Chapman Hall/CRC, Boca Raton 1999.

[67] S. Zitouni,L. Bouzetouta, KH. Zennir, D. Ouchnane; Exponential decay of thermo-elastic Bresse system with distributed delay term. Hacettepe J Math & Stat, 2018; 47(5): 1-15.

[68] S. Zitouni, A. Ardjouni, Kh. Zennir, R. Amiar; Existence and exponential stability of solutions transmission system with varying delay in $\mathbb{R}$. Math. Mor., 2016; Vol. 20(2): 143-161.

[69] S. Zitouni, A. Ardjouni, Kh. Zennir, R. Amiar; Existence and stability of a damped wave equation with two delayed terms in internal feedback. ROMAI J., v.13, no.2(2017):143-163.

[70] S. Zitouni, Kh. Zennir, L. Bouzettouta; Uniform decay for a viscoelastic wave equation with density and time varying delay in $\mathbb{R}^n$. FILOMAT 33, 3(2019): 961-970.