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ملخَّص

في هذه الأطروحة، قمنا بدراسة بعض نماذج المعادلات التفاضلية الجزئية، وبصورة أدق، عالجنا بعض الجمل المقترنة القطعية - القطع المكافئ. من خلال هذه النماذج أخذنا في الاعتبار المشكلات التي تم تناولها و بناءً على ذلك تمكنا من حل مشاكل الوجود والتفرد من جهة، وسلوك الحلول من حيث الاستقرار من جهة أخرى.

في الفصل الأول ، عملنا على دراسة نظام مسامي مرن مع الأخذ في الاعتبار تأثيرات الحرارة المحلية والجزئية حيث يتم فيها انتقال التدفق الحراري عن طريق فورييه.

في الفصل الثاني ، عالجنا نموذج فون كارمان الزائدي إلى جانب تأثيرات حرارة الصوت الثانية حيث ينتج التوصيل عن قانون كاتانيو.

في الفصل الثالث ، درسنا نموذج تشقق عارضة مرنة من نوع تيموشنكو حيث يأخذ بعين الاعتبار وجود انتقال حراري كسري وكذلك تاثيرات اللزوجة الكسرية.

في الفصل الرابع ، يتعلق الامر بحل مشكلة تندرج في إطار نظرية المرونة الحرارية غير الكلاسيكية. حيث تم التعامل مع نموذج فون كارمان الميكانيكي مقترنًا بمعادلة الحرارة التي قدمها قانون جورتن- بيبكين.

الكلمات المفتاحية

انظمة مقترنة, طريقة ليابونوف, انظمة مشتّتة, الوجود الشّامل, نصف زمرة التّقلّصات, الاستقرار, تقنية المضاعف.

Abstract

In this thesis, we study some models of partial differential equations, and more precisely we have treated some hyperbolic-parabolic coupled systems. Through these models, we have taken into consideration problems that had been tackled based on the resolution of the problems of existence and uniqueness on the one hand, and the behaviour of the solutions in terms of stability on the other hand.

In the first chapter, we analysed a porous-elastic system with consideration of the effects of local and micro-local heat, which, by means of flow transfer, results in Fourier's law.

In the second chapter, we treated von Karman's hyperbolic model coupled with the effects of second sound heat where conduction results from Cattaneo's law.

In the third chapter, we studied a model that interprets the shear of a Timoshenko type beam and takes into account the existence of a Gurtin-Pipkin type fractional heat transfer.

In the fourth chapter, the model consists in the study of a problem which falls within the framework of the theory of non-classical thermoelasticity. This problem is presented by von Karman's mechanical model coupled with the heat equation introduced by Gurtin-Pipkin's law.

Keywords

Coupled systems; Lyapunov method; Dissipative systems; Global existence; Contraction semigroup; Stability; Multiplier technique.

Résumé

Dans cette thèse, nous étudions quelques modèles d'équations aux dérivées partielles, et plus précisément nous avons traité certains systèmes couplés hyperboliques-paraboliques. A travers ces modèles, nous avons pris en considération des problèmes qu'on avait abordés se basant ainsi sur la résolution des problèmes d'existence et d'unicité d'une part, et le comportement des solutions en termes de stabilité d'autre part.

Dans le premier chapitre, nous avons analyse un system poreux-élastique avec la prise en compte des effets de chaleur local et micro-local qui, par un transfert de flux abouti à la loi de Fourier.

Dans le deuxième chapitre, nous avons traité le modèle hyperbolique de von Karman couplé avec les effets de chaleur deuxième son ou la conduction est résultante de la loi de Cattaneo.

Dans le troisième chapitre, il s'agit de l'étude d'un modèle qui interprète le cisaillement d'une poutre de type Timoshenko et qui tient compte de l'existence d'un transfert de chaleur fractionnaire de type Gurtin-Pipkin.

Dans le quatrième chapitre, le modèle consiste à l'étude d'un problème qui entre dans le cadre de la théorie de la thermo élasticité non classique. Ce problème est présenté par le modèle mécanique de von Karman couplé avec l'équation de la chaleur introduite par la loi de Gurtin-Pipkin.

Mots-clés .

Systèmes couplés; Méthode de Lyapounov; Systèmes dissipative; Existence globale; Semi-groupe de contraction; Stabilité; Technique du multiplicateur.

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Contents

1

Introduction

Contents

In this thesis, we study some models of coupled hyperbolic-parabolic systems, on which we treated a series of problems. We can give an overview of the contents of chapter (1). So, we will therefore present all the problems we have dealt with in the next section. Then, for reasons of motivation, we present the important review in the literature. Next, we show the basic models used to derive our models. Afterwards, the objectives of this work and also the methodology used were presented. Finally, we present the main lines of the organization of the thesis.

In chapter (2), we will present some essential preliminaries.

1.1 Problematic

Here we are in the process of presenting models and highlighting the problems related to them that have been studied.

In chapter (3), we were interested in studying the following system

$$\begin{aligned}
&\rho u_{tt} = \mu u_{xx} + b\varphi_x - \gamma\theta_x,\\
&J\varphi_{tt} = \delta\varphi_{xx} - bu_x - \xi\varphi - dw_x + m\theta - \beta\varphi_t,\\
&c\theta_t = -\gamma u_{tx} - m\varphi_t - k_1 w_x,\\
&aw_t = k_2 w_{xx} - k_3 w - k_1\theta_x - d\varphi_{tx}.
\end{aligned} \tag{1.1}$$

This system is extremely important from a mathematical and physical perspective, so we note its presence in the priorities of mathematicians for the purpose of developing several areas, including the mechanics of materials and theories of solids. The aforementioned system models the vibration or displacement of a porous-elastic solid with thermal effects in addition to a porous damping. Here the thermal effects are taken through its diffusion in the parts of the local matter, as well as for the very fine parts of the material, so we call it the microtemperature.

We note that in the theory of elastic materials with voids they linked the behavior of the solution to the coupling types (and / or) dissipation mechanisms, the different studies are summarized by the following diagram This diagram can be used in such manner, that is, if we take

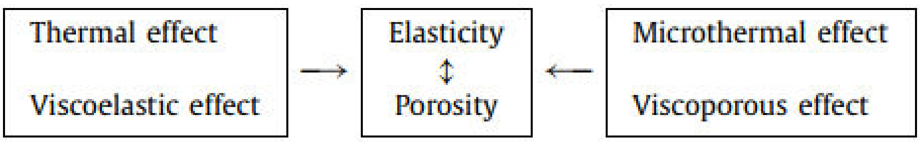


Figure 1.1 – stability shema

at the same time an effect of the right box and another of the left box, so we get the exponential stability. But, if we consider two effects which take place at the same time of a single box, we obtain then a slow decay i.e., the polynomial stability.

In our work, we took the zero thermal conductivity in addition to the presence of porous damping. On the one hand this suggestion is very important because it is contained in several physical issues and on the other hand, we notice a clear difference with the procedures that can be adopted by looking at the schematic diagram. Therefore, the presented problem that requires study is to prove the existence of solutions to the system and to show their behavior, i.e. stability of the solutions, where they can be either exponentially stable or polynomial stable. In some phenomena, work is done to achieve the exponential stability of the systems, as this translates into the presence of very microdeformations in the material that are almost neglected and this is what makes them coherent well, in other phenomena, the polynomial stabilization is the most important so that it can allow a longer period for the cohesion of the materials, but in the presence of ineffective deformations.

In chapter (4), we were interested in studying the following systems

$$\begin{aligned}
&w_{tt}+\gamma_1 w_t-d_1\left[\left(u_x+\frac{1}{2}\left(w_x\right)^2\right)w_x\right]_x+d_2 w_{xxxx}=0,\\
&u_{tt}-d_1\left[\left(u_x+\frac{1}{2}\left(w_x\right)^2\right)\right]_x+\delta\theta_x=0,\\
&\theta_t+q_x+\delta u_{tx}=0,\\
&q_t+\gamma_2 q+\theta_x=0.
\end{aligned}\tag{1.2}$$

Here, we will deal with a non-linear type of thermo-mechanical system, so we can describe the system that models the longitudinal and transversal vibrations of nonlinear displacement of elastic solid, and this is followed by the presence of thermal effects guaranteed by the heat transfer according to the second sound law. It can also be called Cattaneo's law. This heat is released as a result of the previously mentioned displacements, and this is explained by the presence of heat dissipation. As for the approved damping, the friction damping has been taken into account, as it helps in reducing the distorted vibrations of the displacement. Based on the approved principle, we can consider at least two dissipations, one of which affects the mechanical displacement, whether longitudinal or transverse, and the other satisfies the thermal effects. In view of the complete absence of the use of thermal effects of Cattaneo type on this mechanical systems and because we realize that each heat flow has a special effect on the stability. So we used this effect with the presence of a mechanical damper and that to solve some of the presented problems which are represented in proving the existence of solutions as well as knowing the type of stability produced.

In chapter (5), we were interested in studying the following system

$$\begin{aligned}
&\rho_1\varphi_{tt} - \kappa(\varphi_{xx} + \psi_x) = 0, \\
&\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) - \int_0^\infty h(s)A^\sigma\psi(t-s)ds + \delta\theta_x = 0, \\
&\rho_3\theta_t + \frac{1}{\beta}\int_0^\infty g(s)A^\sigma\theta(t-s)ds + \delta\psi_{tx} = 0,
\end{aligned} \tag{1.3}$$

The first and second coupled equations of the system are called the Timoshenko system, while the third equation represents the heat equation. Due to the great importance of linear mechanical systems associated with the heat equation, we dealt in our work to study the system (1.3) which models the transverse displacement and shear angle vibration of the elastic rigid beams in the presence of partial thermal spatial effects of the Gurtin-Pipkin type, more precisely this heat is produced due to the different vibrations. This type of heat flux has been taken into account, which can be derived from the non-classical law of thermal elasticity. Also, we can observe a partial memory affecting the shear angle. Studying this type of model is very important. Therefore, we can summarize the problems presented for this model that require study are proving the existence of solutions, and showing the type of resulting stability and its relationship to the number of stability as well as the fractional variable.

In chapter (6), we were interested in studying the following system

$$\begin{aligned}
&w_{tt} - d_1\left[\left(u_x + \frac{1}{2}(w_x)^2\right)w_x\right]_x + d_2 w_{xxxx} + \alpha w_t = 0, \\
&u_{tt} - d_1\left[\left(u_x + \frac{1}{2}(w_x)^2\right)\right]_x + \delta\theta_x = 0, \\
&\theta_t - \frac{1}{\beta}\int_0^\infty g(s)\theta_{xx}(t-s)ds + \delta u_{tx} = 0.
\end{aligned} \tag{1.4}$$

The system allows the modeling of both longitudinal and transverse displacements of non-linear beams or plates, whereby due to the vibrations they create a thermal transfer according to the Gurtin-Pipkin law. With regard to these mechanical systems combined with heat, we note that the thermal effects are sometimes ineffective in working on the stability of non-linear mechanical systems, except with the presence of auxiliary mechanical dampers. Both mechanical dampers and thermal effects absorb unwanted vibrations and thus provide stability to the beam. We have worked on solving the problem of having solutions as well as showing what kind of stability the model achieves.

Finally, in chapter (7), for more clarifications, we were able to obtain a comprehensive conclusion, that contained some results abridgement and perspectives. We have shown, through the problems raised earlier, some valid questions that require answers. Therefore, this can be found in the chapters presented below.

In the next section, we will present the most important research work that we have relied on.

1.2 Motivations

The topic of our research is of importance and a direct link to various broad scientific fields e.g., in the theoretical and applied physics for controlling the engineering of materials and the recognition of their different properties, and in biology for being able to observe and control some biological fluorescence. We also see it in the mechanics of controlling the beams, arch beams, and plate. Indeed, scientific phenomena can be treated with mathematical modeling through which the corresponding model can be obtained. We can say that the problems that we were interested in treating represented in elasticity/ or thermoelasticity problems of various kinds, whether they were related to materials and their properties, or were related to mechanical systems. It is noticeable that these problems have attracted the attention of researchers from various fields, as previously mentioned.

Through extensive scientific research, we can distinguish types of scientific research in which scientists have dealt with the behavior of temporal decay of systems solutions. e.g., in the one-dimensional case it is certain and which has been shown that the interaction between thermal and mechanical fields leads to the exponential decay of the solution. Hence, the question arises in our minds about the nature of the temporal behavior of solutions when we think about another type of coupling and / or dissipation mechanisms. In our discussion of the problems of elastic materials and their properties through which elastic solids with voids have been discussed and studied, in this context it is worth noting the work of Cowin and Nunziato [1, 2] and Cowin [3] on the theory of porous elastic materials. In simple terms, this can be said to be one of the theories that take into account the inner structure of an body as it has been extensively researched in the theory of void matter in recent years.

Now, for mechanical models, looking at our research, a basic model developed earlier by Timoshenko [4, 5] caught our attention, which depended on the transverse displacement of the beam as well as the rotation angle of the beam strings. Many researchers have been interested in studying the system, and various damping mechanisms have been used to stabilize the vibrations of this system. The results obtained showed that the presence of dissipation for both equations leads to regular stability (exponential or polynomial), this without addressing the values of the constants in the model. This has been explained by Kim and Renardy [6], Feng et al. [7], Raposo et al. [8], Santos [9], Messaoudi and Mustafa [10] and others. However, it was found that stability in the case of only one effective damping in this respect may depend on the values of the system constants. Specifically, if the wave propagation velocities are equal, then stability can be obtained for weak solutions. This was demonstrated by Soufyane and Wehbe [11], Ammar-Khodja et al. [12], Guesmia and Messaoudi [13, 14], Messaoudi and Mustafa [15, 16] and Messaoudi et al. [17], Fernández Sare and Rivera [18], Messaoudi and Said-Houari [19], Rivera and Racke [20,21], Mustafa and Messaoudi [22]. In the opposite case, some researchers have shown that a weaker dissolution rate is obtained for more uniform solutions. In this regard, we quote, among other things, the works of Fernández Sare and Rivera [18], Messaoudi and Said-Houari [19].

In addition to the various influences, for example thermal effects and viscosity effects, we will highlight the works Ieşan [23], Ieşan and Quintanilla [24] through which they had extensive research contributions. These theories are directly related to the problems of elastic materials in mechanical systems, where it must be said that Grot [25] has developed a theory in the thermodynamics of elastic materials with microstructures whose microelements, in addition to microdeformations, possess microtemperatures. Indeed, this theory depends on the continuum mechanics of micromorphic materials, and as a result this can be explained by the fact that the microelements may undergo homogeneous deformations called microdeformations. This link between the theories raised many questions that have culminated in scientific publications working to develop the theory of nanomaterial mechanics on a large scale. We refer the works of Eringen [26, 27], Eringen and Kafadar [28], and we can also recall the contributions of Ieşan [29, 30], Ieşan, and Quintanilla [31]. Historical developments on the subject, as well as references to various contributions can be found in [32]. The mathematical results that were based on the study of displacement models of porous-elastic materials are adopted by many researchers from various scientific fields, so looking at the work [33] that dealt with a one-dimensional system, we conclude through temporal decay analysis that the dissipation due to porous viscosity is not It is sufficient to ensure the exponential stability of the solutions. Likewise, with regard to thermal effects, through the work of Casas and Quintanilla [34] it is clear to us that temperature does not give exponential stability. However, they also showed in the same work that a combination of porous viscosity with thermal effects does indeed produce it.

With regard to stabilization through the thermal effect of Timoshenko systems, Rivera and Racke [35] considered the Timoshenko system whose equations are the beam displacement, rotation angle and difference in temperature. Under the appropriate conditions for the constants of the linear system, they demonstrated many results of exponential decay and non-exponential stability in the case of the different wave velocities of the system. For the thermal effect the heat flux has been given according to the Fourier law. As a result, this theory predicts an infinite speed of heat propagation. That is any thermal disturbance at one point has an instantaneous effect elsewhere in the body. Experiments showed that heat conduction in some dielectric crystals at low temperatures is free of this paradox and disturbances, which are almost entirely thermal, propagate in a finite speed. To overcome this material contradiction, several theories such as thermoelasticity by sound second or thermoelasticity by type III have been merged. For the background related to this theory, we refer the reader to Green and Naghdi [36, 37] and Chandrasekharaiah Review Paper [38]. By relying on thermal diffusion according to the previous laws, the coupled Timoshenko system have been considered, and this has been put forward by a number of authors. Please see the work of Djebabla and Tatar [39–41]. Messaoudi and Said-Houari [42]. Messaoudi and Fareh [43]. Kafini et al. [44, 45] and Kafini [46] also Fatori et al. [47] .

As for the existence of the microtemperature effects, through the article [34] we can say that the coupling temperature and microtemperature effects lead to the exponential stability of the system. Some of these results have recently been extended to generalized thermal elasticity and we refer in this context to works [48, 49]. It is known that the systems to be studied need a dissipative mechanisms. Therefore, by talking about the dissipative mechanisms that help to dampen the systems, it has been discussed in many contributions [34, 34, 49, 50] where many results were extracted enabling us to remember the main conclusions with the help of scheme (Fig. (1.1)). The work with the previous scheme can be translated by taking one effect from the right square and another from the left square, to obtain exponential stability. Or by taking into account two simultaneous effects from one square only, to obtain slow decay (see for example works [49, 50])). Indeed, in this direction it is noteworthy to consider the work [51] where it has been demonstrated that some of the models studied decay polynomially with rates of decay that depends on the regularity of the initial data. Which means and explains that the decay can be very slow provided the initial data is not regular. In addition to the previously mentioned dissipation mechanisms, we also point out that some other dissipation mechanisms (linear boundary feedback or memory type dissipation) have been recently considered (see for example [52, 53]). Now, by looking at the one-dimensional systems related to the theory of elastic solids with voids presented by Nunziato and Cowin in work [2], we can rely on Ieşan's works [23, 29, 32, 54] where he was able to add temperature and precise temperatures also to the linear theory given in the article [30]. It should also be noted recent contributions [55–57] related to three-dimensional thermal problems.

For the stabilization of one-dimensional and multi-dimensional von Karman system, one can refer to Horn and Lasiecka [58, 59], Lasiecka [60, 61] and so on. The study of elastic solid materials is insufficient to determine the characteristics of these materials, (see e.g. [62, 63]), to better understand and satisfy the complete and definitive study it is necessary to introduce the models of linear thermo elastic plates (coupling of the plate and the heat) and the standard linear thermoelastic system (coupling between wave and heat equations). It is known in the literature that models have different properties, in this sense for thermoelastic models one can dampen unwanted vibrations with control of an exponential function but only in certain domains, on the other hand, for the model which results in a coupling of the equation of the plate and the equation of the heat this model is always exponentially stable. In the linear case, D. B. Henry et al. [64] proved that the coupled thermoelastic system is equivalent to that of the decoupled system, so the exponential stability is proven. In this context, Assia Benabdallah et al. [65] studied a von Karman one-dimensional model with thermal effects, where they have derived equations by constituting the new mathematical model, then they determined the new model by a coupling of the parabolic equations modeled according to the classical Fourier law, and finally they proved the existence and the uniqueness of a global solution as well as the exponential stability. Liu et al. [66] studied a one-dimensional full von Karman beam with a thermo-viscoelastic damping, frictional dampings and a delay term in the internal feedback, and proved the well-posedness and general decay result of the system. Bouzettouta and Abdelhak [67] extended the results to the case of the system with distributed delay. For multi-dimensional case, we mention the contribution of Lasiecka [68].

1.3 Background

In this section, we will begin by providing a historical overview of the field of research, as well as presenting the basic models that have been adopted to derive the models studied in the next thesis chapters.

1.3.1 Classical Thermoelasticity

In the theory of thermoelasticity, it was Duhamel who founded for the first time in 1838 the equations of the deformation in an elastic body with temperature gradients, subsequently in 1841 Neumann found the same results. However, this theory was based on the independence of thermal and mechanical effects. Regarding the total strain, it was determined by superimposing that the elastic strain and thermal expansion are caused by the temperature distribution only, which means that this theory therefore did not describe the motion associated with the state thermal, nor did not therefore include the interaction between the strain and the temperature distributions. Therefore, thermodynamic arguments were needed. Thus, in 1857 Thomson used the laws of thermodynamics to determine the stresses and strains in an elastic body in response to varying temperatures. As for Landau and Lifshitz in 1953 developed the classical methods of thermodynamics for the derivation of the coupled equations of thermoelasticity. The thermoelastic equations describe the behavior of a elastic, heat-conducting body. In the classical model, the hyperbolic elastic system is combined with the classical equivalent model of thermal conductivity. This leads to a parabolic coupling system. Thermoelasticity also describes the interactions between elastic stresses and temperature differences. When the proposed materials are present in the linear case of a homogeneous and isotropic medium with a zero of the external body forces and a zero of the external heat, and a Fourier law for the conduction of the internal heat, we can introduce the coupled system as follows

$$\begin{aligned} &u_{tt} - \alpha u_{xx} + \gamma_1 \theta_x = 0, \quad \text{in } \mathbb{R}_+ \times I, \\ &\theta_t - \kappa \theta_{xx} + \gamma_2 u_{tx} = 0, \quad \text{in } \mathbb{R}_+ \times I \end{aligned} \tag{1.5}$$

and in $3D$ by

$$\begin{aligned} &v_{tt} - \mu \Delta v - (\mu + \lambda) \nabla \nabla^T v + \gamma_1 \nabla \theta = 0, \quad \text{in } \mathbb{R}_+ \times \Omega, \\ &\theta_t - \kappa \Delta \theta + \gamma_2 \nabla^T v_t = 0, \quad \text{in } \mathbb{R}_+ \times \Omega. \end{aligned} \tag{1.6}$$

The unknows $u := u(t,x) \in \mathbb{R}, v := v(t,x) \in \mathbb{R}^3$ and $\theta := \theta(t,x) \in \mathbb{R}$ denote the elastic displacement and the temperature difference to the equilibrium state, respectively. Physical properties of the underlying isotropic medium are described by the thermal conductivity $\kappa > 0$, the elasticity modules $\alpha > 0$ or μ and λ with $\mu,\ \lambda + 2\mu > 0$ and the thermoelastic coupling coefficients γ_1 and γ_2 with $\gamma_1\gamma_2 > 0$. The derivation of the classical thermoelasticity model is based on Fourier's law of heat conduction, i.e., the heat flux q is assumed to be proportional to the temperature gradient

$$q = -\kappa \nabla \theta. \tag{1.7}$$

That implies that the heat equation for the coupled theory is a parabolic one, giving rise to the unphysical property that if a sudden change of temperature is made at some point of the heat-conducting body, it will be felt instantly everywhere, though with exponentially small amplitudes at distant points. Hence, we observe an infinite propagation speed of thermal disturbances. Moreover, the temperature of a body is the macroscopic consequence of certain kinds of vibratory motions. Heat is transported by near-neighbor excitation in which changes of momentum and energy on a microscopic scale are propagated as waves.

1.3.2 Porous-thermoelasticity

From the arrival in 1972 of Goodman and Cowen through their research they were able to introduce the theory of porous materials, defeating the classical theory described by the deformation resulting from a contribution of microstructure. Regarding the above theory, we can say that there is another equally applicable theory developed by Goodeman and Cowin based on granular materials valid for porous materials. They introduced higher order stress and body force to account for the energy flow and energy input associated with the time rate of volume fraction. Terms of this type are also contained in the higher order elasticity theories developed by Mindlin, Toupin and Green and Rivlin in 1964. Nunziato and Cowin in 1979 used the same equilibrium equations developed by Goodman and Cowin and have presented a nonlinear theory for the behavior of porous solids. It was only after two years (1981) that Slemrod proposed a first study on thermoelastic coupling. Consequently, we have seen that in the unidimensional case the solutions decay exponentially. Since then, many problems have been studied considering different dissipation mechanisms at the microscopic and / or macroscopic level. In the year 1983, Cowin and Nunziato developed the theory of linear elastic materials with voids to mathematically study the mechanical behavior of porous solids. An extension of this theory to linear thermoelastic bodies was proposed by Iecsan 1986. Moreover, in 2001, he added the elements of microtemperature to this theory.

1.3.2.A Porous-elasticity

The theory of porous materials is an important generalization of the classical theory of elasticity for the treatment of porous solids in which the skeletal materials is thermoelastic and the interstices are void of material. This theory deals with materials containing small pores or voids. The basic premise underlying this theory is the concept that the bulk density is the product of two fields, the matrix material density field and the volume fraction field. In the one-dimensional case, the evolution equations are as follows

$$\rho_0 u_{tt} = t_x, \qquad \rho_0 \kappa \varphi_{tt} = h_x + g. \tag{1.8}$$

Here t is the stress, h is the equilibrated stress, and g is the equilibrated body force. The variables u and φ prepresent the displacement of a solid elastic material and the volume fraction, respectively. The constitutive equations are

$$\begin{aligned} &t = \mu u_x + \beta \varphi, \\ &h = \alpha \varphi_x, \qquad g = -\beta u_x - \xi \varphi, \end{aligned} \tag{1.9}$$

when we assume that the internal energy density is a positive definite form, the constitutive coefficients satisfy the conditions

$$\mu > 0, \quad \alpha > 0, \quad \xi > 0, \quad \xi\mu > \beta^2. \tag{1.10}$$

If we substitute the constitutive equations (1.9) into the evolution equations (1.8), we obtain the field equations. Thus, the field equations of the one-dimensional linear theory of porous elastic solids are

$$\begin{aligned} &\rho_0 u_{tt} = \mu u_{xx} + \beta\varphi_x, \quad \text{in } \mathbb{R}_+ \times I, \\ &\rho_0\kappa\varphi_{tt} = \alpha\varphi_{xx} - \beta u_x - \xi\varphi, \quad \text{in } \mathbb{R}_+ \times I. \end{aligned} \tag{1.11}$$

Equations (1.11) constitute a system of two partial differential equations with two unknown functions u and φ. The constant ρ_0 is the mass density that is assumed positive and κ is the equilibrated inertia that is also assumed positive. Parameters μ, β, α and ξ are the constitutive constants of this theory; they satisfy inequalities (1.10).

Now, in $3D$ we consider the system of porous-elastic materials by the differential equations of low amplitude acoustic waves in elastic materials with voids. In case of homogeneous and isotropic material and in the absence of the external forces, we introduce the following system

$$\begin{aligned} &\rho v_{tt} - \mu\Delta v - (\mu+\lambda)\nabla\nabla\cdot v + \beta\nabla\phi = 0, \quad \text{in } \mathbb{R}_+ \times \Omega, \\ &\rho\kappa\phi_{tt} - \alpha\Delta\phi + \xi\phi + \beta\nabla\cdot u = 0, \quad \text{in } \mathbb{R}_+ \times \Omega, \end{aligned} \tag{1.12}$$

where v is the displacement field, ϕ is the difference of the volume fraction and $\alpha, \beta, \rho, \mu, \lambda, \tau$ and ξ are positive constitutive coefficients.

1.3.2.B Porous-elasticity with thermal effects

When the temperature is effective, we define the entropy η by

$$d\eta_t = q_x, \quad d > 0, \tag{1.13}$$

where q is the heat flux. Then, based on (1.13) and (1.8), the evolution equations will be in the following form

$$\rho_0 u_{tt} = t_x, \qquad \rho_0\kappa\varphi_{tt} = h_x + g, \qquad d\eta_t = q_x. \tag{1.14}$$

Here t is the stress, h is the equilibrated stress, and g is the equilibrated body force.

We can rewrite the constitutive equation (1.9) as follows

$$\begin{aligned} &t = \mu u_x + b\varphi - \beta\theta, \quad \eta = \beta u_x + m\varphi + c\theta, \\ &h = \alpha\varphi_x, \quad g = -bu_x - \xi\varphi + m\theta, \\ &q = k\theta_x. \end{aligned} \tag{1.15}$$

If we introduce the constitutive equations in the evolution equations, we obtain the field equations:

$$\begin{aligned} &\rho u_{tt} = \mu u_{xx} + b\varphi_x - \beta\theta_x, \quad \text{in } \mathbb{R}_+ \times I, \\ &J\varphi_{tt} = \alpha\phi_{xx} - bu_x - \xi\varphi + m\theta, \quad \text{in } \mathbb{R}_+ \times I, \\ &c\theta_t = k^*\theta_{xx} - \beta u_x - m\varphi_t, \quad \text{in } \mathbb{R}_+ \times I, \end{aligned} \tag{1.16}$$

where $J = \rho\kappa$ and $k^* = k/d$.

If the body occupe a domain in $\mathbb{R}^3$, i.e., in $3D$ the system can be given as follows

$$\begin{aligned} &\rho v_{tt} - \mu\Delta v - (\mu + \lambda)\nabla\nabla \cdot v + \beta\nabla\phi - \alpha\nabla\theta = 0, \quad \text{in } \mathbb{R}_+ \times \Omega, \\ &\rho\kappa\phi_{tt} - \alpha\Delta\phi + \xi\phi + \beta\nabla \cdot v + m\theta = 0, \quad \text{in } \mathbb{R}_+ \times \Omega, \\ &c\theta_t = k^*\Delta\theta - \alpha\nabla \cdot v - m\phi_t = 0, \quad \text{in } \mathbb{R}_+ \times \Omega, \end{aligned} \tag{1.17}$$

where v, ϕ and θ are the displacement field, the difference of the volume fraction and the temperature respectively.

1.3.2.C Porous-elasticity with microtemperaure effects

Taking into account the presence of micro thermal effects. Then, in the one-dimensional case, the evolution equations for the theory of elastomeric solids with voids are given as follows

$$\begin{aligned} &\rho u_{tt} = s_x, \quad \rho\kappa\varphi_{tt} = h_x + g, \\ &\rho T_0\chi_t = q_x, \quad \rho\Xi_t = p_x + q - Q. \end{aligned} \tag{1.18}$$

Here, s is the stress, h is the equilibrated stress, g is the equilibrated body force, q is the heat flux, χ is the entropy, p is the first heat flux moment, Q is the mean heat flux, Ξ is the first moment of energy, κ is a coefficient of inertia and T_0 is the absolute temperature in the reference configuration which is assumed positive. The variables u and φ are, respectively, the displacement of the solid elastic material and the volume fraction. We assume that ρ and κ are positive constants. To state the field equations, we need the following constitutive equations

$$
\begin{aligned}
s &= \mu u_x + b\varphi - \beta\theta + \gamma u_{xt} + e_1 w_x + \tau_1 \varphi_t, \\
h &= \delta\varphi_x - (d - k_6)w + \eta\varphi_{tx} + k_1\theta_x, \\
g &= -bu_x - \xi\varphi + m\theta - \tau_2 u_{tx} - \varepsilon_1 w_x - \tau\varphi_t, \\
\rho\chi &= \beta u_x + c\theta + m\varphi, \\
q &= k\theta_x + k_2\varphi_{xt} + k_4 w, \\
p &= -k_8 w_x - e_2 u_{tx} - \varepsilon_2\varphi_t, \\
Q &= (k - k_3)\theta_x + (k_4 - k_5)w + (k_2 - k_7)\varphi_{tx}, \\
\rho\Xi &= -aw - d\varphi_x.
\end{aligned}
\tag{1.19}
$$

Here, θ and w are the temperature and the microtemperature, respectively. When the coupling is considered b, β, m and d must be different from zero, but its sign does not matter in the analysis we propose. If we introduce the constitutive equations (1.19) in the evolution equations (1.18), we obtain the system of field equations

$$
\begin{aligned}
\rho u_{tt} &= \mu u_{xx} + b\varphi_x - \beta\theta_x + \gamma u_{xxt} + e_1 w_{xx} + \tau_1\varphi_{xt} \quad \text{in } \mathbb{R}_+ \times I, \\
J\varphi_{tt} &= \delta\varphi_{xx} - bu_x - \xi\varphi + m\theta - (d - k_6 + \varepsilon_1)w_x + k_1\theta_{xx} \\
&\qquad - \tau_2 u_{xt} - \tau\varphi_t + \eta\varphi_{xxt} \quad \text{in } \mathbb{R}_+ \times I, \\
c\theta_t &= k\theta_{xx} - \beta u_{xt} - m\varphi_t + k_4 w_x + k_2\varphi_{xxt} \quad \text{in } \mathbb{R}_+ \times I, \\
aw_t &= k_8 w_{xx} + (\varepsilon_2 - d - k_7)\varphi_{xt} - k_3\theta_x - k_5 w + e_2 u_{xxt} \quad \text{in } \mathbb{R}_+ \times I,
\end{aligned}
\tag{1.20}
$$

where $J = \rho\kappa$.
In $3D$ the system is given as follows

$$
\begin{aligned}
\rho v_{tt} &= (\lambda + \mu)\nabla\nabla\cdot v + b\nabla\phi - \beta\nabla\theta \quad \text{in } \mathbb{R}_+ \times \Omega, \\
J\phi_{tt} &= \alpha\Delta\phi - b\nabla\cdot v - \xi\phi - d\nabla\cdot w + m\theta \quad \text{in } \mathbb{R}_+ \times \Omega, \\
c\theta_t &= k\Delta\theta - \beta T_0\nabla\cdot v_t - mT_0\phi_t + k_1\nabla\cdot w \quad \text{in } \mathbb{R}_+ \times \Omega, \\
aw_t &= k_6\Delta w + (k_4 + k_5)\nabla\nabla\cdot w - d\nabla\phi_t - k_3\nabla\theta - k_2 w \quad \text{in } \mathbb{R}_+ \times \Omega,
\end{aligned}
\tag{1.21}
$$

where $c = aT_0$. Here ρ is the reference mass density, u is the displacement vector; θ is the temperature($T_0 > 0$); $\lambda, \mu, \beta, a, b, \xi, J, m, d, \alpha, k_i(i = 1, \cdot, 6)$ are constitutive coefficients; w is the microtemperature vector and ϕ is the microstretch. The system satisfayes the following

Clausius-Duhem inequality

$$\begin{aligned} &3k_4 + k_5 + k_6 \geq 0, \qquad k_5 + k_6 \geq 0, \\ &k_6 - k_5 \geq 0, \quad k \geq 0, \quad (k_1 + T_0 k_3)^2 \leq 4T_0 k k_2. \end{aligned} \tag{1.22}$$

1.3.3 Non classical thermolasticity

Nonclassical thermoelasticity theories involving hyperbolic-type heat transport equations admitting finite velocities for thermal signals have been formulated either by incorporating a flow term in Fourier's law or by including the temperature rate among the variables. constitutive. In 1991, Green and Naghdi introduced three new types of thermoelastic theories based on the replacement of the usual entropy inequality by an entropy equilibrium law. In each of these theories, the heat flux is given by a different constitutive assumption. As a result, three theories were obtained and called type I, type II and type III thermoelasticity, respectively. When the type I theory is linearized, we get the classical thermoelasticity system. The systems resulting from thermoelasticity of type III are of dissipative nature whereas those of type II do not support energy dissipation. Efforts to eliminate the paradox of the infinite speed of heat propagation have been underway for over a century. As early as 1867, Maxwell postulated the appearance of a wave-like heat flux, which was called the second sound, while developing a kinetic theory of gases, and suggesting a modification of Fourier's law. In 1917, Nernst speculated on the possibility of the appearance of temperature waves in good thermal conductors, at low temperatures. Whereas in 1947 Tisza predicted the possibility of extremely low heat propagation velocities in liquid helium. In 1941 Landau described the second sound as the propagation of a disturbance in the density of phonons. In his studies of super fluid helium. In 1941, Landau described the second sound as the propagation of a disturbance in the density of phonons. Experimentally, the second sound was first detected in liquid helium by Peshkov in 1944. thus the predictions of Tisza and Landau were verified experimentally by Maurer, Herlin, Pellam and Scott in 1949, and Atkins and Osborne in 1950, and theoretically by Ward, Dingle and Wilks (1951; 1952) and London 1954. In solid helium at certain temperatures and from a theoretical point of view, studies have been made, among others, by Cattaneo (1948; 1958) and Vernotte (1958; 1961) to account for the existence of the second sound.

1.3.3.A Thermal effects of type Green-Nagdi

This theory is based on an analogy between the concepts and equations of the purely thermal and the purely mechanical theories, three types of constitutive equations for heat flow in a stationary rigid solid such that when the respective theories are linearized, type I leads to the usual heat conduction by Fourier's law, type II to a telegraph equation (with a possibly vanishing damping term), whose solution is capable of transmitting waves with finite speed, and type III leads to an equation of Jeffreys type. By using the classical model of thermoelasticity (or thermoelasticity of type I), the thermoelasticity models of type III and II for isotropic media, are given as follows

A – Thermoelasticity of type II We can introduce the model of linear thermoelasticity in which the system is based on the principle of conserving energy dissipation for $1D$ and $3D$ respectively by

$$\begin{aligned} &u_{tt} - \alpha u_{xx} + \gamma_1 \theta_x = 0, \quad \text{in } \mathbb{R}_+ \times I, \\ &\theta_{tt} - \kappa \theta_{xx} + \gamma_2 u_{ttx} = 0, \quad \text{in } \mathbb{R}_+ \times I \end{aligned} \tag{1.23}$$

and

$$\begin{aligned} &v_{tt} - \mu \Delta v - (\mu + \lambda) \nabla \nabla^T v + \gamma_1 \nabla \theta = 0, \quad \text{in } \mathbb{R}_+ \times \Omega, \\ &\theta_{tt} - \kappa \Delta \theta + \gamma_2 \nabla^T v_{tt} = 0, \quad \text{in } \mathbb{R}_+ \times \Omega. \end{aligned} \tag{1.24}$$

B – Thermoelasticity of type III

$$\begin{aligned} &u_{tt} - \alpha u_{xx} + \gamma_1 \theta_x = 0, \quad \text{in } \mathbb{R}_+ \times I, \\ &\theta_{tt} - \kappa \theta_{xx} - \delta \theta_{txx} + \gamma_2 u_{ttx} = 0, \quad \text{in } \mathbb{R}_+ \times I \end{aligned} \tag{1.25}$$

and in $3D$ by

$$\begin{aligned} &v_{tt} - \mu \Delta v - (\mu + \lambda) \nabla \nabla^T v + \gamma_1 \nabla \theta = 0, \quad \text{in } \mathbb{R}_+ \times \Omega, \\ &\theta_{tt} - \kappa \Delta \theta - \delta \Delta \theta_t + \gamma_2 \nabla^T v_{tt} = 0, \quad \text{in } \mathbb{R}_+ \times \Omega. \end{aligned} \tag{1.26}$$

The thermoelasticity models of *type III*, (1.25) and (1.26), formally converge to the ones of *type II* , (1.24) and (1.23), as $\delta \to 0$.

1.3.3.B Thermal effects of type Second sound

In this regard, based on the (articles of Joseph and Priziosi), we can replace (1.7) with the so-called Cattaneo's equation

$$\tau q_t + q = -\kappa \nabla \theta, \tag{1.27}$$

or more general even by a heat-flux equation of Jeffreys type

$$\tau q_t + q = -\kappa \nabla \theta - \tau \kappa_1 \nabla \theta_t. \tag{1.28}$$

In the above equations ((1.27) and (1.28)), $\tau > 0$ denotes the (in general very small) relaxation time and $\kappa_1 > 0$ the effective thermal conductivity. Using Cattaneo's law of heat conduction instead of (1.7) one immediately arrives at the so-called thermoelasticity systems with second sound, given in the linear $1D$ and $3D$ cases by

$$\begin{aligned} &u_{tt} - \alpha u_{xx} + \gamma_1 \theta_x = 0, \quad \text{in } \mathbb{R}_+ \times I, \\ &\theta_t + q_x + \gamma_2 u_{tx} = 0, \quad \text{in } \mathbb{R}_+ \times I, \\ &\tau q_t + q + \kappa \theta_x = 0, \quad \text{in } \mathbb{R}_+ \times I, \end{aligned} \tag{1.29}$$

and in $3D$ by

$$\begin{aligned} &v_{tt} - \mu \Delta v - (\mu + \lambda) \nabla \nabla^T v + \gamma_1 \nabla \theta = 0, \quad \text{in } \mathbb{R}_+ \times \Omega, \\ &\theta_t + \nabla^T q + \gamma_2 \nabla^T v_t = 0, \quad \text{in } \mathbb{R}_+ \times \Omega, \\ &\tau q_t + q + \kappa \nabla \theta = 0, \quad \text{in } \mathbb{R}_+ \times \Omega. \end{aligned} \tag{1.30}$$

The name is due to the problem of second sound, which arose first in studies of Tisza, [Tis38], and Landau, [Lan41], of heat waves in liquid helium II. Further, we note that as the relaxation parameter τ goes to zero, the second sound models (1.29) and (1.30) formally converge to the classical ones in (1.5) and (1.6).

1.3.3.C Thermal effects of type Gurtin-pipkin

We consider the following constitutive equation

$$\beta q + \int_0^\infty g(s) \theta_x(t-s) ds = 0, \tag{1.31}$$

where g, called the memory kernel, is a bounded convex summable function on $[0, \infty)$ of total mass

$$\int_0^\infty g(s) ds = 1.$$

Using Gurtin-Pipkin's law of heat conduction instead of (1.7) one immediately arrives at the so-called thermoelasticity systems with thermal effects of type Gurtin-Pipkin, which is given in the linear $1D$ and $3D$ cases by

$$\begin{aligned} &u_{tt} - \alpha u_{xx} + \gamma_1 \theta_x = 0, \quad \text{in } \mathbb{R}_+ \times I, \\ &\theta_t - \frac{1}{\beta} \int_0^\infty g(s) \theta_{xx}(t-s) ds + \gamma_2 u_{tx} = 0, \quad \text{in } \mathbb{R}_+ \times I, \end{aligned} \tag{1.32}$$

and in $3D$ by

$$\begin{aligned} & v_{tt} - \mu \Delta v - (\mu + \lambda) \nabla \nabla^T v + \gamma_1 \nabla \theta = 0, \quad \text{in } \mathbb{R}_+ \times \Omega, \\ & \theta_t - \frac{1}{\beta} \int_0^\infty g(s) \Delta \theta(t-s) ds + \gamma_2 \nabla^T v_t = 0, \quad \text{in } \mathbb{R}_+ \times \Omega. \end{aligned} \tag{1.33}$$

1.3.4 Mechanical models

In what follows, we will present some mechanical models

1.3.4.A Timoshenko systems

In 1921, Timoshenko through his research found an improvement of an Euler - Bernoulli model including the consideration of shear deformation. This mathematical model is embodied by two partial differential equations resulting from the article by Timoshenko, as regards the Euler-Bernoulli model limited by the study of the transverse vibrations of a beam and allowing the transmission of energy to speeds close to infinity are not suitable for all applications. Also the introduction of the Rayleigh principle based on the inertia of rotation to solve the lack of applications but the model is still insufficient.

Timoshenko beam theory includes the effects of both rotary inertia and shear deformation, it was initially introduced by Stéphane Timoshenko, this model could solve the majority of the applications, this model is defined through the following equations

$$\begin{aligned} & \rho_1 \varphi_{tt} = S_x, \\ & \rho_2 \psi_{tt} = M_x - S, \end{aligned} \tag{1.34}$$

where t is the time, x is the distance along the center line of the beam structure, φ is the transverse displacement, and ψ is the rotation of the neutral axis due to bending. Here, $\rho_1 = \rho A$ and $\rho_2 = \rho I$ where ρ is the density, A is the cross-sectional area, and I is the second moment of area of the cross-sectional area. The corresponding constitutive laws are given by

$$\begin{aligned} & M = EI\psi_x - \delta\theta, \\ & S = \kappa AG(\varphi_x + \psi). \end{aligned} \tag{1.35}$$

In these equations δ denotes the density, E and G are the elastic constants and κ is the shear coefficient.

We can conclude by the previouse equations (1.34) and (1.35) the following coupled hyperbolic system

$$\rho_1 u_{tt} - \kappa(u_x + \psi)_x = 0, \quad \text{in } \mathbb{R}_+ \times I,$$
$$\rho_2 \psi_{tt} - b\psi_{xx} + \kappa(u_x + \psi) = 0, \quad \text{in } \mathbb{R}_+ \times I.$$

Here, u and ψ are the displacement and the shear angle respectively.

1.3.4.B von Kármán system

Periodic vortex erasure has many physical and technical applications. It provides the explanation, as Rayleigh has shown, of aeolian tones, and the exciting forces can be responsible for oscillations of structures which can have serious consequences in case of resonance. Many years later, in 1940, the collapse of the bridge spanning the Tacoma Narrows was caused by resonance from periodic vortices and floating. Flat plates had been used as side walls instead of trusses. These gave rise to vortices and torsional oscillations of the bridge developed. The set of events was somewhat complicated. von Kármán was called in as a consultant to investigate the bridge collapse.

The uniform prismatic beam of length $L.$ is modeled by the following system

$$\begin{aligned} &u_{tt} - \left[u_x + \frac{1}{2}w_x^2\right]_x = 0, \quad \text{in } \mathbb{R}_+ \times I, \\ &w_{tt} + w_{xxxx} - \left[w_x\left(u_x + \frac{1}{2}w_x^2\right)\right]_x = 0, \quad \text{in } \mathbb{R}_+ \times I, \end{aligned} \tag{1.36}$$

where $0 < x < L$ and $t > 0$. $h = \frac{I}{A}$ is a parameter related to the rotational inertia of the beam, where the physical constants are A, the area of a cross section, I its moment of inertia with respect to the $y-$axis. The quantities $u = u(x,t)$ and $w = w(x,t)$ represent, respectively, the longitudinal and transversal displacement of the point x at time t.

1.3.4.C Timoshenko systems with thermal effect

The general form of Timoshenko system with thermal effects can be written as:

$$\begin{aligned} &\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, \quad \text{in } \mathbb{R}_+ \times I, \\ &\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \delta\theta_x = 0, \quad \text{in } \mathbb{R}_+ \times I, \\ &\rho_3\theta_t + q_x + \delta\psi_{tx} = 0, \quad \text{in } \mathbb{R}_+ \times I, \end{aligned} \tag{1.37}$$

for the heat flux vector

$$q : (x,t) \in (0,L) \times \mathbb{R}^+ \mapsto \mathbb{R}. \tag{1.38}$$

we define the following thermal laws:

A – Fourier thermal law When the heat conduction was assumed the classical Fourier

$$\beta q = -\theta_x. \tag{1.39}$$

B – Cattaneo thermal law When the heat conduction was assumed the Cattaneo law

$$\tau q_t = -\beta\theta + \theta_x, \quad \tau > 0. \tag{1.40}$$

C – Gurtin-Pipkin thermal law The constitutive equation related to Gurtin-Pipkin heat conduction law is given by

$$\beta q(t) + \int_0^\infty g(s)\theta_x(t-s)ds = 0, \tag{1.41}$$

where g is called the memory kernel, is a convex summable function on $\mathbb{R}^+$ of total mass

$$\int_0^\infty g(s)ds = 1.$$

D – Fractional Gurtin-Pipkin thermal law Our consideration is the following fractional constitutive equation

$$\beta q(t) - \int_0^\infty g(s)(-\partial_{xx})^{\sigma-\frac{1}{2}}\theta(t-s)ds = 0. \tag{1.42}$$

Appaling the operator $(-\partial_{xx})^{\frac{1}{2}} = \partial_x$ to the previous equation, we get

$$\beta q_x - \int_0^\infty g(s)(-\partial_{xx})^{\sigma}\theta(t-s)ds = 0. \tag{1.43}$$

E – Green-Naghdi thermal law The constitutive equation related to Green-Naghdi heat conduction law of type III is given by

$$\beta q + \theta_x + dp_x = 0, \quad d > 0, \tag{1.44}$$

where

$$p(t) = p(0) + \int_0^t \theta(r)dr. \tag{1.45}$$

1.3.4.D von Kármán systems with thermal effects

The general model of the thermoelastic plate of von Kármán type system with heat flow can be written as follows

$$\begin{aligned} & u_{tt} - \left[u_x + \frac{1}{2}w_x^2\right]_x + \delta\theta_x = 0 \quad \text{in } \mathbb{R}_+ \times I, \\ & w_{tt} + w_{xxxx} - \left[w_x\left(u_x + \frac{1}{2}w_x^2\right)\right]_x = 0, \quad \text{in } \mathbb{R}_+ \times I, \\ & \theta_t + q_x + \delta u_{tx} = 0, \quad \text{in } \mathbb{R}_+ \times I. \end{aligned} \tag{1.46}$$

Through the third equation in system (1.46), the heat flux field (1.38) can be given by the previously mentioned thermal conduction laws (1.39),(1.40),(1.41) and (1.44).

1.4 Objectives

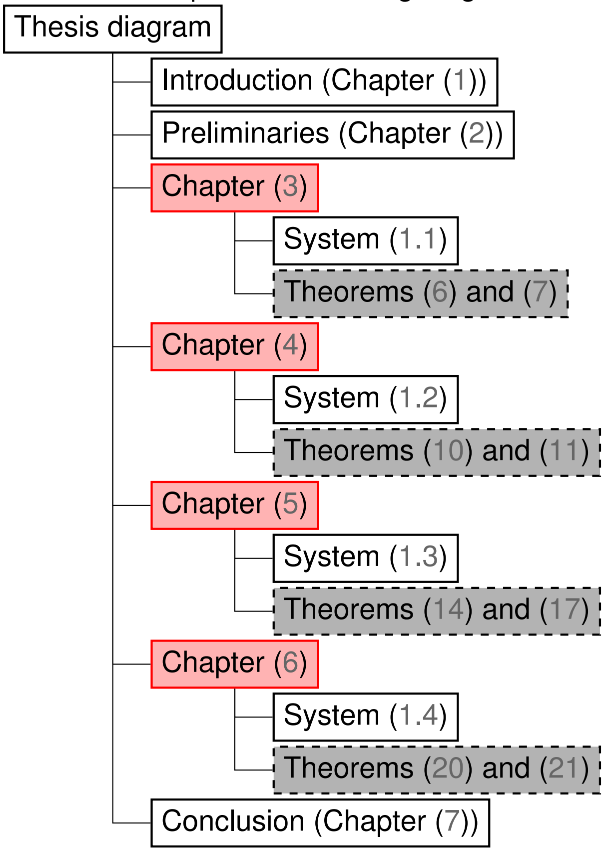
In this section, we will present the main objectives of the thesis. Regarding system (1.1), our goal is to prove the existence of solutions as well as to show the exponential stability in the non-thermal conductivity state and without any requirement on system parameters. Then, for both systems (1.4) and (1.2), we focus on proving the existence of solutions using the semigroup framework as well as showing that the energy functional decays exponentially. Finally, with regard to system (1.3), our aim is to demonstrate the existence of solutions as well as to address the proof of results related to the stability of the system by applying semigroup method in fractional Hilbert space .

1.5 Methodology

The research methodology that we have adopted in our work, which is considered essential in addressing the problems that are posed and intended to be solved on the one hand, and to reach the desired goals on the other hand, has been limited to the Lyapunov method and the semigroup method. These methods are based on principles and concepts. Accordingly they were dealt with the functional analysis, spectral theory and semigroups theory.

1.6 Thesis overview

This thesis respect the following diagram:



2

Basic facts

Contents

In this chapter we will remind some of the basic tools that are useful for our work. A more detailed presentation can be found in the classical books and papers, see e.g., [69, 70, 70–76]

2.1 Hölder inequality

Lemma 1. *Let $p > 1$ and $q > 1$ be two conjugate real numbers, that is to say $p^{-1} + q^{-1} = 1$. Then, for all $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, we have $fg \in L^1(\Omega)$.*

In particular, we have the following cases

A) If $p, q \in]1, +\infty[$. Then, we have

$$\int_\Omega |fg| dx \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

B) If $p = 1, q = +\infty$. Then, we have

$$\int_\Omega |fg| dx \leq \|f\|_{L^1(\Omega)} \|g\|_{L^\infty(\Omega)}.$$

The Cauchy-Schwarz inequality is a special case of the inequality of Hölder in the case $p = q = 2$.

2.2 Poincaré inequality

Lemma 2. *Let Ω be a bounded open in $\mathbb{R}^n$. Then, there exists a constant $c > 0$ such that*

$$\|f\|_{H^1(\Omega)} \leq c \|\nabla f\|_{L^2(\Omega)}, \ \forall f \in H_0^1(\Omega).$$

Then, we deduce that

$$\|f\|_{L^2(\Omega)} \leq c \|\nabla f\|_{L^2(\Omega)}, \ \forall f \in H_0^1(\Omega).$$

2.3 Young inequality

Lemma 3. *Let p and q be two conjugate real numbers in $]1, +\infty[$. Then, for all $a, b \in \mathbb{R}_+$, we have*

$$ab \leq p^{-1}a^p + q^{-1}b^q.$$

In particular for $p = q = 2$, we have

$$ab \leq \varepsilon a^2 + \varepsilon^{-1}b^2, \ \forall \varepsilon > 0.$$

2.4 Integral inequalities

We recall here some known integral inequalities widely used in the stabilization of dissipative and also non-dissipative evolutionary systems. Indeed, several results concerning the estimation of the energy of certain dissipative problems are based on the following Lemmas.

Lemma 4 ([77–79]). *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous non-increasing function, and $g : \mathbb{R}_+ \to \mathbb{R}_+$ a strictly increasing function in the class $C^1(\mathbb{R}_+)$ such that $g(0) = 0$ and $\lim_{t\to\infty} g(t) = +\infty$. Suppose that there exist $p \geq 0$ and $d > 0$ such that*

$$\int_s^{+\infty} g'(t)f^{p+1}(t)dt \leq d^{-1}f^p(0)f(s), \ \forall s > 0.$$

Then,

$$f(t) \leq f(0)\exp(1 - dg(t)), \ \forall t > 0 \text{ if } p = 0$$
$$f(t) \leq f(0)(\frac{1+p}{1+pdg(t)})^{p^{-1}}, \ \forall t > 0 \text{ if } p > 0.$$

Lemma 5 ([80]). *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous non-increasing function such that*

$$\int_s^{+\infty} g(f(t))dt \leq d^{-1}f(s), \ \forall s > 0,$$

where $d > 0$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ a convex strictly increasing with $g(0) = 0$. Then, there exist $c_1, c_2 > 0$ such that

$$f(t) \leq g^{-1}(\frac{h^{-1}(c_1 t)}{c_2 t}), \ \forall t > t_0,$$

where

$$h(s) = \int_s^1 g^{-1}(t)dt, \ \forall 1 > s > 0.$$

2.5 Embeddings of Sobolev spaces

Theorem 1. *Let Ω be a Lipschitz domain. Let $1 \le p < n$ and $q^{-1} = p^{-1} - n^{-1}$. Then, $W^{1,p}(\Omega) \subset L^q(\Omega)$, i.e., the identity mapping from $W^{1,p}(\Omega)$ to $L^q(\Omega)$ is bounded.*

Consider a subspace $C^{k,\mu}(\overline{\Omega})$ of $C^k(\Omega)$, consisting of all such functions, whose $k-$th partial derivatives are $\mu-$Hölder continuous. The norm in this space is introduced through the following formula

$$\|f\|_{C^{k,\mu}(\overline{\Omega})} = \|f\|_{C^k(\Omega)} + \sum_{|\alpha|=k} \sup_{x\neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^\mu},$$

where $D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$ with $|\alpha| = \alpha_1, \cdots, \alpha_n$.

Theorem 2. *Let Ω be a domain with Lipschitz boundary. Let $p > n$ and $\mu = 1 - \frac{n}{p}$. Then $W^{1,p}(\Omega) \subset C^{0,\mu}(\Omega)$.*

For the proof, see [81]. The two above embeddings are only special cases.

Proposition 1. *If the domain Ω in $\mathbb{R}^n$ then, the following embeddings are continuous:*

$$W^{j+m,p}(\Omega) \subset W^{j,q}, \text{ when } p \le q \le \frac{np}{n-mp}.$$

If Ω is a Lipschitz domain, then

$$W^{j+m,p}(\Omega) \subset C^{j,\lambda}(\overline{\Omega}), \text{ for } 0 < \lambda \le m - \frac{n}{p}.$$

For the proof, see [82].

2.6 The Lax-Milgram lemma

Theorem 3. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space equipped with the norm $|u| = \sqrt{\langle u, u\rangle}$, where $u \in H$.*

Let $B : H \times H \to \mathbb{R}$ be bilinear and there exist numbers $\alpha, \beta > 0$ such that for all $u, v \in H$, we have

- *Boundedness/continuity*

$$|B(u,v)| \le \alpha |u||v|,$$

- *Coercivity/ellipticity*

$$B(u,u) \ge \beta |u|^2.$$

Then, for any bounded linear functional $L : H \to \mathbb{R}$ *there exists a unique vector* $v \in H$ *such that for all* $u \in H$

$$Lv = B(u,v).$$

For the proof, see [73].

2.7 Gagliardo-Nirenberg-Sobolev inequality

Let $1 \le p < n$. There exists a constant $C = C(n,p)$ such that for all $u \in C_c^1(\mathbb{R}^n)$ we have

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C \|Du\|_{L^p(\mathbb{R}^n)},$$

where $p^* = \frac{np}{n-p}$ (in other words $p^{*-1} = p^{-1} - n^{-1}$).

2.8 Hille-Yosida Theorem

Let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ be a maximal monotone operator in Hilbert space $\mathcal{H}$. Then, given any $f_0 \in \mathcal{D}(\mathcal{A})$ there exists a unique function

$$f \in C^1(\mathbb{R}_+; \mathcal{H}) \cap C(\mathbb{R}_+; \mathcal{D}(\mathcal{A}))$$

satisfying

$$\begin{aligned} &\frac{df}{dt} + \mathcal{A}f(t) = 0 \text{ on } \mathbb{R}_+, \\ &f(0) = f_0. \end{aligned}$$

Moreover,

$$|f(t)| \le |f_0| \quad \text{and} \quad \left|\frac{df}{dt}\right| = |\mathcal{A}f(t)| \le |\mathcal{A}f_0|, \ \forall t \ge 0.$$

3

Stability of a linear coupled hyperbolic-parabolic system

Contents

3.1 Introduction

In this chapter we have discussed the existence and uniqueness of solutions; we have also shown the stability result of the problem of porous elasticity with microthermal effects. Thus, the system is given as follows

$$\begin{array}{ll} \rho u_{tt} = \mu u_{xx} + b\varphi_x - \gamma\theta_x, & \text{in } (0,1)\times(0,\infty), \\ J\varphi_{tt} = \delta\varphi_{xx} - bu_x - \xi\varphi - dw_x + m\theta - \beta\varphi_t, & \text{in } (0,1)\times(0,\infty), \\ c\theta_t = -\gamma u_{tx} - m\varphi_t - k_1 w_x, & \text{in } (0,1)\times(0,\infty), \\ aw_t = k_2 w_{xx} - k_3 w - k_1\theta_x - d\varphi_{tx}, & \text{in } (0,1)\times(0,\infty), \end{array} \tag{3.1}$$

subject to the following boundary conditions

$$\begin{array}{l} u_x(0,t) = u_x(1,t) = \varphi(0,t) = \varphi(1,t) = 0, \quad t > 0 \\ \theta(0,t) = \theta(1,t) = w_x(0,t) = w_x(1,t) = 0, \quad t > 0, \end{array} \tag{3.2}$$

and the initial conditions

$$\begin{array}{l} u(x,0) = u^0(x),\ u_t(x,0) = u^1(x),\ \varphi(x,0) = \varphi^0(x), \quad x \in (0,\ 1), \\ \varphi_t(x,0) = \varphi^1(x),\ w(x,0) = w^0(x),\ \theta(x,0) = \theta^0(x), \quad x \in (0,\ 1), \end{array} \tag{3.3}$$

where u^0, u^1, φ^0, φ^1, w^0,θ^0 are given functions and the functions u, φ, θ and ω are the displacement of the elastic solid, the volume fraction of the material, the difference of temperature and the microtemperature, respectively. We have imposed a zero heat transfer for the local thermal effects and we will depend only on the thermal diffusion of the micro-heat that has a microscopic effect on the material, and we will also depend on the viscoporosity effect to obtain the desired goals.

From the first equation of (3.1) and the boundary conditions (3.2), we get

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)dx = 0,\ \forall\, t \geq 0. \tag{3.4}$$

Therefore

$$\int_0^1 u(x,t)dx = t\int_0^1 u^1(x)dx + \int_0^1 u^0(x)dx,\ \forall\, t \geq 0.$$

Consequently, if we set

$$\overline{u}(x,t) = u(x,t) - t\int_0^1 u^1 dx + \int_0^1 u^0 dx,\ t \geq 0,\ x \in [0,1],$$

we find

$$\int_0^1 \overline{u}(x,t)dx = 0,\ t \geq 0.$$

Now, from the fourth equation of (3.1) and the boundary conditions (3.2), we obtain

$$\frac{d}{dt}\int_0^1 w(x,t)dx + \frac{k_3}{\alpha}\int_0^1 w(x,t)dx = 0,\ \ \forall\, t \geq 0, \tag{3.5}$$

thus

$$\int_0^1 w(x,t)dx = \left(\int_0^1 w^0 dx\right) e^{-\frac{t}{\alpha}k_3},$$

so, if we put

$$\overline{w}(x,t) = w(x,t) - \left(\int_0^1 w^0 dx\right) e^{-\frac{t}{\alpha}k_3},\ t \geq 0,\ x \in [0,1]\,,$$

we arrive at

$$\int_0^1 \overline{w}(x,t)dx = 0,\ \ t \geq 0.$$

Hence, the use of Poincaré's inequality for u and w is justified and $(\overline{u},\ \varphi, \theta,\ \overline{w})$ satisfies the same equations in (3.1)-(3.2). In what follows we will work with $\overline{u}$ and $\overline{w}$ but, for convenience, we write u and w instead of $\overline{u}$ and $\overline{w}$.

This chapter has been inspired by the work [83], where the first result is the following existence and uniqueness Theorem.

Theorem 4. *Let $U_0 \in D(A)$, Then, the Problem* (3.1)-(3.3) *has a unique solution U such that*

$$U \in C\left(\mathbb{R}_+,\ \mathcal{D}(\mathcal{A})\right) \cap C^1\left(\mathbb{R}_+, \mathcal{H}\right).$$

Moreover if $U_0 \in \mathcal{H}$, then the solution in the following class

$$U \in C\left(\mathbb{R}_+, \mathcal{H}\right).$$

We note that based on the semigroup approach the problem (3.1)-(3.3) can be represented as EDO Cauchy problem which is given by

$$\begin{aligned} &\frac{d}{dt}U(t) = \mathcal{A}U,\ \ t > 0, \\ &U(0) = U_0. \end{aligned}$$

Then, we can use the semigroup theory for the proof of the above result.

We define the energy functional by

$$E(t) := \frac{1}{2}\int_0^1 \left[\rho u_t^2 + J\varphi_t^2 + \mu u_x^2 + c\theta^2 + \alpha w^2 + \delta\varphi_x^2 + \xi\varphi^2 + 2bu_x\varphi\right] dx. \tag{3.6}$$

Then, we have the following stability result

Theorem 5. *Let* $(u, \varphi, \theta, \omega)$ *be a solution of the problem* (3.1)-(3.3)*. Then, the solution* $(u, \varphi, \theta, \omega)$ *decays exponentially, i.e., the energy functional satisfies*

$$E(t) \leq \lambda_1 \exp(-\lambda_2 t), \quad \forall t \geq 0, \tag{3.7}$$

where λ_1 *and* λ_2 *are positive constants.*

For the proof we need to construct a positif Lyapunov functional $F(t)$ equivalent to the energy functional $E(t)$, i.e.,

$$\alpha_1 E(t) \leq F(t) \leq \alpha_2 E(t),$$

for $t > 0$ and some positive constants α_1 and α_2 such that

$$F'(t) \leq -CE(t),$$

where C is a positive constant.

3.1.1 Earlier results

We will present the most important works that have been issued in relation to porous-elastic systems with thermal effects, but before this, it is worth noting that Goodman and Cowin, in the paper [84], proposed an extension of the classical theory of of porous-elasticity. Indeed, they introduced the concept of a continuum theory of granular to interstitial voids materials into the theory of elastic solids with voids. The first contribution was established by Quintanilla quintanilla2003slow, the author proved that the porous-viscosity is not strong enough to stabilize the system exponentially. Later, in [34], the same authors proved that the association of both temperature and microtemparatures stabilized the system exponentially. However, Casas et al in [34], showed that the combination of porous-viscosity and temperature also lacks exponential stability. Likewise, Magana et al. [85] proved that the combination of viscoelasticity with microtemperatures produced exponential stability, whereas the combination of viscoelasticity with temperature lacks exponential stability. In addition, several results about stability have been established, we refer the reader to the works [24, 29, 32, 86, 87]. For the considered damping mechanisms that could control the exponential stability of the system, see references [51, 53, 88–97].

3.1.2 Model derivation

We present the following basic evolution equations

$$\begin{array}{ll} \rho u_{tt} = T_x, & J\varphi_{tt} = H_x + G, \\ \rho\eta_t = q_x, & \rho\tilde{E}_t = P_x - Q. \end{array} \tag{3.8}$$

In (3.8), the functions T is the stress, H is the equilibrated stress, G is the equilibrated body force, q is the heat flux vector, η is the entropy, P is the first heat flux moment, Q is the mean heat flux and $\tilde{E}$ is the first moment of energy. The variables u and φ are, respectively, the displacement of the solid elastic material and the volume fraction. The constitutive equations are

$$\begin{array}{ll} T = \mu u_x + b\varphi - \gamma\theta, & H = \delta\varphi_x - dw, \\ G = -bu_x - \xi\varphi + m\theta - \beta\varphi_t, & \rho\eta = \gamma u_x + c\theta + m\varphi, \\ q = \kappa\theta_x + k_1 w, & P = -k_2 w_x, \\ Q = -k_3 w - k_1\theta_x, & \rho\tilde{E} = -aw - d\varphi_x, \end{array} \tag{3.9}$$

where w denotes the microtemperature vector and k_1, k_2, k_3, μ, δ, ξ, a, κ and c are constitutive constants which are positive. As coupling is considered, b must be different from zero and satisfies $\mu\xi > b^2$. The coefficients γ, m and d are constants that are not necessarily positive. In this work, the thermal effects are considered, so we assume that the thermal capacity c is strictly positive, but to make the problem more interesting we assume that the thermal conductivity κ is zero.

We can obtain the system under study (3.1) by substituting the constitutive equations (3.9) into the evolution equations (3.8). We show that the dissipations due to the effects of the microtemperatures and the porosity of material enough to control the system (3.1)-(3.2) by an exponential function.

3.1.3 Chapter plan

This chapter is organized as follows. In Section (3.3), we study the existence and uniqueness of solutions for the system (3.1)-(3.2) using semigroup techniques. Next, in Section (3.4), we prove the exponential stability of the problem by using the multiplier method.

3.2 Preliminaries

We consider the following spaces:

$$\begin{array}{l} L_*^2(0,1) = \left\{ \Psi \in L^2(0,1)\,,\ \int_0^1 \Psi(x)\,dx = 0 \right\}, \\ H_*^2(0,1) = \left\{ \Psi \in H^2(0,1)\,,\ \Psi_x(0) = \Psi_x(1) = 0 \right\}, \\ H_*^1(0,1) = H^1(0,1) \cap L_*^2(0,1)\,, \end{array} \tag{3.10}$$

Let the Hilbert space defined by

$$\mathcal{H} = H_*^1(0,1) \times L_*^2(0,1) \times H_0^1(0,1) \times L^2(0,1) \times L^2(0,1) \times L_*^2(0,1)\,. \tag{3.11}$$

The Hilbert space (3.11) is equipped with the following inner product

$$\begin{aligned} \left\langle U, \widetilde{U} \right\rangle :=& \mu \int_0^1 u_x \widetilde{u}_x dx + \xi \int_0^1 \varphi \widetilde{\varphi} dx + \rho \int_0^1 v \widetilde{v} dx \\ &+ J \int_0^1 \phi \widetilde{\phi} dx + c \int_0^1 \theta \widetilde{\theta} dx + a \int_0^1 w \widetilde{w} dx \\ &+ \delta \int_0^1 \varphi_x \widetilde{\varphi}_x dx + b \int_0^1 \left(u_x \widetilde{\phi} + \phi \widetilde{u}_x \right) dx, \end{aligned} \tag{3.12}$$

for

$$U = (u, v, \varphi, \phi, \theta, w)^T \in \mathcal{H},$$

and

$$\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\varphi}, \tilde{\phi}, \tilde{\theta}, \tilde{\omega})^T \in \mathcal{H}.$$

3.3 Existence and uniqueness

In this section, we give an existence and uniqueness results for the system (3.1)-(3.2) using the semigroup theory. [98, 99], for the proof of the following Theorem

Theorem 6. *Let $U_0 \in D(A)$, Then, the Problem (3.1)-(3.3) has a unique solution U such that*

$$U \in C\left(\mathbb{R}_+,\, \mathcal{D}(\mathcal{A})\right) \cap C^1\left(\mathbb{R}_+, \mathcal{H}\right).$$

Moreover if $U_0 \in \mathcal{H}$, then the solution in the following class

$$U \in C\left(\mathbb{R}_+, \mathcal{H}\right).$$

3.3.1 The Semigroup approach

First, we denote $U=(u,v,\varphi,\phi,\theta,w)^T$, with $v=u_t$ and $\phi=\varphi_t$.

Then, system (3.1)-(3.2) can be rewritten as follows:

$$\begin{cases} U_t=\mathcal{A}U, \quad t>0, \\ U(0)=U_0=(u_0,\ u_1,\ \varphi_0,\ \varphi_1,\ \theta_0,\ w_0)^T, \end{cases}$$

such that the operator

$$\mathcal{A}:\mathcal{D}(\mathcal{A})\subset\mathcal{H}\longrightarrow\mathcal{H}$$

is defined by

$$\mathcal{A}=\begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 \\ \frac{\mu}{\rho}\partial_x^2(.) & 0 & \frac{b}{\rho}\partial_x(.) & 0 & -\frac{\gamma}{\rho}\partial_x(.) & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ -\frac{b}{J}\partial_x(.) & 0 & \frac{\delta}{J}\partial_x^2(.)-\frac{\xi}{\rho} & -\frac{\beta}{J} & \frac{m}{J} & -\frac{d}{J}\partial_x(.) \\ 0 & -\frac{\gamma}{c}\partial_x(.) & 0 & -\frac{m}{c} & 0 & -\frac{k_1}{c}\partial_x(.) \\ 0 & 0 & 0 & -\frac{d}{a}\partial_x(.) & -\frac{k_1}{a}\partial_x(.) & \frac{k_2}{a}\partial_x^2(.)-\frac{k_3}{a} \end{pmatrix}. \tag{3.13}$$

The domain of $\mathcal{A}$ is given by

$$D(\mathcal{A})=\begin{pmatrix} U\in\mathbb{H}\ /\ u\in H_*^2(0,1)\cap H_*^1(0,1),\ v\in H_*^1(0,1), \\ \varphi\in H^2(0,1)\cap H_0^1(0,1),\ \phi\in H_0^1(0,1), \\ \theta\in H_0^1(0,1),\ w\in H_*^2(0,1)\cap H_*^1(0,1) \end{pmatrix}. \tag{3.14}$$

3.3.2 Proof of Theorem (6)

Proof. Firstly, it is clear that the domain $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$. Then,

$$\langle\mathcal{A}U,\ U\rangle_{\mathcal{H}}=-\beta\int_0^1\phi^2dx-k_2\int_0^1 w_x^2dx-k_3\int_0^1 w^2dx\le 0,$$

from where it follows that the operator $\mathcal{A}$ is dissipative. Now, by using the Lax–Milgram Lemma and classical regularity arguments, we can prove that the operator $I-\mathcal{A}$ is surjective. For each $f=(f_1,\cdots,f_6)\in\mathcal{H}$, we must show that there exists unique $U\in\mathcal{D}(\mathcal{A})$ such that

$$(I-\mathcal{A})U=f,$$

that is translated to the following system

$$
\begin{aligned}
&u - v = f_1, \\
&v - \mu u_{xx} - b\varphi_x + \gamma\theta_x = f_2, \\
&\varphi - \mu u_{xx} - b\varphi_x + \gamma\theta_x = f_2, \\
&\varphi - \phi = f_3, \\
&\phi - \delta\varphi_{xx} + bu_x + \xi\varphi + dw_x - m\theta + \beta\phi = f_4, \\
&\theta - \gamma v_x + m\phi + k_1 w_x = f_5, \\
&w - k_2 w_{xx} + k_3 w + k_1\theta_x + d\phi_x = f_6.
\end{aligned}
\tag{3.15}
$$

Suppose u, φ, w are found with the appropriate regularity. Then, by using equalities $(3.15)_1$ and $(3.15)_1$, we find

$$
\begin{aligned}
&v = u - f_1 \in H^1_*(0,1), \\
&\phi = \varphi - f_3.
\end{aligned}
\tag{3.16}
$$

By using equality (3.16), the system (3.15) can be reduced as follows

$$
\begin{aligned}
&u - \mu u_{xx} - b\varphi_x + \gamma\theta_x = F_1 \in L^2_*(0,1), \\
&(1 + \xi + \beta)\varphi - \delta\varphi_{xx} + bu_x + dw_x - m\theta = F_2 \in L^2(0,1), \\
&\theta + \gamma u_x + m\varphi + k_1 w_x = F_3 \in L^2(0,1), \\
&(1 + k_3)w - k_2 w_{xx} + k_1\theta_x + d\varphi_x = F_4 \in L^2_*(0,1),
\end{aligned}
\tag{3.17}
$$

where

$$
\begin{aligned}
&F_1 = f_2 + f_3, \\
&F_2 = f_4 + (\beta + 1)f_3, \\
&F_3 = f_5 + \gamma f_{1x} + mf_3, \\
&F_4 = f_6 + f_{3x}.
\end{aligned}
\tag{3.18}
$$

We use (3.17) to build the variational formulation, so we multiply the equations by the test functions $(\tilde{u}, \tilde{\varphi}, \tilde{\theta}, \tilde{w}) \in C_c^\infty$ respectively, and so we get the following

$$
B((u, \varphi, \theta, w), (\tilde{u}, \tilde{\varphi}, \tilde{\theta}, \tilde{w})). \tag{3.19}
$$

The bilinear form is defined as follows

$$B : X \times X \longrightarrow \mathbb{R},$$

where

$$X = H^1_*(0,1) \times H^1_0(0,1) \times H^1_0(0,1) \times H^1_*(0,1),$$

and it is given by

$$\begin{aligned} B\left((u,\varphi,\theta,w),(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{w})\right) &= \int_0^1 (u-\mu u_{xx}-b\varphi_x+\gamma\theta_x)\tilde{u}dx \\ &+ \int_0^1 ((1+\xi+\beta)\varphi-\delta\varphi_{xx}+bu_x+dw_x-m\theta)\tilde{\varphi}dx \\ &+ \int_0^1 (\theta+\gamma u_x+m\varphi+k_1w_x)\tilde{\theta}dx \\ &+ \int_0^1 ((1+k_3)w-k_2w_{xx}+k_1\theta_x+d\varphi_x)\tilde{w}dx. \end{aligned} \tag{3.20}$$

By using (3.21), we get

$$\begin{aligned} B((u,\varphi,\theta,w),(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{w})) &= \int_0^1 u\tilde{u}dx + \mu\int_0^1 u_x\tilde{u}_xdx \\ &+ (1+\xi+\beta)\int_0^1 \varphi\tilde{\varphi}dx + \delta\int_0^1 \varphi_x\tilde{\varphi}_xdx \\ &+ \int_0^1 \theta\tilde{\theta}dx + (1+k_3)\int_0^1 w\tilde{w}dx + k_2\int_0^1 w_x\tilde{w}_xdx. \end{aligned} \tag{3.21}$$

The linear form $L : X \times X \longrightarrow \mathbb{R}$ is defined by

$$L(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{w}) = \int_0^1 F_1\tilde{u} + F_2\tilde{\varphi} + F_3\tilde{\theta} + F_4\tilde{w}dx. \tag{3.22}$$

By applying inequality of Cauchy-Schwarz on (3.21), it follows that

$$\left| B\left((u,\varphi,\theta,w),(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{w})\right) \le \nu_0\|(u,\varphi,\theta,w)\|_X\|(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{w})\|_X, \right. \tag{3.23}$$

Then, we have

$$B\left((u,\varphi,\theta,w),(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{w})\right) \ge \nu_1\|(u,\varphi,\theta,w),(u,\varphi,\theta,w)\|^2_X, \tag{3.24}$$

where ν_0 and ν_1 are positive constant.

For the linear form (3.22) we deduce that there exists $\nu_2 > 0$ such that

$$|L(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{w})| \leq \nu_2 \|(\tilde{u},\tilde{\varphi},\tilde{\theta},\tilde{w})\|_X \tag{3.25}$$

By the fact that B and L satisfy the above Lax-Milgram conditions (3.25),(3.23) and (3.24). Then, we deduce that there exists unique vector solution $(u,\varphi,\theta,w) \in X$ for the problem (3.19) and that prove the surjectivity of the operator $I - \mathcal{A}$.

By substituting respectively u, φ, θ and w in system (3.17), we obtain

$$v \in H^1_*(0,1), \qquad \phi \in H^1_0(0,1). \tag{3.26}$$

- If $(\tilde{\varphi},\tilde{\theta},\tilde{w}) \equiv (0,0,0) \in H^1_0(0,1) \times L^2(0,1) \times L^2_*(0,1)$, then from (3.21) and (3.22), we obtain

  $$\int_0^1 (u - \mu u_{xx} - b\varphi_x + \gamma\theta_x)\tilde{u}dx = \int_0^1 F_1\tilde{u}dx, \quad \forall \tilde{u} \in H^1_*(0,1), \tag{3.27}$$

  from equality (3.27) and by the regularity theory, it follows that

  $$u \in H^2_*(0,1),$$

  Hence, by using integration of (3.27), we get

  $$u \in H^2_*(0,1) \cap H^1_*(0,1).$$

- If $(\tilde{u},\tilde{\theta},\tilde{w}) \equiv (0,0,0) \in H^1_*(0,1) \times L^2(0,1) \times L^2_*(0,1)$, then from (3.21) and (3.22), we obtain

  $$\int_0^1 (1+\xi+\beta)\varphi\tilde{\varphi} + \delta\varphi_x\tilde{\varphi}_x + (bu_x + dw_x - m\theta - F_2)\tilde{\varphi}dx = \delta[\varphi_x\tilde{\varphi}]_0^1. \tag{3.28}$$

  Hence, from (3.29) we deduce that $\varphi(0) = \varphi(1) = 0$. Then, we obtain

  $$\varphi \in H^2(0,1) \cap H^1_0(0,1).$$

- If $(\tilde{u},\tilde{\varphi},\tilde{w}) \equiv (0,0,0) \in H^1_*(0,1) \times H^1_0(0,1) \times L^2_*(0,1)$, then from (3.21) and (3.22), we obtain

$$\int_0^1 (\theta + \gamma u_x + m\varphi + k_1 w_x)\tilde{\theta} dx = \int_0^1 F_3\tilde{\theta}, \quad \forall \tilde{\theta} \in H_0^1(0,1), \tag{3.29}$$

by using (3.29) and the regularity theory it follows that

$$\theta \in H_0^1(0,1).$$

- If $(\tilde{u}, \tilde{\varphi}, \tilde{\theta}) \equiv (0,0,0) \in H_*^1(0,1) \times H_0^1(0,1) \times L^2(0,1)$, then from (3.21) and (3.22), we obtain

$$\int_0^1 ((1+k_3)w - k_2 w_{xx} + k_1\theta_x + d\varphi_x)\tilde{w} dx = \int_0^1 F_4\tilde{w} dx, \qquad \forall \tilde{w} \in H_*^2(0,1), \tag{3.30}$$

by using equality (3.30) and the regularity theory, it follows that

$$w \in H_*^1(0,1).$$

We conclude after integrating equality (3.30) that

$$w \in H_*^2(0,1) \cap H_*^1(0,1).$$

Finally, by using Lumer-Phillips Theorem, we deduce that the operator $\mathcal{A}$ is an infinitesimal generator of a linear C_0-semigroup on Hilbert space $\mathcal{H}$. Then, we deduce tha the solusions exists and unique.

□

3.4 Stability result of solutions

In this section, we use energy method to prove that system (3.1)-(3.3) is exponentially stable, can recall this result through the following Theorem

Theorem 7. *Let (u, φ, θ, w) be a solution of the problem determined by system* (3.1)*, initial conditions* (3.3) *and boundary conditions* (3.2)*. Then, the solution (u, φ, θ, w) decays exponentially,* *i.e.,Then, there exist positive constants $\lambda_1,\ \lambda_2$ such that*

$$E(t) \leq \lambda_1 e^{-\lambda_2 t}, \quad \forall t \geq 0. \tag{3.31}$$

Remark 1. *We will indicate by c_0 a general constant that will change from one inequality to another.*

3.4.1 Technical Lemmas

First, we state and prove some technical Lemmas needed in the proof of our result.

Lemma 6. *Let (u, φ, θ, w) be a solution of the problem (3.1)-(3.3) . Then the energy functional $E(t)$, defined by (3.6) satisfies the following estimate*

$$\frac{d}{dt}E(t) \leq -k_2 \int_0^1 w_x^2 dx - k_3 \int_0^1 w^2 dx - \beta \int_0^1 \varphi_t^2 dx \leq 0. \tag{3.32}$$

Proof. By multiplying equations of (3.6) by $u_t,\ \varphi_t,\ \theta$ and w respectively in $L^2(0,1)$, we get

$$\begin{aligned}
&\bullet \quad \rho \int_0^1 u_{tt} u_t dx = \int_0^1 (\mu u_{xx} + b\varphi_x - \gamma\theta_x) u_t dx,\\
&\bullet \quad J \int_0^1 \varphi_{tt}\varphi_t dx = \int_0^1 (\delta\varphi_{xx} - bu_x - \xi\varphi - dw_x + m\theta - \beta\varphi_t)\varphi_t dx,\\
&\bullet \quad c \int_0^1 \theta_t \theta dx = \int_0^1 (-\gamma u_{tx} - m\varphi_t - k_1 w_x)\theta dx,\\
&\bullet \quad a \int_0^1 w_t w dx = \int_0^1 (k_2 w_{xx} - k_3 w - k_1\theta_x - d\varphi_{tx}) w dx.
\end{aligned} \tag{3.33}$$

By using integraion and the boundary conditions, then we obtain

$$\begin{aligned}
&\bullet \quad \frac{\rho}{2}\frac{d}{dt}\int_0^1 u_t^2 dx + \frac{\mu}{2}\frac{d}{dt}\int_0^1 u_x^2 dx + b\int_0^1 \varphi u_{tx} dx = -\gamma \int_0^1 \theta_x u_t dx,\\
&\bullet \quad \frac{J}{2}\frac{d}{dt}\int_0^1 \varphi_t^2 dx + \frac{\delta}{2}\frac{d}{dt}\varphi_x^2 + \frac{\xi}{2}\frac{d}{dt}\int_0^1 \varphi^2 dx + b\int_0^1 u_x\varphi_t dx\\
&\quad = -d\int_0^1 w_x\varphi_t dx + m\int_0^1 \theta\varphi_t dx - \beta\int_0^1 \varphi_t^2 dx,\\
&\bullet \quad \frac{c}{2}\frac{d}{dt}\int_0^1 \theta^2 dx dx = \int_0^1 (-\gamma u_{tx} - m\varphi_t - k_1 w_x)\theta dx,\\
&\bullet \quad \frac{a}{2}\frac{d}{dt}\int_0^1 w^2 dx = \int_0^1 (k_1\theta_x - d\varphi_{tx}) w dx - k_2\int_0^1 w_x^2 dx - k_3\int_0^1 w^2 dx.
\end{aligned} \tag{3.34}$$

By summing up all equality given by (3.34), we get

$$\frac{d}{dt}\left[\frac{1}{2}\left(\rho\|u_t\|_2^2 + J\|\varphi_t\|_2^2 + \mu\|u_x\|_2^2 + c\|\theta\|_2^2 + a\|w\|_2^2 + \delta\|\varphi_x\|_2^2 + \xi\|\varphi\|_2^2\right) + b\int_0^1 u_x\varphi dx\right] = -k_2\int_0^1 w_x^2 dx - k_3\int_0^1 w^2 dx - \beta\int_0^1 \varphi_t^2 dx. \tag{3.35}$$

By recalling the energy functional (3.6) given as follows

$$E(t) := \frac{1}{2}\left(\rho\|u_t\|_2^2 + J\|\varphi_t\|_2^2 + \mu\|u_x\|_2^2 + c\|\theta\|_2^2 + a\|w\|_2^2 + \delta\|\varphi_x\|_2^2 + \xi\|\varphi\|_2^2\right) + b\int_0^1 u_x\varphi dx. \tag{3.36}$$

Then, from (3.6) and (3.36), we deduce the desired inequality (3.32). □

Lemma 7. *Define the following functional*

$$I_1(t) := \rho\int_0^1 u_t u dx,\ t \geq 0. \tag{3.37}$$

Then, for (u, φ, θ, w) *solution of the problem* (3.1)-(3.3)*, the functional* I_1 *satisfies*

$$I_1'(t) \leq -\frac{\mu}{2}\int_0^1 u_x^2 dx + \rho\int_0^1 u_t^2 dx + c_0\int_0^1 \left(\varphi^2 + \theta^2\right) dx,\ \ t \geq 0. \tag{3.38}$$

Proof. Direct computation, using equation $((3.1))_1$ and then integrating by parts, we get

$$I_1'(t) = -\mu\int_0^1 u_x^2 dx + \rho\int_0^1 u_t^2 dx - b\int_0^1 \varphi u_x dx + \gamma\int_0^1 \theta u_x dx,\ \ t \geq 0. \tag{3.39}$$

Now, we can estimate the remaining terms in (3.39) by using Young's inequality as follows

$$\begin{aligned} &\bullet\quad -b\int_0^1 \varphi u_x dx \leq \frac{\mu}{4}\int_0^1 u_x^2 dx + c_0\int_0^1 \varphi^2 dx,\\ &\bullet\quad \gamma\int_0^1 \theta u_x dx \leq \frac{\mu}{4}\int_0^1 u_x^2 dx + c_0\int_0^1 \theta^2 dx. \end{aligned} \tag{3.40}$$

By substituting inequalities in (3.40). Then, we can find the desired estimate (3.38). □

Lemma 8. *Define the following functional*

$$I_2(t) := J\int_0^1 \varphi_t\varphi dx - \frac{b\rho}{\mu}\int_0^1 u_t\left(\int_0^x \varphi(y)\, dy\right) dx + \beta\int_0^1 \varphi^2 dx,\ t \geq 0, \tag{3.41}$$

for (u, φ, θ, w) *solution to the problem (3.1)-(3.2). Then, the functional I2 satisfies, for any* $\varepsilon_1 > 0$, *the following estimate*

$$\begin{aligned} I_2'(t) \leq & -\frac{\delta}{2}\int_0^1 \varphi_x^2 dx - \mu_1 \int_0^1 \varphi^2 dx + \varepsilon_1 \int_0^1 u_t^2 dx \\ & + c_0 \int_0^1 \left(w^2 + \theta^2\right) dx + c_0 \left(1 + \frac{1}{\varepsilon_1}\right) \int_0^1 \varphi_t^2 dx, \quad t \geq 0, \end{aligned} \tag{3.42}$$

where

$$\mu_1 = \left(\xi - \frac{b^2}{\mu}\right).$$

Proof. By differentiating the functional I_2, we obtain

$$\begin{aligned} I_2'(t) = & J \int_0^1 \varphi_{tt} \varphi dx + J \int_0^1 \varphi_t^2 dx - \frac{b\rho}{\mu} \int_0^1 u_t \left(\int_0^x \varphi_t\left(y\right) dy\right) dx \\ & - \frac{b\rho}{\mu} \int_0^1 u_{tt} \left(\int_0^x \varphi\left(y\right) dy\right) dx, \quad t \geq 0. \end{aligned}$$

Then, using the second equation of the system (3.1) to get

$$\begin{aligned} I_2' = & \int_0^1 (\delta\varphi_{xx} - bu_x - \xi\varphi - dw_x + m\theta - \beta\varphi_t)\varphi dx \\ & + J \int_0^1 \varphi_t^2 dx - \frac{b\rho}{\mu} \int_0^1 u_t \left(\int_0^x \varphi_t\left(y\right) dy\right) dx \\ & - \frac{b\rho}{\mu} \int_0^1 u_{tt} \left(\int_0^x \varphi\left(y\right) dy\right) dx, \quad t \geq 0. \end{aligned} \tag{3.43}$$

Hence, by using integration by parts together with the boundary conditions, we get

$$\begin{aligned} I_2'(t) = & -\delta \int_0^1 \varphi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right) \int_0^1 \varphi^2 dx + J \int_0^1 \varphi_t^2 dx \\ & + d \int_0^1 w \varphi_x dx - \frac{b\rho}{\mu} \int_0^1 u_t \left(\int_0^x \varphi_t\left(y\right) dy\right) dx \\ & + m \int_0^1 \theta\varphi dx - \frac{b\gamma}{\mu} \int_0^1 \theta\varphi dx. \end{aligned} \tag{3.44}$$

Now, applying Young's inequality as follows

$$\begin{aligned} \bullet \quad & d \int_0^1 w\varphi_x dx \leq \frac{\delta}{6} \int_0^1 \varphi_x^2 dx + c_0 \int_0^1 w^2 dx. \\ \bullet \quad & -\frac{b\rho}{\mu} \int_0^1 u_t \left(\int_0^x \varphi_t(y) dy\right) dx \leq \varepsilon_1 \int_0^1 u_t^2 dx + \frac{c_0}{\varepsilon_1} \int_0^1 \varphi_t^2 dx. \end{aligned} \tag{3.45}$$

$$\begin{aligned}
&\bullet \quad m\int_0^1 \theta\varphi dx \le \frac{\delta}{6}\int_0^1 \varphi_x^2 dx + c_0\int_0^1 \theta^2 dx.\\
&\bullet \quad -\frac{b\gamma}{\mu}\int_0^1 \theta\varphi \le \frac{\delta}{6}\int_0^1 \varphi_x^2 dx + c_0\int_0^1 \theta^2 dx.
\end{aligned} \tag{3.46}$$

By substituting the estimates (3.45) and (3.46) in equality (3.44), we arrive at

$$\begin{aligned}
I_2'(t) \le &-\frac{\delta}{2}\int_0^1 \varphi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right)\int_0^1 \varphi^2 dx + c_0\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1 \varphi_t^2 dx\\
&+ c_0\int_0^1 \left(w^2+\theta^2\right) dx + \varepsilon_1\int_0^1 u_t^2 dx,
\end{aligned}$$

which is exactly (3.42) with $\mu_1 = \xi - \frac{b^2}{\mu} > 0$. □

Lemma 9. *Define the following functional*

$$I_3(t) := -c\int_0^1 \theta\left(\int_0^x u_t(y,t)\, dy\right) dx,\ t \ge 0, \tag{3.47}$$

Then, for (u,φ,θ,w) *solution of* (3.1)-(3.3)*. Then, the functional* I_3 *satisfies*

$$\begin{aligned}
I_3'(t) \le &-\frac{\gamma}{2}\int_0^1 u_t^2 dx + \varepsilon\int_0^1 u_x^2 dx + \varepsilon\int_0^1 \varphi^2 dx + c_0\int_0^1 w^2 dx\\
&+ c_0\int_0^1 \varphi_t^2 dx + c_0\left(1+\frac{1}{\varepsilon}\right)\int_0^1 \theta^2 dx, \quad t \ge 0.
\end{aligned} \tag{3.48}$$

Proof. By differentiating the functional I_3, we obtain

$$I_3' = -c\int_0^1 \theta_t\left(\int_0^x u_t(y,t)dy\right) dx - c\int_0^1 \theta\left(\int_0^x u_{tt}(y,t)dy\right) dx, \quad t \ge 0. \tag{3.49}$$

Then, by using equations $(3.1)_1$, $(3.1)_3$ and integrating by parts, we find

$$\begin{aligned}
I_3'(t) = &-\gamma\int_0^1 u_t^2 dx - k_1\int_0^1 w u_t dx - c\mu\int_0^1 \theta u_x dx - cb\int_0^1 \theta\varphi dx\\
&+ c\gamma\int_0^1 \theta^2 dx + m\int_0^1 \varphi_t\left(\int_0^x u_t(y,t)\, dy\right) dx,\ t \ge 0.
\end{aligned} \tag{3.50}$$

Therefore, we use Young and Poincaré's inequalities, to get the following estimates

$$\bullet \quad -k_1\int_0^1 w u_t dx \le \frac{\gamma}{4}\int_0^1 u_t^2 dx + c_0\int_0^1 w^2 dx. \tag{3.51}$$

$$\begin{aligned} &\bullet \quad -c\mu\int_0^1 \theta u_x dx \le \varepsilon\int_0^1 u_x^2 dx + \frac{c_0}{\varepsilon}\int_0^1 \theta^2 dx \\ &\bullet \quad -cb\int_0^1 \theta\varphi dx \le \varepsilon\int_0^1 \varphi^2 dx + \frac{c_0}{\varepsilon}\int_0^1 \theta^2 dx. \end{aligned} \tag{3.52}$$

Finally, by using Cauchy-Schwarz inequality together with Young inequality, so the last term of equality (3.50) can be estimated as follows

$$\begin{aligned} m\int_0^1 \varphi_t\left(\int_0^x u_t(y,t)\,dy\right)dx &\le \frac{\gamma}{4}\int_0^1\left(\int_0^x u_t(y,t)\,dy\right)^2 dx + c_0\int_0^1 \varphi_t^2 dx. \\ &\le \frac{\gamma}{4}\left(\int_0^1 u_t dx\right)^2 + c_0\int_0^1 \varphi_t^2 dx \\ &\le \frac{\gamma}{4}\int_0^1 u_t^2 dx + c_0\int_0^1 \varphi_t^2 dx. \end{aligned} \tag{3.53}$$

Now, substituting estimates (3.51)-(3.53) into estimate (3.50), so we obtain the desired inequality (3.48). □

Lemma 10. *Define the following functional*

$$I_4(t) := ca\int_0^1\left(\int_0^x w(y)\,dy\right)\theta dx,\ t \ge 0, \tag{3.54}$$

Then, for u, φ, θ, w solution of the problem (3.1)-(3.3) *and for any $\varepsilon_1 > 0$, the functional I_4 satisfies the following estimate*

$$\begin{aligned} I_4'(t) \le &-\frac{k_1 c}{2}\int_0^1 \theta^2 dx + \varepsilon_1\int_0^1 u_t^2 dx + c_0\int_0^1\left(w_x^2+\varphi_t^2\right)dx \\ &+ c_0\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1 w^2 dx, \quad t \ge 0, \end{aligned} \tag{3.55}$$

Proof. First, the differentiation of the functional I_4, gives

$$\begin{aligned} I_4'(t) = ck_2&\int_0^1 \theta\left(\int_0^x w_{yy}(y)\,dy\right)dx - ck_3\int_0^1 \theta\left(\int_0^x w(y)\,dy\right)dx \\ &- k_1 c\int_0^1 \theta\left(\int_0^x \theta_y(y)\,dy\right)dx - dc\int_0^1 \theta\left(\int_0^x \varphi_{ty}(y)\,dy\right)dx \\ &- \gamma a\int_0^1\left(\int_0^x w(y)\,dy\right)u_{tx}dx - ma\int_0^1\left(\int_0^x w(y)\,dy\right)\varphi_t dx \\ &- k_1 a\int_0^1\left(\int_0^x w(y)\,dy\right)w_x dx,\ t \ge 0. \end{aligned} \tag{3.56}$$

Then, we simplifying equality (3.56) by integrating by parts and using the fact that $\int_0^1 w\,(x)\,dx = 0$, so we get

$$\begin{aligned} I_4'\,(t) = &-k_1 c \int_0^1 \theta^2 dx - a k_1 \int_0^1 w^2 dy + k_2 c \int_0^1 w_x \theta dx \\ &- dc \int_0^1 \varphi_t \theta dx - k_3 c \int_0^1 \left(\int_0^x w dy \right) \theta dx + a\gamma \int_0^1 w u_t dx \\ &- am \int_0^1 \left(\int_0^x w\,(y)\,dy \right) \varphi_t dx, \;\; t \geq 0. \end{aligned} \tag{3.57}$$

By using Young's inequality, we can gives the following estimates

$$\bullet \quad k_2 c \int_0^1 w_x \theta dx \leq c_0 \int_0^1 w_x^2 dx + \frac{k_1 c}{6} \int_0^1 \theta^2 dx. \tag{3.58}$$

also we have

$$\begin{aligned} &\bullet \quad -k_3 c \int_0^1 \left(\int_0^x w dy \right) \theta dx \leq c_0 \int_0^1 w^2 dx + \frac{k_1 c}{6} \int_0^1 \theta^2 dx. \\ &\bullet \quad -dc \int_0^1 \theta \varphi_t dx \leq c_0 \int_0^1 \varphi_t^2 dx + \frac{k_1 c}{6} \int_0^1 \theta^2 dx. \\ &\bullet \quad a \int_0^1 w u_t dx \leq \varepsilon_1 \int_0^1 u_t^2 dx + \frac{c_0}{\varepsilon_1} \int_0^1 w^2 dx. \\ &\bullet \quad -am \int_0^1 \left(\int_0^x w dy \right) \varphi_t dx \leq c_0 \int_0^1 \left(\varphi_t^2 + w^2 \right) dx. \end{aligned} \tag{3.59}$$

By substituting estimates (3.58) and (3.59) into equality (3.57), we get exactly the desired estimate (3.55). □

We are now ready to state and prove the main result.

3.4.2 Proof of Theorem (7)

We define for N, N_1 and $N_2 > 0$ the following functional

$$F(t) := NE(t) + I_1\,(t) + N_1 I_2\,(t) + \frac{4\rho}{\gamma} I_3\,(t) + N_2 I_4\,(t)\,. \tag{3.60}$$

Remark 2. *By using the Young, Poincaré and Cauchy-Schwarz inequalities we can easily prove that the functional energy E is equivalent to the functional F that is, for two positive constants κ_1 and κ_2,*

$$\kappa_1 E\,(t) \leq F(t) \leq \kappa_2 E\,(t)\,, \quad \forall t \geq 0.$$

Based on the previous Lemmas, we can give the proof of the stability result of the problem (3.1)-(3.3) .

Proof. By differentiating equation (3.60), then recalling equations (3.32), (3.38), (3.42), (3.48) and (3.55). So, we get for all $t > 0$

$$\begin{aligned}\frac{d}{dt}F(t) \leq &-C_w \int_0^1 w^2 dx - C_{\varphi_t} \int_0^1 \varphi_t^2 dx - C_{\varphi_x} \int_0^1 \varphi_x^2 dx \\ &- C_\theta \int_0^1 \theta^2 dx - C_{w_x} \int_0^1 w_x^2 dx - C_{u_x} \int_0^1 u_x^2 dx \\ &- C_{u_t} \int_0^1 u_t^2 dx - C_\varphi \int_0^1 \varphi^2 dx,\end{aligned} \tag{3.61}$$

where

$$\begin{aligned}&C_w = Nk_3 - N_1 c_0 - N_2 c_0 \left(1 + \tfrac{1}{\varepsilon_1}\right), \quad C_{\varphi_t} = \beta N - N_1 c_0 \left(1 + \tfrac{1}{\varepsilon_1}\right), C_{\varphi_x} = \tfrac{\delta N_2}{2}, \\ &C_\theta = \tfrac{N_2 k_1 c}{2} - N_1 c_0 - c_0 \left(1 + \tfrac{1}{\varepsilon_2}\right), \\ &C_{w_x} = Nk_2 - N_2 c_0, \qquad\qquad C_{u_x} = \tfrac{\mu}{2} - 4\rho\varepsilon_2, \\ &C_{u_t} = \rho - N_1\varepsilon_1 - N_2\varepsilon_1, \qquad\qquad C_\varphi = N_1\mu_1 - c_0 - 4\rho\varepsilon_2.\end{aligned} \tag{3.62}$$

Now, all these terms (on the right-hand side of (3.61)) become negative if we select our parameters appropriately.

First, choose ε_1 and ε_2 so small that

$$\varepsilon_1 < \frac{\rho}{N_1 + N_2}, \quad \varepsilon_2 < \frac{\mu}{8\rho},$$

and N_1 large enough so that

$$N_1 > \frac{1}{\mu_1}\left(c_0 + 4\rho\varepsilon_2\right).$$

Next, we select N_2 large enough so that

$$N_2 > \frac{2}{k_1 c}\left[N_1 c_0 + c_0\left(1 + \frac{1}{\varepsilon_2}\right)\right].$$

Finally, we choose N large enough so that (3.62) remains valid and, further

$$Nk_3 - c_0 - N_2 c_0 \left(1 + \frac{1}{\varepsilon_1}\right) > 0,$$

$$\beta N - c_0 - N_2 c_0,$$

and

$$Nk_2 - N_2 c_0 > 0.$$

So, we arrive at

$$\begin{aligned} F'(t) &\leq -\beta_1 \int_0^1 \left(u_t^2 + \varphi_t^2 + \theta^2 + w^2 + (u_x + \varphi)^2 + \varphi_x^2\right) dx \\ &\leq -\beta_2 E(t), \end{aligned}$$

for some positive constants β_1 and β_2.

Having in mind the remark on the equivalence of $E(t)$ and $F(t)$ we infer that

$$F'(t) \leq -d_1 F(t), \ t \geq 0, \tag{3.63}$$

where

$$d_1 = \frac{\beta_2}{\kappa_2} > 0.$$

A simple integration of (3.63) gives

$$F(t) \leq F(0)\, e^{-d_1 t}, \ t \geq 0,$$

which yields the desired result (3.31) by using the other side of the equivalence relation again.

□

4

Stability of a nonlinear hyperbolic-parabolic coupled system

Contents

4.1 Introduction

In this chapter we consider the study of a mechanical problem with thermal effects, on the one hand we have concentrated on the existence and uniqueness of the solutions and on the other hand we have shown the result of the stability of the problem, more precisely, this problem represents a von Karman system coupled with the thermal effect of the second sound where the heat flow is given by Cattaneo's law. This system is given as follows

$$\begin{aligned}
&w_{tt} + \gamma_1 w_t - d_1 \left[\left(u_x + \frac{1}{2} (w_x)^2 \right) w_x \right]_x + d_2 w_{xxxx} = 0, \quad \omega \times \mathbb{R}_+,\\
&u_{tt} - d_1 \left[\left(u_x + \frac{1}{2} (w_x)^2 \right) \right]_x + \delta \theta_x = 0, \quad \omega \times \mathbb{R}_+,\\
&\theta_t + q_x + \delta u_{tx} = 0, \quad \omega \times \mathbb{R}_+,\\
&q_t + \gamma_2 q + \theta_x = 0, \quad \omega \times \mathbb{R}_+,
\end{aligned} \tag{4.1}$$

subject to the following boundary conditions

$$\begin{aligned}
&u(0,t) = u(L,t) = w(0,t) = w(L,t) = 0, \quad t \in \mathbb{R}_+,\\
&w_x(0,t) = w_x(L,t) = \theta(0,t) = \theta(L,t) = 0, \quad t \in \mathbb{R}_+,
\end{aligned} \tag{4.2}$$

and the initial data

$$\begin{aligned}
&u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, \quad w(\cdot,0) = w_0, \quad x \in (0,L),\\
&w_t(\cdot,0) = w_1, \quad q(\cdot,0) = q_0, \quad \theta(\cdot,0) = \theta_0, \quad x \in (0,L).
\end{aligned} \tag{4.3}$$

Here, the functions $u(x,t), w(x,t), \theta(x,t)$ and $q(x,t)$ represent, respectively, the longitudinal, the transversal displacements, the temperature difference and the heat flux.

The domain Ω is an interval $(0,L)$ and the coefficients d_1, d_2, δ, γ_1 and γ_2 are positive constants which have a physical meaning.

From the fourth equation of the system (4.1) and the boundary conditions, we easily verify that

$$\tfrac{d}{dt} \int_0^L q(x,t)\,dx + \gamma_2 \int_0^L q(x,t)\,dx = 0.$$

So, if we put

$$\overline{q}(x,t) = q(x,t) - \left(\int_0^L q_0(x)\,dx \right) \exp(-\gamma_2 t),$$

then, by simple substitution, we check that $(w,u,\theta,\overline{q})$ satisfies (4.1) and more importantly we have

$$\int_0^L \overline{q}(x,t)\,dx = 0, \ \ \forall t \geq 0,$$

which justified the use of the inequality of Poincaré for q. From now on, we work with $\overline{q}$ but write q for simplicity.

This chapter has been inspired by the work [100], the first result is the following existence and uniqueness theorem

Theorem 8. *Let $(w, \varphi, u, \psi, \theta, q)^T \in \mathcal{H}$. For any initial condition $U_0 \in \mathcal{H}$, there exists a unique solution for the problem* (4.7) *such that*

$$U \in C\left([0, \infty), \mathcal{H}\right).$$

Moreover, if $U_0 \in D\left(\mathcal{A}\right)$. Then, we have

$$U \in C\left([0, \infty), D\left(\mathcal{A}\right)\right) \cap C^1\left([0, \infty), \mathcal{H}\right).$$

We note that based on the semigroup approach our problem can be represented as a semilinear Cauchy problem which is given by

$$\begin{cases} U_t = \mathcal{A}U + \mathcal{F}\left(U\right), \quad t > 0, \\ U\left(0\right) = \left(w_0, w_1, u_0, u_1, \theta_0, q_0\right), \end{cases} \tag{4.4}$$

Then, we can use the semigroup theory for the proof of the above result. Now, we define the energy functional by

$$E\left(t\right) = \frac{1}{2}\left[\|w_t\|_{L^2}^2 + \|u_t\|_{L^2}^2 + \|\theta\|_{L^2}^2 + \|q\|_{L^2}^2 + d_2\|w_{xx}\|_{L^2}^2 + d_1\left\|u_x + \frac{1}{2}\left(w_x\right)^2\right\|_{L^2}^2\right].$$

We give the following stability result.

Theorem 9. *Let $(u,\ w,\ \theta,\ q)$ be a solution of* (4.1)-(4.3) *where the initial data are given in $\mathcal{H}$. Then, the energy $E(t)$ satisfies*

$$E(t) \leq kE(0)e^{-\xi t}, \quad \forall t \geq 0 \tag{4.5}$$

where k and ξ are two positive constants.

By using the multiplier method, we prove that the dissipation induced by the heat effects of second sound is strong enough to stabilize system (4.1) in the presence of a frictional damping in the first equation of the system.

4.1.1 Earlier results

In this section, we will shed light on the most important studies that dealt with the study of nonlinear dynamic elastic systems presented by von K´arm´an's equations. We can say that the nonlinearity contained in the system comes from the modeling that depends on the theories of oscillation. Moreover, in the following works [66, 101–104], the stability of some systems has been demonstrated, through which we find that there is a correlation between the results of this stability and the effects related to damping on the one hand, as well as temperature on the other hand. Lagnese et al. in paper [105], addressed the problems of existence, uniqueness, and the behavior of solution over time that some effects of damping are being considered, in addition to some other important characteristics, see e.g., [106], and references therein. Benabdallah and Teniou [65] proposed a new model in which a strong coupling of the system was imposed with two thermal equations: one for the longitudinal displacement and the other for the transversal displacement. As a result, the authors have shown that the system is exponentially stable. In paper [107] Djebabla and Tatar proposed a one-dimensional von Karman in which the coupling with the thermal effects is on the longitudinal displacement, where the heat equation according to Green-Naghdi's theory, see references [36, 37, 108]. Similarly to the previous mention work, the authors have shown that the system is exponentially stable.

4.1.2 Chapter plan

This chapter is organized as follows. In Section (4.3) we state and show the well posedness of the system. In Section (4.4), we establish our stability results.

4.2 Preliminaries

We introduce the following spaces

$$L^2_*(0,L) = \left\{ v \in L^2(0,L) : \int_0^L v\,dx = 0 \right\},$$

and

$$H^1_*(0,L) = H^1(0,L) \cap L^2_*(0,L).$$

Also, we give the Hilbert space

$$\mathcal{H} := H^2_0(0,L) \times L^2(0,L) \times H^1_0(0,L) \times L^2(0,L) \times L^2(0,L) \times L^2_*(0,L),$$

the space is equipped with the following inner product

$$\begin{aligned}\left\langle U,\widetilde{U}\right\rangle_{\mathcal{H}} = &\int_0^L \varphi\widetilde{\varphi}dx + \int_0^L \psi\widetilde{\psi}dx + \int_0^L \theta\widetilde{\theta}dx + \int_0^L q\widetilde{q}dx + d_2\int_0^L w_{xx}\widetilde{w}_{xx}dx \\ &+ d_1\int_0^L u_x\widetilde{u}_x dx,\end{aligned} \tag{4.6}$$

where

$$U = (w,\varphi,u,\psi,\theta,q)^T \in \mathcal{H}$$

and

$$\tilde{U} = (\tilde{w},\tilde{\varphi},\tilde{u},\tilde{\psi},\tilde{\theta},\tilde{q})^T \in \mathcal{H}.$$

4.3 Existence and uniqueness

In this section, we prove the existence, uniqueness, and smoothness of solution of problem (4.1)-(4.3) using the semigroup theory, see e.g., [99]. In this direction our main result is given by the following theorem.

Theorem 10. *Let* $(w,\varphi,u,\psi,\theta,q)^T \in \mathcal{H}$. *For any initial condition* $U_0 \in \mathcal{H}$, *there exists a unique solution for the problem* (4.7) *such that*

$$U \in C\left([0,\infty),\mathcal{H}\right).$$

Moreover, if $U_0 \in D(\mathcal{A})$. *Then, we have*

$$U \in C\left([0,\infty), D(\mathcal{A})\right) \cap C^1\left([0,\infty),\mathcal{H}\right).$$

4.3.1 The semigroup approach

We introduce two new dependent variables $\varphi = w_t$, and $\psi = u_t$. Then, system (4.1)-(4.3) takes the form of an abstract first-order evolutionary problem

$$\begin{cases} U_t = \mathcal{A}U + \mathcal{F}(U), \quad t>0, \\ U(0) = (w_0,w_1,u_0,u_1,\theta_0,q_0), \end{cases} \tag{4.7}$$

where $U = (w, w_t, u, u_t, \theta, q)^T$ and the linear operator $\mathcal{A}$ is defined as follows

$$A = \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 \\ -d_2\partial_x^4 & -\gamma_1 I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & d_1\partial_x^2 & -\delta\partial_x & 0 & 0 \\ 0 & 0 & 0 & -\delta\partial_x & 0 & -\partial_x \\ 0 & 0 & 0 & 0 & -\partial_x & -\gamma_2 I \end{pmatrix},$$

and

$$\mathcal{F}(U) = \begin{pmatrix} 0 \\ d_1\left[\left(u_x + \frac{1}{2}(w_x)^2\right) w_x\right]_x \\ 0 \\ \frac{d_1}{2}(w_x)_x^2 \\ 0 \\ 0 \end{pmatrix}.$$

It is clear that $\mathcal{F}(U)$ is a continuous and uniformly Lipschitz operator.

The domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{ \begin{array}{c} U \in \mathcal{H}/\ w \in H^4(0,L) \cap H_0^2(0,L),\ u \in H^2(0,L) \cap H_0^1(0,L) \\ \theta \in H_0^1(0,L),\ q \in H_*^1(0,L),\ \varphi \in H_0^2(0,L),\ \ \psi \in H_0^1(0,L) \end{array} \right\}$$

4.3.2 Proof of Theorem (10)

Proof. First, the domain of $\mathcal{A}$ is dense in $\mathcal{H}$. Next, we show that the operator $\mathcal{A}$ generates a C_0-semigroup in $\mathcal{H}$. For this step, we prove that the operator $\mathcal{A}$ is dissipative.

Let define the vector $U = (w, \varphi, u, \psi, \theta, q)^T$. Then, we have

$$\mathcal{A}U = \begin{pmatrix} \varphi \\ -\gamma_1\varphi - d_2 w_{xxxx} \\ \psi \\ d_1 u_{xx} - \delta\theta_x \\ -q_x - \delta\psi_x \\ -\gamma_2 q - \theta_x \end{pmatrix},$$

it is straightforward to verify that, for any $U \in D(\mathcal{A})$

$$
\begin{aligned}
\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= \int_0^L \left(-\gamma_1\varphi^2 - d_2\varphi w_{xxxx} - \delta\theta_x\psi + d_1\psi u_{xx} - q\theta_x - \gamma_2 q^2\right)\, dx \\
&\quad + \int_0^L \left(d_2 w_{xx}\varphi_{xx} - \theta q_x - \delta\theta\psi_x + d_1 u_x\psi_x\right)\, dx \\
&= -\gamma_1\int_0^L \varphi^2 - \gamma_2\int_0^L q^2\, dx \leq 0.
\end{aligned}
$$

After that, we prove that the operator $(I - \mathcal{A})$ is surjective.

Let define the vector functions $F = (f_1,\ f_2,\ f_3,\ f_4,\ f_5,\ f_6)^T \in H$, we prove that there exists $U = (w, \varphi, u, \psi, \theta, q)^T \in D\left(\mathcal{A}\right)$ satisfying

$$
\left(I - \mathcal{A}\right) U = F. \tag{4.8}
$$

From equation (4.8), we deduce the following equivalent system

$$
\begin{cases}
w - \varphi = f_1 \in H_0^2\left(0, L\right), \\
\varphi + \gamma_1\varphi + d_2 w_{xxxx} = f_2 \in L^2\left(0, L\right), \\
u - \psi = f_3 \in H_0^1\left(0, L\right) \\
\psi - d_1 u_{xx} + \delta\theta_x = f_4 \in L^2\left(0, L\right), \\
\theta + q_x + \delta\psi_x = f_5 \in L^2\left(0, L\right), \\
\left(1 + \gamma_2\right) q + \theta_x = f_6 \in L_*^2\left(0, L\right).
\end{cases} \tag{4.9}
$$

Then, from equations $(4.9)_1$, $(4.9)_3$ and $(4.9)_6$, it yeilds

$$
\begin{aligned}
&\varphi = w - f_1 \in H_0^2\left(0, L\right), \\
&\psi = u - f_3 \in H_0^1\left(0, L\right), \\
&\theta_x = f_6 - \left(1 + \gamma_2\right) q \in L^2\left(0, L\right), \\
&q_x - \left(1 + \gamma_2\right)\textstyle\int_0^x q dx = f_5 - \int_0^x f_6 dx + \delta f_x^3 - \delta u_x \in L^2\left(0, L\right).
\end{aligned} \tag{4.10}
$$

From equality $(4.10)_3$, we find

$$
\theta\left(0, t\right) = \theta\left(L, t\right) = 0,\ \forall\, t \geq 0.
$$

Then, it implies that

$$\theta \in H_0^1(0,L).$$

Now, by the regularity of the elliptic problem we can conclude from $(4.10)_4$ that

$$q \in H_*^1(0,L).$$

Substituting $\varphi,\ \psi$ and q given by $(4.9)_1$, $(4.9)_3$, $(4.9)_5$ and $(4.9)_6$, then we obtain the following system

$$\begin{cases} (1+\gamma_1)w + d_2 w_{xxxx} = g_1 \in L^2(0,L), \\ -d_1 u_{xx} + u + \delta\theta_x = g_2 \in L^2(0,L), \end{cases}$$

where

$$g_1 = f_2 + f_1(1+\gamma_1),$$

$$g_2 = f_3 + f_4.$$

Now, we define the bilinear form B over the Hilbert space

$$V = H_0^2(0,L) \times H_0^1(0,L)$$

as follows

$$\begin{aligned} B\left((w,u),(\tilde{w},\tilde{u})\right) = &\int_0^L \left[(1+\gamma_1)w\tilde{w} + d_2 w_{xx}\tilde{w}_{xx} + u\tilde{u} + \delta\theta_x\tilde{u}\right]\, dx \\ &+ \int_0^L d_1 u_x \tilde{u}_x dx \end{aligned}$$

and the linear form F by

$$F(\tilde{w},\tilde{u}) = \int_0^L (g_1\tilde{w} + g_2\tilde{u})\, dx$$

It is easy to verify that B is continuous and coercive, and F is continuous. So applying the Lax-Milgram theorem, we obtain the existence and uniqueness of a vector

$$(w,u,\theta) \in V$$

, which satisfies

$$B\left((w,u),(\tilde{w},\tilde{u})\right)=F(\tilde{w},\tilde{u}),\quad \forall(\tilde{w},\tilde{u})\in V. \tag{4.11}$$

Now by taking $\tilde{U}=(\tilde{w},0)$ in (4.11), we get

$$\int_0^L(1+\gamma_1)w\tilde{w}+d_2w_{xx}\tilde{w}_{xx}\;dx=\int_0^L g_1\tilde{w}\;dx,\quad \forall\tilde{w}\in H_0^1\left(0,L\right), \tag{4.12}$$

using twice integration by parts in (4.12), we obtain

$$(1+\gamma_1)w+d_2w_{xxxx}=g_1\in L^2\left(0,L\right). \tag{4.13}$$

Consequetly, we get

$$w\in H^4\left(0,L\right)\cap H_0^2\left(0,L\right).$$

Taking $\tilde{U}=(0,\tilde{u})$, then (4.11) reduces to

$$\int_0^L d_1u_x\tilde{u}_x+u\tilde{u}+\delta\theta_x\tilde{u}\;dx=\int_0^L g_2\tilde{u}\;dx,\;\forall\tilde{u}\in H_0^1\left(0,L\right). \tag{4.14}$$

That is

$$d_1u_{xx}=u+\delta\theta_x-g_2\in L^2\left(0,L\right), \tag{4.15}$$

by the regularity of the elliptic problem we can conclude that

$$u\in H^2\left(0,L\right)\cap H_0^1\left(0,L\right)$$

Hence, there exists a unique $U\in D\left(\mathcal{A}\right)$ such that (4.8) is satisfied. Therefore, the operator $\mathcal{A}$ is maximal. From where, we conclude that $\mathcal{A}$ is a maximal monotone operator. □

4.4 Stability result of solutions

In this section we state and prove our result on exponential decay for the nonlinear system (4.1)-(4.3).

Theorem 11. *Let* $(u,\,w,\,\theta,\,q)$ *be a solution of* (4.1)-(4.3) *where the initial data are given in* $\mathcal{H}$.

Then, the energy $E(t)$ satisfies

$$E(t) \leq kE(0)e^{-\xi t}, \quad \forall t \geq 0 \tag{4.16}$$

where k and ξ are two positive constants.

Firstly, we present some technical lemmas.

4.4.1 Technical Lemmas

Lemma 11. *Let $(w,\, u,\, \theta,\, q)$ be the solution of (4.1)-(4.3). Then, the energy functional E, defined by*

$$E\left(t\right) := \frac{1}{2}\left[\|w_t\|_{L^2}^2 + \|u_t\|_{L^2}^2 + \|\theta\|_{L^2}^2 + \|q\|_{L^2}^2 + d_2\left\|w_{xx}\right\|_{L^2}^2 + d_1\left\|u_x + \frac{1}{2}\left(w_x\right)^2\right\|_{L^2}^2\right],$$

satisfies

$$E'(t) = -\gamma_1\left\|w_t\right\|_{L^2}^2 dx - \gamma_2\left\|q\right\|_{L^2}^2 dx, \quad \forall t \geq 0. \tag{4.17}$$

Proof. By multiplying equations of system (4.1) by u_t, w_t, θ and q respectively in $L^2(0,L)$, we gat

$$\begin{aligned}
&\bullet \int_0^L w_{tt}w_t dx + d_2\int_0^L w_{xxxx}w_t dx = d_1\int_0^L \left[\left(u_x + \frac{1}{2}w_x^2\right)w_x\right]_x w_t dx - \gamma_1\int_0^L w_t^2 dx.\\
&\bullet \int_0^L u_{tt}u_t dx = -\delta\int_0^L \theta_x u_t dx + d_1\int_0^L \left[\left(u_x + \frac{1}{2}(w_x)^2\right]_x u_t dx.\\
&\bullet \int_0^L \theta_t\theta dx = -\int_0^L q_x\theta dx - \delta\int_0^L u_{tx}\theta dx.\\
&\bullet \int_0^L q_t q dx = -\gamma_2\int_0^L q^2 dx - \int_0^L \theta_x q dx.
\end{aligned} \tag{4.18}$$

Simplifying (4.18) by using integration by part and the boundary conditions (4.2). Then, we get

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\left[\|w_t\|_{L^2}^2 + \|u_t\|_{L^2}^2 + \|\theta\|_{L^2}^2 + \|q\|_{L^2}^2 + d_2\left\|w_{xx}\right\|_{L^2}^2 + d_1\left\|u_x + \frac{1}{2}\left(w_x\right)^2\right\|_{L^2}^2\right]\\
&\quad = -\gamma_1\|w_t\|_{L^2}^2 - \gamma_2\|q\|_{L^2}^2, \;\; t \geq 0.
\end{aligned} \tag{4.19}$$

By recalling the definition of the energy functional, the equality (4.17) appears directly. □

Lemma 12. *Define the following functional*

$$I_1(t) = \int_\Omega \left(u_t u + \frac{1}{2} w_t w + \frac{\gamma_1}{4} w^2 \right) dx,\ t \geq 0, \tag{4.20}$$

Then, for $(u,\, w,\, \theta,\, q)$ *solution of the problem* (4.1)-(4.3) *and for any* $\varepsilon_1 > 0$, *the functional* I_1 *satisfies*

$$\begin{aligned} \frac{d}{dt} I_1(t) \leq & -d_1 \left\| u_x + \frac{1}{2} (w_x)^2 \right\|_{L^2}^2 - \frac{d_2}{2} \|w_{xx}\|_{L^2}^2 \\ & + \|u_t\|_{L^2}^2 + \frac{1}{2} \|w_t\|_{L^2}^2 + \varepsilon_1 \|u_x\|^2 + \frac{\delta^2}{4\varepsilon_1} \|\theta\|_{L^2}^2, \quad t \geq 0. \end{aligned} \tag{4.21}$$

Proof. Differentiating the functional I_1 and using the first and the second equation of(4.1) we get

$$\begin{aligned} \frac{d}{dt} I_1(t) = & \|u_t\|_{L^2}^2 + \int_\Omega \left[d_1 \left[u_x + \frac{1}{2} (w_x)^2 \right]_x - \delta \theta_x \right] u dx + \frac{\gamma_1}{2} \int_\Omega w_t w dx \\ & + \frac{1}{2} \int_\Omega \left[-\gamma_1 w_t + d_1 \left[\left(u_x + \frac{1}{2} (w_x)^2 \right) w_x \right]_x - d_2 w_{xxxx} \right] w dx + \frac{1}{2} \|w_t\|_{L^2}^2 \\ = & \|u_t\|_{L^2}^2 - d_1 \int_\Omega \left(u_x + \frac{1}{2} (w_x)^2 \right) u_x dx + \delta \int_\Omega \theta u_x dx + \frac{1}{2} \|w_t\|_{L^2}^2 \\ & - \frac{d_1}{2} \int_\Omega \left(u_x + \frac{1}{2} (w_x)^2 \right) w_x^2 dx - \frac{d_2}{2} \|w_{xx}\|_{L^2}^2, \quad t \geq 0. \end{aligned} \tag{4.22}$$

The application of Young's inequality gives

- $$\delta \int_\Omega \theta u_x dx \leq \varepsilon_1 \|u_x\|_{L^2}^2 + \frac{\delta^2}{4\varepsilon_1} \|\theta\|_{L^2}^2. \tag{4.23}$$

Substituting inequality (4.23) in equality (4.22), we get the desired estimate (4.21). □

Lemma 13. *Define the following functional*

$$I_2(t) = \int_\Omega \left(\int_0^x \theta(t, y)\, dy \right) u_t dx, \quad t \geq 0, \tag{4.24}$$

The, for any $\varepsilon_2 > 0$, *the functional* I_2 *satisfies the estimate*

$$\begin{aligned} \frac{d}{dt} I_2(t) \leq & -\frac{\delta}{2} \|u_t\|_{L^2}^2 + \varepsilon_2 \left\| u_x + \frac{1}{2} (w_x)^2 \right\|_{L^2}^2 \\ & + \frac{1}{2\delta} \|q\|^2 + C(\varepsilon_2) \|\theta\|_{L^2}^2 + \varepsilon_2 u_x^2 (L), \quad t \geq 0, \end{aligned} \tag{4.25}$$

where

$$C(\varepsilon_2) = \frac{d_1^2}{4\varepsilon_2}(1+L) + \delta.$$

Proof. Differentiatiing the functional (4.24) along solutions of system (4.1). Then, by taking into account the boundary conditions (4.2), that yields

$$\begin{aligned}\frac{d}{dt}I_2(t) &= \int_\Omega \left(\int_0^x \left[-q_x - \delta u_{tx}\right] dy\right) u_t dx \\ &\quad + \int_\Omega \left(\int_0^x \theta\left(t,y\right) dy\right) \left[d_1 \left[u_x + \frac{1}{2}\left(w_x\right)^2\right]_x - \delta\theta_x\right] dx \\ &= -\delta \left\|u_t\right\|_{L^2}^2 - \int_\Omega q u_t dx + \delta \left\|\theta\right\|_{L^2}^2 \\ &\quad - d_1 \int_\Omega \left(u_x + \frac{1}{2}\left(w_x\right)^2\right) \theta dx + d_1 \left(\int_0^L \theta dx\right) u_x\left(L\right), \quad t \geq 0.\end{aligned} \tag{4.26}$$

Thanks to Young's inequality, we obtain

- $$-\int_\Omega q u_t dx \leq \frac{\delta}{2}\left\|u_t\right\|_{L^2}^2 + \frac{1}{2\delta}\left\|q\right\|_{L^2}^2. \tag{4.27}$$

Similarly, for any $\varepsilon_2 > 0$, we have

- $$d_1 \left(\int_0^L \theta dx\right) u_x\left(L\right) \leq \varepsilon_2 u_x^2\left(L\right) + \frac{d_1^2 L}{4\varepsilon_2}\left\|\theta\right\|_{L^2}^2, \tag{4.28}$$

and also

- $$-d_1 \int_\Omega \left(u_x + \frac{1}{2}\left(w_x\right)^2\right) \theta dx \leq \varepsilon_2 \left\|u_x + \frac{1}{2}\left(w_x\right)^2\right\|_{L^2}^2 + \frac{d_1^2}{4\varepsilon_2}\left\|\theta\right\|_{L^2}^2. \tag{4.29}$$

By substituting the estimates (4.27)-(4.29) into equality (4.26). Then, inequality (4.25) is established. □

Lemma 14. *Define the following functional*

$$I_3\left(t\right) = \int_\Omega \left(\int_0^x q\left(t,y\right) dy\right) \theta dx, \quad t \geq 0, \tag{4.30}$$

let $(u,\, w,\, \theta,\, q)$ *be a solution of* (4.1)-(4.3). *Then, for any* $\varepsilon_3 > 0$, *the functional* I_3 *satisfies*

$$\frac{d}{dt}I_3(t) \leq -\frac{1}{2}\left\|\theta\right\|_{L^2}^2 + \varepsilon_3 \left\|u_t\right\|_{L^2}^2 + C_1\left(\varepsilon_3\right)\left\|q\right\|_{L^2}^2, \quad t \geq 0, \tag{4.31}$$

where

$$C_1(\varepsilon_3) = 1 + \frac{\delta^2}{4\varepsilon_3} + \frac{\gamma_2^2 L}{2}.$$

Proof. A differentiation of the equality (4.30) and the use of the system equations give

$$\begin{aligned}\frac{d}{dt}I_3(t) &= \int_\Omega \left(\int_0^x [-\gamma_2 q - \theta_x]\, dy\right)\theta dx + \int_\Omega \left(\int_0^x q(t,y)\, dy\right)[-q_x - \delta u_{tx}]\, dx \\ &= -\|\theta\|_{L^2}^2 + \|q\|_{L^2}^2 - \gamma_2 \int_\Omega \left(\int_0^x q(t,y)\, dy\right)\theta dx + \delta \int_\Omega q u_t dx, \quad t \geq 0.\end{aligned} \tag{4.32}$$

Hence, by exploiting Young's and Poincaré's inequalities, we obtain

$$\begin{aligned}&\bullet \quad -\gamma_2 \int_\Omega \left(\int_0^x q(t,y)\, dy\right)\theta dx \leq \frac{1}{2}\|\theta\|_{L^2}^2 + \frac{\gamma_2^2 L}{2}\|q\|_{L^2}^2, \\ &\bullet \quad \delta \int_\Omega q u_t dx \leq \varepsilon_3 \|u_t\|_{L^2}^2 + \frac{\delta^2}{4\varepsilon_3}\|q\|_{L^2}^2.\end{aligned} \tag{4.33}$$

Thus, by combining inequalities (4.33) with (4.32), we obtain

$$\frac{d}{dt}I_3(t) \leq -\frac{1}{2}\|\theta\|_{L^2}^2 + \varepsilon_3 \|u_t\|_{L^2}^2 + C_1(\varepsilon_3)\|q\|_{L^2}^2, \quad t \geq 0,$$

which is exactly the desired inequality (4.31). □

we introduce the function

$$m(x) = 2 - \frac{4x}{L}, \ \ x \text{ in } \ \Omega,$$

and also the following functionals

$$\begin{aligned}&\bullet \quad A(t) := \int_\omega u_t m u_x dx. \\ &\bullet \quad B(t) := \int_\omega w_t m w_x dx. \\ &\bullet \quad C(t) := -\int_\omega (\theta - \delta u_x) m q dx.\end{aligned} \tag{4.34}$$

Remark 3. *The previous function $m(x)$ will be used to eliminate the boundary term.*

Lemma 15.

$$I_4(t) = A(t) + B(t) + C(t), \quad t \geq 0. \tag{4.35}$$

Then, for (u,w,θ,q) solution of the problem, the functional I_4 satisfies the following estimate

$$\begin{aligned}\frac{d}{dt}I_4(t) \le & -d_1\left[u_x^2(L)+u_x^2(0)\right]+\left(1+\frac{2d_1}{L}\right)\|u_x\|_{L^2}^2+\frac{2}{L}\|u_t\|_{L^2}^2 \\ & +\left(\frac{2}{L}+\gamma_1\right)\|w_t\|_{L^2}^2+\left(\gamma_1 C_p+\frac{6d_2}{L}\right)\|w_{xx}\|_{L^2}^2+\frac{8d_1}{L}\left\|u_x+\frac{1}{2}(w_x)^2\right\|_{L^2}^2 \\ & +\left(\gamma_2+\frac{2}{L}\right)\|\theta\|_{L^2}^2+\left(\frac{2}{L}+\gamma_2+\gamma_2^2\delta^2\right)\|q\|_{L^2}^2, \quad t\ge 0,\end{aligned} \tag{4.36}$$

where C_p is the Poincaré constant.

Proof. The differentiation of the functional A, by using the second equation of system (4.1) and the integration by parts, leads to

$$\begin{aligned}\frac{d}{dt}A(t) &= d_1\int_\Omega u_{xx}mu_x dx + d_1\int_\Omega w_x w_{xx} m u_x dx \\ &\quad -\delta\int_\Omega \theta_x m u_x dx + \int_\Omega u_t m u_{tx} dx \\ &= \frac{d_1}{2}\left[mu_x^2\right]_{x=0}^{x=L} - \frac{d_1}{2}\int_\Omega m_x u_x^2 dx + d_1\int_\Omega w_x w_{xx} m u_x dx \\ &\quad -\delta\int_\Omega \theta_x m u_x dx - \frac{1}{2}\int_\Omega m_x u_t^2 dx \\ &= -d_1\left[u_x^2(L)+u_x^2(0)\right] + \frac{2d_1}{L}\|u_x\|_{L^2}^2 + d_1\int_\Omega w_x w_{xx} m u_x dx \\ &\quad -\delta\int_\Omega \theta_x m u_x dx + \frac{2}{L}\|u_t\|_{L^2}^2, \quad t\ge 0.\end{aligned} \tag{4.37}$$

Similarly, using the first equation of(4.1), we have

$$\begin{aligned}\frac{d}{dt}B(t) &= -\gamma_1\int_\Omega w_t m w_x dx + d_1\int_\Omega\left(\left(u_x+\frac{1}{2}(w_x)^2\right)w_x\right)_x m w_x dx \\ &\quad - d_2\int_\Omega w_{xxxx} m w_x dx + \int_\Omega w_t m w_{tx} dx \\ &= -\gamma_1\int_\Omega w_t m w_x dx - d_1\int_\Omega\left(\left(u_x+\frac{1}{2}(w_x)^2\right)w_x\right) m_x w_x dx \\ &\quad - d_1\int_\Omega\left(\left(u_x+\frac{1}{2}(w_x)^2\right)w_x\right) m w_{xx} dx + d_2\int_\Omega w_{xxx} m_x w_x dx \\ &\quad + d_2\int_\Omega w_{xxx} m w_{xx} dx - \frac{1}{2}\int_\Omega m_x w_t^2 dx, \quad t\ge 0.\end{aligned} \tag{4.38}$$

Simplifying equality (4.38) by integrating by parts on Ω and taking account the fact that $m_x =$

$-\frac{4}{L}$, then we get

$$\frac{d}{dt}B(t) = -\gamma_1 \int_\Omega w_t m w_x dx + \frac{4d_1}{L}\int_\Omega \left(u_x + \frac{1}{2}(w_x)^2\right) w_x^2 dx$$
$$- d_1 \int_\Omega w_x w_{xx} m u_x dx - \frac{d_1}{2}\int_\Omega w_x^2 w_x m w_{xx} dx$$
$$- \frac{4d_2}{L}\int_\Omega w_{xxx} w_x dx + d_2 \int_\Omega w_{xxx} m w_{xx} dx + \frac{2}{L}\|w_t\|_{L^2}^2 .$$

Now, utilizing a second integration by parts together with the following relation

- $$-\frac{d_1}{2}\int_\Omega w_x^2 w_x m w_{xx} dx = -\frac{d_1}{4}\int_\Omega w_x^2 m \left((w_x)^2\right)_x dx.$$

Then, we can give the following estimate

$$\frac{d}{dt}B(t) = -\gamma_1 \int_\Omega w_t m w_x dx + \frac{4d_1}{L}\int_\Omega \left(u_x + \frac{1}{2}(w_x)^2\right) w_x^2 dx$$
$$- d_1 \int_\Omega w_x w_{xx} m u_x dx - \frac{d_1}{4}\int_\Omega w_x^2 m \left((w_x)^2\right)_x dx + \frac{4d_2}{L}\|w_{xx}\|_{L^2}^2 \tag{4.39}$$
$$- d_2\left[w_{xx}^2(L) + w_{xx}^2(0)\right] + \frac{2d_2}{L}\|w_{xx}\|_{L^2}^2 + \frac{2}{L}\|w_t\|_{L^2}^2 , \quad t \geq 0.$$

Now, taking equality (4.39) and by applying Young's inequality, we obtain

$$\frac{d}{dt}B(t) \leq \left(\frac{2}{L} + \gamma_1\right)\|w_t\|_{L^2}^2 + \gamma_1 \|w_x\|_{L^2}^2$$
$$+ \frac{8d_1}{L}\left\|u_x + \frac{1}{2}(w_x)^2\right\|_{L^2}^2 - d_1 \int_\Omega w_x w_{xx} m u_x dx \tag{4.40}$$
$$+ \frac{6d_2}{L}\|w_{xx}\|_{L^2}^2 , \quad t \geq 0.$$

Finally, direct calculations using $(4.1)_3$, $(4.1)_4$ and integrating by parts we arrive at

$$-\frac{d}{dt}C(t) = -\int_\Omega (\theta_t + \delta u_{tx})\, m q dx - \int_\Omega (\theta + \delta u_x)\, m q_t dx$$
$$= \int_\Omega q_x m q dx - \int_\Omega (\theta + \delta u_x)\, m\,(-\gamma_2 q - \theta_x)\, dx.$$

Also we have

$$
\begin{aligned}
-\frac{d}{dt}C(t) &= -\frac{1}{2}\int_\Omega m_x q^2 dx + \gamma_2 \int_\Omega \theta m q dx - \frac{1}{2}\int_\Omega m_x \theta^2 dx \\
&+ \gamma_2 \delta \int_\Omega q m u_x dx + \delta \int_\Omega \theta_x m u_x dx \\
&= \frac{2}{L}\int_\Omega q^2 dx + \gamma_2 \int_\Omega \theta m q dx + \frac{2}{L}\int_\Omega \theta^2 dx + \gamma_2 \delta \int_\Omega q m u_x dx \\
&+ \delta \int_\Omega \theta_x m u_x dx.
\end{aligned}
$$

By using Young's inequality, we find

$$
\begin{aligned}
-\frac{d}{dt}C(t) \leq \|u_x\|_{L^2}^2 + \left(\gamma_2^2\delta^2 + \gamma_2 + \frac{2}{L}\right)\|q\|_{L^2}^2 \\
+ \left(\gamma_2 + \frac{2}{L}\right)\|\theta\|_{L^2}^2 + \delta \int_\Omega \theta_x m u_x dx.
\end{aligned}
\tag{4.41}
$$

By adding (4.37), (4.40) and (4.41), we arrive at (4.36). □

Remark 4. *As in the work of Djebabla and Tatar [107], the following relation is needed*

$$
\begin{aligned}
\int_\Omega u_x^2 dx &= \int_\Omega \left(u_x + \frac{1}{2}\left(w_x^2\right) - \frac{1}{2}\left(w_x^2\right)\right)^2 dx \\
&\leq 2\int_\Omega \left(u_x + \frac{1}{2}\left(w_x^2\right)\right)^2 dx + \frac{1}{2}\int_\Omega w_x^4 dx \\
&\leq 2\left\|u_x + \frac{1}{2}\left(w_x^2\right)\right\|_{L^2}^2 + c\left\|w_{xx}\right\|_{L^2}^2,
\end{aligned}
$$

where c is a positive constant.

4.4.2 Proof of Theorem (11)

Proof. We define the Lyapunov functional $F(t)$ as follows

$$
F(t) = NE(t) + I_1 + \frac{\gamma_1}{\delta} I_2 + N_1 I_3 + \varepsilon_2 I_4, \quad t \geq 0. \tag{4.42}
$$

By utilizing the previous Lemmas, and taking into account the estimates (4.17), (4.21), (4.25), (4.31), (4.36), then we can obtain

$$\begin{aligned}
\frac{d}{dt}F(t) \leq &-\left[N\gamma_2 - \frac{1}{2\delta} - N_1 C_1\left(\varepsilon_3\right) - \left(\frac{2}{L} + \gamma_2 + \gamma_2^2\delta^2\right)\varepsilon_2\right]\|q\|_{L^2}^2 \\
&-\left[\frac{\delta d_1}{4} - \left(\frac{8d_1}{L} + 3 + \frac{4d_1}{L}\right)\varepsilon_2 - \frac{\delta\varepsilon_1}{2}\right]\left\|u_x + \frac{1}{2}\left(w_x\right)^2\right\|_{L^2}^2 \\
&-\left[\frac{\delta d_2}{8} - \left(\gamma_1 C_p + \frac{6d_2}{L} + c + \frac{2cd_1}{L}\right)\varepsilon_2 - \frac{\delta c\varepsilon_1}{4}\right]\|w_{xx}\|_{L^2}^2 \\
&-\left[\frac{N_1}{2} - C(\varepsilon_2) - \left(\gamma_2 + \frac{2}{L}\right)\varepsilon_2 - \frac{\delta^3}{16\varepsilon_1}\right]\|\theta\|_{L^2}^2 \\
&-\left[N\gamma_1 - \frac{\delta}{8} - \left(\frac{2}{L} + \gamma_1\right)\varepsilon_2\right]\|w_t\|_{L^2}^2\,dx \\
&-\left[\frac{\delta}{4} - \frac{2\varepsilon_2}{L} - N_1\varepsilon_3\right]\|u_t\|_{L^2}^2, \quad t \geq 0.
\end{aligned} \tag{4.43}$$

At this point, we impose the following restrictions on the coefficients. First, we choose

$$\varepsilon_2 < \min\left[\begin{array}{c} \frac{d_2}{16}\left(\gamma_1 C_p + \frac{6d_2}{L} + c + \frac{2cd_1}{L}\right)^{-1}, \\ \frac{d_1\delta}{8}\left(\frac{8d_1}{L} + 3 + \frac{4d_1}{L}\right)^{-1}, \frac{L\delta}{16} \end{array}\right],$$

and, we select N_1 respectively large enough so that

$$\frac{N_1}{2} - C(\varepsilon_2) - \left(\gamma_2 + \frac{2}{L}\right)\varepsilon_2 - \frac{\delta^3}{16\varepsilon_1} > 0,$$

Now, we we pick ε_1 and ε_3 respectively so small that

$$\varepsilon_1 < \tfrac{1}{4}\min\left(\tfrac{d_2}{c}, d_1\right),$$

$$\varepsilon_3 < \tfrac{\delta}{8N_2}.$$

Finally, we choose N large enough so that

$$N\gamma_2 - \tfrac{1}{2\delta} - N_1 C_1\left(\varepsilon_3\right) - \left(\tfrac{2}{L} + \gamma_2 + \gamma_2^2\delta^2\right)\varepsilon_2 > 0,$$

and

$$N\gamma_1 - \tfrac{\delta}{8} - \left(\tfrac{2}{L} + \gamma_1\right)\varepsilon_2 > 0.$$

Therefore, (4.43) takes the form

$$\begin{aligned} \frac{d}{dt}F(t) &\leq -\eta\left[\|w_t\|_{L^2}^2+\|u_t\|_{L^2}^2+\|\theta\|_{L^2}^2+\|q\|_{L^2}^2+d_2\|w_{xx}\|_{L^2}^2+d_1\left\|u_x+\frac{1}{2}(w_x)^2\right\|_{L^2}^2\right] \\ &\leq -CE(t), \quad t\geq 0, \end{aligned} \tag{4.44}$$

for some positive η and C.

On the other hand, we can use the following remark

Remark 5. *By using Young, Poincaré and Cauchy-Schwarz inequalities, we can deduce that the functional energy E is equivalent to the Lyapunov functional F i.e., there exists two positive constants β_1 and β_2 such that*

$$\beta_2 E(t) \leq F(t) \leq \beta_1 E(t), \ \forall t \geq 0. \tag{4.45}$$

By Combining inequality (4.44) and the right hand side of (4.45), we conclude that

$$\tfrac{d}{dt}F(t) \leq -dF(t), \ \forall t \geq 0, \tag{4.46}$$

where d is a positive constant.

By a simple integration of (4.46) we get

$$F(t) \leq F(0)e^{-dt}, \ \forall t \geq 0, \tag{4.47}$$

and thus (4.16) follows from the left hand side of (4.45). □

5

Effect of the fractional order operator on the coupled hyperbolic-parabolic system

Contents

5.1 Introduction

In this chapter we focus on demonstrating the existence and uniqueness of solutions and we also show stability results. The system considered describes a dynamic model that describes the shear of a Timoshenko beam [4]. We will consider the fractional thermal effect acting on the bending moment according to Gurtin-Pipkin's law [109].

The system is given as follows

$$\begin{aligned} &\rho_1\varphi_{tt} - \kappa(\varphi_{xx} + \psi_x) = 0, \\ &\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) - \int_0^\infty h(s)A^\sigma\psi(t-s)ds + \delta\theta_x = 0, \\ &\rho_3\theta_t + \frac{1}{\beta}\int_0^\infty g(s)A^\sigma\theta(t-s)ds + \delta\psi_{tx} = 0, \end{aligned} \tag{5.1}$$

Here, the unknows variables $\varphi, \psi, \theta : (x,t) \in (0,L) \times \mathbb{R}_+ \mapsto \mathbb{R}$ are the transverse displacement, the rotation angle and the relative temperature, as stated respectevely. Furthemore, the operator A represents the derivatives $(-\partial_{xx})$ and σ is a parameter in the interval $[0,1]$.

The coefficients κ, b, δ, β represent the positive coefficients in addition to ρ_i for $i = 1,2,3.$ The aforementioned system (5.1) is complemented with the Dirichlet boundary conditions for φ and θ

$$\varphi(0,t) = \varphi(L,t) = \theta(0,t) = \theta(L,t) = 0, \tag{5.2}$$

in addition to Neumann boundary condition for ψ

$$\psi_x(0,t) = \psi_x(L,t) = 0. \tag{5.3}$$

The system describes a model for elastic beams vibrations. It is the coupling of the shear force and the bending moment acting on the system.

Setting the change of variable as follows

$$\int_0^L \psi(x,t)dx = \Psi(t), \tag{5.4}$$

Then, by integration the second equation of the system (5.1) on $(0,L)$, gives the following equation

$$\rho_2\Psi'' + \kappa\Psi' = 0. \tag{5.5}$$

Hence, if

$$\Psi(0) = \Psi'(0) = 0,$$

then we have

$$\Psi(t) = 0.$$

We deduce that the use of Poincaré's inequality for is justified.

This chapter has been inspired by our work [110].

The first result is the following existence and uniqueness Theorem.

Theorem 12. *Let z be a vector solution of the following Cauchy problem*

$$\begin{cases} z'(t) = \mathcal{A}z(t), \\ z(0) = z_0, \end{cases} \tag{5.6}$$

for every initial datum $z_0 = (\varphi_0, \phi_0, \psi_0, \chi_0, \zeta_0, \theta_0, \eta_0) \in \mathcal{H}$, given at time $t = 0$. Then, there exists a unique solution of the problem (5.1)-(5.3) *such that*

$$z \in C(\mathbb{R}_+; \mathcal{H}).$$

Moreover, if $z_0 \in D(\mathcal{A})$. Then, we have

$$z \in C(\mathbb{R}_+; D(\mathcal{A})) \cap C^1(\mathbb{R}_+; \mathcal{H}).$$

The stability results are translated as follows

- The semigroup $s(t)$ generated by the problem (5.1)-(5.3) is polynomially stable if and only if $\xi_g \neq 0,$ with the rate $t^{-\frac{1}{4-2\sigma}}$.
- The semigroup $s(t)$ generated by the problem (5.1)-(5.3) is polynomially stable if and only if $\xi_g = 0$ and $\sigma \in [0, 1),$ with the rate $t^{-\frac{1}{2-2\sigma}}$.
- The semigroup $s(t)$ generated by the problem (5.1)-(5.3) is exponentially stable if and only if $\xi_g = 0$ and $\sigma = 1,$ where ξ_g is a stability number proved by [111].

5.1.1 Earlier results

In this section, we will present the most important works that are investigated in the field of mechanical systems with thermal effects. Intensive researchs has been carried out in the recent decades to impose the minimum energy dissipations to guarantee the stability of thermoelastic Timoshenko systems. This regard will briefly discuss some of the main results of the systems.

Timoshenko's theory [4] is an improvement of the Euler-Bernoulli theory [112]. Indeed, these systems modeling beams under several vibration. Recently, the system stability of Timoshenko is one of important posed question . Widely speaking, there is different types of damping that have been used to dampen undesirable vibrations, as portrayed by several authors, among them Kim and Renardy [6], Raposo et al. [8], Messaoudi and Mustafa [10], and Tian and Zhang [113]. Subsequently, many results of stability wether (exponential or polynomial) have prevailed in the literature. Consequently, when the term damping occurs in the system, it will give the wave speeds a crucial role in determining the behavior of the solutions at infinite time.

5.1.2 Results about stability numbers

The Timoshenko system without heat effect was initially introduced by Stéphane Timoshenko [4], in which he presented the mechanical model of the beams. Several researchers treated Timoshenko system, among them Soufyane [114] where he showed that the system is exponentially stable under the "equal wave speeds" assumption, i.e.

$$\xi_0 = \frac{\rho_1}{k} - \frac{\rho_2}{b}. \tag{5.7}$$

Note that the authors [20, 83, 115–117] results are mainly based on Soufyane's contributions to the word of theory.

Next, is Timoshenko system with thermal effect. Its general form can be written as follows

$$\begin{aligned} &\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, \\ &\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \delta\theta_x = 0, \\ &\rho_3\theta_t + q_x + \delta\psi_{tx} = 0. \end{aligned} \tag{5.8}$$

According to the equation $(5.19)_3$ the heat flux vector

$$q : (x, t) \in (0, L) \times \mathbb{R}^+ \mapsto \mathbb{R}, \tag{5.9}$$

depends on the following thermal laws. The first law is based on Fourier classical assumption for the heat conduction of the system

$$\beta q + \theta_x = 0. \tag{5.10}$$

The integration of this law in the system (5.19) has been analyzed by Jaime E Muñoz Rivera and Reinhard Racke [35]. The second is Cattaneo law [109], it is given by the following equation

$$\tau q_t + \beta q + \theta_x = 0, \quad \tau > 0. \tag{5.11}$$

As shown by ML Santos et al. [118], the main result of the system (5.19) along with equation (5.11) is exponentially stable only when

$$\xi_\tau = \left(\frac{\rho_1}{\rho_3 k} - \tau\right)\left(\frac{\rho_1}{\kappa} - \frac{\rho_2}{b}\right) - \tau\frac{\rho_1\delta^2}{\rho_3\kappa b} = 0. \tag{5.12}$$

Thirdly, the constitutive equation related to Gurtin-Pipkin heat conduction law [119] is given by the following equation

$$\beta q(t) + \int_0^\infty g(s)\theta_x(t-s)ds = 0, \tag{5.13}$$

where g is the memory kernel, which is a convex summable function on $\mathbb{R}^+$ of the total mass

$$\int_0^\infty g(s)ds = 1.$$

Dell'Oro and Vittorino [111] proved that the system (5.19) in addition to equation (5.13) is exponentially stable only when

$$\xi_g = \left(\frac{\rho_1}{\rho_3\kappa} - \frac{\beta}{g(0)}\right)\xi_0 - \frac{\beta\rho_1\delta^2}{g(0)\rho_3\kappa b} = 0. \tag{5.14}$$

Last but not least, the constitutive equation related to Green-Naghdi heat conduction law of type III [108, 120] is given by this equation

$$\beta q + \theta_x + dp_x = 0, \quad d > 0, \tag{5.15}$$

where

$$p(t) = p(0) + \int_0^t \theta(r)dr. \tag{5.16}$$

Salim Messaoudi and Belkacem Said-Houari [42], have proven that the system (5.19) with equation (5.15) is exponontially stable only when

$$\xi_0 = 0.$$

5.1.3 Contributions

María Astudillo and Higidio Portillo Oquendo [121] added the term memory $\int_0^\infty g(s)(-\partial_{xx})^\theta\psi(t-s)ds$ to the system as follows

$$\begin{aligned}&\rho_1\phi_{tt}-\kappa(\phi_{xx}+\psi_x)=0,\\&\rho_2\psi_{tt}-b\psi_{xx}-\int_0^\infty g(s)(-\partial_{xx})^\theta\psi(t-s)ds+\kappa(\phi_x+\psi)=0.\end{aligned}\tag{5.17}$$

They proved that the system is both exponentially stable only when $\xi_0=0$ and $\theta=1$, and Polynomial stable only when $\xi_0=0$ and $\theta\in[0,1[$. Not to mention that the decay rates are optimal only when $\xi_0\neq 0$. Danese et al. [122] studied the abstract system, but for a fractional operator influencing the coupling

$$\begin{aligned}&\rho_1\phi_{tt}+\kappa A^{1/2}(A^{1/2}\phi+\psi)=0,\\&\rho_2\psi_{tt}+bA\psi+\kappa(A^{1/2}\phi+\psi)-\delta A^\gamma\theta=0,\\&\rho_3\theta_t+A\theta+\delta A^\gamma\psi=0,\end{aligned}\tag{5.18}$$

with A is a self-adjoint positive operator. The existence of suitable energy functionals proved that the system is both exponentially stable only when $\xi_0=0$ and $\gamma=\frac{1}{2}$, and Polynomial stable only when $\xi_0\neq 0$ and $\gamma\in[\frac{1}{2},1]$.

Our work is based on the improvement of the stability results of Timoshenko system with a fractional operator in memory [121], this improvement is based on the addition of a fractional operator in the thermal effect of Gurtin-Pipkin type.

5.1.4 Model derivation

The general form of Timoshenko system with thermal effect can be written as follows

$$\begin{aligned}&\rho_1\varphi_{tt}-\kappa(\varphi_x+\psi)_x=0,\\&\rho_2\psi_{tt}-b\psi_{xx}+\kappa(\varphi_x+\psi)+\delta\theta_x=0,\\&\rho_3\theta_t+q_x+\delta\psi_{tx}=0.\end{aligned}\tag{5.19}$$

According to the equation $(5.19)_3$ the heat flux vector

$$q:(x,t)\in(0,L)\times\mathbb{R}^+\mapsto\mathbb{R}.\tag{5.20}$$

We consider the following fractional constitutive equation

$$\beta q(t) - \int_0^\infty g(s) A^{\sigma - \frac{1}{2}} \theta(t-s) ds = 0, \tag{5.21}$$

when applying the opertor ∂_x to the previous equation, we obtain

$$\beta q_x - \int_0^\infty g(s) A^{\sigma} \theta(t-s) ds = 0, \tag{5.22}$$

substitute equation (5.22) in equation $(5.19)_3$ and add a fractional operator in the memory to obtain the system (5.1).

5.1.5 Chapter plan

This chapter respect the following plan, section (5.2) will introduce some assumptions on the kernels, functional spaces and some characteristics of the fractional operator. Then, section (5.3) will prove the global existence of the solutions. Hence, section (5.4) will show the stability results by using the semigroup method.

5.2 Preliminaries

The semigroup $s(t)$ is exponentially stable on $\mathcal{H}$ if there are $c_1 > 0$ and $c_2 \geq 1$ such that

$$\|s(t)z\|_{\mathcal{H}} \leq c_2 e^{-c_1 t} \|z\|_{\mathcal{H}},$$

and $s(t)$ is stable on $\mathcal{H}$ if

$$\lim_{t \to \infty} \|s(t)z\|_{\mathcal{H}} = 0, \quad \forall z \in \mathcal{H}.$$

5.2.0.A Assumptions

Firstly, we introduce the assumptions on the kernels μ and h as follows

μ is a summable function on $\mathbb{R}_+$ with

$$\int_0^\infty \mu(s) ds = g(0) > 0,$$

in which its relationship with the relaxation function is expressed by

$$\mu(s) = -g'(s),$$

and a requirement g that makes a total mass 1 that is translated to

$$\int_0^\infty s\mu(s)ds = 1.$$

- μ is a nonnegative nonincreasing absolutely continuous function on $\mathbb{R}_+$ such that

$$\mu(0) = \lim_{s\to 0} \mu(s),$$

- There exist $c_2 > 0$ such that the differential inequality

$$\mu'(s) + c_2\mu(s) \leq 0, \quad \forall s > 0. \tag{5.23}$$

  the kernel h of the memory term checks the following hypotheses:

- h is a nonnegative C^1 function satisfying

$$h(0) = \int_0^\infty h(s)ds < b\left(\frac{\pi}{L}\right)^{2(1-\sigma)}.$$

- There exist $c_1 > 0$ such that the differential inequality

$$h'(s) + c_1 h(s) \leq 0, \quad h(0) > 0, \tag{5.24}$$

  that hold almost every $s > 0$.

5.2.0.B Functional spaces

When it comes to the functional spaces, the omision of zero is always mandatory for $\sigma \in [0, 1]$, in which the compactly nested family of Hilbert spaces are

$$H^\sigma = \mathcal{D}(A^{\frac{\sigma}{2}}), \quad H^\sigma_* = \mathcal{D}(A_*^{\frac{\sigma}{2}}), \quad \langle f, g\rangle_{H^\sigma} = \left\langle A^{\frac{\sigma}{2}} f, A^{\frac{\sigma}{2}} g\right\rangle_{\mathrm{L}^2}, \quad \|f\|_{H^\sigma} = \|A^{\frac{\sigma}{2}} f\|_{\mathrm{L}^2}.$$

Not to mention that $\|\cdot\|_{\mathrm{L}^2}$ and $\langle\cdot,\cdot\rangle_{\mathrm{L}^2}$ are both the standard inner norm and the scalar product on the Hilbert space $\mathrm{L}^2(0, L)$. Denote that $H^{-\sigma}$ is the completion of the domain. It simply means that $H^{-\sigma}$ is the dual space of H^σ. Then, if $u \in H^\sigma$ it is possible to write $A^\sigma u$ which means that this element belongs to the dual space $H^{-2\sigma}$ acting as

$$\langle A^\sigma u, v\rangle = \langle u, A^\sigma v\rangle, \quad \forall v \in H^{2\sigma}.$$

Hilbert subspace is introduced by

$$\mathtt{L}^2_*(0,L)=\left\{f\in\mathtt{L}^2(0,L):\int_0^L f(x)dx=0\right\}.$$

For zero-mean functions, along with the Hilbert spaces

$$\mathtt{H}^1_0(0,L)\quad\text{and}\quad\mathtt{H}^1_*(0,L)=\mathtt{H}^1(0,L)\cap\mathtt{L}^2_*(0,L).$$

The memory spaces is introduced by

$$\mathcal{M}_\sigma=\mathtt{L}^2_\mu(\mathbb{R}^+;H^\sigma),\quad \mathcal{N}_\sigma=\mathtt{L}^2_h(\mathbb{R}^+;H^\sigma),$$

with the inner product

$$\begin{aligned}\langle f,g\rangle_{\mathcal{N}_\sigma}&=\int_0^\infty h(s)\left\langle A^{\frac{\sigma}{2}}f(s),A^{\frac{\sigma}{2}}g(s)\right\rangle_{L^2}ds=\int_0^\infty h(s)\left\langle f(s),g(s)\right\rangle_{H^\sigma}ds,\\ \langle f,g\rangle_{\mathcal{M}_\sigma}&=\int_0^\infty \mu(s)\left\langle A^{\frac{\sigma}{2}}f(s),A^{\frac{\sigma}{2}}g(s)\right\rangle_{L^2}ds=\int_0^\infty \mu(s)\left\langle f(s),g(s)\right\rangle_{H^\sigma}ds.\end{aligned}$$

The infinitesimal generators of the right-translation semigroup on $\mathcal{M}$ and $\mathcal{N}$ are the following linear operators

$$\overline{T}\zeta=-\overline{D}\zeta,\quad T\eta=-D\eta,$$

with the domain

$$\begin{aligned}\mathcal{D}(\overline{T})&=\left\{\eta\in\mathcal{N}_\sigma:\overline{D}\zeta\in\mathcal{N}_\sigma,\ \zeta(0)=0\right\}\\ \mathcal{D}(T)&=\left\{\eta\in\mathcal{M}_\sigma:D\eta\in\mathcal{M}_\sigma,\ \lim_{s\to0}\|\eta\|_{H^\sigma}=0\right\},\end{aligned}$$

where D stands for weak derivative with respect to the internal variable $s\in\mathbb{R}_+$. The phase space of the problem (5.1)-(5.3) will be

$$\mathcal{H}=\mathtt{H}^1_0(0,L)\times\mathtt{L}^2(0,L)\times\mathtt{H}^1_*(0,L)\times\mathtt{L}^2_*(0,L)\times\mathcal{N}_\sigma\times\mathtt{L}^2(0,L)\times\mathcal{M}_\sigma,\tag{5.25}$$

with its inner product

$$\begin{aligned}\langle z_1,z_2\rangle_{\mathcal{H}}=\kappa\left\langle\varphi_{1x}+\psi_1,\varphi_{2x}+\psi_2\right\rangle_{\mathtt{L}^2}+\rho_1\left\langle\phi_1,\phi_2\right\rangle_{\mathtt{L}^2}+\left\langle B_*^{\frac12}\psi_1,B_*^{\frac12}\psi_2\right\rangle_{\mathtt{L}^2}\\ +\rho_2\left\langle\chi_1,\chi_2\right\rangle_{\mathtt{L}^2}+\left\langle\zeta_1,\zeta_2\right\rangle_{\mathcal{N}_\sigma}+\rho_3\left\langle\theta_1,\theta_2\right\rangle_{\mathtt{L}^2}+\frac{1}{\beta}\left\langle\eta_1,\eta_2\right\rangle_{\mathcal{M}_\sigma},\end{aligned}$$

for z_1 and z_2 in $\mathcal{H}$.

At last, the characterstics of the fractional operators can be defined as follow

$$A = -\partial_{xx} : \mathtt{H}^2 \subset \mathtt{L}^2 \to \mathtt{L}^2,$$
$$A_* = -\partial_{xx} : \mathtt{H}^2_* \subset \mathtt{L}^2_* \to \mathtt{L}^2_*,$$

on the undermentioned subspaces

$$H^2 = \mathtt{H}^2 \cap \mathtt{H}^1_0,$$
$$H^2_* = \left\{\mathtt{H}^2 \cap \mathtt{L}^2_* : f_x(0) = f_x(L) = 0\right\}.$$

The operators are positive, self-adjoint and have compact inverse. Therefore, the operators A^σ and A^σ_* are bounded for $\sigma \le 0$ and positive self-adjoint for $\sigma \in \mathbb{R}$. The spectrum of the operators are constituted only by positive eigenvalues as shown by λ_n^2 where

$$\lambda_n = \frac{n\pi}{L}, \quad n \in \mathbb{N},$$

the sequences (Π_n) and (Π^*_n) are the corresponding unitary eigenfunctions associated to λ_n. They form a Hilbert's base for the space $\mathtt{L}^2$ or $\mathtt{L}^2_*$, it can be defined as follows

$$\Pi_n(x) = \sqrt{\frac{2}{L}}\sin(\lambda_n x), \quad \Pi^*_n(x) = \sqrt{\frac{2}{L}}\cos(\lambda_n x).$$

For $f \in \mathtt{L}^2$ and $g \in \mathtt{L}^2_*$ it is possible to write

$$f = \sum_{n=1}^{n=\infty} \langle f, \Pi_n\rangle \Pi_n, \quad g = \sum_{n=1}^{n=\infty} \langle g, \Pi^*_n\rangle \Pi^*_n.$$

Furthermore, for $f \in H^{2\sigma+1}$ and $g \in H^{2\sigma+1}_*$, will give the following identity

$$A^{\sigma+\frac{1}{2}} f = \sum_{n=1}^{n=\infty} \lambda_n^{2\sigma+1} \langle f, \Pi_n\rangle \Pi_n, \quad A^\sigma \partial_x g = \sum_{n=1}^{n=\infty} \lambda_n^{2\sigma+1} \langle g, \Pi_n\rangle \Pi^*_n.$$

In correspondence to Parseval's identity the f and g norms will be

$$\|f\|_{H^{2\sigma+1}} = \|A^\sigma_* \partial_x f\|_{\mathtt{L}^2}, \quad \|g\|_{H^{2\sigma+1}_*} = \|A^\sigma \partial_x g\|_{\mathtt{L}^2},$$

and for $\sigma = 0$

$$\|f\|_{H^1} = \|A^{\frac{1}{2}} f\|_{\mathrm{L}^2} = \|\partial_x f\|_{\mathrm{L}^2}, \quad \|g\|_{H^1_*} = \|A_*^{\frac{1}{2}} g\|_{\mathrm{L}^2} = \|\partial_x g\|_{\mathrm{L}^2}.$$

The following represents the conclusion

$$\langle A^\sigma f, g_x \rangle_{\mathrm{L}^2} = - \langle \psi_x, A^\sigma g \rangle_{\mathrm{L}^2}.$$

Introducing the operator

$$B_* = bA_* - \left(\int_0^\infty g(s) ds \right) A_*^\sigma,$$

for $\psi \in \mathcal{D}(A_*)$ to get

$$B_* \psi = \sum_{n=1}^{n=\infty} \lambda_n^{2\sigma} \left(b \lambda_n^{2(1-\sigma)} - \int_0^\infty g(s) ds \right) \langle \psi, \Pi_n^* \rangle \, \Pi_n^*. \tag{5.26}$$

To deduce, the norms $\|A_*^\sigma \psi\|$ and $\|B_*^\sigma \psi\|$ are equivalents. Correspondingly, the space $\mathcal{H}$ is normed by

$$\begin{aligned} \|z\|_{\mathcal{H}}^2 = \rho_1 \|\phi\|_{\mathrm{L}^2}^2 + \kappa \|\varphi_x + \psi\|_{\mathrm{L}^2}^2 + \left\| B_*^{\frac{1}{2}} \psi \right\|_{\mathrm{L}^2}^2 + \rho_2 \|\chi\|_{\mathrm{L}^2}^2 \\ + \|\zeta\|_{\mathcal{N}_\sigma}^2 + \rho_3 \|\theta\|_{\mathrm{L}^2}^2 + \frac{1}{\beta} \|\eta\|_{\mathcal{M}_\sigma}^2 . \end{aligned} \tag{5.27}$$

The following remark plays a crucial role in this work.

Remark 6. *For every $\zeta \in \mathcal{D}(\overline{T})$ and $\eta \in \mathcal{D}(T)$, we define the following identities*

$$2 \langle \overline{T} \zeta, \zeta \rangle_{\mathcal{N}_\sigma} = -\Gamma[\zeta], \quad 2 \langle T \eta, \eta \rangle_{\mathcal{M}_\sigma} = -\Gamma[\eta].$$

where

$$\begin{aligned} \Gamma[\zeta] &= - \int_0^\infty h'(s) \|A^{\frac{\sigma}{2}} \zeta(s)\|_{L^2}^2 ds = - \int_0^\infty h'(s) \|\zeta(s)\|_{H^\sigma}^2 ds, \\ \Gamma[\eta] &= - \int_0^\infty \mu'(s) \|A^{\frac{\sigma}{2}} \eta(s)\|_{L^2}^2 ds = - \int_0^\infty \mu'(s) \|\eta(s)\|_{H^\sigma}^2 ds, \end{aligned}$$

by using both inequalities (5.23) *and* (5.24)*, we deduce the following*

$$\|\zeta\|_{\mathcal{N}_\sigma}^2 \le \frac{1}{c_1} \Gamma[\zeta], \quad \|\eta\|_{\mathcal{M}_\sigma}^2 \le \frac{1}{c_2} \Gamma[\eta].$$

5.3 Existence and uniqueness

This section will be concerned with the existence of global solutions based on the classical Lumer–Phillips Theorem [98, 99]

Lemma 16. *Let $\mathcal{A}$ be a densely defined linear operator on a Hilbert space $\mathcal{H}$. Then $\mathcal{A}$ is the infinitesimal generator of a contraction semigroup $s(t)$ if and only if*

- *$\mathcal{A}$ is dissipative.*
- $Ran(I - \mathcal{A}) = \mathcal{H}$.

5.3.1 The semigroup approach

As in [123], in order to determine the operator $\mathcal{A}$, the following change of variable is needed

$$\eta^t(x, s) = \int_0^s \theta(x, t - \nu) d\nu,$$

which satisfies the Dirichlet boundary condition

$$\eta^t(0, s) = \eta^t(L, s) = 0,$$

and η^t satisfies the following equation

$$\eta^t_t = -\eta^t_s + \theta(t).$$

It checks the explicit representation formula as in [124]

$$\eta^t(s) = \begin{cases} \displaystyle\int_0^s \theta(t - \iota) d\iota, & s \leq t, \\ \eta_0(s - t) + \displaystyle\int_0^t \theta(t - \iota) d\iota, & s > t. \end{cases}$$

Now, the use of the above operator B_*, the relative history of ψ which is defined as

$$\zeta(t, s) = \psi(t) - \psi(t - s)$$

that is introduced in [125], and the above mentioned data allow the writing of the following partial differential system

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_{xx} + \psi_x) = 0, \\ \rho_2\psi_{tt} + B_*\psi + \displaystyle\int_0^\infty h(s)A_*^\sigma\zeta(t,s)ds + \kappa(\varphi_x + \psi) + \delta\theta_x = 0, \\ \zeta_t = \overline{T}\zeta + \psi_t, \\ \rho_3\theta_t + \dfrac{1}{\beta}\displaystyle\int_0^\infty \mu(s)A^\sigma\eta(s)ds + \delta\psi_{tx} = 0, \\ \eta_t = T\eta + \theta. \end{cases} \tag{5.28}$$

By introducing the state vector $z(t) = (\varphi, \phi, \psi, \chi, \zeta, \theta, \eta^t)^*$, where $(\cdot)^*$ represents the transposed vector, the system (5.28) can be written as a Cauchy problem in $\mathcal{H}$

$$\begin{cases} z'(t) = \mathcal{A}z(t), \\ z(0) = z_0, \end{cases} \tag{5.29}$$

for every initial datum

$$z_0 = (\varphi_0, \phi_0, \psi_0, \chi_0, \zeta_0, \theta_0, \eta_0) \in \mathcal{H},$$

given at time $t = 0$, the solution at time $t > 0$ to the problem (5.29) can be written as

$$z(t) = s(t)z_0 = e^{t\mathcal{A}}z_0,$$

where $\mathcal{A}$ is the linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \longrightarrow \mathcal{H}$ which is defined as

$$\mathcal{A}\begin{pmatrix} \varphi \\ \phi \\ \psi \\ \chi \\ \zeta \\ \theta \\ \eta \end{pmatrix} = \begin{pmatrix} \phi \\ \frac{\kappa}{\rho_1}(\varphi_{xx} + \psi_x) \\ \chi \\ -\frac{1}{\rho_2}B_*\psi - \frac{\kappa}{\rho_2}(\varphi_x + \psi) - \frac{1}{\rho_2}\int_0^\infty h(s)A_*^\sigma\zeta(t,s)ds - \frac{\delta}{\rho_2}\theta_x \\ \overline{T}\zeta + \chi \\ -\frac{1}{\beta\rho_3}\int_0^\infty \mu(s)A^\sigma\eta(s)ds - \frac{\delta}{\rho_3}\chi_x \\ T\eta + \theta \end{pmatrix},$$

with the domain $\mathcal{D}(\mathcal{A})$

$$\mathcal{D}(\mathcal{A})=\left\{ z\in\mathcal{H} \;\middle|\; \begin{array}{c} \varphi\in \mathtt{H}^2(0,L),\\ \phi\in \mathtt{H}^1_0(0,L),\\ \psi_x\in \mathtt{H}^1_0(0,L),\\ \chi\in \mathtt{H}^1_*(0,L),\\ B_*\psi+\int_0^\infty h(s)A_*^\sigma\zeta(t,s)ds\in \mathtt{L}^2_*(0,L),\\ \theta\in \mathtt{H}^1_0(0,L),\\ \zeta\in\mathcal{D}(\overline{T}),\quad \eta\in\mathcal{D}(T),\\ \int_0^\infty \mu(s)A^\sigma\eta(s)ds\in \mathtt{L}^2(0,L). \end{array} \right\}.$$

Theorem 13. *The operator $\mathcal{A}$ is the infinitesimal generator of a contraction semigroup*

$$s(t)=e^{t\mathcal{A}}:\mathcal{H}\longrightarrow\mathcal{H}.$$

5.3.2 Proof of Theorem (13)

Proof. The proof is based on an application of (16). It is clear that the linear operator $\mathbf{A}$ is dissipative

$$\Re\left\langle \mathcal{A}z,z\right\rangle_{\mathcal{H}}=\left\langle \overline{T}\zeta,\zeta\right\rangle_{\mathcal{N}_\sigma}+\left\langle T\eta,\eta\right\rangle_{\mathcal{M}_\sigma}=-\frac{1}{2}\Gamma[\zeta]-\frac{1}{2\beta}\Gamma[\eta]\le 0. \tag{5.30}$$

Next, to prove that

$$Ran(I-\mathcal{A})=\mathcal{H},$$

the solution $z\in\mathcal{D}(\mathcal{A})$ of the equation $z-\mathcal{A}z=\tilde{z}$ where $\tilde{z}=(\tilde{\varphi},\tilde{\phi},\tilde{\psi},\tilde{\chi},\tilde{\zeta},\tilde{\theta},\tilde{\eta})^\perp\in\mathcal{H}$, can be written in the following compenents

$$\begin{aligned}
&\varphi-\phi=\tilde{\varphi},\\
&\phi-\frac{\kappa}{\rho_1}(\varphi_{xx}+\psi_x)=\tilde{\phi},\\
&\psi-\chi=\tilde{\psi},\\
&\chi+\frac{1}{\rho_2}B_*\psi+\frac{\kappa}{\rho_2}(\varphi_x+\psi)+\frac{1}{\rho_2}\int_0^\infty h(s)A_*^\sigma\zeta(t,s)ds+\frac{\delta}{\rho_2}\theta_x=\tilde{\chi},\\
&\zeta-\overline{T}\zeta-\chi=\tilde{\zeta},\\
&\theta+\frac{1}{\beta\rho_3}\int_0^\infty \mu(s)A^\sigma\eta(s)ds+\frac{\delta}{\rho_3}\chi_x=\tilde{\theta},\\
&\eta-T\eta-\theta=\tilde{\eta}.
\end{aligned} \tag{5.31}$$

Now, by substituting equations (5.31) and $(5.31)_3$ in ,$(5.31)_4$ and $(5.31)_6$, the obtained system will be

$$
\begin{aligned}
&\varphi-\frac{\kappa}{\rho_1}(\varphi_x+\psi_x)=\tilde{\phi}+\tilde{\varphi},\\
&\psi+\frac{1}{\rho_2}B_*\psi+\frac{\kappa}{\rho_2}(\varphi_x+\psi)+\frac{1}{\rho_2}\int_0^\infty h(s)A_*^\sigma\zeta(t,s)ds+\frac{\delta}{\rho_2}\theta_x=\tilde{\chi}+\tilde{\psi},\\
&\zeta-\overline{T}\zeta-\chi=\tilde{\zeta},\\
&\theta+\frac{1}{\beta\rho_3}\int_0^\infty \mu(s)A^\sigma\eta(s)ds+\frac{\delta}{\rho_3}\psi_x=\tilde{\theta}+\frac{\delta}{\rho_3}\tilde{\psi}_x,\\
&\eta-T\eta-\theta=\tilde{\eta}.
\end{aligned}
\tag{5.32}
$$

The integration of both the equations (5.32)$_3$ and (5.32)$_5$ with $\eta(0)=0$ and $\zeta(0)=0$, will make the following writing possible

$$
\begin{aligned}
&\zeta(s)=(1-e^{-s})\chi+\overline{\Upsilon}(s),\\
&\eta(s)=(1-e^{-s})\theta+\Upsilon(s),
\end{aligned}
\tag{5.33}
$$

where $\overline{\Upsilon}(s)=\int_0^s e^{y-s}\tilde{\zeta}(y)dy$ and $\Upsilon(s)=\int_0^s e^{y-s}\tilde{\eta}(y)dy.$

Substitute (5.33)$_1$ and (5.33)$_2$ in (5.32)$_3$ and (5.32)$_5$ respectively, to obtain

$$
\begin{aligned}
&\varphi-\frac{\kappa}{\rho_1}(\varphi_{xx}+\psi_x)=\vartheta_1,\\
&\psi+\frac{1}{\rho_2}B_*\psi+\frac{\kappa}{\rho_2}(\varphi_x+\psi)+\frac{\overline{\varsigma}}{\rho_2}A_*^\sigma\psi+\frac{\delta}{\rho_2}\theta_x=\vartheta_2,\\
&\theta+\frac{\varsigma}{\beta\rho_3}A^\sigma\theta+\frac{\delta}{\rho_3}\psi_x=\vartheta_3,
\end{aligned}
$$

where $\varsigma=\int_0^\infty \mu(s)(1-e^{-s})ds, \quad \overline{\varsigma}=\int_0^\infty h(s)(s)(1-e^{-s})ds$ and

$$
\begin{aligned}
&\vartheta_1=\tilde{\phi}+\tilde{\varphi},\\
&\vartheta_2=\tilde{\chi}+\tilde{\psi}+\frac{1}{\rho_2}A_*^\sigma\int_0^\infty h(s)\overline{\Upsilon}(s)ds,\\
&\vartheta_3=\tilde{\theta}+\frac{\delta}{\rho_3}A^{\frac{1}{2}}\tilde{\psi}+\frac{1}{\beta\rho_3}A^\sigma\int_0^\infty \mu(s)\Upsilon(s)ds.
\end{aligned}
$$

It is apparent that $\vartheta_1 \in \mathtt{H}^{-1}$. For ϑ_2 is obtained as follows

$$\begin{aligned}\left\|A_*^\sigma \int_0^\infty h(s)\overline{\Upsilon}(s)ds\right\|_{H^1} &\leq \int_0^\infty \int_0^s h(s)e^{y-s}\left\|\tilde{\zeta}(y)\right\|_{H^1} dyds \\ &\leq \int_0^\infty \int_0^s \sqrt{h(s)}e^{y-s}\sqrt{h(y)}\left\|\tilde{\zeta}(y)\right\|_{H^1} dyds \\ &\leq \sqrt{\int_0^\infty h(s)ds}\|\tilde{\zeta}(y)\|_{\mathcal{N}_\sigma},\end{aligned}$$

which implies that $\vartheta_2 \in \mathcal{N}_{-1}$. Hence, for ϑ_3 is acquired by

$$\begin{aligned}\left\|A^\sigma \int_0^\infty \mu(s)\Upsilon(s)ds\right\|_{H^1} &\leq \int_0^\infty \int_0^s \mu(s)e^{y-s}\|\tilde{\eta}(y)\|_{H^1} dyds \\ &\leq \int_0^\infty \int_0^s \sqrt{\mu(s)}e^{y-s}\sqrt{\mu(y)}\|\tilde{\eta}(y)\|_{H^1} dyds \\ &\leq \sqrt{g(0)}\|\tilde{\eta}(y)\|_{\mathcal{M}_\sigma},\end{aligned}$$

which implies that $\vartheta_3 \in \mathcal{M}_{-1}$. Henceforth, for ζ is attained through

$$\begin{aligned}\|\zeta\|^2_{\mathcal{N}_\sigma} &\leq 2\int_0^\infty h(s)ds\,\|\chi\|^2_{H^1} + 2\int_0^\infty h(s)\left\|\overline{\Upsilon}(s)\right\|^2_{H^1} ds \\ &\leq 2\int_0^\infty h(s)ds\,\|\chi\|^2_{H^1} + 2\int_0^\infty \left(\int_0^s e^{y-s}\sqrt{h(y)}\left\|\tilde{\zeta}(y)\right\|_{H^1} dy\right)^2 ds \\ &\leq 2\int_0^\infty h(s)ds\,\|\chi\|^2_{H^1} + 2\left\|\tilde{\zeta}\right\|^2_{\mathcal{N}_\sigma},\end{aligned}$$

this implies that $\zeta \in \mathcal{N}_\sigma$, so, $\overline{T}\zeta = \zeta - \chi - \tilde{\zeta} \in \mathcal{N}_\sigma$. At last, η is given by

$$\begin{aligned}\|\eta\|^2_{\mathcal{M}_\sigma} &\leq 2g(0)\|\theta\|^2_{H^1} + 2\int_0^\infty \mu(s)\|\Upsilon(s)\|^2_{H^1} ds \\ &\leq 2g(0)\|\theta\|^2_{H^1} + 2\int_0^\infty \left(\int_0^s e^{y-s}\sqrt{\mu(y)}\|\tilde{\eta}(y)\|_{H^1} dy\right)^2 ds \\ &\leq 2g(0)\|\theta\|^2_{H^1} + 2\|\tilde{\eta}\|^2_{\mathcal{M}_\sigma},\end{aligned}$$

which implies that $\eta \in \mathcal{M}_\sigma$, to get $T\eta = \eta - \theta - \tilde{\eta} \in \mathcal{M}_\sigma$. According to Lax–Milgram's Theorem there exists a unique (weak) solution. Because it is clear that $\mathcal{D}(\mathcal{A})$ dense in $\mathcal{H}$, so it is possible to announce that the following result is held true

Theorem 14. *Let $z_0 \in \mathcal{H}$. Then, the system* (5.1)-(5.3) *has a unique global weak solutions such* *that* $z \in C(\mathbb{R}_+; \mathcal{H})$.

In conclusion, the semigroup $s(t)$ exists if and only if $\sigma \leq 1$. When the operator $(I - \mathcal{A})$ cannot be onto $\mathcal{H}$ whenever, if $\sigma = 1 + \varepsilon, \quad \varepsilon > 0$, then its inverse $(I - \mathcal{A})^{-1}$ would map the whole space $\mathcal{H}$ onto $\mathcal{D}(\mathcal{A})$. □

Remark 7. *It is important to show that* 0 *belongs to the resolvent set* $\varrho(\mathcal{A})$. *For this, it is suf[U+FB01]cient to prove that the stationary problem* $\mathcal{A}z = \tilde{z}$ *has a solution* $z \in \mathcal{D}(\mathcal{A})$ *and* $\|z\| \leq C\|\tilde{z}\|$ *for some positive constant. From the definition of the operator* $\mathcal{A}$*, this system can be written as*

$$\begin{aligned}
&\phi = \tilde{\varphi},\\
&\kappa(\varphi_{xx} + \psi_x) = \rho_1 \tilde{\phi},\\
&\chi = \tilde{\psi},\\
&B_*\psi + \kappa(\varphi_x + \psi) + \int_0^\infty h(s) A_*^\sigma \zeta(t,s)ds + \delta\theta_x = -\rho_2 \tilde{\chi},\\
&\overline{T}\zeta + \chi = \tilde{\zeta},\\
&\frac{1}{\beta}\int_0^\infty \mu(s) A^\sigma \eta(s) ds + \delta\chi_x = -\rho_3 \tilde{\theta},\\
&T\eta + \theta = \tilde{\eta},
\end{aligned} \tag{5.34}$$

from which it follows that $\zeta(s) = s\tilde{\psi} - \int_0^s \tilde{\zeta}(r)dr, \ \int_0^\infty \mu(s)A^\sigma \eta(s)ds = -\beta(\rho_3\tilde{\theta} + \delta\tilde{\psi}_x)$, *and* (φ, ψ) *must satisfy*

$$\begin{aligned}
&\kappa(\varphi_{xx} + \psi_x) = F_1,\\
&B_*\psi - \kappa(\varphi_x + \psi) = -F_2,
\end{aligned} \tag{5.35}$$

where

$$\begin{aligned}
F_1 &= \rho_1 \tilde{\phi},\\
F_2 &= \int_0^\infty h(s) A_*^\sigma \zeta(t,s)ds + \rho_2\tilde{\chi} + \delta\tilde{\eta}_x - T\eta_x.
\end{aligned}$$

The system (5.35) *can be represented in a variational form*

$$\mathbb{B}(\varphi, \psi; \overline{\varphi}, \overline{\psi}) = \langle F_1, \overline{\varphi}\rangle - \langle F_2, \overline{\psi}\rangle, \tag{5.36}$$

where the sesquilinear form is given by

$$\mathbb{B}(\varphi,\psi;\overline{\varphi},\overline{\psi}) = \kappa\left\langle \varphi_x+\psi, \overline{\varphi}_x+\overline{\psi}\right\rangle + \left\langle B_*^{1/2}\psi, B_*^{1/2}\overline{\psi}\right\rangle. \tag{5.37}$$

As this sesquilinear form is continuous and coercive, by Lax–Milgram's Theorem there exists an unique solution. We can take $(\overline{\varphi},\overline{\psi}) = (\varphi,\psi)$ *in equality* (5.36)*, we get*

$$\begin{aligned}\kappa\|\varphi_x+\psi\|^2 + \|B_*^{1/2}\psi\|^2 &= \rho_1\left\langle \tilde{\phi},\varphi\right\rangle - \rho_2\left\langle \int_0^\infty h(s)A_*^{\sigma/2}\zeta(t,s)ds, A_*^{\sigma/2}\psi\right\rangle \\ &\quad + \rho_2\left\langle \tilde{\chi},\psi\right\rangle + \delta\left\langle \tilde{\eta}_x,\psi\right\rangle - \left\langle T\eta_x,\psi\right\rangle \\ &\le \epsilon(\|\varphi_x\|^2+\|\psi_x\|^2) + C_\epsilon\|\tilde{z}\|^2\end{aligned}$$

for $\epsilon>0$. *Using the inequality* $\|\varphi_x\|^2 \le C(\|\varphi_x+\psi\|^2+\|\psi_x\|^2)$, *the equivalence* $\|\psi_x\| \sim \|B_*^{1/2}\psi\|$ *and fixing* ϵ *small, we get*

$$\|\varphi_x+\psi\|^2+\|\psi_x\|^2 \le C\|\tilde{z}\|^2,$$

which imply that

$$\|z\|^2 \le C\|\tilde{z}\|^2,$$

that is $0\in\varrho(\mathcal{A})$.

5.4 Stability result of solutions

This section exploit a particular results from papers [69, 126, 127], which are presented by the following Theorems

Theorem 15. *Let* $\mathcal{A}$ *be the infinitisimal generator of a contraction semigroup* $s(t)$ *on a Hilbert space* $\mathcal{H}$. *Then, the following are equivalent:*

- $s(t)$ *is exponentially stable.*
- *There exists* $\varepsilon>0$ *such that*

$$\inf_{\lambda\in\mathbb{R}} \|i\lambda z - \mathcal{A}z\|_{\mathcal{H}} \ge \varepsilon\|z\|_{\mathcal{H}}, \quad \forall z\in\mathcal{D}(\mathcal{A}) \tag{5.38}$$

- *The imaginary axis* $i\mathbb{R} \subset \varrho(\mathcal{A})$ *and*

$$\sup_{\lambda\in\mathbb{R}} \|(i\lambda - \mathcal{A})^{-1}\|_{L(\mathcal{H})} < \infty. \tag{5.39}$$

Theorem 16. *Let A be the generator of a c_0-semigroup of bounded operators on a Hilbert space such that $i\mathbb{R} \subset \varrho(\mathcal{A})$. Then, the following are equivalent:*

- *For* $z_0 \in \mathcal{D}(\mathcal{A})$

$$\|z\|_{\mathcal{H}} \leq ct^{-1/\alpha}\|z_0\|_{\mathcal{D}(\mathcal{A})}, \quad \forall t > 0, \tag{5.40}$$

- *For* $\lambda \in \mathbb{R}$

$$\limsup_{|\lambda|\to\infty} |\lambda|^{-\alpha}\|(i\lambda I - \mathcal{A})^{-1}\| < \infty. \tag{5.41}$$

Let $\lambda \in \mathbb{R}$ and $y = (y_1, y_2, y_3, y_4, y_5, y_6, y_7)$. In what follows, the stationary problem $(i\lambda - \mathcal{A})z = y$ will be considered several times in the course of this section.

Note that $z = (\varphi, \phi, \psi, \chi, \zeta, \theta, \eta) \in \mathcal{D}(\mathcal{A})$ is a solution of this problem if the following equations are satisfied:

$$\begin{aligned}
&i\lambda\varphi - \phi = y_1,\\
&\rho_1 i\lambda\phi - \kappa(\varphi_{xx} + \psi_x) = \rho_1 y_2,\\
&i\lambda\psi - \chi = y_3,\\
&\rho_2 i\lambda\chi + B_*\psi + \kappa(\varphi_x + \psi) + \int_0^\infty h(s)A^\sigma\zeta(t,s)ds + \delta\theta_x = \rho_2 y_4,\\
&i\lambda\zeta - \overline{T}\zeta - \chi = y_5,\\
&\beta\rho_3 i\lambda\theta + \int_0^\infty \mu(s)A^\sigma\eta(s)ds + \beta\delta\chi_x = \beta\rho_3 y_6,\\
&i\lambda\eta - T\eta - \theta = y_7.
\end{aligned} \tag{5.42}$$

By using the first part of remark (6) and equality (5.30), we get

$$\begin{aligned}
\frac{c_1}{2}\|\zeta\|^2_{\mathcal{N}_\sigma} + \frac{c_2}{2\beta}\|\eta\|^2_{\mathcal{M}_\sigma} &\leq \frac{1}{2}\Gamma[\zeta] + \frac{1}{2\beta}\Gamma[\eta] = -\Re\left\langle \mathcal{A}z, z\right\rangle_{\mathcal{H}}\\
&= \Re\left\langle (i\lambda - \mathcal{A})z, z\right\rangle_{\mathcal{H}}\\
&\leq \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}.
\end{aligned} \tag{5.43}$$

The use of the following remark is essential

Remark 8:

- *The following embeddings*

$$H^{2\sigma_1} \hookrightarrow H^{2\sigma_2}, \qquad H_*^{2\sigma_1} \hookrightarrow H_*^{2\sigma_2}, \tag{5.44}$$

are continuous for $\sigma_1 > \sigma_2$.

- *In the remaining part of this paper* c *and* c_α *will denote positive constants that will assume different values in different places.*

What follows, represents the main results of this study

Theorem 17. *The semigroup* $s(t)$ *generated by the problem* (5.1)-(5.3) *satisfys the following asymptotic behavior*

- $s(t)$ *is polynomial stable* $\quad\Leftrightarrow\quad \xi_g \neq 0,$ *with the rate* $t^{-\frac{1}{4-2\sigma}}$.
- $s(t)$ *is polynomial stable* $\quad\Leftrightarrow\quad \xi_g = 0$ *and* $\sigma \in [0,1),$ *with the rate* $t^{-\frac{1}{2-2\sigma}}$.
- $s(t)$ *is exponentially stable* $\quad\Leftrightarrow\quad \xi_g = 0$ *and* $\sigma = 1.$

5.4.1 Technical Lemmas

Let $y \in \mathcal{H}$. Suppose that for every $\lambda \in \mathbb{R}$ such that $0 < \alpha < |\lambda|$ there exists a solution $z \in \mathcal{D}(\mathcal{A})$ of the stationary system $(i\lambda I - \mathcal{A})z = y$, then we have the following technical Lemmas

Lemma 17. *There exist a positive constants* c_α *such that*

$$\begin{aligned} &\|\theta\|^2_{H^\sigma} \leq c_\alpha \lambda^2 \left(\|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}} + \|y\|^2_{\mathcal{H}}\right), \\ &\|\theta\|^2_{H^{\sigma-1}} \leq c_\alpha \left(\|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}} + \|y\|^2_{\mathcal{H}} + \frac{1}{\lambda^2}\|\chi\|^2_{H_*^\sigma}\right). \end{aligned} \tag{5.45}$$

Proof. For the first inequality, multiply equation $(5.42)_7$ by θ in the space $\mathcal{M}_\sigma$, use the inequalities of Cauchy-Schwarz, Young and inequality (5.43), to obtain

$$\begin{aligned} \|\theta\|^2_{H^\sigma} &= \frac{i\lambda}{g(0)}\int_0^\infty \mu(s)\langle \eta, \theta\rangle_{H^\sigma} ds - \frac{1}{g(0)}\int_0^\infty \mu(s)\langle T\eta, \theta\rangle_{H^\sigma} ds - \langle y_7, \theta\rangle_{H^\sigma} \\ &\leq \|\theta\|_{H^\sigma}\left(\frac{|\lambda|}{g(0)}\left\|\int_0^\infty \mu(s)A^{\frac{\sigma}{2}}\eta ds\right\|_{L^2} - \frac{1}{g(0)}\int_0^\infty \mu'(s)\|\eta\|_{H^\sigma} ds + \|y_7\|_{H^\sigma}\right) \\ &\leq c\left(\lambda^2\|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}} + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}} + \|y\|^2_{\mathcal{H}}\right), \end{aligned}$$

for $|\lambda| \geq \alpha$ we get the inequality $(5.45)_1$.

For the second inequality, multiply equation $(5.42)_7$ by θ in the space $H^{\sigma-1}$, use the inequalities of Cauchy-Schwarz, Young, (5.43) and take into account that

$$\frac{3\sigma-2}{2} \leq \frac{\sigma}{2},$$

to obtain

$$\begin{aligned}
\|\theta\|^2_{H^{\sigma-1}} &= \frac{1}{i\lambda}\langle y_6, \theta\rangle_{H^{\sigma-1}} + \frac{1}{i\lambda\beta\rho_3}\left\langle \int_0^\infty \mu(s)A^\sigma \eta ds, \theta\right\rangle_{H^{\sigma-1}} - \frac{\delta}{i\lambda\rho_3}\langle \chi_x, \theta\rangle_{H^{\sigma-1}} \\
&\leq \|\theta\|_{H^{\sigma-1}}\left(\frac{1}{|\lambda|}\|y_6\|_{H^{\sigma-1}} + \frac{c}{|\lambda|}\|\chi\|_{H^\sigma_*}\right) + \frac{c}{|\lambda|}\left\|\int_0^\infty \mu(s)A^{\frac{3\sigma-2}{2}}\eta ds\right\|_{L^2}\|\theta\|_{H^\sigma} \\
&\leq \frac{c}{\lambda^2}\left(\|y\|^2_{\mathcal{H}} + \|\chi\|^2_{H^\sigma_*}\right) + \frac{c\epsilon}{\lambda^2}\|\theta\|^2_{H^\sigma} + c\|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}, \\
&\leq \frac{c}{\lambda^2}\left(\|y\|^2_{\mathcal{H}} + \|\chi\|^2_{H^\sigma_*}\right) + c\|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}},
\end{aligned}$$

for $|\lambda| \geq \alpha$ we get the inequality $(5.45)_2$. □

Lemma 18. *There exists a positive constant c_α such that*

$$\begin{aligned}
&\|\chi\|^2_{H^\sigma_*} \leq c_\alpha\left(\|y\|^2_{\mathcal{H}} + \lambda^2\|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}\right), \\
&\|\chi\|^2_{H^{\sigma-1}_*} \leq c\epsilon\|\varphi_x + \psi\|^2_{H^{\sigma-2}} + c_\alpha\left(\|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}} + \|y\|^2_{\mathcal{H}}\right).
\end{aligned} \tag{5.46}$$

Proof. For the first inequality, multiply the equation $(5.42)_4$ by χ in the space $\mathcal{N}_\sigma$, use the inequalities of Cauchy-Schwarz, Young and (5.43), to obtain

$$\begin{aligned}
\left|\int_0^\infty h(s)ds\|\chi\|^2_{H^\sigma_*}\right| &= \left|i\lambda\int_0^\infty h(s)\langle\zeta,\chi\rangle_{H^\sigma}ds - \int_0^\infty h(s)\langle\overline{T}\zeta,\chi\rangle_{H^\sigma}ds - \int_0^\infty h(s)\langle y_5,\chi\rangle_{H^\sigma}ds\right| \\
&\leq \|\chi\|_{H^\sigma_*}\left(|\lambda|\left\|\int_0^\infty h(s)A_*^{\frac{\sigma}{2}}\zeta(s)ds\right\|_{L^2} - \int_0^\infty h'(s)\|\zeta\|_{H^\sigma_*}ds + \int_0^\infty h(s)\|y_5\|_{H^\sigma_*}ds\right) \\
&\leq c\left(\lambda^2\int_0^\infty h(s)\|\zeta\|^2_{H^\sigma_*} - \int_0^\infty h'(s)\|\zeta\|^2_{H^\sigma_*} + \int_0^\infty h(s)\|y_5\|^2_{H^\sigma_*}\right) \\
&\leq c\left(\lambda^2\|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}} + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}} + \|y\|^2_{\mathcal{H}}\right),
\end{aligned}$$

for $|\lambda| \geq \alpha$ we get the inequality $(5.46)_1$.

For the proof of the second inequality, multiply equation $(5.42)_4$ by χ in the space $\mathcal{N}_{\sigma-1}$, use the inequalities of Cauchy-Schwarz, Young and (5.43), to obtain

$$\begin{aligned}\left|\rho_2\int_0^\infty h(s)ds\|\chi\|^2_{H_*^{\sigma-1}}\right| &= \left|\rho_2 i\lambda\int_0^\infty h(s)\left\langle\zeta,\chi\right\rangle_{H_*^{\sigma-1}}ds+\rho_2\int_0^\infty h'(s)\left\langle\zeta,\chi\right\rangle_{H_*^{\sigma-1}}ds\right.\\ &\quad\left.-\rho_2\int_0^\infty h(s)\left\langle y_5,\chi\right\rangle_{H_*^{\sigma-1}}ds\right|\\ &\le\|\chi\|_{H_*^{\sigma-1}}\left(\rho_2\int_0^\infty h(s)\|y_5\|_{H_*^{\sigma-1}}ds-\rho_2\int_0^\infty h'(s)\|\zeta\|_{H_*^{\sigma-1}}ds\right)\\ &\quad+\rho_2\left|i\lambda\int_0^\infty h(s)\left\langle\zeta,\chi\right\rangle_{H_*^{\sigma-1}}ds\right|\\ &\le c\|y\|^2_{\mathcal{H}}+c\|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}+\rho_2\left|i\lambda\int_0^\infty h(s)\left\langle\zeta,\chi\right\rangle_{H_*^{\sigma-1}}ds\right|.\end{aligned}\tag{5.47}$$

For the last term in inequality (5.47), multiply the equation $(5.42)_3$ by $\int_0^\infty h(s)A_*^{\sigma-1}\zeta(s)ds$ in the space $\mathtt{L}^2$ and use inequalities of Cauchy-Schwarz, Young and $(5.45)_2$ to obtain

$$\begin{aligned}\left|\rho_2 i\lambda\int_0^\infty h(s)\left\langle\zeta,\chi\right\rangle_{H_*^{\sigma-1}}ds\right| &= \left|\int_0^\infty h(s)\left\langle B_*\psi+\delta\theta_x+\kappa(\varphi_x+\psi)-\rho_2 y_4,A^{\sigma-1}\zeta\right\rangle_{\mathtt{L}^2}ds\right|\\ &\quad+\left\|\int_0^\infty h(s)A_*^{\frac{2\sigma-1}{2}}\zeta(s)ds\right\|^2_{\mathtt{L}^2}\\ &\le\left|\int_0^\infty h(s)\|\zeta\|_{H^\sigma}ds\left(\delta\|\theta\|_{H^{\sigma-1}}+\kappa\|\varphi_x+\psi\|_{H^{\sigma-2}}\right)\right.\\ &\quad\left.+\int_0^\infty h(s)\left\langle B_*\psi,A_*^{\sigma-1}\zeta(s)\right\rangle_{\mathtt{L}^2}ds\right|\\ &\quad+\left\|\int_0^\infty h(s)A_*^{\frac{2\sigma-1}{2}}\zeta(s)\right\|^2_{\mathtt{L}^2}\\ &\le\left|\int_0^\infty h(s)\left\langle B_*\psi,A_*^{\sigma-1}\zeta(s)\right\rangle_{\mathtt{L}^2}ds\right|\\ &\quad+\left\|\int_0^\infty h(s)A_*^{\frac{2\sigma-1}{2}}\zeta(s)\right\|^2_{\mathtt{L}^2}\\ &\quad+c\epsilon\left(\|\theta\|^2_{H^{\sigma-1}}+\|\varphi_x+\psi\|^2_{H^{\sigma-2}}\right)+c_\epsilon\|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}\\ &\le\left|\int_0^\infty h(s)\left\langle B_*\psi,A_*^{\sigma-1}\zeta(s)\right\rangle_{\mathtt{L}^2}ds\right|+c\epsilon\|\varphi_x+\psi\|^2_{H^{\sigma-2}}\\ &\quad+c\epsilon\|y\|^2_{\mathcal{H}}+c_\epsilon\|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}+\left\|\int_0^\infty h(s)A_*^{\frac{2\sigma-1}{2}}\zeta(s)\right\|^2_{\mathtt{L}^2}.\end{aligned}\tag{5.48}$$

Using the equation $(5.42)_3$, Cauchy-Schwarz and Young inequalities, then by applying the definition of B_* and taking into account that $\frac{2\sigma-1}{2} \leq \frac{\sigma}{2}$, we get

$$\begin{aligned}\left|\int_0^\infty h(s)\left\langle B_*\psi, A^{\sigma-1}\zeta\right\rangle_{\mathrm{L}^2} ds\right| &= \left|\int_0^\infty h(s)\left\langle B_*\left(\frac{\chi+y_3}{i\lambda}\right), A^{\sigma-1}\zeta\right\rangle_{\mathrm{L}^2} ds\right| \\ &= \left|\int_0^\infty h(s)\left\langle bA_* - \left(\int_0^\infty h(s)ds\right)A_*^\sigma\left(\frac{\chi+y_3}{i\lambda}\right), A^{\sigma-1}\zeta\right\rangle_{\mathrm{L}^2} ds\right| \\ &= \left|\frac{b}{i\lambda}\int_0^\infty h(s)\left\langle A_*^{\frac{\sigma}{2}}(\chi+y_3), A^{\frac{\sigma}{2}}\zeta\right\rangle_{\mathrm{L}^2} ds\right. \\ &\quad \left. -\frac{1}{i\lambda}\left(\int_0^\infty h(s)ds\right)\int_0^\infty h(s)\left\langle A^{\frac{2\sigma-1}{2}}(\chi+y_3), A^{\frac{2\sigma-1}{2}}\zeta\right\rangle_{\mathrm{L}^2} ds\right| \\ &\leq \frac{c}{\lambda^2}\|\chi\|^2_{H^\sigma} + c\|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}} + \frac{c}{\lambda^2}\|y\|^2_{\mathcal{H}} \\ &\leq c\|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}} + \frac{c}{\lambda^2}\|y\|^2_{\mathcal{H}}.\end{aligned} \tag{5.49}$$

By adding inequalities (5.49) and (5.48), using the result with the inequality (5.47), so we can get the inequality $(5.46)_2$ for $|\lambda| \geq \alpha$. □

Lemma 19. *For $\gamma \leq \sigma - 1$, there exist positive constant c_α such that*

$$\begin{aligned}\lambda^2\|\phi\|^2_{H^\gamma} &\leq c_\alpha\left(\|\phi\|^2_{H^{\gamma+1}} + \|y\|^2_{\mathcal{H}} + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}\right), \\ \|\phi\|^2_{H^{\gamma+1}} &\leq c_\alpha\left(\lambda^2\|\phi\|^2_{H^\gamma} + \|y\|^2_{\mathcal{H}} + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}\right), \\ \|\phi\|^2_{H^1} &\leq c_\alpha\left(\lambda^2\|\phi\|^2_{\mathrm{L}^2} + \|\psi_x\|^2_{\mathrm{L}^2} + \|y\|^2_{\mathcal{H}}\right).\end{aligned} \tag{5.50}$$

Proof. Multiply equation $(5.42)_2$ by $i\lambda \times \phi$ in the space H^γ, to obtain

$$\lambda^2\|\phi\|^2_{H^\gamma} = i\lambda\left\langle y_2, \phi\right\rangle_{H^\gamma} + \frac{\kappa}{\rho_1}\left(\left\langle Ay_1, \phi\right\rangle_{H^\gamma} + \left\langle A\phi, \phi\right\rangle_{H^\gamma} - i\lambda\left\langle \psi_x, \phi\right\rangle_{H^\gamma}\right), \tag{5.51}$$

by using equation $(5.42)_1$, Young inequality and the consideration that

$$\gamma \leq \sigma - 1, \gamma - \frac{\sigma}{2} + \frac{1}{2} \leq \frac{\gamma}{2}, \gamma \leq \frac{\gamma}{2},$$

we get the following estimate

$$\begin{aligned}\lambda^2\|\phi\|^2_{H^\gamma} &\leq c\left(\|\phi\|^2_{H^{\gamma+1}} + \|y_1\|^2_{H^{\gamma+1}}\right) + \epsilon\|\phi\|^2_{H^{\gamma+1}} + \left|i\lambda\left(\left\langle y_2, \phi\right\rangle_{H^\gamma} + \frac{\kappa}{\rho_1}\left\langle \psi_x, \phi\right\rangle_{H^\gamma}\right)\right| \\ &\leq c\left(\|\phi\|^2_{H^{\gamma+1}} + \|y_1\|^2_{H^{\gamma+1}}\right) + \epsilon\|\phi\|^2_{H^{\gamma+1}} + \epsilon\lambda^2\|\phi\|^2_{H^\gamma} + c\|y\|^2_{\mathcal{H}} + c\|\psi\|^2_{H^\sigma_*} \\ &\quad + \epsilon\lambda^2\|A^{\gamma-\frac{\sigma}{2}+\frac{1}{2}}\phi\|^2_{L^2}.\end{aligned} \tag{5.52}$$

Then, we get

$$\begin{aligned}\lambda^2\|\phi\|^2_{H^\gamma} &\le c\left(\|\phi\|^2_{H^{\gamma+1}} + \|y_1\|^2_{H^{\gamma+1}}\right) + \epsilon\|\phi\|^2_{H^{\gamma+1}} + \epsilon\lambda^2\|\phi\|^2_{H^\gamma} + c\|y\|^2_{\mathcal{H}} \\ &\quad + c\|\psi\|^2_{H^\sigma_*} + \epsilon\lambda^2\|\phi\|^2_{H^\gamma} \\ &\le c\left(\|\phi\|^2_{H^{\gamma+1}} + \|y\|^2_{\mathcal{H}} + \|\psi\|^2_{H^\sigma_*}\right),\end{aligned} \tag{5.53}$$

recalling that

$$\|\psi\|^2_{H^\sigma_*} \le \frac{1}{\lambda^2}\left(\|y\|^2_{\mathcal{H}} + \|\chi\|^2_{H^\sigma_*}\right), \tag{5.54}$$

using the previous inequality, Lemma (16) and inequality (5.53), we deduce the following

$$\lambda^2\|\phi\|^2_{H^\gamma} \le c\left(\|\phi\|^2_{H^{\gamma+1}} + \frac{1}{\lambda^2}\|y\|^2_{\mathcal{H}} + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}\right), \tag{5.55}$$

for $|\lambda| \ge \alpha$ we get the inequality $(5.50)_1$. For the proof of the second inequality, use equation(5.51), Young inequality and Lemma (16), to obtain

$$\begin{aligned}\|\phi\|^2_{H^{\gamma+1}} &= \lambda^2\frac{\rho_1}{\kappa}\|\phi\|^2_{H^\gamma} - i\lambda\frac{\rho_1}{\kappa}\left\langle y_2, \phi\right\rangle_{H^\gamma} + \left\langle y_{1x}, \phi_x\right\rangle_{H^\gamma} \\ &\quad + \left\langle A^{\frac{\sigma}{2}}(y_3+\chi), A^{\gamma+\frac{1-\sigma}{2}}\phi\right\rangle_{\mathrm{L}^2} \\ &\le c\lambda^2\|\phi\|^2_{H^\gamma} + c\|y\|^2_{\mathcal{H}} + c\|\chi\|^2_{H^\sigma_*} \\ &\le c\lambda^2\|\phi\|^2_{H^\gamma} + c\|y\|^2_{\mathcal{H}} + c\lambda^2\|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}},\end{aligned}$$

for $|\lambda| \ge \alpha$ we get the inequality $(5.50)_2$. For the proof of the last inequality, take $\gamma = 0$ in equality (5.51), to obtain directly the inequality $(5.50)_3$. □

Lemma 20. *For $\beta < 0$, there exist positive constants c_α such that,*

$$\begin{aligned}\|\varphi_x + \psi\|^2_{H^\beta} &\le c_\alpha\left(\|\chi\|^2_{H^{\beta+1}} + \|y\|^2_{\mathcal{H}} + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}} + \epsilon\|\phi\|^2_{H^\beta} + \epsilon\|\theta\|^2_{H^{\beta+1}}\right) \\ &\quad + \xi_g\frac{g(0)b\kappa}{\beta\gamma_g}\left|\langle\chi, \phi_x\rangle_{H^\beta}\right|,\end{aligned} \tag{5.56}$$

such that

$$\gamma_g = \frac{g(0)\rho_1}{\beta\rho_3} - \kappa, \quad \textit{and} \quad \xi_g = \left(\frac{\rho_1}{\rho_3\kappa} - \frac{\beta}{g(0)}\right)\xi_0 - \frac{\beta\rho_1\delta^2}{g(0)\rho_3\kappa b}$$

where ξ_0 is defined in (5.7).

Proof. For the proof, we need to construct the following essential three steps:

A – Step 1. Multiply equation $(5.42)_3$ by $(\varphi_x+\psi)$ in the space H^β

$$\begin{aligned}\kappa\|\varphi_x+\psi\|^2_{H^\beta} &= \rho_2\left\langle y_4,(\varphi_x+\psi)\right\rangle_{H^\beta}-\delta\left\langle \theta_x,(\varphi_x+\psi)\right\rangle_{H^\beta}-\left\langle B_*\psi,(\varphi_x+\psi)\right\rangle_{H^\beta}\\ &\quad -i\lambda\rho_2\left\langle \chi,(\varphi_x+\psi)\right\rangle_{H^\beta}-\int_0^\infty h(s)\left\langle A_*^\sigma\zeta,(\varphi_x+\psi)\right\rangle_{H^\beta}\\ &= \rho_2\left\langle y_4,(\varphi_x+\psi)\right\rangle_{H^\beta}-\delta\left\langle \theta_x,(\varphi_x+\psi)\right\rangle_{H^\beta}+\int_0^\infty h(s)\left\langle A_*^\sigma(\psi-\zeta),(\varphi_x+\psi)\right\rangle_{H^\beta}\\ &\quad -b\left\langle A_*\psi,(\varphi_x+\psi)\right\rangle_{H^\beta}-i\lambda\rho_2\left\langle \chi,(\varphi_x+\psi)\right\rangle_{H^\beta},\end{aligned} \tag{5.57}$$

for the last terms in equation (5.57), by using equations $(5.42)_2$ and $(5.42)_3$, we get

$$\begin{aligned} b\left\langle \psi_x,(\varphi_{xx}+\psi_x)\right\rangle_{H^\beta} &= \frac{b\rho_1}{\kappa}\left(\left\langle y_{3x},\phi\right\rangle_{H^\beta}-\left\langle \psi_x,y_2\right\rangle_{H^\beta}-\left\langle \chi,\phi_x\right\rangle_{H^\beta}\right),\\ i\lambda\rho_2\left\langle \chi,(\varphi_x+\psi)\right\rangle_{H^\beta} &= \rho_2\left(\left\langle \chi,y_{1x}\right\rangle_{H^\beta}+\left\langle \chi,\phi_x\right\rangle_{H^\beta}+\left\langle \chi,y_3\right\rangle_{H^\beta}+\|\chi\|_{H^\beta}\right).\end{aligned} \tag{5.58}$$

Now, for the term $\int_0^\infty h(s)\left\langle A^\sigma(\psi-\zeta),(\varphi_x+\psi)\right\rangle_{H^\beta}$, use equations $(5.42)_3$, $(5.42)_5$ and the fact that $\sigma\le-\beta$, to find

$$\begin{aligned}\left|\int_0^\infty h(s)\left\langle A_*^\sigma(\psi-\zeta),(\varphi_x+\psi)\right\rangle_{H^\beta}\right| &= \left|\frac{1}{i\lambda}\int_0^\infty h(s)\left\langle A^{\frac{\sigma}{2}}(y_3-y_5),A^{\beta+\frac{\sigma}{2}}(\varphi_x+\psi)\right\rangle_{\mathrm{L}^2}\right.\\ &\quad\left.+\frac{1}{i\lambda}\int_0^\infty h(s)\left\langle A^{\frac{\sigma}{2}}\overline{T}\zeta,A^{\beta+\frac{\sigma}{2}}(\varphi_x+\psi)\right\rangle_{\mathrm{L}^2}\right|\\ &\le \frac{c\epsilon}{\lambda^2}\|\varphi_x+\psi\|^2_{H^\beta}+c(\|y\|^2_{\mathcal{H}}+\|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}).\end{aligned} \tag{5.59}$$

Substitute the equations (5.58) and the inequality (5.59) in the equality (5.57), to obtain the following inequality

$$\begin{aligned}\kappa\|A^{\frac{1}{2}}\varphi+\psi\|^2_{H^\beta} &\le -\delta\left|\left\langle \theta_x,(\varphi_x+\psi)\right\rangle_{H^\beta}\right|+\left(\frac{b\rho_1}{\kappa}-\rho_2\right)\left|\left\langle \chi,\phi_x\right\rangle_{H^\beta}\right|\\ &\quad -\frac{b\rho_1}{\kappa}\left(\left|\left\langle y_{3x},\phi\right\rangle_{H^\beta}\right|-\left|\left\langle \psi_x,y_2\right\rangle_{H^\beta}\right|\right)\\ &\quad -\rho_2\left(\left|\left\langle \chi,y_{1x}+y_3\right\rangle_{H^\beta}\right|+\|\chi\|^2_{H^\beta}-\left|\left\langle y_4,(\varphi_x+\psi)\right\rangle_{H^\beta}\right|\right)\\ &\quad +c(\|y\|^2_{\mathcal{H}}+\|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}).\end{aligned} \tag{5.60}$$

Note that β satisfies the conditions

$$\beta \leq \frac{\beta}{2} < \frac{\beta+1}{2}, \quad \beta + \frac{1}{2} \leq \frac{1}{2} \tag{5.61}$$

and by using Young inequality, we obtain the following estimates

$$\begin{aligned}
\frac{b\rho_1}{\kappa}\,|\langle y_{3x}, \phi\rangle_{H^\beta}| &\leq \epsilon c\|\phi\|^2_{H^{2\beta}} + c\|y\|^2_{\mathcal{H}} \\
&\leq c\epsilon\|\phi\|^2_{H^\beta} + c\|y\|^2_{\mathcal{H}}, \\
\frac{b\rho_1}{\kappa}\,|\langle \psi_x, y_2\rangle_{H^\beta}| &\leq c\|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}, \\
\rho_2\,|\langle \chi, y_{1x} + y_3\rangle_{H^\beta}| &\leq \epsilon c\|\chi\|^2_{H^{2\beta}} + c\|y\|^2_{\mathcal{H}} \\
&\leq \epsilon c\|\chi\|^2_{H^\beta} + c\|y\|^2_{\mathcal{H}} \\
&\leq \epsilon c\|\chi\|^2_{H^{\beta+1}} + c\|y\|^2_{\mathcal{H}}, \\
\rho_2\,|\langle y_4, (\varphi_x + \psi)\rangle_{H^\beta}| &\leq c\epsilon\|\varphi_x + \psi\|^2_{H^{2\beta}} + c\|y\|^2_{\mathcal{H}} \\
&\leq c\epsilon\|\varphi_x + \psi\|^2_{H^\beta} + c\|y\|^2_{\mathcal{H}}.
\end{aligned} \tag{5.62}$$

By using estimates (5.62), inequality (5.60) and consideration (5.61), we get

$$\begin{aligned}
\|\varphi_x + \psi\|^2_{H^\beta} &\leq -\delta\,|\langle \theta_x, (\varphi_x + \psi)\rangle_{H^\beta}| + \left(\frac{b\rho_1}{\kappa} - \rho_2\right)|\langle \chi, \phi_x\rangle_{H^\beta}| \\
&\quad + c_\alpha\left(\epsilon\|\phi\|^2_{H^\beta} + \|\chi\|^2_{H^{\beta+1}} + \|y\|^2_{\mathcal{H}} + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}\right) \\
&\leq -\delta\left|\left\langle \theta, (\partial_x)^{-1}(\varphi_x + \psi)\right\rangle_{H^{2\beta}}\right| + \left(\frac{b\rho_1}{\kappa} - \rho_2\right)|\langle \chi, \phi_x\rangle_{H^\beta}| \\
&\quad + c_\alpha\left(\epsilon\|\phi\|^2_{H^\beta} + \|\chi\|^2_{H^{\beta+1}} + \|y\|^2_{\mathcal{H}} + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}\right)
\end{aligned} \tag{5.63}$$

B – Step 2. By multiplying in the space $\mathcal{M}_{2\beta}$ the equation $(5.42)_7$ by $((\partial_x)^{-1}(\varphi_x + \psi))$, we get the following

$$\begin{aligned}
g(0)\left\langle \theta, (\partial_x)^{-1}(\varphi_x + \psi)\right\rangle_{H^{2\beta}} &= \left\langle y_7, (\partial_x)^{-1}(\varphi_x + \psi)\right\rangle_{\mathcal{M}_{2\beta}} - \left\langle T\eta, (\partial_x)^{-1}(\varphi_x + \psi)\right\rangle_{\mathcal{M}_{2\beta}} \\
&\quad + i\lambda\left(\langle \eta, \varphi\rangle_{\mathcal{M}_{2\beta}} + \left\langle \eta, (\partial_x)^{-1}\psi\right\rangle_{\mathcal{M}_{2\beta}}\right) \\
&= \left\langle y_7, (\partial_x)^{-1}(\varphi_x + \psi)\right\rangle_{\mathcal{M}_{2\beta}} + \left\langle D\eta, (\partial_x)^{-1}(\varphi_x + \psi)\right\rangle_{\mathcal{M}_{2\beta}} \\
&\quad + \left\langle \eta, (\partial_x)^{-1}(\chi + y_3)\right\rangle_{\mathcal{M}_{2\beta}} + \langle \eta, y_1\rangle_{\mathcal{M}_{2\beta}} + \langle \eta, \phi\rangle_{\mathcal{M}_{2\beta}},
\end{aligned} \tag{5.64}$$

for the last term in the equation (5.64), multiply the equation $(5.42)_6$ by ϕ in the space $H^{2\beta}$, to obtain

$$\langle \eta, \phi \rangle_{\mathcal{M}_{2\beta}} = -\beta\delta \langle \chi, \phi_x \rangle_{H^\beta} + \beta\rho_3 i\lambda \langle \theta, \phi \rangle_{H^\beta} - \beta\rho_3 \langle y_6, \phi \rangle_{H^\beta} . \tag{5.65}$$

Remembering that $\beta < 0$ and also recalling the conditions (5.61), Hence, by the equation (5.65), equation (5.64) gives

$$\begin{aligned}
\left| \frac{g(0)}{\beta} \left\langle \theta, (\partial_x)^{-1}(\varphi_x + \psi) \right\rangle_{H^{2\beta}} \right| &= \left| \frac{1}{\beta} \left(\left\langle y_7, (\partial_x)^{-1}(\varphi_x + \psi) \right\rangle_{\mathcal{M}_{2\beta}} + \left\langle D\eta, (\partial_x)^{-1}(\varphi_x + \psi) \right\rangle_{\mathcal{M}_{2\beta}} \right) \right. \\
&\quad + \frac{1}{\beta} \left(\left\langle \eta, (\partial_x)^{-1}(\chi + y_3) \right\rangle_{\mathcal{M}_{2\beta}} + \langle \eta, y_1 \rangle_{\mathcal{M}_{2\beta}} \right) - \delta \langle \chi, \phi_x \rangle_{H^\beta} \\
&\quad \left. + \rho_3 i\lambda \langle \theta, \phi \rangle_{H^\beta} - \rho_3 \langle y_6, \phi \rangle_{H^\beta} \right| \\
&\leq \left| \frac{1}{\beta} \left(\langle y_7, (\varphi_x + \psi) \rangle_{\mathcal{M}_\beta} + \left\langle D\eta, (A^{\frac{1}{2}}\varphi + \psi) \right\rangle_{\mathcal{M}_\beta} \right) \right. \\
&\quad + \frac{1}{\beta} \left(\left\langle \eta, (\partial_x)^{-1}(\chi + y_3) \right\rangle_{\mathcal{M}_\beta} + \langle \eta, y_1 \rangle_{\mathcal{M}_\beta} \right) \\
&\quad \left. - \rho_3 \langle y_6, \phi \rangle_{H^\beta} - \delta \langle \chi, \phi_x \rangle_{H^\beta} + \rho_3 i\lambda \langle \theta, \phi \rangle_{H^\beta} \right| ,
\end{aligned} \tag{5.66}$$

for $\beta \leq \frac{\beta}{2}, \beta - \frac{\sigma}{2} < \frac{\beta}{2}, \frac{\beta+1-\sigma}{2} < \frac{\beta+1}{2}, \frac{\sigma+\beta}{2} - 1 < \frac{\sigma}{2}, \beta - \frac{\sigma}{2} \leq \frac{\beta}{2}$, the terms of inequality (5.66) are estimated as follows

$$\begin{aligned}
\frac{1}{\beta} \left| \langle y_7, (\varphi_x + \psi) \rangle_{\mathcal{M}_\beta} \right| &\leq c\epsilon \|\varphi_x + \psi\|^2_{H^{2\beta}} + c\|y\|^2_{\mathcal{H}} \\
&\leq c\epsilon \|\varphi_x + \psi\|^2_{H^\beta} + c\|y\|^2_{\mathcal{H}}, \\
\frac{1}{\beta} \left| \langle D\eta, (\varphi_x + \psi) \rangle_{\mathcal{M}_\beta} \right| &= \left| \int_0^\infty \mu(s) \left\langle A^{\frac{\sigma}{2}} D\eta, A^{\beta - \frac{\sigma}{2}}(\varphi_x + \psi) \right\rangle_{L^2} ds \right| \\
&\leq c\epsilon \|\varphi_x + \psi\|_{H^\beta} + c\|y\|_{\mathcal{H}} \|z\|_{\mathcal{H}}, \\
\frac{1}{\beta} \left| \left\langle \eta, (\partial_x)^{-1}(\chi + y_3) \right\rangle_{\mathcal{M}_\beta} \right| &= \left| \int_0^\infty \mu(s) \left\langle A^{\frac{\sigma+\beta}{2} - 1}\eta, A^{\frac{\beta+1-\sigma}{2}} \chi \right\rangle_{L^2} ds \right. \\
&\quad \left. + \int_0^\infty \mu(s) \langle A^{\beta - \frac{1+\sigma}{2}} \eta, A^{\frac{\sigma}{2}} y_3 \rangle_{L^2} ds \right| \\
&\leq c\epsilon \|\chi\|_{H^{\beta+1}} + c \left(\|y\|_{\mathcal{H}} \|z\|_{\mathcal{H}} + \|y\|^2_{\mathcal{H}} \right), \\
\left| \langle \eta, y_1 \rangle_{\mathcal{M}_\beta} \right| &= \left| \int_0^\infty \mu(s) \left\langle A^{\beta - \frac{\sigma}{2}} \eta, A^{\frac{\sigma}{2}} y_1 \right\rangle_{L^2} ds \right| \\
&\leq \|y\|_{\mathcal{H}} \|z\|_{\mathcal{H}} + \|y\|^2_{\mathcal{H}}, \\
\rho_3 \left| \langle y_6, \phi \rangle_{H^\beta} \right| &\leq c\epsilon \|\phi\|_{H^\beta} + c\|y\|^2_{\mathcal{H}},
\end{aligned} \tag{5.67}$$

by using the previous estimates then, the inequality (5.66) will reduce to

$$\begin{aligned}\left|\frac{g(0)}{\beta}\left\langle\theta,(\partial_x)^{-1}(\varphi_x+\psi)\right\rangle_{H^{2\beta}}\right| &\leq \left|-\delta\left\langle\chi,\phi_x\right\rangle_{H^\beta}+\rho_3 i\lambda\left\langle\theta,\phi\right\rangle_{H^\beta}\right| \\ &\quad + c\epsilon\|\varphi_x+\psi\|^2_{H^\beta}+c\|y\|^2_{\mathcal{H}}+c\|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}} \\ &\quad + c\epsilon\|\phi\|^2_{H^\beta}+c\epsilon\|\chi\|^2_{H^{\beta+1}}.\end{aligned} \tag{5.68}$$

C – Step 3. Multiply equation $(5.42)_1$ by θ in the space H^β

$$i\lambda\rho_1\left\langle\phi,\theta\right\rangle_{H^\beta}=\rho_1\left\langle y_2,\theta\right\rangle_{H^\beta}-\kappa\left\langle\theta_x,(\varphi_x+\psi)\right\rangle_{H^\beta}, \tag{5.69}$$

using the conditions (5.61) and by Young inequality, we get

$$\begin{aligned}\left|i\lambda\rho_1\left\langle\phi,\theta\right\rangle_{H^\beta}\right| &\leq \left|\rho_1\left\langle y_2,\theta\right\rangle_{H^{\beta+1}}-\kappa\left\langle\theta,(\partial_x)^{-1}(\varphi_x+\psi)\right\rangle_{H^{2\beta}}\right| \\ &\leq c\|y\|^2_{\mathcal{H}}+\epsilon c\|\theta\|^2_{H^{\beta+1}}-\kappa\left|\left\langle\theta,(\partial_x)^{-1}(\varphi_x+\psi)\right\rangle_{H^{2\beta}}\right|.\end{aligned} \tag{5.70}$$

Noting that

$$\gamma_g=\frac{g(0)\rho_1}{\beta\rho_3}-\kappa,\quad \text{and}\quad \xi_g=\left(\frac{\rho_1}{\rho_3\kappa}-\frac{\beta}{g(0)}\right)\xi_0-\frac{\beta\rho_1\delta^2}{g(0)\rho_3\kappa b}.$$

Taking the multiplication of the inequality (5.68) by $\frac{\rho_1\delta}{\gamma_g\rho_3}$ and the inequality (5.70) by $\frac{\delta}{\gamma_g}$, summing up the result with the inequality (5.63), to obtain

$$\begin{aligned}\|\varphi_x+\psi\|^2_{H^\beta} &\leq \xi_g\frac{g(0)b\kappa}{\beta\gamma_g}\left|\left\langle\chi,\phi_x\right\rangle_{H^\beta}\right|+c\left(\|\chi\|^2_{H^{\beta+1}}+\|y\|^2_{\mathcal{H}}+\|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}\right) \\ &\quad + c\epsilon\left(\|\phi\|^2_{H^\beta}+\|\theta\|^2_{H^{\beta+1}}\right),\end{aligned} \tag{5.71}$$

for $|\lambda|\geq\alpha$ we obtain (5.56). □

Lemma 21. *For $\xi_g\neq 0$, there exist a positive constant c_α such that*

$$\begin{aligned}\|\phi\|^2_{H^{\sigma-2}} &\leq c_\alpha\left(\|y\|^2_{\mathcal{H}}+\|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}\right), \\ \|\phi\|^2_{H^{\sigma}} &\leq c_\alpha\lambda^4\left(\|y\|^2_{\mathcal{H}}+\|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}\right).\end{aligned} \tag{5.72}$$

For $\xi_g = 0$, there exist a positive constant c_α such that

$$\|\phi\|^2_{H^{\sigma-1}} \leq c_\alpha \left(\|y\|^2_{\mathcal{H}} + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}\right), \\ \|\phi\|^2_{H^{\sigma}} \leq c_\alpha \lambda^2 \left(\|y\|^2_{\mathcal{H}} + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}\right). \tag{5.73}$$

Proof. Firstly, for $\xi_g \neq 0$. Taking the first term of equality (5.70), by using Young inequality, we get

$$\xi_g \frac{g(0)b\kappa}{\beta\gamma_g} \left|\langle \chi, \phi_x \rangle_{H^\beta}\right| \leq c_\alpha \left(\|\chi\|^2_{H^{\beta+1}} + \epsilon\|\phi\|^2_{H^\beta}\right), \tag{5.74}$$

substitute the previous inequality in inequality (5.70) and by taking $\beta = \sigma - 2$, we obtain

$$\|\varphi_x + \psi\|^2_{H^{\sigma-2}} \leq c_\alpha \left(\|\chi\|^2_{H^{\sigma-1}} + \|y\|^2_{\mathcal{H}} + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}} + \epsilon\|\phi\|^2_{H^{\sigma-2}} + \epsilon\|\theta\|^2_{H^{\sigma-1}}\right). \tag{5.75}$$

Now, by applying Lemma (17) and Lemma (19), we get

$$\|\varphi_x + \psi\|^2_{H^{\sigma-2}} \leq c_{\alpha,\epsilon} \left(\|y\|^2_{\mathcal{H}} + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}\right) + c\epsilon\|\phi\|^2_{H^{\sigma-2}}, \tag{5.76}$$

using Lemma (19) by taking $\gamma = \sigma - 2$, so we get

$$\begin{aligned} \|\varphi\|^2_{H^{\sigma-1}} &\leq 2\|\varphi_x + \psi\|^2_{H^{\sigma-2}} + 2\|\psi\|^2_{H^{\sigma-2}} \\ &\leq c_{\alpha,\epsilon} \left(\|y\|^2_{\mathcal{H}} + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}\right) + c\epsilon\|\phi\|^2_{H^{\sigma-2}} + \frac{c}{\lambda^2}\|\chi\|^2_{H^{\sigma-2}} \\ &\leq c_{\alpha,\epsilon} \left(\|y\|^2_{\mathcal{H}} + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}\right) + c\epsilon\|\phi\|^2_{H^{\sigma-2}}, \end{aligned} \tag{5.77}$$

and we also have

$$\|\phi\|^2_{H^{\sigma-1}} \leq c\lambda^2\|\varphi\|^2_{H^{\sigma-1}} + \|y\|^2_{\mathcal{H}}. \tag{5.78}$$

Now, applying Lemma (18) by taking $\gamma = \sigma - 2$, we get

$$\begin{aligned} \|\phi\|^2_{H^{\sigma-2}} &\leq \frac{c}{\lambda^2} \left(\|\phi\|^2_{H^{\sigma-1}} + \|y\|^2_{\mathcal{H}} + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}\right) \\ &\leq c \left(\|\varphi\|^2_{H^{\sigma-1}} + \|y\|^2_{\mathcal{H}} + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}\right) \end{aligned}$$

It follows by inequality (5.77) that

$$\|\phi\|_{H^{\sigma-2}}^2 \le c_{\alpha,\epsilon}\left(\|y\|_{\mathcal{H}}^2 + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}\right), \tag{5.79}$$

which is exactly inequality $(5.72)_1$. By taking $\gamma = \sigma - 1$ in Lemma (19), we obtain

$$\begin{aligned} \|\phi\|_{H^{\sigma}}^2 &\le c_{\alpha}\left(\lambda^2\|\phi\|_{H^{\sigma-1}}^2 + \|y\|_{\mathcal{H}}^2 + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}\right) \\ &\le c_{\alpha}\lambda^4\left(\|\phi\|_{H^{\sigma-2}}^2 + \|y\|_{\mathcal{H}}^2 + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}\right) \\ &\le c_{\alpha}\lambda^4\left(\|y\|_{\mathcal{H}}^2 + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}\right). \end{aligned} \tag{5.80}$$

Then, the inequality $(5.72)_2$ appears directly. Secondly, for $\xi_g = 0$. From the inequality (5.59), we can write the following

$$\begin{aligned} \left|\int_0^{\infty} h(s)\left\langle A_*^{\sigma}(\psi-\zeta), (\varphi_x+\psi)\right\rangle_{H^{\beta}}\right| &\le \frac{c\epsilon}{\lambda^2}\|\varphi_x+\psi\|_{H^{2\beta+\sigma}}^2 \\ &\quad + c(\|y\|_{\mathcal{H}}^2 + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}), \end{aligned} \tag{5.81}$$

by using the inequality (5.70) and taking account that $\xi_g = 0$, we obtain

$$\begin{aligned} \|\varphi_x+\psi\|_{H^{\beta}}^2 &\le c_{\alpha}\left(\frac{\epsilon}{\lambda^2}\|\varphi_x+\psi\|_{H^{2\beta+\sigma}}^2 + \|\chi\|_{H^{\beta}}^2 + \|y\|_{\mathcal{H}}^2 + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}\right) \\ &\quad + c_{\alpha}\left(\epsilon\|\phi\|_{H^{2\beta}}^2 + \epsilon\|\theta\|_{H^{\beta+1}}^2\right), \end{aligned} \tag{5.82}$$

now, for $\beta = \sigma - 1$ in inequality (5.82) and by applying Lemma (17), we get

$$\begin{aligned} \|\varphi_x+\psi\|_{H^{\sigma-1}}^2 &\le c_{\alpha}\left(\frac{\epsilon}{\lambda^2}\|\varphi_x+\psi\|_{H^{3\sigma-2}} + \|\chi\|_{H^{\sigma-1}}^2 + \|y\|_{\mathcal{H}}^2 + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}\right) \\ &\quad + c_{\alpha}\left(\epsilon\|\phi\|_{H^{2\sigma-2}}^2 + \epsilon\|\theta\|_{H^{\sigma}}^2\right) \\ &\le c_{\alpha}\left(\frac{\epsilon}{\lambda^2}\|\varphi_x+\psi\|_{H^{3\sigma-2}} + \|\chi\|_{H^{\sigma-1}}^2 + \|y\|_{\mathcal{H}}^2 + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}} + \epsilon\|\phi\|_{H^{2\sigma-2}}^2\right), \end{aligned} \tag{5.83}$$

by applying Lemma (19) and the fact that $\frac{\sigma-2}{2} < \frac{\sigma-1}{2}$, we get

$$\begin{aligned} \|\varphi_x+\psi\|_{H^{\sigma-1}}^2 &\le c_{\alpha}\left(\frac{\epsilon}{\lambda^2}\|\varphi_x+\psi\|_{H^{3\sigma-2}} + \epsilon\|\varphi_x+\psi\|_{H^{\sigma-2}}\right) \\ &\quad + c_{\alpha}\left(\|y\|_{\mathcal{H}}^2 + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}} + \epsilon\|\phi\|_{H^{2\sigma-2}}^2\right) \\ &\le c_{\alpha}\left(\frac{\epsilon}{\lambda^2}\|\varphi_x+\psi\|_{H^{3\sigma-2}} + \|y\|_{\mathcal{H}}^2 + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}} + \epsilon\|\phi\|_{H^{2\sigma-2}}^2\right), \end{aligned} \tag{5.84}$$

by using equation $(5.42)_2$ and by taking $\gamma = 3\sigma - 3$ in Lemma (19), we obtain

$$\begin{aligned}
\|\varphi_x + \psi\|_{H^{3\sigma-2}} &= \|(\varphi_{xx} + \psi_x)\|_{H^{3\sigma-3}} \\
&= \frac{\rho_1}{\kappa} \langle y_{2x}, (\varphi_x + \psi)\rangle_{H^{3\sigma-3}} - \frac{\rho_1 i\lambda}{\kappa} \langle \phi_x, (\varphi_x + \psi)\rangle_{H^{3\sigma-3}} \\
&\leq c\|y\|^2_{\mathcal{H}} + \epsilon\lambda^2 c\|\varphi_x + \psi\|_{H^{3\sigma-3}} + c\|\phi\|^2_{H^{3\sigma-2}} \\
&\leq c\|y\|^2_{\mathcal{H}} + c\|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}} + \epsilon\lambda^2 c\|\varphi_x + \psi\|_{H^{3\sigma-3}} + c\lambda^2\|\phi\|^2_{H^{3\sigma-3}},
\end{aligned} \tag{5.85}$$

substitute the inequality (5.85) in the inequality (5.83), to give

$$\begin{aligned}
\|\varphi_x + \psi\|^2_{H^{\sigma-1}} \leq c_\alpha &\left(\epsilon\|\varphi_x + \psi\|_{H^{3\sigma-3}} + \|y\|^2_{\mathcal{H}} + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}\right) \\
&+ c_\alpha \left(\epsilon\|\phi\|^2_{H^{2\sigma-2}} + \epsilon\|\phi\|^2_{H^{3\sigma-3}}\right),
\end{aligned} \tag{5.86}$$

since $\frac{3\sigma-3}{2} \leq \sigma - 1, \quad \frac{3\sigma-3}{2} \leq \frac{\sigma-1}{2}$, then estimate (5.86) will be

$$\begin{aligned}
\|\varphi_x + \psi\|^2_{H^{\sigma-1}} &\leq c_\alpha \left(\|y\|^2_{\mathcal{H}} + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}} + \epsilon\|\phi\|^2_{H^{2\sigma-2}} + \epsilon\|\phi\|^2_{H^{3\sigma-3}}\right) \\
&\leq c_\alpha \left(\|y\|^2_{\mathcal{H}} + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}} + \epsilon\|\phi\|^2_{H^{2\sigma-2}}\right).
\end{aligned} \tag{5.87}$$

By using the inequality (5.77) and Lemma (19) we get

$$\begin{aligned}
\|\varphi\|^2_{H^\sigma} &\leq 2\|\varphi_x + \psi\|^2_{H^{\sigma-1}} + 2\|\psi\|^2_{H^{\sigma-1}} \\
&\leq 2\|\varphi_x + \psi\|^2_{H^{\sigma-1}} + \frac{c}{\lambda^2}\left(\|\chi\|^2_{H^{\sigma-1}} + \|y\|^2_{\mathcal{H}}\right) \\
&\leq 2\|\varphi_x + \psi\|^2_{H^{\sigma-1}} + c\left(\|y\|^2_{\mathcal{H}} + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}\right),
\end{aligned} \tag{5.88}$$

by substituting inequality (5.87) in inequality (5.88), we get the following

$$\|\varphi\|^2_{H^\sigma} \leq c_\alpha \left(\|y\|^2_{\mathcal{H}} + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}} + \epsilon\|\phi\|^2_{H^{2\sigma-2}}\right). \tag{5.89}$$

Taking $\gamma = \sigma - 1$ in Lemma (18), with the inequality (5.89) and the fact that $\sigma - 1 < \frac{\sigma-1}{2}$, we get

$$\begin{aligned}
\|\phi\|^2_{H^{\sigma-1}} &\leq \frac{c}{\lambda^2}\left(\|\phi\|_{H^\sigma} + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}} + \|y\|^2_{\mathcal{H}}\right) \\
&\leq c\left(\|\varphi\|_{H^\sigma} + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}} + \|y\|^2_{\mathcal{H}}\right) \\
&\leq c\left(\|y\|^2_{\mathcal{H}} + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}} + \epsilon\|\phi\|^2_{H^{2\sigma-2}}\right) \\
&\leq c\left(\|y\|^2_{\mathcal{H}} + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}\right),
\end{aligned} \tag{5.90}$$

for $|\lambda| \geq \alpha$, inequality $(5.73)_1$ is obtained. For $\gamma = \sigma - 1$ in Lemma (18) and by inequality (5.90), we get

$$\begin{aligned} \|\phi\|_{H^\sigma} &\leq c\left(\lambda^2\|\phi\|^2_{H^{\sigma-1}} + \|y\|^2_{\mathcal{H}} + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}\right) \\ &\leq c\lambda^2\left(\|y\|^2_{\mathcal{H}} + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}\right), \end{aligned} \tag{5.91}$$

for $|\lambda| \geq \alpha$ we obtain the inequality $(5.73)_2$. □

Lemma 22. *There exist a positive constant c_α such that*

$$\|B_*^{\frac{1}{2}}\psi\|^2_{L^2} \leq c_\alpha\left(\|y\|^2_{\mathcal{H}} + \|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}\right) + c\|\chi\|^2_{\mathrm{L}^2} + \frac{1}{\lambda^2}\|\phi\|^2_{\mathrm{L}^2} \tag{5.92}$$

Proof. Multiplying equation $(5.42)_4$ by ψ in the space L^2, gives

$$\begin{aligned} \|B_*^{\frac{1}{2}}\psi\|^2_{\mathrm{L}^2} &= \rho_2\left\langle y_4, \psi\right\rangle_{\mathrm{L}^2} - \rho_2 i\lambda\left\langle \chi, \psi\right\rangle_{\mathrm{L}^2} + \delta\left\langle \theta, \psi_x\right\rangle_{\mathrm{L}^2} \\ &\quad - \int_0^\infty h(s)\left\langle A^{\frac{\sigma}{2}}\zeta, A^{\frac{\sigma}{2}}\psi\right\rangle_{\mathrm{L}^2} ds - \kappa\left\langle (\varphi_x + \psi), \psi\right\rangle_{\mathrm{L}^2}, \end{aligned} \tag{5.93}$$

by using the equations $(5.42)_3$ and $(5.42)_6$, Lemma (17) and Young inequality, we get

$$\begin{aligned} |\delta\left\langle \theta, \psi_x\right\rangle_{\mathrm{L}^2}| &= \left|\frac{\delta}{i\lambda}\left\langle \theta, y_{3x}\right\rangle_{\mathrm{L}^2} + \frac{\delta}{i\lambda}\left\langle \chi_x, \theta\right\rangle_{\mathrm{L}^2}\right| \\ &= \left|\frac{\delta}{i\lambda}\left\langle \theta, y_{3x}\right\rangle_{\mathrm{L}^2} + \left\langle \frac{\rho_3}{i\lambda}y_6 - \frac{1}{i\beta\lambda}\int_0^\infty \mu(s)A^\sigma\eta(s)ds, \theta\right\rangle_{\mathrm{L}^2}\right| \\ &\quad - \rho_3\|\theta\|^2_{\mathrm{L}^2} \\ &\leq \frac{c}{\lambda^2}\|\theta\|^2_{H^\sigma} + c\|y\|^2_{\mathcal{H}} + c\|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}} \\ &\leq c\|y\|^2_{\mathcal{H}} + c\|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}. \end{aligned} \tag{5.94}$$

Now, we need to the following estimates

$$\begin{aligned} \rho_2\left\langle y_4, \psi\right\rangle_{\mathrm{L}^2} &\leq c\|y\|^2_{\mathcal{H}} + c\|\psi\|^2_{\mathrm{L}^2}, \\ |\rho_2 i\lambda\left\langle \chi, \psi\right\rangle_{\mathrm{L}^2}| &\leq \rho_2\|\chi\|_{\mathrm{L}^2} + \rho_2\left|\left\langle \chi, y_3\right\rangle_{\mathrm{L}^2}\right| \\ &\leq c\|\chi\|^2_{\mathrm{L}^2} + c\|y\|^2_{\mathcal{H}}, \\ \kappa\left\langle (\varphi_x + \psi), \psi\right\rangle_{\mathrm{L}^2} &= -\left\langle \varphi, \psi_x\right\rangle_{\mathrm{L}^2} + \|\psi\|^2_{\mathrm{L}^2} \\ &\leq \|\varphi\|^2_{\mathrm{L}^2} + \epsilon\|B_*^{\frac{1}{2}}\psi\|^2_{\mathrm{L}^2} + \|\psi\|^2_{\mathrm{L}^2} \\ &\leq \frac{1}{\lambda^2}\left(\|\phi\|^2_{\mathrm{L}^2} + \|y\|^2_{\mathcal{H}}\right) + \epsilon\|B_*^{\frac{1}{2}}\psi\|^2_{\mathrm{L}^2} + \|\psi\|^2_{\mathrm{L}^2}, \end{aligned} \tag{5.95}$$

and also

$$\int_0^\infty h(s)\left\langle A^{\frac{\sigma}{2}}\zeta, A^{\frac{\sigma}{2}}\psi\right\rangle_{\mathrm{L}^2} ds \leq c\epsilon\|\psi\|^2_{H^\sigma} + c\|y\|_{\mathcal{H}}\|z\|_{\mathcal{H}}. \tag{5.96}$$

Summing up estimates (5.95) and (5.96) with inequality (5.94), so inequality (5.92) is obtained.

□

5.4.2 Proof of Theorem (17)

Proof. We wish to apply both Theorem (16) and Theorem (15) to prove the stability results announced in (17). In order to do, we need to show that $i\mathbb{R} \subset \varrho(\mathcal{A})$. We argue by contradiction, assuming that $i\mathbb{R} \not\subseteq \varrho(\mathcal{A})$. Since we already know by Remark (8) that $0 \in \varrho(\mathcal{A})$. Then, we can consider the highest positive number λ_0 such that $(-i\lambda_0, i\lambda_0) \subset \varrho(\mathcal{A})$ which would imply that $i\lambda_0$ or $-i\lambda_0$ are elements of the spectrum $\vartheta(\mathcal{A})$. We suppose $i\lambda_0 \in \vartheta(\mathcal{A})$. Then, we consider a sequence of real numbers λ_n such that for some $\alpha > 0, \alpha \leq \lambda_n < \lambda_0, \lambda_n \to \lambda_0$ and a sequence $z_n = (\varphi_n, \phi_n, \psi_n, \chi_n, \theta_n, \eta_n) \in \mathcal{D}(\mathcal{A})$ with unitary norms such that

$$\|(i\lambda_n - \mathcal{A})z_n\|_{\mathcal{H}} = \|y_n\|_{\mathcal{H}} \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty, \tag{5.97}$$

if $y_n = (y_{1n}, y_{2n}, y_{3n}, y_{4n}, y_{5n}, y_{6n})$, we have

$$\begin{aligned}
&i\lambda_n\varphi_n - \phi_n = y_{1n} \to 0 \quad \text{in} \quad \mathrm{H}^1_0(0,L),\\
&\rho_1 i\lambda_n\phi_n - \kappa(\varphi_{nxx} + \psi_{nx}) = \rho_1 y_{2n} \to 0 \quad \text{in} \quad \mathrm{L}^2(0,L),\\
&i\lambda_n\psi_n - \chi_n = y_{3n} \to 0 \quad \text{in} \quad \mathrm{H}^1_*(0,L),\\
&\rho_2 i\lambda_n\chi_n + B_*\psi_n + \kappa(\varphi_{nx} + \psi_n)\\
&\qquad + \int_0^\infty h(s)A^\sigma\zeta_n(t,s)ds + \delta\theta_{nx} = \rho_2 y_{4n} \to 0 \quad \text{in} \quad \mathrm{L}^2(0,L),\\
&i\lambda_n\zeta_n - \overline{T}\zeta_n - \chi_n = y_{5n} \to 0 \quad \text{in} \quad \mathcal{N}_\sigma,\\
&\beta\rho_3 i\lambda_n\theta_n + \int_0^\infty \mu(s)A^\sigma\eta_n(s)ds + \beta\delta\chi_{nx} = \beta\rho_3 y_{6n} \to 0 \quad \text{in} \quad \mathrm{L}^2(0,L),\\
&i\lambda_n\eta_n - T\eta_n - \theta_n = y_{7n} \to 0 \quad \text{in} \quad \mathcal{M}_\sigma.
\end{aligned} \tag{5.98}$$

We distinguish three different cases:

The first case, for $\xi_g \neq 0$. By using the inequality (5.43), we deduce

$$\|\eta_n\|_{\mathcal{M}_\sigma} \leq c\|y_n\|_{\mathcal{H}}\|z_n\|_{\mathcal{H}} \tag{5.99}$$

$$\|\zeta_n\|_{\mathcal{N}_\sigma} \leq c\|y_n\|_{\mathcal{H}}\|z_n\|_{\mathcal{H}}. \tag{5.100}$$

By using Lemma (21) and interpolation inequality, given that $\frac{\sigma-2}{2} \leq 0 \leq \frac{\sigma}{2}$,

$$\begin{aligned}\|\phi_n\|_{\mathrm{L}^2} &\leq \|\phi_n\|_{H^{\sigma-2}}^{\frac{\sigma}{2}}\|\phi_n\|_{H^\sigma}^{\frac{2-\sigma}{2}} \\ &\leq c\lambda_n^{2-\sigma}\left(\|z_n\|_{\mathcal{H}}^{\frac{1}{2}}\|y_n\|_{\mathcal{H}}^{\frac{1}{2}} + \|y_n\|_{\mathcal{H}}\right).\end{aligned} \tag{5.101}$$

From Lemma (18), Lemma (21) and the inequality (5.76), we have

$$\begin{aligned}\|\chi_n\|_{H_*^{\sigma-1}}^2 &\leq c\epsilon\|\varphi_{nx} + \psi_n\|_{H^{\sigma-2}}^2 + c_{\epsilon,\alpha}\left(\|y_n\|_{\mathcal{H}}\|z_n\|_{\mathcal{H}} + \|y_n\|_{\mathcal{H}}^2\right) \\ &\leq c\epsilon\|\phi_n\|_{H^{\sigma-2}}^2 + c_{\epsilon,\alpha}\left(\|y_n\|_{\mathcal{H}}\|z_n\|_{\mathcal{H}} + \|y_n\|_{\mathcal{H}}^2\right) \\ &\leq c\left(\|y_n\|_{\mathcal{H}}\|z_n\|_{\mathcal{H}} + \|y_n\|_{\mathcal{H}}^2\right),\end{aligned} \tag{5.102}$$

by using Lemma (18), inequality (5.102) and the interpolation inequality, we get

$$\begin{aligned}\|\chi_n\|_{\mathrm{L}^2} &\leq \|\chi_n\|_{H_*^{\sigma-1}}^{\sigma}\|\chi_n\|_{H_*^\sigma}^{1-\sigma} \\ &\leq c\lambda_n^{1-\sigma}\left(\|z_n\|_{\mathcal{H}}^{\frac{1}{2}}\|y_n\|_{\mathcal{H}}^{\frac{1}{2}} + \|y_n\|_{\mathcal{H}}\right).\end{aligned} \tag{5.103}$$

By Lemma (22), using as well the inequalities (5.101) and (5.103), we deduce

$$\begin{aligned}\|B_*^{\frac{1}{2}}\psi_n\|_{\mathrm{L}^2}^2 &\leq c_\alpha\left(\|y_n\|_{\mathcal{H}}^2 + \|y_n\|_{\mathcal{H}}\|z_n\|_{\mathcal{H}}\right) + c\|\chi_n\|_{\mathrm{L}^2}^2 + \frac{1}{\lambda^2}\|\phi_n\|_{\mathrm{L}^2}^2 \\ &\leq c\lambda_n^{2-2\sigma}\left(\|z_n\|_{\mathcal{H}}\|y_n\|_{\mathcal{H}} + \|y_n\|_{\mathcal{H}}^2\right).\end{aligned} \tag{5.104}$$

By using Lemma (17) and Lemma (18), we have

$$\begin{aligned}\|\theta_n\|_{H^{\sigma-1}}^2 &\leq c_\alpha\left(\|y_n\|_{\mathcal{H}}\|z_n\|_{\mathcal{H}} + \|y_n\|_{\mathcal{H}}^2 + \frac{1}{\lambda_n^2}\|\chi_n\|_{H_*^\sigma}^2\right) \\ &\leq c_\alpha\left(\|y_n\|_{\mathcal{H}}\|z_n\|_{\mathcal{H}} + \|y_n\|_{\mathcal{H}}^2\right),\end{aligned} \tag{5.105}$$

using the interpolation inequality to find

$$\begin{aligned} \|\theta_n\|_{\mathrm{L}^2} &\leq \|\theta_n\|_{H^{\sigma-1}}^{\sigma}\|\theta_n\|_{H^{\sigma}}^{1-\sigma} \\ &\leq c\lambda_n^{1-\sigma}\left(\|z_n\|_{\mathcal{H}}^{\frac{1}{2}}\|y_n\|_{\mathcal{H}}^{\frac{1}{2}}+\|y_n\|_{\mathcal{H}}\right). \end{aligned} \tag{5.106}$$

By using equations $(5.42)_1$ and $(5.42)_3$, we also obtain

$$\begin{aligned} \|\varphi_{nx}+\psi_{nx}\|_{\mathrm{L}^2}^2 &\leq c\left(\|\varphi_{nx}\|_{\mathrm{L}^2}^2+\|\psi_n\|_{\mathrm{L}^2}^2\right) \\ &\leq c\left(\frac{1}{\lambda_n^2}\|\phi_{nx}\|_{\mathrm{L}^2}^2+\frac{1}{\lambda_n^2}\|\chi_n\|_{\mathrm{L}^2}^2+\|y_n\|_{\mathcal{H}}^2\right), \end{aligned} \tag{5.107}$$

and by using Lemma (19) and inequality (5.103), so inequality (5.107) will be

$$\begin{aligned} \|\varphi_{nx}+\psi_{nx}\|_{\mathrm{L}^2}^2 &\leq c\left(\|\phi_n\|_{\mathrm{L}^2}^2+\|B_*^{\frac{1}{2}}\psi_n\|_{\mathrm{L}^2}^2+\frac{1}{\lambda_n^2}\|\chi_n\|_{\mathrm{L}}^2+\|y_n\|_{\mathcal{H}}^2\right. \\ &\leq c\lambda_n^{2-2\sigma}\left(\|z_n\|_{\mathcal{H}}\|y_n\|_{\mathcal{H}}+\|y_n\|_{\mathcal{H}}^2\right). \end{aligned} \tag{5.108}$$

Now, we consider $z=(\varphi,\phi,\psi,\chi,\zeta,\theta,\eta)$ the solution of the system $(i\lambda-\mathcal{A})z=y$. Then, the previous estimates $(5.99)_1$,$(5.99)_2$,(5.101),(5.103),(5.104),(5.106) and (5.108) imply that

$$\begin{aligned} \|z\|_{\mathcal{H}}^2 \leq{}& \|\varphi_x+\psi\|_{\mathrm{L}^2}^2+\|\phi\|_{\mathrm{L}^2}^2+\left\|B_*^{\frac{1}{2}}\psi\right\|_{\mathrm{L}^2}^2+\|\chi\|_{\mathrm{L}^2}^2 \\ &+\|\zeta\|_{\mathcal{N}_\sigma}^2+\|\theta\|_{\mathrm{L}^2}^2+\|\eta\|_{\mathcal{M}_\sigma}^2 \\ \leq{}& c\lambda_n^{4-2\sigma}\left(\|z\|_{\mathcal{H}}\|y\|_{\mathcal{H}}+\|y\|_{\mathcal{H}}^2\right), \end{aligned} \tag{5.109}$$

which proves the first part of (17).

The second case, for $\xi_g=0$ and $\sigma\in[0,1)$. By using Lemma (21) and the interpolation inequality, given that $\frac{\sigma-1}{2}\leq 0\leq\frac{\sigma}{2}$

$$\begin{aligned} \|\phi\|_{\mathrm{L}^2} &\leq \|\phi\|_{H^{\sigma-1}}^{\sigma}\|\phi\|_{H^{\sigma}}^{1-\sigma} \\ &\leq c\lambda^{1-\sigma}\left(\|z\|_{\mathcal{H}}^{\frac{1}{2}}\|y\|_{\mathcal{H}}^{\frac{1}{2}}+\|y\|_{\mathcal{H}}\right), \end{aligned} \tag{5.110}$$

combining inequality (5.110) with the previous estimates $(5.99)_1$,$(5.99)_2$,(5.101), (5.103),(5.104),(5.106) and (5.108), we get

$$\|z\|_{\mathcal{H}}^2 \leq c\lambda^{2-2\sigma}\left(\|z\|_{\mathcal{H}}\|y\|_{\mathcal{H}}+\|y\|_{\mathcal{H}}^2\right), \tag{5.111}$$

which is the desired results, so the proof of the second part of (17) is over.
The third case, for $\xi_g = 0$ and $\sigma = 1$. We have $\|z\|_{\mathcal{H}} \leq c\|y\|_{\mathcal{H}}$, by Theorem (15), we deduce that the semigroup is exponentially stable. That completes the proof of Theorem (17). □

6

Nonlinear coupled system in nonclassical thermoelasticity

Contents

6.1 Introduction

In this chapter, we studied a thermoelastic von Karman type system by taking the conduction of the thermal flux according to the Gurtin-Pipkin law [119], the problem is given as follows

$$\begin{cases} w_{tt} - d_1 \left[\left(u_x + \frac{1}{2} (w_x)^2 \right) w_x \right]_x + d_2 w_{xxxx} + \alpha w_t = 0, & x \in (0,1),\, t > 0, \\ u_{tt} - d_1 \left(u_x + \frac{1}{2} (w_x)^2 \right)_x + \delta \theta_x = 0, & x \in (0,1),\, t > 0, \\ \theta_t - \frac{1}{\beta} \int_0^\infty g(s) \theta_{xx}(t-s)\, ds + \delta u_{tx} = 0, & x \in (0,1),\, t > 0, \end{cases} \tag{6.1}$$

we associate the system with the boundary conditions

$$u(x,t)\big|_{x=0}^{x=1} = w(x,t)\big|_{x=0}^{x=1} = w_x(x,t)\big|_{x=0}^{x=1} = \theta(x,t)\big|_{x=0}^{x=1} = 0, \quad t \geq 0, \tag{6.2}$$

and the initial conditions

$$\begin{cases} w(x,0) = w_0(x),\ w_t(x,0) = w_1(x),\ u(x,0) = u_0(x), \\ u_t(x,0) = u_1(x),\ \theta(x,t)|_{t \leq 0} = \theta_0(x,t),\ x \in (0,1), \end{cases} \tag{6.3}$$

where $\theta_0(x,t)$ is a prescribed past history of θ for $t \leq 0$.

This chapter has been inspired by the work [128]. Our main result is given as follows

Theorem 18. *The energy functional $E(t)$ decays exponentially, i.e., the solution energy functional (6.22) satisfies,*

$$E(t) \leq cE(0) \exp(-\omega t), \quad \forall t \geq 0, \tag{6.4}$$

where c and ω are two positive constants independent of t and the initial data, where the energy functional is defined by

$$\begin{aligned} E(t,z) = \frac{1}{2} \Bigg[& \|w_t\|^2_{L^2(0,1)} + \|u_t\|^2_{L^2(0,1)} + d_1 \left\| u_x + \frac{1}{2} w_x^2 \right\|^2_{L^2(0,1)} \\ & + \|\theta\|^2_{L^2(0,1)} + d_2 \|w_{xx}\|^2_{L^2(0,1)} + \frac{1}{\beta} \|\eta\|^2_{\mathcal{M}} \Bigg]. \end{aligned} \tag{6.5}$$

For the proof we need to construct a positif Lyapunov functional $F(t)$ equivalent to the energy functional $E(t)$, i.e.,

$$\alpha_1 E(t) \leq F(t) \leq \alpha_2 E(t),$$

for $t > 0$ and some positive constants α_1, α_2 such that

$$\frac{d}{dt}F(t) \leq -CE(t),$$

where C is a positive constant.

6.1.1 Earlier results

Many researchers were interested in different mathematical models in the fields of physics, engineering, biology,...Etc. In the engineering literature, as well as the PDE and control literature, beam theory is well studied. Mathematically, the notable problems of existence, uniqueness, and asymptotic behavior of solutions (dynamic stability) over time in the linear theory of Euler-Bernoulli, Rayleigh, and Timoshenko beams have been established (see, for instance [4, 112, 129]). Among all these dynamic models, nonlinear beam models such as the von Kármán's beam is the most descriptive of the transverse and longitudinal displacements for the slender bodies vibrating with a significant deviation, this is well detailed in the book of Lagnese and Lions [130]. Based on previous models, the essential principle of existing studies is to reduce unwanted vibrations affecting systems and in this context, it is worth noting that several effects of damping have been considered, including other important characteristics (see for example [131, 132] and the references therein). This study from the semigroup and boundary control point of view permitting the possibility of the stabilization configuration and additionally may allow establishing stability.

Lagnese and Leugering [104] typically derived a model that reflects the effect of stretching on bending, which necessarily leads to nonlinear partial differential equations for the motion of the beam and this, despite the assumption of linear constitutive equations for the bending. Given the above, the authors proposed the following system

$$\begin{cases} v_{tt} - \left[v_x + \frac{1}{2}w_x^2\right]_x = 0, & (x,t) \in (0,L) \times (0,\infty), \\ w_{tt} + w_{xxxx} - hw_{xxtt} - \left[w_x\left(v_x + \frac{1}{2}w_x^2\right)\right]_x = 0, & (x,t) \in (0,L) \times (0,\infty), \end{cases} \tag{6.6}$$

where $0 < x < L$ and $h > 0$ is a parameter related to the rotational inertia of the beam. v and w represent, respectively, the longitudinal and transversal displacement of the point x at time t. They obtained a uniform stabilization of the model by using nonlinear boundary feedback. The system (6.6) opened a wide field for published research, and many research results were issued, the most prominent of which was the work presented by Menzala and Zuazua [101], where the authors took into account the previous model but entered a parameter $\epsilon > 0$ and more accurately the first equation was converted from the system (6.6) into a form of the parametric equation as follows

$$\epsilon v_{tt} - \left[v_x + \frac{1}{2} w_x^2 \right]_x = 0.$$

The authors considered the following dynamical model

$$\begin{cases} \epsilon v_{tt} - \left[v_x + \frac{1}{2} w_x^2 \right]_x = 0, & (x,t) \in (0,L) \times (0,\infty), \\ w_{tt} + w_{xxxx} - h w_{xxtt} - \left[w_x \left(v_x + \frac{1}{2} w_x^2 \right) \right]_x = 0, & (x,t) \in (0,L) \times (0,\infty), \end{cases} \tag{6.7}$$

which is depending on a parameter $\epsilon > 0$, they studied its weak limit as $\epsilon \to 0$. More precisely, they analyzed various boundary conditions and demonstrated that the nature of the limit system is very sensitive to it and as a result, the models they obtained do not correspond to the classical Timoshenko equation that they obtained as a limit in the case of Dirichlet boundary conditions, that is, depending on the type of a particular boundary condition, the nonlinearity of Timoshenko's model may vanish or, by the contrary, may become a nonlinearity concentrated on the extremes of the beam. So, the system (6.7) can coincide with the classical Timoshenko equation [133, 134] only in the particular case of boundary condition, it can be given as follows

$$u_{tt} + u_{xxxx} - h u_{xxtt} - \frac{1}{2L} \left(\int_0^L u_x^2 \, dx \right) u_{xx} = 0.$$

The asymptotic behavior of the coupling between elastic and heat phenomena has been studied by several authors, the linear thermoelastic plate models (coupling of plate and heat) is always exponentially stable (namely the energy approaches zero exponentially when the time approaches infinity), we can say that thanks to the thermal effects introduced into the system, many types of dissipations have been used to the stabilization of the system and this is due to the choice of the type of dissipation weak or strong. When the system (6.6) is coupled with a parabolic heat equation modeled by the Fourier law in the following form

$$\theta_t - \kappa \theta_{xx} = 0.$$

In this context, several papers have been appeared (see articles [65, 132, 135] and references therein), in which it has been proved the exponential stability of the thermo-elastic von Kármán system.

6.1.2 Model derivation

The general model of the thermoelastic beam of von Kármán type system with heat flow can be written as follows

$$\begin{cases} v_{tt} - \left[v_x + \frac{1}{2}w_x^2\right]_x + \delta\theta_x = 0, & (x,t) \in (0,L)\times(0,\infty), \\ w_{tt} + w_{xxxx} - hw_{xxtt} - \left[w_x\left(v_x + \frac{1}{2}w_x^2\right)\right]_x = 0, & (x,t) \in (0,L)\times(0,\infty), \\ \theta_t + q_x + \delta v_{tx} = 0, & (x,t) \in (0,L)\times(0,\infty). \end{cases} \tag{6.8}$$

Through the third equation in system (6.10), the classic Fourier law of thermal conductivity can be given by

$$\beta q + \theta_x = 0.$$

We consider the constitutive equation as follows

$$\beta q(t) + \int_0^\infty g(s)\theta_x(t-s)\,ds = 0. \tag{6.9}$$

By replacing the flow equation (6.9) in the system (6.10), we obtain the system (6.1).

6.1.3 Contributions

The general model of the thermoelastic beam of von Kármán type system with heat flow can be written as follows

$$\begin{cases} v_{tt} - \left[v_x + \frac{1}{2}w_x^2\right]_x + \delta\theta_x = 0, & (x,t) \in (0,L)\times(0,\infty), \\ w_{tt} + w_{xxxx} - hw_{xxtt} - \left[w_x\left(v_x + \frac{1}{2}w_x^2\right)\right]_x = 0, & (x,t) \in (0,L)\times(0,\infty), \\ \theta_t + q_x + \delta v_{tx} = 0, & (x,t) \in (0,L)\times(0,\infty). \end{cases} \tag{6.10}$$

Through the third equation in system (6.10), the classic Fourier law of thermal conductivity can be given by

$$\beta q + \theta_x = 0.$$

In paper [107], Djebabla and Tatar considered a thermoelastic system by coupling the von Kármán system with a heat equation where the flow is given by Green-Naghdi [108, 120]. In theory, these are called thermo-elasticity of the type III and its law is given as follows

$$\beta q + \theta_x + dp_x = 0, \quad d > 0, \tag{6.11}$$

where

$$p(t) = p(0) + \int_0^t \theta(r)\,dr,$$

The authors obtain their system by replacing equation (6.11) in the system (6.10), given as follows

$$\begin{cases} u_{tt} - D_1\left(u_x + \frac{1}{2}(w_x)^2\right)_x + \gamma\theta_{tx} = 0, & (x,t) \in (0,L)\times(0,\infty), \\ w_{tt} + \delta w_t - D_1\left[\left(u_x + \frac{1}{2}(w_x)^2\right)w_x\right]_x + D_2 w_{xxxx} = 0, & (x,t) \in (0,L)\times(0,\infty), \\ \theta_{tt} - l\theta_{xx} + K_2\theta_t + \gamma u_{tx} = 0, & (x,t) \in (0,L)\times(0,\infty), \end{cases} \tag{6.12}$$

where D_1, D_2, δ, K_1, l and γ are positive constants, with the boundary conditions

$$\begin{cases} u = 0,\ w = 0,\ \theta_x = 0,\ x = 0, L,\ t > 0, \\ w_x = 0,\ x = 0, L,\ t > 0. \end{cases} \tag{6.13}$$

They proved the exponential decay of solution under some restrictions on the coefficients and the relaxation function g.

Recently, Wenjun Liu et al. [136] added a viscoelastic memory term and studied the nonautonomous full von Kármán beam. The system studied was as follows

$$\begin{cases} u_{tt} - D_1\left(u_x + \frac{1}{2}(w_x)^2\right)_x + \gamma\theta_{tx} = f(x,t), & (x,t) \in (0,L)\times(0,\infty), \\ w_{tt} + \delta w_t - D_1\left[\left(u_x + \frac{1}{2}(w_x)^2\right)w_x\right]_x + D_2 w_{xxxx} = y(x,t), & (x,t) \in (0,L)\times(0,\infty), \\ \theta_{tt} - l\theta_{xx} + \int_0^t g(t-s)\theta_{xx}(s)\,ds + \gamma u_{tx} = h(x,t), & (x,t) \in (0,L)\times(0,\infty). \end{cases} \tag{6.14}$$

They introduced suitable energy and some Lyapunov functionals, and employed some restrictions on the non-autonomous functions and the relaxation function, as a result of which they showed the asymptotic behavior of the solution and they established a general decay result for the energy, i.e., the system is exponentially or polynomial stable only in special cases.

Motivated by the previous works, in the present work we study the thermoelastic von Kármán type system (6.6) by assuming that the conduction of the thermal flux according to the Gurtin-

Pipkin law [119].

6.1.4 Chapter plan

This chapter respect the following plan. In the next section, we introduce some preliminaries. After that, in section (6.3), we indictae the proof of the existence of the solution by the semigroup method. Then, in section (6.4), we prove the stability results by using the multiplier techniques.

6.2 Preliminaries

In this section, we will present some assumptions and functional spaces.

6.2.1 Assumptions

We assume that the kernel g satisfies the following assumptions:

(i) $\mu(s) = -g'(s)$ is summable on $\mathbb{R}_+$ such that

$$\int_0^\infty \mu(s)\, ds = g(0) > 0,$$

and g has a total mass 1 given by

$$\int_0^\infty s\mu(s)\, ds = 1.$$

(ii) μ is a nonnegative nonincreasing absolutely continuous function on $\mathbb{R}_+$ such that

$$\mu(0) = \left(\lim_{s\to 0} \mu(s)\right) \in (0, \infty).$$

(iii) There exists $\nu > 0$ such that the differential inequality

$$\mu'(s) + \nu\mu(s) \leq 0,$$

holds for almost every $s > 0$.

6.2.2 Functional spaces

We introduce the Sobolev spaces:

$$H_0^1(0,1) \quad \text{and} \quad H_0^2(0,1),$$

and we consider the memory space:

$$\mathcal{M} := L^2_\mu(\mathbb{R}^+; H_0^1(0,1)) := \left\{ f : \mathbb{R}_+ \longrightarrow H_0^1 \,\middle|\, \int_0^\infty \mu(s) \left\| f_x(s) \right\|^2_{L^2(0,1)} ds < \infty \right\},$$

equipped with the inner product:

$$\langle f, g \rangle_{\mathcal{M}} = \int_0^\infty \mu(s) \langle f_x(s), g_x(s) \rangle_{L^2(0,1)} \, ds.$$

The infinitesimal generator of the right-translation semigroup on $\mathcal{M}$ is the linear operator

$$T\eta = -D\eta$$

with domain

$$\mathcal{D}(T) = \left\{ \eta \in \mathcal{M} : D\eta \in \mathcal{M}, \ \lim_{s \longrightarrow 0} \|\eta_x(s)\| = 0 \right\},$$

where D stands for weak derivative with respect to the internal variable $s \in \mathbb{R}_+$.

The phase space that contains the solution of problem (6.1)-(6.3), and is given by the following

$$\mathcal{H} := H_0^2(0,1) \times H_0^1(0,1) \times H_0^1(0,1) \times L^2(0,1) \times L^2(0,1) \times \mathcal{M}, \tag{6.15}$$

normed by

$$\begin{aligned} \|(w, \tilde{w}, u, \tilde{u}, \theta, \eta)\|^2_{\mathcal{H}} = \|w_t\|^2_{L^2(0,1)} + \|u_t\|^2_{L^2(0,1)} + d_1 \left\| u_x + \frac{1}{2} w_x^2 \right\|^2_{L^2(0,1)} \\ + d_2 \|w_{xx}\|^2_{L^2(0,1)} + \|\theta\|^2_{L^2(0,1)} + \frac{1}{\beta} \|\eta\|^2_{\mathcal{M}} . \end{aligned}$$

Remark 9. *For every $\eta \in \mathcal{D}(T)$, the nonnegative functional $\Gamma[\eta]$ is well defined such that*

$$\Gamma[\eta] = -\int_0^\infty \mu'(s) \|\eta_x(s)\|^2_{L^2(0,1)} ds,$$

and satisfy

$$2\langle T\eta, \eta\rangle_{\mathcal{M}} = -\Gamma[\eta].$$

Moreover, we deduce the following inequality by using assumption (iii)

$$\nu\|\eta\|_{\mathcal{M}}^2 \leq \Gamma[\eta].$$

Remark 10. *Using several times the Hölder, Young and Poincaré inequalities to get*

$$\int_0^\infty \mu(s)\|\eta_x(s)\|ds \leq \sqrt{g(0)}\|\eta\|_{\mathcal{M}}.$$

Which will be useful for our purposes.

6.3 The semigroup approach

In this section, we give an existence and uniqueness result for problem (6.1)-(6.3) using the semigroup theory. We need to introduce the auxiliary variable

$$\eta = \eta^t(x,s) : (x,t,s) \in (0,1) \times [0,\infty) \times \mathbb{R}_+ \longmapsto \mathbb{R},$$

then, the integration of the past hystory of θ defined as

$$\eta^t(x,s) = \int_0^s \theta(x, t-\sigma)d\sigma,$$

satisfying the Dirichlet boundary condition

$$\eta^t(0,s) = \eta^t(1,s) = 0,$$

in addition to that the condition

$$\lim_{s\longrightarrow 0} \eta^t(x,s) = 0.$$

Hence, η satisfies the equation

$$\eta_t^t = -\eta_s^t + \theta(t).$$

The above mentioned data allow the writing of the following partial differential system in the unknowns $(w, u, \theta, \eta) = (w(t), u(t), \theta(t), \eta^t)$

$$\begin{cases} w_{tt} - d_1 \left[(u_x + \frac{1}{2} w_x^2) w_x \right]_x + d_2 w_{xxxx} + \alpha w_t = 0, \\ u_{tt} - d_1 \left[u_x + \frac{1}{2} w_x^2 \right]_x + \delta \theta_x = 0, \\ \theta_t - \frac{1}{\beta} \int_0^\infty \mu(s) \eta_{xx}(s) ds + \delta u_{tx} = 0, \\ \eta_t = T\eta + \theta. \end{cases} \tag{6.16}$$

By using Semigroup method (see Pazy book [99]). Setting $(\tilde{w} = w_t, \tilde{u} = u_t)$ and

$$z(t) = (w(t), \tilde{w}(t), u(t), \tilde{u}(t), \theta(t), \eta^t)^T \in \mathcal{H},$$

we view the problem (6.16) as the evolution equation in the Hilbert space $\mathcal{H}$, then the problem can be written as a semilinear Cauchy problem

$$\begin{cases} \frac{d}{dt} z(t) = \mathcal{A} z(t) + \mathcal{G}(z), \\ z(0) = z_0, \end{cases} \tag{6.17}$$

where $z_0 = (w_0, w_1, u_0, u_1, \theta_0, \eta_0)^T \in \mathcal{H}$.

The linear operator $\mathcal{A}$ is defined as

$$\mathcal{A} \begin{pmatrix} w \\ \tilde{w} \\ u \\ \tilde{u} \\ \theta \\ \eta \end{pmatrix} = \begin{pmatrix} \tilde{w} \\ -d_2 w_{xxxx} - \alpha \tilde{w} \\ \tilde{u} \\ d_1 u_{xx} - \delta \theta_x \\ \frac{1}{\beta} \int_0^\infty \mu(s) \eta_{xx}(s) ds - \delta \tilde{u}_x \\ T\eta + \theta \end{pmatrix}, \tag{6.18}$$

and $\mathcal{G}(z)$ defined by

$$\mathcal{G}(z) = \begin{pmatrix} 0 \\ d_1 \left[\left(u_x + \frac{1}{2} (w_x)^2 \right) w_x \right]_x \\ 0 \\ \frac{d_1}{2} (w_x)_x^2 \\ 0 \\ 0 \end{pmatrix}. \tag{6.19}$$

With domain

$$\mathcal{D}(\mathcal{A}) = \left\{ z \in \mathcal{H} \left| \begin{array}{c} w \in H^4(0,1) \cap H_0^2(0,1) \\ \tilde{w} \in H^3(0,1) \cap H_0^2(0,1) \\ u \in H_0^2(0,1) \cap L^2(0,1) \\ \tilde{u} \in L^2(0,1) \\ \theta \in H_0^1 \\ \eta \in \mathcal{D}(T) \\ \int_0^\infty \mu(s)\eta(s)ds \in H^2 \end{array} \right. \right\}. \tag{6.20}$$

Theorem 19. *The operator $\mathcal{A}$ is the infinitesimal generator of a contraction semigroup*

$$S(t) = e^{t\mathcal{A}} : \mathcal{H} \longrightarrow \mathcal{H}.$$

It well known that the solution of problem (6.17) satisfies this integral equation

$$z(t) = S(t)z_0 + \int_0^t S(t-s)\mathcal{G}(z)ds, \ \forall 0 \leq s \leq t. \tag{6.21}$$

It's clearly that $\mathcal{G}(z)$ is locally Lipschitz continuous in $\mathcal{H}$, then the local existence of problem (6.16) is achieved. To obtain a global existence we need an a priori estimate and more precisely to have that $\|z\|_{\mathcal{H}}$ is bounded where the solution exists, and therefore, the global existence has been shown (for more detail see [137]). Finally, we use Gronwall inequality for the proof of solution uniqueness. Therefore, we introduce the following result

Theorem 20. *Let $z_0 \in \mathcal{H}$. Then, the problem* (6.1)-(6.3) *has a unique global weak solutions such that*

$$z(t) \in C(\mathbb{R}_+; \mathcal{H}).$$

Moreover if $z_0 \in \mathcal{D}(\mathcal{A})$, then we have

$$z(t) \in C(\mathbb{R}_+; \mathcal{D}(\mathcal{A})) \cap C^1(\mathbb{R}_+; \mathcal{H}).$$

6.4 Stability result of solutions

In this section we use the multiplier method, so our argument is based on the choice of an appropriate Lyapunov function, at first we give the energy functional of the problem (6.1)-(6.3) such that

$$E(t,z) = \frac{1}{2}\left[\|w_t\|^2_{L^2(0,1)} + \|u_t\|^2_{L^2(0,1)} + d_1\left\|u_x + \frac{1}{2}w_x^2\right\|^2_{L^2(0,1)} \right.$$
$$\left. + \|\theta\|^2_{L^2(0,1)} + d_2\|w_{xx}\|^2_{L^2(0,1)} + \frac{1}{\beta}\|\eta\|^2_{\mathcal{M}}\right], \tag{6.22}$$

Lemma 23. *The energy functional* (6.22) *satisfies, along the solution of* (6.1)

$$\frac{d}{dt}E(t,z) = -\alpha\|w_t\|^2_{L^2(0,1)} - \frac{1}{\beta}\Gamma[\eta]. \tag{6.23}$$

Proof. By multipling equations of system (6.16), respectively by w_t, u_t and θ, using the boundary conditions (6.2), then we obtain (6.23). □

We announce our main result of this section

Theorem 21. *The energy functional $E(t)$ decays exponentially, i.e., the solution energy functional* (6.22) *satisfies,*

$$E(t) \leq cE(0)\exp(-\omega t), \quad \forall t \geq 0, \tag{6.24}$$

where c and ω are two positive constants independent of t and the initial data.

Remark 11. *Along this subsection, we will denote $c > 0$ as a generic constant not associated with ϵ, it changes from inequality to another.*

For the proof, we need to introduce the following subsection of thechnical lemmas.

6.4.1 Technical Lemmas

Define the following functional

$$I(t) := \int_0^1 \left(u_t u + \frac{1}{2}ww_t + \frac{\alpha}{4}w^2\right)dx, \ \forall t \geq 0. \tag{6.25}$$

Lemma 24. *For $z \in \mathcal{H}$ solution to the problem* (6.1)-(6.3)*. Then, the functional I satisfy the estimate*

$$\frac{d}{dt}I(t) + d_1\left\|u_x + \frac{1}{2}w_x^2\right\|^2_{L^2(0,1)} + \frac{d_2}{2}\|w_{xx}\|^2_{L^2(0,1)}$$
$$\leq \|u_t\|^2_{L^2(0,1)} + \|w_t\|^2_{L^2(0,1)} + \sigma_1\|u_x\|^2_{L^2(0,1)} + \frac{c}{\sigma_1}\|\theta\|^2_{L^2(0,1)}, \ \forall t \geq 0, \tag{6.26}$$

where σ_1 is a positive constant.

Proof. By taking the derivative of (6.25), using the equations of the system (6.16), and integrations by part, we obtain

$$\begin{aligned}\frac{d}{dt}I(t) &= \|u_t\|^2_{L^2(0,1)} + \|w_t\|^2_{L^2(0,1)} + \int_0^1 \left(u_{tt}u + \frac{1}{2}w_{tt}w + \frac{\alpha}{2}w_t w\right) dx \\ &= \|u_t\|^2_{L^2(0,1)} + \|w_t\|^2_{L^2(0,1)} - d_1 \left\|u_x + \frac{1}{2}w_x^2\right\|^2_{L^2(0,1)} - \frac{d_2}{2}\|w_{xx}\|^2_{L^2(0,1)} \\ &\quad + \int_0^1 \theta u_x\, dx,\end{aligned} \tag{6.27}$$

Now, using the boundary conditions (6.2) and by exploiting Young's inequality for $\sigma_1 > 0$. Then, the inequality (6.26) is established. □

Define the following functional

$$J(t) := \frac{2}{\delta}\int_0^1 \int_0^x u_t(x)\theta(y)\, dy dx, \quad \forall t \geq 0. \tag{6.28}$$

Lemma 25. *For $z \in \mathcal{H}$ solution to the problem (6.1)-(6.3). Then, the functional I satisfy the estimate*

$$\begin{aligned}\frac{d}{dt}J(t) + \|u_t\|^2_{L^2(0,1)} \leq \sigma_3 \left\|u_x + \frac{1}{2}w_x^2\right\|^2_{L^2(0,1)} + \sigma_2 u_x^2(1) \\ + \frac{c}{\sigma_3}\|\theta\|^2_{L^2(0,1)} + c\Gamma[\eta], \quad \forall t \geq 0,\end{aligned} \tag{6.29}$$

where σ_2, σ_3 are a positive constants.

Proof. By taking the derivative of (6.28), using the equations of the system (6.16), we obtain

$$\begin{aligned}\frac{d}{dt}J(t) &= \frac{2}{\delta}\int_0^1\int_0^x \left(\theta(y)u_{tt}(x) + u_t(x)\theta_t(y)\right) dy dx \\ &= \frac{2\beta}{\delta}\int_0^1\int_0^x\int_0^\infty u_t(x,t)\mu(s)\eta_{xx}(y,s)\, ds dy dx \\ &\quad - 2\int_0^1\int_0^x u_t(x)u_{tx}(y)\, dy dx - 2\int_0^1\int_0^x \theta_x(x)\theta(y)\, dy dx \\ &\quad + \frac{2d_1}{\delta}\int_0^1\int_0^x \theta(y)\left(u_x(x) + \frac{1}{2}w_x^2(x)\right)_x dy dx.\end{aligned} \tag{6.30}$$

By simplifing (6.30), and using the boundary conditions (6.2), we obtain

$$\begin{aligned}\frac{d}{dt}J(t) = -2\left\|u_t\right\|^2_{L^2(0,1)} + 2\left\|\theta\right\|^2_{L^2(0,1)} + \frac{2}{\delta}\int_0^1\int_0^\infty (u_t(x,t)\mu(s)\eta_x(x,s))\, dsdx \\ -\frac{2d_1}{\delta}\int_0^1 (u_x + \frac{1}{2}w_x^2)\theta\, dx + \frac{2d_1}{\delta}u_x(1)\int_0^1 \theta\, dx,\end{aligned} \tag{6.31}$$

By exploiting Young's inequality for $\sigma_2, \sigma_3 > 0$, then, we get the following estimates

$$\bullet \quad \begin{aligned}\frac{2}{\delta}\int_0^1\int_0^\infty (u_t(x,t)\mu(s)\eta_x(x,s))\, dsdx &\leq c\left\|u_t\right\|_{L^2(0,1)}\left\|\eta\right\|_{\mathcal{M}} \\ &\leq \left\|u_t\right\|^2_{L^2(0,1)} + c\Gamma[\eta],\end{aligned} \tag{6.32}$$

$$\bullet \quad \frac{2d_1}{\delta}u_x(1)\int_0^1 \theta\, dx \leq \sigma_2 u_x^2(1) + \frac{c}{\sigma_2}\left\|\theta\right\|^2_{L^2(0,1)}, \tag{6.33}$$

$$\bullet \quad -\frac{2d_1}{\delta}\int_0^1 (u_x + \frac{1}{2}w_x^2)\theta dx \leq \sigma_3\left\|u_x + \frac{1}{2}w_x^2\right\|^2_{L^2(0,1)} + \frac{c}{\sigma_3}\left\|\theta\right\|^2_{L^2(0,1)}. \tag{6.34}$$

Finally, by substituting the estimates (6.32)-(6.34) in the equality (6.30), then, we find the inequality (6.29). □

Remark 12. *As in [35], we introduce the function* $\Pi \in C^1([0,1])$ *satisfying*

$$\Pi(1) = -\Pi(0) = -2,$$

this function is used to handle the boundary term.

Define the following functional

$$J^*(t) := J_1(t) + J_2(t), \quad \forall t \geq 0, \tag{6.35}$$

such that the functionals J_1 and J_2 are given by

$$\begin{cases} J_1(t) := \displaystyle\int_0^1 u_t \Pi u_x\, dx, \\ J_2(t) := \displaystyle\int_0^1 w_t \Pi w_x\, dx. \end{cases} \tag{6.36}$$

Lemma 26. *For* $z \in \mathcal{H}$ *solution to the problem* (6.1)-(6.3)*. Then, the functional* J^* *satisfy the* *estimate*

$$\frac{d}{dt}J^*(t)+d_1\left[u_x^2(1)-u_x^2(0)\right]\le c\left(\left\|u_x+\frac{1}{2}w_x^2\right\|_{L^2(0,1)}^2+\|w_{xx}\|_{L^2(0,1)}^2\right)\\+c\left(\|w_t\|_{L^2(0,1)}^2+\|u_t\|_{L^2(0,1)}^2+\|\theta\|_{L^2(0,1)}^2\right),\quad\forall t\ge 0. \tag{6.37}$$

Proof. By taking the derivative of the functional J_1 using the equations of the system (6.16), and integrations by part, we obtain

$$\begin{aligned}\frac{d}{dt}J_1(t)&=2\|u_t\|_{L^2(0,1)}^2+d_1\int_0^1 u_{xx}\Pi u_x+d_1\int_0^1 w_{xx}w_x\Pi u_x\,dx-\delta\int_0^1\theta_x\Pi u_x\,dx\\&=-d_1\left[u_x^2(1)-u_x^2(0)\right]+2\|u_t\|_{L^2(0,1)}^2+d_1\int_0^1 w_{xx}w_x\Pi u_x\,dx-\delta\int_0^1\theta_x\Pi u_x\,dx,\end{aligned} \tag{6.38}$$

By taking the derivative of the functional J_2, using the equations of the system (6.16), and integrations by part, we obtain

$$\begin{aligned}\frac{d}{dt}J_2(t)&=-\alpha\int_0^1 w_t\Pi w_x\,dx+d_1\int_0^1\left[\left(u_x+\frac{1}{2}w_x^2\right)w_x\right]_x\Pi w_x\,dx\\&\quad-d_2\int_0^1 w_{xxxx}\Pi w_x\,dx+\int_0^1 w_t\Pi w_{tx}\,dx\\&=-d_2\left[w_{xx}^2(1)-w_{xx}^2(0)\right]+2\|w_t\|_{L^2(0,1)}^2-\frac{d_1}{2}\left\|w_x^2\right\|_{L^2(0,1)}^2+6d_2\|w_{xx}\|_{L^2(0,1)}^2\\&\quad-\alpha\int_0^1 w_t\Pi w_x\,dx+4d_1\int_0^1\left(u_x+\frac{1}{2}w_x^2\right)w_x^2\,dx-d_1w_xw_{xx}\Pi u_x\,dx,\end{aligned} \tag{6.39}$$

By simplifing the addition result of (6.38) and (6.39), then, by using the boundary conditions (6.2), we obtain

$$\begin{aligned}\frac{d}{dt}J^*(t)&=-d_1\left[u_x^2(1)-u_x^2(0)\right]-d_2\left[w_{xx}^2(1)-w_{xx}^2(0)\right]+2\|u_t\|_{L^2(0,1)}^2+2\|w_t\|_{L^2(0,1)}^2\\&\quad-\frac{d_1}{2}\left\|w_x^2\right\|_{L^2(0,1)}^2+6d_2\|w_{xx}\|_{L^2(0,1)}^2-\delta\int_0^1\theta_x\Pi u_x\,dx-\alpha\int_0^1 w_t\Pi w_x\,dx\\&\quad+4d_1\int_0^1\left(u_x+\frac{1}{2}w_x^2\right)w_x^2\,dx.\end{aligned} \tag{6.40}$$

By exploiting Young's inequality, we get the desire inequality (6.37). □

The functional

$$R(t):=J(t)+\frac{\sigma_2}{d_1}J^*(t),\quad\forall t\ge 0, \tag{6.41}$$

Lemma 27. *For $z \in \mathcal{H}$ solution to the problem* (6.1)-(6.3)*. Then, the functional R satisfy the estimate*

$$\frac{d}{dt}R(t) + (1 - c\sigma_2)\|u_t\|^2_{L^2(0,1)} \leq (c\sigma_2 + \sigma_3)\left\|u_x + \frac{1}{2}w_x^2\right\|^2_{L^2(0,1)} + \left(\frac{c}{\sigma_3} + \sigma_2\right)\|\theta\|^2_{L^2(0,1)} + c\sigma_2\left(\|w_{xx}\|^2_{L^2(0,1)} + \|w_t\|^2_{L^2(0,1)}\right) + c\Gamma[\eta], \;\; \forall t \geq 0. \tag{6.42}$$

Proof. Taking the derivative of the functional R, by using inequalities (6.30) and (6.37), then estimate (6.42) is given. □

Let the functional

$$K(t) := -\frac{2}{g(0)}\int_0^1\int_0^\infty \mu(s)\eta^t(s)\theta(t)\,dsdx, \;\; \forall t \geq 0. \tag{6.43}$$

Lemma 28. *For $z \in \mathcal{H}$ solution to the problem* (6.1)-(6.3)*. Then, the functional K satisfy the estimate*

$$\frac{d}{dt}K(t) + \|\theta\|^2_{L^2(0,1)} + \|\eta\|^2_{\mathcal{M}} \leq \sigma_4\|u_t\|^2_{L^2(0,1)} + \frac{c}{\sigma_4}\Gamma[\eta], \;\; \forall t \geq 0, \tag{6.44}$$

where σ_4 is a positive constant.

Proof. By taking the derivative of (6.28), using the equations of the system (6.16), and integrations by part, we obtain

$$\frac{d}{dt}K(t) + \|\eta\|^2_{\mathcal{M}} = -\frac{2}{g(0)}\int_0^1\int_0^\infty \mu(s)\theta_t(t)\eta^t(s)\,dsdx + \|\eta\|^2_{\mathcal{M}} + \int_0^1\int_0^\infty \mu(s)\theta(t)\eta^t_t(s)\,dsdx. \tag{6.45}$$

By using the boundary conditions (6.2), we get

$$\frac{d}{dt}K(t) + \|\eta\|^2_{\mathcal{M}} = -2\|\theta\|^2_{L^2(0,1)} + \frac{2}{g(0)\beta}\left\|\int_0^\infty \mu(s)\eta_x(s)\,ds\right\|^2_{L^2(0,1)} + \|\eta\|^2_{\mathcal{M}} - \frac{2\delta}{g(0)\beta}\int_0^1\int_0^\infty \mu(s)u_t(t)\eta_x(s)\,dsdx - \frac{2}{g(0)}\int_0^1\int_0^\infty \mu(s)\theta(t)T\eta(s)\,dsdx, \tag{6.46}$$

By exploiting Young's inequality for $\sigma_4 > 0$, then, we get the following estimates

$$\bullet \quad \frac{2}{g(0)\beta}\left\|\int_0^\infty \mu(s)\eta_x(s)\,ds\right\|^2_{L^2(0,1)} \leq c\,\|\eta\|^2_{\mathcal{M}} \\ \leq c\Gamma[\eta], \tag{6.47}$$

$$\bullet \quad -\frac{2\delta}{g(0)\beta}\int_0^1\int_0^\infty \mu(s)u_t(t)\eta_x(s)\,dsdx \leq c\,\|u_t\|_{L^2(0,1)}\,\|\eta\|_{\mathcal{M}} \\ \leq \sigma_4\,\|u_t\|^2_{L^2(0,1)} + \frac{c}{\sigma_4}\,\|\eta\|^2_{\mathcal{M}} \\ \leq \sigma_4\,\|u_t\|^2_{L^2(0,1)} + \frac{c}{\sigma_4}\Gamma[\eta], \tag{6.48}$$

$$\bullet \quad -\frac{2}{g(0)}\int_0^1\int_0^\infty \mu(s)\theta(t)T\eta(s)\,dsdx = -\frac{2}{g(0)}\int_0^1\int_0^\infty \mu'(s)\eta(s)\theta(t)\,dsdx \\ \leq c\,\|\theta\|_{L^2(0,1)}\sqrt{\Gamma[\eta]} \\ \leq \|\theta\|^2_{L^2(0,1)} + c\Gamma[\eta]. \tag{6.49}$$

Finally, substituting the estimates (6.47)-(6.49) in the equality (6.46), then we find the desire equality (6.44). ☐

We define the Lyapunov functional $\mathcal{F}$ as follows

$$\mathcal{F}(t) := NE(t) + 2I(t) + N_1R(t) + N_2K(t). \tag{6.50}$$

where $N, N_1,$ and N_2 are positive constants to be chosen appropriately later.

Lemma 29. *For N large enough, there exist two positive constants α_1 and α_2 such that*

$$\alpha_1 E(t) \leq \mathcal{F}(t) \leq \alpha_2 E(t), \forall t \geq 0. \tag{6.51}$$

Proof. Let

$$\tilde{\mathcal{F}}(t) := 2I(t) + N_1R(t) + N_2K(t). \tag{6.52}$$

By using Young's and Poincaré's inequalities and the functionals (6.26), (6.42) and (6.44), we obtain

$$|\tilde{\mathcal{F}}(t)| \leq cE(t)$$

Consequently,

$$|\mathcal{F}(t) - NE(t)| \leq cE(t),$$

that is

$$(N-c) \leq \mathcal{F}(t) \leq (N+c)E(t).$$

By choosing N large enough, (6.51) follows. □

Remark 13. *Noting that*

$$\|u_x\|^2_{L^2(0,1)} \leq 2\left\|u_x + \frac{1}{2}w_x^2\right\|^2_{L^2(0,1)} + \frac{1}{4}\|w_{xx}\|^2_{L^2(0,1)}. \tag{6.53}$$

6.4.2 Proof of Theorem (21)

Now, we are ready to prove Theorem (21)

Proof. The derivative of (6.50), bearing in mind (6.23),(6.26),(6.42),(6.44), using Poincaré's inequality and Remark (13), gives

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(t) \leq & -\varpi_1\|u_t\|^2_{L^2(0,1)} - \varpi_2\|w_t\|^2_{L^2(0,1)} - \varpi_3\left\|u_x + \frac{1}{2}w_x^2\right\|^2_{L^2(0,1)} \\ & - \varpi_4\|w_{xx}\|^2_{L^2(0,1)} - \varpi_5\|\theta\|^2_{L^2(0,1)} - N_2\|\eta\|^2_{\mathcal{M}} - \varpi_6\Gamma[\eta],\end{aligned} \tag{6.54}$$

where the canstants ϖ_i (for $i = 1, \cdots, 6$) are defined by

$$\begin{aligned}\varpi_1 &= N_1(1 - c\sigma_2) - 2 - N_2\sigma_4, \\ \varpi_2 &= N\alpha - 2 - N_1c\sigma_2, \\ \varpi_3 &= 2d_1 - N_1(c\sigma_2 + \sigma_3) - 4\sigma_1, \\ \varpi_4 &= d_2 - N_1c\sigma_2 - \frac{\sigma_1}{2}, \\ \varpi_5 &= N_2 - \frac{2c}{\sigma_1} - N_1\left(\frac{c}{\sigma_3} + \sigma_2\right), \\ \varpi_6 &= N\frac{1}{\beta} - N_1c - N_2\frac{c}{\sigma_4}.\end{aligned} \tag{6.55}$$

At this point, we set

$$\sigma_2 = \frac{1}{N_1} \quad \text{and} \quad \sigma_4 = \frac{1}{N_2}, \tag{6.56}$$

and we select N_1 large enough such that

$$N_1 - 3 - c > 0, \tag{6.57}$$

then we choose N_2 large enough such that

$$N_2 - \frac{2c}{\sigma_1} - N_1 \frac{c}{\sigma_3} - 1 > 0. \tag{6.58}$$

Next, we choose σ_1 and σ_3 small enough so that

$$\sigma_1 < 2(d_2 - c), \tag{6.59}$$

and

$$\sigma_3 < \frac{2d_1 - 8(d_2 - c) - c}{N_1}. \tag{6.60}$$

We can choose N big enough such that (6.51) remains valid and

$$N - 2 - c > 0, \tag{6.61}$$

also we have

$$N\frac{1}{\beta} - N_1 c - cN_2^2 > 0. \tag{6.62}$$

Then, ϖ_i (for $i = 1, \cdots, 6$) are all a positive constants. So, by using (6.22), we can deduce that there exist a positive constant ϑ, such that estimate (6.54) takes the following form

$$\begin{aligned} \frac{d}{dt}\mathcal{F}(t) &\leq -\vartheta E(t) - \varpi_6 \Gamma[\eta] \\ &\leq -\vartheta E(t), \quad \forall t \geq 0. \end{aligned} \tag{6.63}$$

By (6.51), we have

$$\mathcal{F}(t) \sim E(t),$$

then this fact can gives the following estimate

$$\frac{d}{dt}\mathcal{F}(t) \leq -\zeta \mathcal{F}(t), \quad \forall t \geq 0. \tag{6.64}$$

For some positive constant $\zeta = \frac{\vartheta}{\alpha_2}$. A simple integration of (6.64) over $(0, t)$ yields

$$\mathcal{F}(t) \leq \mathcal{F}(0) \exp(-\zeta t), \quad \forall t \geq 0. \tag{6.65}$$

Thus, using (6.51) and (6.65), the conclusion of theorem (21) follows. □

7

Conclusion and perspectives

Contents

7.1 Abridgement

Based on the above studies and research we conclude some important results. So, with regard to chapter (3), through the completed study we were able to know the behavior of system (1.1). It can be described as a porous-elastic system with dissipation due to the effects of microtemperature and frictional damping in addition to the absence of thermal conductivity. We showed the exponential stability without any condition on system parameters. This enables us to say that any condition of the system parameters is a sufficient and unnecessary condition for achieving exponential stability, as it does not lack the exponential decay of the energy in the absence of it.

In the chapter (4), we considered a one-dimensional nonlinear mechanical model of type von Karman with thermal effects. For the system (1.2) we relied on Cataneo's law by which the system can dissipate the strong heat produced by the thermal field flow, as well as a damping the friction caused by the longitudinal displacement of the beam, in which case the basic stability hypothesis depends on the existence of at least two dissipasions, one at the level of the mechanical model and the other at the level of The thermal effect, so the system is exponentially stable.

The study of chapter (5) requires the Timoshenko system (1.3). The aforemen tioned system contains two effects represented by the fractional memory and the spatial fractional thermal effect of the Gurtin-Pipkin type. We deduce the weakness of the dampers in controlling the system, and this is explained by the appearance of the stability number as a primary controller, then the fractional order coefficient as a second controller.

For the system (1.4) in chapter (6), we relied on the Gurtin-Pipkin law as a thermal effect, due to the weak dissipation of the term thermal memory and the presence of friction damping resulting from the longitudinal displacement of the beam, and we concluded that the system maintains exponential stability in this case.

7.2 Perspectives

For future works, we suggest studying the following systems

$$
\begin{aligned}
&\rho u_{tt} = \mu u_{xx} + b\varphi_x - \gamma\theta_x,\\
&J\varphi_{tt} = \delta\varphi_{xx} - bu_x - \xi\varphi - d\omega_x + m\theta - \beta g * \varphi_{xx},\\
&c\theta_t = -m\varphi_t - \gamma u_{xt} - k_1\omega_x,\\
&a\omega_t = k_2\omega_{xx} - k_3\omega - k_1\theta_x - d\varphi_{xt},
\end{aligned}
\tag{7.1}
$$

and

$$\begin{aligned}
&\rho_1\varphi_{tt} - k(\varphi_x + \psi + l\omega)_x - k_0 l(\omega_x - l\varphi) = 0,\\
&\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + l\omega) + \gamma\theta_{tx} = 0,\\
&\rho_1\omega_{tt} - k_0(\omega_x - l\varphi)_x + kl(\varphi_x + \psi + l\omega) = 0,\\
&\rho_3\theta_{tt} - \kappa\theta_{xx} + \beta g * \theta_{xx} + \gamma\psi_{tx} = 0.
\end{aligned} \tag{7.2}$$

- For the system (7.1), we have the effect of memory on the porous equation it can be considered as a weak viscoporous effect. From our point of view, this effect cannot control the behavior of the exponential stability of the system but the presence of the effects of microtemperature in addition to the aforementioned effect, we can deduce the stability which is controlled by the nature of the kernel g of the memory term.
- For the system (7.2), where it be called Bresse system. We note that this system can reduces to the classical Timoshenko system when the arch curvature l = 0, The asymptotic stability of one-dimensional Timoshenko system by thermoelasticity of type III was proved by Djebabla and Tatar [39]. From our point of view, this system which is consedered as a generalization of Timoshenko system is exponential and polynomial stable only in special cases i.e., the kernel function is the determinant of the type of stability.

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