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Notation

We take the spacetime metric to have signature $(- + ++)$ that is consistent with this of Misner, Thorne and Wheeler [79]. We shall use geometric units in which gravitational constant G and the speed of light c are set to one, when we deal with equations of general relativity. Latin indices are used to describe four dimensional expression i.e. $a = 0, 1, 2, 3$. Spacelike part of the expression is indexed by Greek characters i.e. $\alpha = 1, 2, 3$. We denote components of Lorentzian metrics by η_{ab}. In general the metric components are symbolized by g_{ab}.

Chapter 1

Introduction

Take us out - James T. Kirk[1]

1.1 Faster Than Light Mechanisms

The immenseness of the interstellar void implies that even if we could accelerate the starship to almost light speed, the exploration of nearby stars, distanced from us by a few light years, would take a few human lifetimes as seen from the Earth. The exploration of the Milky Way which includes over 200 billion stars and is about 100,000 light years across, would involve almost-geological time scales. The nearest large galaxy to our own, Andromeda, is estimated to be 2 million light years away. Although the starship crew would be able to survive the trip because of the slowing down of clocks aboard the starship, after the return they might find nobody to report to, back on Earth. Definitely the traditional space travel technology will not allow us for efficient space exploration so if we want to ever conquer the space we have to look for more sophisticated, than currently existing, travelling means. There are

[1]Captain of starship Enterprise NCC-1701 from Star Trek series.

chances that this can be achieved by utilization of gravitational physics, in particular Einstein theories. One of consequences of Einstein's general theory of relativity is the existence of spacetime shortcuts called wormholes. If they are ever constructed or discovered, they would provide us with practically unlimited possibilities of interstellar transportation. In this and consecutive chapters we will provide foundations for wormhole physics.

Einstein's special theory of relativity is based on two principles. The first says that all inertial observers are equivalent. This means that if one inertial observer investigates some physical law, then any other inertial observer performing the same investigations will see the same law as the first observer, even if the particular measurements differ. The second principle says that the velocity of light is the same in all inertial systems. Therefore special theory does not prohibit faster than light mechanisms. More specifically it categorizes all physical bodies into three classes (Fig 1.1.a):

1. **Subluminal** - the particles that move slower than light, also called tardyons or bradyons. All observed massive particles belong to this class.

2. **Luminal** - luxons, particles that move with the speed of light. They are massless and comprise of the photon, the carrier of electromagnetic field, and graviton, a particle that carries gravitational field.

3. **Superluminal** - the particles that move faster than the speed of light. These include hypothetical tachyons.

We are well accustomed to subluminal and luminal bodies. Superluminal particles were not found yet, possibly because of two reasons: they do not exist or they are not able to interact with normal (subluminal) matter which renders them unobservable. Tachyons differ from other particles in another important way: to slow them down addition of energy is required (and vice versa). Methodological approach to different types of bodies is presented by Kowalczynski in [64]. In this work a free object in flat spacetime endowed with Lorentzian coordinates is assumed to have the world-line $x^a(\sigma)$, where σ is the normalized affine parameter. We start from Minkowski spacetime

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{1.1}$$

and for $x^a(\sigma)$ we set

$$ds^2 = -kd\sigma^2 \tag{1.2}$$

Here the discrete dimensionless parameter k may take the values

1. $k = 1$ in the bradyonic (timelike, subluminal) case
2. $k = 0$ in the luxonic (null, luminal) case
3. $k = -1$ in the tachyonic (spacelike, superluminal) case

Such an approach helps to get rid of the deeply rooted but not entirely correct notion of "negative squared rest mass" when tachyons are considered. Dividing (1.1) and (1.2) by $d\sigma^2$ and defining a four-velocity vector as $u^a = dx^a/d\sigma$ yields

$$-k = -(u^t)^2 + (u^x)^2 + (u^y)^2 + (u^z)^2 \tag{1.3}$$

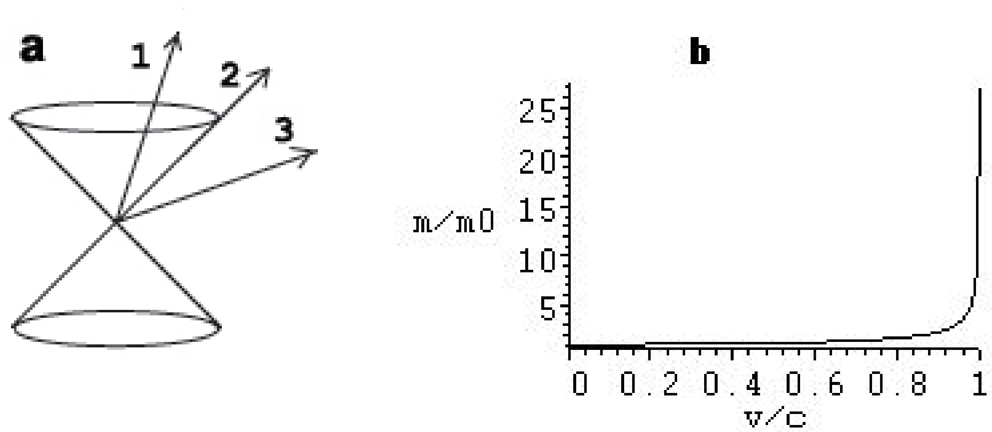

Figure 1.1: (a) Minkowski diagram with worldlines of three different types of particles: 1 - subluminal, 2 - luminal and 3 - superluminal. (b) Dimensionless increase in mass m/m_0 for fast moving object as a function of v/c.

Multiplication of this equation by m^2 brings special relativistic formula for a four-momentum vector $p^a = mu^a$

$$-km^2 = -(p^t)^2 + (p^x)^2 + (p^y)^2 + (p^z)^2 \tag{1.4}$$

We usually denote $\mathbf{p}^2 = (p^x)^2 + (p^y)^2 + (p^z)^2$ and $(p^t)^2 = E^2$ arriving to the general formula for energy

$$E = (\mathbf{p}^2 + km^2)^{1/2} \tag{1.5}$$

which alleviates any need for imaginary mass notion. Another example of tachyon's examination is contained in [16] where it was shown that they may produce a high intensity gravitational field not astronomically observed yet. For an overview of classical tachyon physics see [90].

The theoretical possibility of superluminal travel was analyzed in [14], [15]. It was shown that the existence of a preferred coordinate system and the existence of an instantaneous travel mechanism may allow for superluminal travel, even when travelling matter is ordinary. In similar spirit it is suggested in [37] that spacetime does not have to be Minkowskian despite Lorentzian invariance. In such a case an absolute frame may exist and consequently the speed of light may become just one of many possible critical limiting speeds in a vacuum.

A major obstacle for crossing the speed of light limit in special relativity is the relativistic mass increase. For a particle with the rest mass m_0 moving at speed v relativistic mass is described by

$$m(v) = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{1.6}$$

If we try to accelerate this particle to the speed of light, the relativistic mass would increase to infinity as depicted in Fig 1.1.b. We would have to commit an infinite amount of energy for this task, so even if superluminal speed is possible, it cannot be achieved by simple "brute force".

Another impediment caused by superluminal effects is causality violation. For a superluminal particle we can always find a reference frame in Minkowski space such that a particle is travelling backwards in time [97]. Let us assume that the particle with speed $v > c$ is sent from point A, reflects at point B and comes back to the sender at point C as in Fig 1.2.a.

Then in barred frame moving with velocity u with respect to the first frame we have

$$\bar{x} = \gamma(x - ut) \tag{1.7}$$

$$\bar{t} = \gamma(t - ux/c^2) \tag{1.8}$$

where $\gamma = \sqrt{1 - u^2/c^2}$. Since $t = x/v$ we can write

$$\bar{x} = \gamma x(1 - u/v) \tag{1.9}$$

$$\bar{t} = \gamma t(1 - uv/c^2) \tag{1.10}$$

If we set $u > c^2/v$ then in the barred frame the particle will be seen as travelling backwards in time ($\bar{t} < 0$) (see Fig 1.2.b). This will not necessarily imply the causality violation as long as we require that the particle will come back in the future light cone of the first frame. However if the above situation is possible, then by appropriate Lorentzian boost we can arrive to the state depicted in Fig 1.2.c, where the particle goes backwards in time to arrive back in the past cone of the first frame producing a closed time loop and this way violating causality. Such a situation we treat as unphysical and avoid it by denying existence to superluminal particles.

Special relativity is valid only when we assume that there is no gravitation present, or in other words spacetime is flat. This is not the case when general relativity is considered, since here gravitation is fully embraced by the theory. It is not considered as a separate force field, but rather as an aspect of spacetime structure [113]. The laws of physics are governed by two different, than in special relativity, principles. The principle of general covariance states, that only spacetime quantities that can appear in the physical law equation are the metric g_{ab} and quantities derived from it. The second principle requires that when g_{ab} becomes flat, i.e. Lorentzian η_{ab}, the special relativity equations must be valid. The consequence of this approach is that from the global point of view the spacetime is curved and the curvature is determined by matter content of spacetime. Curved spacetime is described by Riemannian geometry and

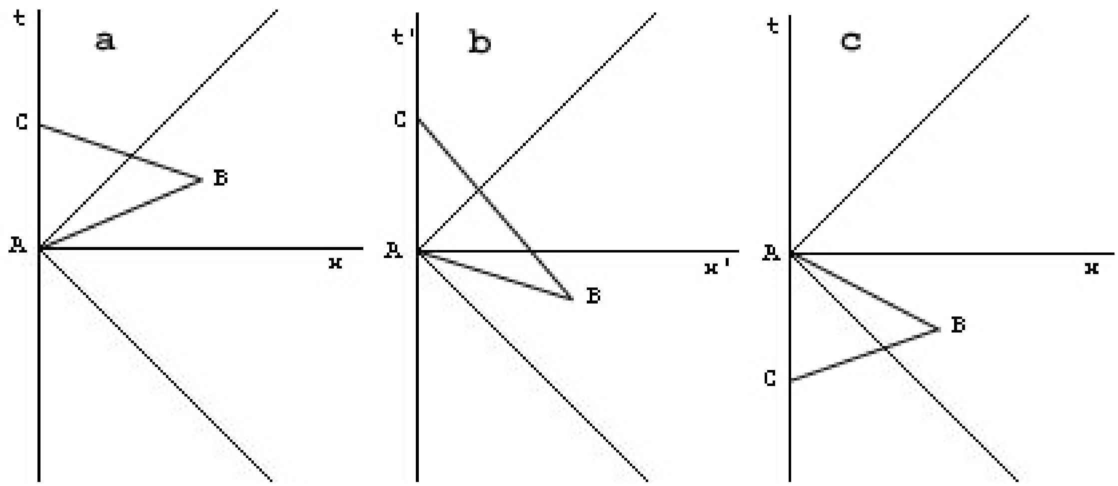

Figure 1.2: (a) Superluminal particle's path launched from point A, reflected from point B and returning to the launcher at point C. (b) The same superluminal particle as seen from another reference frame. (c) Closed time loop is possible when superluminal speed is allowed.

the relation of curvature to matter content is embodied in Einstein field equations.

Curved spacetime gives rise to effects that may result in faster than light (FTL) travel not contradicting special relativity limitations. One of the first such solutions was discovered by Gödel [35]. It describes a universe in which the gravitational tendency for collapse is balanced by the centrifugal force due to rotation of the universe. The light cones inside of the Gödel universe are tilted by a rotation-induced twist in spacetime and this causes a breakdown of the segregation of past and future. A traveller taking a trip into her future, moving along a circular path around the rotating universe, ends up in her own past, without breaking the light speed barrier. If our universe was spinning at the appropriate rate we might have experienced FTL effects already. However such rotation would produce measurable anisotropy in relic radiation that was not confirmed by any of the measurements.

We do not have to rotate the entire universe in order to obtain FTL effect. It was shown by Tipler [100] that a stationary infinitely long cylinder of dust spinning in such a way that its gravitational attraction is balanced by centrifugal forces may cause the light cones to tilt. When the surface speed of the cylinder exceeds $\frac{1}{2}c$ the closed timelike curves allowing for FTL effects will appear. This solution has major engineering problems - Tipler's cylinder quickly collapses into singularity due to powerful gravitational forces. Also the existence of infinitely long objects is very questionable.

The Kerr black hole possesses the spacetime that contains closed timelike curves and their presence is related to the ringlike structure of the singularity. When the singularity is surrounded by an event horizon, after crossing such, one can travel backwards in time. However returning to the starting point seems impossible - the traveller may easily emerge in a different universe. If the singularity is naked then FTL travel is possible, but because of cosmic censorship conjecture we usually consider this spacetime as unphysical.

Cosmological FTL mechanism was suggested by Gott [54]. In certain theories of high-energy physics when spontaneous symmetry breaking occurs, for example in early universe, there may still exist areas where this did not happen yet. In those places the energy is highly concentrated, creating exotic object called cosmic string. If two infinitely long parallel cosmic strings move apart at high speed, the light cones will tilt to allow for creation of closed timelike curves. There are a number of problems with physical reasonability of this solution, however in principle the cosmic string leads to FTL effect.

Recently Alcubierre proposed another FTL mechanism that would allow a spaceship to travel with superluminal speed - a warpdrive [2]. The travel speed is never superluminal locally, but rather the metric is engineered in such a way that the spaceship is carried along with its distortion. The space in front of the spaceship is contracted, whereas the space behind is expanded. The expansion/contraction rate determines the speed of the bubble containing the spaceship. Alternative version of the warp drive was proposed by Natario [86]. The physical reasonability of warpdrive is limited by the fact that such a mechanism requires a substantial amount of exotic energy (chapter 3) for creating spacetime distortion. Lobo and Visser [78] have shown that the net exotic energy stored in the warp fields must be a significant fraction of the mass of the spaceship.

General relativity admits more solutions which allow for existence of FTL effects. Besides ones already mentioned one can add ringholes [36], Deutch-Politzer time machine [67], [24], [89], gravitational shock waves [97], Krasnikov tube [30], [66] and a number of others. However the widest area of research of FTL mechanisms belongs to the most promising of them all - the wormhole.

1.2 Historical Perspective

First hints pointing towards wormhole physics were made by Flamm [31] in 1916. Nineteen years later Einstein and Rosen [28] conceived the concept of the "bridge" joining two disparate parts of space time via a narrow channel. If we make the coordinate change $v^2 = r - 2M$ in Schwarzschild coordinates

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.11}$$

then we obtain the Schwarzschild solution in the Einstein-Rosen form

$$ds^2 = -\frac{v^2}{v^2 + 2M} dt^2 + 4(v^2 + 2M)dv^2 + (v^2 + 2M)^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.12}$$

where the new coordinate v is in $(-\infty, +\infty)$. By using this coordinate we can avoid singularities at $r = 0$ and $r = 2M$. The region $r \in [2M, +\infty)$ is covered twice and the region where v is near zero we may interpret as a bridge connecting two asymptotically flat regions. However the coordinate change just cloaks the fact that we are still dealing with Schwarzschild spacetime that possesses the horizon, a one-way membrane. Thus crossing the Einstein-Rosen bridge is equivalent to crossing inside of the black hole horizon, with no chance for return. Another feature of the Einstein-Rosen bridge is that it is dynamic. It's circumference expands from zero to the maximum value and back to zero in such a rapid fashion, that we will not be allowed to cross the bridge even when moving at the speed of light.

Next major development in wormhole science came in 1950's, started by Wheeler. Together with Misner he was creating a framework in which classical physics would be explained with the help of Riemannian geometry of nontrivial topology manifolds. In the paper [80] the word “wormhole” was introduced for the first time :

> *There is a net flux of lines of force through what topologists would call a handle of the multiply-connected space and what physicists might perhaps be excused for more vividly terming a “wormhole”.*

In 1963 the rotating black hole solution was introduced by Kerr. This led to the discovery of the construction similar to Einstein-Rosen bridge for rotating spacetimes in cases when the rotation is slow. This solution carries the same limitations as in the Schwarzschild case - the horizons and big tidal forces. On the other hand, fast rotating black hole allows for fast transfer between two asymptotically flat regions. However in this case the traveller is not able to choose the destination. Also rotating Kerr black holes have a property that is called a naked singularity - the singularity is visible to the outside world since there is no horizon. The existence of naked singularity is frowned upon by cosmic censorship conjecture stating that all physically reasonable spacetime are globally hyperbolic, i.e. apart from a possible initial singularity (such as big bang) no singularity is ever visible to any observer [113]. There is still no experimental evidence for or against the validity of this statement, moreover there are known exact solutions that violate it, unless appropriate restrictions are set. Cosmic censorship conjecture prohibits the fast rotating Kerr black hole solution and with it the possibility of fast travel within it.

The most significant and still continuing period of development in wormhole physics started in 1988 with the publication of paper [81] by Morris and Thorne. It was proven there that in principle it may be possible to engineer spacetimes that would contain traversable wormholes. The major point of their approach was to find the spacetime that does not contain horizons, then moulding such a spacetime in order to obtain FTL effect. The derivation was followed by analysis of physical reasonability of energy-momentum tensor linked to this spacetime. It was found that Morris-Thorne wormholes in general violate classical energy conditions (chapter 3).

However the energy conditions seem not to be universal in the physical world since there are a number of violations already discovered while other physical effects were analyzed. Therefore this development seems to be the most promising one of all approaches to wormhole designs.

1.3 Concept of a Wormhole

A wormhole is a hypothetical shortcut for travel between points in the universe, or even between two different universes, as depicted in Fig 1.3. It has two entrances/exits called “mouths” that are connected to each other by a tunnel called the “throat”. The throat may be very short, but the wormhole traveller may be able to cover very large distances from the point of view of the outside observer. There are a number of general arguments against the existence of wormholes. These objections do not come strictly from general or special relativity, but are a consequence of what we usually call the “physical reasonability”. The major one is called the chronology protection conjecture and was proposed by Hawking [40] who summarized it in the following way:

> *It seems that there is a Chronology Projection Agency which prevents the appearance of closed timelike curves and so makes the universe safe for historian*

The relevance of chronology protection in case of wormholes comes from their convertibility to time machines which can be achieved by linking two wormhole mouths [82]. Visser in [106] provided a three-step procedure for achieving this task: wormhole ac-

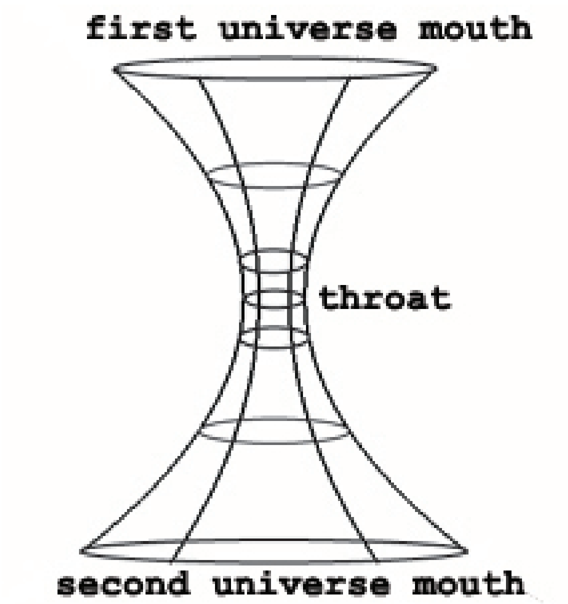

Figure 1.3: Embedding diagram for a wormhole that connects two different universes.

quisition, introducing a time shift between two mouths of the wormhole and bringing the wormhole mouths close together. More generally one can say that the wormhole construction leads to closed timelike curves. In classical physics notions of causality and chronology are built into the theory itself and any violation of them would be considered unphysical. General relativity does not assert that the chronology is preserved from the global point of view, only locally where the spacetime is considered Minkowskian. The chronology projection conjecture is strengthened by paradoxes that would appear if the conjecture was false [107]:

1. Consistency paradox: An attempt to change the past and so modify the entity that is trying to do the change (like killing the grandfather of the time traveller).

2. Bootstrap paradox: Causing the effect to be its own cause (a time traveller gives blueprints of the time machine to its inventor in the past).

Contemporary researched wormholes can be classified into a number of groups. We can start classification from the type of manifold the wormhole resides in. Lorentzian wormhole can be described as one where Lorentzian, or pseudo-Riemannian metric with signature $(-1, 1, 1, 1)$ is used. Since this metric is applied in the description of spacetime and the physical world, we may assume that Lorentzian wormholes are going to be of major scientific interest. In contrast we can consider Euclidean metric with the signature $(1, 1, 1, 1)$ which is used in quantum gravity theories.

Another approach to wormhole classification will involve the volume the wormhole resides in. Microscopic wormholes have size of the Planck length order. On the other side of spectrum we may find macroscopic wormholes. Those would have potential for use as FTL transport mechanisms.

We may classify the wormhole according to it's duration. Here are quasipermanent wormholes, persistent for noticeable length of time and transient wormholes, that are extremely short lived. When considering quasipermanent wormholes we would like to have their lifetime being longer than the time needed for crossing.

The wormhole is linking two regions in spacetime. When both regions belong to the same universe, the wormhole is called intra-universe. For wormhole linking sections situated in two different universes we reserve the term inter-universe.

The traversable wormholes that we are going to consider further in this work are Lorentzian, macroscopic, quasipermanent and intra-universe as well as inter-universe. We will add another view that categorizes the wormhole regarding the type of metric we use. Spherically symmetric metric is applied mostly to simplify calculations, since the wormhole does not require this symmetry. However in general we can have static (symmetric with regard to time reversal) and dynamic metric. The dynamic metric we will further split into radially expanding or contracting and rotating.

1.4 Current State of Research

After the initial work by Morris and Thorne there appeared a substantial activity in area of wormhole physics. Research took off in a number of directions, with major ones as follows:

Generalization of Morris-Thorne framework.
The original results [81] were based on restrictive assumptions, like spherical symmetry, being static or having suitability for human travel. Soon a number of generalizations were introduced. Visser [105] applied thin-shell formalism in order to construct cubic and then polyhedral wormholes, which do not have spherical symmetry constraint. Visser and Hochberg in [48] and [110] generalized the concept of the throat by defining it as 2-dimensional hypersurface of minimal area taken in one of the constant-time spatial slices. General class of solutions describing a spherically symmetric wormhole system was obtained by DeBenedictis and Dias [10]. Inflating wormhole model allowing for pulling traversable wormhole out of the quantum foam, was proposed by Roman [91]. Conformal approach to wormhole physics was described in Kar [57], Kar and Sahdev [58] and then it was generalized in [6]. The general form of a stationary, axially symmetric rotating wormhole was first described by Teo [98]

and Khatsymovsky [59], then further refinements together with detailed analysis of energy condition violations were done by Kuhfitting [73]. Wormhole solutions inside cosmic strings were found by Clement [22]. Thermodynamical properties and entropy of wormholes were discussed by Hong and Kim in [52].

Stress-energy tensor supporting wormholes.

This direction includes analysis of the state of energy conditions in wormhole, exotic matter needed for wormhole support and the quantification of it. Solutions representing non-static wormholes that require little exotic matter were considered by Wang and Letelier [114]. A bound on exotic energy densities in the wormhole with application of quantum inequalities was derived by Ford and Roman in [33]. Discussion of a wide class of wormhole solutions that meet these constraints was given by Kuhfitting [72]. The proof that the violation of null energy condition is a generic and universal feature of traversable wormhole was presented by Hochberg and Visser in [49] and [50]. Olum [87] provided general proof of the fact that any superluminal travel requires negative energies. A measure for total amount of energy condition violating matter in spacetime as well as a way to minimize this violation was proposed by Visser, Kar and Dadhich in [111] and [112]. Another approach to exotic matter quantification was given by Nandi, Zhang and Kumar in [83] and [84]. Lobo [77] described it in the cosmological and dust shells context. Constraints on exotic energy density needed for general dynamic wormhole were found by Ida and Hayward [53]. A possibility of opening the wormhole with arbitrary small amounts of exotic energy was considered by Kuhfitting in [71] and [74]. Some restrictions on stress-energy tensors in rotating wormhole supported by perfect or anisotropic fluid were obtained by Bergliaffa and Hibberd [12]. An essay reviewing some of the recent progress in the area of energy conditions and wormholes with the discussion centered on quantum inequalities was written by Roman [92]. The backreaction to the traversable wormhole spacetime by the scalar field or electric charge with exact solution was given by Kim and Lee [63]. Also Kim and Kim studied the static traversable wormholes coupled to quadratic scalar fields [62].

Cosmological Wormholes.

Here the traversable wormholes in the context of cosmological models and nonzero cosmological constant are considered. Kim [60] analyzed the Friedmann-Robertson-Walker model with a traversable wormhole. Choudhury and Pendharkar [21] constructed a model where the central core of the universe is a wormhole. A spacetime of two open universes connected by a Lorentzian wormhole was constructed by Li [76]. Investigation of macroscopic traversable wormholes in (2+1) and (3+1) dimensions with cosmological constant was done by Delgaty and Mann in [25]. DeBenedectis and Das [10] found a general wormhole class with cosmological constant. Moreover a detailed analysis of traversable wormholes with cosmological constant was presented by Lemos, Lobo and Oliviera in [75].

Wormholes in other gravitational theories.

Brans-Dicke wormholes were considered by Visser and Hochberg in [48] as well as by Agnese and Camera in [1]. A wormhole solution with torsion in Brans-Dicke gravity was obtained by Anchordoqui [3]. Shen et al [95] provided wormhole solutions in Kaluza-Klein theory. Barcelo and Visser [8] working in a brane world scenario developed a notion of “brane surgery” and analyzed dynamics of traversable wormholes. Wormhole throats in the presence of fully antisymmetric torsion were discussed by Hochberg and Visser [49]. Hayward, Kim and Lee [46] considered two-dimensional gravity dilaton model supporting the existence of wormholes. Nandi, Zhang and Kumar in [85] considered limits on the wormhole size in the context of Einstein massless minimally coupled scalar field (EMS) theory.

Wormholes and causality.

Conversion of a wormhole into time machine was proposed by Morris, Thorne and Yurtsever [82]. Inevitability of such a process during the interaction of a wormhole with the surrounding matter and with the external gravitational field was proven by Frolov and Novikov in [34]. For an extensive overview of this subject see [107]. Existence of closed timelike curves in wormholes and convertibility may lead to wormhole collapse as suggested by Hawking [40].

Wormholes and black holes.

There is a strong possibility that wormholes and black holes are in fact very similar objects. A new framework for unification of dynamic wormholes with black holes was proposed by Hayward [43]. A number of conversion mechanisms between wormholes and black holes was given [44] by the same author, as well as wormhole solutions supported by pure ghost radiation that collapse after the radiation is switched off [45].

Koyama and Hayward [65] presented an analytical solution which describes the construction of a traversable wormhole from Schwarzschild black hole.

Observable effects of wormholes.

Wormholes inhabiting the universe may produce visible effects. Cramer et al [23] suggested that a wormhole mouth embedded in high mass density may cause gravitational lensing effect and the resulting light enhancement can be observed. Torres et al, examined the scenario of extragalactic microlensing for natural wormholes in [102] and [103]. Anchordoqui et al [5] investigated a number of time profiles of gamma ray bursts searching for observable signatures produced by microlensing events related to natural wormholes. Safonova, Torres and Romero [104] provided computer simulations of exotic mass microlensing effects that can be produced by large wormholes and methods for finding wormholes in the observational microlensing experiments.

Chapter 2

Morris-Thorne Framework

Sorry to bother you, Kip....Would you give me advice? - Carl Sagan to Kip Thorne[1]

2.1 Desired Properties of Traversable Wormhole

Attempts of superluminal bridge creation usually suffered one or more drawbacks. Major obstacles included existence of horizons, enormous tidal forces, bridge instability, high radiation levels, too small bridge size or no possibility of destination control. There arose a need for a more unorthodox approach. Usual method of solving the Einstein equations would be to assume an existence of matter for the source of the stress-energy tensor. Then the equations of state would be derived for the ten-

[1]Despite nonchalant disregard of science fiction literature by a substantial portion of scientific community one has to admit that there is a large synergy between fiction and science. Wormhole physics is a child of such synergy. As described in [99], Kip Thorne was approached by astrophysicist and writer Carl Sagan with help regarding accuracy of gravitational physics in one of his sci-fi novels titled "Contact". The main hero of the story was supposed to plunge into a black hole, travel for one hour in hyperspace and then reemerge near star Vega, 26 light years away from Earth. Kip Thorne realized that this scenario is impossible according to black hole physics, but he provided another solution that seemed to be much more plausible - the wormhole. This approach was incorporated into the final version of the novel. Soon after this the first serious scientific paper about wormholes, which Thorne wrote together with his student Morris, appeared.

sion and pressure as a function of the energy density. These together with the field equations would provide the geometry of the spacetime described by the metric g_{ab}. Morris and Thorne approach in [81] differs substantially from this procedure. Firstly, they provided a list of properties the traversable wormhole should have. They are:

1. The metric should be both spherically symmetric and static.
2. The solution must everywhere obey the Einstein field equations.
3. The solution must have a throat that connects two asymptotically flat regions of spacetime.
4. There should be no horizon present, since such would prohibit two way travel.
5. The tidal gravitational forces must be reasonably small.
6. The crossing time from the point of view of both traveller and observer must be acceptably short.
7. The stress-energy tensor generating the wormhole must be physically reasonable.
8. The wormhole should be stable.
9. It must be possible to assemble the wormhole.

Secondly a diagonal stress-energy tensor was assumed and by use of Einstein field equations its components were found.

Above presented properties constitute a set of blueprints that would allow design of a simple geometry for workable traversable wormhole. We can generalize those blueprints in many ways. For example the spherical symmetry constraint is not really necessary since the wormhole should have the ability to assume an arbitrary shape. We use this and being static limitations just to simplify the calculations. A wormhole need not be traversable only by human beings in order to be useful. We may want to send non-biological object through, or just use photons for fast signalling (for example for communication purposes). Thus the tidal forces as well as the crossing time constraints can be substantially relaxed. Even if we are not able to assemble the wormhole with contemporary technology, we may hope that in future the engineering means will be adequately developed, or that wormholes were created during the formation of the universe.

2.2 Metric and Einstein Field Equations

A solution of the equations describing some system is stationary if there exists a special coordinate system in which it is time independent, i.e. all metric terms are not explicit functions of time. Being static means even more: the solution cannot be evolutionary in any way in some coordinate system. In particular the static metric is invariant under a time reversal with respect to any origin of time. Thus all static metric terms of type $g_{t\alpha}$ where $t \neq \alpha$ (t is a time coordinate, α is spacelike coordinate) must vanish in some coordinate system. Mathematically we can say that a space-time is stationary if and only if it admits a timelike Killing vector field. A space-time we call static if additionally this Killing vector field is hypersurface orthogonal.

Spherical symmetry means that there exists a privileged point such that the system is invariant under spatial rotations around this point. Since the reflection, or a coordinate reversal is a special case of rotation, we cannot have terms of mixed type $g_{\alpha\beta}$ $(\alpha \neq \beta)$. Mathematically we would say that a space time is spherically symmetric only if it admits three linearly independent spacelike Killing vector fields K^{α} whose orbits are closed and which satisfy [26], [27]

$$[K^{\alpha}, K^{\beta}] = \epsilon_{\alpha\beta\gamma} K^{\gamma} \tag{2.1}$$

where $\epsilon_{\alpha\beta\gamma}$ is an antisymmetric tensor such that $\epsilon_{123} = 1$. Since the unit line element of the 2-sphere is $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$ we take our static and spherically symmetric metric to be in general form [79]

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.2}$$

where $\Phi(r)$ and $\Lambda(r)$ are functions of the radial coordinate only. Further specification of the g_{tt} and g_{rr} metric components depends on the way we want to compute and interpret the outcome. If the considered geometry contains a horizon, which is the physically nonsingular surface where $g_{tt} = 0$, we may choose

$$e^{2\Phi(r)} = 1 - \frac{2M}{r}, \quad e^{2\Lambda(r)} = \left(1 - \frac{2M}{r}\right)^{-1} \tag{2.3}$$

which describes the Schwarzschild black hole, or

$$e^{2\Phi(r)} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \quad e^{2\Lambda(r)} = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} \tag{2.4}$$

which describes Reissner-Nordstrøm spacetime - a charged black hole.

The geometry that does not have a horizon must have nonzero g_{tt}. Thus we may set the metric as

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.5}$$

where $\Phi(r)$ is finite for all values of r. The functions $\Phi(r)$ and $b(r)$ will play an important role in future considerations. $b(r)$ determines the spatial shape of the wormhole so we shall call it the "shape function". $\Phi(r)$ determines the gravitational redshift, so we shall call it the "redshift function". The radial coordinate r covers the range $[r_0, +\infty)$ where r_0 defines the wormhole's throat radius.
Using the Einstein equations

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R = 8\pi T_{ab} \tag{2.6}$$

the strategy for obtaining the wormhole equation of state will be as follows:

1. From the metric (2.5) we obtain the connection coefficients according to

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}(g_{db,c} + g_{dc,b} - g_{bc,d}) \tag{2.7}$$

2. We compute Riemann curvature tensor components from

$$R^a_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{ec}\Gamma^e_{bd} - \Gamma^a_{ed}\Gamma^e_{bc} \tag{2.8}$$

3. We simplify the results by rotating the basis to proper reference frame

$$\begin{aligned} &\mathbf{e}_t \rightarrow e^{\Phi}\hat{\mathbf{e}}_t, \quad \mathbf{e}_r \rightarrow (1 - b/r)^{-1/2}\hat{\mathbf{e}}_r, \\ &\mathbf{e}_\theta \rightarrow r\hat{\mathbf{e}}_\theta, \quad \mathbf{e}_\phi \rightarrow r\sin(\theta)\hat{\mathbf{e}}_\phi \end{aligned} \tag{2.9}$$

4. The Riemann tensor is contracted to Ricci tensor by use of

$$\hat{R}_{ab} = \hat{R}^c_{acb} \tag{2.10}$$

5. The curvature scalar is obtained from the formula

$$\hat{R} = \hat{g}^{ab}\hat{R}_{ab} \tag{2.11}$$

6. We calculate the Einstein tensor from

$$\hat{G}_{ab} = \hat{R}_{ab} - \frac{1}{2}R\hat{g}_{ab} \tag{2.12}$$

to obtain (we will drop the function parameters if this does not lead to confusion)

$$\hat{G}_{tt} = \frac{b'}{r^2} \tag{2.13}$$

$$\hat{G}_{rr} = -\frac{b}{r^3} + 2\left(1 - \frac{b}{r}\right)\frac{\Phi}{r} \tag{2.14}$$

$$\hat{G}_{\theta\theta} = \hat{G}_{\phi\phi} = \left(1 - \frac{b}{r}\right)\left(\Phi'' - \frac{b'r - b}{2r(r-b)}\Phi' + (\Phi')^2 + \frac{\Phi'}{r} - \frac{b'r - b}{2r^2(r-b)}\right) \tag{2.15}$$

where "prime" denotes differentiation with respect to r. See Appendix A for Maple derivation of this result.

7. We assume the stress-energy tensor to be in the form

$$\hat{T}_{tt} = \rho(r), \quad \hat{T}_{rr} = -\tau(r), \qquad \hat{T}_{\theta\theta} = \hat{T}_{\phi\phi} = p(r) \tag{2.16}$$

where $\rho(r)$ is the total density, $\tau(r)$ is the radial tension, $p(r)$ is the lateral pressure, and all mixed values of T_{ab} are null. This choice is compatible with the expression for Einstein tensor, i.e. its only nonzero components are diagonal ones.

8. By use of expressions for G_{ab} and T_{ab} in the Einstein equation (2.6) we obtain the equations of state

$$\rho = \frac{b'}{8\pi r^2} \tag{2.17}$$

$$\tau = \frac{1}{8\pi r^2}\left[b/r - 2(r-b)\Phi'\right] \tag{2.18}$$

$$p = \frac{r}{2}\left[(\rho - \tau)\Phi' - \tau'\right] - \tau \tag{2.19}$$

We fulfilled the two first requirements in Morris-Thorne framework by getting to a solution for spherically symmetric metric with use of Einstein field equations. Exploiting the derived equations of state we can manipulate $b(r)$ and $\Phi(r)$ to obtain the metric which describes the wormhole. Next we derive the detailed expressions for the components of stress-energy tensor of the wormhole.

2.3 Traversability Conditions

Now we will consider how to obtain the configuration in which two asymptotically flat regions can be connected. For this purpose let's look at the geometry of three-dimensional space at a fixed time. Because of spherical symmetry it is sufficient to analyze only the equatorial slice. The equation (2.5) will then reduce to the two-dimensional metric

$$ds^2 = \frac{dr^2}{1 - b(r)/r} + r^2 d\phi^2 \tag{2.20}$$

We can visualize the slice by embedding (2.20) into three-dimensional Euclidean space which is described by cylindrical coordinates z, r and ϕ.

$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2 \tag{2.21}$$

The embedded surface is axially symmetric and this way it can be described by the function of r called $z(r)$. Manipulation of the last equation gives

$$ds^2 = \left[1 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + r^2 d\phi^2 \tag{2.22}$$

Comparison of (2.20) and (2.22) yields the equation for the embedding surface

$$\frac{dz}{dr} = \pm\left(\frac{r}{b(r)} - 1\right)^{-1/2} \tag{2.23}$$

This equation gives the reason for calling $b(r)$ the shape function. According to the third item in Morris-Thorne framework we should set b in such a way that the geometry has a minimum radius r_0 defined as a throat, at which the embedded surface is vertical. Also far away from the throat the space should be asymptotically flat, that corresponds to dz/dr vanishing at infinity.

Because of the divergence of dz/dr at the throat r is not always the best coordinate to describe the wormhole's shape. We may consider the proper radial distance l (as measured by static observers) instead. The expression for proper radial distance can be deduced from (2.20)

$$l(r) = \pm\int_{r_0}^{r} \frac{dr'}{\sqrt{1 - b(r')/r'}} \tag{2.24}$$

The lower limit of integration is the radius of the throat whereas the maximum upper one we set as the radius of wormhole's mouth. To get reasonable travelling conditions we require that $l(r)$ is well defined and finite through spacetime. This implies that the $1 - b(r)/r$ is never negative.

Obtaining the minimal radius can be done by consideration of the inverse of the embedding function $z(r)$. Since from (2.23) we have

$$\frac{dr}{dz} = \pm \left(\frac{r}{b(r)} - 1 \right)^{1/2} \tag{2.25}$$

and the condition for extremum is nullification of first derivative, the shape function must have property $b(r_0) = r_0$. We want to have a minimum at this point so the embedding surface flares-out to infinity. This will be satisfied if the second derivative of $r(z)$ is strictly positive, i.e.

$$\frac{d^2r}{dz^2} = \frac{b - b'r}{2b^2} > 0 \tag{2.26}$$

at or near the throat.

The fourth item in the framework indicates that the horizons should be absent from the wormhole's solution. The horizon is the physically nonsingular surface at which the g_{tt} metric element vanishes. In case of the wormhole, from (2.5) we can conclude that avoidance of the horizon can be accomplished by setting a condition on the redshift function as $\Phi(r) < \infty$ since then $e^{2\Phi(r)}$ can never be nullified.

Now lets consider a radially moving traveller with velocity u^a inside of wormhole. Let ζ be the vector separation between two parts of traveller's body. These two points do not have the same acceleration because of inhomogeneity of gravitational field, and this difference δa is referred to as the tide. The tidal effect is dependent on Riemann tensor

$$\delta a^a = -\hat{R}^a_{bcd} u^b \zeta^c u^d \tag{2.27}$$

In traveller's reference frame $u^a = \delta^a_t$ and ζ is purely spatial there, $\zeta^t = 0$. Thus, since the Riemann tensor is antisymmetric in its first two indices, we can define the spatial vector z by $\zeta^a = (0, z^\alpha)$ and then, noting that $(\alpha, \beta = r, \theta, \phi)$

$$\delta a^\alpha = -\hat{R}_{\alpha t \beta t} z^\beta \tag{2.28}$$

Since we have already got the expressions for the Riemann tensor for the observers in proper reference frame, we could use this result to obtain details of (2.28) by application of Lorentzian transformation. Let's denote the set of basis vector in the new frame as $(\hat{\mathbf{e}}_0, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$. Then

$$\begin{aligned} \hat{\mathbf{e}}_0 &= \gamma(\hat{\mathbf{e}}_t \mp \beta \hat{\mathbf{e}}_r), \\ \hat{\mathbf{e}}_1 &= \gamma(\mp \hat{\mathbf{e}}_r + \beta \hat{\mathbf{e}}_t), \\ \hat{\mathbf{e}}_2 &= \hat{\mathbf{e}}_\theta, \\ \hat{\mathbf{e}}_3 &= \hat{\mathbf{e}}_\phi \end{aligned} \tag{2.29}$$

where $\beta = v/c$, $\gamma = (1-\beta^2)^{-1/2}$, and the Riemann tensor components are

$$\hat{R}_{1010} = -\left(1 - b/r\right)\left(-\Phi'' + \frac{b'r - b}{2r(r-b)}\Phi' - (\Phi')^2\right) \tag{2.30}$$

$$\hat{R}_{2020} = \hat{R}_{3030} = \frac{\gamma^2}{2r^2}\left(\beta^2(b' - b/r) + 2(r-b)\Phi'\right) \tag{2.31}$$

If we want to get reasonably safe tidal forces this result has to be of g/l order, with l being the height of the traveller and $g \approx 9.81 m/s^2$ being the gravitational acceleration on the Earth surface.

Let the radial velocity of traveller as seen by the static observer be v. Then the proper radial distance as in (2.24) is linked to the radial velocity with the help of (2.5) by

$$v = \frac{dl}{e^\Phi dt} \tag{2.32}$$

which gives the traverse time measured by static observer as

$$\Delta\tau = \int \frac{dl}{ve^{\Phi}} \tag{2.33}$$

The proper traverse time of the journey recorded by the traveller's clock amounts to

$$\Delta t = \int \frac{dl}{v\gamma} \tag{2.34}$$

with γ defined above. Both these times must not exceed acceptable values, for example they are set by Morris and Thorne as less than one year.

The assumption (2.16) in conjunction with flare-out condition (2.26) and equations of state (2.17) - (2.19) brings an important constraint on the stress energy tensor. The difference between the energy density and tension, made dimensionless by dividing the result by the absolute value of energy density is

$$\frac{\rho - \tau}{|\rho|} = \left[\frac{b'}{r^2} - \frac{b}{r^3} + \frac{2(b-r)\Phi'}{r^2}\right]\left(\frac{r^2}{|b'|}\right) = \frac{-b/r + b' + 2(b-r)\Phi'}{|b'|} \tag{2.35}$$

We can explicitly use (2.26) in this equation to obtain

$$\frac{\rho - \tau}{|\rho|} = -\frac{2b^2}{r|b'|}\frac{d^2r}{dz^2} + 2(b-r)\frac{\Phi'}{|b'|} \tag{2.36}$$

At the throat radius r_0 we found out previously that $b(r_0) = r_0$ is one of necessary factors for satisfying flare out conditions. Thus because of Φ finiteness, and following boundness of Φ' we can discard the second factor of the equation and since the second derivative is positive at the throat we can write

$$\frac{\rho_0 - \tau_0}{|\rho_0|} < 0 \tag{2.37}$$

where the index 0 indicates that we are operating in immediate throat surroundings. Above result is central to the wormhole analysis since it indicates that the tension has to be greater than the mass-energy density and this undermines the physical reasonability of stress-energy tensor. It is widely recognized that such a situation is impossible when we are dealing with classical matter only. The generalization of this statement is called “energy condition” and we are going to commit the whole next chapter to its description and analysis. We give name “exotic” to the matter that exhibits property (2.37).

Two last items in Morris-Thorne framework, the stability and wormhole construction are becoming rather elusive when considering above findings. We currently know very little about the nature of exotic matter, its resistance to small, as produced by traveller or large, as generated by external bodies perturbations, so analysis of wormhole stability seems to be rather premature. We can create small amounts of exotic energy, for example in Casimir effect [20], but do not have enough engineering skills for moulding it into a traversable wormhole.

2.4 Wormhole Examples

Traversable wormholes described in previous sections are dependent on two independent and freely specifiable functions: the shape function denoted as $b(r)$ and the redshift function denoted by $\Phi(r)$. In order to get more insight into the wormhole physics we can consider wormholes with additional restrictions. This approach may become necessary when we deal with more complex dynamic wormholes in consecutive chapters. Some simple solutions for traversable wormhole are given in [82] and [106]. We will consider here two useful models: "zero radial tides" and "zero density".

2.4.1 Zero radial tides

In this case we assume $\Phi(r) = 0$, so the wormhole does not have any radial tides and consequently the horizon cannot be generated. The metric becomes

$$ds^2 = -dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.38}$$

and the Einstein field equations solved in proper reference frame yield equations of state

$$\rho = \frac{b'}{8\pi r^2} \tag{2.39}$$

$$\tau = \frac{b}{8\pi r^3} \tag{2.40}$$

$$p = \frac{b - b'r}{16\pi r^3} \tag{2.41}$$

The example of the simple shape function $b(r) = \sqrt{r_0 r}$ is considered in [81]. We will assume a little bit of a different approach using the shape function defined by

$$b(r) = r_0/r \tag{2.42}$$

From the equations of state we can conclude that material used for wormhole is everywhere exotic since

$$\rho - \tau = \frac{-r_0}{4\pi r^4} \tag{2.43}$$

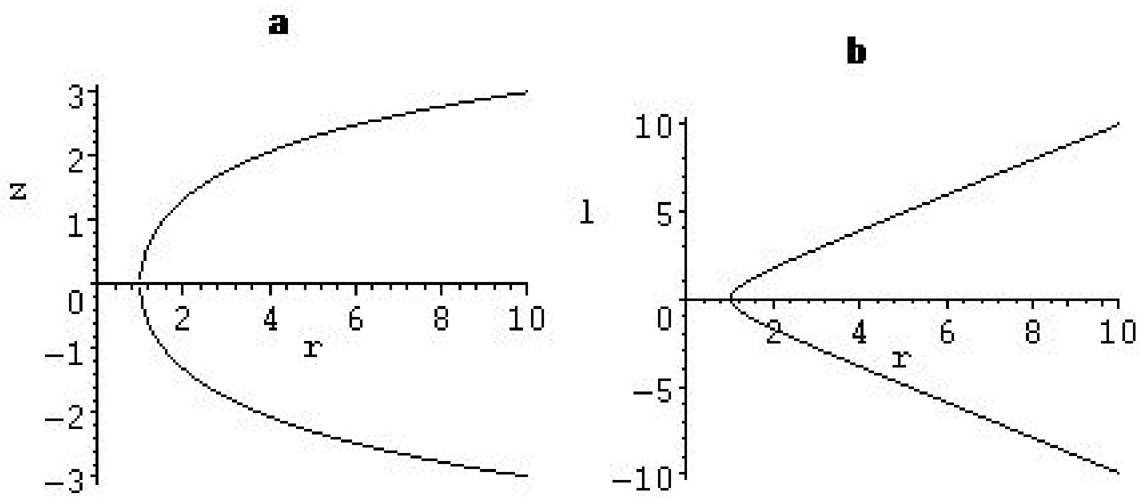

Figure 2.1: (a) Embedding diagram $z(r)$ for zero tides wormhole. (b) Proper distance l as a function of r coordinate.

but exotic matter density quickly falls off with the radius. The embedding function $z(r)$ can be obtained by reversing and integrating the equation (2.23)

$$z(r) = \pm \int_{r_0}^{r} \frac{d\bar{r}}{\sqrt{\bar{r}^2/r_0 - 1}} = \pm r_0 \ln \frac{r + \sqrt{r^2 - r_0^2}}{r_0} \tag{2.44}$$

This function is depicted in Fig 2.1(a) where we set $r_0 = 1$. The proper radial distance as measured by static observer can be obtained from (2.24)

$$l(r) = \frac{r^2 - r_0^2}{\sqrt{r - r_0^2/r}} \tag{2.45}$$

Here $l(r) \approx r$ when $r \gg r_0$, as presented in Fig 2.1(b). The radial tides are null since all factors in (2.30) depend on Φ. Nonradial tides (2.31) reduce to

$$\hat{R}_{2020} = \hat{R}_{3030} = \frac{-\gamma^2 \beta^2 r_0}{r^4} \tag{2.46}$$

being the most prominent at r_0 then falling off rapidly when the coordinate r grows. Slow motion through the wormhole will render $\beta \ll 1$ which makes the tides negligible

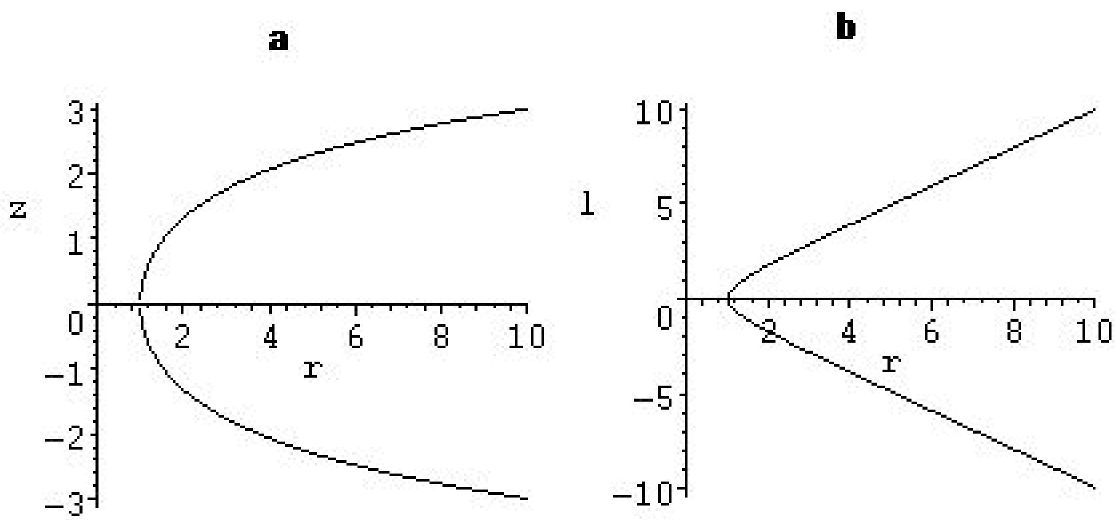

Figure 2.2: (a) Embedding diagram $z(r)$ for zero density wormhole. (b) Proper distance l as a function of r coordinate.

enough.

2.4.2 Zero density

Here we set the shape function to be constant $b(r) = r_0$. By substituting $r_0 = 2M$ one can make the observation that the metric is analogous to non-traversable Schwarzschild black hole, but since we do not allow Φ to slip to infinity, this configuration is traversable. Thus the metric of "zero density" wormhole is described by

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - r_0/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.47}$$

The Einstein equations take the form

$$\rho = 0 \tag{2.48}$$

$$\tau = \frac{1}{8\pi r^2}\left[r_0/r - 2(r - r_0/r)\Phi'\right] \tag{2.49}$$

$$p = (1 - \frac{r_0}{r})\left[\Phi'' + \Phi'(\Phi' + \frac{1}{r})\right] + \frac{r_0}{2r^2}\left(\Phi' + \frac{1}{r}\right) \tag{2.50}$$

and the exoticity can be described as

$$\rho - \tau = -\frac{1}{8\pi r^2}\left[r_0/r - 2(r - r_0/r)\Phi'\right] \tag{2.51}$$

i.e. is purely made of negative pressure. For simple case of constant Φ the wormhole needs twice less exotic matter at the throat than in "zero tidal" example. The embedding function is

$$z(r) = \pm \int_{r_0}^{r} \frac{d\bar{r}}{\sqrt{\bar{r}/r_0 - 1}} = 2\sqrt{r_0(r - r_0)} \tag{2.52}$$

This, and proper radial distance functions are depicted in Fig 2.2 (a) and (b). The tides are

$$\hat{R}_{1010} = -\left(1 - r_0/r\right)\left(-\Phi'' + \frac{r_0}{2r(r - r_0)}\Phi' - (\Phi')^2\right) \tag{2.53}$$

$$\hat{R}_{2020} = \hat{R}_{3030} = \frac{\gamma^2}{2r^2}\left(\beta^2(-r_0/r) + 2(r - r_0)\Phi'\right) \tag{2.54}$$

and for constant Φ they reduce to lateral part only.

Chapter 3

Energy Conditions

Dear Mike. It follows immediately from Proposition 9.2.8 of the book by Hawking and Ellis, plus the Einstein field equation, that any wormhole requires exotic material to hold it open. Sincerely... - Don Page to Mike Morris [1]

3.1 Pointwise and Averaged Energy Conditions

In general we can compute the Einstein tensor G_{ab} for any given metric and using the Einstein equations proclaim it to be the stress-energy tensor $8\pi T_{ab}$ describing the matter fields content of spacetime. Although formally correct, this approach might yield matter fields that do not occur in reality. To avoid this situation we impose conditions on the stress-energy tensor that are compatible with experimental data. The simplest condition arises from the assumption that energy is conserved, that in terms of stress-energy tensor can be expressed as

$$T^{ab}_{;a} = 0 \tag{3.1}$$

[1]Thorne [99] describes in detail the process of discovery of modern wormhole physics. One of highlights was that after receiving the preprint of [81] Don Page, by application of global methods, was immediately able to reach conclusions that took Morris and Thorne two years to work out.

This can be seen as a consequence of Einstein's field equations and Bianchi identities. Another generic feature that seems to be shared by most of physical matter fields is that energy densities are almost always nonnegative. To investigate this observation we will assume that components of the stress-energy tensor can be set in form (Type I in [42])

$$T^{ab} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 \\ 0 & 0 & p_2 & 0 \\ 0 & 0 & 0 & p_3 \end{bmatrix} \tag{3.2}$$

with respect to an orthonormal basis (e_0, e_1, e_2, e_3) with timelike e_0. This is the general case of the stress-energy tensor for fields with non-zero mass and for zero mass fields with exception of some special types of radiation. The energy density, as measured by an observer whose world-line has unit tangent vector e_0 is represented by ρ. The principal pressures in the three spacelike directions e_α are p_α.

The weak energy condition (WEC) is the assertion that for any timelike vector V^a

$$T_{ab}V^aV^b \geq 0 \tag{3.3}$$

meaning that the energy density seen by any observer is non-negative. This condition plays an important role in the proof of singularity theorems. In terms of principal pressures WEC can be described as

$$\rho \geq 0, \quad \rho + p_\alpha \geq 0 \tag{3.4}$$

The strong energy condition (SEC) says that for any timelike vector V^a

$$(T_{ab} - \frac{1}{2}Tg_{ab})V^aV^b \geq 0 \tag{3.5}$$

where T is the trace of the stress-energy tensor $T = T_{ab}g^{ab}$. When considering principal pressures we can write

$$\rho + p_\alpha \geq 0, \quad \rho + \sum_\alpha p_\alpha \geq 0 \tag{3.6}$$

If the SEC is satisfied, then it is ensured that gravity is always an attractive force.

The dominant energy condition (DEC) is the assertion that for any timelike vector V^a

$$T_{ab}V^aV^b \geq 0 \tag{3.7}$$

and $T_{ab}V^a$ is not spacelike. This may be interpreted as saying that any observer will see the local energy density as non-negative and that the local energy flow vector is non-spacelike. In other words DEC is equivalent to WEC with the requirement that pressure should not exceed energy density.

$$\rho \geq 0, \quad -\rho \leq p_\alpha \leq \rho \tag{3.8}$$

The WEC and SEC by continuity imply null energy condition (NEC) in the limit when V^a becomes a null vector k^a

$$T_{ab}k^ak^b \geq 0 \tag{3.9}$$

Expressed in terms of energy density and principal pressures NEC is equivalent to

$$\rho + p_\alpha \geq 0 \tag{3.10}$$

From all above energy conditions NEC is the weakest one in use. In general we notice two chains of energy condition dependence:

$$DEC \Rightarrow WEC \Rightarrow NEC \tag{3.11}$$

and

$$SEC \Rightarrow NEC \tag{3.12}$$

The assumption that energy conditions are satisfied by every kind of classical matter leads in general relativity to many interesting theorems [9]. Amongst them are singularity theorem, positive mass theorem, superluminal censorship theorem, topological censorship theorem, no-hair theorems and various constraints on black hole surface gravity.

It is possible to demonstrate that a number of classical and quantum fields can violate all of the pointwise energy conditions (next section). It was suggested by Tipler [101] that averaging the pointwise energy conditions over an observer's geodesics should be used in order to obtain physically reasonable solutions of Einstein field equations. Three averaged energy conditions (AEC) are currently in use. The averaged weak energy conditions (AWEC) hold on a timelike curve γ if

$$\int_\gamma T_{ab} V^a V^b d\tau \geq 0 \tag{3.13}$$

where γ denotes timelike curve, V^a is tangent vector and τ is observer's proper time. Here the energy density seen by the observer is not measured at a single point but is averaged over the observer's worldline. It is obvious that the existence of negative energy densities is allowed on some portions of the worldline so long as there is compensating positive energy density at some other worldline interval.

In similar fashion we can define averaged null energy condition (ANEC).

$$\int_\gamma T_{ab} k^a k^b d\lambda \geq 0 \tag{3.14}$$

where γ now denotes null curve parameterized by λ and k^a is tangent vector to this curve. Although AWEC may imply ANEC in many cases, in general spacetime those conditions are independent.

The last of commonly used averaged energy conditions is averaged strong energy condition (ASEC). It holds on a timelike curve γ if

$$\int_\gamma (T_{ab}V^aV^b + \frac{1}{2}T)d\tau \geq 0 \tag{3.15}$$

A new set of energy constraints on energy density, called quantum inequalities (QI), was introduced by Ford and Roman [32] in 1995. They limit magnitude and duration of negative energy density seen by an inertial observer in Minkowski spacetime. Let's consider a free, minimally coupled scalar field in an arbitrary quantum state. Renormalized expectation value of stress-energy tensor we denote by $< T_{ab} >$, observer's velocity by u^a. Then $< T_{ab}u^au^b >$ is the expectation value of local energy density in observer's frame of reference and QI states that

$$\int_{-\infty}^{\infty} \frac{< T_{ab}u^au^b > d\tau}{\tau^2 + \tau_0^2} \geq -\frac{3}{32\pi^2\tau_0^2} \tag{3.16}$$

Here τ is the observer's proper time and τ_0 is the characteristic width of the Lorentzian sampling function (the sampling time). The physical content of this formula is that the more negative energy density in the interval τ_0, the shorter the duration of the interval must be. An inertial observer cannot see arbitrarily large negative energy densities which last for arbitrarily long periods of time, which would be an excellent situation for maintaining a traversable wormhole. The biggest concern with QI is that they were derived from Minkowski space quantum field theory and they may not hold when the curved background is applied [33]. For example Krasnikov [70] by creating an explicit example shows that QI break down when a conformal scalar field in the two-dimensional de Sitter space is considered. Other objections that can be applied to QI are listed in [77] and the strong defence of QI validity is presented in [92].

An interesting consequence of pointwise and averaged EC is the topological censorship theorem. It states that the topology of any physically reasonable isolated system (i.e. where EC and AEC hold) is shrouded. That is a spacetime may contain isolated topological structures such as wormholes, but there is no method by which this structure can be observed [93]. Let γ_0 be a causal curve with past and future endpoints lying in an asymptotically flat region. It was proved in [56] that

> **Topological Censorship Theorem:** *If any asymptotically flat globally hyperbolic spacetime satisfies ANEC then every causal curve from past null infinity to future null infinity is deformable to* γ_0.

Thus to have a chance to actively probe the wormhole structure (or traverse it) we have to contravene one of the theorem assumptions.

3.2 Violations

If energy conditions strictly hold, we would have no hope to ever construct a traversable wormhole in the way described previously. However, there exist a number of theoretical and experimental examples on both classical and quantum level indicating that EC can be broken in some cases. The most often quoted example in literature of such a possibility are scalar fields. They play an important role in many theoretical models, but the convincing experimental confirmation is still to come. The only scalar fields detected are the scalar mesons - pions π, kaons K and a range of resonances. None of them is elementary, they are quark-antiquark bound states [109]. Responsible for electroweak symmetry breaking Higgs particle is the next candidate for a scalar field, unfortunately it was never observed. More exotic scalar fields include explaining lack of strong CP violation axions, extending general relativity Brans-Dicke scalar and coming from string theory dilatons - all of them being just theoretical constructs, nonetheless important enough to take seriously.

Let's consider the classical free minimally-coupled massive scalar field related to meson π described by the Lagrangian density

$$\pounds = -\psi_{;a}\psi_{;b}g^{ab} - \frac{m^2}{\hbar^2}\psi^2 \tag{3.17}$$

We derive equations of motion for the field by integrating the Lagrangian density in order to obtain action, then vary the action with respect to the field and this way arrive at Lagrangian equations. The stress-energy tensor is obtained in similar way, however we vary metric rather than the field. In our case

$$T_{ab} = -\frac{\partial \pounds}{\partial g^{ab}} + \frac{1}{2}\pounds g_{ab} \tag{3.18}$$

that leads to

$$T_{ab} = \psi_{;a}\psi_{;b} - \frac{1}{2}g_{ab}\left(\psi_{;c}\psi_{;d}g^{cd} + \frac{m^2}{\hbar^2}\psi^2\right) \tag{3.19}$$

Using this result in definition of SEC (3.5) we obtain

$$\left(T_{ab} - \frac{1}{2}Tg_{ab}\right)V^aV^b = (V^a\psi_{;a})^2 - \frac{1}{2}\frac{m^2}{\hbar^2}\psi^2 \tag{3.20}$$

This quantity is easy to make negative, for example when $V^a = (1,0,0,0)$ with ψ time independent. Thus SEC fails when the scalar fields with minimal coupling are considered.

Adding a non-minimal coupling complicates the derivation significantly. Following the reasoning in [109] we arrive at the expression for stress-energy tensor

$$T_{ab} = \frac{1}{1-8\pi\xi\psi^2}\left(\psi_{;a}\psi_{;b} - \frac{1}{2}g_{ab}(\nabla\psi)^2 - g_{ab}V(\psi) - \xi[2(\psi\psi_{;b})_{;a} - 2g_{ab}(\psi\psi_{;c})^{;c}]\right) \tag{3.21}$$

where ξ is the coupling and $V(\psi)$ describes the potential. If this potential is positive then we can find that SEC is again violated. But here we can get even more insight into the nature of scalar fields. NEC is satisfied when minimal coupling is considered, but this is not the case for nonzero coupling constant. Using LHS of (3.9) and expression above we get

$$T_{ab}k^a k^b = \frac{1}{1-8\pi\xi\psi^2}\left[(k^a\psi_{;a})^2 - \xi[2k^a k^b(\psi\psi_{;b})_{;a}]\right] \tag{3.22}$$

We extend k to be a geodesic vector field so that $k^a k^b_{;a} = 0$ and we set λ to be the affine parameter, that implies $k^a\nabla_a = d/d\lambda$. Then follows

$$T_{ab}k^a k^b = \frac{1}{1-8\pi\xi\psi^2}\left[\left(\frac{d\psi}{d\lambda}\right)^2 - \xi\left(\frac{d^2\psi^2}{d\lambda^2}\right)\right] \tag{3.23}$$

For any local extremum of ψ on local geodesic

$$T_{ab}k^a k^b = -\frac{1}{1-8\pi\xi\psi^2}\left[\xi\left(\frac{d^2\psi^2}{d\lambda^2}\right)\right] \tag{3.24}$$

can be made negative by manipulating ξ and ψ. Thus we may conclude that scalar fields have the ability to contravene pointwise NEC when coupling is nonzero.

We can extend above analysis towards average null energy conditions. Lets consider a complete null geodesic and compute the LHS of (3.14). This brings us to

$$\int T_{ab}k^a k^b d\lambda = \int \frac{1}{1-8\pi\xi\psi^2}\left[\left(\frac{d\psi}{d\lambda}\right)^2 - 2\xi\frac{d}{d\lambda}\left(\psi\frac{d\psi}{d\lambda}\right)\right] d\lambda \tag{3.25}$$

Integration by parts and discarding the boundary terms yield

$$\int T_{ab}k^a k^b d\lambda = \int \frac{1-8\pi\xi(1-4\xi)\psi^2}{(1-8\pi\xi\psi^2)^2}\left(\frac{d\psi}{d\lambda}\right)^2 d\lambda \tag{3.26}$$

This integral can be negative, and ANEC can be violated in the region

$$\xi(1-4\xi)\psi^2 > 1 \tag{3.27}$$

The most known physical result rendering EC false that was confirmed experimentally is Casimir effect [20]. In its simplest form it is the interaction of a pair of neutral, parallel conducting planes due to the disturbance of the vacuum of the electromagnetic field. The origin of this effect is purely quantum - classical electrodynamics does not predict it. The stress-energy tensor of Casimir effect can be described as

$$T^{ab} = \frac{\pi^2\hbar}{720d^4}(\eta^{ab} - 4z^a z^b) \tag{3.28}$$

where z is a normal vector with respect to the plates and d is the distance between them. Since the energy density is negative, i.e.

$$\rho = \frac{-(\pi^2\hbar)}{720d^4} \tag{3.29}$$

we can immediately conclude that WEC and DEC are violated (although the effect is very small). Now, because $\rho + p_z < 0$ we experience NEC and SEC violation. It can be shown [107] that averaged energy conditions will break down as well.

One can generalize above result by applying topology rather than physical planes to distort the vacuum. Visser [107] gives the example of periodic boundary condition imposition on the whole universe. He assumed that the universe has periodicity a in the z direction, which is equivalent to the assumption that it is rolled up into a cylinder of circumference a. For quantum field the z component of momentum is constrained by

$$k_z = n\frac{2\pi}{d} \tag{3.30}$$

and this gives us stress-energy tensor in "Topological Casimir Effect"

$$T^{ab} = \frac{\pi^2 \hbar}{45 d^4}(\eta^{ab} - 4z^a z^b) \tag{3.31}$$

which leads to the violation of all energy conditions.

Less speculative and currently intensively researched cosmological phenomenon that leads to EC violation is the existence of non-zero cosmological constant Λ. Writing Einstein field equations (2.6) with Λ as

$$R_{ab} - \frac{1}{2}g_{ab}R + g_{ab}\Lambda = 8\pi T_{ab} \tag{3.32}$$

and moving the last term on the LHS over to the RHS of the equation, we obtain "total" stress-energy tensor made of two parts

$$T^{ab}_{total} = T^{ab} - g_{ab}\Lambda \tag{3.33}$$

The term containing Λ is often identified with the vacuum energy density. The net cosmological constant is the sum of a number of disparate contributions including potential energies from scalar fields and zero-point fluctuations of each field theory degree of freedom, as well as a bare cosmological constant [19]. During early epochs Λ might have a significant value leading to violations of energy conditions.

Another EC violating result was obtained in [17]. The inflating universe may be locally approximated by Robertson-Walker metric

$$ds^2 = a^2(\eta)(d\eta^2 - dx^2) \tag{3.34}$$

where η is conformal time, related to cosmic time by $dt = ad\eta$. The Hubble parameter can be defined by

$$H = \frac{a'}{a^2} \tag{3.35}$$

where the derivative is with respect to η. It follows from differentiation that

$$H' = (aa'' - 2a'^2)/a^3 \tag{3.36}$$

We use a normalized null vector $k^a = (1, n)/a^2, |n| = 1$ to construct the LHS in (3.9). Since

$$R_{ab}k^a k^b = T_{ab}k^a k^b \tag{3.37}$$

we get

$$T_{ab}k^a k^b = -\frac{2}{a^6}(aa'' - 2a'^2) = -\frac{2}{a^3}H' = -\frac{2}{a^2}H' \tag{3.38}$$

From this equation we can see that the growth of the Hubble parameter will induce NEC violation. To get deeper into the nature of such behavior we notice that the Hubble parameter can be approximated as [38]

$$H^2 = \frac{8\pi}{3}\left(\frac{\dot{\phi}}{2} + V(\phi)\right) \tag{3.39}$$

During inflation the second term is strongly dominant and quantum fluctuations will cause variation in $V(\phi)$. Thus H will oscillate, causing its derivative to change sign. This implies that $T_{ab}k^ak^b$ in (3.38) will occasionally become negative and thus NEC will not hold.

There exist a number of other physical conditions causing EC violations. One could list Hawking evaporation [39], Hawking-Hartle vacuum [108], moving mirrors [13] and a couple of others [107]. The quantum-induced violations are usually very small, typically of order of $\hbar$, and it is questionable if they suffice to support traversable wormhole. To evaluate the worth of quantum effects for wormhole design one has to know how much of exotic matter is needed.

3.3 Quantification of exotic matter

The process of exotic mater quantification is not yet particulary well understood. Originally the exotic mater was defined as the matter where $\rho - \tau$ is negative. We may try to generalize this notion by use of various energy conditions described in this chapter. For example we may propose that for a null vector k^a the amount of NEC exotic matter density is

$$\rho_{NEC} = T_{ab}k^ak^b \tag{3.40}$$

if $T_{ab}k^ak^b$ is negative. Similar definition would apply for other EC types. The next step in generalization would be use of averaged energy conditions. In the ANEC (3.14) case the expression

$$\int_\gamma T_{ab}k^ak^b d\lambda \tag{3.41}$$

will give the measure of EC violation. To evaluate this integral along radial geodesic we have to express λ in terms of radial coordinate r. We start from the observation that for an affine parametrization the inner product of Killing vector and the tangent vector is constant and it can be arbitrarily set to one [107]. Thus

$$K^a k_a = K^a g_{ab} k^b = g_{tt} k^t = g_{tt} \frac{dt}{d\lambda} = 1 \tag{3.42}$$

and that, since in metric (2.5) we have $g_{tt} = e^{2\Phi}$, gives us the relationship between parameter λ and time coordinate

$$d\lambda = e^{2\Phi} dt \tag{3.43}$$

as well as expression for the null vector as seen by the external to the wormhole observer

$$k^t = e^{-2\Phi} \tag{3.44}$$

In the proper reference frame above vector becomes

$$\hat{k}^t = e^{-\Phi} \tag{3.45}$$

Also let's note that from the metric (2.5) follows that for radial null curve

$$e^{\Phi} dt = \frac{dr}{\sqrt{1 - b/r}} \tag{3.46}$$

Combining those results into (3.41) yields the expression for ANEC integral along the radial geodesic

$$\int_\gamma T_{ab} \frac{e^{-\Phi} dr}{\sqrt{1 - b/r}} = \int_\gamma (\rho - \tau) \frac{e^{-\Phi} dr}{\sqrt{1 - b/r}} \tag{3.47}$$

This is a line integral and therefore not quite adequate to describe the amount of violation. One should instead try to construct a volume integral. There exist two approaches to the solution of this problem. In [112] it was proposed that straight integration over the volume should be done with an appropriate choice of the integration measure $4\pi r^2 dr$ or $\sqrt{g}drd\theta d\phi$. Since from (2.17) and (2.18) follows that

$$\rho - \tau = \frac{1}{8\pi r}\left(1 - \frac{b}{r}\right)\left[\ln\left(\frac{e^{2\Phi}}{1 - b/r}\right)\right]' \tag{3.48}$$

and the volume integral becomes, after integration by parts and inclusion of both asymptotic regions

$$X_{ANEC1} = \int(\rho - \tau)dV = \left[(r - b)\ln\left(\frac{e^{2\Phi}}{1 - b/r}\right)\right]_{r_0}^{\infty} - \int_{r_0}^{\infty}(1 - b')\left[\ln\left(\frac{e^{2\Phi}}{1 - b/r}\right)\right]dr \tag{3.49}$$

where we defined X_{ANEC1} as a measure of exotic mass.

More fundamental approach was taken in [84]. It was proposed that the volume integral should fulfill three conditions:

1. The integration measure must be natural, that is, it must be such that the resulting total mass is the same in any coordinate system with its own natural measure.

2. If the role of exotic matter is played by some scalar field, then the integrated exotic mass for a given solution must reproduce, at least to first order, the scalar "charge" present in the solution.

3. The mass must obey some conservation principle.

The integration measure was identified here as $\sqrt{-g}d^3x$ and in the static configuration the total volume of ANEC violating matter present was proposed as

$$X_{ANEC2} = \int_{r_0}^{\infty} (\rho - \tau)\sqrt{-g}d^3x \tag{3.50}$$

where r_0 is the isotropic throat radius. All above conditions are satisfied when this integral is used.

One can calculate X_{ANEC2} for some specific wormhole solutions. For example for $\Phi = 0$ and $b(r) = r_0$ we obtain

$$X_{ANEC2} = -\frac{r_0}{2}\left[\ln\left(-r_0/2 + r + r\sqrt{1 - r_0/r}\right)\right]_{r_0}^{\infty} \tag{3.51}$$

This expression is diverging when r goes to infinity indicating that an infinite amount of exotic matter is required to maintain the wormhole. We may conclude that this result prohibits the existence of "zero tides" and "zero density" wormholes. In case of "zero tides" wormhole with $b(r) = r_0/r$ the integral yields

$$X_{ANEC2} = -\pi\sqrt{r_0} \tag{3.52}$$

and this in [83] is identified with the charge of some exotic matter field.

Chapter 4

Evolving Wormholes

Einstein's equations, in some sense, were like a Trojan horse. On the surface, the horse looks like a perfectly acceptable gift, giving us the observed bending of starlight under gravity and a compelling explanation of the origin of the universe. However, inside lurk all sorts of strange demons and goblins, which allow for possibility of interstellar travel through wormholes and time travel. - Michio Kaku

4.1 Types of Evolution

There exists a possibility that wormhole construction is easier on the quantum level than on the classical one, as investigated by Wheeler [80]. This wormhole would not be traversable since quantum fluctuations in spacetime metric live in distances of Planck length order which is much too small comparable to any macroscopic object. However one could imagine that with the presence of sufficiently advanced technology the quantum wormhole can be pulled out of the spacetime foam, enlarged and adjusted to the size and shape adequate for interstellar travel [81]. For this purpose we need to analyze the physics of wormholes that are radially dynamic. Another reason for evolving wormhole investigation comes from the fact that by changing the shape function and consequent change in the throat radius we may try to influence the amount of exotic matter needed for wormhole maintenance or even try to avoid violation of energy conditions.

In following sections we will consider three popular approaches to evolving wormholes, all of them related to cosmological physics: conformal transformation of wormhole metric, inflation of its spatial part and embedding the traversable wormhole metric into Friedmann-Robertson-Walker (FRW) universe. For analysis of wormhole construction with the cosmological constant present see [25] and [75].

Conformal transformation of the wormhole was considered by Kar [57] in order to find out if within classical general relativity a class of nonstable not violating energy conditions wormholes could exist. It was found that evolving geometry can support a wormhole and the WEC violation can be avoided for arbitrarily large intervals of time.

Roman [91] analyzed Morris-Thorne type wormhole embedded in an inflationary background, with all non-temporal components of the metric tensor multiplied by a factor of the form $e^{2\chi t}$, where χ is related to cosmological constant. It was proven there that a wormhole can be enlarged to traversable size, however violation of WEC could not be avoided. At most the exotic matter needed for wormhole maintenance can be minimized at the later stage of inflation.

After inflation the universe undergoes an evolution that is usually described by one of FRW models. As in [60] we will investigate whether insertion of the wormhole into expanding universe will allow suspension of the need for exotic matter.

4.2 Conformal Approach

The conformal transformation technique is used in general for bringing the points at infinity to a finite position and hence analyze the causal structure of infinity. We start with a metric g_{ab} describing some physical spacetime and introduce conformal metric $\overline{g}_{ab}$ that is related to the former by

$$\overline{g}_{ab} = \Omega^2 g_{ab} \tag{4.1}$$

where Ω is the conformal factor, a smooth, finite and strictly positive function [113]. The distances within $\overline{g}_{ab}$ are different than when described by g_{ab} but the null geodesics do not change.

We will use (4.1) to evolve the Morris-Thorne wormhole (2.5) as described in [57] and [58]. The general case leads to considerable complexity so we limit our interest to "zero tidal" instance. The metric is going to be

$$ds^2 = \Omega^2(t)\left[-dt^2 + \frac{dr^2}{1-b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{4.2}$$

Again we perform transformation to the proper reference frame

$$\begin{aligned} \mathbf{e}_t \to \Omega(t)\hat{\mathbf{e}}_t, \quad \mathbf{e}_r \to \Omega(t)(1-b/r)^{-1/2}\hat{\mathbf{e}}_r, \\ \mathbf{e}_\theta \to \Omega(t)r\hat{\mathbf{e}}_\theta, \quad \mathbf{e}_\phi \to \Omega(t)r\sin\theta\hat{\mathbf{e}}_\phi \end{aligned} \tag{4.3}$$

and solve Einstein equations with diagonal stress-energy tensor (appendix B), obtaining

$$\Omega(t)^{-4}r^{-2}[3\dot{\Omega}^2(t)r^2 + \Omega(t)^2 b'(r)] = 8\pi\rho(t,r) \tag{4.4}$$

$$\Omega(t)^{-4}r^{-3}[\dot{\Omega}^2(t)r^3 - 2\Omega(t)\ddot{\Omega}(t)r^3 - \Omega^2(t)b(r)] = -8\pi\tau(t,r) \tag{4.5}$$

$$\frac{1}{2}\Omega(t)^{-4}r^{-3}[2\dot{\Omega}^2(t)r^3 - 4\Omega(t)\ddot{\Omega}(t)r^3 - \Omega^2(t)b'(r)r + \Omega^2(t)b(r)] = 8\pi p(t,r) \quad (4.6)$$

Here overdot denotes the derivative with respect to time t, the prime denotes the derivative with respect to radial coordinate r.

We would like to get to the situation where energy conditions are satisfied. Let's consider details of WEC which are $\rho \geq 0$, $\rho - \tau \geq 0$, $\rho + p \geq 0$. They consecutively imply (for simplicity we drop the function arguments)

$$\frac{b'}{8\pi r^2} \geq -3\left(\frac{\dot{\Omega}}{\Omega}\right)^2 \quad (4.7)$$

$$2\left(\frac{\dot{\Omega}}{\Omega}\right)^2 - \frac{\ddot{\Omega}}{\Omega} \geq \frac{b - b'r}{16\pi r^3} \quad (4.8)$$

$$2\left[2\left(\frac{\dot{\Omega}}{\Omega}\right)^2 - \frac{\ddot{\Omega}}{\Omega}\right] \geq \frac{-b - b'r}{16\pi r^3} \quad (4.9)$$

The equation (4.8) can be rewritten with "flare-out" term (2.26) as follows

$$F(t) = 2\left(\frac{\dot{\Omega}}{\Omega}\right)^2 - \frac{\ddot{\Omega}}{\Omega} \geq \left(\frac{b - b'r}{2b^2}\right)\frac{2b^2}{16\pi r^3} \quad (4.10)$$

where we defined $F(t)$ for future convenience. Since the "flare-out" has to be always greater than zero and $2b^2/r^3$ is nonnegative, $F(t)$ has to be positive and in this way values of Ω are constrained. One can notice that this constraint will make the condition (4.9) satisfied as well. In general it is possible to show that (4.10) cannot be satisfied on the whole domain providing that Ω is positive and bounded. This means that although wormholes can be created without exotic matter, their lifetime is limited since $F(t)$ sooner or later will became negative. Fig 4.2 represents graphs

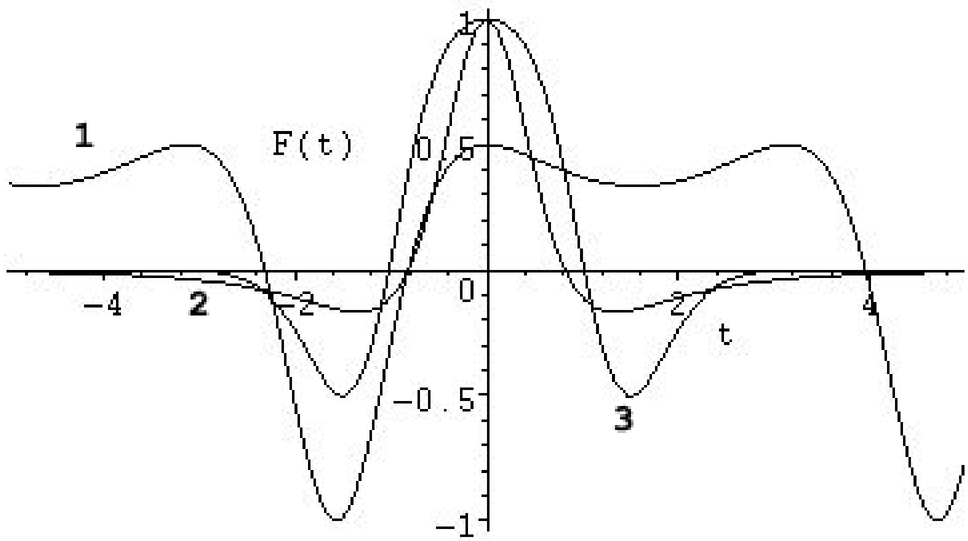

Figure 4.1: Function $F(t)$ for three choices of Ω: 1 - periodic, 2 - polynomial and 3 - exponential.

of $F(t)$ for three arbitrarily chosen conformal factors that are strictly positive and bounded. Their shape can be modulated by application of constant multipliers.

$$\Omega_1 = \sin(t) + 2 \tag{4.11}$$

$$\Omega_2 = \frac{2 + t^2}{1 + t^2} \tag{4.12}$$

$$\Omega_3 = e^{-t^2} + 1 \tag{4.13}$$

We observe that the function $F(t)$ undergoes changes of sign depending on the time coordinate, which indicates that the energy conditions are violated for some periods of time, when $F(t) < 0$.

One can interpret (4.8) in terms of Hubble constant H and deceleration parameter q as described in [48]. For this purpose we will approach conformal wormhole like it was an expanding universe in which the scale factor is described by $\Omega(t)$. Lets relate conformal time t to comoving time T via the relation

$$\Omega dt = dT \tag{4.14}$$

then (4.2) becomes

$$ds^2 = -dT^2 + \Omega^2(t)\left[\frac{dr^2}{1-b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{4.15}$$

Hubble constant can be expressed as the ratio of the time derivative of the scale factor and scale factor itself [88]. Thus in our case

$$H = \frac{1}{\Omega}\frac{d\Omega}{dT} = \frac{1}{\Omega}\frac{dt}{dT}\dot{\Omega} = \frac{1}{\Omega^2}\dot{\Omega} \tag{4.16}$$

where overdot denotes differentiation with respect to conformal time. Deceleration parameter which measures the deviation of the growth of the scale factor from linearity is defined [88] by

$$q = -\Omega\frac{d^2\Omega}{dT^2}\left(\frac{d\Omega}{dT}\right)^{-2} \tag{4.17}$$

and this after short calculation gives

$$q = 1 - \frac{\Omega\ddot{\Omega}}{\dot{\Omega}^2} \tag{4.18}$$

The insertion of above derived results for H and q into the expression (4.10) yields

$$F(t) = H^2\Omega^2(1+q) \tag{4.19}$$

which indicates that energy conditions can be suspended providing that $q > -1$ and the Hubble parameter is large.

4.3 Inflated Wormhole

In this section we will analyze a wormhole with time-dependent inflationary background. The metric is obtained by multiplication of the spatial part of Morris-Thorne wormhole by de Sitter scale factor $e^{2\chi t}$, as suggested in [91]

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\chi t}\left[\frac{dr^2}{1-b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{4.20}$$

where χ is a constant. We will use inflation to enlarge the microscopic wormhole to size that allows traversability. Let us consider a sequence of time slices where $t =$ const. At $t = 0$ the metric is identical to the static wormhole metric. When the inflation starts, the radial proper length is given by

$$l(t) = \pm e^{\chi t}\int_A^B \frac{dr'}{\sqrt{1-b(r')/r'}} \tag{4.21}$$

where A and B are any two points at $t =$ const. We can introduce a new barred coordinate for each $t =$ const slice as

$$\bar{r} = e^{\chi t} r \tag{4.22}$$

This is a simple coordinate transformation, it can be seen as the rescaling of the r coordinate on each slice of constant time. By application of the embedding procedure we can show that the inflated wormhole will have the same size and shape relative to the barred coordinate system, if the shape function behaves as

$$\bar{b}(\bar{r}) = e^{\chi t} b(r) \tag{4.23}$$

The flareout condition near the throat will be

$$\frac{d^2\bar{r}}{d\bar{z}^2} = \left(\frac{\bar{b} - \bar{b}'\bar{r}}{2\bar{b}^2}\right) = e^{-\chi t}\left(\frac{b - br'}{2b^2}\right) \tag{4.24}$$

which is equivalent to the static case multiplied by decaying factor.

The formal solution of Einstein equations for the inflated wormhole is included in appendix C. We will limit ourselves to consideration of "zero tides" wormhole, where $\Phi = 0$. The Einstein equations in the proper reference system give

$$3\chi^2 + \frac{b'}{r^2} e^{-2\chi t} = 8\pi\rho \tag{4.25}$$

$$-3\chi^2 - \frac{b}{r^3} e^{-2\chi t} = -8\pi\tau \tag{4.26}$$

$$-3\chi^2 - \frac{b - b'r}{2r^3} e^{-2\chi t} = 8\pi p \tag{4.27}$$

and thus the "absolute" exoticity of wormhole's matter is

$$\rho - \tau = \frac{b'r - b}{8\pi r^3} e^{-2\chi t} \tag{4.28}$$

Assuming the same shape function as in subsection 2.4.1 we see that

$$\rho - \tau = \frac{2r_0}{8\pi r^4} e^{-2\chi t} \tag{4.29}$$

which is the expected result. At the beginning of inflation the amount of exotic matter needed is analogous to the static wormhole, but with inflation progressing we need it less and less. Contradictory to the conformal wormhole example, we cannot find that the energy conditions satisfaction and wormhole existence are in agreement with each other.

4.4 Cosmological Wormhole

We can generalize inflated wormhole results by consideration of wormhole embedded in FRW model of the universe. Such a model is based on three assumptions, namely [27]:

1. **The cosmological principle:** At each epoch the universe presents the same aspect from every point, except for local irregularities.

2. **Weyl's postulate:** The particles of the substratum lie in spacetime on a congruence of timelike geodesics diverging from a point in the finite or infinite past.

3. **General Relativity:** Special relativity augmented by Einstein field equations.

These postulates lead to the metric of the FRW form

$$ds^2 = -dt^2 + R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{4.30}$$

where R(t) is the scale factor of the universe and k can be +1, 1, or 0, the sign of the curvature of spacetime. Following [60] we can construct a cosmological wormhole by combining two spacetime metrics: FRW spacetime with static wormhole metric (2.5)

$$ds^2 = -e^{\Phi(r)}dt^2 + R^2(t)\left[\frac{dr^2}{1-kr^2-b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{4.31}$$

The solution of Einstein equations with diagonal stress-energy tensor for "zero tides" wormhole in proper reference frame (appendix D) gives

$$r^{-2}R^{-2}(3r^2\dot{R}^2 + 3kr^2 + b') = 8\pi\rho \tag{4.32}$$

$$r^{-3}R^{-2}(-2r^3R\ddot{R} - r^3\dot{R}^2 - kr^3 - b) = -8\pi\tau \tag{4.33}$$

$$\frac{1}{2}r^{-3}R^{-2}(-4r^3R\ddot{R} - 2r^3\dot{R}^2 - kr^3 - b'r + b) = 8\pi p \tag{4.34}$$

Now we can consider exoticity of the cosmological wormhole matter. From (4.32) and (4.33) follows that $\rho - \tau \geq 0$ only if

$$\dot{R}^2 - R\ddot{R} + k \geq \frac{b - b'r}{16\pi r^3} \geq 0 \tag{4.35}$$

We could check the possibility of wormhole existence against various cosmological models (for their overview see [29]). Einstein-de Sitter model assumes $k = 0$ and $\Lambda = 0$. Here R can be expressed as

$$R(t) = At^{2/3} \tag{4.36}$$

where A is a constant. The LHS of (4.35) gives

$$\dot{R}^2 - R\ddot{R} + k = \frac{2}{3}A^2 t^{-2/3} \tag{4.37}$$

This function is strictly decreasing but never reaching zero for finite t. Thus we cannot expect that wormholes without exotic matter arise in Einstein-de Sitter universe. Another approach which assumes $k = 0$ and $\Lambda \neq 0$ yields

$$R(t) = Be^{Ct} \tag{4.38}$$

where B is a constant and C is a function of Λ. This is called de Sitter model and LHS of (4.35) yields always zero, which indicates that no wormhole can exist at all. Similar result can be obtained for Milne universe, where $R(t) = t$ and $k = -1$.

Chapter 5

Rotating Wormholes

Solutions to field equations can be found which exhibit virtually any type of bizarre behavior - Frank Tipler

5.1 Rotating Gravitational Fields

A vast majority of macroscopic objects in the universe rotate. Thus it seems to be reasonable to believe that rotating wormhole, if the natural wormholes exist at all, would be a common exotic object as well. In order to analyze rotating wormhole we firstly describe a general behavior of gravitational field generated by rotating bodies. We assume that the gravitational field is generated by a statically rotating object, that possesses axial symmetry about the axis of rotation, which in turn passes through the center of the object. We take for granted that there exists a time-like coordinate t and an angular coordinate ϕ respectively of which the metric and matter components are independent. Thus the metric and stress-energy tensor are functions of two other coordinates, r and θ. The considered object rotates in the ϕ direction, so in order to avoid rotation direction change when the time reversal occurs we have to set $g_{tr} = g_{t\theta} = 0$. Similarly we have to get rid of invariance under the rotation direction reversal, $g_{r\phi} = g_{\theta\phi} = 0$, otherwise the object's rotation would change when this transformation is performed. Finally we arrive at

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 + g_{ij}dx^i dx^j \tag{5.1}$$

where i, j are to be summed over values r, θ. Above metric is uniquely determined up to coordinate transformation of (r, θ), and this transformation can be used when the metric is adjusted for the particular problem we are currently solving.

Above result can be derived more rigorously by use of Killing vectors [55]. The spacetime of a physical object is stationary and axially symmetric when the metric possesses two linearly independent Killing vectors: everywhere timelike ξ (responsible for generating invariant time translation) and everywhere spacelike η (generating invariant rotation with respect to ϕ). Also the orbits of η have to be closed. It can be shown that these vectors commute, so one can introduce coordinates (t, ϕ) such that

$$\xi = \frac{\partial}{\partial t}, \quad \eta = \frac{\partial}{\partial \phi} \tag{5.2}$$

and by considering orthogonal transitivity obtain (5.1).

The most known result when considering rotating gravitational fields is the Kerr solution. It describes the axially symmetric rotating black hole which can be characterized by two parameters: the mass M and angular momentum J. In Boyer-Lindquist coordinates the metric of rotating black hole is [94]

$$ds^2 = -\frac{\Delta - a^2\sin^2\theta}{\rho^2} - 2a\frac{2Mr\sin^2\theta}{\rho^2}dtd\phi + \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta d\phi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$

(5.3)

where

$$\Delta = r^2 - 2Mr + a^2 \tag{5.4}$$

$$\rho^2 = r^2 + a^2 \cos^2\theta \tag{5.5}$$

and $a = J/M$. The Kerr solution, due to rotational geometry, has a number of features that distinguishes it from the Schwarzschild one. For example, instead of a point singularity in the static case, there appears a ring singularity. Kerr black hole also has two surfaces of infinite redshift at $r = M \pm (M^2 - a^2 \cos^2\theta)^{1/2}$, and in the case $a^2 < M^2$ two event horizons at $r = M \pm (M^2 - a^2)^{1/2}$. Beyond the outer event horizon lies region called the ergosphere. It allows for extraction of black hole's rotational energy - the Penrose process. It is easily illustrated by considering a particle sent into the ergosphere, where it breaks up into two particles. First particle has negative energy and falls onto Kerr black hole while second one escapes to infinity with energy greater than the energy of initial particle. The extracted energy must come at the expense of the rotational energy of the black hole.

A particularly useful form of rotational metric arises when it is cast into spherical coordinates by requiring that $g_{\theta\theta} = g_{\phi\phi}/\sin^2\theta$ [98]

$$ds^2 = -N^2 dt^2 + e^{\mu} dr^2 + r^2 K^2 [d\theta^2 + \sin^2\theta (d\phi - \omega dt)^2] \tag{5.6}$$

Here the N, μ, K and ω are called gravitational potentials and they are functions of (r, θ) because of the direct relationship with g_{ab} in (5.1). The quantity ω is the angular velocity $d\phi/dt$ of the particle infalling from infinity into the rotating object.

This particle is dragged by the influence of gravity so that it acquires an angular velocity in the same sense as rotating object - an effect called "dragging of inertial frames". The factor rK determines proper radial distance measured from the origin. N, if it can ever be nullified, will indicate the existence of an event horizon. This can be seen from analysis of metric's determinant.

5.2 Rotating Wormhole

The metric (5.6) can be cast into the form describing a wormhole created in the spirit of Morris and Thorne by use of substitution

$$e^{\mu} = (1 - b(r,\theta)/r)^{-1} \tag{5.7}$$

where $b(r,\theta)$ corresponds to the shape function described in chapter 2. Thus we have

$$ds^2 = -N^2 dt^2 + \frac{dr^2}{1 - b/r} + r^2 K^2 [d\theta^2 + \sin^2\theta (d\phi - \omega dt)^2] \tag{5.8}$$

It is immediately visible that if we stop the wormhole rotation (ω=0), set $K = 1$ and get rid of θ dependence, the metric will converge to form (2.5). However there is one major difference when we compare the static and rotating objects which is exhibited by the existence of nonzero off-diagonal components in stress-energy tensor. In particular $T_{t\phi}$ characterizes the rotation of the matter distribution.

The process of solution of Einstein field equations can be simplified when we use the orthonormal tetrads defined by

$$\Theta^0 = N dt \tag{5.9}$$

$$\Theta^1 = \left(1 - \frac{b}{r}\right)^{-1} dr \tag{5.10}$$

$$\Theta^2 = rKd\theta \tag{5.11}$$

$$\Theta^3 = rK\sin\theta(d\phi - \omega dt) \tag{5.12}$$

which yield

$$ds^2 = -(\Theta^0)^2 + (\Theta^1)^2 + (\Theta^2)^2 + (\Theta^3)^2 \tag{5.13}$$

and then use Cartan calculus method. An example of such approach is described in appendix E.

General analysis of the EC near the throat of rotating wormhole contained in [73] and [98] gives two major results:

1. There is always a violation of energy condition, so rotation does not alleviate the need for exotic matter.

2. The exotic matter can be moved around the throat, so that some class of in-falling observers would not encounter it.

The complexity of rotating wormhole solution is lowered by the simplification of describing it gravitational potentials. The approach presented by Teo [98] deals with the wormhole with

$$N = K = 1 + \frac{(4a\cos\theta)^2}{r}, \quad b = 1, \quad \omega = \frac{2a}{r^3} \tag{5.14}$$

and here a is the angular momentum. In this case the throat at $r = 1$ has proper radius $r_0 = 1 + (4a\cos\theta)^2$ and its shape is dumbbell-like.

If the rotational speed of the wormhole is substantial, g_{tt} factor may become positive in some region outside of the throat. This indicates the presence of an ergoregion where particles can no longer remain stationary with respect to infinity. Since from (5.14) follows that

$$g_{tt} = -N^2 + r^2K^2\omega^2\sin^2\theta = \left(1 + \frac{(4a\cos\theta)^2}{r^2}\right)\left(\frac{(2a)^2\sin^2\theta}{r^4} - 1\right) \tag{5.15}$$

the ergoregion occurs when

$$r^2 = |2a\sin\theta| > 1 \tag{5.16}$$

It cannot completely surround the throat because in this case it would necessarily intersect the event horizon at the poles, whose existence is prohibited by the wormhole framework. One can anticipate that by bombarding the rotational wormhole with particles we may be able to exert some control over its parameters.

Similar analysis can be performed for the rigid rotating wormhole as done by Kim in [61]. The gravitational potentials are assumed to have the values

$$N = K = 1, \quad b = \frac{b_0}{r^2}, \quad \omega = const \tag{5.17}$$

Here the shape of the throat is a sphere, like in the case of static wormhole. The g_{tt} factor is easily shown to be

$$g_{tt} = -1 + \omega^2r^2\sin^2\theta \tag{5.18}$$

The ergoregion appears when

$$r = \frac{1}{|\omega\sin\theta|} \tag{5.19}$$

and has a cylindrical shape.

The stress-energy tensor of the rotating wormhole was discussed by Bergliaffa and Hibberd [12]. A fluid with anisotropic stresses was taken as a matter source and providing that

$$T_{ab} = (\rho + p)u_a u_b + p\Theta_{ab} + \Pi_{ab} \tag{5.20}$$

where the metric Θ_{ab} is given by (5.13), u_a is the four-velocity of the fluid and Π_{ab} is the stress tensor satisfying

$$\Pi^a_a = 0, \quad \Pi_{ab}u^b = 0 \tag{5.21}$$

the conditions on Einstein tensor were derived. If we assume $\Pi_{ab} = 0$ then the fluid is usually described as "perfect" and $\hat{G}_{ab}$ must satisfy ("hat" means here that we are in rotated reference frame)

$$\hat{G}_{rr} - \hat{G}_{\theta\theta} = 0 \tag{5.22}$$

$$\hat{G}_{r\theta} = 0 \tag{5.23}$$

$$\hat{G}^2_{t\phi} = (\hat{G}_{tt} + \hat{G}_{\theta\theta})(\hat{G}_{rr} - \hat{G}_{\phi\phi}) \tag{5.24}$$

In general case the anisotropic fluid must obey

$$\hat{G}_{tt} + \hat{G}_{\theta\theta} \geq 2\hat{G}_{03} \tag{5.25}$$

and

$$\left(\frac{u_t}{u_\phi}\right) > 1 \tag{5.26}$$

It was shown in [12] that the rotating wormhole described by gravitational potentials (5.6) cannot be generated by neither perfect nor anisotropic fluids. Thus more realistic source of rotating wormhole has to be found.

Chapter 6

Summary

People like us, who believe in physics, know that the distinction between past, present, and future is only a stubbornly persistent illusion. - Albert Einstein

This thesis has examined a number of aspects of static and dynamic wormholes. We started from a review of faster than light effects that are consistent with Einstein field equations. Then the scope was narrowed down to analysis and description of wormhole features. The historical perspective was presented, then the major concepts, classifications and most important directions of research.

Contemporary wormholes take their origin from the work done by Morris and Thorne, here dubbed as Morris-Thorne framework. We describe it in detail providing the analysis of a static and spherically symmetric case. Ways to achieve traversability as well as engineering issues are considered, also some examples for specific solutions are presented.

The major problem encountered during wormhole engineering is violation of energy conditions. They are not an inherent part of general relativity but rather additional limitation on stress-energy tensor that assure physical reasonability of solutions to Einstein field equations. Their detailed description is provided, then a number of physical effects that violate energy conditions is given. Attempts to measure the violation and approaches to exotic mass quantification are reviewed.

Treating wormholes as dynamic objects give hopes of alleviation or at least lessening of some tough constraints set on wormhole engineering. We analyze conformal approach, wormholes in inflationary scenario as well as wormholes in cosmological context. Then effects in rotating bodies and analysis of axially symmetric rotating wormholes is provided.

A number of computer algebra with Maple 8.0 and manual calculations were performed and those are contained in appendices. A list of references to wormhole related articles is provided in bibliography section.

Issues discussed in this thesis are mostly very new and not yet fully settled in contemporary science. It may be worthwhile to make a few final points that summarize the validity of the approach and give indications regarding the future directions of wormhole physics.

1. The fundamental underlying theory that leads to wormhole solution - general theory of relativity - is well tested and understood.

2. The biggest obstacle to wormholes existence, namely violation of energy conditions, seems to be less important as experiments on the micro and macro levels indicate that energy conditions are not necessarily an universal phenomena.

3. There is a strong indication that the amount of exotic matter needed for worm-

hole support may not have to be so huge as thought a decade ago. Introduction of dynamics to wormholes usually makes this amount even smaller.

4. Ultimately the answer to the wormhole existence issue has to come from the laboratory experiment or astrophysical observation.

5. Many questions surrounding the behavior of wormholes may be answered by a theory that does not fully exist yet - quantum gravity.

Appendix A

Einstein Tensor for Morris-Thorne Wormhole

```
> with(tensor):
```

Create expression for the metric

```
> coord := [t, r, theta, phi]:
> g := array(symmetric,sparse, 1..4, 1..4):
> g[1,1] := -exp(2*Phi(r)):
> g[2,2] := 1/(1-b(r)/r):
> g[3,3] := r^2:
> g[4,4] := r^2*sin(theta)^2:
> MTmetric := create([-1,-1], eval(g)):
```

Compute Riemann tensor

```
> invMTmetric := invert(MTmetric, 'detg'):

> D1g := d1metric(MTmetric, coord):

> D2g := d2metric(D1g, coord):

> Christoff := Christoffel1(D1g):

> Riem := Riemann(invMTmetric, D2g, Christoff):

> RiemRaised := raise(invMTmetric, Riem, 1):

> RiemannComponents := get_compts(RiemRaised):
> map(proc(x) if
> RiemannComponents[op(x)]<>0 then
> x=simplify(RiemannComponents[op(x)], radical)
> else NULL
> fi end,[indices(RiemannComponents)]):
```

Compute Ricci, Curvature Scalar and Einstein tensor.

```
> RICCI := Ricci(invMTmetric, Riem):

> RS := Ricciscalar(invMTmetric, RICCI):

> Estn := Einstein(MTmetric, RICCI, RS):
```

Find transformation routine to the proper reference frame

```
> h := array(symmetric,sparse, 1..4, 1..4):

> h[1,1] := exp(Phi(r)):

> h[2,2] := (sqrt(1-b(r)/r))^(-1):

> h[3,3] := r:

> h[4,4] := (r*sin(theta)):

> hmetric := create([1,-1], eval(h)):
```

```
> invhmetric := invert(hmetric, 'deth'):
```

Rotate Riemann tensor to the proper reference frame

```
> ProperRiemann := change_basis(RiemRaised, hmetric, invhmetric):

> ProperRComponents := get_compts(ProperRiemann):
> map(proc(x)
> if ProperRComponents[op(x)]<>0 then
> x=simplify(ProperRComponents[op(x)], radical, power)
> else NULL
> fi
> end,[indices(ProperRComponents)]):
```

Rotate Einstein tensor to the proper reference frame

```
> ProperEinstein :=
> simplify(change_basis(Estn, hmetric, invhmetric),
> radical, power);
```

$$ProperEinstein := \mathrm{table}([compts =$$

$$\left[-\frac{\frac{d}{dr}\,\mathrm{b}(r)}{r^2}\,,\,0\,,\,0\,,\,0\right]$$

$$\left[0\,,\,\frac{2\,r\,(\frac{d}{dr}\,\Phi(r))\,\mathrm{b}(r)+\mathrm{b}(r)-2\,(\frac{d}{dr}\,\Phi(r))\,r^2}{r^3}\,,\,0\,,\,0\right]$$

$$\Big[0\,,\,0\,,\,\frac{1}{2}(-2\,(\tfrac{d}{dr}\,\Phi(r))\,r^2+r\,(\tfrac{d}{dr}\,\Phi(r))\,\mathrm{b}(r)+(\tfrac{d}{dr}\,\mathrm{b}(r))\,r-\mathrm{b}(r)-2\,(\tfrac{d^2}{dr^2}\,\Phi(r))\,r^3$$

$$+\,2\,(\tfrac{d^2}{dr^2}\,\Phi(r))\,r^2\,\mathrm{b}(r)-2\,(\tfrac{d}{dr}\,\Phi(r))^2\,r^3+2\,(\tfrac{d}{dr}\,\Phi(r))^2\,r^2\,\mathrm{b}(r)$$

$$+\,(\tfrac{d}{dr}\,\Phi(r))\,(\tfrac{d}{dr}\,\mathrm{b}(r))\,r^2)/r^3,\,0\Big]$$

$$\Big[0\,,\,0\,,\,0\,,\,\frac{1}{2}(-2\,(\tfrac{d}{dr}\,\Phi(r))\,r^2+r\,(\tfrac{d}{dr}\,\Phi(r))\,\mathrm{b}(r)+(\tfrac{d}{dr}\,\mathrm{b}(r))\,r-\mathrm{b}(r)-2\,(\tfrac{d^2}{dr^2}\,\Phi(r))\,r^3$$

$$+\,2\,(\tfrac{d^2}{dr^2}\,\Phi(r))\,r^2\,\mathrm{b}(r)-2\,(\tfrac{d}{dr}\,\Phi(r))^2\,r^3+2\,(\tfrac{d}{dr}\,\Phi(r))^2\,r^2\,\mathrm{b}(r)$$

$$+\,(\tfrac{d}{dr}\,\Phi(r))\,(\tfrac{d}{dr}\,\mathrm{b}(r))\,r^2)/r^3\Big],$$

$$index\_char = [-1,\,-1]$$

$$])$$

Appendix B

Einstein Tensor for Conformal Wormhole

```
> restart;
```

```
> with(tensor):
```

Create expression for the metric with conformal factor

```
> coord := [t, r, theta, phi]:
```

```
> g := array(symmetric,sparse, 1..4, 1..4):
```

```
> g[1,1] := -Omega(t)^2:
```

```
> g[2,2] := Omega(t)^2/(1-b(r)/r):
```

```
> g[3,3] := Omega(t)^2*r^2:
```

```
> g[4,4] := Omega(t)^2*r^2*sin(theta)^2:
```

```
> MTmetric := create([-1,-1], eval(g)):
```

Compute Riemann tensor

```
> invMTmetric := invert(MTmetric, 'detg'):
```

```
> D1g := d1metric(MTmetric, coord):

> D2g := d2metric(D1g, coord):

> Christoff := Christoffel1(D1g):

> Riem := Riemann(invMTmetric, D2g, Christoff):

> RiemRaised := raise(invMTmetric, Riem, 1):

> RiemannComponents := get_compts(RiemRaised):
> map(proc(x)
> if RiemannComponents[op(x)]<>0 then
> x=simplify(RiemannComponents[op(x)], radical)
> else NULL
> fi end,[indices(RiemannComponents)]):
```

Compute Ricci, Curvature Scalar and Einstein tensor.

```
> RICCI := Ricci(invMTmetric, Riem):

> RS := Ricciscalar(invMTmetric, RICCI):

> Estn := Einstein(MTmetric, RICCI, RS):
```

Find transformation routine to the proper reference frame

```
> h := array(symmetric,sparse, 1..4, 1..4):

> h[1,1] := Omega(t):

> h[2,2] := Omega(t)/sqrt(1-b(r)/r):

> h[3,3] := Omega(t)*r:

> h[4,4] := Omega(t)*r*sin(theta):

> hmetric := create([1,-1], eval(h)):

> invhmetric := invert(hmetric, 'deth'):
```

Rotate Riemann tensor to the proper reference frame

```
> ProperRiemann := change_basis(RiemRaised, hmetric, invhmetric):

> ProperRComponents := get_compts(ProperRiemann):
> map(proc(x)
> if ProperRComponents[op(x)]<>0 then
> x=simplify(ProperRComponents[op(x)], radical, power)
> else NULL
> fi
> end,[indices(ProperRComponents)]):
```

Rotate Einstein tensor to the proper reference frame

```
> ProperEinstein := simplify(change_basis(Estn, hmetric, invhmetric),
> radical, power);
```

$$
\mathit{ProperEinstein} := \mathrm{table}([\mathit{index\_char} = [-1, -1],
$$

$$
\mathit{compts} = \left[\begin{array}{c}
\dfrac{-3\,\%1\,r^2 - \Omega(t)^2\,(\frac{d}{dr}\,\mathrm{b}(r))}{\Omega(t)^4\,r^2}\,,\,0\,,\,0\,,\,0 \\[2ex]
0\,,\,\dfrac{-r^3\,\%1 + 2\,r^3\,\Omega(t)\,(\frac{d^2}{dt^2}\,\Omega(t)) + \Omega(t)^2\,\mathrm{b}(r)}{\Omega(t)^4\,r^3}\,,\,0\,,\,0 \\[2ex]
0\,,\,0\,,\,\dfrac{1}{2}\,\dfrac{-2\,r^3\,\%1 + 4\,r^3\,\Omega(t)\,(\frac{d^2}{dt^2}\,\Omega(t)) + \Omega(t)^2\,(\frac{d}{dr}\,\mathrm{b}(r))\,r - \Omega(t)^2\,\mathrm{b}(r)}{\Omega(t)^4\,r^3}\,,\,0 \\[2ex]
0\,,\,0\,,\,0\,,\,\dfrac{1}{2}\,\dfrac{-2\,r^3\,\%1 + 4\,r^3\,\Omega(t)\,(\frac{d^2}{dt^2}\,\Omega(t)) + \Omega(t)^2\,(\frac{d}{dr}\,\mathrm{b}(r))\,r - \Omega(t)^2\,\mathrm{b}(r)}{\Omega(t)^4\,r^3}
\end{array}\right]
$$

$$
])
$$

$$
\%1 := (\tfrac{d}{dt}\,\Omega(t))^2
$$

Appendix C

Einstein Tensor for Inflating Wormhole

```
> with(tensor):
```

Create expression for the metric

```
> coord := [t, r, theta, phi]:

> g := array(symmetric,sparse, 1..4, 1..4):

> g[1,1] := -exp(2*Phi(r)):

> g[2,2] := exp(2*chi*t)/(1-b(r)/r):

> g[3,3] := exp(2*chi*t)*r^2:

> g[4,4] := exp(2*chi*t)*r^2*sin(theta)^2:

> MTmetric := create([-1,-1], eval(g)):
```

Compute Riemann tensor

```
> invMTmetric := invert(MTmetric, 'detg'):

> D1g := d1metric(MTmetric, coord):
```

```
> D2g := d2metric(D1g, coord):

> Christoff := Christoffel1(D1g):

> Riem := Riemann(invMTmetric, D2g, Christoff):

> RiemRaised := raise(invMTmetric, Riem, 1):

> RiemannComponents := get_compts(RiemRaised):
> map(proc(x)
> if RiemannComponents[op(x)]<>0 then
> x=simplify(RiemannComponents[op(x)], radical)
> else NULL
> fi end,[indices(RiemannComponents)]):
```

Compute Ricci, Curvature Scalar and Einstein tensor.

```
> RICCI := Ricci(invMTmetric, Riem):

> RS := Ricciscalar(invMTmetric, RICCI):

> Estn := Einstein(MTmetric, RICCI, RS):
```

Find transformation routine to the proper reference frame

```
> h := array(symmetric,sparse, 1..4, 1..4):

> h[1,1] := exp(Phi(r)):

> h[2,2] := exp(chi*t)*(sqrt(1-b(r)/r))^(-1):

> h[3,3] := exp(chi*t)*r:

> h[4,4] := exp(chi*t)*(r*sin(theta)):

> hmetric := create([1,-1], eval(h)):

> invhmetric := invert(hmetric, 'deth'):
```

Rotate Riemann tensor to the proper reference frame

```
> ProperRiemann := change_basis(RiemRaised, hmetric, invhmetric):

> ProperRComponents := get_compts(ProperRiemann):
> map(proc(x)
> if ProperRComponents[op(x)]<>0 then
> x=simplify(ProperRComponents[op(x)], radical, power)
> else NULL
> fi
> end,[indices(ProperRComponents)]):
```

Rotate Einstein tensor to the proper reference frame

```
> ProperEinstein := simplify(change_basis(Estn, hmetric, invhmetric),
> radical, power);
```

$ProperEinstein :=$ table([$index\_char$ = [−1, −1],
$compts$ =

$$\left[\frac{(-3\,\chi^2\,e^{(2\,\chi\,t)}\,r^2 - e^{(2\,\Phi(r))}\,(\frac{d}{dr}\,\mathrm{b}(r)))\,\%1}{r^2}\,,\ -2\,e^{(-\chi\,t-\Phi(r))}\,\sqrt{\frac{r-\mathrm{b}(r)}{r}}\,\chi\,(\tfrac{d}{dr}\,\Phi(r))\,,\ 0\,,\ 0\right]$$

$$\left[-\,2\,e^{(-\chi\,t-\Phi(r))}\,\sqrt{\frac{r-\mathrm{b}(r)}{r}}\,\chi\,(\tfrac{d}{dr}\,\Phi(r))\,,\right.$$

$$\frac{\%1\,(3\,\chi^2\,e^{(2\,\chi\,t)}\,r^3 + 2\,r\,(\frac{d}{dr}\,\Phi(r))\,e^{(2\,\Phi(r))}\,\mathrm{b}(r) + e^{(2\,\Phi(r))}\,\mathrm{b}(r) - 2\,(\frac{d}{dr}\,\Phi(r))\,e^{(2\,\Phi(r))}\,r^2)}{r^3}\,,\ 0$$

$$\left.,\ 0\right]$$

$$\left[0\,,\ 0\,,\ \frac{1}{2}\%1(6\,\chi^2\,e^{(2\,\chi\,t)}\,r^3 - 2\,(\tfrac{d}{dr}\,\Phi(r))\,e^{(2\,\Phi(r))}\,r^2 + r\,(\tfrac{d}{dr}\,\Phi(r))\,e^{(2\,\Phi(r))}\,\mathrm{b}(r)\right.$$

$$+\,e^{(2\,\Phi(r))}\,(\tfrac{d}{dr}\,\mathrm{b}(r))\,r - e^{(2\,\Phi(r))}\,\mathrm{b}(r) - 2\,(\tfrac{d^2}{dr^2}\,\Phi(r))\,e^{(2\,\Phi(r))}\,r^3 + 2\,(\tfrac{d^2}{dr^2}\,\Phi(r))\,e^{(2\,\Phi(r))}\,r^2\,\mathrm{b}(r)$$

$$-\,2\,(\tfrac{d}{dr}\,\Phi(r))^2\,e^{(2\,\Phi(r))}\,r^3 + 2\,(\tfrac{d}{dr}\,\Phi(r))^2\,e^{(2\,\Phi(r))}\,r^2\,\mathrm{b}(r) + (\tfrac{d}{dr}\,\Phi(r))\,e^{(2\,\Phi(r))}\,(\tfrac{d}{dr}\,\mathrm{b}(r))\,r^2)/r^3,$$

$$\left.0\right]$$

$$\left[0\,,\ 0\,,\ 0\,,\ \frac{1}{2}\%1(6\,\chi^2\,e^{(2\,\chi\,t)}\,r^3 - 2\,(\tfrac{d}{dr}\,\Phi(r))\,e^{(2\,\Phi(r))}\,r^2 + r\,(\tfrac{d}{dr}\,\Phi(r))\,e^{(2\,\Phi(r))}\,\mathrm{b}(r)\right.$$

$$+\,e^{(2\,\Phi(r))}\,(\tfrac{d}{dr}\,\mathrm{b}(r))\,r - e^{(2\,\Phi(r))}\,\mathrm{b}(r) - 2\,(\tfrac{d^2}{dr^2}\,\Phi(r))\,e^{(2\,\Phi(r))}\,r^3 + 2\,(\tfrac{d^2}{dr^2}\,\Phi(r))\,e^{(2\,\Phi(r))}\,r^2\,\mathrm{b}(r)$$

$$\left.-\,2\,(\tfrac{d}{dr}\,\Phi(r))^2\,e^{(2\,\Phi(r))}\,r^3 + 2\,(\tfrac{d}{dr}\,\Phi(r))^2\,e^{(2\,\Phi(r))}\,r^2\,\mathrm{b}(r) + (\tfrac{d}{dr}\,\Phi(r))\,e^{(2\,\Phi(r))}\,(\tfrac{d}{dr}\,\mathrm{b}(r))\,r^2)/r^3\right]$$

])

$\%1 := e^{(-2\,\chi\,t-2\,\Phi(r))}$

Appendix D

Einstein Tensor for Cosmological Wormhole

```
> with(tensor):
```

Create expression for the metric

```
> coord := [t, r, theta, phi]:
> g := array(symmetric,sparse, 1..4, 1..4):
> g[1,1] := -1:
> g[2,2] := R(t)^2/(1-k*r^2-b(r)/r):
> g[3,3] := R(t)^2*r^2:
> g[4,4] := R(t)^2*r^2*sin(theta)^2:
> MTmetric := create([-1,-1], eval(g)):
```

Compute Riemann tensor

```
> invMTmetric := invert(MTmetric, 'detg'):
> D1g := d1metric(MTmetric, coord):
```

```
> D2g := d2metric(D1g, coord):

> Christoff := Christoffel1(D1g):

> Riem := Riemann(invMTmetric, D2g, Christoff):

> RiemRaised := raise(invMTmetric, Riem, 1):

> RiemannComponents := get_compts(RiemRaised):
> map(proc(x)
> if RiemannComponents[op(x)]<>0 then
> x=simplify(RiemannComponents[op(x)], radical)
> else NULL
> fi end,[indices(RiemannComponents)]):
```

Compute Ricci, Curvature Scalar and Einstein tensor.

```
> RICCI := Ricci(invMTmetric, Riem):

> RS := Ricciscalar(invMTmetric, RICCI):

> Estn := Einstein(MTmetric, RICCI, RS):
```

Find transformation routine to the proper reference frame

```
> h := array(symmetric,sparse, 1..4, 1..4):

> h[1,1] := 1:

> h[2,2] := R(t)*(sqrt(1-k*r^2-b(r)/r))^(-1):

> h[3,3] := R(t)*r:

> h[4,4] := R(t)*(r*sin(theta)):

> hmetric := create([1,-1], eval(h)):

> invhmetric := invert(hmetric, 'deth'):
```

Rotate Riemann tensor to the proper reference frame

```
> ProperRiemann := change_basis(RiemRaised, hmetric, invhmetric):

> ProperRComponents := get_compts(ProperRiemann):
> map(proc(x)
> if ProperRComponents[op(x)]<>0 then
> x=simplify(ProperRComponents[op(x)], radical, power)
> else NULL
> fi
> end,[indices(ProperRComponents)]):
```

Rotate Einstein tensor to the proper reference frame

```
> ProperEinstein := simplify(change_basis(Estn, hmetric, invhmetric),
> radical, power);
```

$$\mathit{ProperEinstein} := \mathrm{table}([\mathit{index\_char} = [-1, -1],$$
$$\mathit{compts} =$$
$$\left[\begin{array}{c} \dfrac{-3\,(\frac{d}{dt}\,\mathrm{R}(t))^2\,r^2 - 3\,k\,r^2 - (\frac{d}{dr}\,\mathrm{b}(r))}{r^2\,\mathrm{R}(t)^2}\,,\,0\,,\,0\,,\,0 \\ 0\,,\,\dfrac{2\,\mathrm{R}(t)\,r^3\,(\frac{d^2}{dt^2}\,\mathrm{R}(t)) + r^3\,(\frac{d}{dt}\,\mathrm{R}(t))^2 + k\,r^3 + \mathrm{b}(r)}{\mathrm{R}(t)^2\,r^3}\,,\,0\,,\,0 \\ 0\,,\,0\,,\,\dfrac{1}{2}\,\dfrac{4\,\mathrm{R}(t)\,r^3\,(\frac{d^2}{dt^2}\,\mathrm{R}(t)) + 2\,r^3\,(\frac{d}{dt}\,\mathrm{R}(t))^2 + 2\,k\,r^3 + (\frac{d}{dr}\,\mathrm{b}(r))\,r - \mathrm{b}(r)}{\mathrm{R}(t)^2\,r^3}\,,\,0 \\ 0\,,\,0\,,\,0\,,\,\dfrac{1}{2}\,\dfrac{4\,\mathrm{R}(t)\,r^3\,(\frac{d^2}{dt^2}\,\mathrm{R}(t)) + 2\,r^3\,(\frac{d}{dt}\,\mathrm{R}(t))^2 + 2\,k\,r^3 + (\frac{d}{dr}\,\mathrm{b}(r))\,r - \mathrm{b}(r)}{\mathrm{R}(t)^2\,r^3} \end{array}\right]$$
$$])$$

Appendix E

Cartan Calculus and Rotating Wormhole

Cartan calculus method is an equivalent of orthonormal basis or tetrad method expressed in the notation of differential forms. It is an alternative to coordinate component method [113]. All the Einstein tensors until now (see previous appendices) used the latter approach, but in the case of rotating wormhole the outcome of that method is quite unwieldy. Cartan calculus approach is explained for example in [79] and is made of following steps:

1. Select appropriate tetrad for given metric as a set $(\Theta^0, \Theta^1, \Theta^2, \Theta^3)$

2. Calculate the exterior derivatives $d\Theta^i$

3. Calculate connection 1-forms ω^i_k from

   $$d\Theta^i = -\omega^i_k \wedge \Theta^k \tag{E.1}$$

   using the relations

   $$\omega^0_\alpha = \omega^\alpha_0 \tag{E.2}$$

$$\omega^{\alpha}_{\beta} = -\omega^{\beta}_{\alpha} \tag{E.3}$$

to simplify the process.

4. Use Cartan structural equations to get curvature 2-forms Ω^i_j

$$\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j \tag{E.4}$$

5. Read off the components of the Riemann curvature tensor from

$$\Omega^i_j = -\frac{1}{2} R^i_{mnj} \Theta^m \wedge \Theta^n \tag{E.5}$$

As an example of application of Cartan calculus to rotating wormhole we consider rotating object described by the metric analogous to "zero tides" case with $K = 1$ and constant angular speed

$$ds^2 = -dt^2 + e^{2\mu} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta (d\phi - \omega dt)^2 \tag{E.6}$$

The tetrad is chosen in following manner

$$\Theta^0 = dt \to dt = \Theta^0 \tag{E.7}$$

$$\Theta^1 = e^{\mu} dr \to dr = e^{-\mu}\Theta^1 \tag{E.8}$$

$$\Theta^2 = r d\theta \to d\theta = \frac{1}{r}\Theta^2 \tag{E.9}$$

$$\Theta^3 = r\sin\theta(d\phi - \omega dt) \to d\phi = \frac{1}{r\sin\theta}\Theta^3 + \omega\Theta^0 \tag{E.10}$$

By use of (E.1) and (E.3) we obtain nonzero connection 1-forms

$$\omega^2_1 = \frac{e^{-\mu}}{r}\Theta^2, \quad \omega^1_2 = -\frac{e^{-\mu}}{r}\Theta^2 \tag{E.11}$$

$$\omega_1^3 = \frac{e^{-\mu}}{r}\Theta^3, \quad \omega_3^1 = -\frac{e^{-\mu}}{r}\Theta^3, \tag{E.12}$$

$$\omega_2^3 = \frac{\cot\theta}{r}\Theta^3, \quad \omega_3^2 = -\frac{\cot\theta}{r}\Theta^3 \tag{E.13}$$

The next step involves calculations of curvature 2-forms by use of (E.4). All the nonzero ones are as follows

$$\Omega_2^1 = -\frac{\mu' e^{-2\mu}}{r}\Theta^1 \wedge \Theta^2, \quad \Omega_1^2 = \frac{\mu' e^{-2\mu}}{r}\Theta^1 \wedge \Theta^2 \tag{E.14}$$

$$\Omega_3^1 = -\frac{\mu' e^{-2\mu}}{r}\Theta^1 \wedge \Theta^2, \quad \Omega_1^3 = \frac{\mu' e^{-2\mu}}{r}\Theta^1 \wedge \Theta^2 \tag{E.15}$$

$$\Omega_3^2 = -\frac{1}{r^2}(1 - e^{-2\mu})\Theta^2 \wedge \Theta^3, \quad \Omega_2^3 = \frac{1}{r^2}(1 - e^{-2\mu})\Theta^2 \wedge \Theta^3 \tag{E.16}$$

where prime denotes differentiation with respect to r coordinate. Those results, together with the fact that for Riemann curvature tensor $R^d_{abc} = -R^d_{bac}$ yield

$$R^1_{122} = \frac{2\mu' e^{-2\mu}}{r}, \quad R^1_{212} = -\frac{2\mu' e^{-2\mu}}{r} \tag{E.17}$$

$$R^2_{121} = -\frac{2\mu' e^{-2\mu}}{r}, \quad R^2_{211} = \frac{2\mu' e^{-2\mu}}{r} \tag{E.18}$$

$$R^1_{133} = \frac{2\mu' e^{-2\mu}}{r}, \quad R^1_{313} = -\frac{2\mu' e^{-2\mu}}{r} \tag{E.19}$$

$$R^3_{131} = -\frac{2\mu' e^{-2\mu}}{r}, \quad R^3_{311} = \frac{2\mu' e^{-2\mu}}{r} \tag{E.20}$$

$$R^2_{233} = -\frac{2}{r^2}(1 - e^{-2\mu}), \quad R^2_{323} = \frac{2}{r^2}(1 - e^{-2\mu}) \tag{E.21}$$

$$R^3_{232} = \frac{2}{r^2}(1 - e^{-2\mu}), \quad R^3_{322} = -\frac{2}{r^2}(1 - e^{-2\mu}) \tag{E.22}$$

Now we can obtain Ricci tensor components from the fact that $R_{ab} = R^d_{adb}$. In our case the diagonal elements of R_{ab} are

$$R_{00} = R^0_{000} + R^1_{010} + R^2_{020} + R^3_{030} = 0 \tag{E.23}$$

$$R_{11} = R^0_{101} + R^1_{111} + R^2_{121} + R^3_{131} = -\frac{4\mu' e^{-2\mu}}{r} \tag{E.24}$$

$$R_{22} = R^0_{202} + R^1_{212} + R^2_{222} + R^3_{232} = \frac{2}{r^2}[1 - (1 + \mu' r)e^{-2\mu}] \tag{E.25}$$

$$R_{33} = R^0_{303} + R^1_{313} + R^2_{323} + R^3_{333} = \frac{2}{r^2}[1 - (1 + \mu' r)e^{-2\mu}] \tag{E.26}$$

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