

Abbreviations and Acronyms

APOS	Action, Process, Objects and Schemas
B.C.E	Before the Common Era (just the same as B.C. meaning Before Christ)
CAPS	Curriculum Assessment Policy Statement
CASS	Continuous Assessment
D9	District 9
DoE	Department of Education
FATHOM	A dynamic statistical computer software that analyses data
FET	Further Education and Training
GET	General Education and Training
NCS	National Curriculum Statement
NSC	National School Certificate
Procept	Processes and concept
SPSS	Statistical Package for Social Sciences
UCLES	University of Cambridge Local Examination Syndicate
ZPD	Zone of Proximal Development

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CHAPTER ONE

Introduction

This chapter introduces the research study and the context in which it is engendered. The context of the study is described in (1.1), the background is outlined in (1.2 and the central problem of the study is described in (1.3)). The purpose of the study, research questions, aims, significance and assumptions follow in sections (1.4), (1.5), (1.6), (1.7) and (1.8) respectively. An analysis of the key concepts is presented in (1.9) and the chapter concludes laying out the structure of the thesis in (1.10).

1.1 THE CONTEXT OF THE STUDY

The new political dispensation in South Africa, after the first democratic elections in 1994, and the move from the defeated Apartheid regime to an inclusive constitutional democracy, brought in several changes of the curriculum for learners at primary and secondary school levels. The most recent of these changes was the introduction, in January 2012, of the Curriculum Assessment Policy Statement (CAPS), an amendment to the National Curriculum Statement (NCS), which had originally come into effect in 2008 (Department of Education (DoE), 2002; 2008). The new policy was intended for the curriculum to be more accessible to teachers and it replaced learning outcomes and assessment standards with topics, aims, objectives and skills (DoE, 2012b) among other things. In mathematics, the topic of linear programming was completely removed from the syllabus and sections of analytical geometry and Euclidian geometry previously examined in the third and elective of the three examination papers was merged into the two compulsory examination papers. This resulted in the topic of circle geometry and some sections of analytical geometry, which were formerly elective, becoming compulsorily. The topics of functions

and transformation of functions remained unchanged, both in content and organization. These two topics are dealt with separately in the Intermediate and Senior phases of the General Education and Training (GET)¹ band. In the Intermediate phase, the topic of functions covers input and output values in function (or flow) diagrams and the topic of transformations covers reflection, translation, rotation, enlargement and reduction, while salient concepts like line symmetry and tessellations are highlighted. These concepts are dealt with in the contexts of triangles and quadrilaterals. During the Senior phase, the study of functions involves input and output values in the function diagram, as well as rules for patterns and their equivalent forms such as verbal, tables, formulae, number sentences, equations and Cartesian plane graphs. The last two equivalent forms are introduced in Grades 8 and 9 respectively. The topic of transformations covers line symmetry of geometrical shapes, reflection, translation, rotation, enlargement and reduction. Learners are required to perform transformations on squared papers.

During the Further Education and Training (FET)² band, functions are studied in greater depth and their types and characteristics are highlighted. Transformations no longer make up a separate topic but are studied as they manifest as transformations of functions. The concepts are the same as those dealt with in the previous grades, except for the introduction of stretch/compression. The objects that result when transformations of functions are mapped differ in the higher grades. It is necessary that learners understand the concept of transformation of two-dimensional shapes first before they can apply these transformations to functions. In the old NCS syllabus, learners covered the three isometric transformations (reflection, translation and rotation), and enlargement in triangles during the GET Senior and FET phases, and transformation of functions during the FET phase. The current NCS–CAPS syllabus for the GET Senior phase still covers transformations such as reflection, translation, rotation, enlargement and reduction in two-dimensional shapes. These transformations are more comprehensive and prepare learners more thoroughly for

¹ The GET band covers the first 10 grades of school education and consists of the Foundation phase (Grades R to 3), the Intermediate phase (Grades 4 to 6), and the senior phase (Grades 7 to 9).

² The FET band is the second level of education. It consists of Grades 10, 11 and 12.

functions during the FET phase. It no longer has transformations as a separate topic for the FET phase. However, a shortcoming of the current NCS–CAPS syllabus is that it does not cover the concept of stretch before it is applied to functions. Teachers need to introduce stretch to learners before they can move on to deal with mapping of functions. Not all teachers do so competently.

1.1.1 GET - NCS – CAPS Syllabi Relative Objectives

During the GET Senior phase, learners are introduced to transformations of plane shapes. This topic lays the foundation for transformations of functions because, logically, one cannot transform something without understanding what transformation is. The NCS–CAPS syllabus objectives for transformations for the 3 grades of the GET Senior phase, are outlined below:

Grade 7

For a learner to be considered adequately competent, he/she must be able to:

- recognize, describe and perform reflection, translations and rotations with geometric figures and shapes on squared paper;
- identify and draw lines of symmetry in geometric figures;
- draw enlargements and reductions of geometric figures on squared paper and compare them in terms of shape and size.

(DoE, 2012a (NCS-CAPS-Senior), pages 57-8).

Grade 8

For a learner to be considered adequately competent, he/she must be able to:

- recognize, describe and perform transformations with points on a co-ordinate plane, focusing on:
 - reflecting a point about the x-axis or y-axis;
 - translating a point within and across quadrants;

- recognize, describe and perform transformations with triangles on a co-ordinate plane, focusing on the co-ordinates of the vertices when:
 - reflecting a triangle about the x-axis or y-axis;
 - translating a triangle within and across quadrants;
 - rotating a triangle around the origin;
- use proportion to describe the effect of enlargement or reduction on the area and perimeter of geometric figures.

(DoE, 2012a (NCS-CAPS-Senior), 2012 page 105-6).

Grade 9

For a learner to be considered adequately competent, he/she must be able to:

- recognize, describe and perform transformations with points, line segments and simple geometric figures on a co-ordinate plane, focusing on:
 - reflection about the x-axis or y-axis;
 - translation within and across quadrants;
 - reflection about the line $y = x$;
- identify what the transformation of a point is, if given the co-ordinates of its image;
- use proportion to describe the effect of enlargement or reduction on the area and perimeter of geometric figures;
- investigate the co-ordinates of the vertices of figures that have been enlarged or reduced by a given scale factor.

(DoE, 2012a (NCS-CAPS-Senior), pages 120-1; 132; 137-8).

1.1.2 FET - NCS – CAPS Syllabi Related Objectives

During the FET phase, some aspects of transformation geometry are no longer studied as a separate topic, but the ideas or concepts are explicitly applied when mappings functions, i.e. in transformations of functions. The topic of transformation of functions is approached with verbal descriptions, graphical representations on the Cartesian plane and symbolical representations using algebraic formulae or coordinate mapping.³ As mentioned above, a learner does not come across the concept of stretch, or the component of dilation and compression, in the GET Senior phase before it is applied to functions in the FET phase. This is a gap that needs to be bridged by future curriculum developers.

The NCS–CAPS syllabus objectives for the 3 grades of the FET phase, are outlined below:

Grade 10

At the end of the topic, a competent Grade 10 learner should be able to state the effects of different values of a , and q on the function equations of:

- a straight line $f(x) = ax + q$;
- a hyperbola $g(x) = a \frac{k}{x} + q$;
- a parabola $h(x) = ax^2 + q$;
- an exponential function $i(x) = ab^x + q (b > 0)$;
- and trigonometric functions $j(x) = a \sin x + q$, $k(x) = a \cos x + q$ and $l(x) = a \tan x + q$.

(DoE, 2012b (NCS-CAPS-FET), 2012 page 25).

³ Neither syllabus specifically prohibits the use of matrix operators and computer programmes. Some teachers might use them as expanded opportunities.

Grade 11

At the end of the topic, a competent Grade 11 learner should be able to:

- state the effects of the parameters k and p on graphs of $y = f(kx)$ and $y = f(x + p)$ for various functions $y = f(x)$;
- identify the characteristics of various functions and draw sketch graphs;
- identify the equations of graphs from given information;
- interpret sketch graphs.

(DoE, 2012b (NCS-CAPS-FET), 2012page 33).

Grade 12

The Grade 12 content does not include transformations of functions, but they are continually assessed as learners head for the summative examinations.

1.1.3 Aggregating the NCS – CAPS Syllabi Objectives for Transformations of Functions and the Aims of the Mathematics in the FET Phase.

In the process of conceptualizing and developing skills for the process of transformation of functions, learners are expected to demonstrate knowledge and skills:

- Adding a constant to any function $f(x)$ to get $f(x) + c$ translates its graph vertically upward by c units with no change in shape.
- Subtracting a constant from any function $f(x)$ to get $f(x) - c$ translates its graph vertically downward by c units with no change in shape.
- Multiplying any function $f(x)$ by a constant a to get $af(x)$ stretches its graph vertically by a factor a (a is a positive integer).

- Multiplying any function $f(x)$ by a constant $\frac{1}{a}$ to get $\frac{1}{a}f(x)$ compresses its graph vertically by a factor $\frac{1}{a}$.
- Multiplying the independent variable, x , in a function $f(x)$ by a to get $f(ax)$ compresses its graph horizontally by a factor $\frac{1}{a}$.
- Multiplying the independent variable, x , in a function $f(x)$ by $\frac{1}{a}$ to get $f(\frac{1}{a}x)$ stretches its graph horizontally by a factor a (a is a positive integer).
- In general the effect of multiplying by $a(a \neq 1)$ in any equation $y = f(x)$ to get $y = af(x)$ stretches (or compresses) $y = f(x)$ vertically with x-axis invariant.

The above-mentioned skills are exemplified in the following statements:

- If a is positive, then the resultant graph is just stretched. For example multiplying $y = 2^{-x}$ in $f(x)$ by 4 to give $y = g(x) = 4[2^{-x}]$ has the effect of stretching the graph of $y = f(x)$ vertically, with the x-axis invariant, by the factor 4. This is true for all $x = n$, $g(n) = 4f(n)$.
- If a is negative, then the resultant graph is first stretched and then reflected about the x-axis before stretching. For example, multiplying $y = x^2$ by -2, the graph $y = -2x^2 + 1$ is the graph $y = x^2$ stretched vertically by a factor 2, moved upwards by 1 and then reflected about $y = 1$ (in other words, stretched by a factor 2, reflected about $y = 0$ and moved up by 1 unit) (Laridon, Barnes, Jawurek, Kitto, Pike, Myburgh, Rhodes-Houghton, Scheiber, Sigabi, & Wilson, 2006, pages 95–128; Pike, Barnes, Jawurek, Kitto, Laridon, Myburgh, Rhodes-Houghton, Sasman, Scheiber, Sigabi, & Wilson., 2011a, pages 157–19; Pike et al, 2011b).
- A special case is multiplying $y = \cos x$ by a to get $y = a \cos x$ which has the effect of stretching $y = \cos x$ vertically, with $y = 0$ (the x-axis) invariant, by factor a . For example multiplying x by 2 to get $y = 2 \cos x$ has the effect of stretching

(compressing) the graph of $y = \cos x$ horizontally by factor $\frac{1}{2}$ (factor 2), and multiplying x by $\frac{1}{2}$ to get $y = \frac{1}{2} \cos x$ has the effect of stretching (compressing) the graph of $y = \cos x$ horizontally by 2 (factor $\frac{1}{2}$).

- The effect of a in the equation $y = f(ax)$ is to stretch (to compress) $y = f(x)$ horizontally by factor $\frac{1}{a}$ (by factor a). If a is negative, then the resultant graph is reflected about the y-axis. For example the statements hold for the graphs $y = \sin x$, $y = 2 \sin x$ and $y = \sin 2x$.
- The graphs of $y = a f(x)$ and $y = -a f(x)$ are mirror images (reflections) of each other about the x-axis and $y = f(ax)$ and $y = f(-ax)$ are mirror images of each other about the y-axis.
- The combined effect of a , b and q in $y = a f(bx) + q$ is a vertical stretch of $y = f(x)$, with the x-axis invariant, by the factor a , then a horizontal stretch, with the y-axis invariant, by a factor b , followed by a vertical translation of q units (Laridon et al, 2006, pages 95–128; Pike et al, 2011a, pages 157–194).

The teaching of transformations of functions is in line with the aims of the NCS–CAPS–FET curriculum document which stresses that learners should be exposed to mathematical experiences that give them the opportunity to develop mathematical reasoning and creative skills in preparation for the abstract mathematics they will encounter in university courses. It is imperative, therefore, that teachers and educators:

- include the description of graphical relationships and representation of mathematical objects;
- help learners to develop mental processes that enhance logical reasoning, critical thinking, accuracy and problem solving that will contribute to decision-making;

- teach learners the ‘how’, ‘when’ and ‘why’ to support learning procedures and proofs with a good understanding of why they are important, and leave learners well equipped to use their knowledge in later life;
- develop mathematical language skills and terminologies for analysis, evaluation and critiquing of conclusions;
- develop mathematical process skills like identifying, investigating, problem solving, creativity and critical thinking;
- teach learners to use spatial skills and properties of shapes and objects to identify, pose and solve problems creatively and critically;
- teach learners to communicate appropriately by using descriptions in words, graphs, symbols, tables and diagrams.

All these aims should be taken into consideration when developing the topic of transformations of functions for the FET phase.

According to the constructivist theory, when learning a concept, learners reconstruct the knowledge about that concept to their level of understanding and the resultant knowledge structure is not always the exact replica of concept definition stated in the books but is something related to it. This new structure may be correct, incorrect or somewhere in-between, but it forms the learner’s concept image. The learner’s cognitive system now uses that concept image to work out the cognitive tasks without consulting the original concept definition. An incorrect concept image may give rise to an undesirable solution to the problem, while a correct concept image could be expected to give a desirable solution. It is necessary, therefore, that educators investigate what learners may have as concept images, in order to facilitate the formation of appropriate concept images. An example of this type of facilitation would be to give learners many correct examples of solutions to a problem, with the expectation that one of the examples will be picked up as the learner’s concept image. This could lessen the likelihood of misconceptions about the concept being formed. When learning about transformations of functions, for example, understanding could be improved if a teacher works through many correct examples of reflection,

translation and stretch with the learners, to facilitate the formation of appropriate images of these concepts in learners' minds.

1.2 THE BACKGROUND OF THE STUDY

This research study looks at how Grade 11 learners interpret the effects of parameters in the transformation of function $y = f(x)$ into the function $y = af(x + p) + q$, in a step-by-step, simple-to-complex manner. It focuses on how learners understand the transformation concepts of reflection, translation and stretch that are covered in the FET mathematics curriculum, and attempts to determine what concept images learners have about these concepts as they manifest on functions. The act of transforming (mapping) a function and identifying a transformation responsible for mapping a function are some of the most crucial skills that learners should acquire in mathematics during the FET phase of the South Africa school system and at equivalent levels of education elsewhere in the world. These skills are, in most instances, not easy to master. In the South African context, the acquisition of such skills should begin in Grade 10, the beginning of FET syllabus, which is where learners are first introduced to functions and their transformations. Transformation of functions is one of the topics where concepts and skills are continuous in three successive grades. The skills taught in Grade 10 are developed and reinforced in Grade 11 and Grade 12, in preparation for the national terminal/summative examination for matriculation. As the Grade 12 examinations are both a basis for skilled and semi-skilled employment and a springboard to tertiary education, successful learners are expected to have mastered basic concepts and skills like that of transformations of functions in order to be able to master the more advanced concepts they will then encounter in further education and training.

It is worrying that the Department of Education examiners report during roadshow⁴ presentations, year after year, that examination candidates performed poorly in

⁴ Report-back sessions where Grade 12 teachers are given feedback on how learners performed in the preceding year's summative examinations.

transformation-related topics in the national summative examinations (DoE, 2011; 2012; 2013; 2014). The Analysis of Candidates' Responses published by the Department of Education cites the inability to interpret graphical representations, inability to do graphical representations (DoE, 2013; 2014), inability to identify functions, incomplete description of transformations, incorrect verbal statement of rules, and an inability to differentiate between various transformations (DoE, 2013) as some of the main problem areas where skills are lacking, as observed from exam scripts. Learners rewrote words from the question as reasons (confused), used brackets improperly, and made mistakes with directions (left, right, up or down) or units (DoE, 2014). The reasons suggested for these shortcomings were language barriers, lack of theoretical understanding of basic concepts involved, and a lack of courage to attempt higher order questions (DoE, 2013; 2014). Several recommendations were made, such as educators stressing the rules of transformation, doing practical examples of transformations, emphasizing the notation, linking transformations to graphs, exposing learners to all aspects of this section (including sketching, interpreting equations and graphs, emphasizing shifted functions etc.), testing theory through questioning, repeated testing, and emphasis on teaching the 'why, what, if, how and when' (DoE, 2013; 2014). Frustrated with poor performance, the examiners suggested the re-training of current mathematics teachers in this topic as a necessary intervention. They foresee that re-training would enable teachers to improve their learners' acquisition of competence skills in mathematical processes, logical reasoning and creative thinking, as elaborated in the Mathematics Learning Area Statement (DoE, 2002; 2012b). This poor performance is of great concern to educators since mathematical proficiency among their learners is a compelling necessity. The purpose of this study is to investigate the barriers to mastery of the concepts and skills involved in transformations of functions faced by Grade 11 mathematics learners.

As a mathematics educator in one of the South African high schools, this researcher has noticed the challenges some learners encounter with transformations of functions. Apart from some learners showing attributes of having short concentration spans, many of them seem to concentrate in learning procedures to be followed before understanding concepts behind those procedures. It has proved extremely difficult, in my experience, to enable

many of the learners at FET band to understand concepts relationally and acquire the problem - solving skills required for transformations of functions. This study stems from our concern with the difficulties experienced by learners, and we hope that it will make a meaningful contribution to the debate around how students grapple with understanding concepts of transformations of functions, and how the resultant thinking and reasoning influences their use of those concepts, given that transformations of functions occupy a significant space in both continuous assessment (CASS)⁵ and summative assessment of FET mathematics in South Africa.

1.2.1 Weighting of transformation of functions

The process of transformation of a function entails the mapping of the whole function or certain points of the function from their original positions onto new positions or images, using some well-defined rule. Transformation-related topics contribute significantly to both the FET continuous assessment (CASS) and the summative National Senior Certificate (NSC)(Matriculation) examination for the FET band. The assessment guidelines and examination projection for the old NCS (DoE, 2008; 2009a) listed contributions of about 0.23 for Paper 1 and about 0.2 for Paper 2. The current NCS–CAPS–FET syllabus examines transformations of functions in Paper 1 (*Functions and Graphs*) and in Paper 2 (*Trigonometry*). The first NCS–CAPS examinations were due to be written in 2014. The image on Figure 1.1 below shows a screenshot, extracted from the CAPS document (DoE, 2012b p 55), of the weighting of topics in the current CAPS assessment guideline and examination projection:

⁵ Continuously planned assessment through a process of identifying, gathering and interpreting information about the performance of learners, using various forms. It involves four steps: generating and collecting evidence of achievement; evaluating this evidence; recording the findings and using this information to understand and assist in the learner’s development to improve the process of learning and teaching (DoE, 2012b p 51).

Figure 1.1: The weighting of topics in the NCS-CAPS syllabus.

In Grades 10, 11 and 12, 25% of the final promotion mark is a year mark and 75% is an examination mark.

All assessments in Grade 10 and 11 are internal while in Grade 12 the 25% year mark assessment is internally set and marked but externally moderated and the 75% examination is externally set, marked and moderated.

Mark distribution for Mathematics NCS end-of-year papers: Grades 10-12

PAPER 1: Grades 12: bookwork: maximum 6 marks

Description	Grade 10	Grade 11	Grade 12
Algebra and equations (and inequalities)	30 ± 3	45 ± 3	25 ± 3
Patterns and sequences	15 ± 3	25 ± 3	25 ± 3
Finance and growth	10 ± 3		
Finance, growth and decay		15 ± 3	15 ± 3
Functions and graphs	30 ± 3	45 ± 3	35 ± 3
Differential Calculus			35 ± 3
Probability	15 ± 3	20 ± 3	15 ± 3
TOTAL	100	150	150

PAPER 2: Grades 11 and 12: theorems and/or trigonometric proofs: maximum 12 marks

Description	Grade 10	Grade 11	Grade 12
Statistics	15 ± 3	20 ± 3	20 ± 3
Analytical Geometry	15 ± 3	30 ± 3	40 ± 3
Trigonometry	40 ± 3	50 ± 3	40 ± 3
Euclidean Geometry and Measurement	30 ± 3	50 ± 3	50 ± 3
TOTAL	100	150	150

Note:

- Modelling as a process should be included in all papers, thus contextual questions can be set on any topic.
- Questions will not necessarily be compartmentalised in sections, as this table indicates. Various topics can be integrated in the same question.
- Formula sheet must not be provided for tests and for final examinations in Grades 10 and 11.

4.5 Recording and reporting

- Recording is a process in which the teacher is able to document the level of a learner's performance in a specific assessment task.
 - It indicates learner progress towards the achievement of the knowledge as prescribed in the Curriculum and Assessment Policy Statements.

[Extracted from the CAPS document (DoE, 2012b p 55)]

It can be seen that transformations of functions forms part of the topics *Functions and Graphs* and *Trigonometry*. The weightings are 35% ± 3% and 40% ± 3% in Papers 1 and 2 respectively.

The NCS curriculum had some minor variations of the calculated contributions for the period 2008 to 2013. The calculated contributions of functions, graphs and transformations (excluding differential calculus and cubic functions) are shown in the table below:

TABLE 1.1: The calculated weighting of functions, graphs and transformations in some National Senior Certificate (NSC) examinations papers in South Africa

Year	Examination	Paper 1	Paper 2	Combined
2008	November	0.31	0.22	0.27
2009	November	0.32	0.25	0.28
2010	November	0.37	0.25	0.31
2011	November	0.35	0.28	0.31
2012	November	0.38	0.17	0.28
2013	November	0.27	0.26	0.26
2014	November	0.39	0.12	0.25

Table 1.1 shows the significant contribution of functions, graphs and transformations (of between 25% and 31% overall) to the national school certificate examinations in South Africa. It is important to observe that in 2014, the first assessment under the NCS–CAPS syllabus, the major contribution is found in Paper 1, and the contribution to Paper 2 has decreased. It is against this background that this study values the topics and intends to investigate how learners interpret the transformation concepts of reflection, translation and stretch, and how they use these forms of transformation to map functions. This examination is intended to reveal different learners’ mathematical thinking and reasoning, and deduce how such thinking and reasoning influences their understanding of the concepts in question.

1.3 THE PURPOSE STATEMENT

The purpose of this study is to investigate how learners develop their understanding of transformations of functions and to analyse their understanding of the concepts involved. In particular, the study investigates how Grade 11 learners understand reflection, translation and stretch as isolated concepts, and how they understand these concepts as they manifest in the transformation of functions through their representations. Researches done about problems of understanding and conceptualisation in mathematics mostly focused on university students (see section 3.2). From this juncture, the fact that students at university have such problems with understanding, conceptualisation and mathematical reasoning highlights the need for interventions to begin at an earlier stage in their education, thus the reason why this research study focuses on secondary school learners is that the necessary strategies can be implemented before students enter university. Grade 11 was preferred against Grade 10 for their longer period of exposure to transformations, functional graphs and effects of parameters on transforming functions and against Grade 12 the latter were perceived to be busy due to preparation for matriculation through school based assessments and later the terminal / summative examinations. We intend to identify various concept images and to highlight misconceptions about the targeted concepts of transformation and mathematical reasoning within the learners' conceptual frameworks about transformation of functions. It is necessary to extend the understanding of knowledge structures that are created in learners' minds as they learn, interpret and represent concepts so as to develop appropriate intervention strategies to effectively correct learner misconceptions. Interventions should assist learners in arriving at the correct conceptual understanding (Mathematics Learning Study Committee, 2001) of the transformation concepts. Transformations of functions and their symbolical and graphical representations play an important part in mathematics and its applied disciplines, so a correct understanding of the concepts is important to learners' success while at school and later in tertiary education.

1.4 THE PROBLEM STATEMENT

As discussed in section 1.2, mathematics learners encounter many challenges in interpreting the effects of parameters that bring about transformations of functions from $y = f(x)$ to $y = af(x + p) + q$. Cognitive conflict could be a factor in learners' memories where the concept images of the mappings reflect differently from what is implied by the formal concept definitions (Tall & Vinner, 1981). This research aims to uncover what types of concept images learners have. There is some doubt as to whether academically challenged learners, particularly those whose home language is different from the language of education, interpret and use the concepts involved (reflection, translation and stretch) as they are meant to, whether they think and reason proficiently in mathematics, and whether, in the end, they construct a coherent and correct view of these mathematical concepts or just take them as separate pieces of abstract knowledge. Basing on personal constructivism or *cognitive constructivism* theory (Piaget, 1963), children develop personal cognitive structures and capabilities as they learn, which help them construct their own understanding of concepts in different ways such as through exploring, observing, listening, touching etc. This guarantees the existence of concept images when learning such concept as reflection, translation and stretch (including compression) of function for they have examples in the empirical world, e.g. your image as you look at yourself in the mirror and flipping pages when reading a book for a reflection, sliding objects on a conveyor belt for translation and stretching stockings as you fit your feet and sheen, just to mention a few. Textbooks and teachers very often present these and other mathematical concepts abstractly and without connections with other previously learnt concepts or to the empirical world in which we live. Learners frequently try to simplify the concepts, to their level of understanding, in their own language. The possibility of misconceptions arising in this process is great. The unfamiliar abstractness and subject specific rigor possibly lead learners to think that mathematics is a difficult learning area/subject. The learners' concept images are the main constitution of this study. The magnitudes of their variations from formal definitions may determine the presence of cognitive conflict factors. Learners who have such potential cognitive conflict factors in

their concept images may be challenged by the formal theory and may find it difficult to operate correctly with the theory. Very often Mathematics knowledge is presented in books abstractly and rigorously and as without connections with other concepts and the empirical world we live in. It is prudent for mathematics learners to have their own interpretations and use their own language to explain and reason out the concepts other than just try to stick to the abstract and rigorous ways of presentations used in textbooks. The abstractness and rigor may produce challenges for learners and make them think that Mathematics is a difficult subject.

The identification and analysis of learners' concept images is the major component of this study. The magnitude of their possible variations from formal definitions may indicate the presence of cognitive conflict factors. Learners who have potential cognitive conflict factors in their concept images may be challenged by the formal theory and may find it difficult to apply the theory correctly. It is important for mathematics learners to develop their own interpretations and use their own language to explain and reason out the concepts being studied rather than just trying to stick to the abstract and rigorous method of presentation used in textbooks. Although some research has been done on learners' concept images and informal reasoning related to mathematical concepts (Tall & Vinner, 1981; Viholainen, 2008), research has not been carried out on the concept images of high school learners and in the field of transformation concepts applied to functions (See section 3.2). Tall & Vinner, and Viholainen's studies focused on university students in terms of limits and continuity, and derivation and differentiability, respectively. The literature review revealed, for example, a similar study done in the Netherlands, but this focused on a single learner as he used pencil and paper visualisations and the analysis of concepts and computer aided displays (Borba and Confrey, 1996).

The summative examiners for South African NSC Matriculation examinations, during road show report-back sessions, drew attention to poor performance by learners in South Africa generally, and in Gauteng Province in particular, with respect to transformation geometry and related topics. They recommended in-service training in the topics for current mathematics teachers as a possible way to improve results in future (DoE, 2011). Teacher training could facilitate learner acquisition of competence skills in mathematical

processes, reasoning and creative thinking (DoE, 2008). Tall (1991) argues that learners must develop their own approaches to mathematics learning that facilitate their intellectual growth and formation of knowledge structures and that take account of the thinking process they have. According to Pinto and Tall (2002), concept images can help learners compress information into single tables, which they can invoke later when recalling concept definitions. This scholarship highlights the need to tap into learners' concept images and mathematical reasoning with regards to the process of translating, reflecting and stretching functions.

1.5 THE RESEARCH QUESTIONS

This research study addressed the following research question:

What are Grade 11 mathematics learners' concept images and what is their mathematical reasoning on transformations of functions?

The main research question was examined by means of *exploratory* and *descriptive* research directions, each containing a number of sub-questions:

The *exploratory* direction addressed the following:

- *What are the mathematics learners' verbal, graphical and symbolical images of reflection, translation and stretch of functions?*
- *What are the reasons given by learners to justify their concept images?*

These exploratory sub-questions attempted to answer the following specific mini-questions:

- What are the concept images of reflection of functions?
- What are the concept images of translation of functions?
- What are the concept images of the stretch of functions?

An exploration of the variables of the category type was conducted to address these questions. The objective was to identify images (verbal, graphical or symbolical) of each of the three transformations and assess how competent the learners were in dealing with the transformations.

The *descriptive* direction addressed the following:

- *Are the learners' concept images and mathematical reasons coherent and representative of formal definitions?*

This sub-question also attempted to answer the following mini-questions:

- To what extent are the learners' concept images competently representative of the formally defined concepts?
- How are the learners' concept images related to formal definitions of these three concepts? (Are there contradictions or not?)
- Does the learners' reasoning about concept images relate to the formal concept definitions?

To answer these questions, learners' competences to use representations were addressed, as was their ability to argue or reason formally or informally, explain, interpret formal definitions, and use their interpretation successfully in reasoning. Did they have problem solving abilities, were the arguments precise or more explicit, and were the conceptions of transformations clear? Did learners have the ability to use interpretations successfully in their reasoning? Did they use formal or informal definitions or use both simultaneously? Did they make the correct use of visuals?

1.6 THE AIMS OF THE STUDY

The aims of this research study were to:

- investigate and assess learners' competencies in defining, identifying and representing reflection, translation and stretch of functions;

- investigate the concept images that students have built after learning, interpreting and representing concepts of reflection, translation and stretch of functions;
- assess how coherent learners' concept images are with reference to the formal definitions of reflection, translation and stretch of functions;
- assess how learners use their concept images to explain, justify, argue and reason in the processes of reflecting, translation and stretching functions;
- assess the link between learners' explanations, justifications, arguments and reasoning using their concept images and those given by formal definitions in the processes of reflecting, translation and stretching functions.

The study was aimed at establishing what concept images learners form as a result of learning transformations of functions, and what mathematical skills, abilities and reasoning learners acquired when dealing with representations or illustrations of those concepts. It needed to compare learners' understanding and reasoning of transformation concepts with those implied by the formal definitions of the concepts, and then suggest how classroom activities around learning such concepts could be improved.

1.7 THE SIGNIFICANCE OF THE STUDY

This research study is intended to benefit mathematics educators, mathematics student teachers, mathematics teacher educators and, to some extent, educationists and mathematicians. Kilpatrick (1993) states that an educational research study may belong to one or more of the following three categories: those studies that attempt to have a direct influence on teaching practices by providing ideas and material for teachers to use and suggesting activities teachers might conduct; those that suggest new ways to understand students' thinking and events in the classroom (indirect influence); and those that attempt to develop the terms and the framework in which mathematics education is portrayed in publications (also indirect influence). This study fits best into the second category, as highlighted by its aims, but it also fits into the first category, as will be outlined by the

recommendations. The results of this study were expected to contribute to the theory of learning transformations of functions, as well as to the theory of teaching more generally, by highlighting how learners interpret translation, reflection and stretch in transforming functions, and also by identifying the misconceptions some learners might have when working with the concepts. Like all good educational research, it has both a practical and a theoretical relevance, vis-à-vis the practice of teaching the concepts of reflection, translation and stretch of functions by broadening or deepening the understanding of how learners learn (Sierpiska, 1993). The research outcomes could contribute to relational mathematics learning and realistic mathematics education. Results from the study may also provide insight for mathematics teacher educators designing programmes to enable student teachers to improve their knowledge of, and ability to teach, transformations of functions. The researcher also anticipated building new knowledge about concept images of reflection, translation and stretching of functions.

1.8 ASSUMPTIONS OF THE STUDY

The study is based on the assumption that correct mathematical statements are partly determined by identifying them with formal axioms, as in accordance with the ideals of Hilbert (1862 - 1943) which value objectivity, abstractness and independence of empirical reality in mathematical concepts. On the other hand, construction of knowledge by learners is not independent of empirical reality, as in accordance with the paradigms of relativism and socio-constructivism. In this study, formal definitions would be taken as the standard against which learners' views about concepts and their associated reasoning would be evaluated. Any concept definition or concept image that is different from the formal definition and formal representation will be considered to be a misconception.

1.9 DESCRIPTIONS OF KEY TERMS AND CONCEPTS

The following are the key terms used in this study: concept image; mathematical reasoning; coherence of concept images; functions; functional representations; transformation of functions; reflection; translation and stretch.

1.9.1 Concept Image

A *concept image* is “all the cognitive structure in the individual’s mind that is associated with a given concept” (Tall & Vinner, 1981:151). It may be a collection of vague conceptions about that concept, with or without connections to the formal concept definition. The concept image may also be in the form of mental images or interpretations based on representations or other properties or processes involved in the manifestations of the concept (Viholainen, 2008). The concept images of transformation of functions (reflection, translation and stretch) are the prime focus of this study (see sections 2.1 and 2.2).

1.9.2 Mathematical Reasoning

Mathematical reasoning is the individually created meaning or interpretation of a concept by extrapolation from an existing knowledge structure (Viholainen, 2008). This creation or interpretation of a concept is dependent on the context in which the concept is used. Mathematical reasoning is a specialized informal reasoning focused on mathematical concepts for mathematically-orientated scholars. It may consist of illusions of mathematical concepts or an attempt to represent mathematical concepts by other real life concepts. Viholainen (2008) views informal reasoning as resulting from visual or physical interpretations of mathematical concepts, and formal reasoning as exact reasoning based on axioms, definitions and theorems. This view is under examination in this study.

1.9.3 Coherence of Concept Image

Coherence of concept image is the way a concept image is organized and linked to formal definitions. According to Viholainen (2008), highly coherent concept images have the following attributes:

- There is a clear personal conception about the concept.
- There are well-connected conceptions, representations and mental images about the concept.
- There are no internal contradictions within the concept image.
- There are no conceptions that contradict formal mathematical axioms.

One of the objectives of this study to assess the coherence of learners' concept images of the transformation of functions.

1.9.4 A Function

A *function*⁶ is a mapping that involves either a one-to-one correspondence or a many-to-one correspondence (Tapson, 2006 p.10), between two sets of numerical values. The symbols $y = f(x)$ indicate a function involving a single variable x that produces a mapping from x -values to y -values. Examples of functions are $f(x) = 3x + 1$; $f(x) = 2x^2 + 5x + 3$ and $f(x) = \cos x$. Functions are the objects on which the concepts of reflection, translation and stretch will be tested during the course of this study to determine learners' concept images.

⁶ Sometimes defined sets are used for the domain and co-domain of functions.

1.9.5 A Functional Representation

Functions can be expressed by means of various visual *functional representations* including symbolic or algebraic formulae, plotted Cartesian graphs, input-output tables, flow diagrams or set diagrams. Markmann (1999) considers the term *representation* to include the represented world of elements to the representing world, and to be a process that uses the information in the representing world. Visual representations play an important role in communicating mathematical concepts in the teaching and learning of mathematics (Elia, Gagatsis, & Deliyianni, 2005). It is necessary for learners to be able to recognize concepts in various types of representation and to be able to manipulate them within these representations and translate them across systems.

1.9.6 A transformation

A *transformation* is a mapping of a set of points onto a second set of points using a well-defined operation (Lewis, 2002). It involves the mapping of a point, a function, a geometrical shape or their representations (objects) from their original positions or forms into new positions or images using a well-defined rule. The simplest transformations are *isometric transformations* (also called congruencies or rigid motions), which change the position of an object while preserving the dimensions (size and shape). These consist mainly of reflections (flips), translations (slides) and rotations (turns). Glide reflections (flip-slide-flip or footprints) combine translation and reflection. Another group is the *non-isometric transformations* (non-congruencies or non-rigid motions). Non-rigid motions change the dimensions, either size, or shape, or both. With *similar transformations* the size is not preserved, but the shape and proportionality of the corresponding lengths are. Similar transformations may be enlargement (dilation) or reduction. With *affine transformations* the object only preserves parallelism. Shear (where area is preserved) and stretch (where area is not preserved) are examples of affine transformations. With *projective transformations* collinearity of points and the concurrency of line are

preserved. Enlargement is an example for this type of transformation, but this is a special case where corresponding sides are parallel. In general, the corresponding sides in a projective transformation are not parallel. A *topological transformation* of a plane figure occurs if the closure (or non-closure), orientability, and relative position of corresponding points are preserved. The South African FET mathematics syllabus focuses on the four isometric transformations, with the inclusion of enlargement and stretch. It is important to note that isometrics are a subset of similarities, which are a subset of affinities, etc. This approach to teaching transformations is known as Klein's Erlangen Approach (de Villiers, 1993).

1.9.7 Transformation of a Function

A *transformation of a function* consists of mapping (almost all) the points of a function onto new positions using a well-defined rule or operation (Lewis, 2002). With reflection and stretch there may be one or more (but still very few) points that have remained stationary after the transformation. In the CAPS syllabus, only translation, reflection, and stretch are covered, and so these will form the focus of this study:

Reflection

*Reflection*⁷ (sometimes referred to as *flip*) is a transformation or mapping which produces a mirror image of the same function and is sized as the original (Laridon et al, 2006). The axis of reflection is halfway between every point and its corresponding image point and is also called the line of symmetry or the mirror line. If a point is its own reflection, then it is on the axis of reflection. In the reflection, any two corresponding points in the original function and the image function are both the same distance from the line of symmetry,

⁷ The reflection is described by giving the position of the fixed line. *Translation* and *reflection*, together with rotation, are isometric transformations (that preserve all the geometrical properties of a figure). For example, translating a parabola changes the positions of all its points by moving them the same distance in the same direction, and reflecting a parabola about a line other than its axis of reflection changes the positions of all its points except those on that axis.

and a line drawn between those points would be perpendicular to that mirror line (Tapson, 2006).

Translation

*Translation*⁸ (sometimes referred to as *slide*) is a transformation or mapping that changes the position of points by sliding them to other positions (Laridon et al, 2006). Every point of the original function can be joined to its corresponding point in the image function by a set of straight lines which are all parallel and equal in length (Tapson, 2006).

Stretch

*Stretch*⁹ is a transformation or mapping that increases the distance between parallel lines, by the same factor, in one direction. In real life, stretchable objects have elasticity like those made of rubber.

1.10 STRUCTURE OF THE THESIS

This research thesis is organized into the following chapters:

Chapter 1 – Introduction

This chapter introduces the reader to the study problem and describes its context. It also includes the background to the study, the problem statement, the purpose statement, the

⁸ Translation is described by the direction and length of the sliding movement (a vector).

⁹ Stretch is one of the affine transformations (only the parallelism of corresponding lines is preserved). In polygons, stretch does not preserve area. A one-way stretch multiplies the original distance from the fixed line by the stretch factor. A two-way stretch of identical stretch factors in different directions results in an enlargement.

research question, the aims and significance of the study, assumptions, and key terms and concepts.

Chapter 2 – Conceptualising Concept Images

This chapter provides a platform for defining the term *concept image* and outlines its characteristics. It also provides models, sourced from the literature, which illustrate conceptual development in learners, and adapts these facts to design a model applicable to this study.

Chapter 3 – Theoretical Framework and Literature Review

The chapter looks at the theories related to concept images and mathematical reasoning that underpin this study. The second part of the chapter reviews other research that deals with issues of conceptualizing in mathematics and science and how learners understand the concepts involved.

Chapter 4 – Methodology

This chapter describes the empirical process of the study. It outlines the research design adopted, population, sampling procedure, data collecting instruments, as well as data processing and analysis procedures.

Chapter 5 – The Data and its Analysis

This chapter presents the raw data and analyses it using a variety of methods: description, frequency tables and bar charts.

Chapter 6 – Relating Results to Research Questions

This chapter is designed to provide empirical answers to the research questions. This is done by matching the findings to the research questions, and evaluating how successful the data is in providing answers.

Chapter 7 – Summary of the Study, Conclusion and Recommendations

The chapter summarises the purpose of the study, the methodology used for the study, and presents the main findings, the conclusion and recommendations.

CHAPTER TWO

Conceptualising Concept Image

This chapter deals with how concept images are conceptualized in this study. The first section extracts the meaning of the term concept image and what it constitutes. It looks at models formulated by other researchers about concept image definition, formation and development. The models focus on conceptual understanding, visualization, mathematical thinking and reasoning, formal and informal deductions from concept definition, as well as the formation of concept images. A model is then developed for this study, which will illustrate the possible stages of conceptual understanding of the transformation of functions. The last section of this chapter attempts to stimulate debate on the logical process of mapping functions. Discussion from a number of different perspectives aims at strengthening the understanding of the idea of concept images.

2.1 WHAT CONCEPT IMAGES ARE AND WHAT THEY CONSTITUTE

Tall and Vinner define a *concept image* as “all the cognitive structure in the individual’s mind that is associated with a given concept” (1981:152). While Davis (1984) uses the phrase *concept frame*, referring to much the same idea, this study will use the phrase *concept image* in preference to the alternative. Concept images are central to studies on advanced mathematical thinking (Tall & Vinner, 1981; Tall, 1991; 1995). A mathematical concept image may not necessarily be an isometric duplication of the formal concept definition *per se*, but may take the form of vague conceptions, mental images or interpretations about that concept, based on abstract imagination or attempts to

make representations, reveal properties, or explain processes involved in that concept, with or without connections to its formal definition (Viholainen, 2008).

The concept image reveals the way an individual learner thinks about a concept, and this may be different from both how other learners think about the concept and from how the concept is formally defined. Depending on whether learners understand the concepts fully, have misconceptions about them, or harbour internal conflicting views within their own minds, the concept images may or may not be coherent across a group of learners (Viholainen, 2008).

In the process of reflecting on why many people have difficulty or are even incapable of understanding mathematical concepts, Rosken and Rolka (2007) formulated two types of mathematical conceptions as determinant factors of how people understand mathematics: the objective and subjective mathematical conceptions. The former conceptions are based on unique characteristics that every mathematical concept has. The concepts may provide different possibilities for the cognitive architecture offered to an individual and the restrictions they entail. The latter conceptions are based on the individual's habits, which may or may not adequately accommodate the formation of mathematical concepts.

This analysis is helpful but it overlooks how the concepts are defined in textbooks, the language used, the rigor involved and how learners interpret all this. Mathematicians and scientists frequently use terms that are unfamiliar to learners and that may make it difficult for learners to redefine concepts in their own terms comfortably. The following questions are pertinent and point to gaps in my knowledge about how learners understand the information available to them in textbooks:

- Are the definitions uniform?
- Do learners understand the language the same way?
- Can learners simplify the terms to explainable forms that are comfortable to use and recall?
- Do they relate the concepts defined in the same contexts (intra or extra-mathematical)?

For learners to understand concepts, they need to be able to define them using their own, possibly simpler, terms, and make representations understandable to themselves. The model of conceptualization provided by Tall and Vinner (1981) gives a good picture of the interplay between subjectivity (the concept images) and objectivity (the formal definition). The model by Vinner (1983), which is a schematic version of the Tall and Vinner (1981) model, is provided in section 2.8 below.

Tall (2005) points out that a concept image may contain traces of the concept definitions or maybe be contained in the concept definitions. This indicates that concept images may take the form of mini-definitions, alternative definitions, vague conceptions, naïve definitions, queer meanings or explanations, visual images, or interpretations based on representations, properties or the processes involved in transformation concepts applied to functions. They may be correct, partially correct or incorrect, but as they are the concept images held by learners', we must accept them as given.

Presmeg (1986) identifies pictorial, pattern, memory, kinaesthetic and dynamic as five kinds of concept imagery. Pictorial imagery is described as the mind pictures that are dependent on thoughts and language, and pattern imagery as the spatial relationships between concepts, while memory imagery refers to mental images resulting from experiences that are not necessarily pictorial. Kinaesthetic and dynamic images as those involving physical activities and movement (see also Love, 1995). Presmeg's formulations are compatible with the definition of concept images in this study since they all deal with the conceptions, accurate or inaccurate, that individual learners have of concepts. The concept images may also be alternative definitions, mini-definitions, visuals or physical meanings of the concepts, or the relationship between that particular concept and other mathematical concepts. It is a common understanding that concept imagery is influenced by how learners understand mathematical concepts in relation to the empirical world.

2.2 FORMATION OF CONCEPT IMAGES AND MATHEMATICAL REASONING

The question of how concept images are formed is debatable. As memory structures, concept images do not just come passively into the mind, they form through mental activity and internal arguments within the learners' minds, and they may be a result of the thinking processes and the logical deductions we refer to as mathematical reasoning. Concept images can be created through learners' reflection on previously seen physical or mental objects that are related somehow to the concept. Learners may create meanings or interpretations of concepts by extrapolating from their existing knowledge structures. Viholainen (2008) refers to this as personal interpretations of formal concept definitions. These interpretations of concepts may depend upon the contexts used before or the apparent context in which they are currently in use. As Viholainen puts it, learners grapple with imagistic ideas of concepts to translate formal definitions into informal representations (2008). Some learners construct concept images through thought experiments that respond to the syntax of the definitions and give imaginative meaning to the formal definitions.

Bodner (1986) states that sense perceived information and cognitive structures exist almost permanently within the minds of learners and, to promote this, learners should be persuaded to relate new knowledge to other relevant concepts and propositions they already know, and should desist from rote learning.¹⁰ Shumba, Ndofirepi and Gwirayi (2012) also emphasize this by referring to Ausubel's (1963; 1978) meaningful learning idea and highlighting the fact that the most important influencing factor in the process of learning is what the learner already knows – well-performing learners learn by building new knowledge on their pre-existing cognitive structures.

Ogunniyi (2000) states that concept (image) formation is a *reflective, creativity-a-complex physiological/logico-metalogical process*, similar to natural selection or the

¹⁰ Rote learning refers to the process where new knowledge is acquired by verbatim memorization or rehearsing.

dominant–recessive phenomenon articulated by Charles Darwin and Gregory Mendel, respectively, where knowledge that survives decay supersedes the rest and becomes the pillar of the concept (imagery). This process uses conscious and subconscious intelligences of exploring formal and informal experiences to derive meanings, understandings and appreciations. Learners have to “negotiate and navigate a complex array of conflicting states” to achieve clarity of learnt ideas (Ogunniyi, 2000).

Duval (1998) links concept image formation to mathematical reasoning. Mathematical reasoning develops from three epistemological components of cognitive processes: *construction*, *visualization*, and *reasoning*. These three components may be connected and interrelated. *Construction* is where tools are used, for example, to make models and this leads to visualization. *Visualization* is not only of objects, but also refers to visual representations of mathematical statements. The clarity of a constructed image depends on connections between relevant mathematical properties and the constraints of the tools being used. Visualization may enhance *reasoning* although it may not for some specific visualized images. Although reasoning is enhanced by visualization, reasoning can also develop independently of construction or visualization. Duval’s idea that understanding and mathematical reasoning enhance the formation of concept images only provides one aspect of the dialectical relationship between concept images and mathematical reasoning and suggests that teaching should emphasize mathematical reasoning (1998).

Some authors espouse the opposite idea, that concept images enhance conceptual understanding and mathematical thinking/reasoning (Usiskin, 1987; Vinner, 1983; Fischbein, 1987; Tall, 1988; Vinner and Dreyfus, 1989). However, both viewpoints indicate a probable dialectical relationship between concept images and mathematical reasoning.

Although concept images and visuals have value for teachers in illustrating concepts to learners, they have even more value for learners in enhancing the understanding of various mathematical concepts (Usiskin, 1987). Therefore concept images are like butter spread in-between two slices of bread – they are used by teachers to facilitate conceptual understanding during instruction and are also used by learners to support conceptual

understanding. Vinner (1983) is of the opinion that concept images are more important than concept definitions when it comes to handling concepts realistically. Fischbein (1987) singles out visual images as being very important for organizing data into meaningful structures in that they act as a guiding factor for analytical development of solutions in problem solving. This seems likely because formal definitions are often not very clear or explanatory, and hence they may not enhance understanding for learners. If correctly linked to a concept, concept images facilitate a meaningful engagement with learning activities. The formal concept definitions may remain passive or forgotten, whereas concept images are always evoked in the process of reflective thinking (Tall & Vinner, 1981).

Holistic and concrete translations of mathematical concepts into concept images are very important in creative mathematical thinking and conceptual understanding. Tall states that when learners encounter old concepts in new contexts, it is the concept images, with all the abstractions made from earlier contexts, which respond to the task at hand (1988). He continues by pointing out that if learners do not have concept images, then a structured approach to learning a topic is unlikely to be successful (1988). Integrating the concept imagery gives learners a richer experience, which can facilitate the formation of more coherent concepts. Lack of conceptual understanding of, for example, transformations of functions, can lead to misinterpretations and misrepresentations of some aspects of the concepts involved in the topic, and this can result in the formation of incorrect images.

2.3 HOW THE APOS MODEL EXPLAINS LEARNING CONCEPTS THROUGH MATHEMATICAL THINKING AND REASONING

APOS model (Dubinsky, 1991) helps to explain how learners construct their understanding of concepts. Its main features are mental *Actions*, *Processes*, *Objects* and *Schemas*. The model is connected with Piaget's idea of reflective abstractions that have to happen during learning. The model is explained more detailed, component to component, in section 2.9 below. Mathematical thinking entails making appropriate connections, in the mind,

between the definitions of various mathematical concepts and their visual or other representations, which may be either formal or informal. Vinner and Dreyfus (1989) stress that learners might not understand a concept in depth if they do not match the concept image and the concept definition appropriately. Mathematical reasoning may come as antecedent, on one hand, or as a follow-up, on the other hand, of what has been conclusively thought and /or understood about a mathematical action. Mathematical reasoning based on concept images may be entirely separate from mathematical reasoning based on formal concept definitions (Vinner, 1991). The former is a dialectical process between figural (graphical or symbolic) and conceptual aspects of concepts and involves the interdependence of concept images and concepts themselves (Mariotti, 1995). Concept imagery, according to Mariotti (1995), helps to build mental schemas for learners and helps them to develop mature ideas of concepts and explore and verify how these concepts work. Concept images combine mental actions with mental objects and continue refining the images to allow learners to arrive at the concept more exactly. According to Pinto & Tall, learners can use concept images for reasoning, for interpreting definitions of terms, for exploring the concepts through thought experiments, and also for reconstructing their own understandings of concept definitions. A concept image can be information compressed into a single diagram, which learners evoke later when recalling definitions (2002). Graphical or symbolical images of concepts, like those in transformations of functions, can support the reconstruction of a learner's understanding of formal definitions.

Visuals or physical representations can be classified as *analogical* (Eysenck & Keane, 1987) or *active* (Pinto & Tall, 1999). They are analogical if they reflect properties of concepts and active if they show how concepts work. For example, to fully understand the algebra of transformations, learners should concentrate on symbols (formulae) as well as other forms of concept images, e.g. picture and action. This duality of **process** and **concept** is formulated by Gray and Tall as the notion of '**procept**' (1994). Concept imagery can be translated into formal linguistic terms and can facilitate the interplay between thought experiment and formal definition (Pinto & Tall, 2002).

Mathematical reasoning is, at times, informal, where it does not entirely depend on formal definitions of the concepts in question, although it may be influenced to a certain extent by

them. It may be exogenic (reality-centred) or endogenic (mind-centred). It is exogenic if it results from visual interpretations of the concept, and endogenic if it results from mental thoughts or interpretations of a concept. A learner might not necessarily reason mathematically about a concept using the mathematical language of the formal definitions, but might instead create his or her own words or ideas to explain the concepts. The words and the precision of mathematical reasoning depend on the individual's understanding of the concepts in use and also on his or her meta-linguistic competence, which is the ability to reflect on the structural and functional features of concepts. Sound mathematical reasoning allows for a lifelong retention of mathematical concepts and their applications (Pinto & Tall, 2002). Understanding concepts like reflection, translation and stretch, and their application to functions, requires not only instrumental understanding, but also the relational understanding of their meanings, and connections to other mathematical concepts and ideas (Skemp, 1976; 1989). Visual or mental interpretations and representations lead to the formation of concept images in individual learners' minds.

2.4 CONCEPTUAL UNDERSTANDING

Conceptual understanding is essential for learning, but what exactly the term *understanding* means, how it is achieved, and how it is measured, is not self-evident. Some scholars in the field of the psychology of learning mathematics, such as Skemp and Dreyfus, agree that conceptual understanding is the restating and redefining process that occurs in the learners mind about the concepts being learnt. Skemp differentiates between two types of understanding, namely *instrumental* and *relational* understanding. Instrumental understanding is “knowing rules without reasons”, while relational understanding is “knowing both what to do and why” (Skemp, 1976 p16). Dreyfus is of the opinion that the mental processes that occur and interact in learner understanding may be derivatives of the sequencing of learning activities that teachers use during instruction (1991) and learners' experiences thereafter in trying to follow-up, may contribute as well.

Kilpatrick, Swafford and Findell (2001) rate conceptual understanding as the most important of the five strands of mathematical proficiency.¹¹

Conceptual understanding is critically important for the effective learning of mathematics, and developing conceptual understanding in learners is every mathematics educator's goal. The Mathematics Learning Study Committee (2001) defines conceptual understanding as the comprehension of mathematics concepts, operations and relations, that is, the integrated and functional grasp of mathematical ideas. Learners who achieve conceptual understanding should have sufficient understanding of the concepts to work intelligently and productively with them. Learners are then able to identify and adopt the common features of the examples and this reinforces their understanding of the abstract concepts. If they adopt features that are not part of the abstract concept, then there will be interference which gives rise to some misconceptions.

2.5 MATHEMATICAL REASONING AND ITS IMPORTANCE

Mathematical knowledge (concepts) presented in its formal form is usually abstractly but not broadly explanatory thus it may not promote immediate conceptual understanding. Mathematical reasoning is a means of constructing meaning using what is presented formally. Formal mathematical reasoning is based on direct mapping from definitions, axioms or previously proven theorems. Informal mathematical reasoning is based on an individual's own visual or physical interpretations of mathematical concepts.

Visualization is considered a key component of reasoning (Arcavi, 2003). The formal reasoning around certain concepts is a direct mapping from the definition of the concept in question. Informal mathematical reasoning and concrete interpretations using visuals

¹¹ Their other four strands are procedural fluency, strategic competence, adaptive reasoning and productive disposition.

on paper or mental images, about a particular concept, are forms of concept images. Visual representations thus play an important role in communicating mathematical ideas (Elia, Gagatsis & Deliyianni, 2005). Learners do not only need to visualize and interpret concepts but also to manipulate them within the framework of their representations and even mix them with elements of the formal system.

Learners form concept images of concepts in their minds that they use when doing investigations and thought provoking activities, using mathematical reasoning whenever that is needed. Transformations of functions are one of the mathematics learning areas that stimulate the development of concept images for mathematical reasoning. This is because transformation procedures require spatial visualization skills in the quest to understand relationships between original and image functions, whether formally or informally constructed. Transformations of functions are a rich source of material for the development of mathematical reasoning skills. Geometrical representations and investigations add excitement and insight to the learning of these transformations through inductive and deductive reasoning and spatial visualization in one or more dimensions. Algebraic approaches, with or without coordinates, if done before geometric representations and investigations have been studied, result in learners resorting to memorizing rather than exploring and discovering the underlying properties (Strutchens, Harris & Martin, 2001).

The starting point for thinking and reasoning mathematically is the interpretation of definitions and properties of the mathematical concepts involved. A learner keeps these relevant facts in mind and tries to use them by making appropriate connections between the definitions and properties and visual and other representations, as well as other concepts, for example. Vinner and Dreyfus (1989) point out that learners might not understand a concept in depth if they do not tie the concept image and concept definition appropriately. Lack of conceptual understanding of transformations of functions can lead to misinterpretations and misrepresentations of some of the ideas about the concepts in the topic. The potential for imagery and visualization (concept images) to enhance the understanding of various areas of mathematics has been noted by Usiskin (1987), among others. Fischbein (1987) states that visual images are able to organize data into

meaningful structures, and they are also an important factor in guiding analytical problem solving. Holistic and concrete interpretations are very important in creative mathematical thinking and conceptual understanding because mathematical knowledge presented in a formal form may not be broadly explanatory and thus may not promote understanding.

Viholainen, (2008) describes informal reasoning as being based on visuals or physical interpretations of mathematical concepts, and formal reasoning as exact reasoning based on axioms, definitions and previously proven theorems. Learners grapple with imagistic ideas to translate ideas into formal definitions and informal representations (concept imagery). Some learners construct the concepts through thought experiments that may respond to the syntax of the definition, but which may also give an imagined meaning for the definition.

2.6 AQUISITION OF CONCEPT IMAGES AND MATHEMATICAL REASONING

Working through appropriate examples has a positive effect on the formation of concept images. Vinner (2011) highlights the importance of examples in learning mathematical concepts saying that it is by their use that concepts and conjecture are formed and verified.

According to Pinto, a good learner has his/her own strategies of learning mathematics (1998). For example, some learners extract meaning¹² and others build from their own imagery and give meaning to definitions by producing highly refined images that support their formal arguments (Pinto & Tall, 2002). The latter learners do not force cognitive

¹² Learners extract meaning by beginning with formal definitions and constructing properties by logical deductions. This goes hand-in-hand with Dubinsky's APOS theory (1991) in which multi-quantified statements are grasped by working from the inner quantifier outwards, and Pinto and Tall's discussion of the conversion of the predicate (as a process) into a statement (as a mental object) (2002).

processes but progress by refining and reconstructing existing imagery in a form that they can use to reconstruct the formal theory. These are the learners who are successful at mathematical reasoning and problem solving. This scenario bears comparison with how learners in mathematics interpret the concepts of transformation geometry and also how they engaged in learning its content. Computers can also assist learners to develop appropriate concept images. Comparing learner's interpretations and the formal definitions of transformation concepts to judge the accuracy of learner's concept images, forms an important part of this research study.

2.7 REVISITING THE FACTS ABOUT CONCEPT IMAGE AND ITS COHERENCE

We have already discussed Tall and Vinner's definition of a concept image as an individual cognitive structure associated with a given concept (1981). They also consider concept images to be a collection of vague conceptions about the concept, with or without connections to its definition. Viholainen is of the opinion that concept images are mental images or interpretations of the concept based on different kinds of representations about the properties or processes that involve it, and considers concept images to be connected to an individual learner's personal way of understanding the concept (2008). Concept images may contain concept definitions or may be contained in a concept definition (Tall, 2005). Vinner (1991) points out that mathematical reasoning based on concept images may be entirely separate from reasoning based on concept definitions. Individuals create meaning or interpretations of a concept by means of their existing knowledge structure and this can be referred to as a personal interpretation of a formal concept definition (Viholainen, 2008). This interpretation of a concept may be dependent upon the apparent context or the context in which it is in use. Concept images include all the conceptions, whether accurate or inaccurate, that an individual has about a concept. They may be definitions, visual or physical meanings, or the relationship between a concept and other mathematical concepts, for example. Concept images influence how learners understand

mathematical concepts because learners cannot understand any concept in isolation to the concrete world. Learners' misconceptions around concepts relating to transformations of functions may derive from the learners' incorrect images of the concepts. In order for learners to understand the concepts involved in transformations of functions, they should possess both theoretical and practical understanding of the concepts of reflection, translation and stretch. They should be able to represent the transformation concepts in multiple ways – practically, diagrammatically and symbolically. For example, reflection about the x-axis can be presented by using a mirror (practically), counting squares (diagrammatically), changing the sign of the x-coordinates (theoretically), using matrices (symbolically) or by using a computer programme.

Boas van Emde (1981) suggests that all concepts should be introduced in a fashion that facilitates understanding, beginning with several examples and then generalizing to end with some form of an abstraction. This statement assumes that learners will be able to identify and adopt the common features of the examples and will be able to then understand the abstract concepts implied. They may, however adopt features that are not part of the abstract concept, in which case misconceptions are likely to arise. It is not always easy for learners to achieve conceptual understanding from the outset. Some learners understand concepts only after acquiring procedural skills in using the concepts, i.e. by first learning to follow symbolic rules, then arriving at a fuller understanding later. For example in advanced parts of mathematics learners need functional understanding or procedural fluency at first, with the possibility of future refinement or revision of the concepts as and when they progress further. Some learners use concept representations for formal definitions, forming generic pictures covering many possible cases of their imaginations. These learners could be said to see the general within the specific (Mason & Pimm, 1984) as they experiment in their thoughts. Some learners combine mental imagery, its verbal equivalent, and its ensuing properties to make a cognitive unit (Barnard & Tall, 1997; Pinto & Tall, 1999).

Translation from visual to verbal forms suggests a possible method of moving from visual mathematics to formal mathematics. Seeing the general in a particular image (Mason &

Pimm, 1984) gives meaning to the corresponding formal definition and uses links between imagery and formalism to formulate and prove theorems.

Visualization (via sensations or the imagination) and spatial skills are essential to conceptual understanding, particularly in transformations of functions. In mathematical thinking, learners need to make appropriate connections between the definitions of concepts and their visual representations. Vinner and Dreyfus (1989) point out that learners might not understand a concept in depth if they do not tie the *concept image* to *concept definition* appropriately. Good teachers help learners to make such connections. Lack of conceptual understanding of transformations of functions, in particular, and mathematics in general, leads to misinterpretations and misrepresentations of some ideas about the concepts. For example, in the report by University of Cambridge Local examination Syndicate (UCLES) (1989), an international examination board for some Common Wealth countries¹³, candidates confused reflection with rotation, and stretch with enlargement. Therefore, it is best that learners understand these transformation concepts through the images they create of them, be they pictorial or concrete representations, regardless of the definitions. It is necessary to ensure, however, that learners do not misconstrue or over generalize those representations and build misinterpretations and misconceptions.

Visualisation and spatial skills have a lot of value. They can enhance a global and intuitive view and understanding of various areas of mathematics (Bishop, 1989; Fischbein, 1987; Usiskin, 1987). Fischbein (1987) points out that visual images can be organized into a meaningful structure and they can also play an important role in analytically developing a problem solution. Bishop (1989) reiterates that it is valuable to emphasize visual representation in all aspects of teaching in the mathematics classroom. Hershkowitz (1989) claims that good visualization is a necessary tool for concept formation. While many mathematics educators recommend the use of visuals in classroom (Bishop, 1983; Usiskin, 1987), it should be noted that traditional methods of having learners copy

¹³ Common Wealth countries are predominantly former British colonies.

diagrams and properties from chalk or white boards, and making them do repetitive exercises, are potentially frustrating for many learners because of poor conceptual understanding.

As mathematics needs precise concept definition and accurate interpretation of concepts, it is imperative to establish how mathematics learners interpret the concepts of transformation of functions, how they engage in the learning of its content, and how they see the relationship between their concept images, their practical use of the images, and the implied meanings of the formal definitions of the concepts. A teacher needs a thorough knowledge of the various possible mental images learners form in their minds, whether they are simple complex, pictorial or symbolic, or in tabular or diagrammatic form, for example. Even if the concepts are formally defined in textbooks, each learner may use these concepts in their own particular form or interpretation. There may be gaps between formally defined concepts and cognitively processed and conceived concepts in the way they are stored in the learners' memory structures. The form in which learners use their concept images may be modified to suit their own experiences. They may have refined the meanings and interpretations to match their own levels of manipulation and communication. When concepts are manipulated, there may be some associated processes that affect their meaning and usage. As Tall and Vinner state, we need to know the resultant cognitive structure, or the concept image. It may be a mental picture and associated properties and processes, which are "built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures" (1981:152). They may be evoked concept images that are in the process of formation. Different learners may have their own personal concept definitions (Tall & Vinner, 1981) and reconstructions which may relate, to a greater or lesser extent, to the formal concept definition. Personal concept definitions may form when learners put their own words and explanations to the evoked concept image. The personal concept definitions may sometimes be in conflict with the formal concept definition, and this could lead to misconceptions that impede meaningful learning. Learners who have misconceptions may develop negative attitudes towards the topic and the subject in general. Some learners believe that transformations of functions is a difficult topic, but this notion could be

dispelled if the teacher is knowledgeable about the types of concept images that promote understanding and is able to assist learners in forming appropriate imagery.

Various models have been developed to explain the ways in which learners construct their understanding of concepts: Vinner's model (1983), Dubinsky's APOS model (1991), Sfard's theory (1991), and Harel and Kaput's model (1991). Only the first three are relevant to this research study, and these will be dealt with in the sections that follow.

2.8 VINNER'S MODEL FOR CONCEPT DEFINITIONS AND CONCEPT IMAGES FRAMEWORK

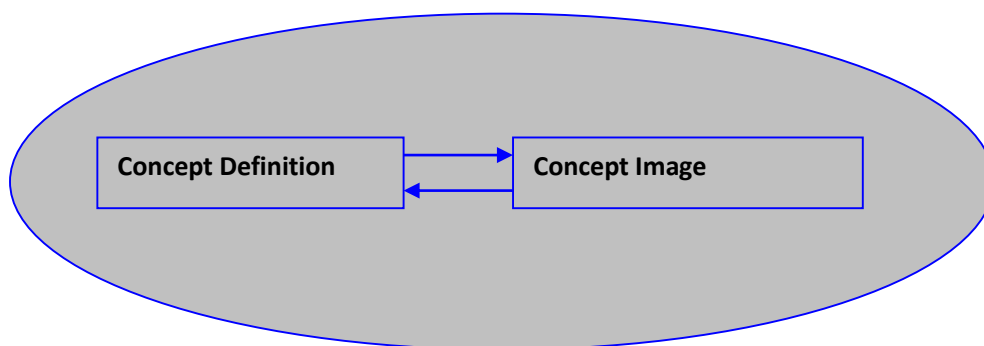
Vinner's model (1983) is the schematic version of the model first described by Tall and Vinner (1981). The model attempts to explain concept definition and the concept image relationship or framework. It assumes the existence of two different cognitive substructures in the learner's mind: one for concept definition and the other for concept image formation. These are considered void as long as no meaning is associated with the concept name. There can be some interaction between the substructures although they are formed independently. If, for example, a teacher introduces the concept of reflection as a *mirror image*, then a learner might have a concept image of a reflection as any object and its inverted (reflected) image. According to this concept image, an object and its reflected image are always on opposite sides of the axis of reflection (mirror line). The learner may not take the time to explore all the possible different positions of the axis of reflection, some of which cannot be modelled by a mirror. This concept image may need to be adjusted or changed to include the situation where the axis of reflection passes through the original object, in which case the image is found on both sides of the axis of reflection. For example, a mirror with only one reflecting surface, which is positioned somewhere on a plane shape, cannot reflect the whole plane shape as part of what is facing the dull surface of the mirror. Furthermore, a mirror cannot pass through a solid object so this scenario would be less than meaningful to the learner. Because the model of a reflection as a mirror image has its

limitations, for some learners the formal definition of reflection would not have been fully assimilated, thus the concept definition substructure would remain incomplete.

When a concept is first introduced by its formal definition, the concept image substructure is void initially, and begins to fill in as examples are given and explained. One model example may not be enough to explain a concept, so, to facilitate a more complete understanding, a teacher would have to give enough examples to reflect important aspects of the concept definition. If learners have too few examples to relate, misconceptions may arise from the limitations of the model.

A two-way interplay or interaction of the concept definition and the concept image substructures results in long-term concept image formation. This model could be used to explain how learners either acquire or fail to acquire mathematical skills like creative thinking and logical reasoning. These skills are very pertinent to transformations of functions as they influence understanding and use of the concepts involved. The illustration in figure 2.1 shows the interplay or interaction between a concept definition and a concept image.

Figure 2.1: A model for long-term concept image formation



This interaction between concept definition and concept image is relevant to those activities in which learners contextualize the concepts of translation, reflection and stretch, interpreting them in terms of his or her world of understanding, be it empirical (real life)

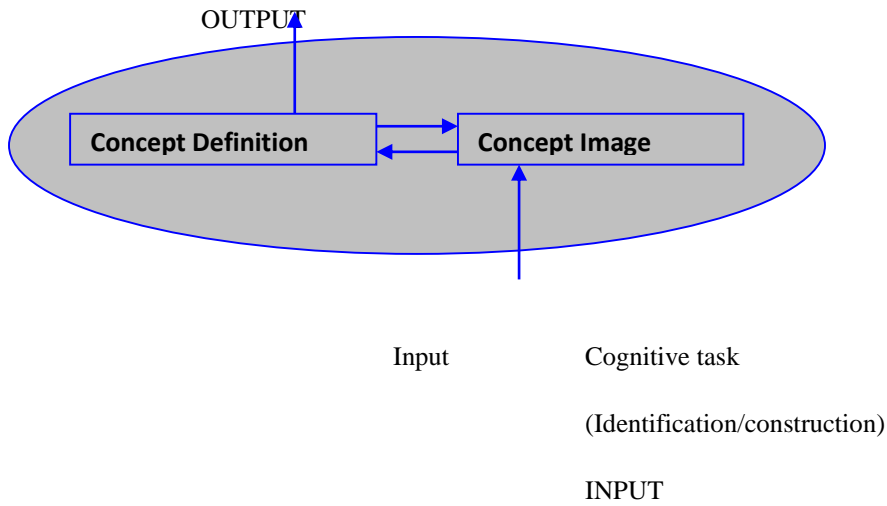
or imaginary. It can take more than one attempt for a learner to form a comprehensive concept image. The amount of time and effort necessary depends on the complexity of the concept. The concept images for translation and reflection of functions are likely to take less time to form than that of stretching a function because translation and reflection are less complex and correlate more readily with real life examples.

Behaviourist-oriented teachers may think that concept images form easily through rehearsal of the concept definitions (diagrammatically, this would be a single direction process, where the arrow points from concept definition to concept image). This forces learners to mechanically memorize the concept definitions. Both the concept definition and the concept image substructures may remain void, thus making the learner likely to forget quickly or to suffer information decay, which could result in a negative attitude towards learning. Some learners who experience only this type of learning may struggle in mathematics and other sciences.

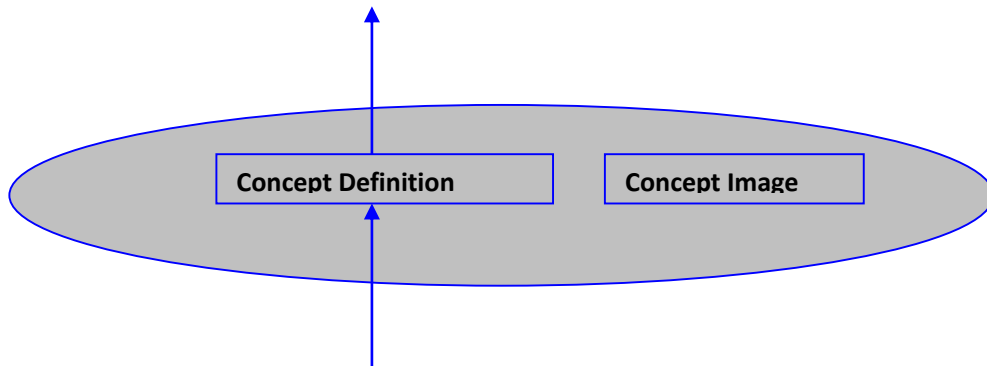
When faced with a cognitive task in problem solving, both the concept definition and the concept image substructures must be activated in the learners' minds. That is why there is an introduction of the input and output arrows to the diagram of the model (see below). The inputs refer to any of the causes that evoke cognitive processes, for example, the mention of the concept, identification of it from various others, or cognitive tasks involving the concept. In the context of this study, the inputs may be questions asking learners to identify transformations of functions that have taken place, to identify the images that correspond to specific transforming functions, to find the pre-images of given functions, or to illustrate verbal transformations of functions through various representations (symbolical, graphical, tabular or in the form of flow diagrams). The outputs might be achievements, intellectual behaviours or attitudes, solutions to the problems posed, or illustrations, among others. In general, they are the answers to the questions posed. The intellectual process involved in the solution of a task is illustrated schematically by the models in figure 2.2.

Figure 2.2: A model for expected intellectual processes in problem solving.

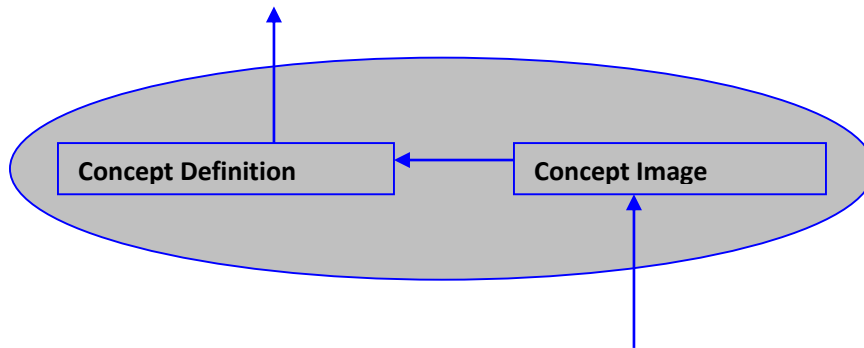
(a) *Input, concept image – concept definition interplay then output.*



(b) *Formal deduction from concept definition.*



(c) *Deduction from intuitive thought.*

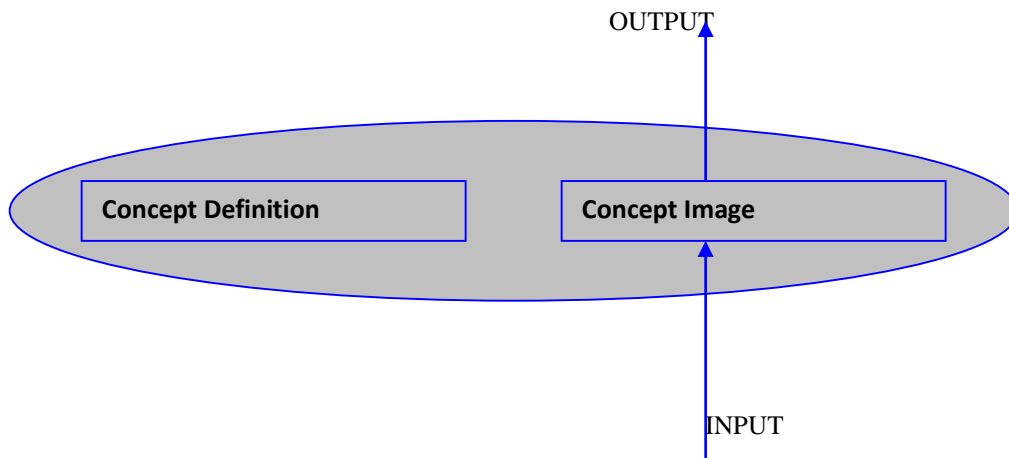


[Source: Tall (1991); Diagram adapted from Vinner (1983).]

In model (a), the process is started by an input (e.g. cognitive task) in which the concept is identified. The concept image is then evoked causing interplay between concept image and concept definition, then a deduction is made directly from the formal definition to an output. In model (b) the input evokes the concept definition directly, which produces the solution without even consulting the concept image. Model (c) is like model (a) but without the interplay between concept definition and concept image. Here the concept image informs the concept definition which, in turn, produces a solution.

Common to all these three models is that when the system reacts to a posed problem, the solution develops after consulting with the concept definition. This is the desired situation but, unfortunately, it does not always happen in reality. The cognitive system does not act against its nature (the empirical world) by forcing itself to consult concept definitions instead of concept images, or by working out a solution to a cognitive task from the concept definitions. Once concept images are formed, they are the ones to be consulted in problem solving. This demonstrates how important correct concept images are. A learner with misconceptions will always make the same mistakes unless his or her concept image is corrected convincingly. The more realistic model for how the process occurs in reality is the one given in figure 2.3 below:

Figure 2.3 : A model for the realistic intuitive response.



[Diagram adapted from Vinner (1983). Source: Tall (1991).]

This may not be the model of choice, but it is what happens in practice. Once concept images are formed, they are the ones that are consulted to solve problems. The cognitive system acts on the concept images and works out the cognitive tasks without consulting the concept definition, even if that substructure is non-void. The everyday thought habit, the concept image, takes over and the respondent is unaware of the need to consult the formal definition. The important issue is whether or not the reference to the concept image substructure is successful and correct. An incorrect concept image can be expected to give rise to an undesirable solution to the problem, if any, while a correct concept image can be expected to give a desirable solution. It is imperative, therefore, that mathematics teachers facilitate learners' formation of appropriate concept images. Working through as many examples as possible could do this, as any one of the examples could be picked up and form the learner's concept image. Numerous examples to illustrate reflection, translation and stretch will facilitate the formation of correct concept images for these concepts in the minds of learners.

2.9 DUBINSKY'S A.P.O.S. MODEL OF CONCEPTUAL FORMATION.

Dubinsky's APOS model (1991) is one of the theories in mathematics education that helps to explain how learners construct their understanding of mathematical concepts. It provides a model for how learners construct mental *Actions*, *Processes* and *Objects* and organize these into *Schemas* to make sense of mathematical concepts and solve problems. The theory is also used, when analysing data, to organize the learners' responses to tasks and provide the language to communicate ideas about learning results. This model connects strongly with the reflective abstractions involved in learning, which were theorised by Piaget. It has been extended into advanced mathematical thinking about how students understand basic mathematics concepts (Asiala, Brown, Kleiman & Mathews, 1998). This model provides objective explanations for student difficulties across a broad range of mathematics concepts and suggests ways of overcoming them (Trigueros & Ursini, 1997),

thereby providing pedagogical strategies that lead to marked improvements in students' learning of abstract concepts (Artigue, 1998).

The APOS model can be applied to the language of communication of ideas about teaching (Dubinsky, 1995) and it has been successfully applied to the teaching of transformations of functions (Breidenbach, Dubinsky, Hawly & Nichols, 1992; Carson, 1998; Dubinsky & Harel, 1992).

2.9.1 How the APOS model explains learning and understanding concepts in mathematics in relation to reflection, translation and stretch of functions

Actions are manoeuvres by learners to soften abstract concepts explicitly or mentally to perform operations, e.g. the term 'mirror' make it easy to visualize reflection mentally, the term 'slide' simplifies the action of translating an object and the simile of 'pulling an elastic band' or 'compressing a spiral spring' concretizes stretch in its two orientations.

Processes are repeated actions and reflective thoughts upon actions, to the extent of having internal constructions, which learners can perform mentally with minimal thinking. They can reverse or combine performances with other processes.

Objects are constructed from processes. Learners become aware of them as part a group, e.g. the term reflection encompasses reflections about the x-axis, y-axis, $y = x$, $y = -x$, or any other line. Similarly, translation can be horizontal (left or right), vertical (up or down) or oblique, and stretch has pull (vertical or horizontal) or compression (vertical or horizontal).

Schemas are collections of actions, processes and objects and other schemas that are linked by the same principles to form a framework in the individual's mind. Once a learner has a schema of transformations, he or she can identify a reflection, a translation or a stretch and can work with them out, given a problem situation. The framework of a schema must be coherent both explicitly and implicitly. With schemas in place, learners are less likely to

fear mathematics, in general, and any question that applies knowledge about transformations of functions, polygons or solids, in particular.

These four components are presented as having a hierarchy, but in reality the implementation of the elements may be in any order.

The APOS model makes provision for the analysis of data. The presence or absence of specific mental constructions can be connected to learners' successes and failures when doing mathematical tasks. The difference between complete and incomplete performances can be assessed by reference to mental constructions of actions, processes, objects and/or schemas to explain why some learners do better than others on a specific task. The APOS model enables us to make predictions about learners' likely success or failure when faced with mathematical concepts and problem situations. The decompositions of schemas in terms of mental constructs are ways of organising hypotheses about how the learning of mathematical concepts takes place. These descriptions also provide a language for talking about such hypotheses.

2.10 SFARD'S MODEL OF CONCEPT FORMATION

Sfard's (1991) model of concept formation has three stages: *interiorization*, *condensation* and *reification*:

- *Interiorization* occurs when a learner acquaints herself or himself with a mathematical concept and the processes concerning it.
- *Condensation* occurs when the learner sees the concept in relation to other concepts and is able to see the connections between these concepts.

The first and second stages are operational or process-oriented.

- *Reification* occurs when the learner has built up a comprehensive picture of the mathematical concept. At this stage, the development achieves a structural level and

the concept is understood as an object, a structure or a product, which can be subject to new operations (Sfard, 1991).

Sfard's model of concept formation is comparable to that of Harel and Kaput (1991), although they refer to the reification stage as the learner having developed a *conceptual entity*. Dubinsky uses the term *encapsulation* in preference to *reification* (1991). Dubinsky's and Sfard's models are related to and appear to be simpler than Piaget's model of cognitive constructivism (See section 3.1.1 below).

2.11 A NEW MODEL

This section presents a new ideal model of the process of understanding the concepts of transformations of functions developed by this research study. It has four developmental stages, which are linked to Dubinsky's APOS model (1991) and Sfard's model (1991) and it may be considered a direct application of the Tall and Vinner (1981) model. The model promotes the formation of correct concept images.

STAGE 1: *Verbal definition*

Learners first receive the information about transformations of functions verbally via the teachers' introductory lesson, demonstrations and/or simulations. This stage lays the foundation for conceptual understanding. It would be almost impossible for learners to enter the next stage of graphical representation if they have not understood what they are to represent. Understanding the verbal definitions equates to Sfard's stage of *interiorization* (1991) and opens the way for *actions* as per Dubinsky's APOS model (1991).

STAGE 2: *Graphical representation*

In order to reinforce the verbal definitions, learners need to identify and represent the transformations by means of Cartesian graphs. This stage equates to the *action* stage in the APOS model. Learners have to learn every action involved in the transformations, be they counting squares or using mathematical instruments to reflect or translate each point of the

original function. The three transformations dealt with in this study, reflection, translation and stretch, become *actions* physically and mentally when mapping functions to other functions. At lower levels of understanding, learners engage with every action involved in isolation from the previous action and the following action. For example, the learner acts on one point of the transformation at a time, until the whole function has been transformed.

STAGE 3: *Algebraic representation*

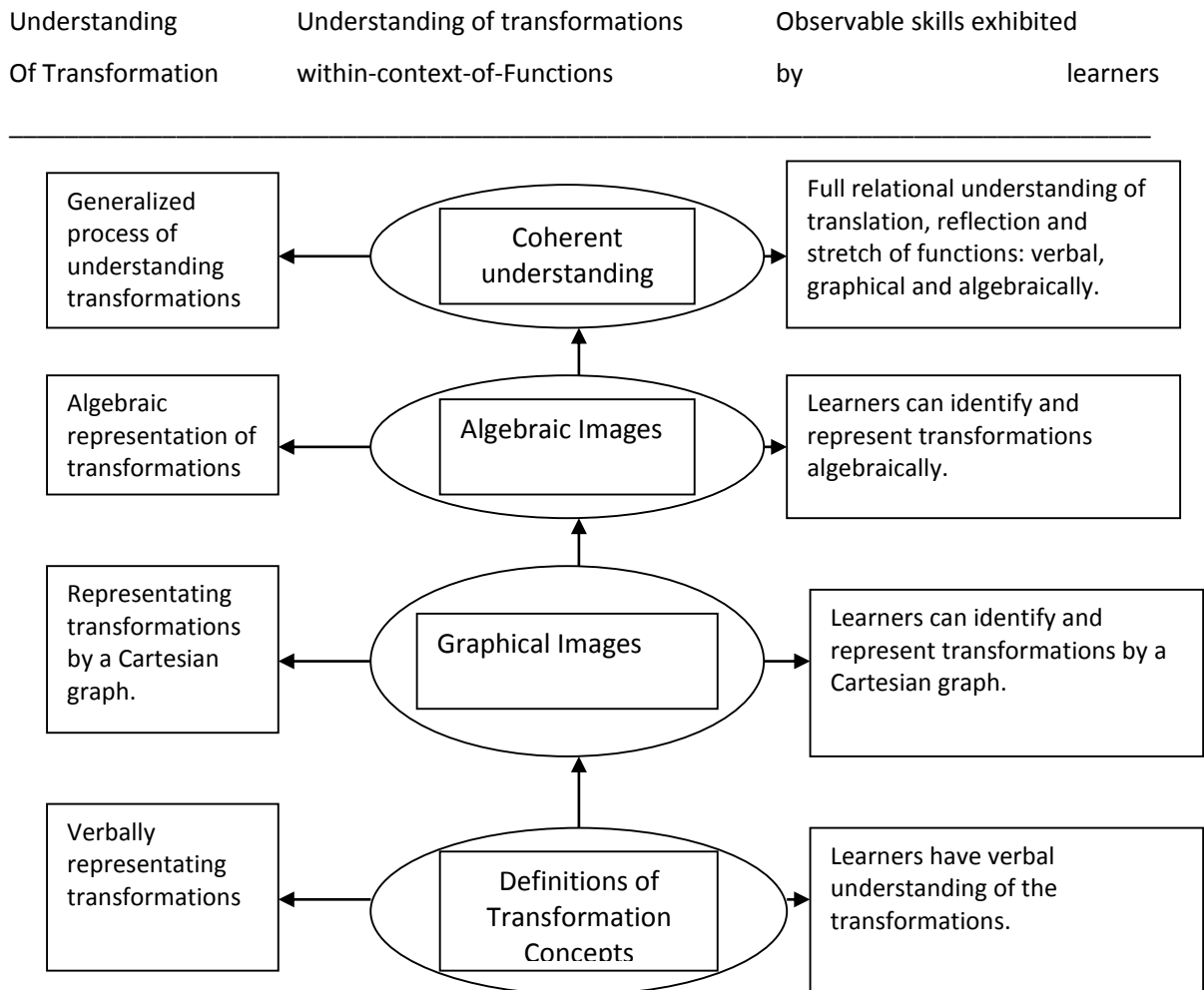
The third stage occurs when learners identify and represent transformations algebraically with a formula. The formulae take two possible forms: showing effects of parameters on the original function, or showing how the two coordinates of a generic point in the original function transform to the image function. This stage corresponds to the *process* stage in Dubinsky's model (1991) or *condensation*, according to Sfard (1991). The early phase of this stage may be characterized by computing and manipulating points without reasoning about processes, but just by uncoiling the given algorithms for the synthesizing of transformations. After a learner has repeated the actions and reflected upon them, he or she may begin to internalize the actions and connect them to form a *process*. The learner is then able to transform the whole object at once. If a learner has a *process* level understanding of transformation concepts he or she can also imagine the image of a transformation without actually performing the action and can reverse the steps of the transformation to get a reverse process/transformation.

STEP 4: *Coherent understanding*

At this stage the learner understands transformations of functions fully and coherently, i.e. verbally, graphically and algebraically and without confusing one with another. This is a generalized *process* of understanding transformations at *object* and *schema* levels (Dubinsky, 1991) or the *reification* stage (Sfard, 1991). Objects represent linkages or connections between processes. If a learner has an objective understanding of transformations of functions, he or she can operate two or more transformations successively or simultaneously, and can also do the reverse easily. The learner can also compare and contrast them correctly. A *schema* is a more advanced stage of relational conceptualization where a learner is able to understand and perform two or more

transformations that have taken place. Schematization, in this context, refers to the ability to do something with little or no thinking at all, i.e. some of the links between concepts and their manifestations are established automatically. Figure 2.4 below summarizes the steps of understanding of transformations of functions.

Figure 2.4: A model for developmental understanding of transformations of functions.



2.12 CONCLUDING REMARK

The analysis of concept images and the presentation of various models serve as a basis for this study because they provide a platform from which to explore images of the mathematical concepts of transformations of functions and related reasoning in that section of mathematics. The following chapter is a literature review about scholarship relevant to this research study.

CHAPTER THREE

Theoretical Framework and Literature Review

The theoretical framework of this study is influenced by the dual cocktail of constructivist theory of learning and the cognitive process of conceptual development. As such, this chapter focuses on the reviews of the proponents and supporters of the two theoretical frameworks. The section for literature review looks at what other researchers worked on similar studies relating to conceptualisation.

3.1 THEORETICAL FRAMEWORK

Learning has been explained from different theoretical perspectives, such as behaviourist, cognitivist or constructivist. A learning theory can function as a lens through which facts about how learning takes place are viewed and it normally influences what is seen and not seen about the facts. Learning theories help us to interpret facts, for example, good learning processes are likely to result in appropriate concept image formation within the learner's mind. Concept images can be interpreted using learning theories.

Contemporary psychology of mathematics education is centred on the constructivist and cognitivist philosophies. The formation of concept images by learners can be explained through these philosophies. Constructivist theories of thinking and reasoning can be traced back as far as Giambattista Vico in the 1700s (Glaserfeld, 1984), but Piaget and Vygotsky, writing in the 1970s, are considered the first true constructivist scholars with regards to education. Constructivism is now considered to have two major streams: personal constructivism, of which the major proponents are Piaget and Von Glaserfeld (Glaserfeld's views of learner independence in learning being more radical than Piaget's),

and social-cultural constructivism, of which the major proponent is Vygotsky. Piaget tends to associate learning with mental development (see section 3.1.1). Constructivism's basic premise is that learners are constructors or creators of their own knowledge under the teacher's guidance (Glaserfeld, 1984). Concept images develop as learners construct knowledge, mentally or socially, about a particular formally defined concept. A discussion of the two subdivisions of cognitive constructivism, personal constructivism (Piaget and Von Glaserfeld) and social-cultural constructivism (Vygotsky), follows.

3.1.1 Cognitive constructivism and concept images

Piaget's (1963) formulation of personal constructivism, i.e. *cognitive constructivism*, postulates that children develop personal cognitive structures and capabilities as they learn, that help them to construct their own understanding of reality (concepts) in various ways through exploring, observing, listening, touching etc. These sensory activities, referred to as active learning or learning by doing, link new knowledge to previously learnt knowledge. Piaget did not use the term concept images, but he referred to such ideas in other words. According to Piaget (1985), under the umbrella of *equilibration*, learners are empowered to transform and reorganize their cognitive constructs (schemas) constantly through *assimilation*,¹⁴ *accommodation*¹⁵ and *disequilibrium*.¹⁶ Piaget refers to the permanency of results that derive from an individual's coordination of experienced data and the subsequent co-ordination with the world that lies between the senses.

¹⁴ Organization of new experiences with current understanding or logical structure.

¹⁵ Reflection and organization of current understanding to integrate new experiences.

¹⁶ New experiences that contradict current understanding leading to accommodation of such knowledge.

3.1.2 Radical Constructivism and Concept Images

Von Glasersfeld refers to concept images as conceptual structures. His version of constructivism is regarded as being more radical than Piaget's because he maintains that knowledge is individually created and adjudicated and that experience is what brings forth knowledge claims. According to Von Glasersfeld (1984), knowledge consists of conceptual structures (concept images) that act as epistemic (knowledge) agents and it is actively built by the thinking individual through the senses or any other communication forms experienced within the individual learner's minds. The emphasis is on active involvement of learners in the process of learning. Von Glasersfeld did not rate social interactions among learners as being important to knowledge building. He focused on the learners' individuality and said that knowledge is functional and adaptive such that learners need to assimilate and accommodate new knowledge into pre-existing schemas for easy and meaningful learning. Von Glasersfeld reasoned that even if new knowledge does not fit into the pre-existing schemas, equilibrium can still occur, but this requires some adjustment of concepts to enable sensory insights for accommodation. This would lead to conceptual structures (concept image) forming.

3.1.3 Social-cultural constructivism and concept images

Vygotsky refers to concept images using the term *knowledge structure*. His primary notion of *social-cultural constructivism* insists that knowledge construction does not happen in the mind of a learner. It stresses that knowledge acquisition and construction happen as a learner interacts with his or her surroundings (1986). This points to the importance of interaction with other people in the school context for learning concepts, both *dialogue* with other learners and *assistance* from teachers and fellow learners. Dialogue aids understanding of concepts, and assistance from others strengthens the learning process

within a learner's *zone of proximal development (ZPD)*¹⁷. *Readiness-to-learn* and *scaffolding*¹⁸ are two of the factors that influence learning within the ZPD (Vygotsky, 1978). Scaffolding is built through the learner-support materials or tools that are used. These could be in the form of hints or advice that prompts reflection, coaching, articulation of different ideas, or making links between every day and formal concepts. All these pathways facilitate concept image formation.

3.1.4 Social-cultural Constructivism as viewed by Ernest

Ernest (1991) is concerned with the nature of mathematics and how it is taught and learnt within the society of learning. He emphasizes the role of teachers in communicating mathematical concepts to learners and checking conception by means of testing and assessment. He is of the opinion that personal mathematical knowledge (i.e. concepts, theorems, algorithms, objectives and other mathematical truths) and explicit mathematical knowledge representations are products of educational research and cultural products created by humans. He was widely criticised, especially for the claim that mathematical theorems are truths and that these truths cannot be corrigible or revisable but are naturally infallible. His other controversial position was that mathematics is socially constructed and accepted, where the acceptance is purely on the basis of group agreement. Ernest argues that mathematical knowledge creation, communication and justification happened in historical communities that lived with traditions of mathematical practice that were based on certain criteria for acceptability. The traditions included acceptable forms of presentation, reasoning and consistency. But these ignored the dynamics of development of mathematical concepts, theories and rules of acceptance. New views come and are debated critically and new consensus reached.

¹⁷ The ZPD is the conceptual knowledge gap between the learner's level of competence and the level expected by teachers or the syllabus.

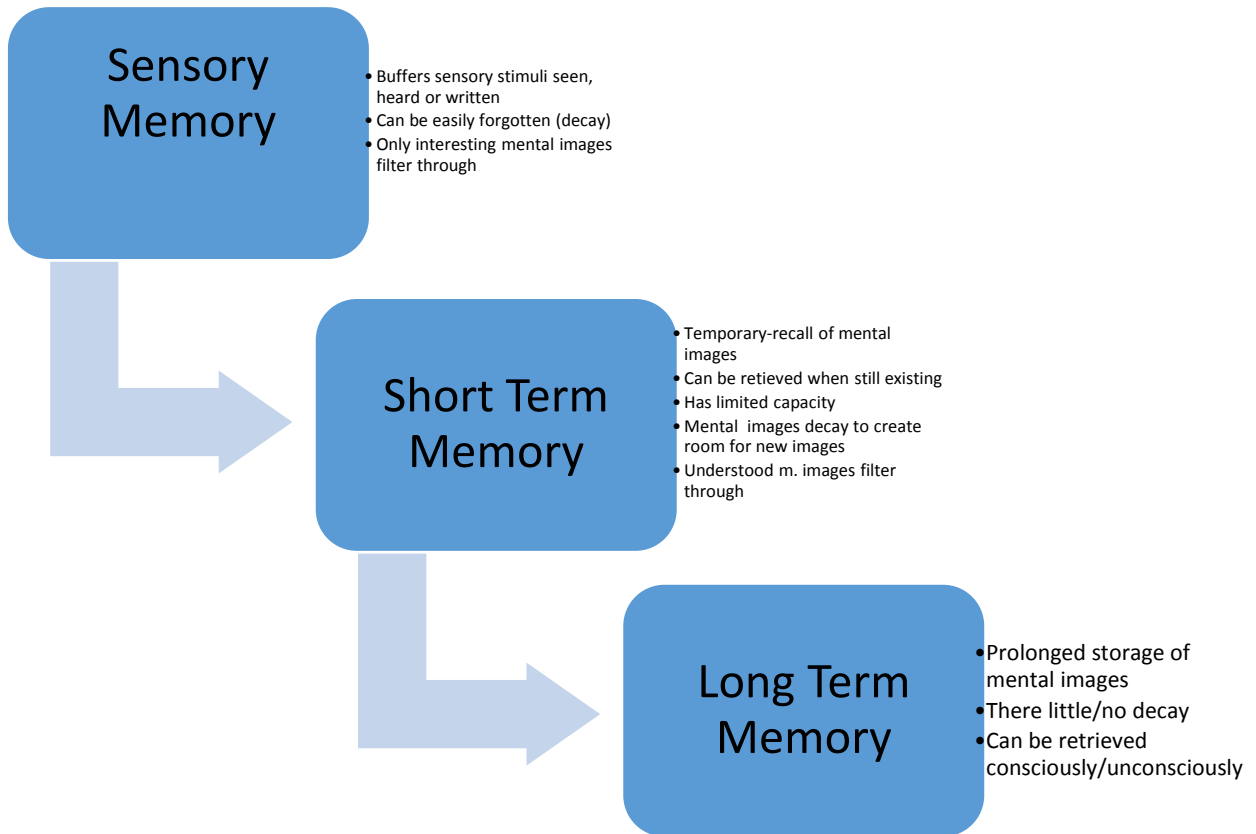
¹⁸ Scaffolding refers to the cognitive 'apprenticeship'.

3.1.5 Models of human memory structures

Concept image formation is a salient product of information processing within the memory structures. The study of memory structures and the explanation of the memory process date back to the fourth century BCE when the Greek philosopher Aristotle used a simile of wax impressions to describe memory structures, as he considered them to be copies of reality that individuals store and retrieve later (Tulvin, 1983). The assumption he made was that whatever is remembered is a simple copy of what was originally experienced in reality. This view was soon superseded since people normally remember part, but not all, of what they experience so, in most cases, remembering is an attempt to reconstruct what was experienced (Atkinson & Shifrin, 1968 as cited in Khateeb, 2008).

According to Tulvin, Atkinson and Shifrin described a model for human memory that consists of three sub-memories, namely, *sensory memory*, *short-term memory* and *long-term memory*, in their research into how learnt information is processed (1983). The linkage between the sub-memories is illustrated in figure 3.1. In this model, the sensory memory buffers sensory stimuli (information) from the iconic (visual), echoic (auditory) and haptic (touch) channels. The important information filters from sensory memory to short-term memory. This happens only if the content is interesting to the learner; otherwise it quickly decays and is lost. The short-term memory is for temporary-recall information and, because of its limited capacity, it is characterized by rapid information decay. Within the short-term memory is the working memory, which determines what to pay attention to and process. Working memory holds on to speech and sound information temporarily through the phonological loop. It creates mental (concept) images or solves visual and spatial problems through the visuo-spatial sketchpad and controls attention systems through the central executive (Tulvin, 1983; Khateeb, 2008).

Figure 3.1: The Atkinson and Shifrin (1968) model for human memory

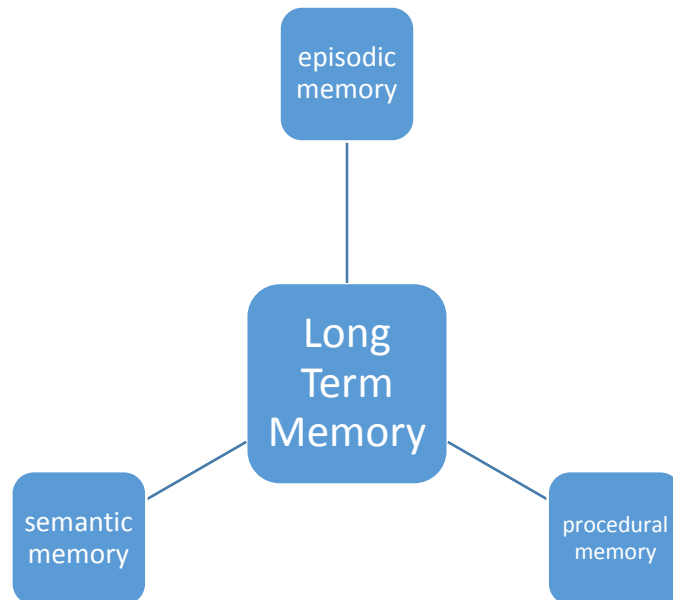


[Source: Tulvin (1983)]

According to Atkinson and Shifrin (1968, cited in Khateeb, 2008), the long-term memory is characterised by prolonged storage of important information that has travelled through the working memory, therefore there is little information decay. In Tulvin's (1983) model of long-term memory there are *semantic*, *episodic* and *procedural* memory structures, which this research study correlates with concept images, for they store information and allow it to be recalled explicitly or implicitly. It is important, therefore that learners use proper learning styles and strategies that take information directly to the long-term

memory. Effective teaching facilitates such styles and strategies of learning. This structure is summarized in Figure 3.2 below.

Figure 3.2: Tulvin model for long-term memory structure.



[Adapted from Tulvin (1983)]

According to Tulvin (1983), semantic memory structures consist of acquired facts, concepts and skills from learning, episodic memory consists of events and experiences, and procedural memory is a form of step-by-step procedures, psychomotor skills and algorithms. It is therefore logical that a combination of these memories influences the formation of concept images. Khateeb (2008) indirectly warned against rehearsal¹⁹ as it cannot store information in the long-term memory. It is unfortunate, therefore, that most learners learn in this less than effective way (DoE, 2013; 2014). Better ways of storage should be encouraged, for example, those that involve continuous use and schematic learning or process-oriented learning. Tulvin (1983, in Khateeb, 2008) states that decay,

¹⁹ Repeated exposure to information or singing jingles.

interference and some emotional factors negatively affect long-term memory, while prompting may retrieve information stored in the long-term memory by recall²⁰ or recognition.²¹ In mathematics education today prompting can be done by testing or posing problems.

The visuo-spatial sketchpad is important in creating mental images and in the solution of visual and spatial problems. The information that is remembered is highly dependent upon the way in which it was processed (Tulvin, 1983). The processing of new information depends heavily upon memory of past experience. Schemas develop that link often-encountered familiar situations to guide in the understanding and memory of the new events.

3.2 LITERATURE REVIEW

Documented research on mathematical conceptualization has been done since the second half of the 20th century, as it is an ever topical issue. Conceptualisation studies in education focus mainly on mathematical reasoning, concept images, cognitive conflicts and learning catastrophes. Significant research in the study of conceptualisation was done by Tall in the 1970s and by Vinner and Hershkowitz ,Tall and Vinner, Presmeg, Usiskin, Fischbein and others in the 1980s,and the bulk of the information now available about conceptualisation was published in the 1990s (Dreyfus, 1991;Gray & Tall, 1994; Sfard, 1994; Eysenck & Keane, 1997; Duval, 1998; Markmann, 1999) and the first decade of the 21st century (Thompson, 2000; Akkock & Tall, 2002; Pinto & Tall, 2002; Elia, Gagatsis & Deliyanni, 2005; Tall, 2005; Viholainen, 2008; Gagatsis, Panaoura, Elia, Stamboulidis and Spyrou ,2010; and Tsamir, Tirosh, Levenson, Barkai and Tabach, 2014).

²⁰ Reproducing as it is found in the source.

²¹ By means of cues.

The paragraphs below outline similar researches to this one, in methodology and objectives, which were done about conceptualising mathematical concepts.

Tall investigated students' understanding of the meaning of terms such as complex number, real number, limit, continuous, infinity, and proof (1977a). The study was carried out using questionnaires and follow-up interviews. The students gave conflicting explanations of the terms, as Skemp (1976) also found when researching relational and instrumental understanding. The study identified difficulties, cognitive conflicts and catastrophes (misconceptions) in learning mathematical concepts and obtaining the necessary skills in the process of restructuring schemas for logical understanding of ideas. Tall (1977a) hypothesizes that such problems in understanding develop during the process of instruction, but stops short of blaming teachers. The understanding of mathematics seems to occur in spurts, alternating sense and confusion, thereby tasking the brain to restructure already existing schemas and work with dynamic flow. Establishing clarity and ensuring permanent understanding of mathematical ideas is a demanding task for teachers. Cognitive conflicts and learning catastrophes occur for many learners, which is why this study is researching the state of learners' concept images and mathematical reasoning. Tall (1977a) advises not to underestimate the role of the teacher, for he or she facilitates schematic restructuring for learners. The teacher's use of programmed learning, work cards, and other tools, together with the voice, helps unblock lines of thought that could potentially lead to conflicts and catastrophes. Competent teachers immediately identify these at the moment of occurrence. The current study deals with Grade 11 learners, who are the year before the end-of-high-schooling examinations so the teachers get informed about such experiences in the mathematics teaching profession.

Tall (1977b) explains reports on the investigation done with Warwick University students doing mathematical proofs on limits of sequences. He presents a qualitative description of the mental activity that happens when new concepts are formed. He uses a rather complex model that focuses on *attractors*, which link flows between concepts and

schemas, and *repellers*, which hold concepts and schemas apart, but that are involved in the same topic. The data used to illustrate the investigated experience was collected from students by means of a test task with follow-up questionnaires. The study by Tall revealed some difficulties, cognitive conflicts and catastrophes in learning the mathematical concepts and skills used by learners to build up schemas for understanding ideas logically. In this study, Tall discusses how teachers are tasked with identifying conflicts and smoothing them out suitably. Learners may not be able to identify their own learning problems, so a teacher has to use the ‘art and science of teaching’ (investigation and observation skills) to identify learners’ individual difficulties and assist in removing possible conflicts, giving a clear exposition of the major mathematical ideas (1977b).

Tall and Vinner were the first scholars to emphasize *concept image* and *concept definition* after the terms were first introduced by Vinner and Hershkowitz (1980). They distinguish concept images from concept definitions in the context of limits of sequences, series and functions, and continuity in functions. They define concept images as “the total cognitive structure in the individual’s mind that is associated with a given concept” (1981:152). Tall and Vinner regard concept definitions as words that are used, in books or scientific articles, to specify a particular concept. A concept definition may be learnt by an individual in rote fashion. In order to investigate students’ concept images for limits and continuity, Tall and Vinner (1981) used questionnaires, casual observations, and follow-up questionnaires and interviews. The students were asked to explain and work through some examples that had missing intermediate working stages, and were asked to define ‘limit’. The researchers found that even though most students could not define limit correctly, they had their own concept image of a limit, which was enough for them to attempt examples. Students used words like ‘approaches’, ‘gets close to’ and ‘tends to c ’, for instance. The concept image for continuity of a function given by some students was the idea of a graph without gaps or one drawn freely without lifting the pencil from the paper. The research study that forms the basis of this thesis uses the term concept image with the same meaning as Tall and Vinner (1981) do. It uses a diagnostic test and follow-up interviews to determine learners’ concept images and mathematical reasoning with respect to transformations of functions.

In another study conducted in 1986, Tall analysed, from a single activity task, the relationship between the definition of a tangent and students' concept images of a tangent to a piecewise function graph of

$$y = \begin{cases} x & \text{for } -x < 0 \\ x+x^2 & \text{for } -x \geq 0 \end{cases}$$

Only a third of the students in the study had correct tangent images, and Tall emphasized that individuals build up their mental imagery of concepts in ways that may not always be coherent and consistent (1988). The difficulties that arise, when students learn mathematics, do not necessarily stem from a lack of aptitude on the part of students, as was shown by Tall's study, they are a widespread human phenomenon. This research study is being performed against this backdrop.

Borba and Confrey (1996) examined students' construction of transformations of functions (translation, reflection, and stretch) in what they called a multiple representational environment. They started with visualization exercises investigating the implications of visual changes of points up to algebraic symbolism. The researchers gave instructions, asked questions, and described and interpreted the students' actions as they did tasks with paper and pencil and using an Apple Macintosh computer with Function Probe software. The researchers concluded that visual reasoning, i.e. seeing graphical transformations on the plane, is a powerful form of cognition and that it is essential that teachers give students adequate time, opportunity, and resources to make constructions, investigations, conjectures and modifications. The researchers also emphasized that students develop effective strategies of enquiry when presented with an environment supporting the use of multiple representations. It is for this reason that the current study focuses on concept images, mathematical reasoning and transformation in graphs.

Weber (2002) analysed how students arrive at an understanding of exponential and logarithmic functions. He interviewed students three weeks after they had first learnt the concepts, asked them how they went about computing the concepts, and questioned them about the functions' properties. His findings were that, while students could compute exponents and logarithms, only a few of them could reason about the processes involved,

e.g. exponentiation. Guided by the first two aspects (action and process) of Dubinsky's APOS theory (1991), Weber proposed a set of theoretical constructions that students could use in future to understand these concepts,

Pinto and Tall (2002) used longitudinal observation and follow-up interviews to assess how students construct formalisms from their own visuo-spatial imagery in the context of limits of sequences. The premise was that students use reflective abstractions as mental processes to construct meaning from quantified statements through visuo-spatial imagery, i.e. using strategies consonant with Dubinsky's APOS theory. The two researchers observed students at work and tape-recorded, transcribed and analysed in depth interviews they conducted with the students. They found that students refined their own understanding of objects to represent and translate convergence of sequences into images and actions. They connected the students' learning strategies to the theory of natural learning (Duffin & Simpson, 1993, 1994) and Dubinsky's APOS theory (1991).

Nyikahadzoyi (2006) assessed student teachers' knowledge and concepts of functions using open-ended, task-based and reflective interviews in a case study of six final-year Zimbabwean student teachers studying for a certificate in teaching secondary school level mathematics. The study was done over a period of three months and it ranged over subject-matter knowledge and pedagogical-content knowledge for the concept of a function, as well as the underlining pedagogical reasons for the student teachers' choices of the contexts used to teach the concept. The majority of the student teachers were found to have a process-conception of a function and a few of them gave set-theoretic definitions. The students' notion of a function was mostly confined to real number sets and they did not think of considering other mathematical objects (for example, the differential operator and the determinant function) as functions.

Viholainen (2008) conducted a study at six Finnish universities using mathematics education student teachers between the middle and the final phase of their university studies as subjects. The study was conducted using a written practical task and interviews to investigate informal and formal understanding of concepts of derivatives and differentiability and how the students used informal and/or formal reasoning in problem

solving situations where these concepts were needed. It showed that connecting formal and informal reasoning was a challenge for the students, the majority of whom tended to avoid using definitions when solving problems. This tendency hindered reasoning in the problem solving processes and sometimes led to conclusion errors. However, some students were able to use definitions when asked to do so. As a result of the study, Viholainen recommends teaching mathematics in a way that supports the students' development of coherent knowledge structures, which is perceived to strengthen the understanding of connections between informal and formal representations. The fact that students at university have such difficulty with mathematical reasoning highlights the need for interventions to begin at an earlier stage in their education, which is why this research study focuses on secondary school learners so that the necessary strategies can be implemented before students enter university

Gagatsis, Panaoura, Elia, Stamboulidis and Spyrou (2010) explored students' constructed definitions for the concept of axis of reflection for a function. They used a test with nine tasks involving various forms, representations and problem-solving activities, to collect data. The students had difficulty giving clear definitions as well as resolving the algebraic, graphical and tabular tasks. The researchers cited lack of flexibility in the use of a variety of approaches to conceptualize the axis of reflection.

Using open-ended, written, test questions and semi-structured interviews, Bayazit (2011) investigated how student teachers understood connections between algebraic and graphical representations of functions, how they developed the function concept, and how they used it thereafter. The researcher explored flexibility in switching between algebraic and graphical representations of functions, and the vertical development of the process-object conception of functions in various contexts. The results indicated that teachers depended more upon algebraic expressions in their thinking and reasoning than graphical (Cartesian) representations. Bayazit (2011) recommends the process-object conception as being useful in promoting more successful mathematical reasoning.

Tsamir, Tirosh, Levenson, Barkai and Tabach (2014) conducted a research study that involved teachers practicing concept images and concept definitions of triangles, circles

and cycles. They asked teachers to define these concepts in their own words and identify the associated geometric representations. The teachers were also required to identify examples and non-examples of the concepts. The teachers were able to use correct and precise mathematical language in defining the concepts, and were also able to identify examples and non-examples of triangles and circles but some had difficulty when dealing with cylinders.

Sepeng (2014) carried out a study in urban townships in South Africa, where learners are from an array of multi-cultural backgrounds, using tests and focus groups discussions to investigate learners' tendencies when solving real-world problems in mathematics. The study revealed that learners draw on their cultural knowledge when constructing justifications and solutions to problems. Sepeng concluded that teachers need to incorporate out-of-school real-world knowledge in formal classroom mathematics to boost learners' mathematical reasoning skills and use of common sense when solving problems.

Our research study has similar characteristics to most of the reviewed studies as well as some significant differences. All focus on students/learners' mathematical thinking and reasoning in the construction of knowledge. They assess visual reasoning and cognition, and identify challenges, difficulties, cognitive conflicts and catastrophes in the process of learning mathematical concepts, developing skills, and building schemas for understanding ideas logically. This research study will explore Grade 11 learners' understanding of the concepts of translation, reflection, and stretch in relation to functions, by mapping at least one point of a function from its original position onto new position(s) (or images) using a well-defined rule and multiple verbal, graphical and symbolical representations. As explained earlier in section 1.4, Grade 11 learners were preferred for the study against the younger Grade 10 with shorter period of exposure to transformations, functional graphs and effects of parameters on transforming functions and against older Grade 12 for the syllabus section is not part of content to be dealt with as new and that they are perceived to be busy due to preparation for matriculation. It has just been mentioned above that researches done about mathematical problems of understanding and conceptualisation were mostly focused on university students so it a fair deal to do with secondary school learners. The term *concepts images* will be used to refer to learners' representations and

other forms of interpretation. Their explanations and justifications, or attempts at such, will be referred to as *mathematical reasoning*. Various concept images result when learning transformations of functions, and mathematical skills and abilities to think and reason mathematically, develop.

What sets this study, apart from the others reviewed above, is its geographical focus, the stage of educational development of the subjects, the specific context of the topic, some methodologies, and the extensiveness of its scope. This study has practical and theoretical relevance and it is intended not only to promote debate around how students understand and think, but also to improve classroom learning activities and have a positive impact on the practice of teaching the concepts in focus and to broaden and deepen the understanding of mathematical teaching and learning

CHAPTER FOUR

Methodology

The main goal of this study is to explore and describe how Grade 11 learners interpret and represent three concepts involved in transforming functions. The empirical component of the study was a diagnostic test taken by 96 Grade 11 mathematics learners from three schools in Johannesburg East district, South Africa. The diagnostic test was triangulated with follow-up interviews with the 14 learners who achieved a higher than 30% achievement score for the test. Although the number of Grade 11 pupils at the schools totalled more than 96, the number of participants was limited because of absenteeism and non-consent. This chapter covers the design of the study, the description of instruments used for testing and interviews, and the methodology followed in collecting, recording, summarizing, analysing and presenting the data. It also gives a description of the participants, sampling procedures, and strategies used to ensure reliability and validity of the research process.

4.1 THE RESEARCH DESIGN

An exploratory mixed method design (Creswell & Plano Clark, 2011) was the model used in this research study to collect both quantitative and qualitative data. This design was considered the most likely to provide the opportunity for open-mindedness on the part of both the researcher and the prospective readers of this thesis, through insights and questioning. Although an exploratory mixed method design facilitates the collection and analysis of quantitative data so that salient interesting results or cases can be selected to form the basis of a more profound qualitative study, slight adjustments and modifications

were made to that design in this case. In this study, a diagnostic test collected both quantitative and qualitative data simultaneously, and the results then lead to the selection of participants for a further qualitative study through interviews.

The research design adopted and adapted for this study was influenced, in many ways, by similar studies focusing on mathematical reasoning, concept images, cognitive conflicts and learning catastrophes (Tall, 1977a; 1977b; Tall and Vinner, 1981; Tall, 1988; Pinto and Tall, 2002; Nyikahadzoyi, 2006; Viholainen, 2008) reviewed in Chapter 2 above. They were mainly practical tasks, structured or open-ended questionnaires, structured or open-ended interviews, or casual or longitudinal observations. This study adopted and adapted some of those methodologies as its goals were similar to those of the studies reviewed, but applied them to a study of high school learners.

Both the quantitative and the qualitative data were collected from the same respondents and the same problem situation, so that a clearer understanding of the problem could be gained than from just one data type. The main methodology of the study is qualitative analysis, but quantitative justifications are used in places to support the qualitative results. A large group of 96 learners wrote the diagnostic test from which quantitative and qualitative data were converted into achievement scores, and then learners whose achievement score was greater than 30% were interviewed. The interviews produced qualitative data for triangulation purposes.

The exploratory mixed method design facilitated an in-depth study of words, phrases, statements and spatial diagrams (visual images) communicated by the participating learners. These were taken as artefacts of concept images as learners reasoned and demonstrated their understanding of translating, reflecting and stretching functions. This grounded theory approach facilitated making comparisons of learned experiences from three sample schools. In addition, it was hoped to collect a wide variety of information pertaining to learners' understanding of transformations of functions, thereby opening debate for generalizing the findings from the three samples and constructing relevant new knowledge.

The data collection was done between April 2012 and October 2013 during, and soon after school hours, in such a way that it did not jeopardize learners' schoolwork. Participating learners had been taught transformation geometry, functional graphs and the effects of parameters on functional graphs during formal school lessons before they participated in the study.

4.2 SAMPLING PROCEDURE

Initially, the study plan was to sample more than 100 respondents out of a total of 110 Grade 11 Mathematics learners from three high schools. Since no artificial sampling strategy was used to select participants for the diagnostic test, only 96 learners ended up participating. These were those Grade 11 mathematics learners present on the day the diagnostic test was conducted, and who had returned the consent forms signed by their parents or guardians. The three high schools were purposively sampled, because of proximity to the researcher, out of 10 high schools cluster of Sandton in Johannesburg East District (D9). Johannesburg East District (D9) is one of the 15 Gauteng education districts and has 37 high schools out of 114 schools in Johannesburg metropolitan city. The metropolitan city of Johannesburg has the bulk of the 196 high schools in South Africa's Gauteng Province. The Johannesburg East District (D9) was chosen for convenience and accessibility during the research study because one of the schools was where the researcher worked and the other two were neighbouring schools to the first one. The first school (Sample A) contributed 30 out of 36 mathematics learners it had. The second and third schools (Samples B and C) had 42 out of 48 and 24 out of 26 mathematics learners respectively. Ninety-six learners were a manageable number of study subjects for an in-depth study to provide meaningful conclusions. The study tested learners in Grade 11 because they had a longer period of experience with transformation geometry, functional graphs and the effects of parameters on transforming functions, compared to Grade 10 learners, and this meant that they were likely to have well-formed

concept images and mathematical reasoning. Grade 12 learners were not tested because that grade is the final year of secondary schools in South Africa and a research study could have disrupted matriculation examination preparations. The 14 interviewees were selected on the basis of having scored over 30% in the diagnostic test and being present on the days that interviews took place. The 30% threshold was used based on the fact that 30% is the promotion mark for mathematics learners to proceed to the next grade. Basically a learner is perceived to have acquired some minimum mathematical skills when he or she achieved above 30% mark.

4.3 THE PARTICIPANTS

The participants in the diagnostic test activities were 96 Grade 11 learners taking the subject mathematics (not the subject mathematical literacy, which is an alternative option for South African high school learners, who do not have transformations nor functions in their syllabus). Of the 96 learners who participated in the diagnostic test, 58 were girls and 38 were boys. The sample size was appropriate for the purpose of the study because it could give enough data to draw some conclusions.

The participants' ages ranged from 15 to 19 years, and the majority of them were aged 16 to 18. The detailed age distribution is given in table 4.1 below:

TABLE 4.1: Age distribution of the participants, in their samples.

Sample	Gender	15 years	16 years	17 years	18 years	19 years	Total
A	Female	0	4	15	2	0	21
A	Male	0	0	5	3	1	9
B	Female	1	8	16	1	1	27
B	Male	1	4	8	1	0	14
C	Female	0	1	5	4	0	10
C	Male	1	5	5	4	0	15
Total		3	22	54	15	2	96

The mean age for the combination was 16.9 years, whereas the median and modal ages were both 17 years.

In terms of race (which is not relevant for the results of this study), the majority of learners, 93%, were African, 5% were mixed race (or ‘coloured’ in the official terminology of the former apartheid regime) and 2% were of Asian descent. The Africans were of different ethnic cultures, namely, Ndebele, Sepedi, Sotho, Swati, Tswana, Venda, Xhosa, and Zulu, and there were a few learners from African countries other than South Africa. The medium of instruction used for mathematics in the sampled schools is English (see 4.3). Although the South African constitution states that all learners have the right to receive education in the official language(s) of their choice in a public education institution (National Education Policy Act, 1996; South African Schools Act, 1996), in practice it is difficult for schools, particularly those that are under-resourced, to accommodate learners’ diversity of home languages and so in urban areas English is the medium of instruction in many South African secondary schools. In the three schools under discussion, there was no chance of a learner switching teaching language because two of the mathematics teachers were South Africans of Indian descent and one was a Zimbabwean national, and none of the three is able to teach mathematics in any official South African language other than English.

The schools were former model C schools (in the terms of the former Apartheid regime’s hierarchy of school types) and they charged fees of between R14 000 and R16 000 per year. The schools’ administrative officers informed the researcher that fees were paid either by learners’ parents, foster parents or the biological parents’ employers. The three schools were relatively well resourced and had perceived better standards than those in run-down inner city or poor township areas.

4.4 DATA COLLECTION INSTRUMENTS

The two data collecting instruments used were a diagnostic test (see Appendix A) and a follow-up face-to-face interview (see Appendix B). The model for the instruments and

methods was adopted and adapted from previous similar studies as outlined in section 4.1, above. A full description of the instruments is provided in the sub-sections that follow.

4.4.1. The diagnostic test

The document used for the written diagnostic test is presented in Appendix A below. This diagnostic test was prepared by this researcher as the main instrument and was used to measure learners' concept images and mathematical reasoning about the transformation of functions. The test had to be subjected to validity and reliability tests (see section 4.5 below). The test was to be completed within an hour, using only a pen or pencil. The answers were written in the space provided on the test sheet.

At the top of the first page of the test sheet, learners were required to fill in their names, gender and ages. Names were needed for possible later interview call-ups, and gender and age were required for demographic analysis. The learners were told that if they were not comfortable giving their real names, they could use pseudonyms. A summary of learners' demographic information is presented in section 4.2 and Table 4.1, above. Immediately below this information on the test sheet were instructions to learners about filling in the required demographic information, and the need to answer all questions or provide a reason when unable to answer. The instructions were followed by 10 questions, some with at most three sub-questions. Learners were to think critically, explain, illustrate, evaluate and justify the mathematical concepts and relationships they built.

The first objective of the study was to investigate concept images built by learners as they learn, interpret and represent the concepts of translation, reflection and stretch of functions, and Question 1 required learners to define those concepts in their own words. The objective of obtaining concept definitions continued to be addressed by other questions, in other ways. Question 2 used drawn graphs of quadratic, exponential and cubic functions, and learners were required to illustrate a translation, a reflection, and a stretch of these three graphs respectively. This question was an alternative way of asking

Question 1. Learners were expected to show consistent knowledge of the three transformations of functions by demonstrating the skills necessary to represent them as they had defined them in their answer to the first question.

Questions 3 and 4 showed drawn graphs of hyperbolas, exponential functions and cubic functions, and, on the basis of the definitions, learners were required to interpret transformations from algebraic representation, then describe fully and illustrate them. They were to interpret, algebraically or symbolically, the translation, reflection, and stretch, and then perform graphical representations. The two requirements were alternatives, asking for the same transformation but applied to a different function. This pairing continued for the rest of the test where each odd-numbered question was paired with an even-numbered question, which facilitated split-half and alternative form reliabilities (the results for the reliability tests are shown in Appendix 3).

Questions 5 and 6 focused on stretching of trigonometric functions, given by both algebraic representations and algebraic formulae, in pursuing the same purpose as Questions 3 and 4. Questions 7 and 8 had verbal descriptions of transformations and learners were required to write the corresponding algebraic formulae for each transformation. Questions 9 and 10 showed functional graphs with illustrations of their image graphs after transformation and learners were required to identify the transformations that had taken place and describe them fully. Questions 3 to 10 mainly addressed the descriptive research question and its sub-questions. From the learners' answers, the relationship between their concept images and the formal definitions of the three transformations was evident. The precision in learners' explanations, arguments and reasoning, could then be measured against the formal definitions.

4.4.2 The follow-up clinical interview

The follow-up clinical interview was intended to further scrutinize learners' conceptions and interpretations of the central concept images and mathematical thinking, and clarify gaps uncovered in the test responses.

Appendix B shows the interview guide used in this research study. The same definitions asked for in the test were asked for in the interview. The design of the guide was semi-closed or semi-structured, because, despite the presence of specific themes, the interviews would inevitably deviate slightly from the plan in practice, depending on the interviewees' views and progress, so some degree of openness was allowed in changing the sequence of themes and the depth covered (Kvale, 1996). The interview questions were partly predetermined, as per the interview schedule and partly generated during interviews in response to the learners' answers.

The interviews were conducted in English, and the language used was as simple as possible to be easily understood by the learners. A voice recorder was used so that interviews could be replayed if necessary to ensure accurate transcription. The verbatim transcript of the interviews appears in Appendix C.

Learners were instructed to ask for a question to be repeated or asked in an alternative form if they had not understood it well. The learners were sometimes asked to repeat their responses for clarity and sometimes to illustrate what they said. This measure was taken to assure reliability of the information exchanged. Each interview took about 15 to 20 minutes depending on the precision of the learners' explanations and the levels of competence. The interviewees were asked to explain, sometimes with graphical illustrations, the concepts of translation, reflection and stretch of functions (see Question 1 in Appendix B. The same issues were addressed by the questions in the test (see Question 1 in Appendix A) and this research study's exploratory question and first research sub-question. The interviewees were also given transformations in the form of algebraic representations and asked to describe them fully. This replicated what was asked in Questions 3, 4, 5, and 6 of the test, except without Cartesian graphs. These activities mainly addressed the descriptive research question and its mini-questions. Finally, learners were asked to describe fully the transformations shown by the positions of the original and the image Cartesian graphs. The discussions in the last two tasks were strongly connected to the descriptive sub-question and its mini-questions. A detailed analysis of the interviews was done from the verbatim transcription of the recordings (see Chapter 5).

4.5 VALIDITY AND RELIABILITY CHECKS

4.5.1 Psychometric validity for the study

The exploratory mixed method design (Creswell & Plano Clark, 2011) was deemed suitable for providing both quantitative and qualitative data. The procedure used in this research study was influenced by, and adopted and adapted from, previous studies with similar goals (see section 4.1). The original procedure's success in several other studies made it psychometrically valid for this study because the chances of it being successful in these circumstances were high. Quantitative and qualitative data were drawn from the same problem situation and the same respondents, to achieve a clearer understanding of the problem. A slight variation from the original procedure was that no quantitative data analysis was done first to select respondents for a more profound qualitative study. Instead, the diagnostic test collected quantitative and qualitative data simultaneously, and then the resultant scores were used to select participants for a further qualitative study done by means of interviews.

4.5.2 Content and construct validity for the diagnostic test and follow-up interview

The following measures were taken to construct the most appropriate diagnostic test and interview schedule which would achieve the objectives of the study and test within the scope of the NCS–CAPS syllabus for Grade 11. Some copies of the test form and the section of the Grade 11 syllabus that deals with transformations of functions were given to two university educators and two other high school teachers to comment on whether the questions were suitably clear and accessible to Grade 11 learners and were addressing the syllabus objectives sufficiently. They were requested to make suggestions for improvement

and also to assess according to the Likert scale, how thoroughly the test examined the full scope of syllabus content for transformations of functions to the required depth. One educator from each of the two categories responded. Appendix D contains the reviewers' comments. By counting the 'yes' answers and 'no' answers and calculating percentage compliance, the high school teacher indicated a 77.5% compliance with syllabus objectives while the university educator indicated 86.6% compliance. The components ratings were correlated. The Spearman rank order correlation (r ranks or Spearman's ρ) was calculated and found to be 0.99 (see Appendix D) showing very strong positive monotonous correlation between the educators' rating of how the test (the main instrument) examines the scope of the syllabus for transformations of functions. This gave the instrument the necessary content and construct/factual validities. The interview design satisfied the same objectives as the test, therefore the validity of the test implied that of the interview. The pilot study (see section 4.6) had the purpose of validating these instruments.

4.5.3 Reliability of the diagnostic test

In order to check the reliability of the written test form and the interview schedule before the main data was collected, a small-scale preliminary study was done with 30 learners (see the pilot study in sub-section 4.5.6). The items in the test, as stated before, were designed in the following way so that internal consistence could be testable. Each even-numbered question asked the same thing as the odd-numbered question that preceded it but in a different form to allow split-half correlation. Outlier questions were removed and the resulting scatter plot for success rates showed a strongly positive correlation with the coefficient 0.9 (for $r^2 = 0.81$) using the FATHOM computer program (see Figure 4.1 below). A retest was done with 10 learners, and their scores were processed through the Spearman-Brown prophecy formula. The smaller sample of 10 gave a psychometric reliability or r -value of 0.86. The internal consistency with the bigger sample of 30, after the removal of outliers, was estimated to be $r = 0.79$ using the Kuder-Richardson Formula 21.

A follow-up interview was done with two learners and the verbatim transcript was subjected to credibility and dependability checks (see section 4.5.5). The research instruments were then deemed ready for the main study.

4.5.4 Acceptability of the instrument

After the pilot test, oral feedback was invited from learners. There were mixed opinions about the fairness of the test. Some learners confessed that the content had been covered in class but their memory of it had faded. Others said the activity reminded them to study more. Generally learners felt the test was challenging as they lacked an in-depth understanding of the concepts covered.

4.5.5 Credibility and dependability of the prepared interview questions

This was done through reflection of learners' responses to the pilot diagnostic test and then inviting comments from colleagues about the prepared interview schedule questions. Two voice records of preliminary interviews were given to colleagues to transcribe and then compared with those done by the researcher to check on the consistency or reliability of data obtained. The colleagues approved the method

4.5.6 The pilot study

The pilot study (referred to in sub-section 4.5.3), involved a class of 30 Grade 11 mathematics learners (21 girls and 9 boys). It was done at one of the high schools in the sample the year that preceded the one when the main study was conducted. The learners who participated in the pilot study did not participate in the main study because they were

now in Grade 12 meaning that they would not qualify. The consent forms for these learners had been signed by parents/guardians and returned. The learners were found to be aged between 16 and 19 years, and their medium of instruction in the school was English. The learners' proficiency in spoken English was generally good although not always fluent enough for academic communication. The aim of the pilot study was to improve and refine the research instruments, and to make sure that they were valid and consistent in capturing learners' understanding of the concepts of reflection, translation and stretch, in relation to functions and their verbal, graphical and symbolical representations.

Learners were given a question paper that they answered within an hour. The responses on each of the answer scripts were then assessed. Marking for the pilot diagnostic test was done with codes, not ticks or crosses, so that if learners saw their scripts later during follow-up interviews, they had no idea of whether their answers had been correct or incorrect. The codes were also used in the pilot data analysis. Questions posed during the clinical face-to-face interviews were generated in response to some of the learners' answers and were to solicit further clarification about gaps found in those answers. Not all learners were interviewed, the interviewees were selected if their answers to the test were interesting and the scores they had obtained were equal to or more than 30%, which is the promotion mark if learners have to proceed to grade 12. The codes used to mark learners' scripts in the pilot study are given below in section 4.6.1.1.

4.5.7 Data analysis and interpretation

The data from the both pilot and main study were to be examined qualitatively and quantitatively. The data were detailed descriptions and evidence (words, graphs and formulae) from the learners' reflections on how they understood reflection, translation and stretch concepts and their manifestations in functions. The frequencies of similar responses from both the diagnostic test and the verbatim transcripts of the voice records from the interview informed the qualitative descriptions. The words extracted from the

test would evidences to authenticate the claims. Thereafter, interpretations was based on the expressions and evidences for concept images and mathematical reasoning.

Quantitative data analysis was done with the assistance of a computer program: Statistical Package for Social Sciences (SPSS) or FATHOM. The data cleaning process was effected to ensure accurate data to facilitate better comparisons of tendencies, similarities and differences and also the formation of ultimate conclusions.

4.6.1.1 Assessment criteria for responses to the written pilot test

Task responses were classified using the following assessment criteria:

- Verbal or word descriptions of the concept were coded with the letter V.
- Graphic representations of the concept were coded with the letter G.
- Symbolical representations of the concept were coded with the letter S.

In this classification, an answer was placed into a class if at least one criterion of the class in question was fulfilled.

Class 0: Unclassified

- No verbal description of the concept (V0).
- No graphic representation of the concept (G0).
- No symbolical (algebraic) representation of the concept or answer (S0).
- A failure to answer the question with or without explanation (V0/G0/S0).

Class 1: Misconception

- Verbal description that does not resemble the concept (V1).
- Graphic representation that does not resemble the concept (G1).
- Symbolical (algebraic) representation that does not resemble the concept or the expected answer (S1).

Class 2: Partial or ambiguous conception

- Verbal description with some aspects of the concept but lacking accuracy (V2).
- Graphic or visual representations with little understanding but some correct aspects of the concept (G2).
- Symbolical (algebraic) representation with little understanding but some correct aspects of the concept or the answer (S2).

Class 3: Correct conception or interpretation of concept

- Verbal description that reflects the correct formal or informal meaning of the concept (V3).
- Graphic or visual representation showing complete understanding of the concept (G3).
- Symbolical (algebraic) representation showing full conceptual understanding (S3).

The values V0, V1, V2, V3; G0, G1, G2, G3; S0, S1, S2, S3, depending on assessment criteria, were counted, and their frequencies, were recorded as success rates in Appendix F. Altogether they were 90 pieces of data. The success rates for the odd-numbered questions and those for the even-numbered questions were paired, item-to-item, and are presented in Table 4.2 below.

TABLE 4.2: Success rates of similar pilot question items to check for consistency.

Question /item	a	b	c	a	b	c	a	b	a	c
Q1	0	1	1	4	4	3	13	22	13	4
Q2	1	0	4	2	4	5	17	18	10	3

Question /item	a	b	a	b	a	b	a	b	a	a	a	b	a	b	b
Q3	4	10	5	16	12	9	6	6	10	9	10	10	7	1	1
Q4	8	7	7	12	9	14	4	6	4	1	9	8	12	1	0

Question/ item				
Q5	5	15	9	1
Q6	5	15	5	5

Question/ item						
Q7	17	6	4	3	13	2
Q8	20	5	5	0	15	0

Question /item	a	b	c	a	b	a	b	c	a	a	b	c	a	b	c	a	b	c
Q9	10	24	8	15	2	5	1	2	0	3	25	24	1	6	0	0	4	0
Q10	10	21	10	15	4	4	0	0	1	5	20	23	4	7	0	0	6	0

Quest /item	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	c	d
Q11	9	12	13	15	21	23	28	19	8	4	0	4	7	4	2	7	10	7	9
Q12	1 3	11	11	12	23	21	28	22	8	6	4	7	2	6	2	3	6	9	8

Question/ item	a	b	c	d	a	b	c	d	a	b	c	d
Q11(cont)	1	1	0	2	3	2	10	2	1	2	0	2
Q12(cont)	5	1	0	0	3	9	6	3	0	2	0	5

The two sets of pilot data were then correlated, the scatter plot drawn, and the correlation coefficient stated. Figure 4.1 below shows that information.

4.6.1.2 Summary of results

The graph in Figure 4.1 shows the FATHOM produced scatter plot and the ‘least square’ regression line with equation $y = 0.877x + 0.97$ for split – half correlation of success rates for the pilot results of the diagnostic test.

There is a strong positive correlation with coefficient $r = 0.9$ ($r^2 = 0.81$).

Figure 4.1: Scatter plot showing split-half correlation of success rates of 30 participant learners in the pilot diagnostic test.

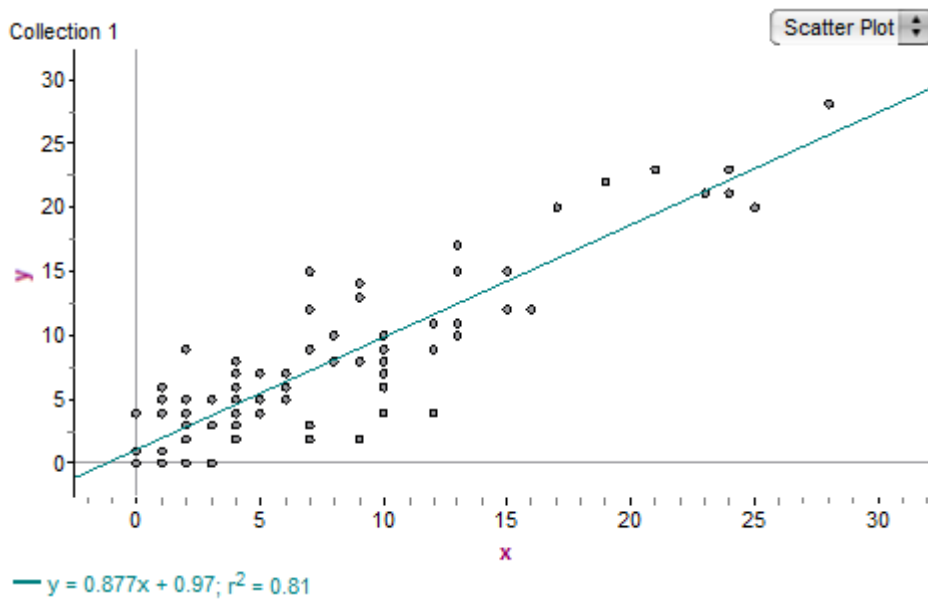


Figure 4.1 and the correlation coefficient of 0.9, show that the odd and even-numbered questions were consistently reliable, therefore the diagnostic test questions were suitable for determining learners' abilities to define, identify and represent transformation concepts on functions, consistently. Sections 4.6.1.3 to 4.6.1.12 go on to look at how learners in the pilot study performed in the activities cited in the study objectives.

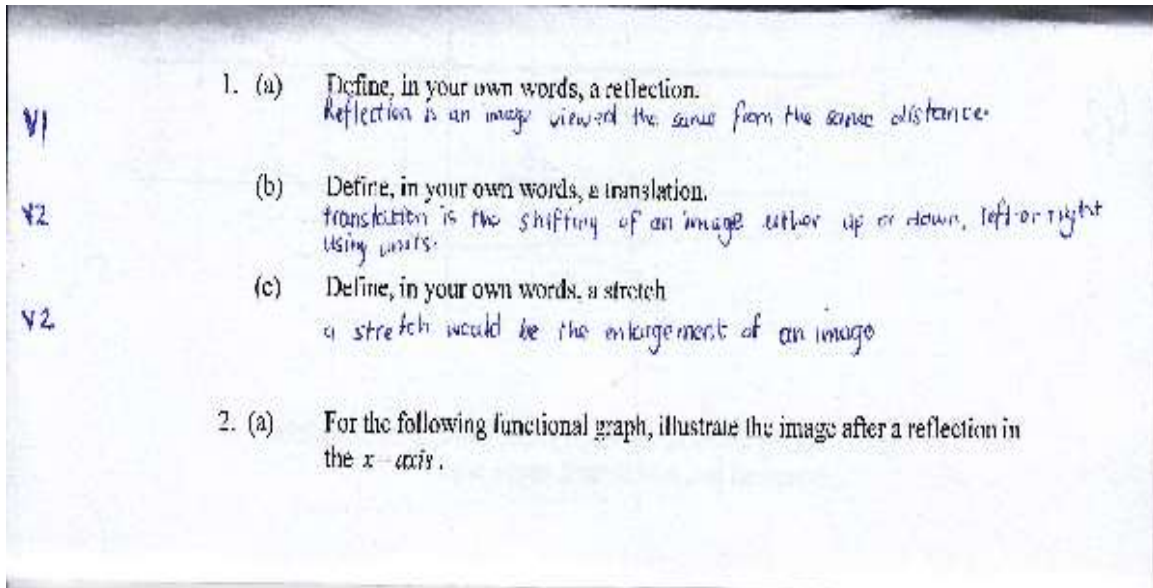
4.6.1.3 Verbal definition of reflection, translation and stretch

The tabulated information (see Table 4.2 above) indicates that the majority of the learners could attempt a definition of translation, reflection, and stretch, which is an indication that they understood what Question 1 required. Those who failed to define the concepts correctly, failed due to a lack of knowledge and not due to misconstruing the question.

- Of the 30 learners, 13 (43.3%) accurately defined reflection, 13 (43.3%) gave incomplete definitions of reflection, and 4 (13.3%) gave definitions showing misconceptions about reflection.
- Of the 30 learners, only 3 (10%) could define translation accurately, 22 (73.3%) gave incomplete definitions, and 4 (13.3%) gave definitions showing misconceptions about translation.
- Of the 30 learners, 4 (13.3%) gave accurate definitions of stretch, 22 (73.3%) gave incomplete definitions, and 3 (10%) gave definitions showing misconceptions about stretch.

All the responses given by learners indicated that their responses could be useful for the proposed major study. An example of a learner's definition of the concepts translation, reflection, and stretch is given in Vignette 4.1.

Vignette 4.1: Example of learner's verbal images of reflection, translation and stretch in the pilot study.



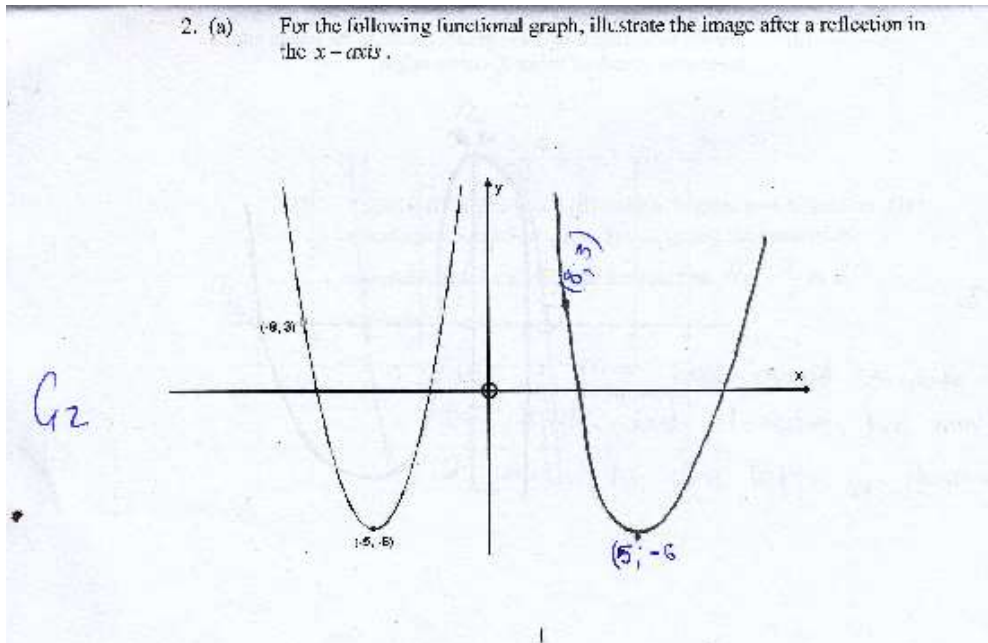
4.6.1.4. Graphical representations of reflection, translation and stretch

Eighteen of the sample (60%) drew the translated image well. A significant number (30%) drew queer graphs. Three of the learners (10%) exhibited serious misconceptions. Eighteen of them (60%) had slight misconceptions while 5 (16.6%) had serious misconceptions. Three learners (10%) answered well whilst 4 learners (13.3%) did not attempt to answer the question.

Seventeen out of 30 learners (56.6%) could carry out a reflection, but did it about the incorrect axis. Ten of them (33.3%) did it imperfectly. Eighteen drew the reflected image about the y -axis although the question required them to reflect about the x -axis. One learner reflected about $y = -6$. Only one left the question unanswered.

An example of a learner's graphical representation of transformation concepts is given in Vignette 4.2.

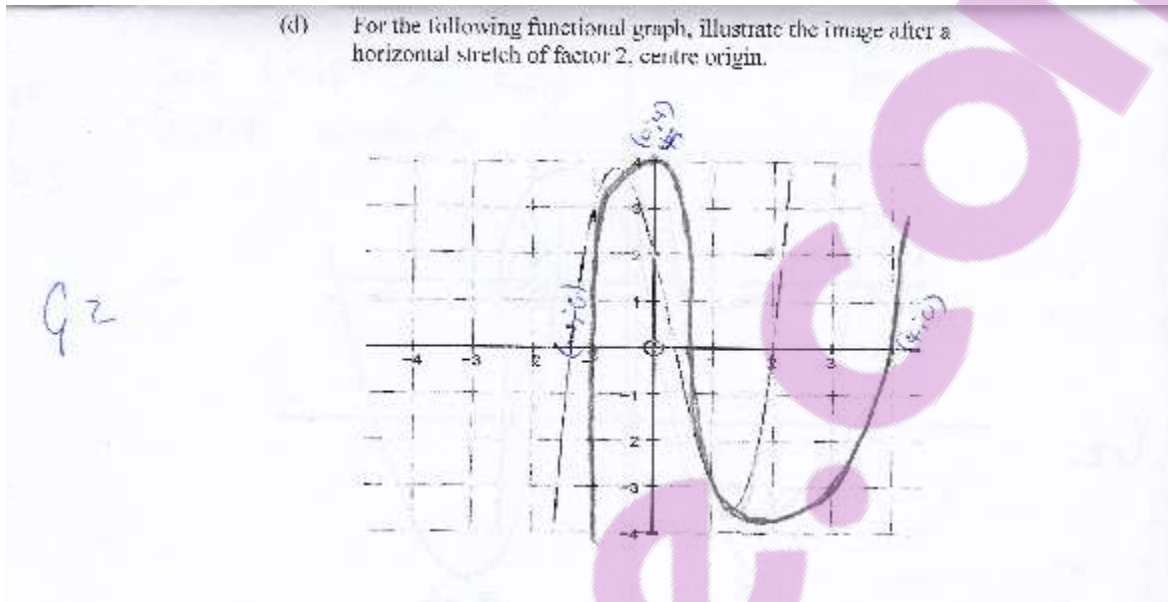
Vignette 4.2: Example of a learner's misconceived graphical representation of a reflection in the pilot study.



4.6.1.5 Drawing the graphs

The success ratings for the task indicated that most learners understood that they were to make graphical presentations of translation, reflection and stretch. Those who failed to represent the function correctly failed due to a lack of knowledge of the correct graph and not because they hadn't understood that the question required a graph. The example in Vignette 4.2 also confirms this. Learners seem to find it more difficult to effect stretch graphically although they knew that they were to draw an image after a stretch (see Vignette 4.3 below).

Vignette 4.3: Example of a learner's misconceived graphical representation of a stretch in the pilot study.



4.6.1.6 Verbal descriptions and graphical representations from symbolically given functions

Learners gave descriptions of concepts and represented them graphically. Misconceptions were evident and some learners left blank spaces (see Appendix F). These were associated with a lack of knowledge rather than misunderstanding the question, as a significant number of learners indicated on the question paper and during interviews.

4.6.1.7 Drawing the image and stating the transformation involved

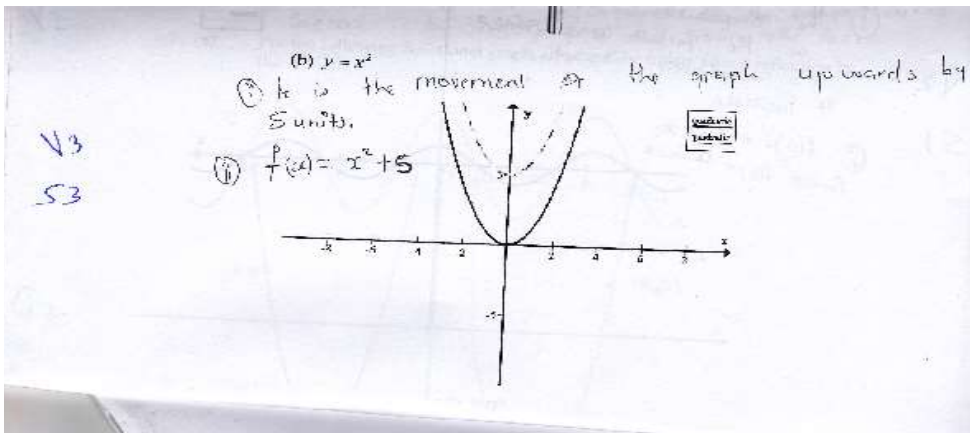
The results in the Table 4.2 above indicate that learners were able to draw images and state the transformations involved. Very few learners gave correct answers however, and

misconceptions and blank spaces were apparent. Again these were associated more with a lack of knowledge than misunderstanding the question. This could be seen from the inscriptions some learners wrote on the question paper.

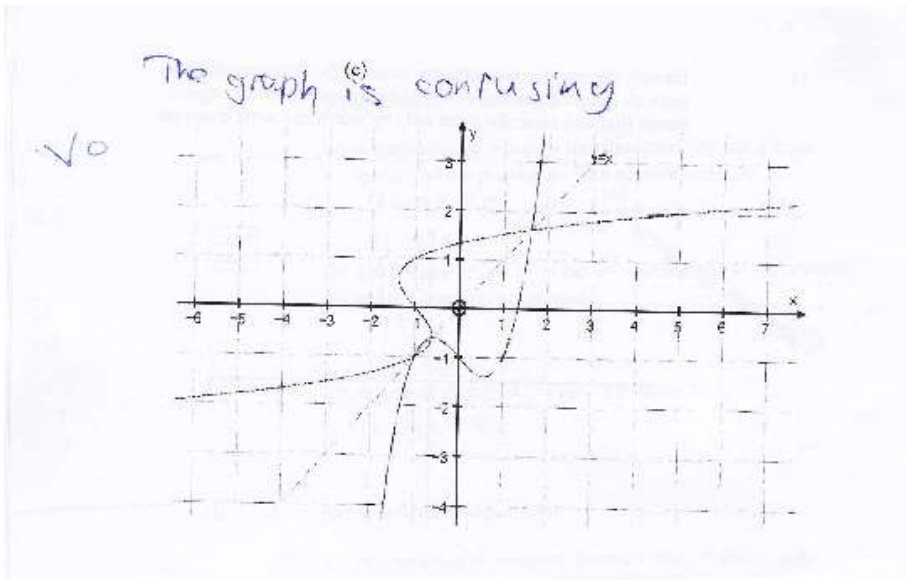
4.6.1.8 Stating and illustrating the transformation involved from verbal descriptions

Learners stated the transformations and represented them graphically (see Vignette 4.4). The number of misconceptions and blank spaces was higher than for previous questions. This was because of the higher level of skills the question required, and a lack of knowledge, rather than misunderstanding the question. Some learners stated as such on the question paper and during interviews (see Vignette 4.5).

Vignette 4.4: Example of learner's represented attempt to identify a transformation in the pilot study.



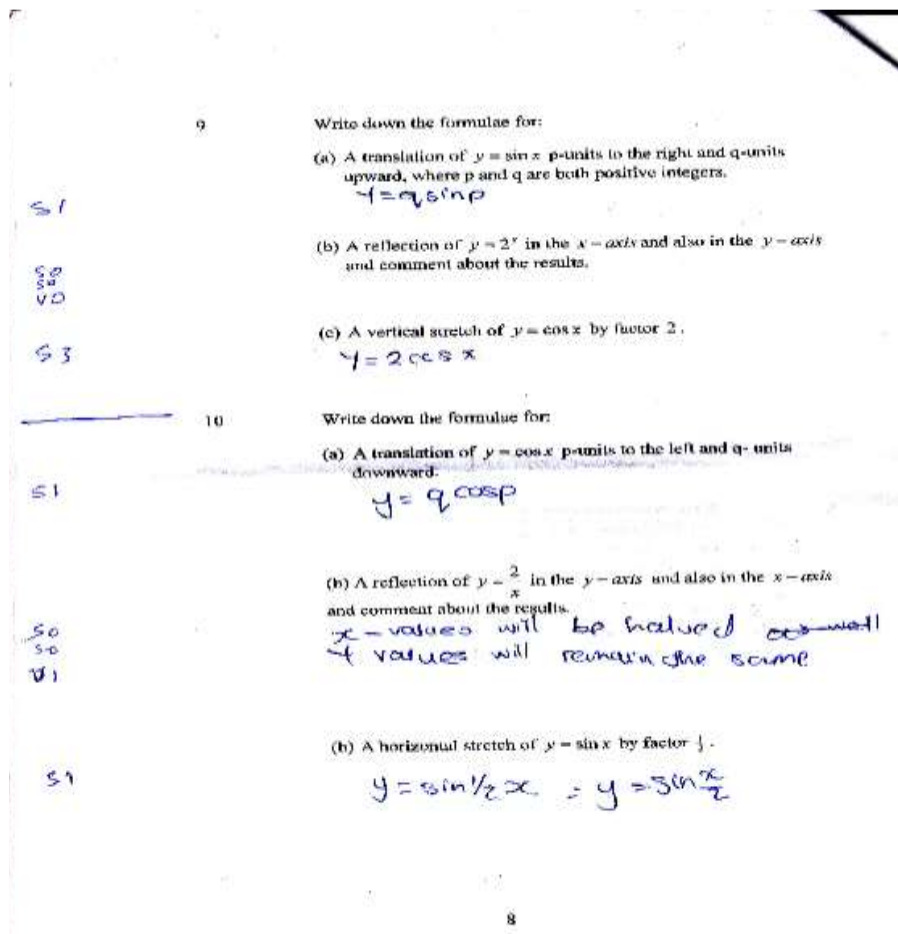
Vignette 4.5: Example where a learner expressed having difficulty with a task in the pilot study.



4.6.1.9 Writing the formulae from verbal descriptions

The majority of learners left some of the answer spaces blank while a significant number gave erroneous formulae. For some questions, such as 9(a), 9(b) verbal, 10(b) verbal and 10(c), no correct answers were given. Some learners stated on the question paper and during interviews that they find algebra difficult. Some of the blank spaces were associated with insufficient time to finish the test. Vignette 4.6 shows one learner's attempt at writing formulae.

Vignette 4.6: Example showing a learner's represented attempt to write a formulae in the pilot.



4.6.1.10 Identifying transformation(s) that map a function to an illustrated image

Most of the learners left the answer spaces blank for Questions 11 and 12. Some who answered the questions had difficulty and revealed misconceptions. The misconceptions in these questions were mostly associated with a lack of knowledge and not with misunderstanding the questions. Some blank spaces were associated with insufficient time to finish the test, as some learners indicated on the question paper and during interviews.

4.6.1.11 Projection of main study results based on the pilot study

The pilot study revealed that some learners find it challenging to define reflection translation and stretch, and make graphical representations of them. We would expect inconsistencies in learners' constructed definitions and their competency when dealing with representations of translation, reflection, and stretch, to reveal the depth of their knowledge of the concepts. Whatever they used to demonstrate their knowledge, whether correct or incorrect, could be considered their concept image. We appreciate that many learners are unable to express exactly what they understand about concepts in words, but they did their best with the terms and words at their disposal. It is, therefore, sometimes a very difficult task to interpret what learners mean when they attempt to define concepts in their own words. Their concept images may differ slightly or significantly from the accepted definition. From these preliminary results, we recommend interviews and think-aloud protocols as a necessary way of spanning the understanding gap from both the researcher's and the learner's perspective.

4.6.1.12 Adjustments of the research instruments

The following adjustments were made to the instruments for the main study, based on the findings of the pilot study:

- The number of questions was reduced from 12 to 10.
- Question 7 was merged with Question 9 to become the new Question 7.
- Question 8 was merged with Question 10 to become the new Question 10.
- Questions 11 and 12 were relabelled as Questions 9 and 10.
- Sub-questions were reduced from a maximum of 4 to a maximum of 2 per question, by leaving out already-tested aspects. This was done so that more learners would be likely to finish the test in the time allotted.

4.7 ETHICAL CONSIDERATIONS

The research complied with prescribed ethical considerations, such as informed consent, confidentiality and ethical clearance. Participating learners' parents/guardians gave their consent for the method of data collection by signing consent forms provided by the researcher.

4.7.1 Informed consent

The designed consent forms clearly stated that voice recordings, photographs and videotapes may be taken as part of the data collection process. The consent forms were issued and returned, signed, before the research began.

4.7.2 Confidentiality

The research process adhered to the highest levels of confidentiality. The data collected were confidential and anonymous and were only used for the stated purposes of this study. The names of participants and their schools do not appear anywhere in the report. Only the names of the district, the province, the city and the country in which the study was conducted appear. All the requirements of the ethics committee were met.

4.7.3 Ethical Clearance

After the validity and reliability checks were done and before the instruments and methodologies were applied, the instruments and the data collecting procedures was sent to the university's ethics committee for clearance. Ethical approval was granted, allowing the study to proceed (see Appendix J, below).

CHAPTER FIVE

The Data and its Analysis

This chapter presents learners' descriptions of how they understand the concepts of reflection, translation and stretch and how these are manifest in functions. Evidence in the form of learners' own words from the diagnostic test and the interviews, as well as vignettes of the test answer sheets, are provided to authenticate the claims made. In addition, this chapter provides another form of visual evidence using frequency categories of similar responses from the diagnostic test. Finally, a summary of the information gathered from the interview responses is presented. The interviews served a triangulation function. Samples of the verbatim transcripts of the voice recordings from the interviews appear at the end of this report as Appendix C. Categories of similar responses were quantified to enable frequency counting and tables of results were drawn up. The coding scheme and data entry method were tested during the pilot study. The analysis of resultant frequency data was carried out using the computer program Statistical Package for Social Sciences (SPSS). The analysis necessitated a data cleaning process. Comparisons of frequency tables enabled identification of tendencies, similarities and contrasts and provided the basis from which to draw conclusions. Ages or age range and gender comparisons formed part of the demographic analysis, which was presented in Chapter 4, section 4.2.

5.1 LEARNERS' RESPONSES TO THE DIAGNOSTIC TEST, AND THE RELATIVE FREQUENCY ANALYSIS OF CONCEPT IMAGES

A multi-task diagnostic test was the main instrument used for this study. The 96 learners in the full sample group were required to answer ten questions, with, at most, three constituent parts, in words or by means of visuals, within a one-hour time limit. The procedure leading up to writing the test is outlined above in section 4.4.1. An account of learner responses, the descriptive analysis of results, and evaluation of concept images from learner responses to the diagnostic test, is given below in section 5.1 (subsection 5.1.1 to subsection 5.1.5). Answers were assessed for correctness and then used as a basis for comparing formal definitions and properties of the mappings of functions. The learners' responses in the three sample groups (A, B and C) were analysed separately and comparatively in frequency tables and vertical multi-bar charts. The horizontal multi-bar chart at the end of each analysis shows a comparative summary of learners' responses from the group as a whole.

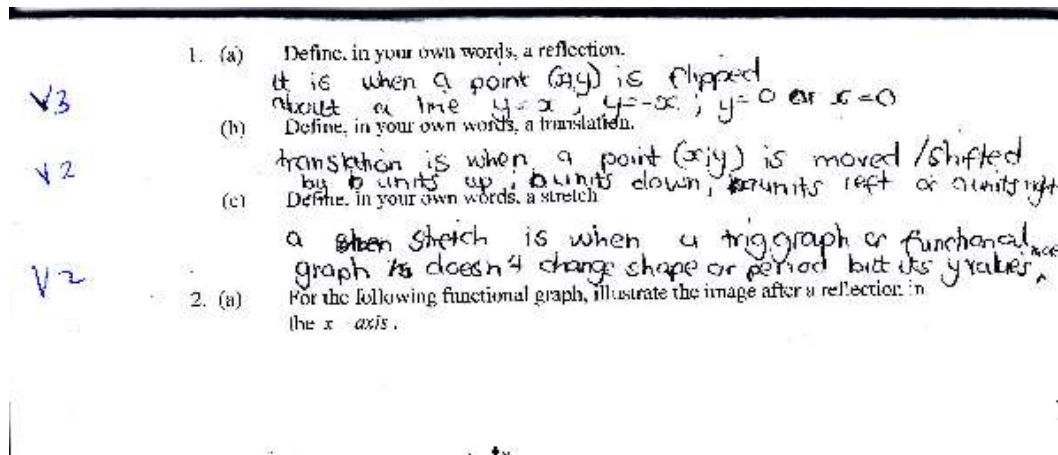
5.1.1 Verbal descriptions of concepts that transform functions

Question 1 read as follows:

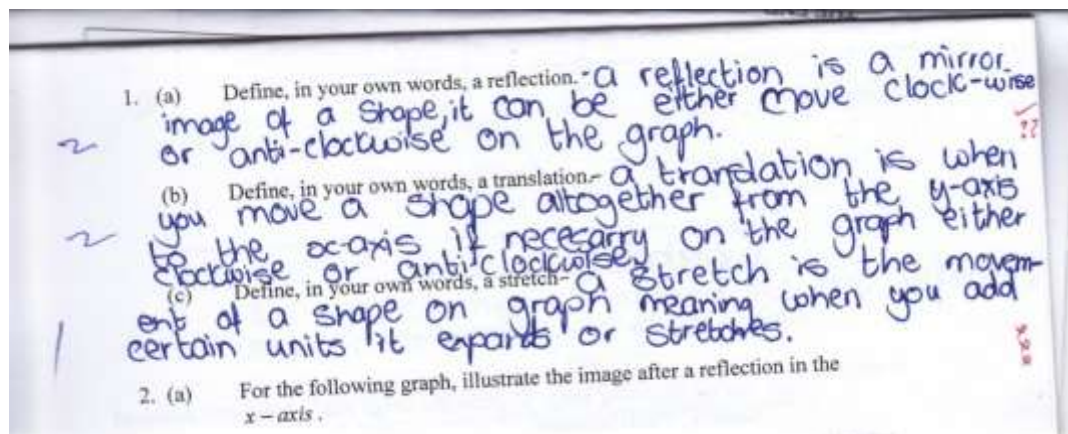
- a) *Define, in your own words, a reflection.*
- b) *Define, in your own words, a translation.*
- c) *Define, in your own words, a stretch.*

This question required learners to write their verbal images (definitions) of the three transformations (reflection, translation and stretch) as they manifest on functions. It was insufficient for learners to define a concept using just its name. Vignettes 5.1 and 5.2, below, provide examples of how learners defined the three transformation concepts.

Vignette 5.1: Example of learner's definition of reflection, translation and stretch concepts



Vignette 5.2: Another example of learner's definition of transformation concepts



Some of the terms used by learners to define transformations were appropriate although they differed from the formal definitions in some cases (see Vignettes 5.1 and 5.2 above). Below is a question-to-question analysis of the learners' responses.

5.1.1.1 Learners' verbal images of reflection (Question 1a)

Reflection is formally defined as “a mapping that produces mirror images of points in lines or in polygons about a particular line called an axis of reflection”²² (Tapson, 2006).

²² Sometimes referred to as mirror line or line of symmetry.

Learners in the sample groups also used terms like *flipped*, *inverted image* or *symmetrical image* in place of the term *mirror image*, as could be expected when giving their own definitions of reflection. Most of the learners struggled to express their definitions in correct English, and some definitions suffered from a lack of precision. However, after careful scrutiny of the responses, the following categories of definitions were constructed for assessment:

- (a) A *correct* definition of reflection, which refers to the two aspects of reflection, namely, the relative positions of the original object, and its image with respect to the axis of reflection.
- (b) A *partly correct* definition, which refers to only one of the two aspects of reflection mentioned in (a), above.
- (c) An *incorrect definition*, which is either so broad that it applies to transformations in general and not reflection specifically, or is not related to reflection at all.
- (d) *Did not attempt to answer*, where the learner left blank the space provided for the answer.

Some learners were able to give a *correct* description of what a reflection is. The examples below are given using the learner's own words:

- It is a transformation of an image about a line where the shape is mirrored on the other side of the line.
- The formation flipped or that flips to another place about a certain axis or line.
- It is when a graph is transformed through a line of symmetry. It produces a mirror image of that graph.
- It is when a specific shape or line has a mirror like image about the x -axis or y -axis.
- The mirror image of a shape about the y - or x -axis.
- It is a repetition of an object across the y -axis or x -axis not changing the size or shape but changing the coordinates.
- A mapping that produces a mirror image of a point, line or polygon about a particular line.

- A copy of an image that is exactly the same as the original. It is about a certain line e.g. $x=0$.

Most of the *incomplete* definitions did not refer to the axis of reflection. The examples below are given using the learners' own words, and the researchers reasons why they are considered incomplete are given in brackets:

- It is creating a mirror image of a particular object (No mention of relative position of the original and image function in respect to reflection).
- It is when an image is flipped (No mention of axis of reflection).
- It is a mirror image of an object or a shape (No mention of axis of reflection).
- It is the image, which is symmetrical and exactly the same, on the other side (No mirror line).
- It is the same graph just on different side depending on where the graph is reflected (No mirror line).
- It is an image produced from an original picture (No mention of how is it produced).
- It is an exact replica of the object (No mention of how is it replicated).
- It is repetition of an image across the x -axis or the y -axis not changing the size or shape but changing the coordinates (No mention of how the image comes about).
- When a graph makes an image about the line $x=0$ or $y=0$ (No source of the image).
- When an image is mirrored in a specific direction (No mirror line).
- An object showing on another set accurately (No mention of how it is showing).

Some of the definitions given by learners were *incorrect*. The examples below are given using the learner's own words, and the reasons why they are considered incorrect are given in brackets:

- It is the image that is exactly the same as the ordinary (All transformations have images. The term ordinary is ambiguous).
- It is a way of changing the position of diagrams on a graph (Not clear how – all transformations can change position).
- An image of a structure, object or picture (It could be any transformation).

- An exact replica of the object (No mention of how it is replicated).
- Reflection is an image viewed the same from same distance (Not clear how it is different from images of other transformations).
- Transformation in which an image of the original object is shown (No indication of what the image is and how it is formed).
- It is an image of a structure, object or picture or shape (No mention of how the image is formed).
- It is an object showing on another set accurately (Not clear what the other set is like).
- Plotting points juxtaposed to each other (Meaning not clear).
- A glance of the same picture (No second picture mentioned and what it looks like).
- It is the way of changing the position of diagrams on a graph (No aspect differentiating it from other transformations).
- It is an act of casting back an image so it can be reflected (Not clear).

Of the 96 learners in the full sample group, only 47 (49%) defined reflection correctly, mentioning both its two aspects, and 38 (40%) defined reflection incompletely, mentioning only one aspect. Of the incorrect definitions, 6 (6%) were too general, and 5 (5%) showed misconceptions about reflection. No learners left the answer space blank. The frequency count can be seen in Table 5.1, section 5.1.1.2 below.

5.1.1.2 Frequency analysis of verbal definitions of reflection of a function

In order to have a clearer comparative picture of how learners defined reflection, a frequency table for the responses (Table 5.1) and a bar graph (Figure 5.1) were created.

Table 5.1: Evaluation frequencies of learners' verbal images of reflection of a function (Question 1a) (n=96)

Assessing the Concept Image (Reflection)		Frequency				Relative Frequency	Cumul. Percentage
		Sample A	Sample B	Sample C	Sub-total		
a	Correct image	13	24	10	47	0.49	100.0
b	Partly correct image	13	14	11	38	0.40	51.0
c	Incorrect image	4	4	3	11	0.11	11.0
d	Did not attempt	0	0	0	0	0	0
TOTAL		30	42	24	96	1.00	

Table 5.1 shows that most learners (89%) had valid or partially correct ideas about reflection although some found it difficult to define it well or convincingly. This was the same in all three samples. The multiple-bar chart for samples A, B and C is shown below, in Figure 5.1.

Figure 5.1: Graph showing frequencies of learners' verbal images of reflection of functions (n=96)

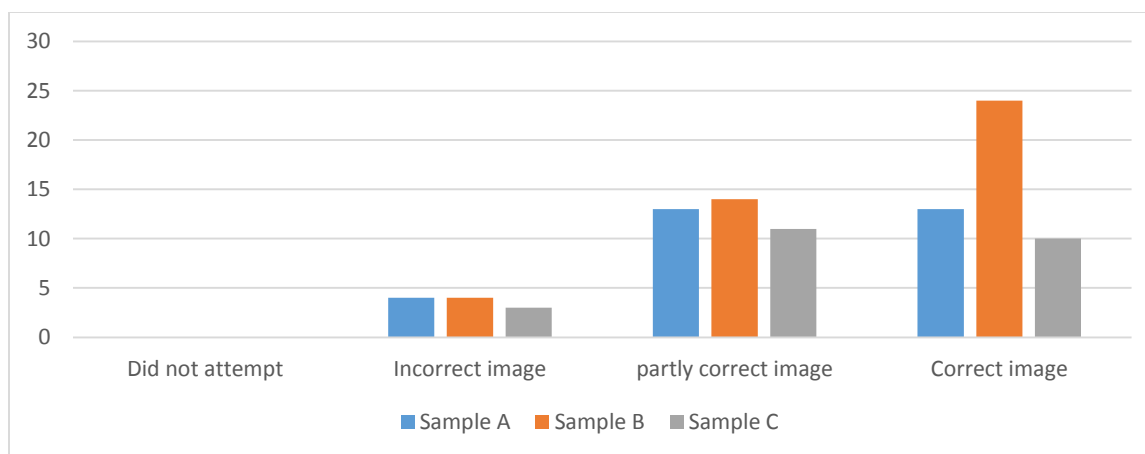


Figure 5.1 displays learners' verbal definitions of reflection within each of the samples. The first bar (blue) refers to Sample A, the middle bar (orange) to Sample B, and the last bar (grey) to Sample C, in the categories of *Correct Image*, *Partly Correct Image*, *Incorrect Image*, and *Did not Attempt*. It can be seen from the graph that all learners attempted to define the concept reflection. The highest percentage of learners (49%) had correct concepts of reflection, while 40% had partly correct ideas about reflection. The lowest number in each case (11%) is of learners who had misconceptions.

5.1.1.3 Learners' verbal images of translation (Question 1b)

Translation is formally defined as “a mapping that changes position of a point, line or a polygon by sliding it in a specific direction through a specific distance” (Tapson, 2006). This description has two aspects, namely that points move the same distance, and movement is in a common direction. Both of these aspects need to be stated for the definition to be considered complete. The process of evaluating learner definitions was complicated due to the varied language used, and the sometimes imprecise descriptions, hence careful scrutiny was necessary. The same parameters were used to assess the answers as in section 5.1.1.1:

- (a) A *correct* definition of translation, which refers to both aspects of translation namely (i) displacement (distance) and (ii) specific direction (or just ‘in a straight line’ displacement).
- (b) A *partly correct* definition which refers to only one of the two aspects of translation mentioned in (a).
- (c) An *incorrect definition* which is either so broad that it applies to transformations in general and not translation specifically, or is not related to translation at all.
- (d) *Did not attempt to answer*, where the learner left blank the space provided for the answer.

The learners' descriptions that were accepted as *correct* included at least one of the following terms: *displacement*, *movement*, *slide*, *change of position* and *shift*, for a specific

distance, followed by *upward, downward, to the left, to the right* or *in a straight line/specific direction*.

Examples of *correct* definitions given by learners are shown below:

- The movement of all points of a graph in a particular factor either up, down, left or right.
- When every point of the body moves the same distance in the same direction.
- When the original object is moved certain units up or down, left or right.
- Transformation that moves points or shapes the same distance in a common direction.
- Is to move or shift an image to certain units up/down or to left/right.
- It is a transformation of an image either going up, or down, left or right by certain units.
- The way of changing the position of diagrams with given units either upwards or downwards.
- Translation is when a point $(x;y)$ is moved/shifted by units up, units down, units left or units right.

Examples of *partly correct* definitions and the reasons why they are considered incomplete (in brackets) are given below:

- Movement of a graph upwards, downwards, or to the right or to the left (No emphasis on same distance).
- When a graph has been shifted either upward or downward or sideways (No mention of distance).
- It is moving a graph through a slide i.e. to the left or right, downwards or upwards (No mention of distance).
- It is the transformation of shifting an image to the left, right, up or down (No mention of distance).
- A transformation that moves points in a common direction (No indication of same distance).
- It moves points or shapes in the same direction (No indication of same distance).

- It is when an object or shape or graph is moved to the right or left, front or back (No indication of same distance).
- It is when a graph moves vertically or horizontally, does not change shape (No indication of specific distance).

Examples of *incorrect* definitions and the reasons why they are considered incorrect (in brackets) are given below:

- Shift of image from its original point/place/coordinates to another (No mention of what the shift is and how far).
- Movement of a diagram across the y -axis and the x -axis changing the coordinates but not the image (Not different from other transformations).
- Shifting or moving to certain positions (Not different from other transformations).
- Moving an image of a graph to a different position from where it was (Neither direction nor distance mentioned).
- Moving the object to the next point, from one position to the other (Neither direction nor distance mentioned).
- The repeat of a diagram in a graph (Not specific of where and how).
- Movement of a shape along a Cartesian plane with no change in shape or size (All congruencies do that).
- Movement that occurs when you rearrange objects (Not clear or specific).
- When a mirror image moves certain units from its original position (Term *mirror image* inappropriate, no direction, no distance).
- When a figure moves towards point A to B, whether it rotates or moves up down or left and right (Ambiguous).
- A type of transformation whereby an object can either be reflected or rotated i.e. the object and the image are not of the same distance (Ambiguous).
- When a point is moved around changing in position or size. (Change of size is inappropriate).
- When a point is moved around a number of degrees, does not change shape but its coordinates, (No degrees involved in translation).

- When a point is moved, when an image is resized (Word *resized* is inappropriate).
- When points are flipped for e.g. $(x; y) \rightarrow (y; -x)$ (Word *flipped* is inappropriate and the formula is for a 90° anticlockwise rotation).
- Change in position by a point in a plane diagram e.g. $(x; y) \rightarrow (-y; x)$ (Formula is for a 90° clockwise rotation).
- Change of coordinates $(A(x; y) \rightarrow A'(-x; y)$ (Formula is for a reflection about the y -axis).
- Is when you enlarge part of a drawing in which you add (Word *enlarge* is inappropriate).
- It is an image of shape that is reflected upon the x -axis or y -axis (Word *reflected* is inappropriate).
- It is when the graph moves as a whole across throughout the set of axis e.g. $(x; -y)$ or $(-x; -y)$ (Formula for reflection in y -axis).
- It is when you move a shape altogether from the y -axis to the x -axis if necessary on the graph either clockwise or anticlockwise (Facts mixed up).
- It is the movement of the image and how it is moved or translated or rotated (Ambiguous and use of word *rotated* inappropriate).
- It is when someone interprets a certain language to others that don't understand (Linguistic instead of mathematical context).
- It is when a figure moves towards a point A from B whether it rotates or moves up, down or left or right (Ambiguous).
- Is an object moved around a number of degrees but it does not change its shape. Only its coordinates. (Appropriate for rotation).

Of the 96 learners in the sample, 76 learners (79%) used at least one of the correct descriptive terms. However, of those, only 25 (26%) could define translation completely, 51 (53%) gave *partly correct* definitions using accepted terms, 15 (16%) gave *incorrect definitions* and 5 learners (5%) *did not attempt* to answer. Complete information of frequencies is given below in section 5.1.1.4 and Table 5.2.

5.1.1.4 *Frequency analysis for verbal definitions of translation of a function*

In order to have a clearer comparative picture of how learners defined translation, a frequency table for the responses (Table 5.2) and a bar graph (Figure 5.2) were created.

Table 5.2: Evaluation frequencies of learners' verbal images of translation of a function (Question 1b) (n=96)

Assessing the Concept Image (Translation)	Frequency				Relative Frequency	Cumul. Percent
	Sample A	Sample B	Sample C	Sub-total		
a Correct image	1	14	10	25	0.26	100.0
b Partly correct image	24	18	9	51	0.53	74.0
c Incorrect image (misconception)	4	7	4	15	0.16	21.0
d Did not attempt	1	3	1	5	0.05	5.0
TOTAL	30	42	24	96	1.00	

Table 5.2 shows that most learners (85%) had the correct idea about translation but half of those were unable to provide accurate definitions. Learners in sample A were the most unable. The multiple bar chart for samples A, B and C is shown in Figure 5.2 below.

Figure 5.2: Graph showing frequencies of learners’ verbal images of translation of a function (n=96)

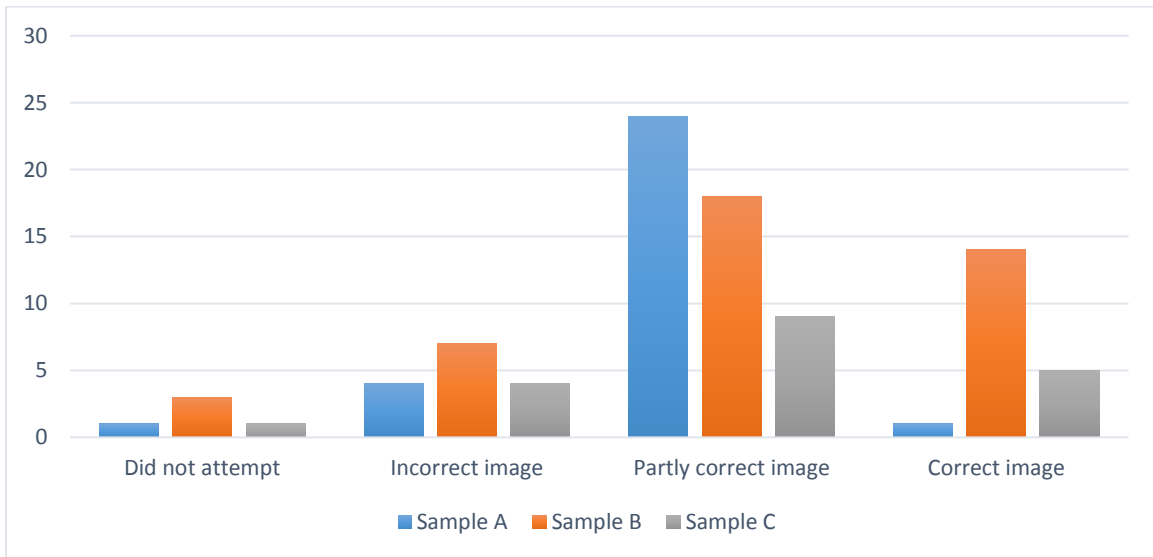


Figure 5.2 displays learners’ verbal definitions of translation per sample. In all three samples, a few learners did not attempt to define the concept reflection. The numbers of learners who had misconceptions was greater than for reflection. A few more learners had partly correct ideas about translation than reflection, and the number of learners who had the correct idea about translation was less than for reflection.

5.1.1.5 Learners’ verbal images of stretch (Question 1c)

Stretch is formally described as “a mapping that changes position of all points outside a particular line (invariant line) away from that line or towards that line in a specific given scale” (Tapson, 2006). It can be either an outward stretch or an inward stretch (compression/contraction) in the x -direction or y -direction (i.e. away or towards the x -axis or the y -axis). For the definition to be complete, there are three aspects that have to be included, namely, (i) the invariant line, (ii) movement of points which are outside the invariant line, away from or towards the invariant line, and (iii) scale (proportion of distance of original point from invariant line to that of its image from the invariant line). Many learners had difficulty defining stretch and it was also more complex for the

researcher to evaluate and categorize learners' definitions. The same categories as in sections 5.1.1.1 and 5.1.1.3 were used to evaluate the definitions given:

- (a) A *correct* definition of stretch, which refers to all three aspects of the concept.
- (b) A *partly correct* definition which refers to only one or two of the three aspects of stretch mentioned in the definition above.
- (c) An *incorrect definition* which is either so broad that it applies to transformations in general and not stretch specifically, or is not related to stretch at all, or is ambiguous.
- (d) *Did not attempt to answer*, where the learner left blank the space provided for the answer.

For a learner's description to be considered *correct* it had to include at least one of the following terms: *expansion/contraction, extension/compression, increase/decrease in size, points move apart/closer, making longer/shorter, widening/narrowing, lengthening/shorting, fatten/make slim, pull/squeeze on both ends, enlarge/shrink, spacing/bring points closer*, all followed by *a factor, specific scale factor, certain scale factor*.

Examples of *partly correct* definitions of stretch are given below:

- Enlargement of the graph or coordinates by a factor.
- Enlargement of the graph by a specific scale factor.
- The type of transformation whereby an object is enlarged by a certain factor depending on what is given.
- It is when a graph is expanded in a certain scale.
- A shape is increased by a certain factor vertically or horizontally.
- Enlarging a graph by means of spacing the points by a given ratio.
- An expansion of a graph depending on the factor.
- When a graph is pulled up at the top and down at the bottom end or when a graph is pulled horizontally on both ends.
- When the graph increases whether upwards or downwards.
- Dooming of a graph on image. (Not sure of what the learner meant)

- It is an enlargement of a graph/shape either negative (smaller/reduction) and positive (enlargement) of the shape/graph.
- An expansion or compression of a common translation.
- A graph that opens up wide, left and right, it stretches.
- When the graph is lengthened/widened horizontally or vertically.
- To pull a shape from its original to a narrow figure vertically or horizontally.
- When the graph is lengthened/shortened horizontally or vertically.
- When a graph on the Cartesian plane is pulled vertically and horizontally.
- When a graph is widened horizontally or vertically.
- When a graph is made bigger through stretching it either horizontally or vertically.
- When you pull something on both ends.
- It is to extend something vertically or horizontally to another point.
- It is when a graph is expanded on both sides.
- It is when a figure is made bigger horizontally and vertically.
- Making something larger/bigger or smaller.
- Enlarging or shrinking an object.
- When something gets pulled, making it longer.
- Making the graph larger and longer than the original.
- When an image is widened.
- Spacing the points by a given ratio.
- Enlarging the image using factors by multiplying all values by the factor.
- Graph is made to appear longer than its usual length.

Some examples of definitions considered *incorrect* are given below:

- A transformation where a graph's x -axis or y -coordinates are moved.
- It is the drawing of a shape on the Cartesian plane.
- It is when an image moves horizontally or vertically on the axis.
- It is when a trig graph or a functional graph doesn't change shape or period but its y -values.
- Is a free hand estimated drawing, an incorrect drawing. Inaccurate drawing.

- Stretch is a fable semi-conductor (The researcher has no idea what the learner's frame of reference for this response was).
- Extension of the graph by changing the graph by adding or subtracting.
- Something that is drawn without rules (Not clear).
- It is when a figure has been distorted (Not clear what sort of distortion).
- A straight line that is 180 degrees (Meaningless).
- It is a freehand drawing that involves only the main coordinates (Meaningless).

Of the 96 learners in the sample, 75 (78%) gave at least one of the aspects of stretch, 26 (27%) gave largely *correct* definitions, but 49 (51%) gave only *partly correct* definitions. A total of 13 (14%) learners had *incorrect* definitions about stretch and 8 (8%) *did not attempt* an answer. It is evident from these results that learners found it more difficult dealing with the concept of stretch than with reflection or translation. None of the definitions given by learners mentioned an invariant line. The frequencies of these results are shown in Table 5.3 in section 5.1.1.6 below.

5.1.1.6 Frequency analysis for verbal definitions of a stretch of a function

Table 5.3: Evaluation frequencies of learners' verbal images of a stretch of a function (Question 1c) (n=96)

	Assessing the Concept Image (Stretch)	Frequency				Relative Frequency	Cumul. Percent
		Sample A	Sample B	Sample C	Sub-total		
a	Correct image	4	14	8	26	0.27	100.0
b	Partly correct image	22	19	8	49	0.51	73.0
c	Incorrect image (misconception)	3	6	5	13	0.14	22.0
d	Did not attempt	1	3	4	8	0.08	8.0
	TOTAL	30	42	24	96	1.00	

Table 5.3 shows that most learners (78%) had some idea of stretch although the majority could not define it completely. The graph below gives a comparative picture.

Figure 5.3: Graph showing frequencies of learners' verbal definitions of a stretch of a function (n=96)

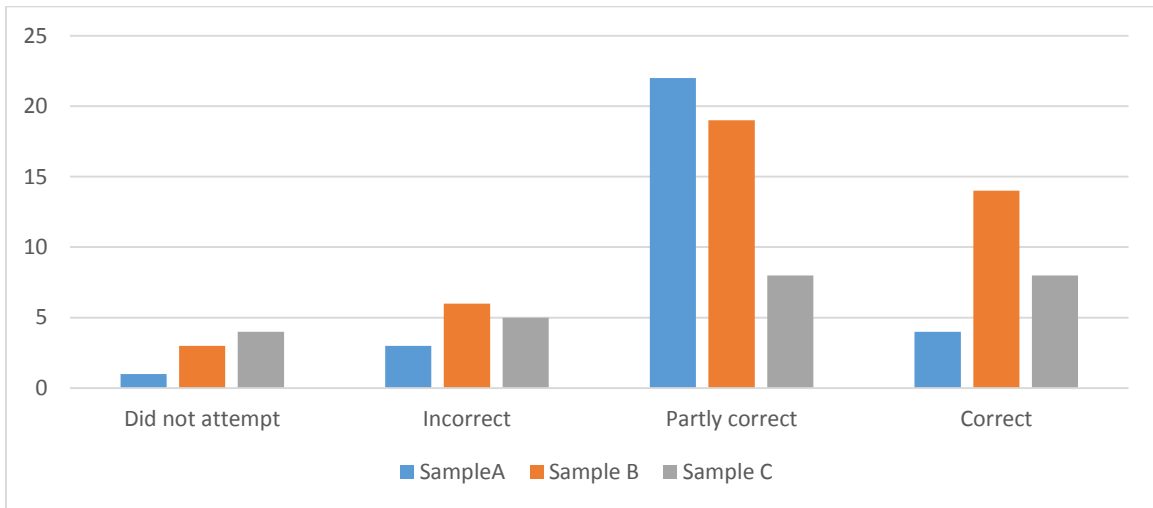


Figure 5.3 displays learners' verbal definitions of stretch. The graph shows that, in all three samples, only slightly more than half the learners (51%) had a *partly correct* idea about stretch, and even fewer (27%) could accurately define it.

The horizontal compound bar chart below (Figure 5.4) correlates and summarizes information from the clustered column multi-bar charts, to enable a comparison of the information about learners' concept images of the three transformations.

Figure 5.4: Graph comparing frequencies of *Correct*, *Partly correct*, *Incorrect*, and *Did not attempt* evaluations about definitions of transformations (n=96)

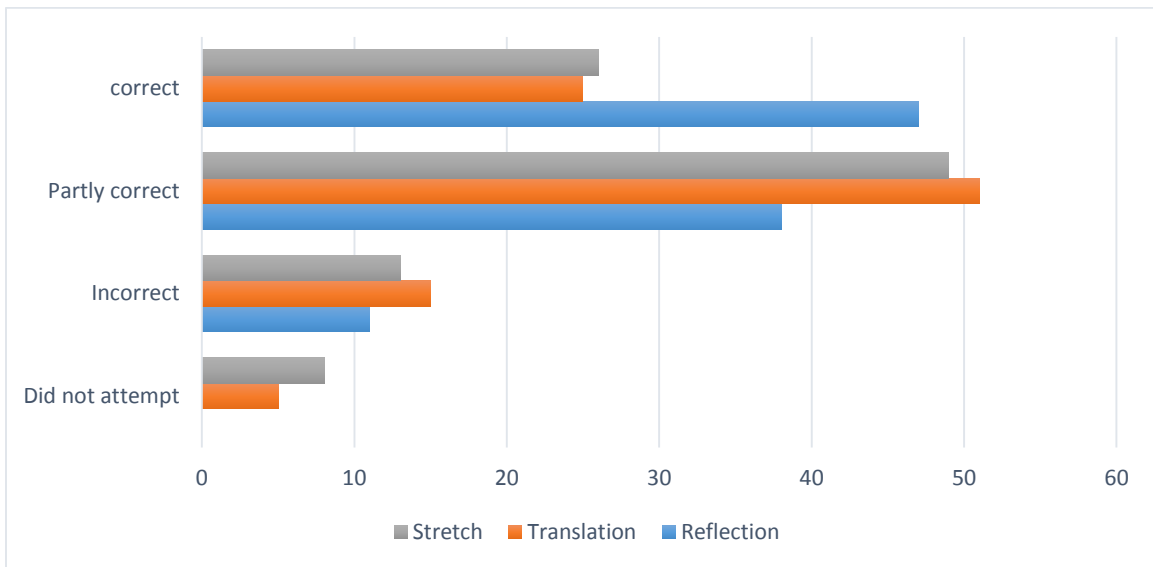


Figure 5.4 displays a cross comparison of learners responses in each evaluative category. The graph is positively skewed²³ and suggests that, in all three samples, learners have some idea of transformation concepts but have difficulty describing them accurately. Reflection is better understood than translation or stretch.

5.1.2 Graphical Interpretations and Representations

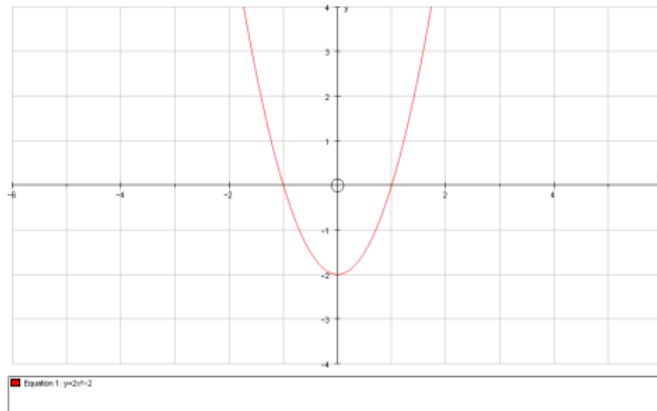
Question 2 of the diagnostic test required learners to draw graphical images of the three transformations (reflection, translation and stretch) as they manifest on some given functions. Graphical images are visuals with mathematical meaning, and in this study, learners were required to draw them on the Cartesian plane. How learners responded to questions of interpreting or drawing such images is discussed below.

²³ A statistical distribution where most scores are lower 50% on the scale

5.1.2.1 Learners' graphical images of reflection (Question 2a)

The question read as follows:

For the following graph, illustrate the image after a reflection in the x -axis.



The correct image is an 'n' shaped parabola with maximum point at (0;2) intersecting the original 'u' shaped parabola at (-2;0) and (2;0). Of the 96 learners, 65 (68%) could reflect about the x -axis as required while 19 learners (20%) reflected incorrectly about the y -axis. Three learners (3%) reflected about lines other than the axes, 6 learners (6%) drew diagrams that were not reflections, and 6 learners (6%) did not attempt to answer the question.

The list below describes some of the partly correct (PC) and incorrect (I) images drawn by learners:

- An image of reflection about the y -axis followed by a reflection about the x -axis or vice versa (I).
- An image of reflection in the y -axis (I).
- An image of a translation in the direction of x followed by a reflection in the x -axis or vice versa (I).
- A reflection in the x -axis followed by a translation 10 units to the right and 6 units downward (PC).

- An image of translation 10 units to the right and 6 units upward followed by a reflection in the x -axis (I).
- A rotation of 90° clockwise about $(0;-2)$, also about $(0;0)$ (I).
- A reflection in $y=-3$ followed by a translation to the right (PC).
- A translation to the right or downwards by 1 unit (I).
- A reflection in $y=-2$ or in $y=-6$ (PC).

A comparison of the results is provided in Table 5.4 and Figure 5.5, below.

Table 5.4: Evaluation frequencies of learners' graphical images of reflection of a function (Question 2a) (n=96)

Assessing the Concept Image (Reflection)	Frequency				Approx. Percent	Cumul. Percent
	Sample A	Sample B	Sample C	Sub- total		
a Correct image	10	35	20	65	68.0	100.0
b Partly correct image	17	1	1	19	20.0	32.0
c Incorrect image (misconception)	2	2	2	6	6.0	12.0
d Did not attempt	1	4	1	6	6.0	6.0
TOTAL	30	42	24	96	100.0	

Table 5.4, above, shows that the majority of learners (68%) could reflect the graph correctly. A number (20%) reflected about a line other than the prescribed line and 6% did not reflect the function but drew some other figure. A total of 6% did not draw anything.

Figure 5.5: Graph showing frequencies of learners' graphical images of reflection of a function (n=96)

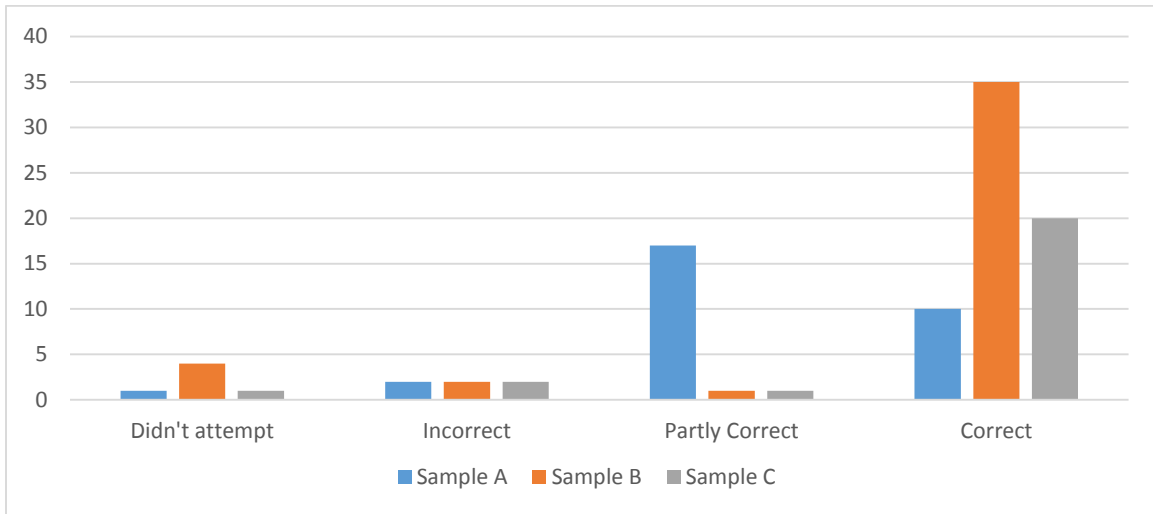
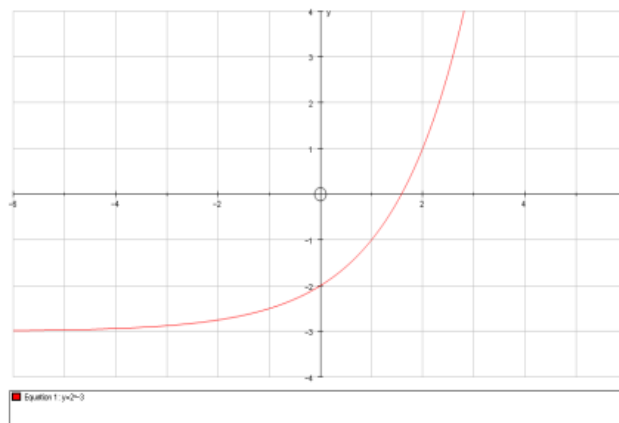


Figure 5.5 displays learners' graphical representations of reflection across the three samples. The graph shows that many learners could do the graphical representation well but the difference in abilities between Sample A and Sample B was significant.

5.1.2.2 Learners' graphical image of translation (Question 2b)

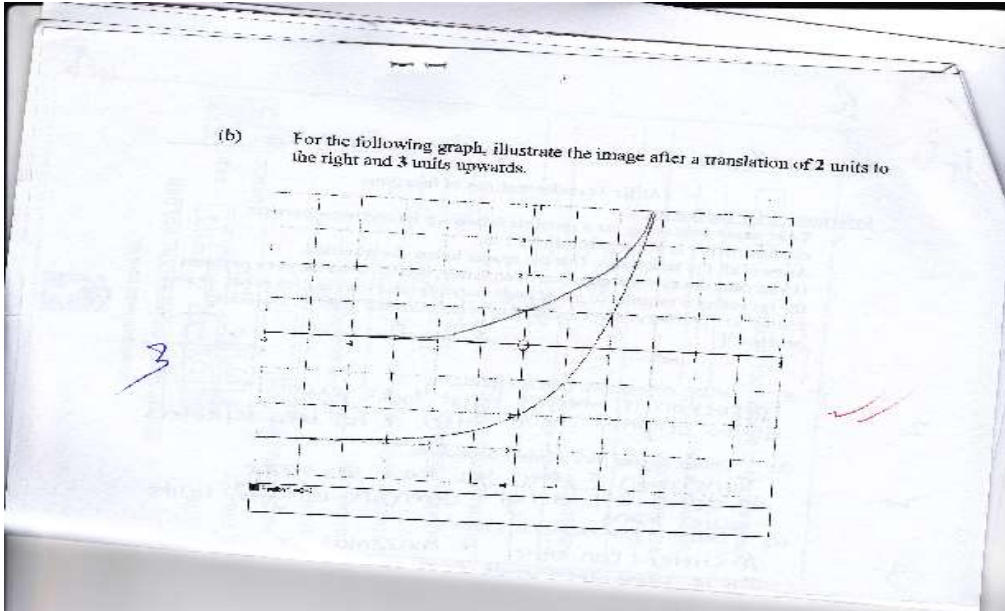
The question read as follows:

For the following graph, illustrate the image after a translation of 2 units to the right and 3 units upwards.



This question required learners to slide the exponential graph two units to the right and then three units upwards or vice versa. Most learners (66%) performed the translation fairly correct image although some lacked complete accuracy. The remaining 34% had either partly correct, incorrect or blank. Examples of learners' correct and incorrect diagrams are given in Vignettes 5.3 and 5.4 respectively.

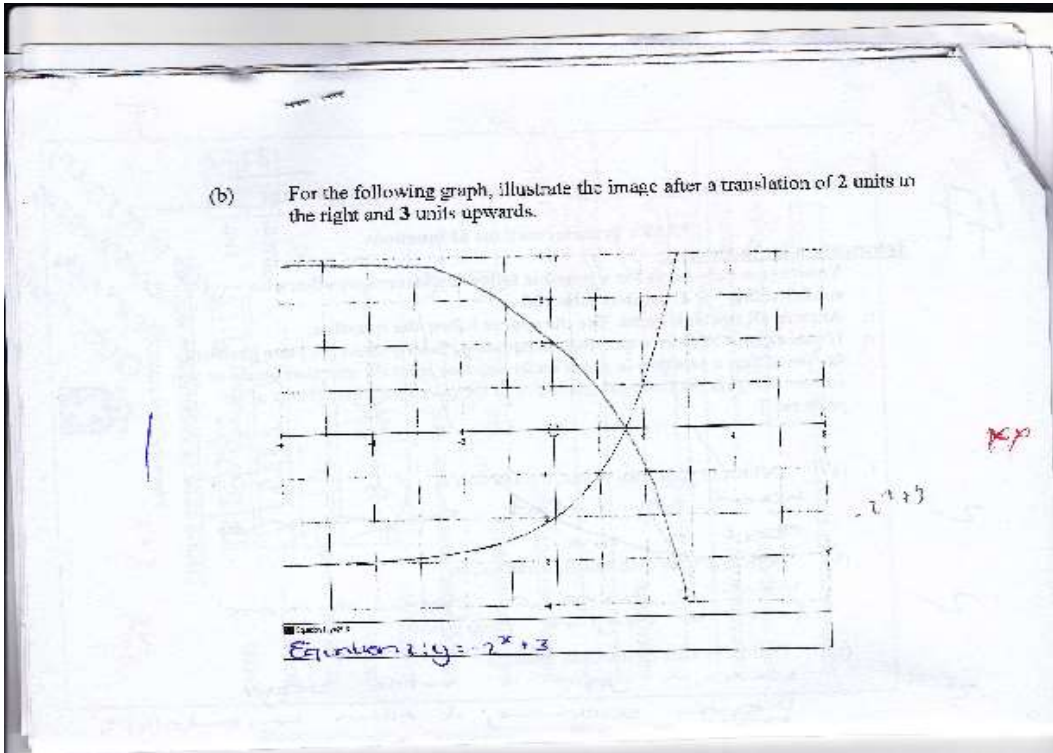
Vignette 5.3: Example of learners' correct image of a translation of a function



The list below indicates some of the partly correct (PC) and incorrect (I) images drawn by learners:

- Translated vertically upwards only (six learners)(PC).
- Translated horizontally to the right only (five learners) (PC).
- Translated upwards and to the left (three learners) (PC).
- Reflected about y -axis then upward translation (two learners) (I).
- Translated vertically downwards and to the right (PC).
- Rotated 90° clockwise, centre at the origin (I).
- Stretched in the x -direction with $x = -5$ invariant (I).
- A line $y=x-2$ (I).
- Upward translation through 4 units (PC).

Vignette 5.4: Example of learner's misconceived graphical images of translation of a function



The misconception illustrated in Vignette 5.4 is a reflection about the x -axis.

Sixty-three learners (66%) performed the translation accurately while 20 (23%) did the directions correctly but were not accurate with the units. Five learners (5%) did not attempt a drawing, but did not give a reason. Six learners had misconceptions about how to illustrate translation graphically. This information is shown in full in Table 5.5 and Figure 5.6, below.

Table 5.5: Evaluation frequencies of learners' graphical images of translation of a function (Question 2b) (n=96)

Assessing the Concept Image (Reflection)	Frequency				Approx. Percent	Cumul. Percent
	Sample A	Sample B	Sample C	Sub-total		
a Correct image	19	28	16	63	66.0	100.0
b Partly correct image	9	8	5	22	23.0	34.0
c Incorrect image (misconception)	1	2	2	6	6.0	11.0
d Did not attempt	1	4	1	5	5.0	5.0
TOTAL	30	42	24	96	100.0	

From this Table 5.5, it can be seen that a fairly high percentage of learners had graphical images of translating a function that were correct (66%). Inaccuracies are common (23%) and many misconceptions were evident. Learners who did not attempt a drawing amounted to 5% of the sample. The graphical representation of these results is shown in Figure 5.6, below.

Figure 5.6: Graph showing frequencies of learners' graphical images of translation of a function (n=96)

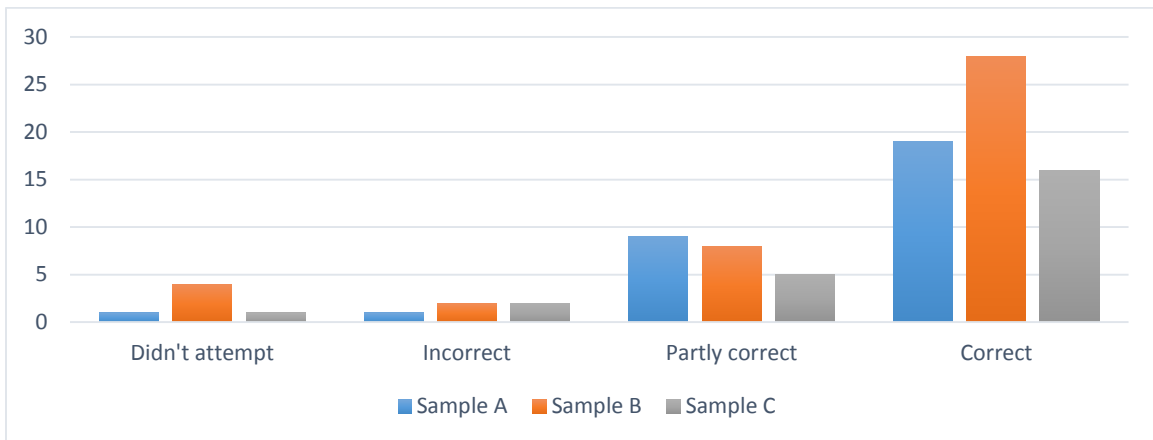
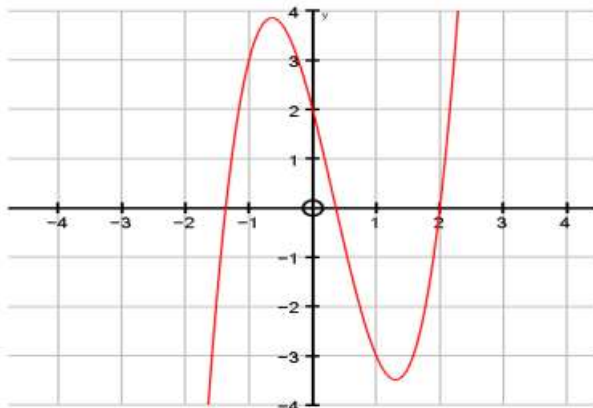


Figure 5.6 displays learners' graphical representations of translation per sample. The graph shows that many learners understood graphical translation. A significant number lacked accuracy, however.

5.1.2.3 Learners' graphical images of compression (Question 2c)

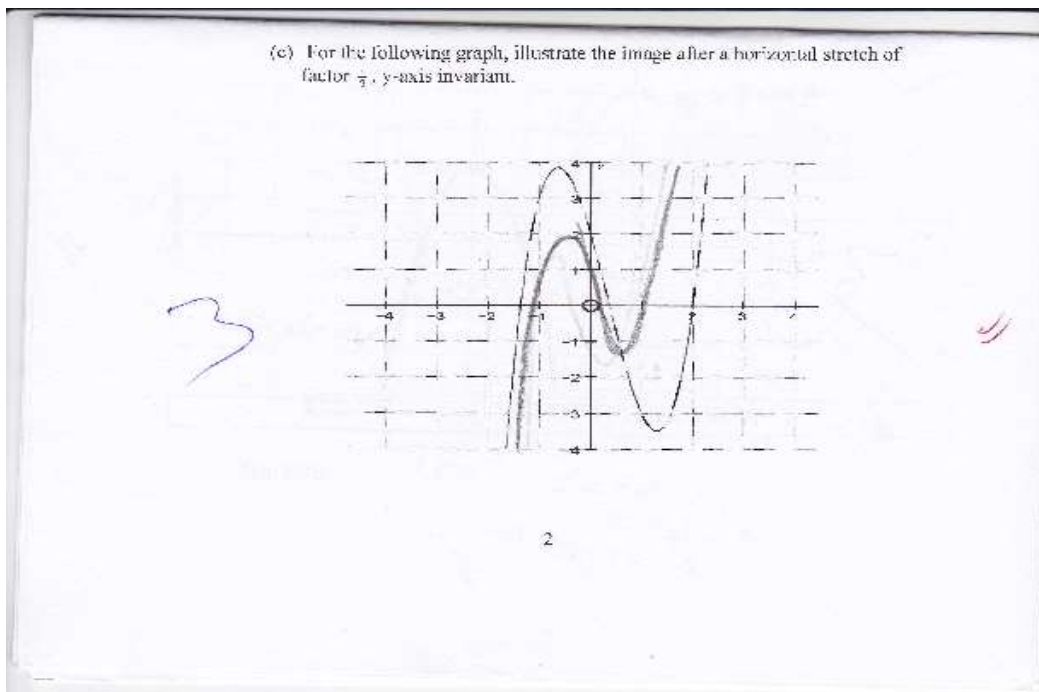
The question read as follows:

For the following graph, illustrate the image after a horizontal stretch of factor $\frac{1}{2}$, y-axis invariant.

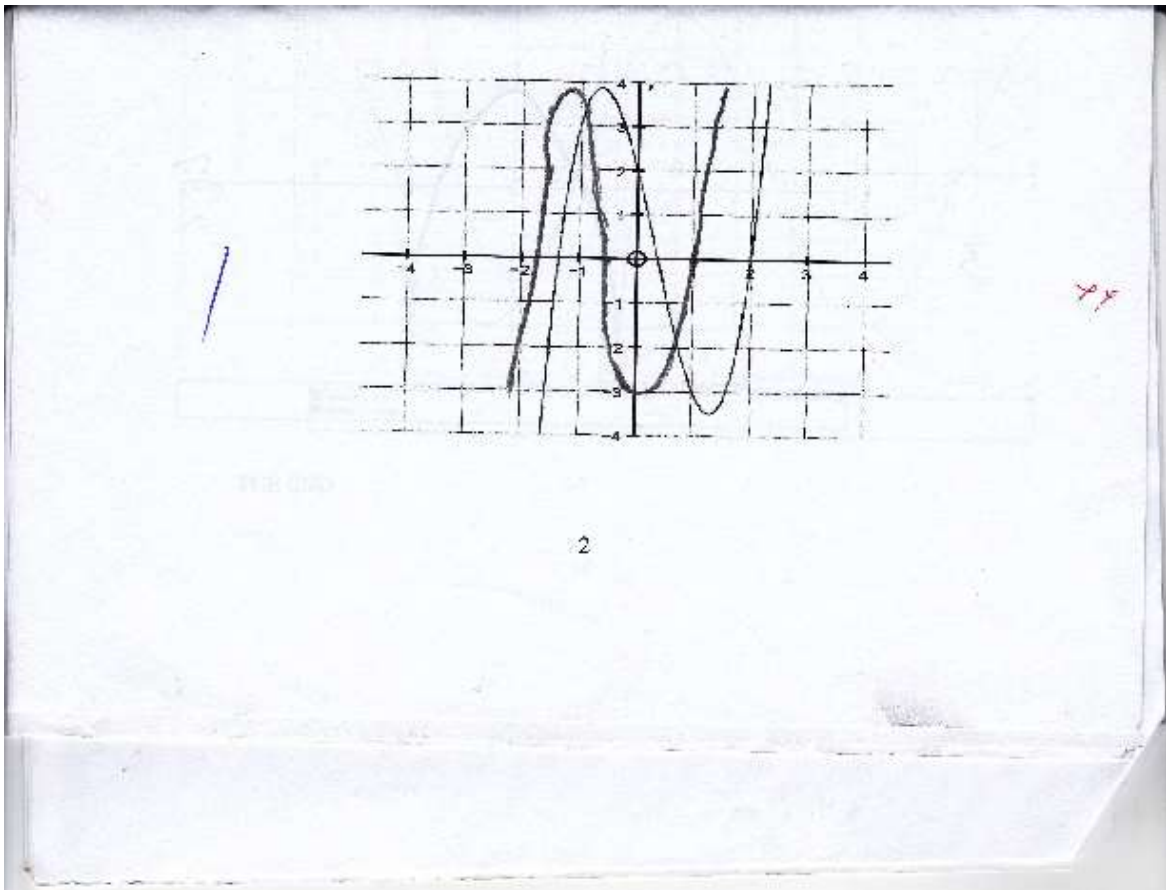


The question required the learners to compress a cubic function graph horizontally by factor $\frac{1}{2}$ with the y -axis invariant. The equation of the function was not given. Eleven learners (11.5%) performed the stretch accurately, while 35 (36.5%) produced images that did not show a stretch significantly different from the original. Twenty-three (24%) learners did not attempt a drawing, giving reasons such as ‘*I don’t understand*’, ‘*can’t figure out what is needed*’, ‘*don’t know the equation, so can’t figure out where image lies after stretch*’ and ‘*cannot interpret the concept referred to in the question*’, although some did not give a reason why they had left the answer space blank. Examples of learners’ representations are given in Vignettes 5.5 and 5.6, below.

Vignette 5.5: Example of a learner’s partly conceived graphical image of stretch of a function



Vignette 5.6: Example of a learner's misconceived graphical representation of stretch of a function



Learner misconceptions, some partly correct (PC) and others totally incorrect (I), were evident from graphs drawn as described below:

- Translating to the left (twelve learners, see Vignette 5.6)(I).
- Translated to the right (two learners)(I).
- Reflecting about the y -axis (three learners)(I).
- Reflecting about the x -axis (one learner)(I).
- Horizontally pulling/outward stretch (eleven learners)(PC).

- Vertically pulling/outward stretch (two learners)(PC).
- Anticlockwise rotation (one learner)(I).
- Vertical compression/inward stretch (two learners, see Vignette 5.5)(PC).
 - One learner who drew two incorrect images stated ‘*I am not sure what the word invariant means*’ (I).
 - Horizontal outward stretch instead of compression (PC).
 - Vertical compression instead of horizontal compression (PC).

Learners seem to find it more difficult to provide stretch images graphically than translation or reflection images. This may be due to the fact that the concept of stretch is not as clear to them as the other two concepts are, possibly because the concept does not have adequate coverage in the NCS–CAPS syllabus (see section 1.1). Table 5.6 and Figure 5.7, below, present visual representations of the results obtained.

Table 5.6: Evaluation frequencies of learners’ graphical images of stretch of a function (Question 2c) (n=96).

Assessing the Concept Image (Stretch)	Frequency				Approx. Percent	Cumul. Percent
	Sample A	Sample B	Sample C	Sub-total		
a Correct image	4	3	4	11	11.5	100.0
b Partly correct image	17	15	3	35	36.5	88.5
c Incorrect image (misconception)	3	18	6	27	28.0	52.0
d Did not attempt	6	6	11	23	24.0	24.0
TOTAL	30	42	24	96	100.0	

The results in Table 5.6 show that very few learners could implement stretch (11.5%). If the graphical images for the three mappings are compared, it is clear that stretch is the most difficult for learners to represent graphically (see also Figure 5.7).

Figure 5.7: Graph showing frequencies of learners' graphical images of stretch of a function (n=96)

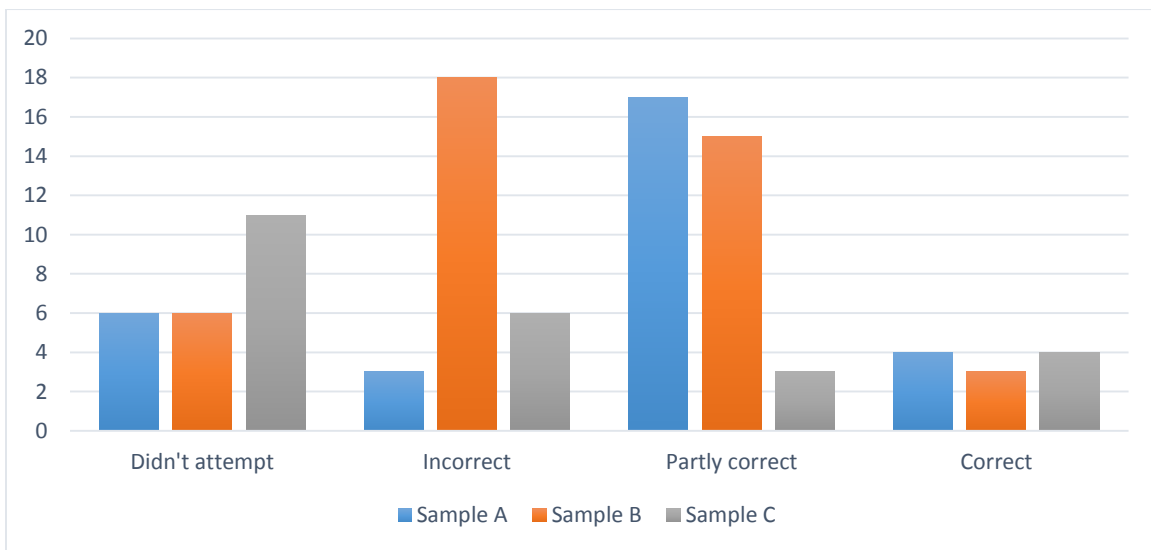
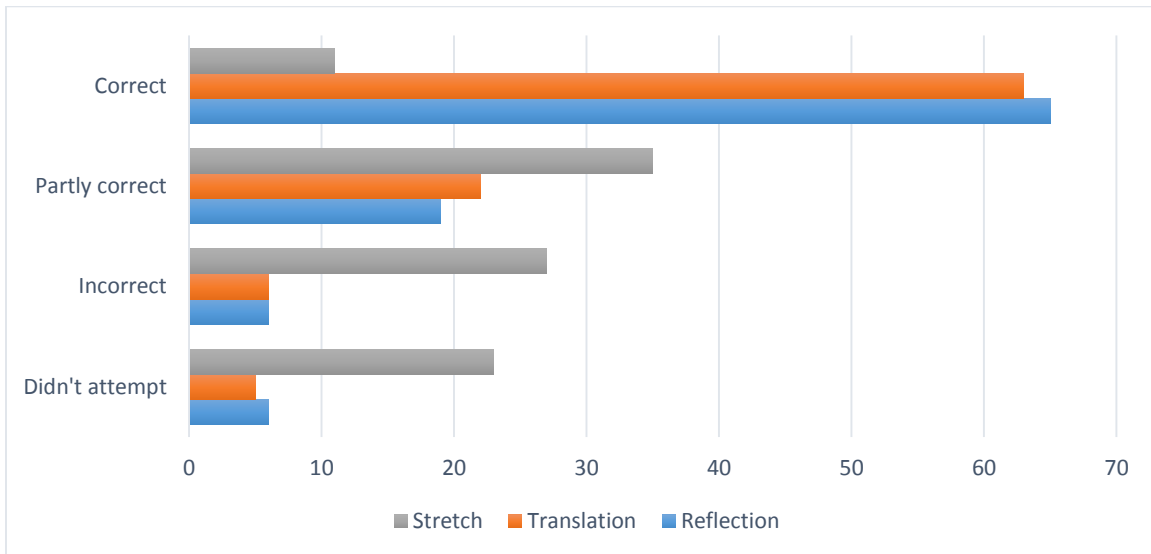


Figure 5.7 shows that learners' graphical representations of a horizontal stretch are extremely problematic. Most learners could not represent stretch correctly and more learners did not attempt to answer this question than any of the previous questions. Stretch is an issue which mathematics teachers need to note and attempt to correct. Figure 5.8, below, gives a cross comparison of the transformations.

Figure 5.8: Graph comparing frequencies of Correct, Partly correct, Incorrect and Did not attempt evaluations about graphical images of stretch of a function (n=96)



From Figure 5.8, it can be seen that the comparison of learners' graphical representations across the concepts shows that stretch is the most difficult graphical representation for learners to master, and reflection is the easiest.

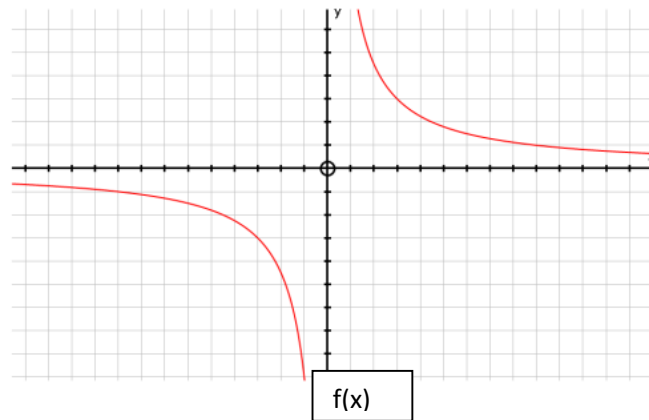
5.1.3 Graphical Interpretations from Symbolical Representations

Question 3 in the diagnostic test required learners to recognize transformation concepts from symbolical or algebraic images and illustrate them graphically.

5.1.3.1 Learner ability to recognize reflection from a symbolical image (Question 3a)

The question read as follows:

Illustrate, on the diagram, the image of $f(x)$ when it transforms to $f(-x)$. Describe fully the transformation involved.



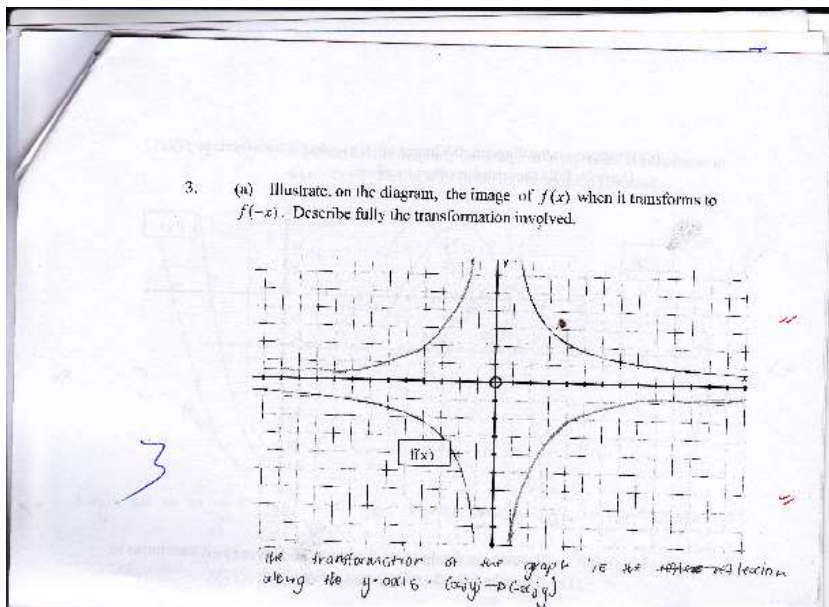
The question required the learners to recognize and describe the concept of reflection about the y -axis from a symbolical image. The function involved was a hyperbola but its equation was not given. Reflecting this function about the axes is relatively easy because there is no difference in the image whether reflected about the x -axis or the y -axis. For that reason it was expected that most learners, if not all, would be able to reflect correctly. Fifty-five (57%) learners drew the correct image in both the second and fourth quadrants (see example in Vignette 5.7) and described it correctly. Twenty-five (26%) drew an incomplete correct image in one of the two quadrants. Of these, 11 learners (11%) provided the correct description (a reflection about the y -axis) while 7 stated that it is a reflection about the x -axis. Nine learners (9%) did not draw the image, and some gave only written descriptions, which were mostly misconceptions.

There were very few totally correct responses. Some responses considered correct (C) and partly correct (PC) were as follows:

- Correct illustration. Reflection in x -axis; reflection in y -axis; 180 clockwise / anticlockwise rotation (PC).
- Correct illustration. Translated through x -axis (PC).

- Correct illustration. Points are reflected about the x-axis (PC).
- Reflection about x-axis. Correct illustration (PC).
- Correct illustration. Reflection along y-axis like $(x;y) \rightarrow (-x;y)$ (C).
- Incomplete illustration, only in the fourth quadrant (PC).
- Correct illustration, no description (PC).
- Correct illustration, translation along the y-axis (PC).
- Incomplete illustration (second quadrant), 'clockwise transformation' $(-x;-y) \rightarrow (-x;y)$ (PC).
- Correct illustration, "f(x) reflected on y-axis over x-axis" (C).
- Correct illustration, "The image of f(x) has been reflected" (C).

Vignette 5.7: Example of learner's correct graphical image $f(-x)$



Some misconceptions, evident in learners' answers are listed below:

- It is a reflection in $y=x$ (I).
- Translation through x -axis (I).
- Translation across $y=x$ (I).
- The image could have resulted from a translation or a rotation and reflection (I).
- States that it is translation in $y=-x$ (I).
- Reflected about line $y= -x$ (I).

Table 5.7: Evaluation frequencies of learners' abilities to recognize a symbolical image of a reflection of a function in the y -axis (Question 3a) (n=96)

Assessing the Concept Image (Reflection)	Frequency				Approx. Percentage	Cumulative Percentage
	Sample A	Sample B	Sample C	Sub-total		
d Did not attempt	4	3	2	9	9.0	9.0
c Incorrect image (misconception)	3	3	1	7	7.0	16.0
b Partly correct image	12	3	10	25	26.0	42.0
a Correct image	11	33	11	55	58.0	100.0
TOTAL	30	42	24	96	100.0	

A correct drawing of the image on the Cartesian plane would indicate that a learner could recognize reflection correctly from its symbolical image i.e. $f(-x)$. Table 5.7 shows that most learners (84%) could identify a reflection, although 26% of them could not draw the correct image. Small percentages (7% and 9%) had incorrect images or could draw no image at all.

Figure 5.9: Graph comparing frequencies of learners' abilities to recognise a symbolical image of a reflection of a function in the y-axis (n=96)

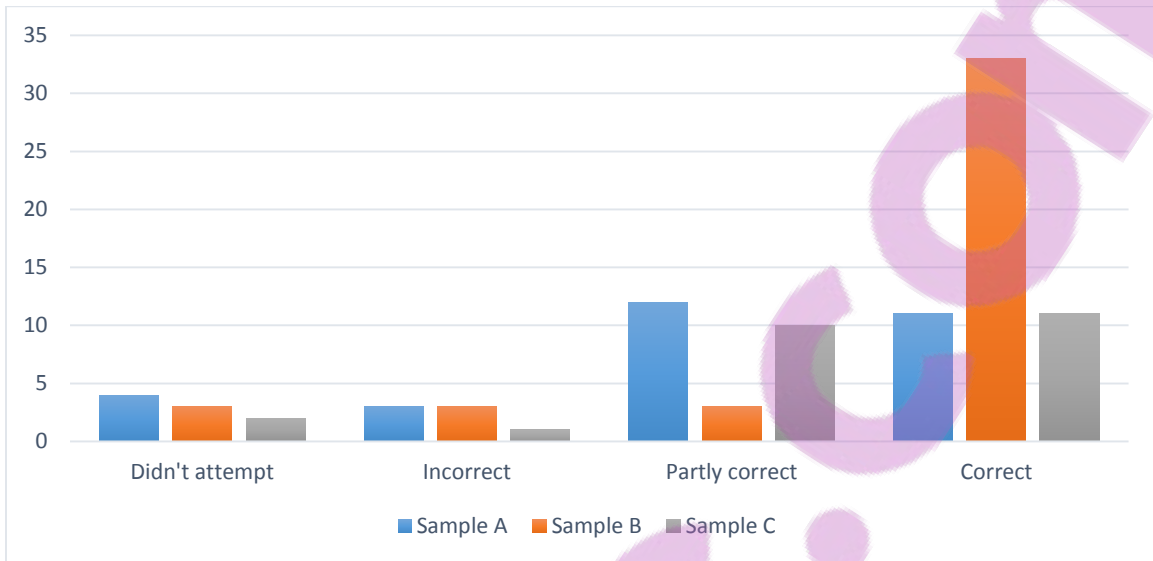


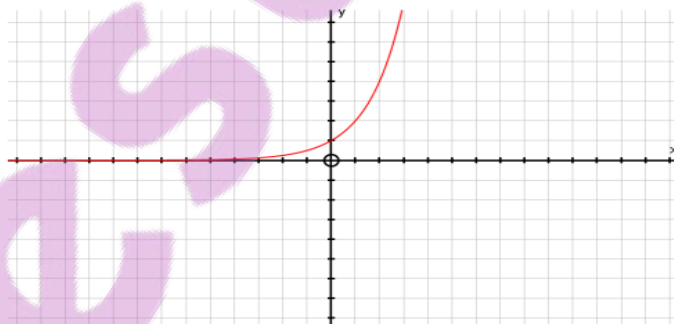
Figure 5.9 displays learners' ability to recognise reflection in the y-axis from its symbolical image. The graph shows that most learners could recognise the reflection formula.

5.1.3.2 Learner ability to recognize translation from a symbolical image (Question 3b)

The question read as follows:

Illustrate, on the diagram, the image of $f(x)$ when it transforms to $f(x+3)-2$. Describe fully the transformation involved.

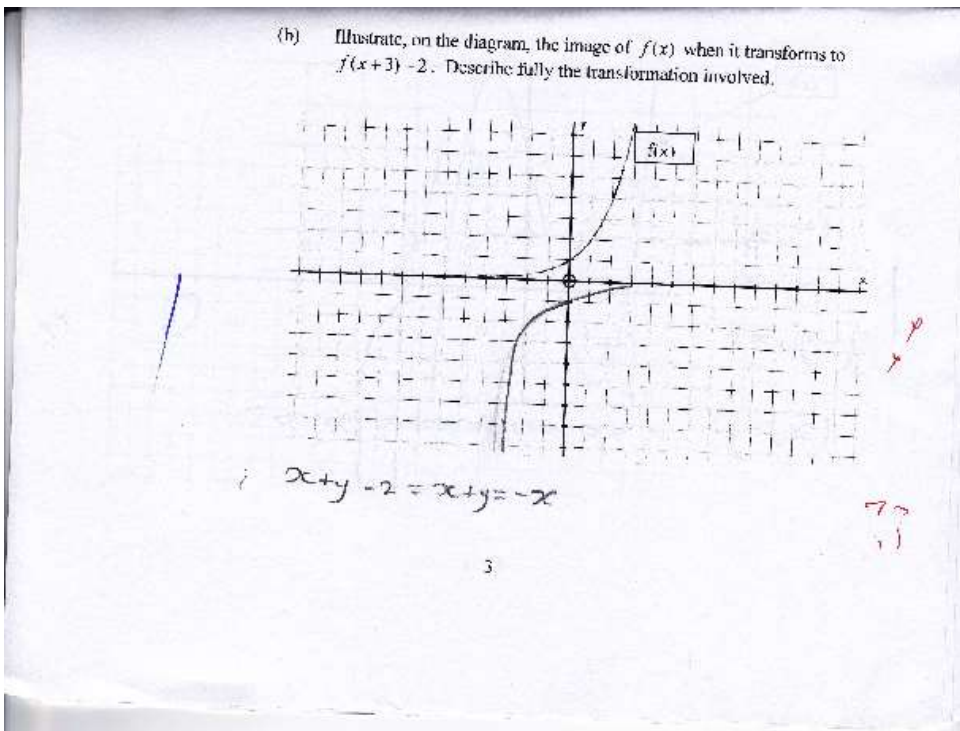
$f(x)$



The question required learners to recognize a two-way translation from its symbolical (algebraic) image and then effect it on a given exponential graph. The resultant direction would be three units to the left and two units downwards. Only thirteen learners (14%) had both the illustration and the description correct. Thirteen learners (14%) had the correct illustration but gave the wrong description, while 18 learners (19%) had the incorrect illustration but gave the correct description. Twenty-four learners (25%) left the question blank. Some learners gave reasons for not answering such as ‘*I don’t understand the question*’, ‘*can’t interpret the question*’, ‘*cannot interpret the concept referred to in the question, so I don’t understand*’ and ‘*I don’t understand when the function $f(x)$ becomes $f(x+3)$* ’. The remaining 59 learners (61%) had misconceptions about how to illustrate translation graphically.

An example of a learner illustration is given in Vignette 5.8, below.

Vignette 5.8: Example of learner’s misconceived graphical image of $f(x+3) - 2$



Learners exhibited a variety of misconceptions. The most common, shown by 19 (20%) of learners, was translating 3 units to the right instead of left. The ‘plus’ sign transforming the

x -variable confused learners because the correct direction is negative, which seems the opposite orientation to what they might intuitively expect.

Correct descriptions or illustrations were as follows:

- Was shifted vertically 2 units downwards and shifted horizontally 3 units to the left. Correct illustration (C).
- Shifted 3 units to the left and 2 units downwards. No illustration (PC).
- Graph was shifted 3 units to the right and 2 units downwards. Inaccurate illustration (PC).
- Translation where the x -coordinate move to the right 3 units and y -coordinate move down 2 units (C).
- The graph moved to the left and moved down (PC).

Other learner misconceptions, as written and/or illustrated, are listed below:

- Translated 3 units vertically and 2 units horizontally (I).
- Reflected about x -axis and described it as 'reflection in $y=1$ ' (I).
- Reflected about the y -axis and no description (I).
- 90° or 180° clockwise rotation (I).
- Vertical translation only (I).
- Translated to the right. No illustration (I).
- $(x;y) \rightarrow (x+3; y-2)$ (I).
- Translating 3 units to the right and 2 units downwards (I).
- Described it as 'enlargement because there is a scale factor' (I).
- Described it as glide reflection and no illustration (I).

Table 5.8: Evaluation frequencies of learners' abilities to recognize symbolical image of a translation of a function (Question 3b) (n=96)

Assessing the Concept Image (Reflection)	Frequency				Approx. Percent	Cumul. Percent
	Sample A	Sample B	Sample C	Sub- total		
a Correct image	1	7	5	13	14.0	100.0
b Partly correct image	12	15	4	31	32.0	86.0
c Incorrect image (misconception)	8	10	10	28	29.0	54.0
d Did not attempt	9	10	5	24	25.0	25.0
TOTAL	30	42	24	96	100.0	

A correct drawing of the image on the Cartesian plane would indicate that a learner could recognize translation correctly from its symbolical image. The symbolical image of the translation given was $f(x+3)-2$ which is three units to the left and two units downwards. Table 5.8, above, shows that only a few learners (14%) could identify the translation correctly. This result was consistent across each sample.

Figure 5.10: Graph showing frequencies of learners' abilities to recognise symbolical image of a translation of a function (n=96)

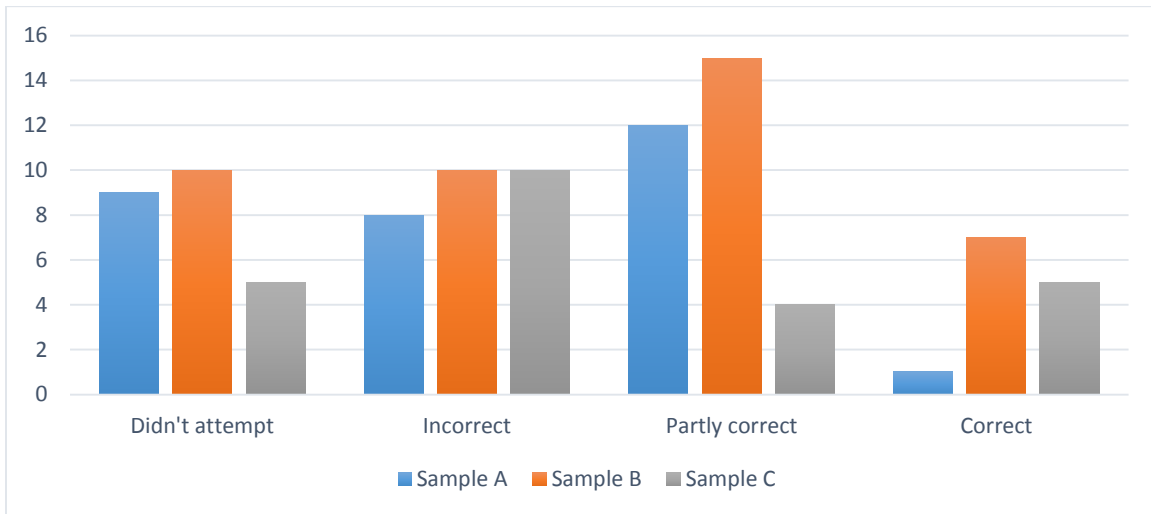
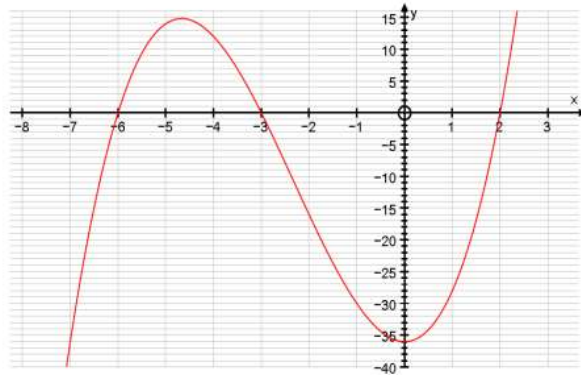


Figure 5.10 displays learners' ability to recognise translation from its symbolical image. The graph shows that very few learners recognised this translation correctly. Although the highest number of responses were in the *Partly correct* category, a significant number of responses were in the *Incorrect* or *Did not answer* categories

5.1.3.3 Learner ability to recognize inward stretch from a symbolical image (Question 3c)

The question read as follows:

Illustrate, on the diagram, the image of $f(x)$ when it transforms to $f(2x)$. Describe fully the transformation involved.



$f(x)$

The question required learners to recognize compression from a symbolic image and then squeeze a cubic function graph horizontally, reducing it by factor half with the y -axis invariant. The equation of the function was not shown. No learner was able to provide both the correct illustration and the correct description. Eleven learners (11%) illustrated the function correctly and 4 (4%) had the correct description. Most learners (89%) either had misconceptions (48 learners, or 50%) or left the question unanswered (37 learners, or 39%). The majority of learners with misconceptions either widened the graph horizontal (outward stretch) by factor 2 or increased the amplitude by factor 2.

Some correct and partly correct responses were as follows:

- Stretch of factor 2. No direction. Correct illustration (PC).
- Correct illustration. Incorrect description: ‘stretched by factor 2’ (PC).
- Correct illustration. ‘Horizontal compression of scale factor 2’ (C).
- Compressed image. No description (PC).

A variety of learner misconceptions, which could be interpreted logically by the researcher, were as follows:

- Translating to the right by 2 units or translating upwards (I).
- Stretched outwards and no description (I).
- Stating that it is a rotation over x and y -axis (I).
- Reflecting about the x -axis (I).
- Translating upwards by 2 units (I).

- Compressing vertically (I).

Table 5.9 and Figure 5.11, below present the frequency count for learners' competences.

Table 5.9: Evaluation frequencies of learners' abilities to recognize symbolical image of a stretch of a function (Question 3c) (n=96)

Assessing the Concept Image (Reflection)	Frequency				Approx. Percent	Cumul. Percent
	Sample A	Sample B	Sample C	Sub- total		
a Correct image	0	3	1	4	4.0	100.0
b Partly correct image	0	3	4	7	7.0	96.0
c Incorrect image	15	22	11	48	50.0	89.0
d Did not attempt	15	14	8	37	39.0	39.0
TOTAL	30	42	24	96	100.0	

A correct drawing of the image on the Cartesian plane would indicate that a learner could recognize stretch correctly from its symbolical image. Table 5.9, above, shows that very few learners (4%) were able to draw correct images. A few others (7%) had some understanding but failed in terms of accuracy. The majority (89%) drew incorrect images or did not answer the question.

Figure 5.11: Graph showing frequencies of learners' abilities to recognise symbolical image of a stretch of a function (n=96)

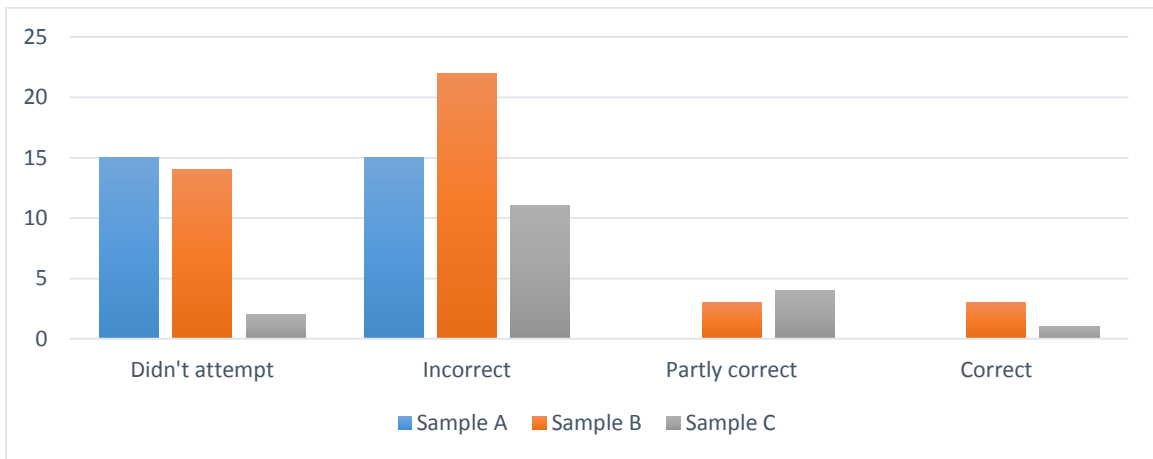


Figure 5.11 displays learners' ability to recognize stretch (compression) from its symbolical representation. The graph shows that very few learners were able to recognize and draw the image from a symbolical representation.

Figure 5.12: Graph comparing frequencies of *Correct*, *Partly correct*, *Incorrect*, and *Did not answer* evaluations about symbolical images of transformation of functions (n=96)

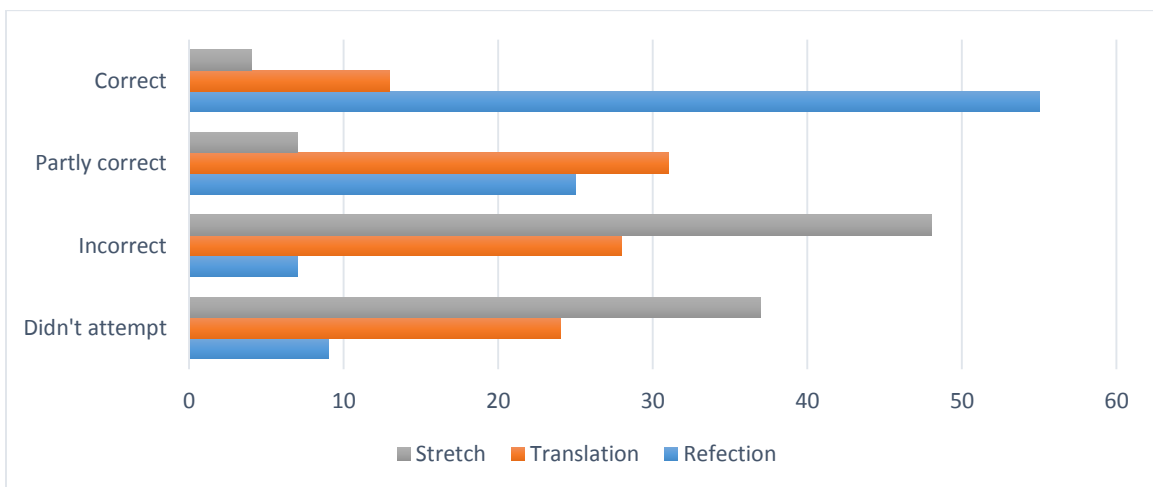


Figure 5.12 displays a cross comparison between learners' *Correct*, *Partly correct*, *Incorrect*, and *Did not attempt* responses for transformations from symbolical representations.

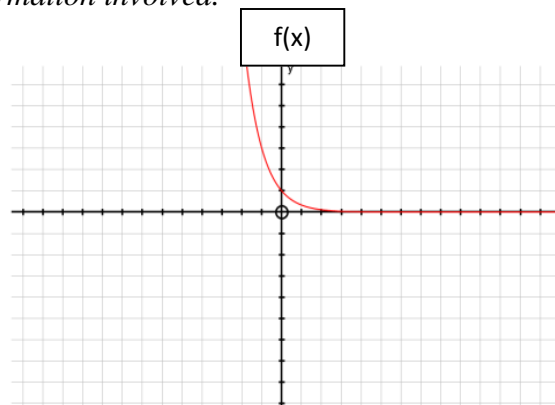
The results demonstrate that learners recognize reflection most easily of the three transformations covered in this study, and have greatest difficulty recognizing stretch.

Question 4 sought to assess learners' interpretations of transformations from symbolical representations or formulae. It also required learners to recognize transformation concepts from symbolical or algebraic images.

5.1.3.4 Learner ability to recognize reflection about the x -axis from symbolical images (Question 4a)

The question read as follows:

Illustrate, on the diagram, the image of $f(x)$ when it transforms to $-f(x)$. Describe fully the transformation involved.



The question required learners to recognize the effect of the negative sign as a reflection and describe that concept of reflection about the x -axis from its symbolical image. The function involved was exponential but its equation was not given. Thirty-nine learners (40%) provided the correct drawing and the correct description while 17 learners (18%)

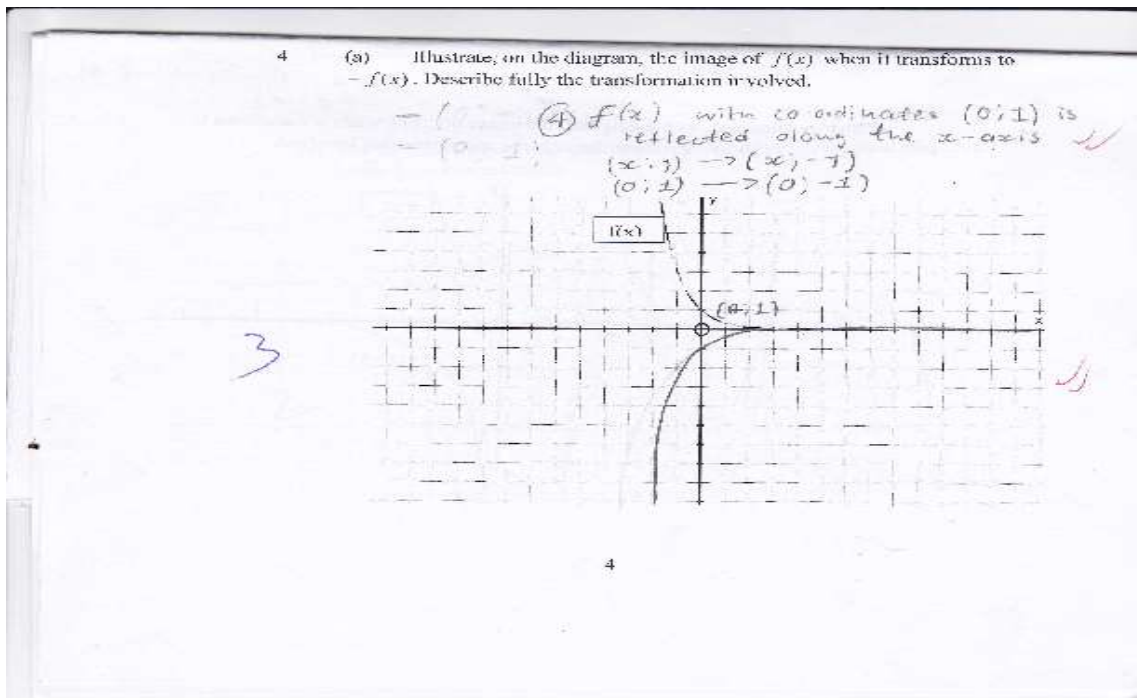
had only the illustration correct. Two learners (2%) gave the correct description. Twenty-five learners had some misconceptions and 15 learners left the question unanswered.

Some correct and partly correct responses were as follows:

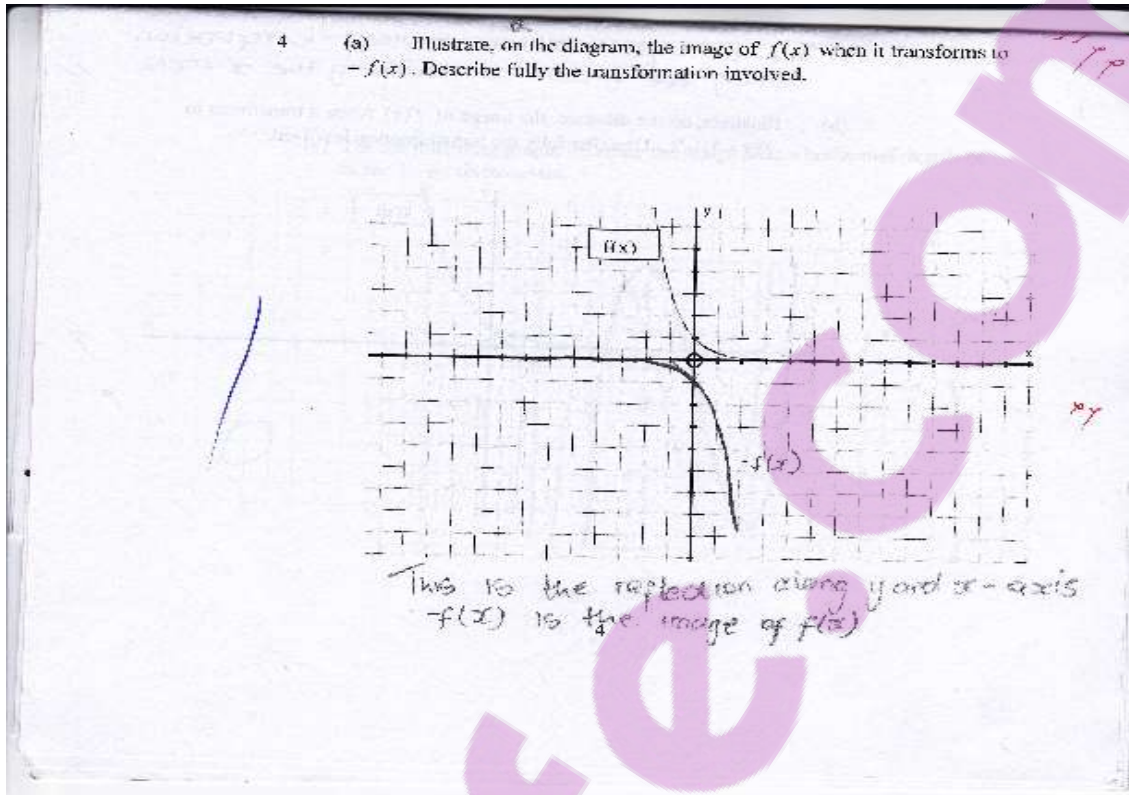
- Reflection on x-axis, correct description, correct illustration (C).
- Reflect in x-axis, no description, correct illustration (C).
- $f(x)$ with coordinates $(0;1)$ is reflected along the x-axis. $(x;y) \rightarrow (x;-1)$; $(0;1) \rightarrow (0;-1)$ (PC).
- Correct illustration, 'reflection about $y=0$. $(x;y) \rightarrow (x;-y)$ (C).
- Correct illustration, 'the image of $f(x)$ was reflected through the x-axis'(C).
- Correct illustration, 'the graph will have a negative, $f(x)$ was reflected in x-axis' (C).
- Correct illustration, 'reflection of image along the x-axis' (C).
- Correct illustration, 'the graph reflects about the x-axis/ $y=0$ ' (C).

Examples of illustrations by learners are shown in Vignettes 5.9 and 5.10, below:

Vignette 5.9: Example of learners' correct image of $-f(x)$



Vignette 5.10: Example of learners' misconceived image of $-f(x)$



Learners had other misconceptions besides that shown in Vignette 5.10. The most common of the misconceptions was that learners reflected the graph about the y-axis.

Other misconceptions demonstrated by learners are listed below:

- Reflected in $y = x$ (PC).
- Rotated 180° about the origin (I).
- Translated vertically (I).
- Translated horizontally (I).
- 'It is reflection of the graph 90 degrees clockwise' (I).
- 'Glide reflection' (I).
- 'Shifted to the left or downwards' (I).

The frequency count of the competences is given below in Table 5.10, and Figure 5.13.

Table 5.10: Evaluation frequencies of learners' abilities to recognize symbolical image of a reflection of a function about x-axis (Question 4a) (n=96)

Assessing the Concept Image (Reflection)	Frequency				Approx. Percent	Cumul. Percent
	Sample A	Sample B	Sample C	Sub-total		
d Did not attempt	7	4	2	13	14.0	14.0
c Incorrect image (misconception)	7	10	8	25	26.0	40.0
b Partly correct image	6	7	6	19	20.0	60.0
a Correct image	10	21	8	39	40.0	100.0
TOTAL	30	42	24	96	100.0	

The results show a consistently better understanding of reflection than other transformations. Thirty-nine learners (40%) had both illustrations and descriptions correct. Nineteen learners (20%) had one of the two aspects of the answer correct. Thirty-eight (40%) learners had completely incorrect responses or did not answer the question. The bar graph below provides a comparative picture of the competences in the frequency table above.

Figure 5.13: Graph showing frequencies of learners' abilities recognise symbolical image of reflection of a function about the x -axis. (n=96)

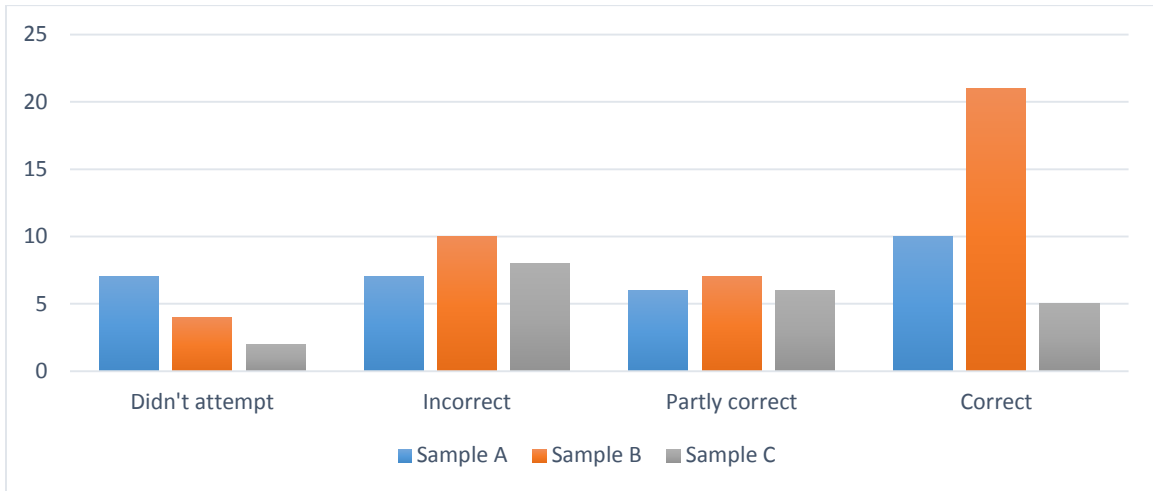
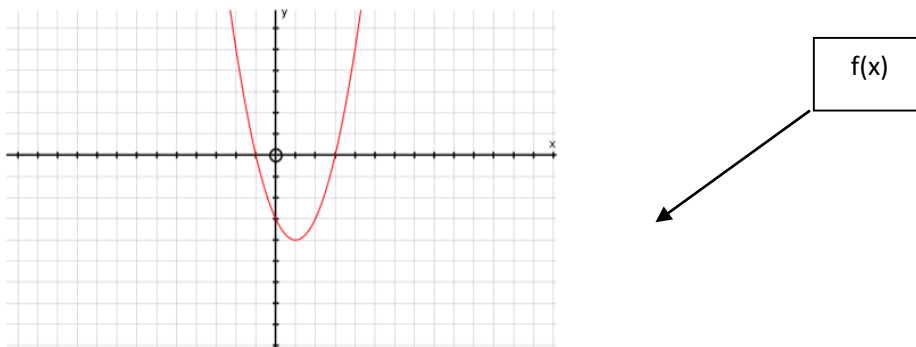


Figure 5.13 displays learners' ability to recognize reflection about the x -axis from its symbolical image. The graph shows that there were significant numbers of learners who had no understanding of the transformation.

5.1.3.5 Learner ability to recognize a two-way translation from symbolical image (Question 4b)

The question read as follows:

Illustrate, on the diagram, the image of $f(x)$ when it transforms to $f(x-2)+3$. Describe fully the transformation involved.



The question required learners to recognize a two-way translation and then slide the given parabola (quadratic graph) two units to the right and three units upwards. Only ten learners (10%) had both the illustration and the description correct. Eight learners (8%) recognized that it is a translation to the right, and thirty-one learners (32%) interpreted the upward translation correctly. Thirty-two learners (33%) mistakenly interpreted the horizontal translation as a slide to the left instead of to the right. Seventeen learners (18%) did not provide an image.

Examples of correct and partly correct conceptions were:

- Translated 2 units right and 3 units up (C).
- Move 2 units to the right and three units upwards (C).
- Translated 3 units upwards and 2 units to the left, incorrect illustration (PC).
- Translated 2 units to the left and 3 units upwards, (majority of learners) (PC).
- Translated to the right (PC).

Other example of learners' misconceptions, as written and/or illustrated, are listed below:

- Translated 3 units downwards and 2 units to the left. (I).
- Translated two units downwards and three units to the right (I).
- Vertical translation of one unit and horizontal stretch (I).
- 'This is a stretch of two units to the left horizontally' (I).
- Translated 3 units to the right and stretched by 2 units (I).
- 'It transforms or moves two units down and 3 units to the right' (I).
- ' $f(x)$ lies on (3;3). It moves three units to the right on the x -axis and 1 unit to the right on the y -axis' (I).
- 'It will go sideways' (I).
- Reflected about the x -axis then translated upward (I).

Table 5.11 and Figure 5.14, below, show the frequencies of learners' abilities in recognizing a two-way translation from a formula.

Table 5.11: Evaluation frequencies of learners' abilities to recognize symbolical image of a two-way translation of a function (Question 4b) (n=96)

Assessing the Concept Image (Reflection)	Frequency				Approx. Percent	Cumul. Percent
	Sample A	Sample B	Sample C	Sub- total		
a Correct image	1	6	3	10	10.0	100.0
b Partly correct image	13	18	6	37	39.0	90.0
c Incorrect image	12	10	10	32	33.0	51.0
d Did not attempt	4	8	5	17	18.0	18.0
TOTAL	30	42	24	96	100.0	

This question is a follow-up to question 3a (compare with Table 5.8). There is no significant difference between the data in the two tables. Learners were to recognize a two-way translation i.e. a slide of two units to the right and three units upwards. The table shows that ten learners (10%) demonstrated accurate recognition. This is a small percentage compared to those who had difficulty providing a satisfactory response. Sample A was the most challenged. Figure 5.14, below, shows the levels of learners' correctness.

Figure 5.14: Graph showing frequencies of learners' ability to recognise symbolical image of a two-way translation of a function (n=96)

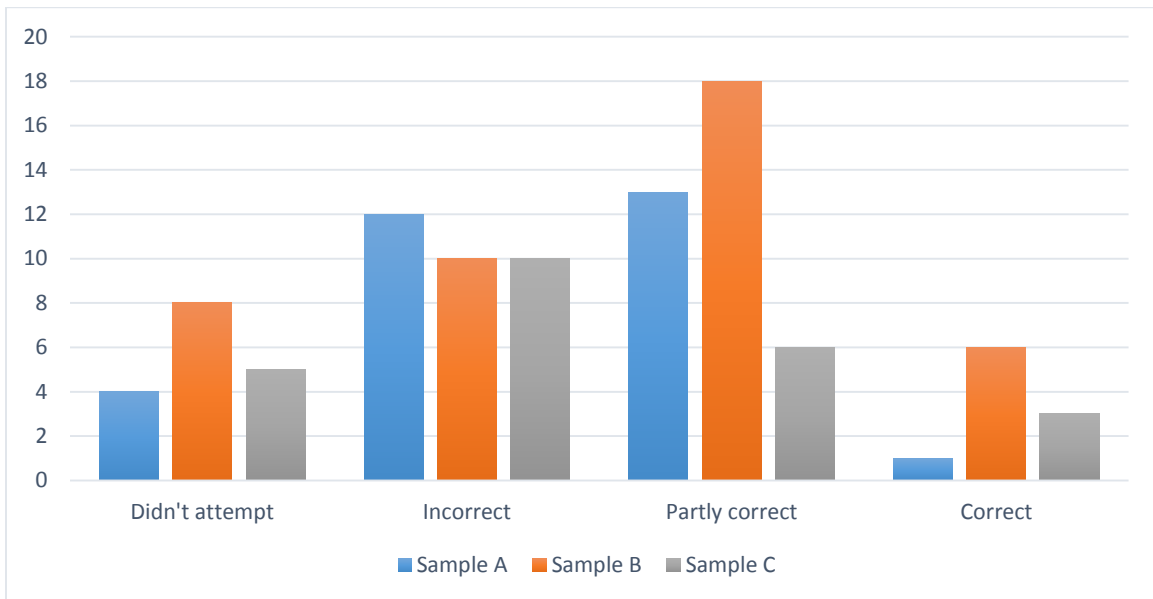


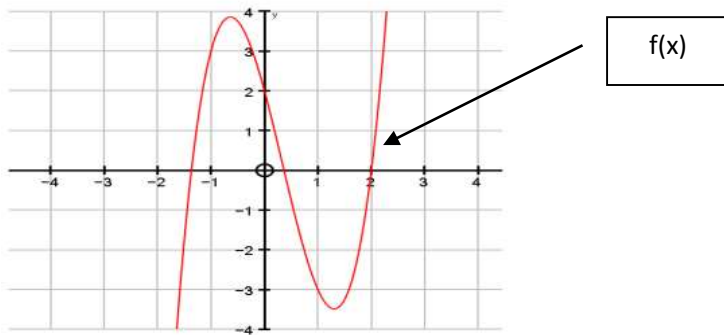
Figure 5.14 shows learners' ability to recognize a two-way translation from its symbolical image. A significant number of learners did not attempt to answer, many had incorrect or partly correct answers and very few were able to provide entirely correct answers.

The graph confirms the claim that stretch was difficult for learners to define, recognize and represent.

5.1.3.6 Learner ability to recognize a horizontal stretch (outward) from its symbolical image (Question 4c)

The question read as follows:

Illustrate, on the diagram, the image of $f(x)$ when it transforms to $f(\frac{1}{2}x)$. Describe fully the transformation involved.



The question required learners to recognize a horizontal extension (outward stretch/pull) of a cubic function graph increasing it by factor 2 with the y-axis invariant. The equation of the function was not given. Only nine learners (9%) had both the illustration and description correct. Twenty-one learners (22%) illustrated the function correctly but had incorrect descriptions. The rest of the learners either demonstrated misconceptions (44%), or left the question unanswered (25%). The majority of the misconceptions were a horizontal compression (inward stretch) of the graph by factor $\frac{1}{2}$ or a decrease in the amplitude by factor $\frac{1}{2}$. Eight learners (8%) drew diagrams with no significant difference from the original diagram.

Some examples of correct and partly correct conceptions were as follows:

- Stretch of factor $\frac{1}{2}$, no direction indicated. Incorrect illustration (PC).
- $(x;y) \rightarrow (\frac{1}{2}x; y)$ (PC).
- The graph has been stretched by the scale factor of $\frac{1}{2}$ (PC).
- Horizontal stretch of factor $\frac{1}{2}$, correct illustration (PC).
- Stretched by $\frac{1}{2}$, correct illustration (PC).
- Correct illustration. ‘The image $f(x)$ has been stretched by $\frac{1}{2}$ vertically’ (PC).
- ‘Compressed vertically by factor 2’ (PC).

- ‘The whole diagram stretches by a $\frac{1}{2}$ ’, (PC).
- ‘The diagram will stretch horizontally by factor $\frac{1}{2}$ ’, (PC).

Some examples of learner misconceptions are listed below:

- Rotation of 180° (I).
- The graph of $f(x)$ was increased by an amplitude of $\frac{1}{2}$ (I).
- Translation to the right/left (I).

The responses of learners in the performance groups are shown in Table 5.12 and Figure 5.15, below.

Table 5.12: Evaluation frequencies of learners’ abilities to recognize symbolical image of an outward horizontal stretch of a function (Question 4c) (n=96)

Assessing the Concept Image (stretch)	Frequency				Approx. Percent	Cumul. Percent
	Sample A	Sample B	Sample C	Sub-total		
a Correct image	1	8	0	9	9.0	100.0
b Partly correct image	9	8	4	21	22.0	91.0
c Incorrect image (misconception)	14	18	10	42	44.0	69.0
d Did not attempt	6	8	10	24	25.0	25.0
TOTAL	30	42	24	96	100.0	

This question, which required learners to recognize outward stretch, was a follows up of Question 3c (compare with Table 5.9), which required learners to recognize a compression. The results are slightly different here because more learners succeeded in recognizing

outward stretch than compression, but very few learners (9%) managed to illustrate and describe the function correctly. The majority (69%) of learners provided incorrect answers and some left the answer space blank. In general, stretch (and compression) were more challenging than the other transformations of functions. The results are represented graphically in Figure 5.15, below.

Figure 5.15: Graph showing frequencies of learners' abilities to recognise a symbolical image of horizontal outward stretch of a function (n=96)

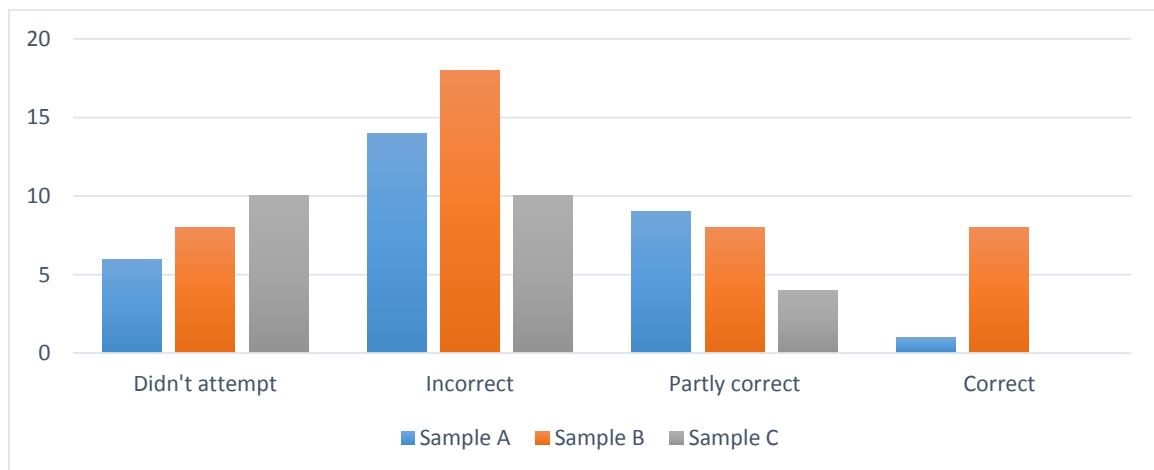


Figure 5.15 displays learners' ability to recognize a horizontal outward stretch from its symbolical image. As in Figure 5.14, *did not attempt*, and *incorrect* responses outnumbered *partly correct* and correct responses.

The graph above confirms that stretch is difficult for the learners in the sample groups. The graph in Figure 5.16, below, compares response frequencies for different transformations of functions.

Figure 5.16: Graphical comparing frequencies of *Correct*, *Partly correct*, *Incorrect*, and *Did not answer* evaluations about recognising symbolical images of transformations of functions (n=96)

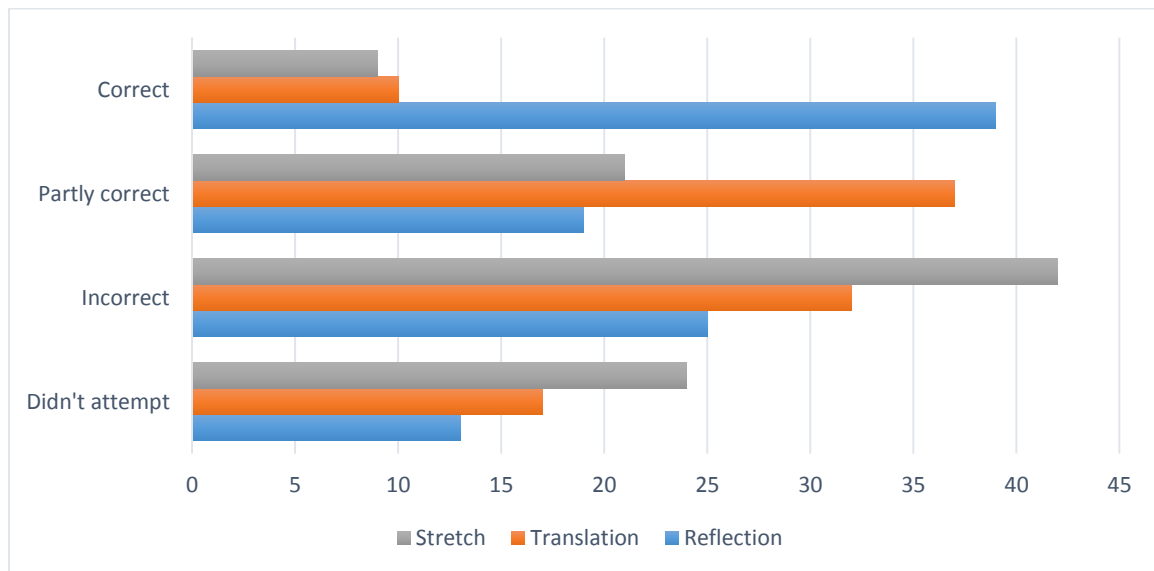


Figure 5.16 displays the cross comparison of *correct*, *partly correct*, *incorrect*, and *did not answer* responses to recognition of transformations from symbolical representations. The transformation with the most *did not attempt* responses was stretch and this type of transformation also had the most *incorrect* responses. This confirms that learners had the most difficulty recognizing stretch. The transformation with the most *partly correct* responses was translation, and the graph confirms that reflection was the transformation that was most easily recognized correctly.

- 5.1.4 $y = \cos x$ for $y = 3\cos x$ and viceversa.

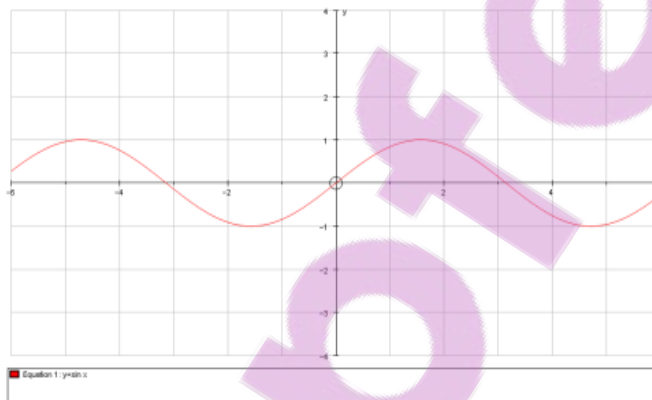
Graphical Interpretations from Symbolical Representations

Question 5 addressed how learners interpret symbolical images or formulae. Below, the results of how learners responded are presented.

5.1.4.1 Learner ability to recognize the concept of vertical outward stretch (pull) from a symbolical image (Question 5)

The question read as follows:

Describe fully the transformation performed if $f(x) = \sin x$ below transforms to $5f(x) = 5\sin x$. Sketch below to illustrate.



The question required learners to recognize a vertical extension (pull) of the sine function graph increasing amplitude five times with the x -axis invariant (away from the x -axis). The equations of the original function and the image function were all given. Only twenty-one learners (22%) had both the illustration and the description correct. Six learners (6%) illustrated correctly but had no descriptions and four learners (4%) had correct descriptions but no illustrations. Thirty learners (31%) had neither illustration nor description correct (misconceptions) and thirty-five learners (37%) left the answer space blank. Some learners drew strange diagrams that were difficult to interpret.

Some correct and partly correct conceptions were as follows:

- ‘The amplitude changed in a stretch’ (PC).
- The graph will stretch vertically (PC).
- Vertical stretch of factor 5 (C).
- Its vertical stretch (PC).
- The graph will move 5 units upwards, it’s a vertical stretch (PC).
- Stretched vertically by 5 units (PC).
- The graph increased in size, expanded or stretched (PC).
- The graph is slightly stretched and appears to be bigger than that of the standard one (PC).

Some misconceptions, which could be interpreted by the researcher, are listed below:

- Reflection about the x -axis (I).
- Enlargement by scale factor 5. No illustration (I).
- Graph is enlarged by 5 units (I).
- Translated 1 unit to the right (I).
- Horizontal translation (I).
- Vertically compression (I).
- Translated upwards (I).
- Pulled to one side upwards (I).
- Contraction (I).
- Drew straight lines (I).

The misconceptions bore so little relation to the actual transformation that, in most cases, the only conclusion to be drawn was that learners were guessing. The frequency of results is displayed in Table 5.13, below.

Table 5.13: Evaluation frequencies of learners' abilities to recognise symbolical image of a vertical (outward) stretch of a function and illustrate it (Question 5) (n=96)

Assessing the Concept Image (Stretch)	Frequency				Approx. Percent	Cumul. Percent
	Sample A	Sample B	Sample C	Sub-total		
a Correct image	9	8	4	21	22.0	100.0
b Partly correct image	4	4	2	10	10.0	78.0
c Incorrect image	9	13	8	30	31.0	68.0
d Did not attempt	8	17	10	35	37.0	37.0
TOTAL	30	42	24	96	100.0	

Learners were to describe the transformation from $f(x)$ to $5f(x)$ as applied on the sine function. A stretch illustration was to accompany this description. Twenty-one learners (22%) did it correctly and 10 (10%) had one of the two required aspects missing. The remainder had misconceptions about the transformation (31%) or did not attempt an answer (37%). The distribution of these results is similar across the three samples (see Figure 5.17, below).

Figure 5.17: Graph showing frequencies of learners' abilities to recognise symbolical image of a vertical (outward) stretch of a function and illustrate it (n=96)

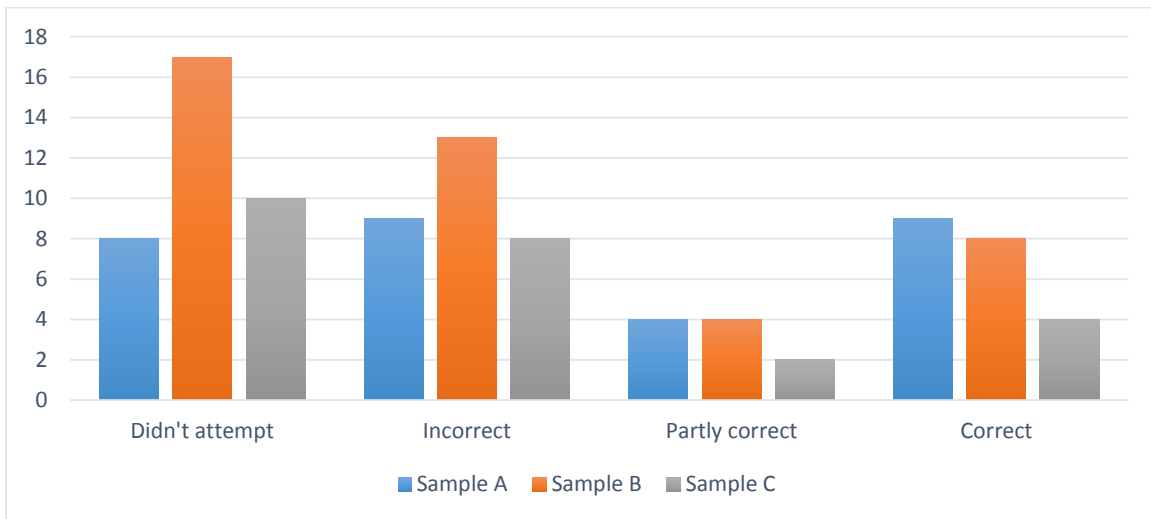


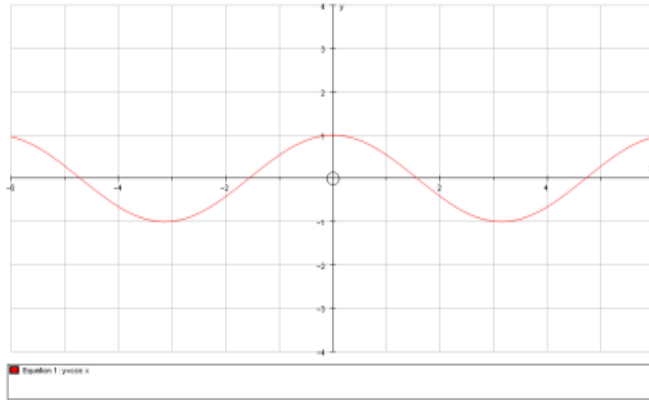
Figure 5.17 displays learners' ability to recognize and illustrate vertical stretch (outward) from the symbolical image. The distribution of frequencies varies from Figure 5.15, which illustrates horizontal stretch (outwards). More learners could recognise vertical stretch than horizontal stretch, but the results show that stretch, generally, was more difficult for learners to recognize than reflection or translation.

Question 6 also addressed the interpretation of symbolical images. It sought to determine consistency of learner recognition of transformations from formulae.

5.1.4.2 Learner ability to recognize the concept of horizontal stretch (a compression) from a symbolical image (Question 6)

The question read as follows:

Describe fully the transformation performed if $f(x) = \cos x$ below transforms to $f(2x) = \cos 2x$. Sketch below to illustrate.



The question required the learners to recognize a horizontal compression of the cosine function graph decreasing period by half with the y-axis invariant. The equations of the original function and the image function were all given. Only five learners (5%) gave both the illustration and the description correctly. Thirty-five (36%) learners had misconceptions, the most common of which was effecting a horizontal extension. Forty-four learners (46%) left the answer space blank, giving reasons such as ‘*I don’t understand what the question requires*’, ‘*I don’t get the concept in question*’. Some learners drew strange diagrams that were difficult to interpret.

Some correct and partly correct conceptions were as follows:

- It has been reduced by the factor 2 horizontally (PC).
- The graph will be compressed horizontally, compressed vertically by the factor 2 (C).
- The period changes to $\frac{360}{2} = 180$, see illustration (C).
- You will be stretching the graph with the factor 2 (C).

Some of the learner misconceptions, which could be interpreted, by the researcher are listed below:

- The amplitude changed (I) (Horizontal compression does not affect the amplitude).
- Translation of $f(x) = \cos 2x$. (I)
- The graph will be two times wider (I) (This is outward stretch) .

- Reflection about the x -axis (I).
- Vertical stretch (I).
- Translation upwards (I).
- A straight line (I).
- Vertical compression (I).
- No significant difference from the original (I).

The frequency table below summarises the overall responses by learners.

Table 5.14: Evaluation frequencies of learners' abilities to recognise symbolical image of a horizontal compression of a function and illustrate (Question 6) (n=96)

Assessing the Concept Image (Reflection)	Frequency				Approx. Percent	Cumul. Percent
	Sample A	Sample B	Sample C	Sub- total		
a Correct image	0	4	1	5	5.0	100.0
b Partly correct image	5	5	2	12	13.0	95.0
c Incorrect image	11	14	10	35	36.0	82.0
d Did not attempt	14	19	11	44	46.0	46.0
TOTAL	30	42	24	96	100.0	

In this question, learners were to describe and illustrate the transformation from $f(x)$ to $f(2x)$ as applied to the cosine function. This is a horizontal compression. The table shows that very few learners (5%) were able to describe and illustrate the transformation correctly. A few more (13%) either described correctly or illustrated correctly. This situation was similar for all three samples. The majority of learners had great difficulty with this question, 82% drew incorrect images or did not provide an answer. The graphical perspective of these results is displayed in Figure 5.18, below.

Figure 5.18: Graph showing frequencies of learners' abilities to recognise symbolical image of a horizontal compression of a function and illustrate it (n=96)



Figure 5.18 displays learners' ability to recognize and illustrate horizontal compression from the symbolical image. The graph shows similar results to the others about stretch. Figure 5.19 compares results for recognition of vertical and horizontal stretch.

Figure 5.19: Graph comparing frequencies of *Correct*, *Partly correct*, *Incorrect* and *Did not attempt* evaluations about recognising vertical or horizontal stretch of a function and illustrate it

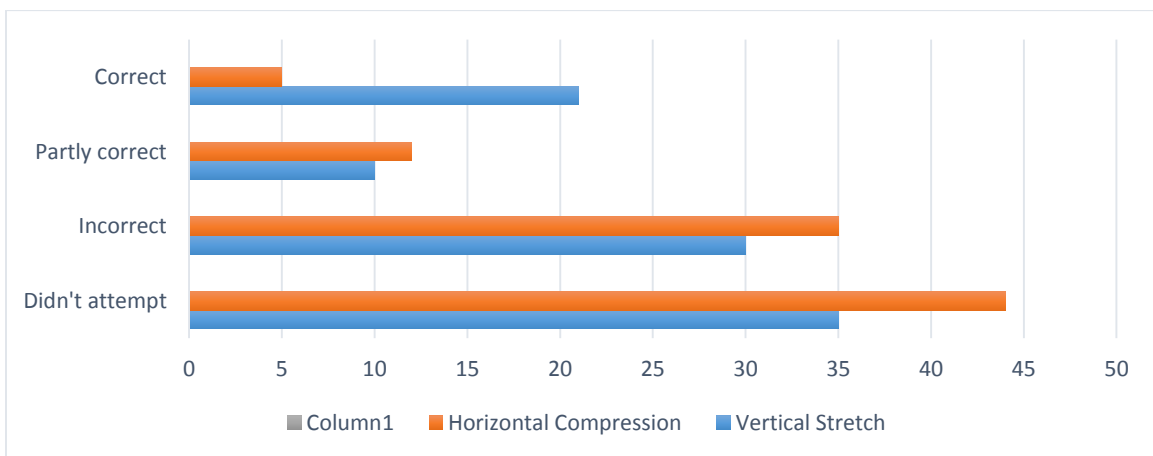


Figure 5.9 displays a cross comparison between *correct*, *partly correct*, *incorrect* and *did not answer* for stretch (vertical vs horizontal). It is evident from the graph that horizontal stretch was more difficult for learners to recognise than vertical stretch.

5.1.5 Symbolical Interpretations from verbal descriptions

One of the aims of the study was to assess a range of learner competencies when working with transformations. Question 7 addresses learners' competence with writing formulae.

5.1.4.3 Learner symbolical images for a two-way translation (Question 7a)

The question read as follows:

Write down the formula for a translation of $y = \sin x$ 4 units to the right and 3 units upward.

The question required learners to write the formula as $y = \sin(x - 4) + 3$. Only four learners (4%) were able to give the correct formula. Twenty-two learners (23%) gave the equation as $y = \sin(x + 4) + 3$ thereby showing a misconception of the horizontal translation direction. They associated the 'plus' sign with translation upwards and to the right. A total of 69 learners (72%) had misconceptions. Many misconceptions related to misunderstanding the resultant direction of the translation and the incorrect use of brackets. Twenty-three learners (24%) left the answer space blank. Some examples of learner responses for formulae are shown in Vignettes 11 and 12 below.

Vignette 5.11: Example of learners' correct symbolical image of various transformations

7.

(a) Write down the formula for a translation of $y = \sin x$ 4-units to the right and 3-units upward.

$y = 3 \sin(x+4)$

(b) Write down the two separate formulae for reflections of $y = 2^x$ in the x -axis and also in the y -axis. Compare and contrast the results.

x -axis	y -axis
$y = 2^{-x}$	$y = -2^x$
x has a negative sign	The base has a negative sign

(c) Write down the formula for a vertical stretch of $y = \cos x$ by factor 2.

$y = 2 \cos x$

8.

(a) Write down the formula for a translation of $y = \cos x$ 4-units to the left and 3-units downward.

$y = 3 \cos(x+4)$

(b) Write down the two separate formulae for reflections of $y = \frac{2}{x}$ in the y -axis and also in the x -axis. Compare and contrast the results.

y -axis	x -axis
$y = -\frac{2}{x}$	$y = \frac{-2}{x}$
x has a negative sign	The numerator has a negative sign

(c) Write down the formula for a horizontal stretch of $y = \sin x$ by factor $\frac{1}{2}$.

$y = \sin \frac{1}{2}x$

Some examples of misconceptions displayed by learners are listed below:

- $y = 3 \sin(x + 4)$
- $y = \frac{\sin x + 4}{3}$

- $y = 3 \sin 4x$
- $y = 3 + \sin x + 4$
- $y = 4 \sin x + 3$
- $3y = \sin 4x$ or $y = \frac{\sin 4x}{3}$
- $y = \sin 4x + 3$
- $y = 3(x + 4)$
- $(x + 4)(y - 3)$
- $(x; y) \rightarrow (x + 3; y + 4)$
- $y = 3 \sin 4$
- $y = \sin 3x$
- $y = (x + 4) + 3$
- $y = (x + 3) - 4$
- $y = \sin(x + 3) + 4$
- $y = 3 \sin x$
- $y = (4 \sin) + 3$
- $y = \sin - x$
- $y = (x - 4) + 3$
- $y = 3 \sin x + 4$
- $\sin x = (x + 4; y + 3)$
- $x + 4 + 3$
- $y = (\sin x + 4) + 3$
- $y = a(\sin x + 4) + 3$
- $y = 3(\sin x + 4)$
- $y = 4 \sin 3x$
- $y = 4 \sin(x + 3)$
- $y = \sin(x + 4) + 3$ or $f(x + 4) + 3$
- $y = a(x + 4)^2 + 3$

From these responses it is evident that various misconceptions related to the incorrect positioning of brackets. The incorrect interpretation of symbols for operations was another frequent misconception. Table 5.15, below, presents the resultant competencies by means of frequencies.

Table 5.15: Evaluation frequencies of learners' symbolical images of a two-way translation of a function (Question 7a) (n=96)

Assessing the Concept Image (Reflection)	Frequency				Approx. Percent	Cumul. Percent
	Sample A	Sample B	Sample C	Sub-total		
a Correct image	0	2	2	4	4.0	100.0
b Partly correct image	5	15	4	24	25.0	96.0
c Incorrect image (misconception)	16	18	11	45	47.0	71.0
d Did not attempt	9	7	7	23	24.0	24.0
TOTAL	30	42	24	96	100.0	

Table 5.15 shows that very few learners (4%) could interpret symbolically correctly. A large percentage of learners (72%) had misconceptions, and 23 learners did not answer the question at all. The use of brackets is evidently little understood by learners and so is one of the major causes of misconceptions. The graph comparing competencies in different samples is shown in Figure 5.20, below.

Figure 5.20: Graph showing frequencies of learners' symbolical images of a two-way translation of a function

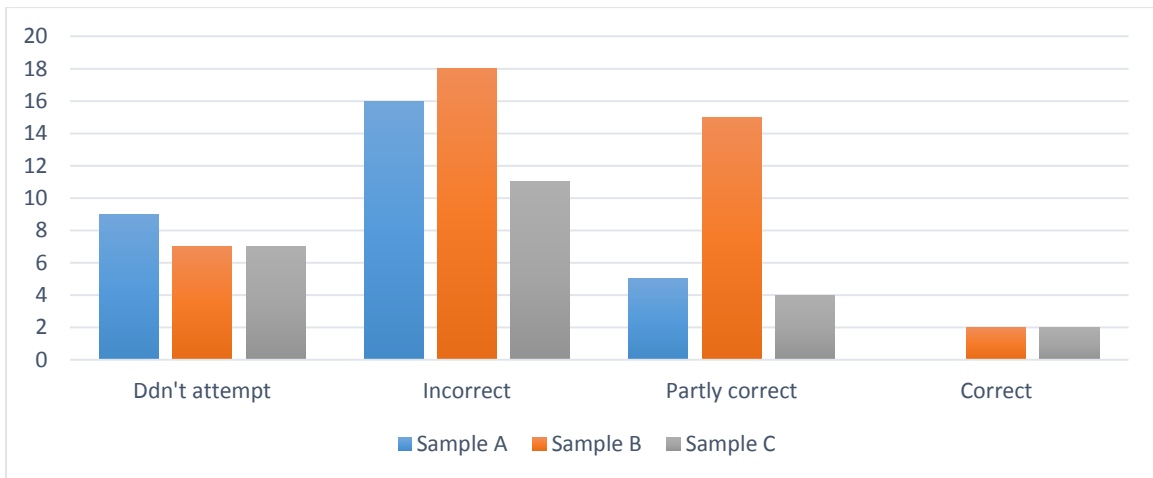


Figure 5.20 displays learners' symbolical images for two-way translation. Results presented earlier in this document show that single translation was the most well understood, identified and represented transformation for Grade 11 learners. But for two-way translation, the situation was different. Very few learners could write the formula correctly and in sample A, no learner was able to draw it correctly. The numbers of learners in the *incorrect* and *did not attempt* categories was much larger than for a single translation.

The following section, 5.1.4.4, deals with learner competency relating to reflection.

5.1.4.4 Learner symbolical images for reflections about the axes (Question 7b)

The question read:

Write down the two separate formulae for reflections of $y = 2^x$ in the x -axis and also in the y -axis. Compare and contrast the results.

The question required learners to write the formulae as $y = -2^x$ for reflection in the x -axis and $y = 2^{-x}$ (or $y = (\frac{1}{2})^x$) for reflection in the y -axis. Nineteen learners (20%) had both formulae correct although some swapped the one for the x -axis with the one for the y -axis. Two learners (2%) had one of the equations correct and the other incorrect. Reasons learners gave for their answers included:

- ‘the negative sign is on the power when you reflect in the x -axis and the negative sign is on the base when you reflect in the y -axis’.
- ‘the graph reflect about the x -axis and intersect at (2;1)’.
- ‘the asymptote will be the same. The two graphs will be diagonally opposite to each other. $-y = 2^{-x}$ is negative and $y = 2^{-x}$ is positive in the y -axis’
- ‘in the y -axis the image will be positive and so as in the x -axis’.
- ‘the graph will have the difference of 2 in both x and y ’.
- ‘ $-y = 2^{-x}$, the image will face the opposite direction of the object’.
- ‘the reflection is the same on opposite sides’.
- ‘reflection at a 180° point clockwise’.
- ‘it will be in the 3rd quadrant’.

Fifty-five learners (57%) left the answer space blank. The reasons for not answering the question included ‘*I can’t differentiate between the formulas, I get confused*’, ‘*I don’t understand*’, ‘*I don’t know*’, ‘*I do not understand the question*’, ‘*Not really sure about this*’ and ‘*Don’t understand the question at all*’. Misconceptions were mostly about the position of the negative sign.

Some misconceptions demonstrated by learners included the following:

- $y = 2^x$ unchanged.
- $y = x + 2$ and $y = x - 2$
- $y = 2^{x-1}$
- $-x = 2^x$

- $2x = 2^y$
- $y = 2^x - x - y$
- $-y = 2^{-x}$

Some examples of learner responses for formulae are shown in Vignettes 5.11 above and 5.12 below.

Table 5.16: Evaluation frequencies of learners' symbolical images of reflection of a function in both the x and y -axes (Question 7b) (n=96)

Assessing the Concept Image (Reflection)	Frequency				Approx. Percent	Cumul. Percent
	Sample A	Sample B	Sample C	Sub-total		
a Correct image	5	7	7	19	20.0	100.0
b Partly correct image	0	0	2	2	2.0	80.0
c Incorrect image (misconception)	6	9	5	20	21.0	78.0
d Did not attempt	19	26	10	55	57.0	57.0
TOTAL	30	42	24	96	100.0	

Learners were to reflect $y = 2^x$ symbolically about the x -axis and the y -axis. The expected symbolical images are $y = -2^x$ in the x -axis and $y = 2^{-x}$ (or $y = (\frac{1}{2})^x$) in the y -axis. Nineteen learners (20%) had both the expected symbolical images correct and two (2%) interchanged the x and y -axes. The majority (78%) had incorrect images or did not provide images at all. The value ratios were similar across the three samples.

Figure 5.21: Graph showing frequencies of learners' symbolical images of reflection of a function in both x- and y-axes

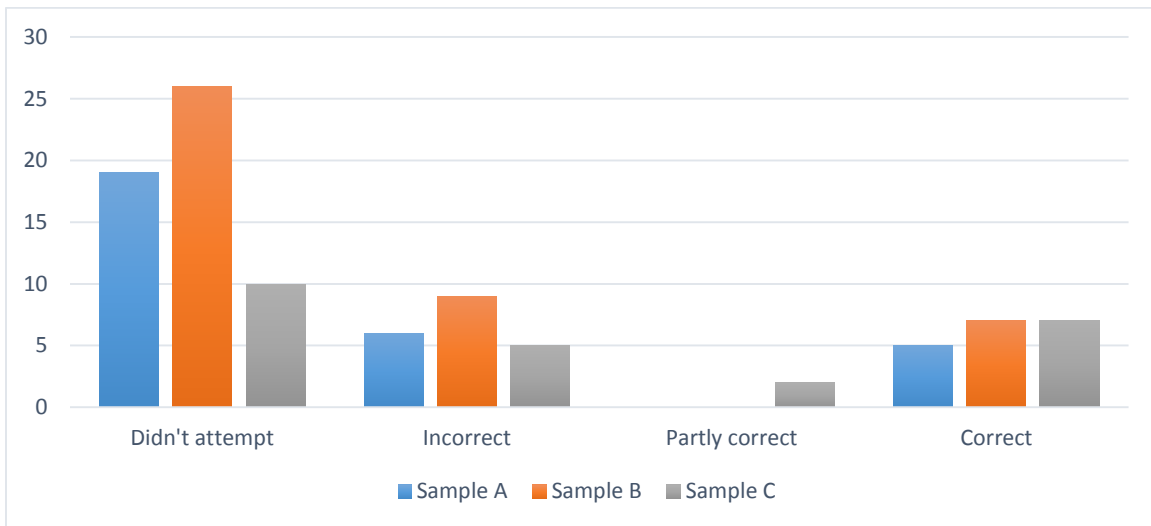


Figure 5.21 displays learners' symbolical images for reflection in both axes. Interestingly unlike the results for other questions, here there was a very small number of responses in the *partly correct* category, so it seems that in this case learners either knew the answer (a small percentage) or did not understand the requirement at all (the majority).

5.1.4.5 Learner symbolical images for a vertical stretch (outward stretch) (Question 7c)

The question read as follows:

Write down the formula for a vertical stretch of $y = \cos x$ by factor 2.

The question required the learners to write the formula as $y = 2 \cos x$. Thirty-nine learners (41%) gave the correct formula. Seventeen learners (18%) mistakenly gave the equation as $y = \cos 2x$ thereby showing a misconception or confusion with horizontal compression. The other frequent misconception was giving the equation as $2y = \cos x$ (7 learners, or 7%). Three learners (3%) showed that they lacked confidence in their answer by giving the correct answer as well as $y = \cos 2x$. Twenty-four learners (25%) left the answer space blank. The only reason given by a learner for the lack of an answer was: '*I don't understand*

what the question needs'. Some examples of learner responses for formulae are shown in Vignettes 5.11 above and 5.12 below.

Other misconceptions showed by learners are listed below:

- $xfy \rightarrow (2x;2y); y = \cos x + 2; (x; y) \rightarrow \begin{cases} y \cos x + x \sin y \\ y \cos x + 2 \sin x \end{cases}$
- $\cos x + 2$
- $y = \cos 2$
- $f(x) = \cos x + q$

Many learners make the excuse 'I don't understand...' What they did not acknowledge is that while understanding is improved by good teaching, it also requires effort on their own part. The frequency of results is given in Table 5.17, below.

Table 5.17: Evaluation frequencies of learners' symbolical images of a vertical (outward) stretch of a function (Question 7c) (n=96)

Assessing the Concept Image (Stretch)	Frequency				Approx. Percent	Cumul. Percent
	Sample A	Sample B	Sample C	Sub-total		
a Correct image	16	18	5	39	41.0	100.0
b Partly correct image	2	3	1	6	6.0	59.0
c Incorrect image	4	12	11	27	28.0	53.0
d Did not attempt	8	9	7	24	25.0	25.0
TOTAL	30	42	24	96	100.0	

Learners were to write the symbolical image for the vertical stretch of $y = \cos x$ by factor 2. The image would be $y = 2 \cos x$. The table shows that vertical stretch was better understood symbolically than horizontal stretch (Table 5.14). A larger number of learners

gave the correct formula than those who gave an incorrect one or who failed to provide an answer. This can be seen on the graph in Figure 5.22, below.

Figure 5.22: Graph showing frequencies of learners' symbolical images of a vertical (outward) stretch of a function

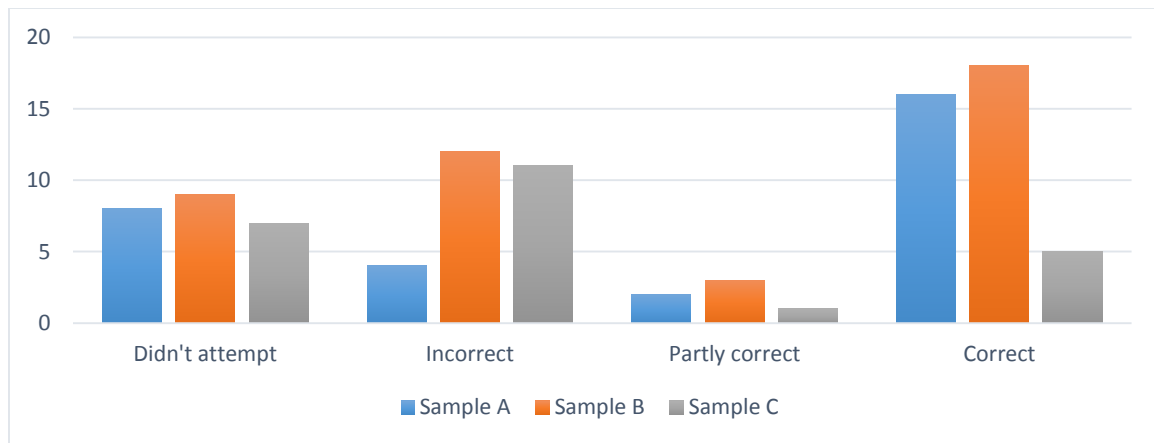


Figure 5.22 displays the results for symbolical images of a vertical (outward) stretch. The graph shows that many learners, especially in samples A and B, could write the correct formula. However, a significantly large number of learners wrote incorrect formulae or did not attempt an answer. A comparison of the results of learners' competencies with respect to symbolical images of vertical stretch, two-way translation and reflection is presented graphically in Figure 5.23.

Figure 5.23: Graph comparing frequencies of *Correct*, *Partly correct*, *Incorrect*, and *Did not attempt* evaluations about symbolical images of transformations

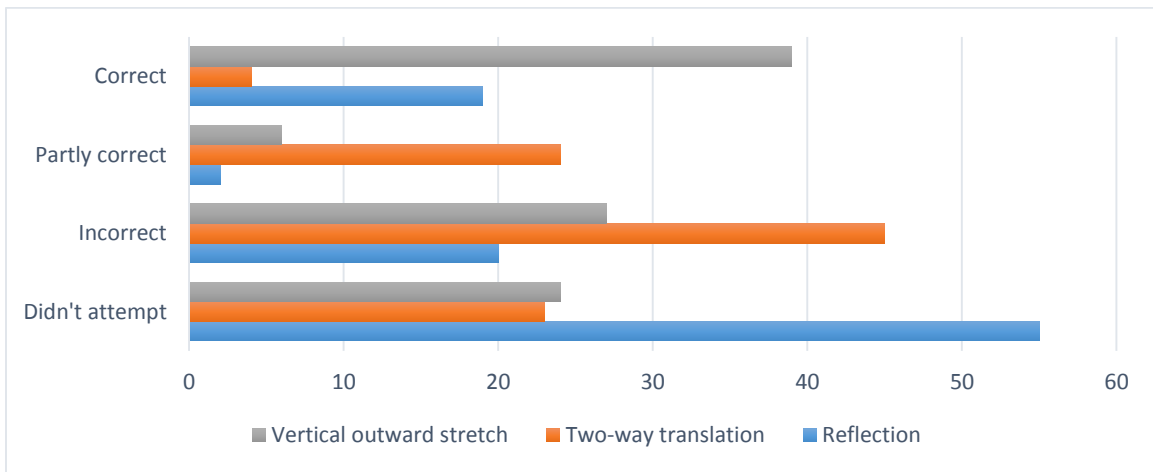


Figure 5.23 displays a cross comparison of *correct*, *partly correct*, *incorrect*, and *did not answer* responses for symbolical images of transformations. The graph shows that the most failures to provide an answer were for reflection, the most *incorrect* answers were for two-way translation, the most *partly correct* answers were also for two-way translation, and the most *correct* answers were for vertical stretch.

Question 8 addresses learners' competence in dealing with symbolical images and formulae of transformations. It sought to assess learner consistency when writing formulae from verbal descriptions using the cosine function. A discussion of this question follows in section 5.1.4.6.

5.1.4.6 Learner symbolical images for translation (Question 8a)

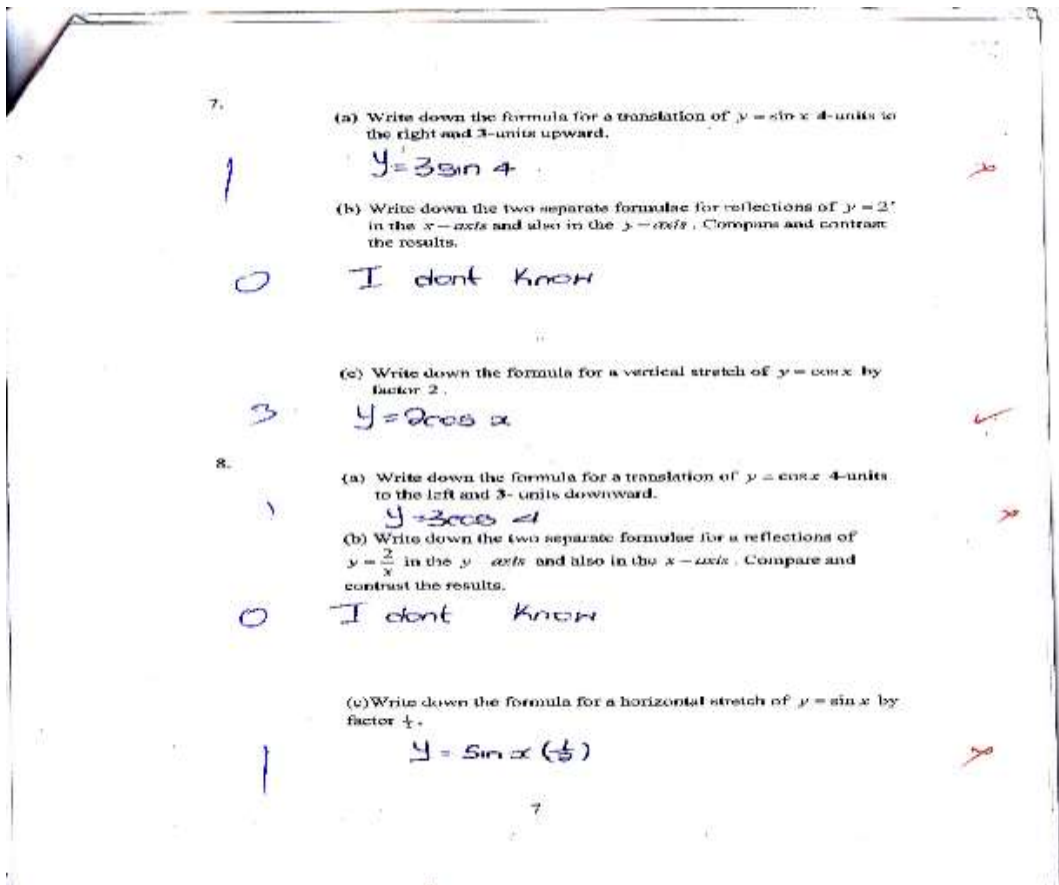
The question read:

Write down the formula for a translation of $y = \cos x$ 4 units to the left and 3 units downward.

The question required the learners to write the formula as $y = \cos(x + 4) - 3$. Only two learners (2%) gave the formula correct. Fifteen learners (16%) gave the equation as $y = \cos(x - 4) - 3$ thereby showing a common misconception of the horizontal translation

direction. This again shows confusion about what effect the '+' or '-' signs have when it comes to translation along the horizontal. Learners got the one direction wrong just like in Question 7a. Thirty-one learners (32%) left the answer space blank. Some examples of learner responses are shown in Vignettes 5.11 above and 5.12 below. Other examples can be found on Appendix D.

Vignette 5.12: Example of learners' misconceived symbolical image of a function and the reasons for not attempting to answer



Some misconceptions showed by learners are listed below:

- $y = 3 \cos(x - 4)$
- $y = \cos \frac{(x-4)}{3}$
- $y = 3 \cos 4x$

- $y = -4 \cos x - 3$
- $y = (0; (x + 4) - 3$ (No closing bracket)
- $y = -3 \cos(-4)$
- $y = -4 \cos(x - 3)$
- $y = -4x(3x)$
- $y = \cos - 4x - 3$
- $y = -4 \cos - 3x$
- $y - 3 = \cos(x - 4)$
- $3y = \cos 4x$
- $y = (x + 4)3$
- $(x + 4)(y - 3)$ (Not a function because of missing '=' sign)
- $4 - y = \cos 3 - x$
- $y = \cos 4 - 3$
- $-3y = \cos - 4$
- $y = \cos 3x$
- $y = (x - 4) - 4$
- $y = \cos 4x$
- $y = \cos 4$
- $\cos x - 4) - 3$ (Not a function because of missing '=' sign)
- $y = 3 \cos 4$
- $y = (x - 4) - 3$
- $y = -3 \cos(x + 4)$
- $y = \cos 4x - 3$
- $y = -4 \cos x - 3$
- $f(\cos x - 4) - 3$ (Not a function because of missing '=' sign)
- $-3 \cos x - 4$ (Not a function because of missing '=' sign)

- $y = \frac{a}{x-4} - 3$ (Value of a needed)
- $f(x) = \cos(-4x) - 3$
- $\cos x \rightarrow (x-4; y-3)$
- $f(x) = (-4; +3)$
- $y = \cos(4-3)$
- $y = \cos x(-4; -3)$
- $y = -4 \cos(3-x)$
- $y = \cos 4x + 3$
- $y = -3(\cos x - 3)$
- $f(x) = \cos x + 2 - 3$
- $y = a(\cos x - 4) - 3$ (Value of a needed)
- $a = \frac{y}{x-4} + 3$ (Value of a needed)

Many of these misconceptions used brackets incorrectly, moved the translation in the opposite direction to the translation required, or omitted something. The frequency distribution for competencies in this question is shown in Table 5.18, below.

Table 5.18: Evaluation frequencies of learners' symbolic images of two-way translation of a function (Question 8a) (n=96)

Assessing the Concept Image (Translation)	Frequency				Approx. Percent	Cumul. Percent
	Sample A	Sample B	Sample C	Sub- total		
a Correct image	1	0	1	2	2.0	100.0
b Partly correct image	4	10	1	15	16.0	98.0
c Incorrect image (misconception)	15	19	14	48	50.0	82.0
d Did not attempt	10	13	8	31	32.0	32.0
TOTAL	30	42	24	96	100.0	

Learners were to translate $y = \cos x$ 4 units to the left and 3 units downwards. The table shows that few learners (2%) responded correctly. The majority (66%) had misconceptions and 32% did not attempt an answer. The information shown here is similar to that shown by Table 5.15 which also dealt with two-way translation. The graphical representation is presented in Figure 5.24, below.

Figure 5.24: Graph showing frequencies of learners' symbolical images of a two-way translation of a function

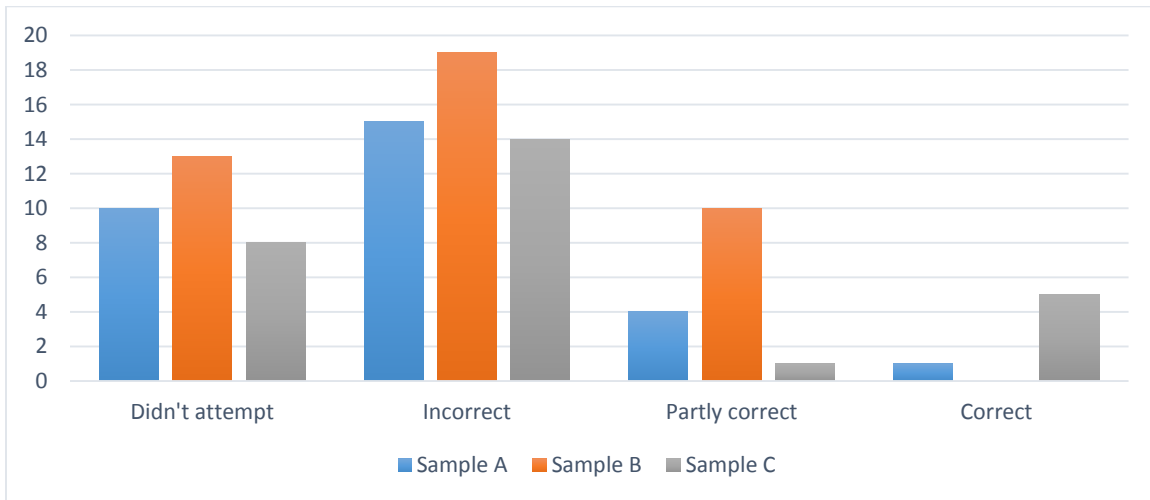


Figure 5.24 displays learners' symbolical images of a two-way translation. Learner performance in answering this question was appalling. Very few gave the correct formula, a small number gave partly correct responses and the majority of the learners either gave an incorrect formula or did not attempt to answer.

5.1.4.7 Learner symbolical image for reflection (Question 8b)

The question read as follows:

Write down the two separate formulae for reflections of $y = \frac{2}{x}$ in the y-axis and also in the x-axis. Compare and contrast the results.

The question required learners to write the formulae as $y = \frac{2}{-x}$ or $y = -\frac{2}{x}$ or $y = \frac{-2}{x}$

for reflections about either of the axes. Twenty-two learners (23%) had both the formulae correct, which result in the same for both the x-axis and the y-axis, The expression given by learners included: 'the results are the same, the negative sign is in the same place', 'x has negative sign, the numerator has negative sign', 'they both have an error at zero', 'on

the x -axis it will be a reflection of the other part of the quadrant', 'the reflection is on opposite side', 'x-values will be halved, y-values will remain the same', 'the two graphs will exchange positions', 'image will be reflected', 'the graph will reflect positively on the x -axis' and 'to get the value of y , it has to be 2 divided by the x integers'. Sixty learners (63%) left the answer space blank. The reasons for not answering were given as: 'I don't understand', 'I don't know', 'I need more information' and 'I don't know what to do and solve it'. Some examples of learner responses for formulae are shown in Vignettes 5.11 and 5.12 above. Additional examples can be found in Appendix D. A common misconception was not putting the negative sign in (eleven learners, or 11%).

Some of the misconceptions learners demonstrated are listed below:

- $y = 2x$
- $y = -2x$
- $y = x^{-x}$ (Difficult to understand the learner's mind-set here)
- $y = \frac{2}{x} - 1$
- $y = \frac{x}{2}$; $2y = \frac{x}{2}$
- $x = \frac{2}{y}$
- $y = \frac{2}{0}$; $o = x^{-2}$ (An 'undefined' case)

The frequencies for the competences are given in Table 5.19, below, and Figure 5.25.

Table 5.19: Evaluation frequencies of learners' symbolical images of a reflection of a function in both x- and y-axes (Question 8b) (n=96)

Assessing the Concept Image (Reflection)	Frequency				Approx. Percent	Cumul. Percent
	Sample A	Sample B	Sample C	Sub- total		
a Correct image	6	9	7	22	23.0	100.0
b Partly correct image	0	0	1	1	1.0	77.0
c Incorrect image (misconception)	8	2	6	16	17.0	76.0
d Did not attempt	16	31	10	57	59.0	59.0
TOTAL	30	42	24	96	100.0	

Learners were expected to reflect the function $y = \frac{2}{x}$ about both x and y -axes. The symbolical image is virtually the same for both regardless of which axis one reflects about. It can be written in three forms $y = \frac{2}{-x}$ or $y = \frac{-2}{x}$ or $y = -\frac{2}{x}$. The majority of learners were unable to respond to this question and left the answer space blank (59%), while 23% of the learners gave the correct images and 18% gave completely incorrect images. The distribution was approximately the same across all three sample groups. The results are not significantly differently from the ones shown in Table 5.16 for Question 7b, which had the same objectives. The following graph, Figure 5.24, serves for comparison.

Figure 5.25: Graph showing frequencies of learners' symbolical images of reflection of a function in both x- and y-axes

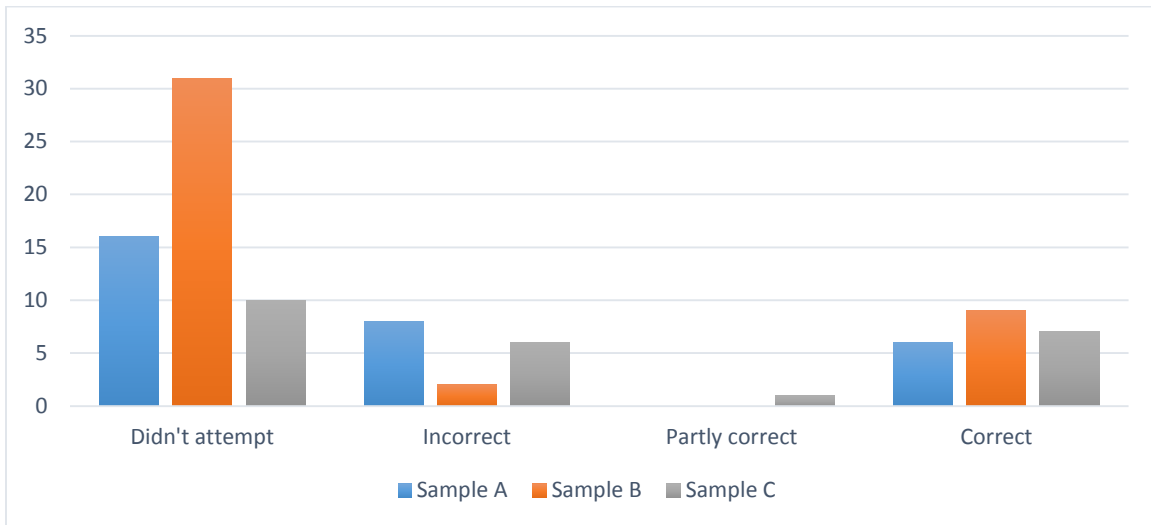


Figure 5.24 displays learners' symbolical images of reflection in both axes. The graph shows that very few learners could write the formulae correctly after reflecting in the x -axis and y -axis and the majority of learners could not even begin to provide an answer. The assumption is that the learners who did not attempt an answer did not know how to reflect.

5.1.4.8 Learner symbolical images for stretch (Question 8c)

The question read as:

Write down the formula for a horizontal stretch of $y = \sin x$ by factor $\frac{1}{2}$

The question required learners to write the formula as $y = \sin 2x$. No learner (0%) was able to give the correct formula. Frequent misconceptions were the equations $y = \sin \frac{1}{2}x$ (26 learners, or 27%) and $y = \frac{1}{2}\sin x$ (23 learners, or 24%). Thirty-nine learners (41%) left the answer space blank. The reasons given by learners for not answering included: '*I don't know*', '*I cannot interpret the concept referred to*' and '*could not interpret*'. Many learners did not state a reason. Some examples of learner responses for formulae are shown in Vignettes 5.11 and 5.12 above. Other examples can be seen in Appendix D.

Other misconceptions shown by learners are listed below:

- $(y + \frac{1}{2}) = \sin x$
- $y = \sin \frac{1}{x}$
- $y = \sin 30$ (Missing units)
- $y = \sin(x + \frac{1}{2})$
- $y = \sin x + \frac{1}{2}$
- $-\frac{1}{2}f(x) = \sin x$
- $y = \frac{2}{\sin x} + \frac{1}{2}$
- $y = \sin \frac{1}{2} = 8,73$

The frequency count for the results are shown in Table 5.20 and the graphical visual is displayed in Figure 5.25.

Table 5.20: Evaluation frequencies of learners' symbolical images of a horizontal stretch of a function (Question 8c) (n=96)

Assessing the Concept Image (stretch)	Frequency				Approx. Percent	Cumul. Percent
	Sample A	Sample B	Sample C	Sub-total		
a Correct image	0	0	0	0	0.0	
b Partly correct image	0	0	0	0	0.0	
c Incorrect image (misconception)	20	25	12	57	59.0	100.0
d Did not attempt	10	17	12	39	41.0	41.0
TOTAL	30	42	24	96	100.0	

Learners were to write a formula for a horizontal stretch of factor $\frac{1}{2}$. The required formula was $y = \sin 2x$. None of the learners, across all three samples, wrote the correct image. Fifty-seven (59%) learners had misconceptions and 39 learners (41%) left the answer space blank. If compared with Table 5.17, where the question had a similar objective, it is clear that learners were more challenged by horizontal stretch than vertical stretch.

Figure 5.26: Graph showing frequencies of learners' symbolical images of horizontal stretch of a function

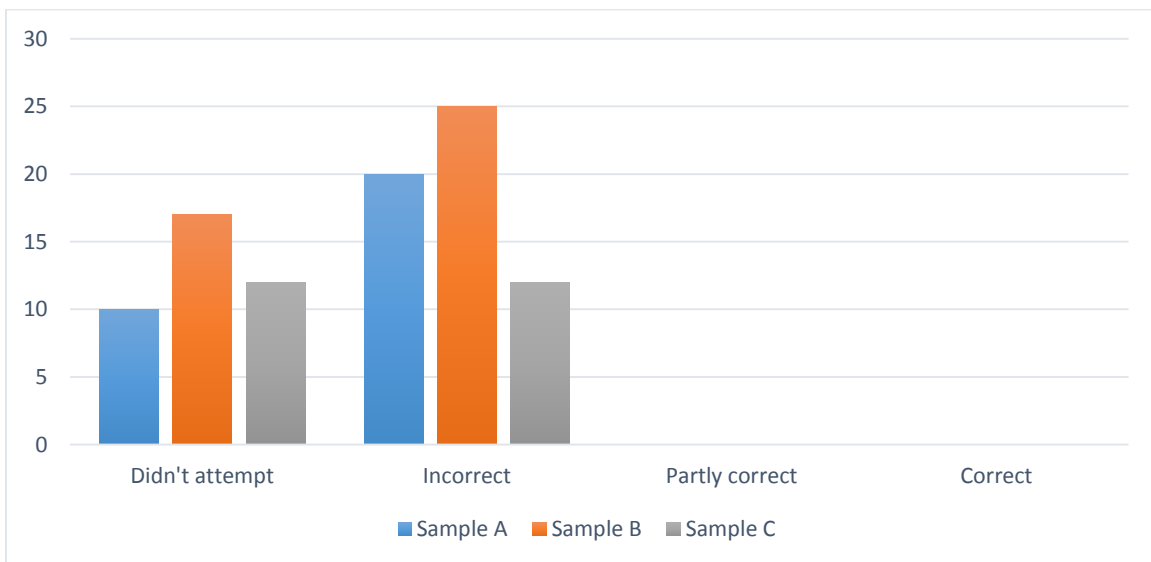


Figure 5.25 displays learners' symbolical images of horizontal stretch. The graph shows that learners had little understanding of horizontal stretch because they either gave an incorrect formula or did not attempt to answer. A comparison of the frequencies for competence for all three transformations is displayed in Figure 26, below.

Figure 5.27: Graph comparing frequencies of *Correct*, *Partly correct*, *Incorrect*, and *Did not attempt* evaluations about symbolical images of transformations of functions

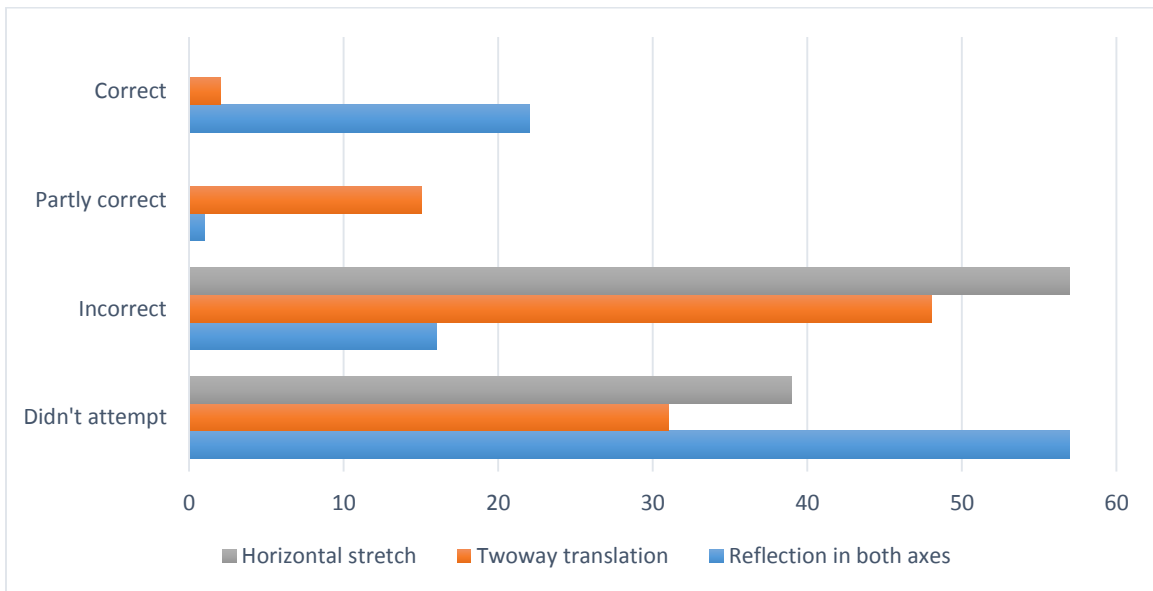


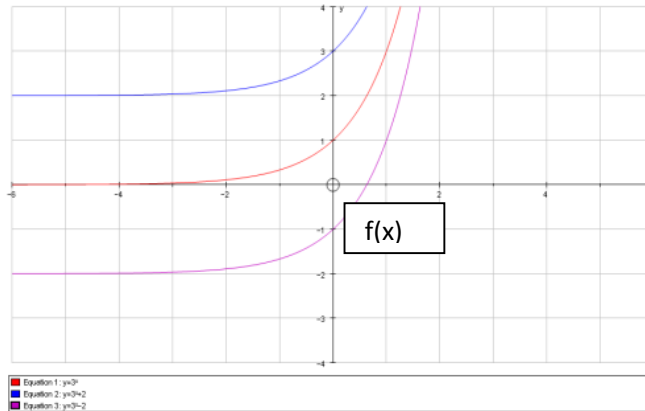
Figure 5.26 displays the cross comparison between the *correct*, *partly correct*, *incorrect*, and *did not attempt* responses for symbolical images of transformations. From the graph it is evident that no learner was able to give the correct formula for horizontal stretch, very few learners gave the correct formula for a two-way translation, but 22 learners answered the question on reflection on both axes correctly. The majority of learners answered incorrectly or did not attempt to answer.

5.1.5 Verbal Descriptions from Graphical Representation

Question 9 addresses learner competencies in recognizing transformations from graphical images. The question read as:

Identify the transformations that mapped $y = f(x)$ to the illustrated images. Describe the transformations fully by words and label their formulae, if possible.

5.1.5.1 Learner ability to recognize vertical translation from graphical images (Question 9a)



The question required learners to recognize and describe two vertical translations: one 2 units upwards and the other 2 units downwards. Thirty-one learners (32%) gave both the descriptions and the formulae correctly. Nineteen learners (20%) recognized that these were translations but were not accurate in saying by how much.

The correct conception was $y = 3^x \pm 2$, vertical translations 2 units upward and 2 units downwards. Thirty-five learners got at least one of them correct.

Misconceptions by learners included those listed below:

- Translation $(x; y) \rightarrow (x - 2)$
- Increase to $2f(x)$ and decrease to $-2f(x)$ (These are stretches)
- $(x; y) \rightarrow (x + 2; y - 2)$
- $(-2; 2)$ $(2; -2)$

The first response listed above translated horizontally instead of vertically, the second stretched and reflected, and the third translated obliquely. The fourth misconception cannot be explained. Their frequency distribution of responses to this question are displayed in Table 5.21, below.

Table 5.21: Evaluation frequencies of learners' ability to recognize graphical image of vertical translation of a function (Question 9a) (n=96)

Assessing the Concept Image (translation)	Frequency				Approx. Percent	Cumul. Percent
	Sample A	Sample B	Sample C	Sub-total		
a Correct image	3	16	12	31	32.0	100.0
b Partly correct image	10	7	2	19	20.0	68.0
c Incorrect image	11	7	2	20	21.0	48.0
d Did not attempt	6	12	8	26	27.0	27.0
TOTAL	30	42	24	96	100.0	

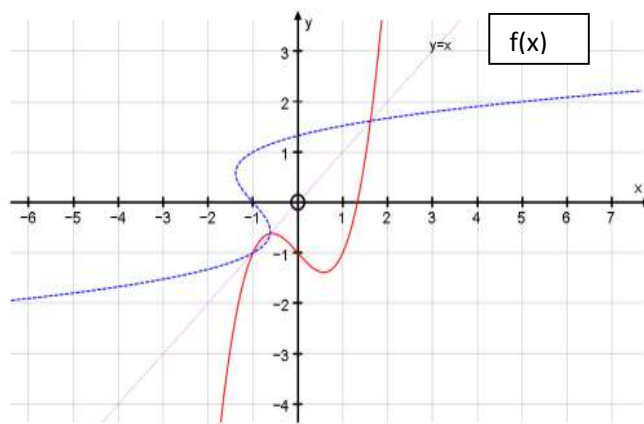
Learners were to identify, describe and write the required formula. Thirty-one learners (32%) managed to answer correctly. Twenty-six learners could not answer the question at all and 39 (41%) had misconceptions. Sample B had more learners who were unable to answer than either of the other two samples. The graphical visual for these results are shown in Figure 5.27, below.

Figure 5.28: Graph showing frequencies of learners' ability to recognise graphical images of a vertical translation of a function



Figure 5.27 displays learners' ability to recognize vertical translation from graphical images. The graph shows that there were a significant number of learners (46) who could neither identify nor describe vertical translation. About a third (32%) were able to answer correctly and the rest had some inaccuracies.

5.1.5.2 Learner ability to recognize reflection in $y = x$ from graphical images (Question 9b)



The question required learners to recognize and describe the concept of reflection about the line $y = x$. The function involved a cubic but neither its name nor its equation were mentioned. Learners could only use verbal descriptions, or symbols if they were very competent. The correct transformation formula, using symbols, is $(x; y) \rightarrow (y; x)$. It was enough, however, for learners to give a verbal description. Twenty-two learners (23%) gave the correct description. Most of these stated that there is a reflection in $y = x$. One of these learners said that there is an exchange of coordinates, which means the same as the symbolic alternative mentioned above. One learner stated it was an inverse graph of the first, which is a more perceptive answer than would be expected from a Grade 11 learner. This represents an extended opportunity. Eighteen learners (19%) gave incomplete descriptions stating that it is a reflection without giving further information. Fifty-two learners (54%) left the answer space blank and the remaining four had serious misconceptions.

The misconceptions shown by some learners are listed below:

- Reflection through x -axis.
- 45° Anticlockwise rotation.
- Reflection on the y -axis.
- 90° clockwise rotation.
- $(x; y) \rightarrow (-x; y)$ (This is a reflection in the y -axis).
- Reflection and rotation.

Table 5.22, below, presents the frequency of the competences and Figure 5.28 is a graphical visual.

Table 5.22: Evaluation frequencies of learners' ability to recognise graphical image of a reflection of a function in $y = x$ (Question 9b) (n=96)

Assessing the Concept Image (Reflection)	Frequency				Approx. Percent	Cumul. Percent
	Sample A	Sample B	Sample C	Sub-total		
a Correct image	10	7	5	22	23.0	100.0
b Partly correct image	7	10	1	18	19.0	77.0
c Incorrect image (misconception)	0	1	3	4	4.0	58.0
d Did not attempt	13	24	15	52	54.0	54.0
TOTAL	30	42	24	96	100.0	

Table 5.22 shows that many learners recognized reflection but had some doubt about whether it can be done about any line other than the axes. The performance was similar across the three samples. Figure 5.28, below, gives a comparison of the competences.

Figure 5.29: Graph showing frequencies of learners' ability to recognise graphical images of a reflection of a function in $y=x$

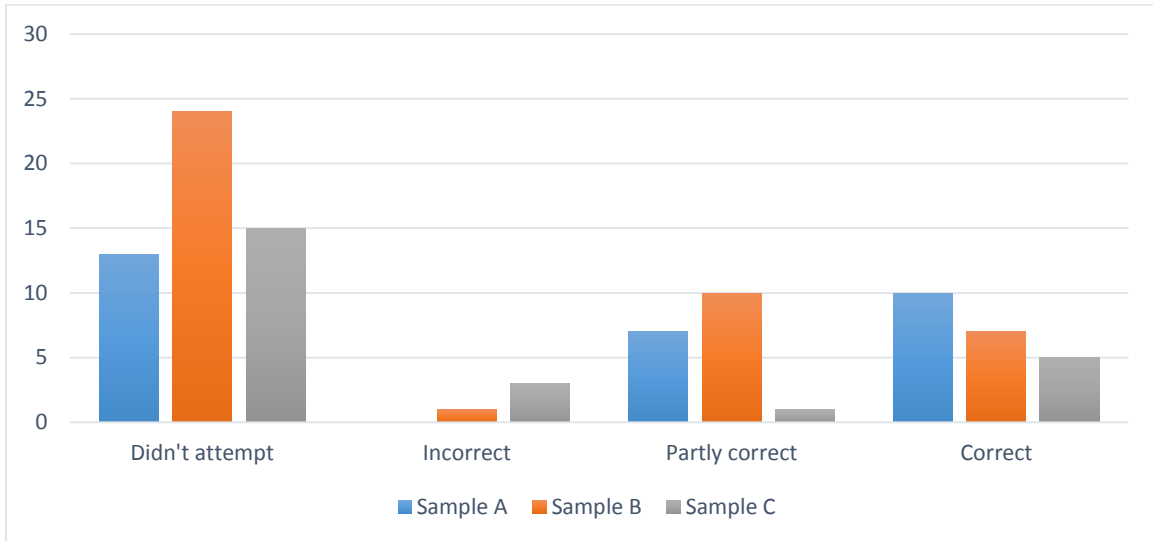
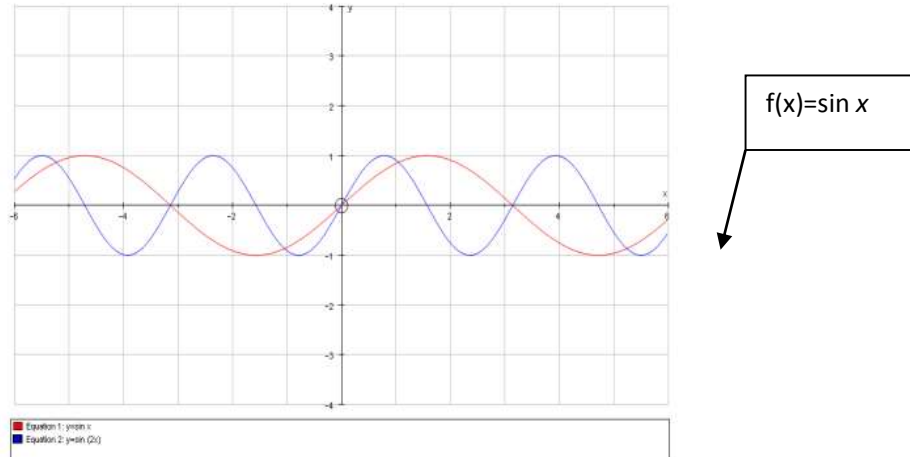


Figure 5.28 displays learners' ability to recognise reflection in $y=x$, without the equation, from a graphical visual alone. From the graph, it is evident that a significant number of learners had some understanding of what was required: 23% gave correct descriptions of a reflection in $y=x$, and a similar number (19%) gave incomplete descriptions (for example, correctly stating that it is a reflection without elaborating further). However, although the number of learners who gave incorrect descriptions was small (4%), a large number of learners (54%) did not attempt an answer at all.

5.1.5.3 Learner ability to recognize compression (inward stretch) from graphical images (Question 9c)



The question required the learners to recognize a horizontal compression of factor $\frac{1}{2}$. Learners were to identify the transformation using either the formula or a verbal description. Thirteen learners (13%) could identify and describe the compression correctly from the graphical representation, and 26 (27%) gave appropriate descriptions without identifying the transformations fully. Forty-three learners (45%) left the answer space blank. The most common misconception demonstrated by learners was a horizontal expansion of the graph by factor 2.

The correct (C) and partly correct (PC) conceptions given by learners were:

- $y = \sin 2x$ has been reduced by factor 2 (PC).
- $y = \sin 2x$ (C).
- Multiplied by the factor $\frac{1}{2}$ (PC).
- Compression by factor 2 (PC).
- The graph is compressed by factor 2. $y = \sin x \rightarrow y = \sin 2x$ (C).
- Compressed horizontally by factor 2 (PC)
- $y = \sin 2x$ has been compressed horizontally (PC).

Misconceptions in learners' responses included those listed below:

- Stretch of factor 2.
- Translation.
- Reflection along x -axis.
- Vertical stretch.
- Reflection and translation 2 units down.
- Vertical reduction.
- 2 is added (Where should it be added?)
- Reduction of factor $\frac{1}{2}$: $f(x) = \frac{1}{2} \sin x$.
- $y = -2 \sin x$.
- $y = 2 \sin x$.

The frequency count for responses to this question is represented in Table 5.23, below, and the graphical visual in Figure 5.9.

Table 5.23: Evaluation frequencies of learners' abilities to recognise and describe graphical image of a horizontal compression of a function (Question 9c) (n=96)

Assessing the Concept Image (stretch)	Frequency				Approx. Percent	Cumul. Percent
	Sample A	Sample B	Sample C	Sub-total		
a Correct image	3	8	2	13	13.0	100.0
b Partly correct image	10	12	4	26	27.0	87.0
c Incorrect image (misconception)	6	2	6	14	15.0	60.0
d Did not attempt	11	20	12	43	45.0	45.0
TOTAL	30	42	24	96	100.0	

The required response is a horizontal compression of factor $\frac{1}{2}$ and learners could use either a formula or words in their answer, or both. From the large number of learners who gave partly correct statements, the table shows that a significant number of learners could identify the transformations but lacked accuracy in describing it.

Figure 5.30: Graph showing frequencies of learners' ability to recognise and describe graphical image of a horizontal compression of a function

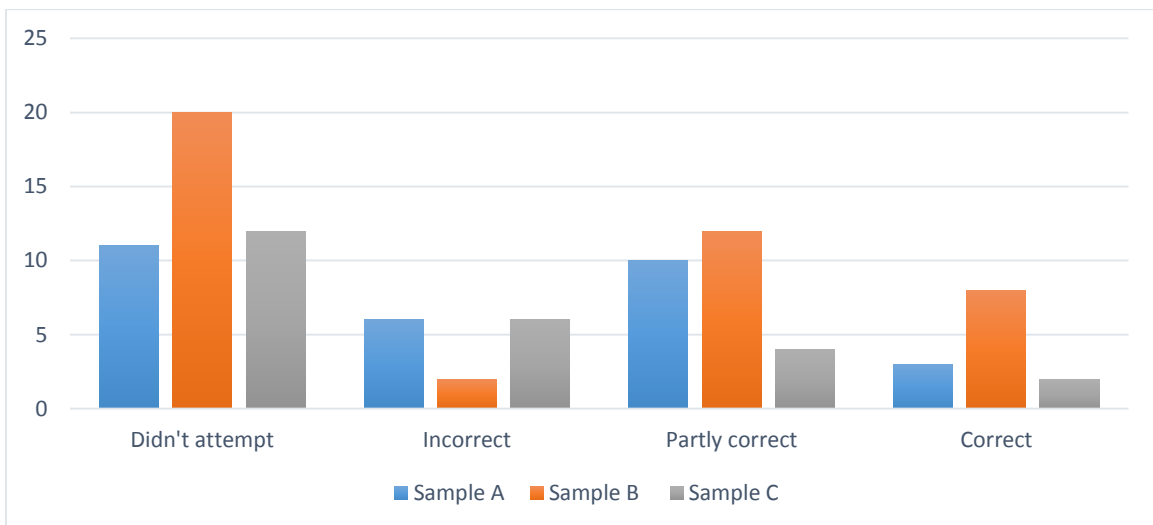


Figure 5.29 displays learners' ability to recognize and describe horizontal compression from the graphical visual. The graph suggests that although some learners (13%) could positively identify and describe compression from the graphical representation correctly, many were insufficiently accurate in their responses (27%). A large number of learners (45%) were not able to attempt an answer at all. Figure 5.30, below, compares competences in recognizing transformations.

Figure 5.31: Graph comparing frequencies of *Correct*, *Partly correct*, *Incorrect*, and *Did not attempt* evaluations about recognising and describing graphical images of transformations of functions

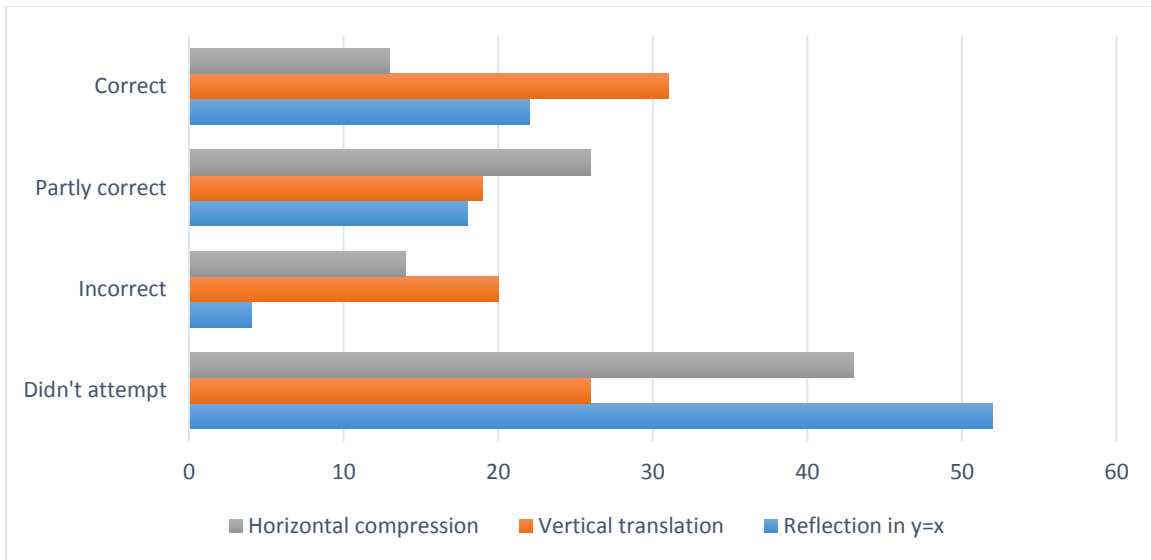


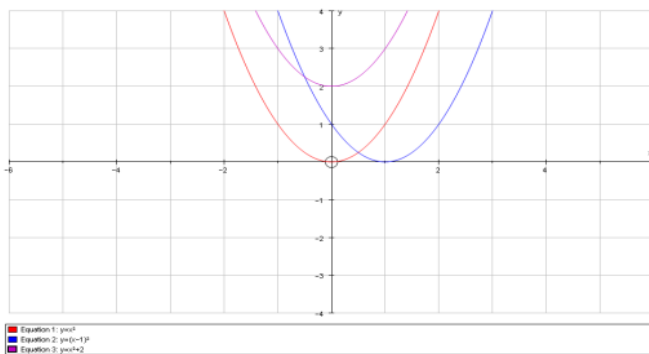
Figure 5.30 displays a cross comparison of *correct*, *partly correct*, *incorrect*, and *did not attempt* for recognition of transformations. The graph shows that most learners had great difficulty identifying the transformations with the exception of vertical translation. Few learners could recognise all three transformations.

Question 10 served as a consistency check in conjunction with the previous question, which also dealt with learner competence in recognizing transformations from graphical images.

The question read as:

Identify the transformations that mapped $y = f(x)$ to the illustrated images. Describe the transformations fully by words and label their formulae, if possible.

5.1.5.4 Learner ability to recognize the concepts of translation from graphical images (Question 10a).



$f(x)=x^2$

The question required learners to recognize and describe translations, one of them 2 units upwards and the other 1 unit to the right. The required formulae are: $y = x^2 + 2$ and $y = (x - 1)^2$. This question was designed to test consistency as it re-tested aspects asked in Question 9. Thirteen learners (13%) gave both the correct descriptions and the correct formulae. Twenty-six (27%) gave one description and formula correctly. Thirteen learners (13%) gave one correct description and ten learners gave one formula correctly. Forty-three learners (45%) left the answer space blank. Reasons for not attempting an answer included: ‘*I don’t know what is needed*’ and ‘*I don’t understand it*’. Most learners who failed to answer the question did not give a reason.

Some of the correct (C) and partly correct (PC) conceptions were follows:

- Translation in y-axis and x-axis $y = x^2 + 2$; $y = x^2 - 1$ (PC)
- Translation 2 units upwards, 1 unit to the right. $y = x^2 - 1$; $y = (x - 1)^2$ (PC)
- Shifted 1 unit to the right; shifted 2 units upwards (C).
- Stretch $y = (x - 1)^2$; translation $y = x^2 + 2$ (mostly correct)
- $y = (x - 1)^2$ and $y = x^2 + 2$, correct description (C).
- Translation $y = (x - 1)^2$; stretch $y = x^2 + 2$ (C).

- The graph is moved 2 units up $y = x^2 + 2$; graph is shifted 1 unit to the right $y = x^2 - 1$ (PC).
- $y = x^2$ is moved 2 units up and become $y = x^2 + 2$ and it moved one unit to the right and become $y = (x - 1)^2$ (C).

Some learners had misconceptions about the descriptions and others about the formulae.

Some of the misconceptions learners demonstrated are listed below:

- Rotation of $f(x) = x^2$ (I).
- Reflection and translation of 2 units (I).
- Reflection about $y = 0$ (I).
- Transformation about the y-axis on the same set of axes or symmetry(I).
- Translation of $y = (x + 2; y + 1)$ (I).

The results are presented in Table 5.24, below.

Table 5.24: Evaluation frequencies of learners' ability to recognise graphical images of vertical and horizontal translation of a function (Question 10a) (n=96)

Assessing the Concept Image (translation)	Frequency				Approx. Percent	Cumul. Percent
	Sample A	Sample B	Sample C	Sub-total		
a Correct image	3	8	2	13	13.0	100.0
b Partly correct image	10	12	4	26	27.0	87.0
c Incorrect image (misconception)	6	2	6	14	15.0	60.0
d Did not attempt	11	20	12	43	45.0	45.0
TOTAL	30	42	24	96	100.0	

Learners were to describe the transformations and write the appropriate formulae. Table 5.24 shows that many learners had difficulty answering the question. Thirteen learners (13%) answered correctly and 26 (27%) gave partly correct answers. There is some inconsistency with the results obtained for Question 9, shown in Table 5.21, which had the same objectives. Horizontal translation was the transformation learners struggled with the most.

Figure 5.32: Graph showing frequencies of learners' ability to recognise graphical images of vertical and horizontal translations of a function

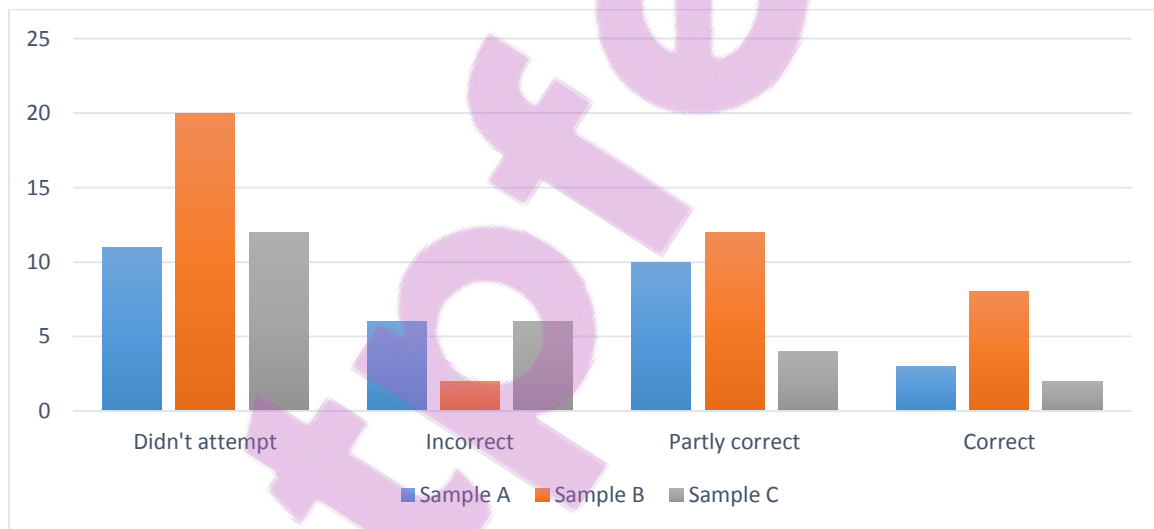
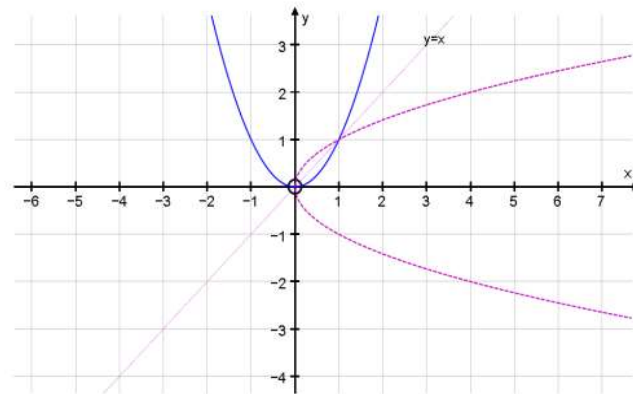


Figure 5.31 displays learners' ability to recognize vertical and horizontal translation from visual graphical images. The graph shows that very few learners (13%) were able to provide both the correct descriptions and the correct formulae. Twenty-six learners (27%) gave a correct description or a correct formula. Fourteen learners (15%) had completely incorrect descriptions and formulae. A large number of learners (45%) did not attempt to answer the question.

5.1.5.5 Learner ability to recognize reflection from graphical images
(Question 10b).



f(x)

The question required learners to recognize and describe the concept of reflection about the line $y = x$. The question also re-tested aspects tested earlier, to assess consistency. The function involved was not named and its equation was not given. Learners would also be *correct* if they described the transformation as a clockwise rotation of 90° or gave the transformation formula as $(x; y) \rightarrow (y; x)$, even though rotation was beyond the scope of this study. Thirty-five learners (37%) gave correct descriptions either in words or using a formula. Of these learners, 13 stated there is a reflection in $y = x$, 5 said there is an exchange of coordinates (which means the same as $(x; y) \rightarrow (y; x)$), and 17 stated that the second graph rotated 90° clockwise. Eleven learners noticed that the transformation was reflection but some of this group could add nothing further and others added incorrect aspects. Fifty learners (52%) left the answer space blank and gave reasons for not attempting to answer including: ‘cannot interpret’, ‘I don’t understand what the question needs’, ‘I don’t understand’, ‘the graph is confusing’ and ‘I don’t have any idea, the shape is confusing’.

Some of the correct (C) conceptions were:

- An inverse of $f(x)$ which is $f^{-1}(x)$ reflection on the line $y = x$ (C).
- They have interchanged the values of x and y and multiplied the new x value by -1. $y^{-1} = ax$ (C).

- The transformation is reflection. The graph reflect on the line $y = x$ (C).
- Reflection. Graph has been reflected on $y = x$ axis (C).
- Reflection about $y = x$ line (C).
- Rotation 90^0 clockwise: $(x;y) \rightarrow (y;-x)$ (C).
- The image of $f(x)$ has been rotated 90^0 clockwise (C).
- $F(x)$ rotated 90^0 in a clockwise direction (C).

The misconceptions demonstrated by learners are listed below:

- Reflection about the x -axis (I).
- $x = \pm\sqrt{y}$. (I)
- $y^{-1} = ax$ (PC).
- The image was vertical and it transformed horizontally (I).
- Translation to the left (I).
- $(x; y) \rightarrow (y;-x)$ (I).
- Reflection in $y = -x$ (I).
- $y = (-x;-y) \rightarrow (y; x)$ (I).
- Rotation and enlargement (I).
- Rotation on the line $y = x$ (I).
- Rotation (PC).
- Anticlockwise rotation (I).
- Reflected and rotated about the y -axis (I).
- The image of $f(x)$ is stretched out vertically (I).

The frequency distribution of these results is shown in Table 5.25, below, and the graph in Figure 5.32 provides a visual representation.

Table 5.25: Evaluation frequencies of learners' ability to recognise graphical image of a reflection of a function in $y = x$ (Question 10b) (n=96)

Assessing the Concept Image (Reflection)	Frequency				Approx. Percent	Cumul. Percent
	Sample A	Sample B	Sample C	Sub- total		
a Correct image	6	6	1	13	14.0	100.0
b Partly correct image	9	10	3	22	23.0	86.0
c Incorrect image (misconception)	5	2	4	11	11.0	63.0
d Did not attempt	10	24	16	50	52.0	52.0
TOTAL	30	42	24	96	100.0	

The function is quadratic $y = ax^2$ where $a > 0$. The graph's position allowed for 'a 90° clockwise rotation' as an alternative correct response. Learners were not expected to give this alternative as it extended beyond the scope of the Grade 11 syllabus. Table 5.25 shows no significant difference from Table 5.22, the results obtained for Question 9b, which had the same objectives as Question 10b.

Figure 5.33: Graph showing frequencies of learners' ability to recognise graphical image of a reflection of a function in $y=x$

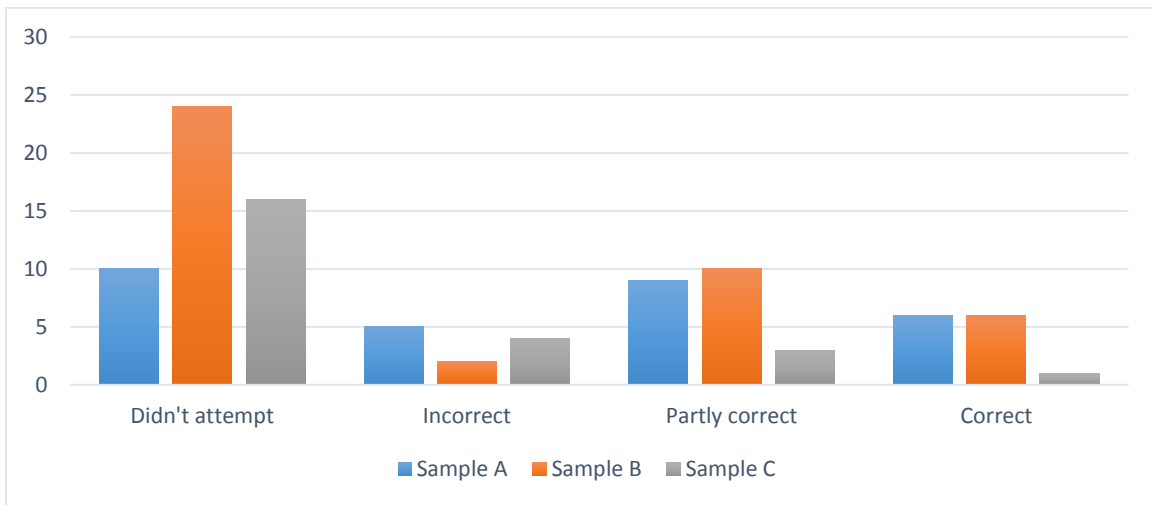
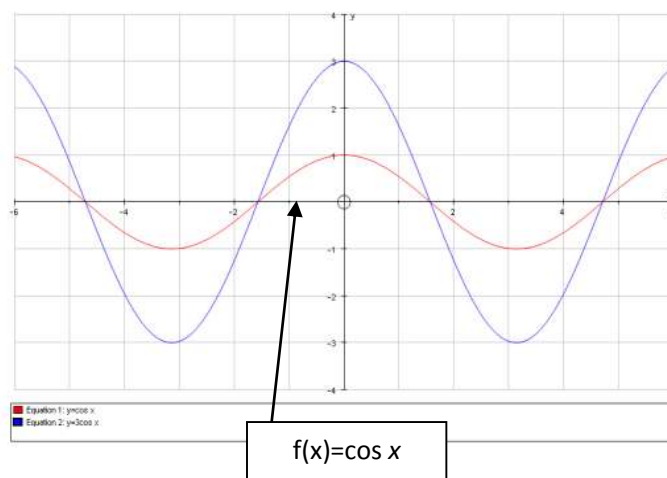


Figure 5.32 displays learners' ability to recognize reflection in $y=x$ from a visual image. The graph shows that although few learners gave completely correct responses, a significant number (37%) were able to give the correct description either in words or using a formula. Sixty-one learners were unable to answer the question correctly (63%), and most of these (52%) did not attempt an answer at all.

5.1.5.6 Learner ability to recognize stretch from graphical images (Question 10c)



The question required learners to recognise a vertical outward stretch (pull) of factor 3 with the x -axis invariant. The formula required is $y = 3\cos x$. Learners who could identify the

transformation either used the formula or gave a description in words. Twenty learners could correctly identify and describe the transformation from the graphical representation and gave the correct formula. Six learners gave the correct description but not the correct formula, while 4 learners gave the correct formula but not the correct description. Forty-six learners (48%) left the answer space blank. The number of learners who did not attempt to answer was large, and while this was most likely due to a lack of knowledge about the material covered, other factors may also have contributed. For example, learners may have been frustrated by the increasing difficulty of the questions or may have run out of time. The reasons for not attempting an answer included the following: *'I don't understand'*, *'I don't know the transformation'* and *'I don't know the formula, so don't understand it'*. Some of the misconceptions probably stemmed from a lack of familiarity with the subtleties of language use, for example, a learner saying that the amplitude increased 'by' 3 units instead of 'to' 3 units. This misconception may have been reflected in the formula given as well, for example using $y = 2 \cos x$, instead of $y = 3 \cos x$.

Some of the correct (C) and partly correct (PC) conceptions were follows:

- Amplitude increased by 3. $f(x) = 3 \cos x$ (PC).
- $y = 3 \cos x$. Stretched vertically by 3 units (PC).
- Stretch $y = 3 \cos x$ (C).
- $y = 3 \cos x$. It has stretched by scale factor 3 (C).
- $y = \cos x$ for $y = 3 \cos x$ and viceversa (C).
- Stretch by 3 units vertically (PC).
- $y = \cos x$ has been stretched vertically (3 units) which will be $y = 3 \cos x$ (PC).

Some other misconceptions demonstrated by learners are listed below:

- $y = \cos(x + 3)$ (I).
- $y = \cos x + 3$ (I).
- $y = \cos x + 2$, 2 units upwards (I).

- y increased by 2 $\therefore 2y = \cos x$ (I).
- Stretched horizontally (I).
- Stretched by a factor 2 (No direction) (PC).
- Moved two units up $2y = \cos x$ (I).
- The image of $f(x)$ is reduced towards the y -axis and enlarged 2 units upwards/horizontally (I).
- It's a reflection (I).
- The image of $f(x) = \cos x$ has been compressed by 2 units vertically and 2 units horizontally (I).
- The image has been compressed to $f(x) = 1\cos x$ (I).
- Enlargement: $f(x) = \cos 2x$ (I).
- The reflection is increased by a factor given $f(x) = y + 2$ (I).
- The graph has been compressed by 2 units (I).

The frequency count for these results is displayed in Table 5.26, below, and the graphical representation can be found in Figure 5.33.

Table 5.26: Evaluation frequencies of learners' ability to recognise graphical image of a vertical stretch of a function (Question 10c) (n=96)

Assessing the Concept Image (stretch)	Frequency				Approx. Percent	Cumul. Percent
	Sample A	Sample B	Sample C	Sub-total		
a Correct image	6	9	5	20	21.0	100.0
b Partly correct image	3	6	1	10	10.0	79.0
c Incorrect image (misconception)	9	4	7	20	21.0	69.0
d Did not attempt	12	23	11	46	48.0	48.0
TOTAL	30	42	24	96	100.0	

The learners were to give the formula $y = 3\cos x$ or describe in words an outward stretch that increases the amplitude threefold. The differences between these results and those of Question 9c in Table 5.23 are not significant. The table shows that learners were familiar with vertical stretch and answered the question accurately.

Figure 5.34: Graph showing frequencies of learners' ability to recognise graphical image of a vertical stretch of a function



Figure 5.33 displays learners' ability to recognize vertical stretch from graphical images. The graph shows that a small number of learners (21%) could positively identify, describe and write the correct formula for the transformation from the graphic representation, but few (10%) were able to give either a correct formula or a correct description. The largest number of learners (48%) didn't attempt to answer this question.

A comparison graph in Figure 5.34 indicates how learners dealt with different transformations.

Figure 5.35: Graph comparing frequencies of *Correct*, *Partly Correct*, *Incorrect*, and *Did not attempt* evaluation about recognition graphical images of transformations of functions

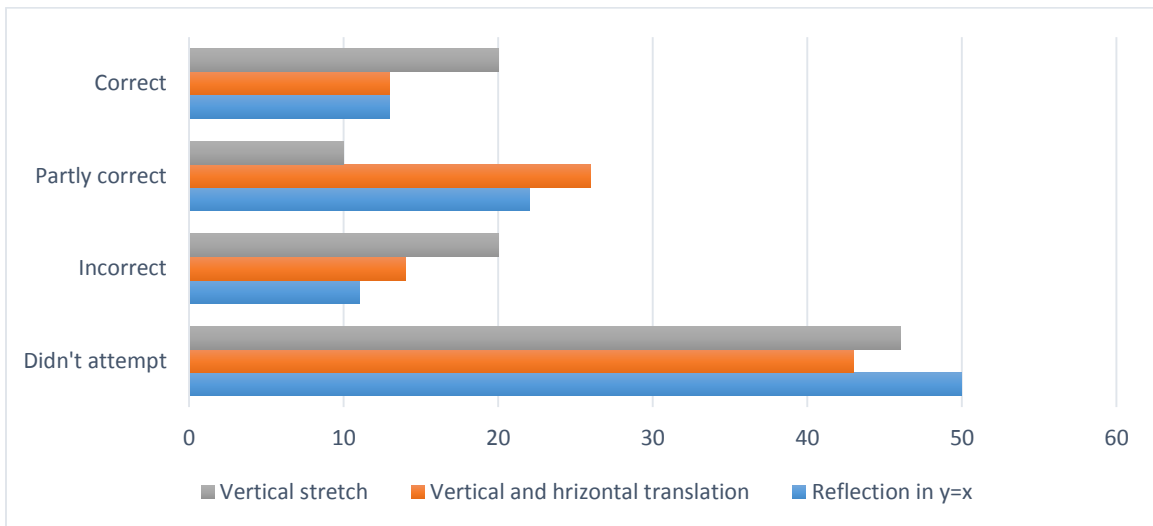


Figure 5.34 displays a cross comparison of *correct*, *partly correct*, *incorrect*, and *did not answer* results for recognition of transformations. It shows that most learners had difficulty in recognizing transformations.

5.2 LEARNER MATHEMATICAL REASONING AS ASSESSED FROM THE FOLLOW-UP INTERVIEW VERBATIM TRANSCRIPTS

Interviews were done as a follow-up on the responses to the diagnostic test to generate greater understanding of the reasoning behind learners' conceptions of the transformation of functions, and to fill in any gaps that were not addressed by the diagnostic test. Learners who scored more than 30% for the diagnostic test, and who were present at school the days the interviews were conducted, were interviewed by the researcher. A total of 14 learners were interviewed.

It was difficult for the researcher to grasp learners' mathematical thinking and reasoning from their verbal expressions in the interviews, which is why a careful detailed analysis of the verbatim transcription was required. Some learners deliberately avoided communicating their process of reasoning, and some were not aware they were reasoning, or their conceptions were based on procedural rather than relational understanding. Part of the problem was the absence of a clear cognitive–realistic partnership. Whilst errors were not always easily identified in verbal expressions of mathematical thinking, they were frequently evident in what learners wrote down. It is for this reason that the interviewer insisted learners write down their explanations as part of the diagnostic test and then followed up with interviews so that learner cognitive thinking could be assessed from two different types of evidence.

Some of the interviewer's questions and the learners' verbal responses are presented below:

Question 1: “**How do you feel about the topic of transformations of functions?**” The statements underlined in the excerpts below indicate where learners expressed difficulty with the concepts of transformations of functions.

***Interviewee 1:** The topic is fine but the things like where it gets to the $g(x)$, $f(x)$, g of $h(x)$ and the like, sometimes working out the answer is the issue, working out the issue of the different functions and how to get to the final stage, but otherwise everything is fine.*

***Interviewee 2:** It's difficult, I don't understand them ... The graphs! I don't understand them ... The functions in general are difficult for me.*

***Interviewee 3:** Personally it's something I enjoyed but it's challenging but with practice you tend to perfect it.*

***Interviewee 4:** I find it difficult sir, I am struggling.*

***Interviewee 5:** Yoh sir, to tell you the truth I hate it, I hardly understand it sir.*

Interviewee 6: *The topic is quiet short ... I'm not comfortable because I don't like it at all sir. I think its lack of interest in the topic ... This topic is not interesting. It's boring. It is challenging.*

Interviewee 7: *Partly I can do and the other part I gave up when I study alone and then I ask my other friends.*

Many of the learners who responded to this search question expressed the difficulty they experienced when dealing with the topic. This situation confirms what the Roadshow presenters said about learners being challenged by transformation of functions (DoE, 2009b; 2010; 2011).

Question 2: This search question was guided by three sub-questions:

“What do you understand by a reflection?”

“What do you understand by a translation?”

“What do you understand by a stretch?”

Interviewee 1: *Reflection is when you take something from one side and is reflected about a line of symmetry ... Translation is when the co-ordinates of the points move either upside or down depending on the function or the given co-ordinate moved to, let's say you have y and x-axis and they tell you that it has to be y-1 and x+2 then you move it accordingly around in the corresponding place ... A stretch diagram is when a graph either is stretched according to the x-axis or either reduced or enlarged on the x-axis.*

Interviewee 3: *It's when, eemh, that is, a sketch or no, it's not a sketch, an image is drawn on the one side of a Cartesian plane and then you draw it on the other side and is reflected back on the y-axis or the x-axis. ... (Stretch) It's when you make the image bigger. You stretch it! (The learner moves hands apart, away from each other demonstrating). Yes like if you draw here and you stretch it by 5 or 10 it will be like huge.*

Interviewee 4: *A reflection is like is a mirror image of something. Maybe it's like something standing here and then what you see on the opposite side is exactly the same as what's in front ... Translation, I think, is moving an object like from one place to another. Maybe if*

it's in the 1st quadrant of a Cartesian plane moving it to the 3rd ... Stretching is like making an object bigger that you can reflect or take it out sideways, vertically or horizontally.

Interviewee 6: Translation is moving, the movement of one image onto the other side of the Cartesian plane. It depends whether you are translating it by how many points, up or down, left or right. That's what I understand by a translation. A reflection is a mirror image of one image onto the other part of the Cartesian plane as well depending on reflection either on the x or the y-axis. Stretch is expanding an image.

Interviewee 7: Ok. Reflection is the mirror image of a shape or a graph about a line, like on a Cartesian plane. You can get an object that is reflected about the y-axis, about the x-axis, about a line which is $y=x$ and about a line which is $y=-x$. Translation is when you shift, the movement of an object from its original space, which it originally occupied. Like you can shift upwards or downwards, to the left or the right. Yoh stretch, stretch, I can't clearly define it but I know what it is.

Interviewee 12: Sometimes they can tell you to translate maybe by a formula or reflect on the y-axis. Let's say they have given you something like any function and you translate this function on the x-axis sir. If you say that, you must actually put a minus on the y-axis (value). Am I right sir? Or usually maybe given two points x and y then they say translate it in the y-axis and then if you translate it it's going to be minus x to y that's what I understand. Or maybe they say rotate 180° the point maybe clockwise let's say from the 1st quadrant rotate 180° the image is going to be x-y but it's on the other quadrant.

The learners' responses show that most of them understood something about the concepts of reflection, translation and stretch but they lacked correct and rigorous terms to define them. Some learners used the term 'image' in an ambiguous manner, using it to refer to an object before transformation and also correctly as an image after transformation (see interviewee 6). Other learners gave accurate definitions, but using their own words (see interviewee 7). Some learners also connected their definitions to symbolical representations (see interviewee 12) by talking about putting a minus sign on x or y values. Most definitions of stretch did not make any distinction between it and enlargement. One

of the learners in the sample used the term stretch to define stretch without being able to explain it fully in his/her own words (see interviewee 1)

Question 3: The third search question was about study effort. “How have you tried to understand the concepts in transformations of functions?”

Interviewee 7: When Mr. ... (teacher) teaches and gives us homework every day about what we learnt in class, he is giving us not because he is trying to be funny, but he is giving it so that he can see what we can do on our own.

This is evidence that some learners depend totally on the teacher for direction. Interviewee 7 attributes all conceptual effort to the teacher. There is no understanding of a schematic approach²⁴ to learning these concepts and no sign of creative and logical thinking and reasoning (DoE, 2002; 2012).

Question 4: The fourth search was about symbolical representation. “Identify and justify the transformation from $y = f(x)$ to: $y = f(-x)$; $y = -f(x)$; $y = 3f(x)$; $y = \frac{1}{2}f(x)$; $y = f(x)-3$; $y = f(x+3) - 2$; $y = f(2x)$; $y = f(\frac{1}{2}x)$; $y = f(x-2)+3$; and $y = -3f(2x)+4$ ”.

Interviewee 2: $-f(x)$ from $y = f(x)$ is a translation. This one is a stretch ($y = 3f(x)$) ... $y = \frac{1}{2}f(x)$ is a stretch again, $y = f(x)-3$ is a translation ... $y = f(x+3) - 2$ is a translation. I'm not quite sure about this one. I think $y = f(2x)$ it's a stretch, $y = f(\frac{1}{2}x)$ is a stretch again, $y = f(x-2) + 3$ is a translation and $y = -3f(2x) + 4$ a translation (No justification to support these identifications was given).

Interviewee 4: $y = f(x)$ to $y = f(-x)$ is a reflection about the y-axis, $y = -f(x)$ is a reflection about the x-axis. Here it is. (Showing a correct illustration). If $y = f(x)$ results in $y = 3f(x)$, then the transformation is a stretch. Horizontal. No, it's horizontal and vertical. Vertically. It is stretched 3 units vertically. This other is also a stretch ($y = \frac{1}{2}f(x)$), horizontally. This one moved 2 units down and 3 units up ($y = f(x-2) + 3$). I am confused.

²⁴ Learning through finding links between concepts and looking for relevance of these concepts to real life

Interviewee 5: Oh. 1st one is a translation? This one is a rotation, rotation about the line $y=x$. This is a 90° rotation basically but it's a rotation about line $y=x$. This that transforms the function $y=f(x)$ to $y=f(2x)$ is enlarged. No, this is stretched by factor 2.

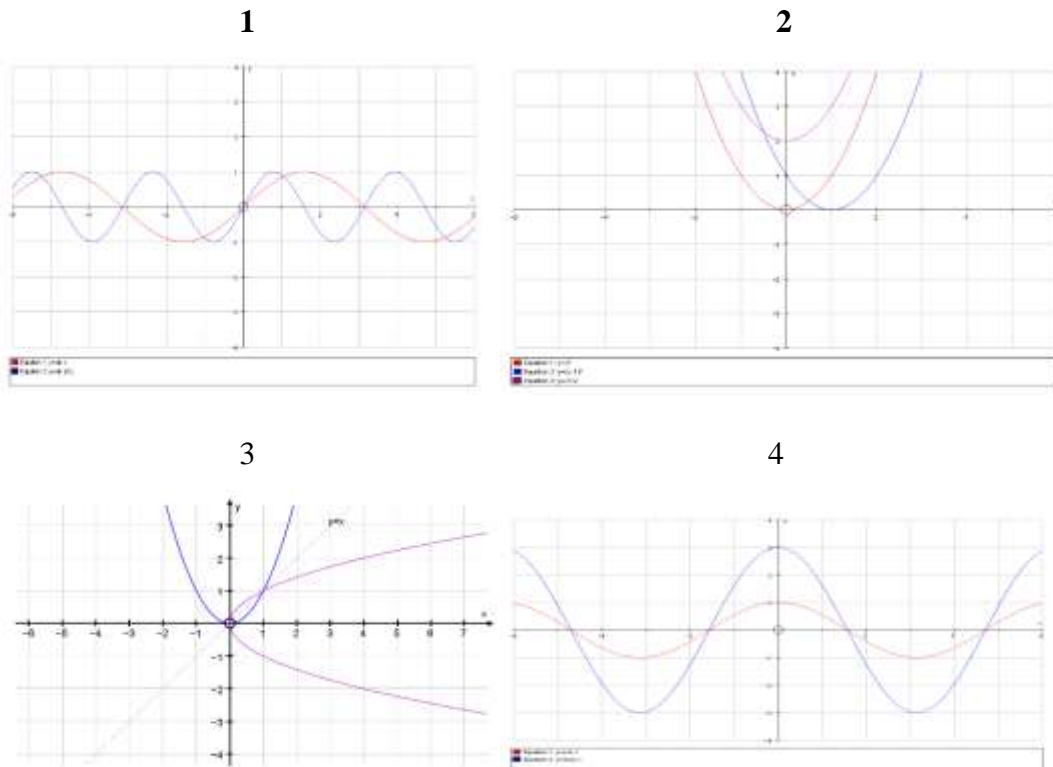
Interviewee 6: Ah, I think its translation because $f(x)$ only changes, and then if you say $-f(x)$ it means you are taking this to the other side, its actually like down, so that means downwards. I think this one means that it's stretching because it's actually moving it 2 units to the left through the x -axis and then the y -axis value 3 units. It's stretching because you taking your scale, if you have 1cm scale you reducing it to half. Oh this is actually reflected because this part comes to this side, the original is this one, it's the big one, oh its compressing vertically, compressing because it was a bit bigger, now it has been compressed becoming a bit smaller or narrower. Number 4 ($y = \frac{1}{2}f(x)$) is a reflection, no not a reflection, its translation because they are moving the object. The other one didn't go up, didn't compress I've forgotten the word. Like it became smaller. Ok compression. Number 6 ($y = f(x+3)$), sir, it's flipped. No it was rotated anticlockwise like 90° anticlockwise.

Interviewee 7: Yoh, ah well I think this one ($y = f(x)-3$) is a translation. This one ($y = 3f(x) + 4$) is enlarged by 3 units and moved to the left. Ok the $y = -f(x)$ is a reflection about the y -axis and $f(x + 3) - 2$ is a translation which is whether the image is expanded by 3 units and moved 2 units to the left. Oh the transformation? This ($f(x-2) + 3$) is a glide reflection and it's moved 2 units to the left and expanded by 3 units. Number 5 is rotated I think 90° clockwise that's number 5.

Many misconceptions surfaced as learners attempted to identify transformations from symbolical representations. Examples of the misconceptions were: mistaking reflection for translation (see interviewees 2, 5 and 6); mistaking a stretch for a reflection (see interviewee 6); mistaking a horizontal translation to the right for horizontal translation to the left (see interviewee 4). The last-mentioned misconception seems to stem from a misunderstanding about the impact of the minus sign, which learners can intuitively associate with a negative direction of the x -variable. There is a lack of precision in descriptions of concepts or concepts are expressed incompletely. Examples of

incompleteness were: failure to state direction of stretch as vertical or horizontal (see interviewees 2, 6); failure to state orientation of stretch as compression or extension (see interviewee 2); failure to fully state a composite transformations. There was a serious misconception that there was a rotation about a line $y = x$ (see interviewee 5). Learners also struggled with identifying composite transformations from symbolical representations like $y = f(x - 2) + 3$ and $y = -3f(2x) + 4$. Some mentioned only one component of the transformation (see interviewees 4, 6 and 7) and others mistook a horizontal compression for a horizontal outward stretch, and vice versa (see interviewees 5, 6).

Question 5: The fifth search question required identification of transformations that mapped a drawn graph from one position to another. “Given the original function $y = f(x)$ (in blue) and the transformed graph (in red), tell me the transformations that are involved in the mappings”.



Interviewee 6: This (Number 1) is a glide reflection and it's moved a unit to the left and expanded by 2 units. Number 2 is translated a unit to the right and 2 units up. This one

(Number 3) is rotated, I think, 90° clockwise. This one (Number 4) is expanded by 2 units, it's expanded by the factor of 2.

Interviewee 10: I can say this has transformed. Because this one is $y = \sin x$ then from there the unidentified line of the graph is $y = \cos x$, sir. I believe that $f(x)$ has been translated 2 units up, sir and also one unit to the right sir (The learner only referred to the first two graphs).

Interviewee 11: This one (Number 1) is $y = \sin x$, maybe it has been reduced by 2. For this one (Number 2), the first has been shifted by one unit to the right and the second is shifted vertically by two units upwards. But sir, I find it difficult because ... well, it's not really difficult but it's how I'm supposed to understand the representations. The staff sometimes confuses me, so I need more practice. I don't have a problem with it but I'm the problem (The learner only commented about the first two graphs).

Interviewee 12: From this graph sir they have reduced the x values by 2 for it is to be smaller, but I don't know which word to use but they have reduced the x value. $F(x)$ is the original it's transformed to that one and from here they have moved the graph upwards 2 units and then from here to here moved 1 unit to the left then this one 2 units upwards. This one (Number 3), sir, the original graph is this one. So I think this graph is the image of this one. The image after 90° rotation. This is the original graph. The original is cosine and as a result it was enlarged sir. Maybe increased by a factor of 3 from here to there. Stretched but I don't know by how many.

Interviewee 13: The turning point has been increased between 0 and 2, on the 2nd one the graph has shifted 1 unit to the right. I think its! Yoh, sir I'm not familiar with this type of transformation (Number 3). By the value of a is now 3, not 1, $y = a \cos bx$. Oh it's there, its 3 (Number 4).

Interviewee 14: (Number 1) I think it's stretching. Stretching to the stretch factor of 2. (Number 2) Function of x which is a parabola. It moves to that one. It's a translation shifted 2 units up and the other one 1 unit to the right. (Number 3) This line $y = x$. It has been stretched by the scale factor of 3 downwards because this is the initial one. The

original one is this one, oh it's that one. The image is the bigger one. Oh, it has been stretched and is bigger than the other one, then the other one to the stretch factor of 3.

Learners' responses show that some of them had misconceptions when interpreting symbolical representations and when stating the transformation(s) in full. Examples of these are: mistaking a horizontal stretch for a glide reflection²⁵ (see interviewee 6); mistaking a sine graph for a cosine graph (see interviewee 10) and mistaking reflection in $y = x$ for stretch (see interviewee 14). Some learners had correct ideas of the transformations taking place but lacked accurate words and completeness. For example, they used words like 'expansion' and 'enlargement' instead of 'stretch' and did not indicate the direction or orientation (see interviewees 4, 13 and 14). Other learners did not attempt to answer and confessed to having difficulty (see interviewee 11). Some of the challenges faced by learners indicated a lack of proper study skills.

Question 6: What really are the challenges you face when dealing with this topic?

Interviewee 2: Like when you have a straight line and hyperbola on one Cartesian plane, which is the one that gives me much of the problem?

Interviewee 10: I believe it's because of the interaction with the teacher and also my understanding when I read the book, I find it difficult to understand what they are saying, like ah! I don't know how to put it. ... I believe it's not meant to be difficult but when I read it, like the theory and the explanation, I don't know how to do it. When it comes to the questioning and the background I don't know why or how it comes because I didn't understand the theory given that it dates back on my Grade 10 teacher.

These responses indicated that learners lacked adequate relational understanding of functions and their transformations. Their study skills may be insufficient or that the topics were inadequately taught, tested and revised in class.

²⁵ Glide reflection is translation and reflection in one transformation.

5.3 CONCLUDING REMARKS

The data presentation in this chapter is mainly descriptive and it provides evidence on learners' thoughts and expressions of how they understand the concepts of translation, reflection and stretch, and how these manifestation with respect to functions. There was a wide range of learner responses. Some learners had an understanding of the topics which form the focus of this study and were able to provide correct answers when tested. Many learners had partially formed or incomplete ideas about transformations of functions and too many did not seem to understand the concepts involved. The evidence for these claims has been provided through an analysis of learners' test responses and the verbatim transcripts of their own words obtained during the interviews. The frequency tables and bar graphs show the patterns within four categories of response from the diagnostic test. The interview responses were used for the purpose of triangulating results and also to expose the level of mathematical reasoning possessed by learners. Mathematical reasoning was found to be inadequately rigorous and possibly revealed a lack of background knowledge of the concepts.

CHAPTER SIX

Relating Research Findings to Research Questions

The qualitative and quantitative results of this study were outlined in Chapter 6. The detailed descriptions and evidence of learners' reflections of how they understood reflection, translation, and stretch concepts, and their manifestations in functions, constituted the qualitative results. Verbatim transcripts of the voice recordings from the interviews also contributed to the qualitative results. In order to authenticate what is claimed from the data presentations, evidence was provided in the form of learners' exact words extracted from the test answer papers and interviews. Interpretations based on the expressions and evidence for concept images and mathematical reasoning were made. Data coding of the answer papers into categories of similar responses produced quantitative results. This coding facilitated frequency counting by learner response categories from the diagnostic test. A data cleaning process, which refined frequency figures for each question, was carried out before a descriptive statistical analysis was done with the assistance of the computer program Statistical Package for Social Sciences (SPSS). Comparison of frequency tables enabled conclusions to be drawn. Ages or age ranges and gender comparisons, which form part of the demographic analysis, were presented earlier in Chapter 4. This chapter interprets the research findings in terms of the research questions, and the benefits of this research study for educational practice are made clear.

6.1 EXPLORING LEARNERS' CONCEPT IMAGES OF REFLECTION, TRANSLATION AND STRETCH OF FUNCTIONS

The main research question, as stated in Chapter 1, read: ***What are Grade 11 mathematics learners' concept images and mathematical reasoning on transformations of***

functions? This question sought to determine learners' verbal, graphical and symbolical interpretations of reflection, translation and stretch of functions. In order to address this topic explicitly and implicitly in sufficient depth, the question was divided into two parts: the *exploratory* part and *descriptive* part, which were also further divided into mini-questions (see section 1.5).

The results and findings follow the same format and arrangement of the exploratory and descriptive mini-questions. The subsections below present the results of the study.

6.1.1 Responding to the first research sub-question

The primary research sub-question of this study reads: *What are mathematics learners' verbal, graphical and symbolical images of reflection, translation and stretch of functions?* Question 1 of the diagnostic test was designed to provide evidence to answer this question, and was reinforced by information gained from the subsequent questions (2 to 10). The qualitative information obtained from responses to the questions and the frequencies derived from the quantitative analysis, contribute to the holistic scope of the results obtained from the research and are presented below.

6.1.1.1 Learners' concept images of reflection

Reflection is formally defined as a mapping that produces mirror images of points in lines or in polygons about a particular line called an axis of reflection (Tapson, 2006). When using their own words, learners' described reflection as a *flip*, an *inverted image* or a *symmetrical image*. These terms are all compatible with the formal definition.

Other concept images provided by learners included those listed below:

- a mapping,
- mirror image,
- flipped formation,
- flipped or flips,
- line of symmetry,

- image mirrored/flipped across the x -axis or y -axis or $y = \pm x$,
- a repeated object across the y -axis or x -axis,
- not changing the size or shape but changing the coordinates,
- an exact copy of another object,
- graph for object and image,
- similar object and images,
- a glance of the same picture.

These various definitions of the same concept show us the diversity of ways in which learners conceptualized the action of reflecting, the relationship between the original object and its image after reflection, and the combined plane occupied by the original object and its image. This evidence supports the claim that although learners were under the guidance of a teacher, they constructed and created their own knowledge (Glaserfeld, 1984; 1989; 1991). Tall (1991) had a similar view in which learners had to develop their own approaches and skills to learn mathematics so as to facilitate their internal growth and these facilitate the formation of knowledge structures and thinking processes.

From the wide range of images of reflection provided by learners, we can identify where they understood concepts successfully and where there were shortcomings. Because English was only a second, third or fourth language for most of the learners in the study group, language limitation was likely to be one of the major factors impacting on successful conception, and official departmental examination reports have identified this problem (DoE, 2013; 2014). This limitation manifested both in learners' ability to understand the question being asked and in their ability to express themselves accurately and precisely in their answer. However, on the positive note, many of the definitions of reflection provided by learners were complete as they contained the two essential properties of reflection: correct description and the relative positions of the original object and its image with respect to the axis of reflection. However, there were some incomplete definitions, containing only one aspect of the definition of reflection, and a number of total misconceptions. Departmental examination reports have identified such problems as being

wide spread, citing cases where examination candidates gave incomplete answers or offered the term ‘reflection’ and no more when asked to define the concept (DoE, 2013).

Examples of misconceptions displayed by learners in the diagnostic test are shown in section 5.1.1.3.

No learner failed to give a definition. Misconceptions were evident where the definition provided by the learner was too neutral, applying equally well to transformations other than reflection, or was meaningless. A lack of conceptual understanding (Tall, 1988) or limited and even erroneous conceptions about the concept (Tall & Vinner, 1981; Vinner, 1991) were evident in many learners’ inability to interpret graphs from equations or vice versa, but poor language skills also played a part (DoE, 2013).

6.1.1.2 The concept images of translation of functions

Formally, a translation is referred to as a mapping that changes position of a point, line or a polygon by sliding it in a specific direction through a specific distance (Tapson, 2006). This definition implies that points move an equal distance, and that the movement is in a common direction. The definitions given by learners varied in content, language used and accuracy. Some of the learners’ descriptions included, as key terms, words like *displacement, movement, slide, change of position* and *shifting, for a specific distance*, followed by *upward, downward, to the left, to the right* or *in a straight line/specific direction*.

Some examples of definitions provided by learners in the study sample are shown in section 5.1.1.4.

Most learners (77%) used at least one of the acceptable descriptive terms. Some of the definitions were complete because they included both displacement (distance) and specific direction (or just *in a straight line* displacement) but others were incomplete because they provided only one of the two aspects of the definition. The departmental examiners’ reports also refer to this shortcoming, which indicates that it maybe was a widespread problem not limited to the learners in the study sample (DoE, 2013). Some definitions provided by

learners were misconceptions because they were too neutral or did not anchor with the concept translation.

Examples of definitions that exhibited misconceptions are also stated in section 5.1.1.4.

Some of these misconceptions correspond to examples cited in past departmental examination reports (DoE, 2013; 2014) which stated that learners commonly make mistakes with direction (left, right, up or down) and units, and also use terms that are not relevant to the concept, which indicates the presence of misconceptions (Bishop, 1986; DoE, 2014). Some learners left the answer spaces blank and did not write anything.

6.1.1.3 Concept images of the stretch of functions

The formal description of a stretch is a mapping that changes the position of all points outside a particular line (invariant line) away from that line or towards that line in a specific given scale (Tapson, 2006). It can be either an outward stretch or an inward stretch (compression/contraction) in the x -direction or y -direction (i.e. away or towards the y -axis or the x -axis). The three important aspects of a stretch are the invariant line, movement of points that are outside the invariant line, and the stretch factor (ratio of distance of original point from invariant line to that of its image from the invariant line).

The concept images about stretch offered by learners included words and phrases such as *expansion or contraction of an object, extension or compression of an object, increase or decrease in size of an object, points moving apart or closer in a plane, making something longer or shorter, widening or narrowing something, lengthening or shortening something, making something fat or slim, pulling or squeezing something on both ends, enlarging or shrinking something, spacing points from each other or bringing them closer*, followed by *a factor or specific scale factor or certain scale factor*. None of the learners' definitions or concept images included mention of an invariant line.

Examples of statements provided by learners about stretch are listed in section 5.1.1.5.

Evidence from learners' responses in the written diagnostic test, when subjected to qualitative and quantitative analysis, points to the fact that stretch was the most challenging

concept for learners to define. Very few definitions were complete, having all three of the aspects required. Many definitions were incomplete and the majority were misconceptions. This corresponds to what provincial examiners found and reported on (DoE, 2013; 2014). Incorrect definitions were either too neutral, ambiguous, not at all related to stretch, or incorrect in terms of meaning. A lack of conceptual understanding (Tall, 1988) was evident in many cases, and this was compounded by the language barrier students faced (DoE, 2013; 2014). The fact that many learners left the answers spaces blank could indicate that they had no concept image of stretch at all or were completely ignorant of the study area (Tall, 1988).

6.2 RESPONDING TO THE SECOND RESEARCH SUB-QUESTION

The second research sub-question addresses the descriptive part of the main research question:

- *Are the learners' concept images and mathematical reasons coherent and representative of formal definitions?*

The term coherent is defined, in section 1.9, as something being organized and having no internal contradictions. According to Viholainen (2008), highly coherent concept images are clear, well-connected, are correct representations and mental images about the concept without internal contradictions, and excluding conceptions that contradict formal mathematical axioms. On the other hand concept images might not be coherent in learners' minds or might have internal conflicts (Viholainen, 2008) but, at the same time, they might contain traces of appropriate concept definitions (Tall, 2005).

The following questions, which were designed to determine the coherence of learners' concept images, formed part of the sub-question:

- To what extent are learners' concept images competent and representative of the formal concept definitions?

- How are learners' concept images related to the formal definitions of the three concepts that form the basis of this study? (Are there contradictions or not?)
- Does learners' reasoning about concept images relate to the formal concept definitions?

To answer these questions, the results were assessed in terms of the success rates of the following competence variables:

- ability to define the concepts,
- ability to use and interpret visuals (graphs) related to the concepts,
- ability to interpret and formulate symbolical abstractions of the concepts,
- ability to identify the concepts when they manifest explicitly or implicitly,
- ability to argue or reason formally or informally, or both,
- ability to demonstrate problem solving skills,
- having clear conceptions of the transformations.

These criteria were formulated to assist in analysing the ways in which learners' concept images are organized and linked to formal definitions.

6.2.1 Coherence of learners' concept images

The verbal images of the concepts of reflection, translation and stretch were discussed during the presentation of data from Question 1 (see section 5.1.1). Questions 2 to 9 were designed to reinforce results obtained from Question 1 and discover how coherent and consistent the concept images were by asking learners to represent and interpret the concepts graphically, and formulate abstract symbolical images or formulae from them. In this way, learners were to show that they were able to identify the concepts whether they existed explicitly or implicitly. Questions 2 to 9 would reveal how successful learners' conceptions of the concepts were or if they were subject to misconceptions.

Table 6.1, below, shows the success rates and the competences regarding learners' ability to define the concepts of reflection, translation and stretch and their ability to interpret concepts and use visuals (graphs) to demonstrate their understanding.

Table 6.1: Success rates for learners’ ability to define and represent reflection, translation and stretch

Item	Competence Variables or Skills	Success Rates	Mean
1a	Defining reflection	89%	
1b	Defining translation	85%	
1c	Defining stretch	78%	84%
2a	Illustrating reflection graphically	68%	
2b	Illustrating translation graphically	66%	
2c	Illustrating stretch graphically	11.5%	48%

From this table, it can be seen that the success rates for defining reflection, translation and stretch were 89%, 85% and 78% respectively, that is, the results fall within the 75th and 90th percentiles, averaging in the 84th percentile. This demonstrates that learners were fully aware of the three concepts. Some learners’ concept definition statements were coherent, even if they used their own words rather than the formal definitions of transformations of functions, or a combination of both. Many learners were handicapped by a lack of English language proficiency, as it is not their first language, and this can lead to frustration and despondency relating to their inability to communicate the ideas they have in their minds, precisely. As a result, contradictions, lack of precision and various misconceptions were evident. Most learners were found to be somewhere between the extremes of complete coherence and complete misconception on this continuum.

Table 6.1 also shows that the learners’ success rates for the ability to interpret concepts and use visuals (graphs) and to demonstrate understanding of the concepts as 68%, 66%

and 11.5% for reflection, translation and stretch respectively. The calculated average for the three concepts is 48.5%. The interpretation and use of stretch was evidently more challenging than of the other two concepts. The percentages shown for reflection and translation are above 50% but the percentage for stretch is very low. The responses that the learners produced in the test were most likely based on the direct recall (remembering) of what was taught in class, but while learners' were fairly successful in defining concepts, they were less successful in illustrating the specific images asked for in the test questions. However, some learners were able to use their intuitions (reflective thoughts) in their interpretations and visuals.

Table 6.2: Success rates for learners' ability to interpret symbolical images of the concepts reflection, translation and stretch.

Item	Competence Variables or Skills	Success Rates	Mean
3a	Interpret symbolical abstraction of a reflection in the y -axis	85%	
4a	Interpret symbolical abstraction of a reflection in the x -axis	60%	
3b	Interpret symbolical abstraction of a 2-way translation	56%	
4b	Interpret symbolical abstraction of a 2-way translation	49%	
3c	Interpret symbolical abstraction of a horizontal compression	11%	
4c	Interpret symbolical abstraction of a horizontal stretch	39%	50%
5	Interpret symbolical abstraction of a vertical stretch	32%	
6	Interpret symbolical abstraction of a horizontal compression	18%	25%

From Table 6.2, it is evident that the success rate for interpreting reflection from formulae was 85% for reflection about the y -axis $f(-x)$, and 60% for reflection about the x -axis $-f(x)$. From this we can conclude that, regardless of the functions used, reflection about the y -axis is better identified than reflection about the x -axis. Reflection about the y -axis

was tested using a hyperbola whilst reflection about the x-axis was tested using an exponential graph. The higher success rate with the hyperbola might be because reflecting it about either of the two axes gave the same result. The fact that both are reflections was clear to learners because of the minus sign, hence the success rate of more than 50%.

The success rates when comparing interpretations of two oblique translations, $f(x+3)-2$ and $f(x-2)+3$, were 56% and 49% respectively. The functions used were exponential and a parabola in that order. There was no significant difference between the two rates of success, but why was the success rate so low? It appeared that learners had problems when dealing with formulae for horizontal translations. They misinterpreted the '+' and '-' signs as indicators of positive and negative directions, respectively.

The success rates when comparing interpretations of two stretches, a horizontal compression $f(2x)$ and a horizontal extension $f(\frac{1}{2}x)$, were 11% and 39% respectively. The below 50% success rates indicate a lack of knowledge about the interpretation of horizontal stretch and confusion about the effects of coefficients. Many learners seemed to think that a coefficient greater than 1 means extension, and a coefficient less than 1, but greater than 0, means compression. In fact, it is the opposite. This misconception might have resulted from a faulty conception of the concept of stretch by the learner. This does not mean that a failure on the teacher's part is underestimated, for this 'rule' could easily be learned 'off by heart' if the teacher communicated it as such, and learners could use it as a tool to develop their concept image.

Questions 5 and 6 were designed to compare success rates of the interpretation of vertical and horizontal stretches when an integer appears in the equation, i.e. $y = 5 \sin x$ against $y = \cos(2x)$. The correct answer could be a statement that there is a vertical stretch of factor 5 and a horizontal compression of factor $\frac{1}{2}$. The success rates were 32% and 18% respectively. The less than 50% success rates indicate that stretch was a major challenge to learners. The horizontal stretch was more challenging than the vertical one because for the horizontal, the factor is the multiplicative inverse of the coefficient of the variable in

the formula, whilst for the vertical stretch the factor is exactly as it appears. What many learners had in common was that they combined their own conceptions of aspects of transformations with those found in the formal definitions, but it is evident from Table 6.2 that there was little coherence in learners' concepts, and inconsistencies were apparent in their understanding of transformations of functions.

Table 6.3: Success rates for learners' ability to formulate and identify symbolical images of the concepts reflection, translation and stretch.

Item	Competence Variables or Skills	Success Rates	Mean
7a	Formulate symbolical abstraction of a reflection in both axes using $y = \frac{2}{x}$	23%	22.2%
8a	Formulate symbolical abstraction of a 2-way translation	29%	
7b	Formulate symbolical abstraction of a reflection in both axes using $y = 2^x$	22%	
8b	Formulate symbolical abstraction of a reflection in both axes using $y = \frac{2}{x}$	23%	
7c	Formulate symbolical abstraction of a different 2-way translation	18%	
8c	Formulate symbolical abstraction of a vertical stretch	41%	
9b	Formulate symbolical abstraction of a horizontal compression	0%	
10b	Identify a reflection as it transforms a cubic function graph	37%	39.5%
9a	Identify a reflection as it transforms a parabola	37%	
10a	Identify a vertical translation as it transforms an exponential graph	52%	
9c	Identify vertical and horizontal translations of a parabola	40%	
9c	Identify a horizontal compression of factor 0.5 of a sine graph	40%	
10c	Identify a vertical stretch of factor 3 of a cosine graph	31%	
	Mean success rate for all three transformations		43%

Table 6.3, above, shows that the success rates for learners' ability to formulate symbolical abstractions were 22% for reflecting $y = 2^x$ about both x - and y -axes, 23% for reflecting $y = \frac{2}{x}$ about both x - and y -axes, 29% for translating $y = \sin x$ by 4 units to the right and 3 units upwards, 18% for translating $y = \cos x$ by 4 units to the left and 3 units downwards, 41% for stretching $y = \cos x$ by factor 2 vertically, and 0% for compressing $y = \sin x$ by factor $\frac{1}{2}$ horizontally. The first set of required formulae were $y = -2^x$ for reflection in the x -axis and $y = 2^{-x}$ (or $y = (\frac{1}{2})^x$) for reflection in the y -axis. The second set of required formulae were $y = \frac{2}{-x}$ or $y = -\frac{2}{x}$ or $y = \frac{-2}{x}$ for reflections about either of the axes.

The success rates were almost equal despite the differences in the functions tested. It could be deduced that writing the formulae for reflection was more challenging for learners than interpreting the formulae (compare 22% and 23% to 85% and 60%). The third set of formulae were $y = \sin(x - 4) + 3$ and $y = \cos(x + 4) - 3$. The success rates here were less than 30%, highlighting the difficulties that learners had in abstracting reflection into symbolic operations or objects. It is evident that writing formulae was consistently more challenging for learners than interpreting formulae (compare 29% and 18% to 56% and 49%). Similarly, horizontal translation along the x -direction was a greater challenge than vertical translation along the y -direction.

The last set of formulae were $y = 2 \cos x$ and $y = \sin 2x$. The success rates for these formulae reaffirmed the fact that writing formulae was more challenging for learners than interpreting formulae (compare 41% and 0% to 39% and 11%).

Horizontal compression was challenging both to interpret and to formulate. For the last set of formulae, a frequent misconception was changing the position of 2 for the image functions or replacing 2 by $\frac{1}{2}$. Many other misconceptions were evident from the incorrect use of brackets, swapping the directions of translation, or omitting necessary aspects of the answer.

Success rates for learners' ability to identify reflections about $y = x$ for graphs of two different functions were both 37%. That these two tasks had equal success rates was not surprising, but what was surprising is that they were below 50%. Learners' responses frequently lacked the complete description – they correctly identified reflection but without elaborating further. More than 50% of the learners left the answer spaces blank for this question and a considerable number had misconceptions such as misidentifying the transformation, reflecting about the incorrect axis (x -axis and y -axis), or putting down $(x; y) \rightarrow (-x; y)$, a translation instead of $(x; y) \rightarrow (y; x)$, a reflection. The rotation $(x; y) \rightarrow (-y; x)$ could also be considered correct, depending on the situation. Some learners gave reasons for failing to answer the question, such as stating that they did not understand the question (or the concept), they did not know how to interpret the graph in symbols, or they were confused.

Success rates for learners' ability to identify vertical and horizontal translations were 52% and 40%. Although these rates were similar, it became surprising that they were both so low for what were relatively clear and trivial transformations. Some learners demonstrated misconceptions when, despite correctly recognizing a transformation as a translation, they could not accurately say by how much the translation was effected. Other learners tried unsuccessfully to use symbols, as did, for example, the students who answered $(x; y) \rightarrow (x - 2)$ (which does not indicate the effect on y) or $(x; y) \rightarrow (x + 2; y - 2)$ (which is neither vertical nor horizontal, so is oblique) or gave the completely inappropriate and illogical answer $(-2; 2) (2; -2)$. Learners showed misconceptions about vertical and horizontal translation either about the description or about the formulae, as evidenced by the following answers: reflection and translation of 2 units; reflection about $y = 0$ (incorrect transformation); and translation of $y = (x + 2; y + 1)$ (effect on x omitted).

Success rates for learners' ability to identify stretches were 40% and 31%. In their identification of the stretches, learners used either the formula or the verbal description. Some descriptions matched the stretch but were not entirely accurate. Misconceptions in trying to identify the horizontal compression of factor $\frac{1}{2}$ were: a horizontal expansion of the graph by factor 2; a translation; reflection along the x -axis; vertical stretch; reflection

and translation 2 units down; vertical reduction; 2 added (where should it be added?); $f(x) = \frac{1}{2}\sin x$; $y = -2\sin x$; and $y = 2\sin x$. For the vertical stretch of factor 3, some learners' descriptions matched the transformation but they could not come up with the correct formula. Other learners had the formula correct but gave an incorrect description. An example of a misconception that probably stemmed from inaccurate use of language was a learner saying that the amplitude increased 'by' 3 units instead of 'to' 3 units' (because the amplitude increased by 2). Examples of misconceptions evident in the formulae given by learners were $y = 2\cos x$ instead of $y = 3\cos x$; $y = \cos(x + 3)$; $y = \cos x + 3$; $y = \cos x + 2$ (2 units upwards); $2y = \cos x$; $f(x) = \cos 2x$; and $f(x) = y + 2$. Examples of misconceptions evident in the descriptions given by learners included *stretched horizontally*; *stretched by a factor 2* (no direction given); *moves two units up*; *the image of $f(x)$ is reduced towards the y-axis and enlarged 2 units upwards/horizontally*; *it's a reflection*; *the image of $f(x) = \cos x$ has been compressed by 2 units vertically and 2 units horizontally*; *the image has been compressed to $f(x) = 1\cos x$* ; *it is enlargement*; *the reflection is increased by a factor given*; and *the graph has been compressed by 2 units*.

The success rates averaged out at 43%, which shows that learners do not have adequate mastery of concepts and hence they perform poorly in tests (DoE, 2011). Competency skills training should be a daily activity when teaching learners to reason logically and think creatively (DoE, 2002).

6.2.2 The relationship of learners' concept images to formal definitions

Most learners used their expressions consistently as they reasoned about transformations of functions. Some learners combined everyday words with formal definitions of transformations of functions to good effect. The ability to do this successfully varied

according to the learner's language proficiency, but the more proficient a learner was in English, the closer his or her definition was to the formal definition, and vice versa.

CHAPTER SEVEN

Summary of the Study, Conclusions and Recommendations

7.1 Summary of the Study

The summary of the study has the following sections: purpose of the study, methodology used in the study, and findings of the study.

7.1.1 Purpose of the study

The purpose of this study was to give an empirical description of how a sample of Grade 11 learners grapple with the concepts involved in transformations of functions. In particular, the study unveiled concept images and misconceptions that learners have about these concepts and the reasoning behind them. An understanding of the knowledge structures created in learners' minds and reflected by them in this study will assist teachers in thinking proactively about corrective intervention strategies to promote the formation of appropriate concept images and prevent learners forming misconceptions. The aims of the study were therefore to investigate the concept images that learners build as they learn, interpret and represent concepts of reflection, translation and stretch of functions. The researcher investigated how coherent learners' concept images were and how learners used their concept images to explain, justify, argue and reason in the processes of translating, reflecting and stretching functions. The study also assessed the relationship between learners' reasoning using their own concept images and the concepts provided by the formal definitions taught in class, and suggests ways of improving classroom activities of learning such concepts. This research study is primarily intended to benefit mathematics educators in high schools, but mathematics student teachers,

mathematics teacher educators and, to some extent, educationists and mathematicians, would also benefit from the findings presented here. The study suggests ways to understand learners' thinking and reasoning backed by events in the classroom (Kilpatrick, 1993). The results of this study are expected to contribute to teaching and learning transformations of functions by highlighting learners' interpretations and misconceptions of translation, reflection and stretch of functions. Like all good educational research, it has both a practical and a theoretical relevance, in that it is directed at having a positive influence on the practice of teaching concepts of translation, reflection and stretch of functions by broadening or deepening the understanding how learners learn (Sierpinska, 1993). It contributes to relational mathematics learning and realistic mathematics education. The results reported here provide insights for mathematics teacher educators. Such insights should be useful in designing relevant programmes for their students (student teachers). Following this, it is envisaged that in future the student teachers would be able to design appropriate and effective lessons about transformation of functions. The results contribute to knowledge construction in respect of concept images on translation, reflection and stretching of functions.

7.1.2 Methodology used in the thesis

The study used an explanatory mixed method design (Creswell & Plano Clark, 2011) to collect both quantitative and qualitative data. The quantitative and qualitative data were both drawn from the same problem situation and the same respondents so as to achieve a clearer understanding of the problem. The explanatory mixed method design was adjusted and modified slightly to facilitate collection of data using the test and interview methods and enhance the subsequent quantitative and qualitative analysis. A diagnostic test was the main instrument used for collecting both quantitative and qualitative data, and its results determined the selection of learners for interviews. Ninety-six learners wrote the diagnostic test and the 14 learners who scored more than 30% in the test were interviewed. Data collection was done between April 2012 and October 2013 during and after school hours, after the participating learners had been taught transformation geometry, functional graphs and the effects of parameters on functional graphs. It was timed so as

not to jeopardize learners' commitment to their schoolwork or put additional pressure on them close to exams.

No artificial sampling strategy was used to select the 96 Grade 11 mathematics learners who participated in the study. They were those who had returned the completed consent forms signed by their parents, and were present at school on the day the diagnostic test took place. The three sample schools were selected, for convenience, in the Johannesburg East District (D9), Gauteng Province, in the Republic of South Africa. The question paper for the written diagnostic test is presented in Appendix 1 below. This diagnostic test was delivered to learners by the researcher during a mathematics 'double lesson' or after school hours, because the test's one hour time limit exceeded the time span of a 'normal' daily lesson. No additional tools besides pencil or pen and the test form were allowed during the test. Answers were done in the space provided on the test paper.

Fourteen learners were interviewed based on their having scored above 30% in the diagnostic test and being present at school on the day interviews were conducted. The interview form (see Appendix 2) served as a guide as individual interviews varied depending on the interviewees' views and progress, hence the design was semi-closed or semi-structured. This follow-up clinical²⁶ interview was intended to provide an opportunity to scrutinize learners' conceptions, interpretations, concept images or mathematical thinking in greater depth and it allowed for clarification of gaps discovered in the task responses. The interviewees were asked to explain verbally, and to illustrate graphically, the concepts of translation, reflection and stretch of functions. A voice recorder was used during interviews and the verbatim transcript of these appears in Appendix 4 of this report. Each interview is between 15 and 20 minutes in duration, depending on the precision of the learners' explanations and the levels of competence.

After being subjected to psychometric (section 4.5.1), content and construct validity checks (section 4.5.2), the research instruments were used in a small-scale pilot study (section 4.6.1). The tasks in the test were subjected to split-half correlation, test-retest, variance

²⁶ Meant to diagnose challenges and also discover other forms of conceptions

components analysis and Spearman–Brown prophecy formula. The internal consistency was analysed using Cronbach’s Coefficient Alpha (α). The results using the three coefficients suggested good internal consistency. The pilot study results were subjected to the same sort of analysis done subsequently on the main study, to enhance the validity of the instruments. Adjustments of underestimations or overestimations to best suit the study were carried out.

Qualitative analysis of the main study data was done through detailed descriptions and evidence (words, graphs and formulae) of the learners’ reflections on how they understand translation, reflection and stretch concepts and their manifestations in transformations of functions. The frequencies of similar test responses and the verbatim transcripts of the voice interviews informed the qualitative descriptions. Thereafter, interpretations based on the expressions and evidences for concept images and mathematical reasoning were made. Responses were categorized and similar categories of responses were coded on the answer papers to produce frequency counts and tables for quantitative analysis. The analysis of the resultant data was done using SPSS.

7.1.3 Findings of the study

A summary of findings, and corresponding recommendations directed mainly at mathematics educators at high school level, are given below:

FINDING 1: Some learners were found to have the following relevant concept images about reflection: a flip; an inverted image; a symmetrical image; mirror image; flipped formation; flips; image mirrored/flipped across the x -axis or y -axis or $y = \pm x$; a repeated object across the y -axis or x -axis. These concept images loosely correspond with the formal definition of a reflection. Learners gave other images of the concept of reflection that relate to its properties but are not entirely accurate: line of symmetry; not changing the size or shape but changing the coordinates; an exact copy of another object; graph for object and image; similar object and images; a glance of the same picture.

FINDING 2: A considerable number of learners described translation as a displacement; a movement; a slide; a change of position or a shifting; all for a specific distance, followed

by a direction upwards; downwards; to the left; to the right or in a straight line/specific direction. These concept images loosely correspond to the formal definition of a translation. Other related concept images given were: the movement of all points of a graph in a particular factor either up, down, left or right; movement of points the same distance in the same direction; movement of certain units up or down, left or right; shift of shape or point in the same distance in a common direction; movement of a graph upwards, downwards, or to the right or to the left; movement of a shape from a set of coordinates to another; a graph moving vertically or horizontally; shift of image from its original point/place/coordinates to another; movement of a diagram across the y -axis and the x -axis changing the coordinates but not the image; moving a graph to a different position from where it was.

FINDING 3: Concept images about stretch given by learners were: an expansion or contraction of an object; an extension or compression of an object; an increase or decrease in size of an object; points moving apart or closer to one another in a plane; making something longer or shorter; widening or narrowing something; lengthening or shortening something; making something fat or slim; pulling or squeezing something on both ends; enlarging or shrinking something; spacing points from each other or bringing them closer; all followed by a factor; or specific scale factor; or certain scale factor. None of the above concept images included mention of an invariant line. Learners expressed no clear difference in understanding of stretch and enlargement.

FINDING 4: Of the 96 learners sampled, 85 learners (89%) had correct ideas of what is meant by reflection, 76 (79%) had correct ideas of what is meant by translation and 75 (78%) had correct ideas of what is meant by stretch. Their verbal definitions made sense, but their lack of proficiency in English and less-than-effective communication skills prevented the full expression of their ideas. Educators should be aware, and appreciate, that learners frequently have their own ways of defining concepts that are similar to, but not exactly the same as, formal definitions. However, many learners had difficulty in defining these concepts, especially stretch, which they confused with enlargement. Learners' difficulty with language meant that their definitions suffered from a lack of precision and

correctness. Many learners had a very loose interpretation of the concepts or revealed misconceptions.

FINDING 5: Most learners could define the transformation concepts better than they could represent them graphically or symbolically (compare the answers to Question 1 in section 5.1.1 and 2 in section 5.1.2).

FINDING 6: Reflection and translation were generally easier for learners to recognize, illustrate graphically (or visually) or represent symbolically than stretch was. Reflection was the easiest but not significantly easier than translation. The challenges of working with stretch emanated from the difficulty learners had defining stretch.

FINDING 7: Learners recognised reflection about the y -axis from the formula better than reflection about the x -axis.

FINDING 8: Learners recognised vertical translation from the formula better than horizontal translation. Most learners were prevented from correctly identifying horizontal translation by a misunderstanding of the operative signs (plus '+' and minus '-'). They thought that '+' means slide to the right and '-' means slide to the left, whereas the opposite is true.

FINDING 9: Learners recognised vertical stretch from the formula better than horizontal stretch. Most learners misunderstood the effect the coefficient of the variable x had when they attempted to identify horizontal stretch. They thought that a positive coefficient greater than 1 meant pull or extend the function, and that a positive coefficient less than 1 meant compress the function, whereas the opposite is true.

FINDING 10: Learners recognised and described reflections about the coordinate axes better from graphical representations than reflections about oblique mirror lines like $y = \pm x$.

FINDING 11: Two-way translations were much more difficult for learners to identify, illustrate graphically and illustrate symbolically, than one-way translations. The horizontal components of translations were the main stumbling block. In addition, two-way

translations were easier for learners to define and identify than to illustrate graphically or symbolically.

FINDING 12: Learners found it easier to reflect a hyperbola than an exponential graph (Question 3a versus 4b), and easier to translate an exponential graph than a parabola (Questions 3b versus 4b and 9a versus 10a). Stretching a cubic graph and stretching a trigonometric graph had no significant difference in difficulty, but both were badly done as were reflecting a cubic graph and reflecting a parabola.

FINDING 13: Learners in the research sample had not adequately mastered the concepts of transformation, and so they displayed many misconceptions about the concepts and could not perform the assessment tasks very well.

7.2 Conclusion

The study of concept images allowed for an analysis of the cognitive processes influencing the learning of transformations. It revealed the connection between learner concepts images about definitions of the concepts of translation, reflection and stretch, and their graphical and symbolical representations. Learners who had difficulty answering the diagnostic test successfully showed a lack of background and topical knowledge or were handicapped by a lack of proficiency in English, the language of instruction, which meant they were unable to understand fully the questions being asked, and were unable to express their understanding of the concepts precisely and accurately

It was found that learners have their own ways of defining concepts, their concept images, that are not necessarily congruent in meaning with the formal definitions, but that are none-the-less related to them. So it should be appreciated that learners are the best definers of what they learn – teachers can only be conceptual catalysts. Some of the learners' definitions were accurate, some were incomplete and others exhibited misconceptions. This was expected and is part of the natural landscape of the learning process. There were several inconsistencies in the students' constructed definitions, and their competency in using representations of reflection, translation and stretch as another way of showing knowledge about the concepts, was uneven. Teachers need to follow the recommendations

given in this report and other related studies to improve their practices and their learners' understanding of the curriculum content.

7.3 Recommendations

Recommendation 1: Teachers need to be aware of the diversity of ways in which learners think of the action of reflecting, translating and stretching, and the terms they use to describe them, how they compare the original objects to images after transformations, and the combined plane occupied by the original object and its image. Teachers need to make accurate assessments of the successes and shortcomings learners display in the quest to master mathematical concepts. This assessment needs to take the limitations of language proficiency into account, as, in grappling with a language of expression that is not the learners' first language, learners frequently fail to express themselves accurately and precisely.

Recommendation 2: Teachers need to take note not only of the complete definitions given by learners, but also the incomplete as these can be used as springboards for correcting misconceptions or redefining the concepts. Discussing possible misconceptions with the class can also be helpful, but this should be done carefully lest some learners conceptualise these instead of the correct conceptions. It would then take much effort to correct the resulting misconceptions.

Recommendation 3: Teachers need to draw a clear line between the properties of stretch and enlargement, and emphasize the need to include the invariant line in the definition.

Recommendation 4: Teachers should exercise patience by encouraging learners to define the concepts for themselves first, before they, the teachers, redefine and correct misconceptions. This is in keeping with the theoretical premise of cognitive constructivism. Teachers also need to help learners understand the formal language of mathematics and encourage them to use it when expressing definitions of terms so that the correct language forms part of the learners' concept images. Teachers should be aware that misconceptions form during the process of teaching and learning about concepts, mostly during

introductory lessons. Giving examples of other learners' misconceptions could assist learners in avoiding and overcoming the possibility of misconceptions when they first come across new concepts. Some misconceptions arise from partial understanding of the concepts due to learners' short concentration spans and teachers should structure lessons in such a way to prevent this being a factor, as far as possible. Teachers need to be aware that misconceptions persist if not corrected and this affects the future development of secondary concepts. Despite the possibility of developing misconceptions and inconsistencies, learners may be the best definers of concepts for themselves. Sometimes words fall short of expressing exactly what the learners understand about concepts, but within their means, learners find words or terms to the best of their ability, as was observed in the interview process. It is a difficult task to interpret what learners mean when they define concepts for themselves, and it often needs more than written expressions for a teacher to understand what learners have in their minds. Having dialogues or think-aloud protocols with learners would enable teachers to assist learners in defining and explaining mathematical concepts accurately and precisely, and if necessary, teachers could have one-on-one discussions with individual learners about the concept he/she is attempting to define, so that remedial sessions could be arranged, if necessary.

Recommendation 5: Teachers should use more than one method when teaching transformations of functions. Practically-oriented and process-oriented instructions, with practical examples, would improve the images of the concepts that learners develop. During and after learning new concepts and skills, learners need to practice regularly to reinforce their newly acquired knowledge. Regular practice tests and constructive feedback from the teacher will benefit students and enable them to form appropriate concept images.

Recommendation 6: To remove confusion around the effect of “-” signs, more practice is needed. Spiral testing of this knowledge could be done to remind learners of that property.

Recommendations 7: Educators need to connect their teaching to the world around us, and show learners how mathematics relates to other subjects and to their lives beyond the school gates, so that they are aware of, and appreciate, the broader role it plays. Educators should help learners move from concrete to abstract thinking and develop essential problem solving skills through application.

7.4 Possible Further Study

Teaching / learning process is not static. Its dynamics is mostly dictated by changes in technology. There is a need to design modes of using available and incoming technology to influence effective teaching and learning of topics like transformations of functions and other related topics and to investigate how best teacher should design mathematics lessons in order to attract the attention of today's young learners. In this age of technology, good teaching should incorporate all possible forms of technology. Gone are the days when we prohibited learners from using cell phones, smartphones, i-phones, i-pods, tablets and other technological devices during tuition. Rather, teachers should investigate how these devices can be used for teaching and learning, and utilize them fully to their own and the learners' advantage in learning concepts and skills.

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APPENDICES

APPENDIX A: THE DIAGNOSTIC TEST: TRANSFORMATIONS OF FUNCTIONS

Your name _____ (first name only). Sex _____ Age _____

TASK: Transformations of functions

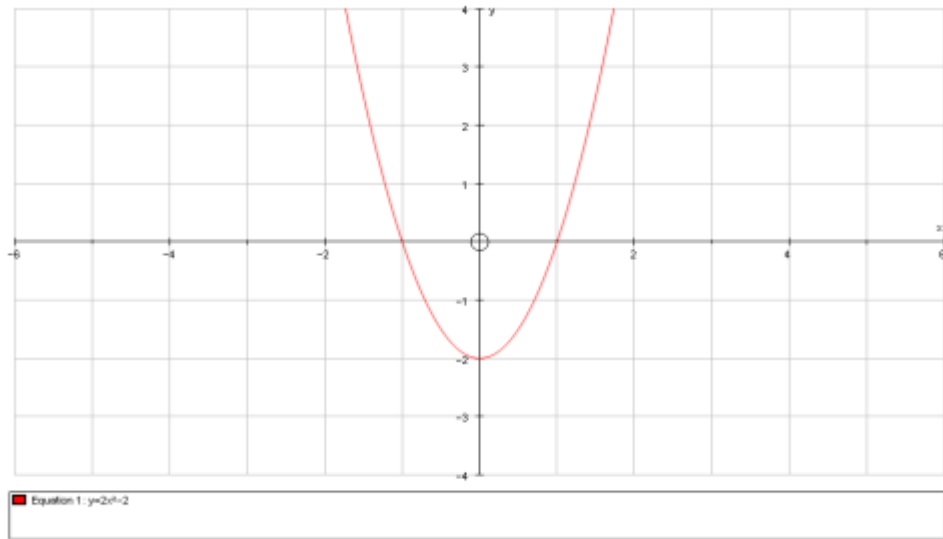
Information for the learner

- **Your name only saves for a possible follow-up interview; otherwise confidentiality is absolutely adhered to.**
- **Answer all the task items. Use the spaces below the question.**
- **If you cannot answer a question adequately, briefly describe your problem for inventing a solution [e.g. *not understanding what the question needs, or you cannot interpret the concept referred to in the question? Specify any other problem.*]**

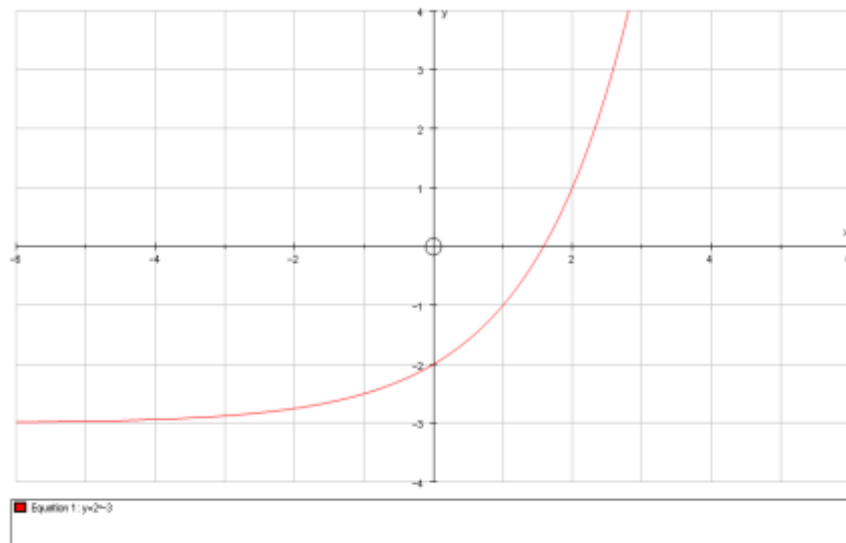
1. (a) Define, in your own words, a reflection.

 - (b) Define, in your own words, a translation.

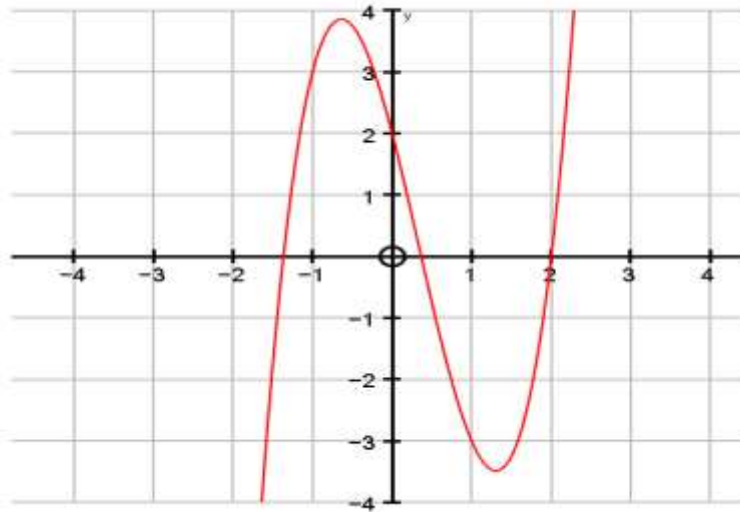
 - (c) Define, in your own words, a stretch
2. (a) for the following graph, illustrate the image after a reflection in the x -axis.



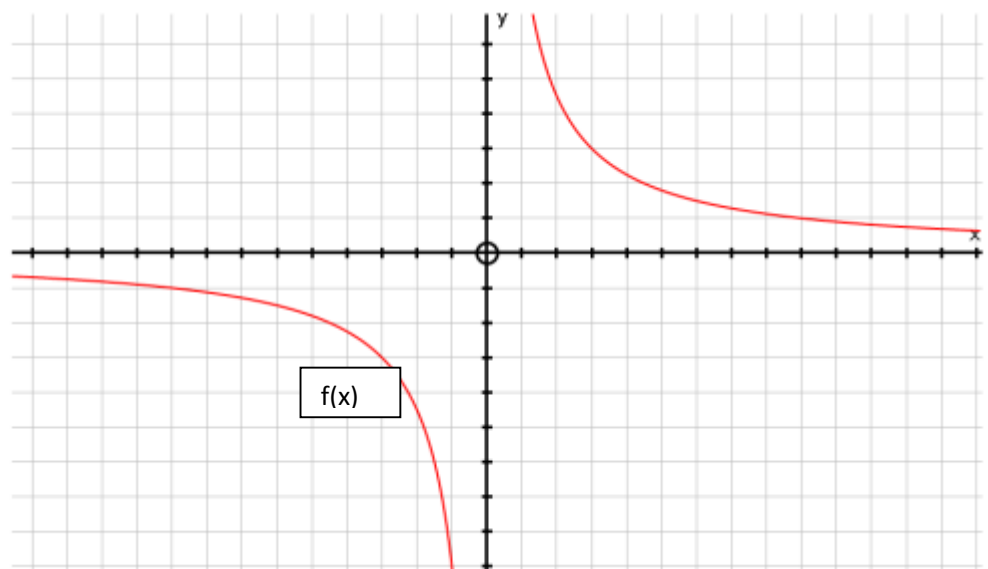
(b) For the following graph, illustrate the image after a translation of **2** units to the right and **3** units upwards.



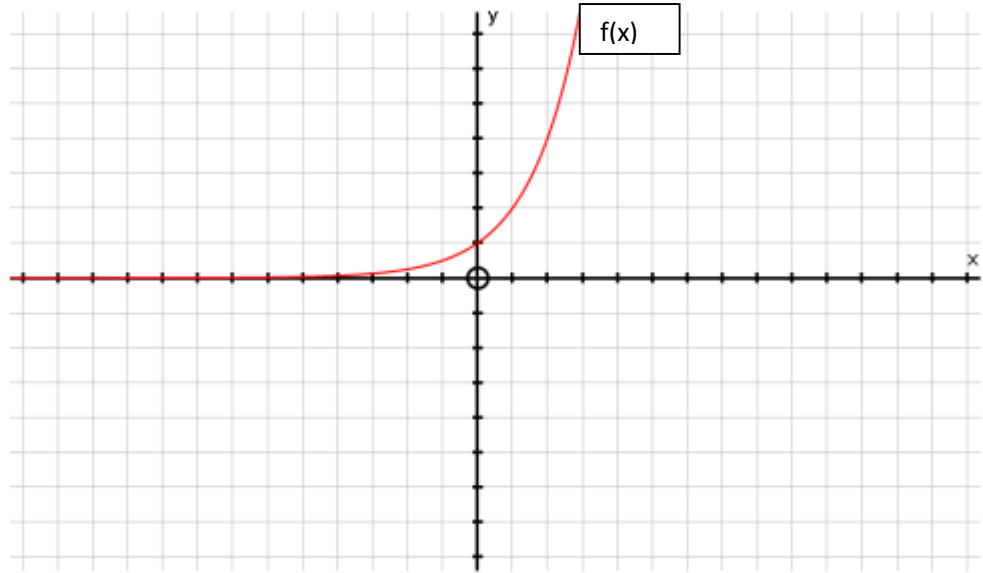
- (c) For the following graph, illustrate the image after a horizontal stretch of factor $\frac{1}{2}$, y-axis invariant.



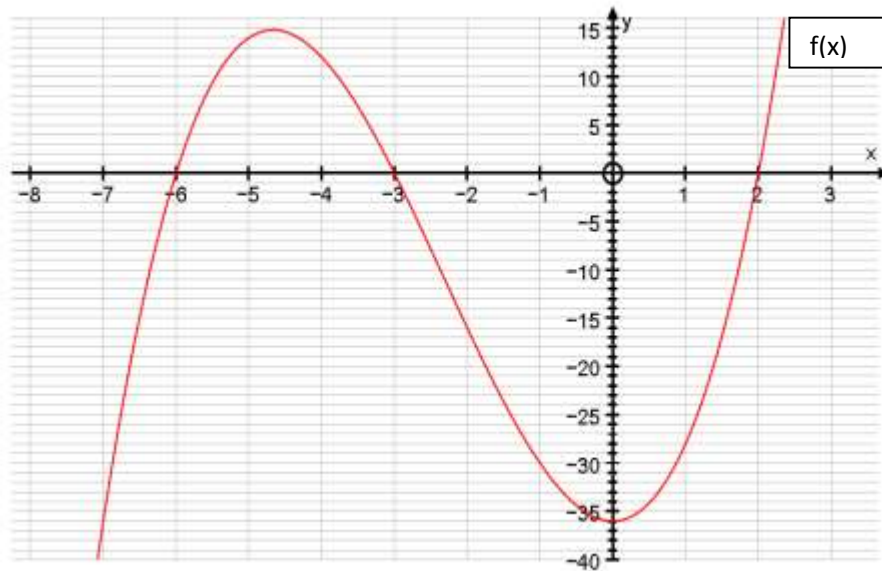
3. (A) Illustrate, on the diagram, the image of $f(x)$ when it transforms to $f(-x)$. Describe fully the transformation involved.



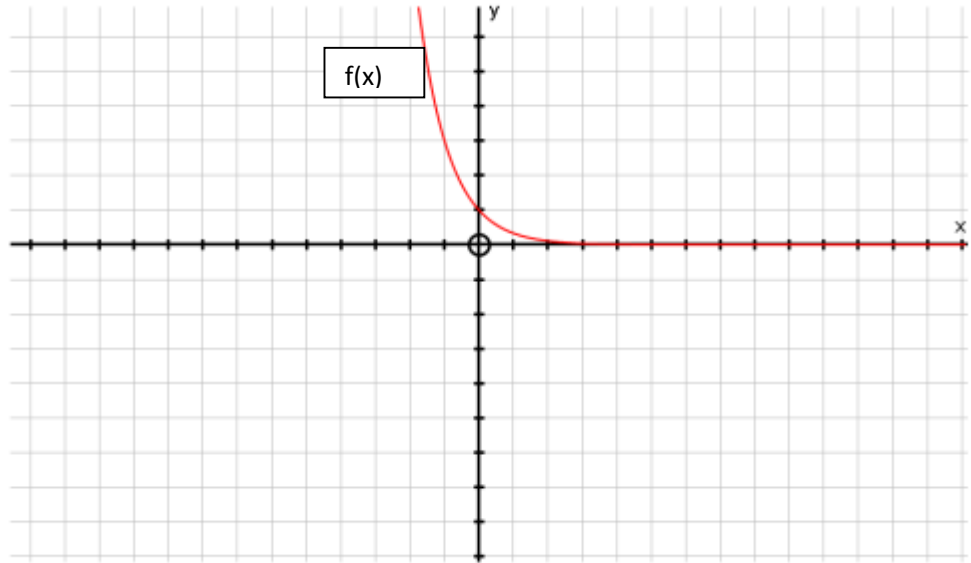
- (b) Illustrate, on the diagram, the image of $f(x)$ when it transforms to $f(x+3) - 2$. Describe fully the transformation involved.



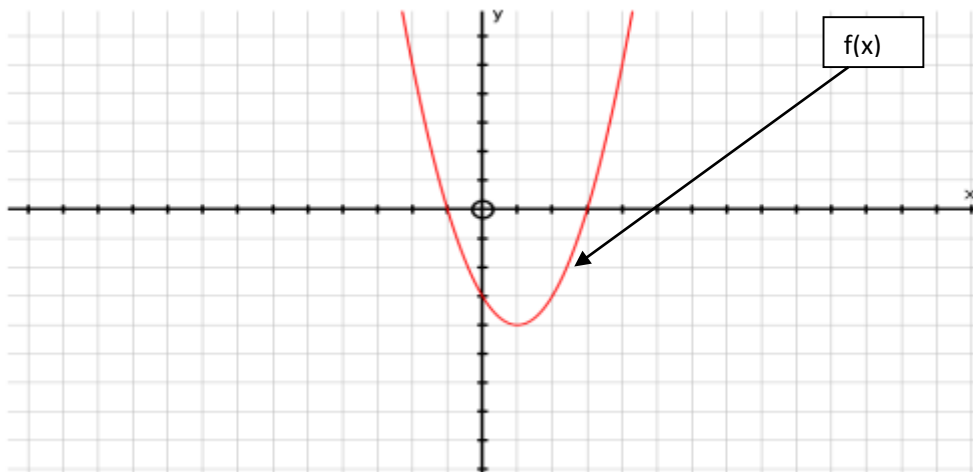
- (c) Illustrate, on the diagram, the image of $f(x)$ when it transforms to $f(2x)$. Describe fully the transformation involved.



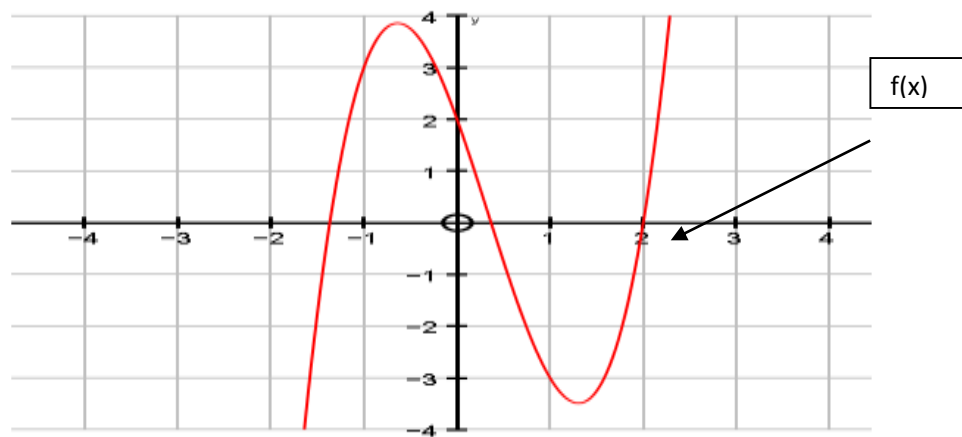
- 4 (a) Illustrate, on the diagram, the image of $f(x)$ when it transforms to $-f(x)$. Describe fully the transformation involved.



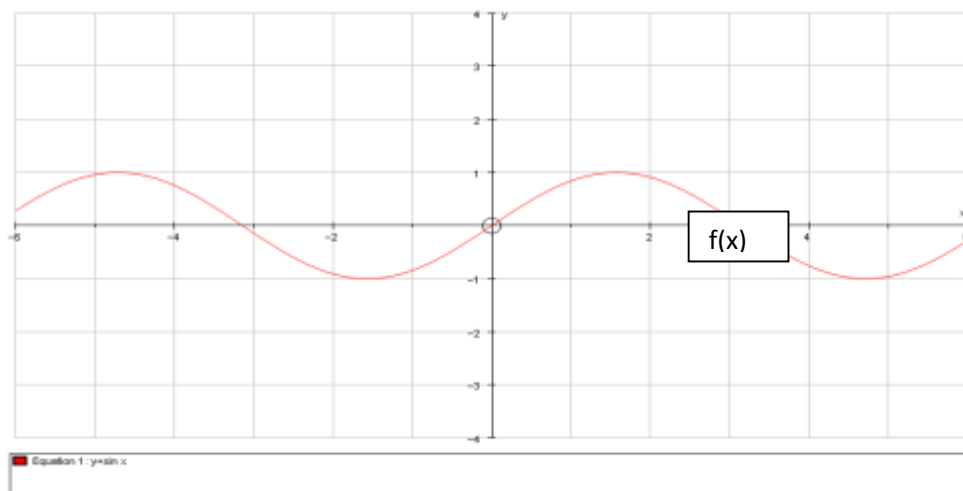
- (b) Illustrate, on the diagram, the image of $f(x)$ when it transforms to $f(x-2)+3$. Describe fully the transformation involved.



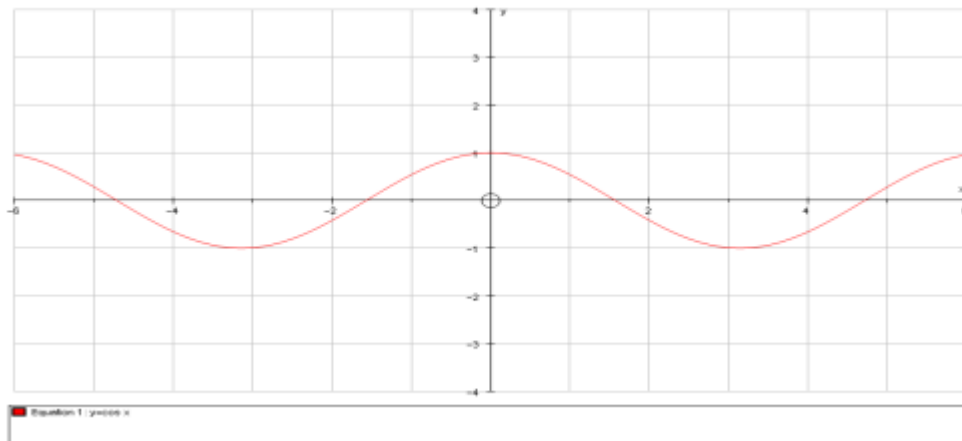
- (c) Illustrate, on the diagram, the image of $f(x)$ when it transforms to $f(\frac{1}{2}x)$. Describe fully the transformation involved.



5. Describe fully the transformation performed if $f(x) = \sin x$ below transforms to $5f(x) = 5 \sin x$. Sketch below to illustrate.



6. Describe fully the transformation performed if $f(x) = \cos x$ below transforms to $f(2x) = \cos 2x$. Sketch below to illustrate.



7.

- (a) Write down the formula for a translation of $y = \sin x$ **4**-units to the right and **3**-units upward.

- (b) Write down the two separate formulae for reflections of $y = 2^x$ in the x -axis and also in the y -axis. Compare and contrast the results.

(c) Write down the formula for a vertical stretch of $y = \cos x$ by factor 2.

8.

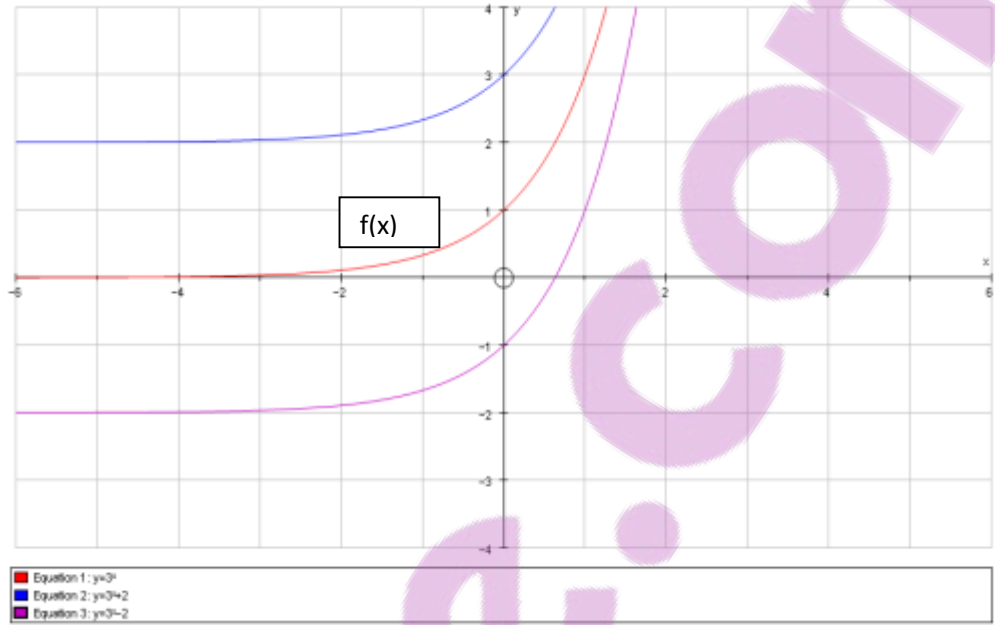
(a) Write down the formula for a translation of $y = \cos x$ **4**-units to the left and **3**- units downward.

(b) Write down the two separate formulae for a reflections of $y = \frac{2}{x}$ in the y -axis and also in the x -axis. Compare and contrast the results.

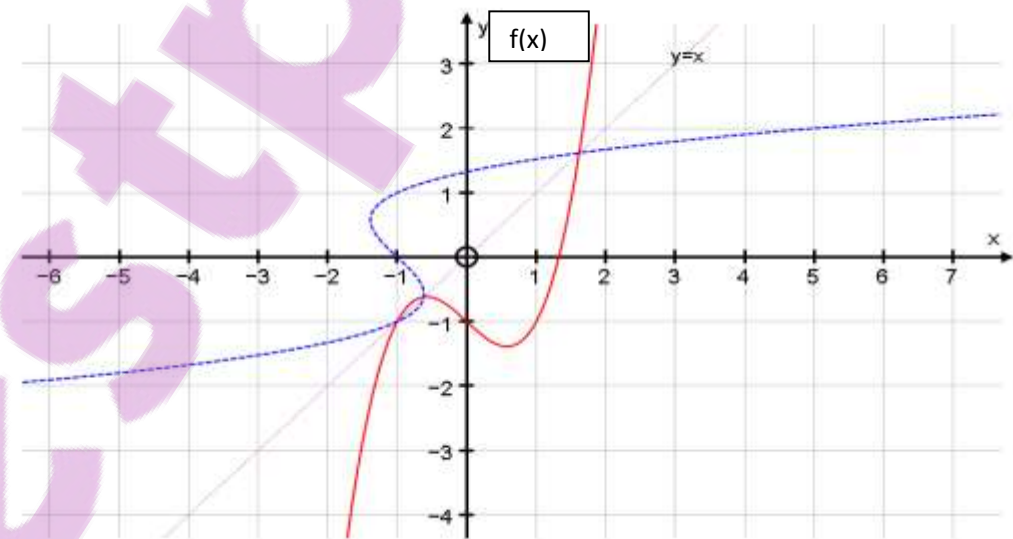
(c) Write down the formula for a horizontal stretch of $y = \sin x$ by factor $\frac{1}{2}$.

9. Identify the transformations that mapped $y = f(x)$ to the illustrated images. Describe the transformations fully by words and label their formulae.

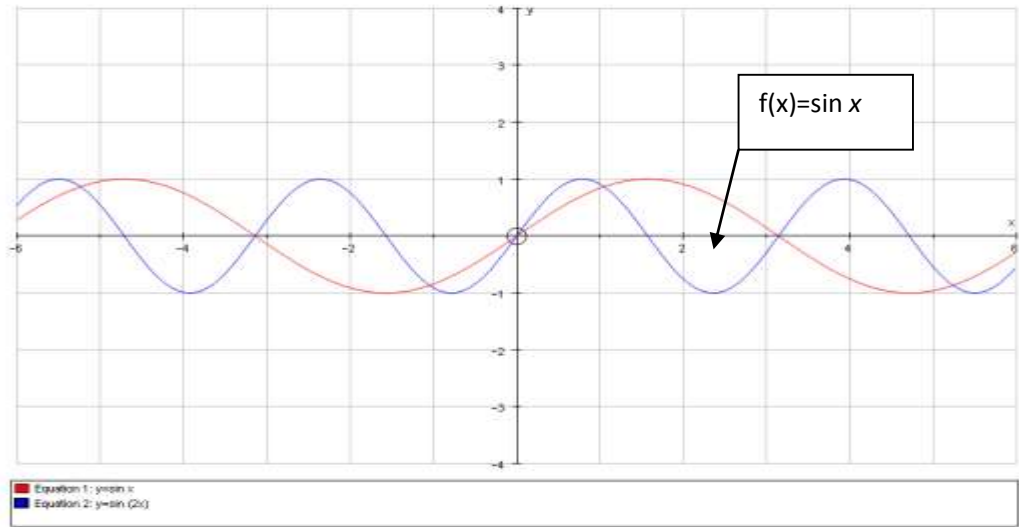
(a)



(b)

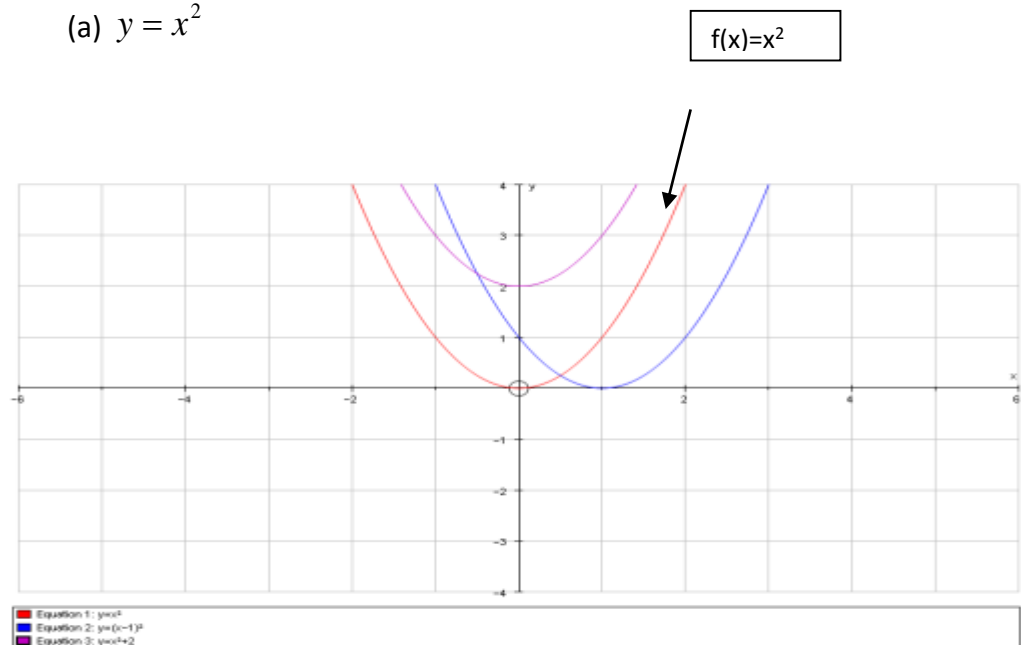


(c) $y = \sin x$

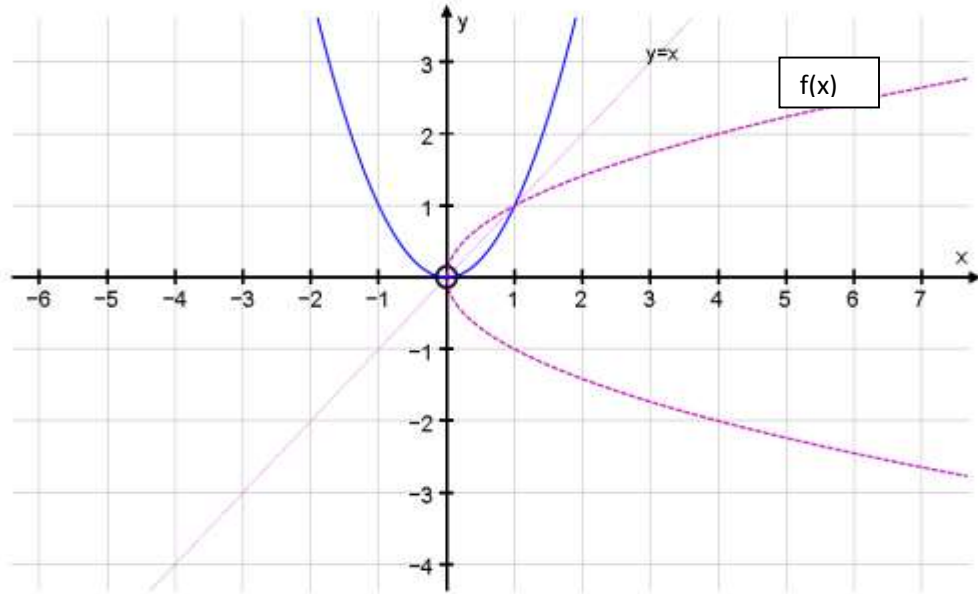


10. Identify the transformations that mapped $y = f(x)$ to the illustrated image. Describe the transformations fully by words and write down their formulae.

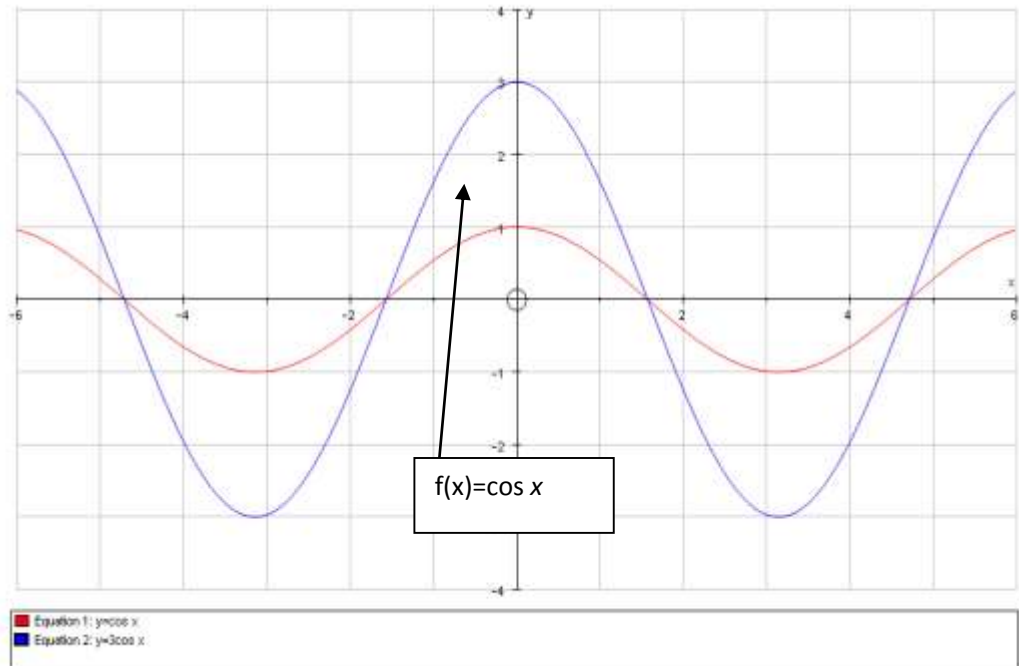
(a) $y = x^2$



(b)



(d) $y = \cos x$



APPENDIX B: INTERVIEW GUIDE

Introduction

This interview follows after the written tasks are marked. It seeks further clarification, mathematical thinking and reasoning behind the answers given by the learners in the task. This document is only a guideline, so the interview questions may be modified in the process in response to the learners' answers. It is open ended.

The atmosphere in the interview

After the motivation of the interview, the rapport that would be established through general talk about the activities done and the topic of transformation of functions, the following questions would guide generally on the interview.

1. Explain, with illustrations, the concepts of translation, reflection or stretch in, relation to function and their graphs.
2. Given a function $y = f(x)$, describe fully the transformation represented by the formulae:

$$y = f(-x);$$

$$y = -f(x);$$

$$y = 3f(x);$$

$$y = f(2x);$$

$$y = \frac{1}{2}f(x);$$

$$y = f\left(\frac{1}{2}x\right);$$

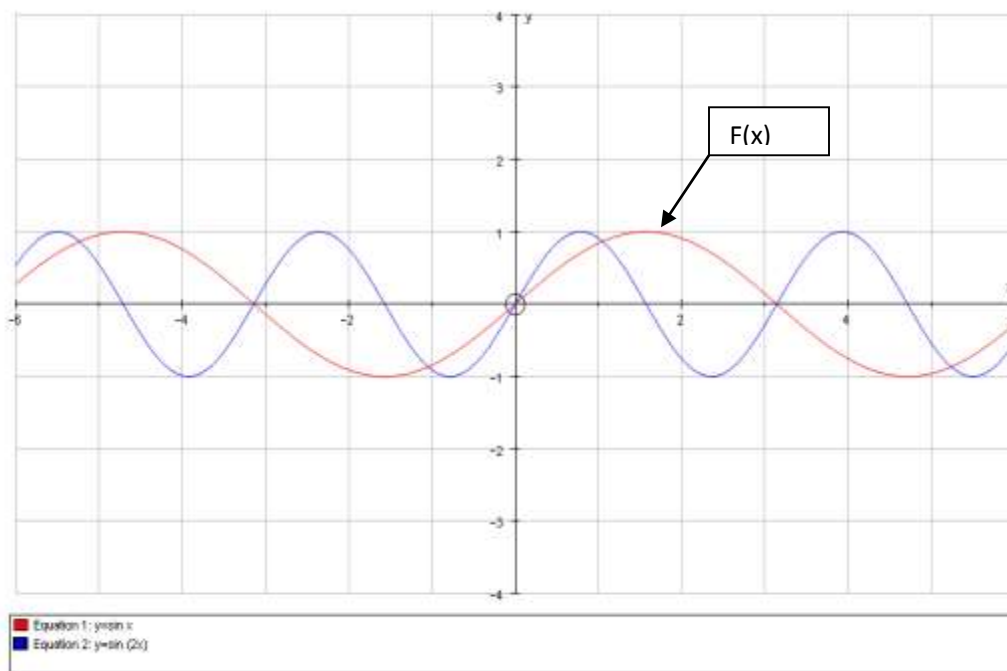
$$y = f(x) - 3;$$

$$y = f(x - 2) + 3;$$

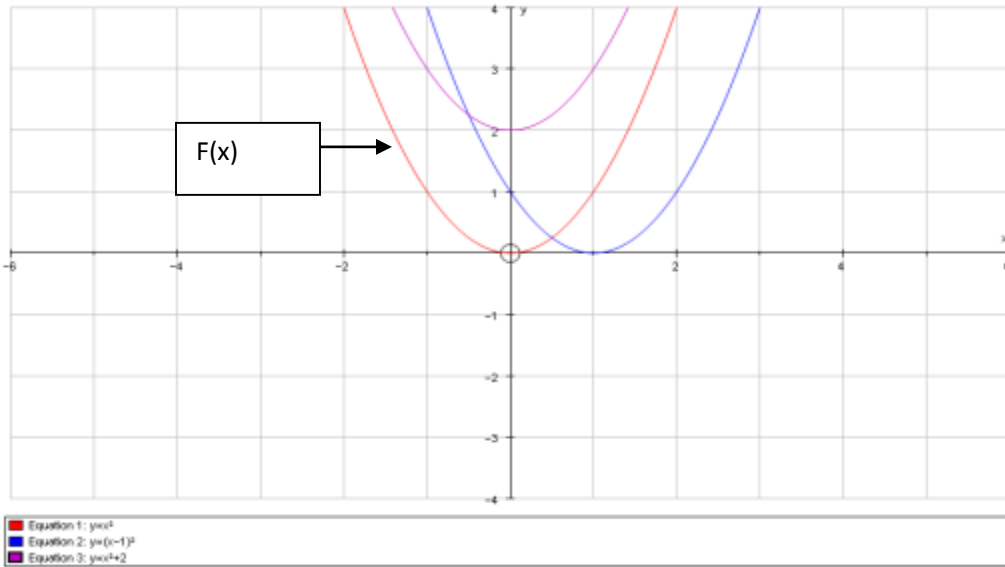
$$y = f(x + 3) - 2;$$

$$y = -3f(2x) + 4.$$

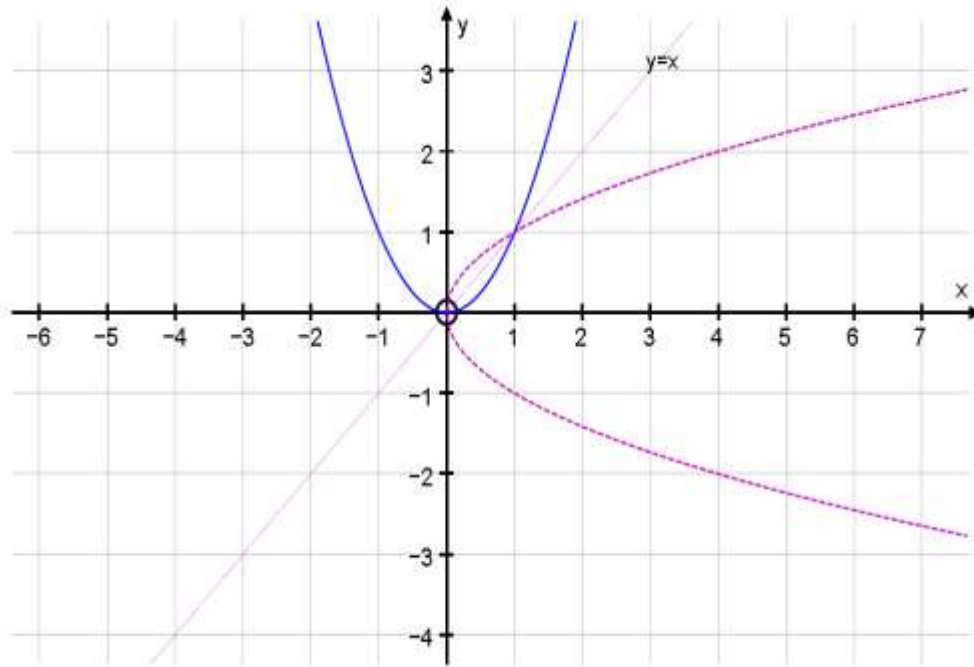
3. Describe the transformation that has mapped $f(x)$ to the other.



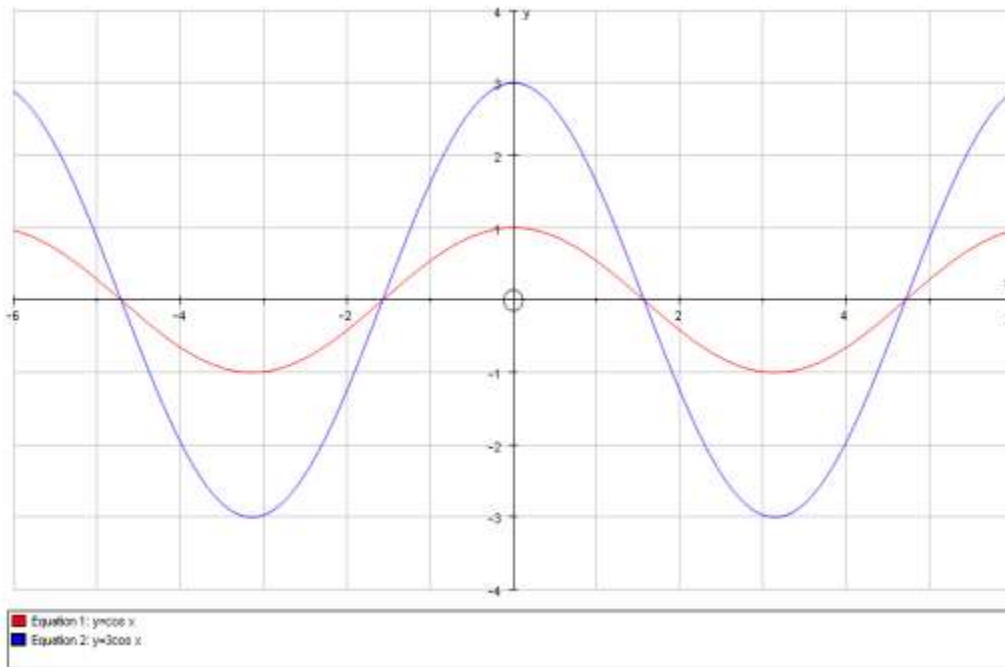
4. Describe the transformations that mapped $f(x)$ to the other two.



5. Describe the transformation below.



6. Describe the transformation that mapped the graph of $y = \cos x$ below.



APPENDIX C: SAMPLE OF THE INTERVIEW VERBATIM

In this account the interviewer's words are in bold and the interviewee's words are not bold. The verbatim was taken from voice recordings after several replays.

Interviewee 1.

So can you tell me how you feel about the topic?

The topic is fine but the things like where it gets to like the $g(x)$, $f(x)$, g of $h(x)$ and the like, sometimes working out the answer is the issue, working out the issue of the different functions and how to get to the final stage, but otherwise everything is fine.

Briefly tell me what you understand by a stretch?

A stretch diagram is when a graph either is stretched according to the x – axis or either reduced or enlarged on the x - axis.

What about a reflection?

Reflection is when you take something from one side and is reflected about a line of symmetry.

What about a translation?

Translation is when the co-ordinates of the points move either up or down depending on the function or the given co-ordinate altitude moved to, let's say you have y and x – axis and they tell you that it has to be $y-1$ and $x +2$ then you move it accordingly around in the corresponding place.

Then if you are supplied with the formula for all the transformations, are you able to identify which transformation has taken place?

Yes.

If we are to transform the $y = f(x)$ to another. If we transform to $y = -f(x)$, then can you identify the transformation that has taken place? Is it a translation, reflection or a stretch?

That's a translation, $-f(x)$ from $y=f(x)$ is a translation.

What about this one? A transformation to $y=3f(x)$,

This is a stretch.

What kinds of transformations are these?

$Y = \frac{1}{2} f(x)$ is a stretch again, $y=f(x)-3$ is a translation. $Y=f(x+3) - 2$ is a translation.

What is the difference between the two translations?

I'm not quite sure about this one.

Ok. Then, the next group.

Then $y=f(2x)$ it's a stretch, $y=f(\frac{1}{2}x)$ is a stretch again, $y=f(x-2) +3$ is a translation and $y=-3f(2x) +4$ a translation.

All right thank you. Let's say you are given diagrams, are you able to identify the transformations that have taken place? The $f(x)$ is this one labelled.

Yes sir, this is the long wave and then the short wave is the transformation. Yes, this is a stretch transformation.

In which direction?

On the horizontal plane

And how big is the stretch? You need to say the factor.

2, sir. The factor is 2?

All right lets go on. Describe the transformation that map $f(x)$ to the other two.

Oh. 1st one is a translation? This one is a rotation, rotation about the line $y=x$,

Oh, ok, go on.

This is a 90 degrees rotation basically but it's a rotation about line $y=x$.

Ok, yes. Now for this one that is crossing y-axis at 3 and -3, what do you say about it?

It's a stretch of the factor of 3.

In which direction?

Vertical direction.

All right! Thank you very much for your co-operation. Have a good day.

interviewee 2

May you tell me generally what you understand about the concept of reflection?

Reflection is when a point is reflected from the 1st quadrant or reflected through the y - axis or x-axis.

Tell me also about the concept of translation with respect to a function. Be it a hyperbola or any shape of function on a Cartesian plane.

If I translate it, I can translate it through the y – axis or through x – axis. It's complicated.

Let's say you are given a function and a general formula after a transformation, are you able to identify the transformation by its formula?

Yes

Then can you describe the transformation, which transformed the function $y=f(x)$ to $y=f(2x)$, what kind of transformation will that be?

Again, this is enlarged. No, this is stretched by factor 2.

Say something about the direction

It is in the direction of x , I think. But sir. I find it difficult because... well, it's not really difficult but it's how I'm supposed to understand the representation. The staff sometimes confuses me, so I need more practice. I don't have a problem with it but I'm the problem.

Which concept maybe is more problematic?

Like when you have a straight line and hyperbola on one Cartesian plane, which is the one that gives me much of the problem?

Ok! Thank you very much

Pleasure!

interviewee 3.

Tell me how you find the topic of functions and their transformations?

It's difficult, I don't understand them

What exactly don't you understand?

The graphs! I don't understand them.

You don't understand the graph? Then if the graphs are not there but there are transformations, do you understand?

Yes. Without graphs?

Is it the problem of the graph itself or the representation of the function?

No it's the chapter, the functions in general are difficult for me.

Does that mean you cannot recognize the function or you are not able to transform the function?

I can recognize it and know that this is the function but I have to work hard for me to figure out its transformation or whether I have to solve for it or draw a graph.

Ok right, let's say the function is given and is represented by a graph. You are to reflect that function graphically, will you be able to?

Hah, if I use a calculator I will.

How do you use a calculator to reflect?

Mmh! If I'm given an equation and they say draw a graph using the equation, yes I would. You can do it obviously! Yah, I use a calculator. If I know the rules.

Tell me the rules

I didn't study for this.

Do you need to study first or you need to know this always?

I need to study.

What about stretch, explain any idea about it?

Mmh, aah, those ones I don't know hey.

Right lets go back to basics again. If we say a function is reflected, what do you understand by that?

That there was an original image, but when an image is formed from an original image through what? I don't know how to explain it.

What about translating, let's say you are to translate this function vertically, use a diagram or whatever to demonstrate this translation?

I know how to do this but I don't know how to explain them, translation

Then draw the diagram: express it diagrammatically or use it in algebraic formula or whatever, whichever is best for you.

Which is which? I know that if you tell me that or if they give me a certain triangle that has been translated 2 units to the right and 3 units down, I know how to figure out the position where it's going to be after the translation.

Ok. (The learners is hesitant and does not do anything). **Any questions?**

No I don't have any questions

Interviewee 4.

Can you please say what you understand by the concept of reflection?

It's when, eemh, that is, a sketch or no, it's not a sketch, an image is drawn on the one side of a Cartesian plane and then you draw it on the other side and is reflected back on the y – axis or the x –axis.

Is it? Can you show it by a sketch diagram or anything?

Anything? Can I draw something?

Yes any sketch diagram.

All right this is the image after the reflection

What about the translation? Explain and then illustrate it.

Yoh! I don't know sir, I'm lost on translation

You are lost?

Yes translation I'm lost ok if I knew what translation was I would show you.

Ok! What about a stretch?

It's when you make the image bigger,

That's a stretch? Bigger how?

You stretch it! (Moving hand apart, away from each other). Yes like if u draw here and you stretch it by 5 or 10 it will be like huge.

Ok that's a stretch, can you just label it. On number 2 we have got a formula.

Yes, sir. I've got some problems on functions.

Right, can you indicate to me what! We are moving from $y = f(x)$. If we go to $y = f(-x)$ what transformation is that?

It's a reflection about the y-axis.

What about if we go to $y = -f(x)$?

A reflection about the x-axis. Here it is. (Showing a correct illustration)

Right, when you have $y = 3f(x)$ what do you think it is?

Stretch.

What type of stretch? Describe fully, stretch has got a direction.

Horizontal. No, it's horizontal and vertical.

What do you mean?

Vertically.

Then how big is the stretch?

3 vertically, it stretched 3 units vertically

Right, how about this one?

It's also a stretch, horizontally. It moved 2 units down and 3 units up.

How do you know?

I am confused.

Interviewee 5.

Can you briefly describe what you understand by a reflection?

A reflection is like is a mirror image of something. Maybe it's like something standing here and then what you see on the opposite side is exactly the same as what's in front.

Ok! What about translation?

Translation, I think, is moving an object like from one place to another. Maybe if it's in the 1st quadrant of a Cartesian plane moving it to the 3rd.

All right, what about a stretch?

Stretching is like making an object bigger that you can reflect or take it out sideways, vertically or horizontally

All these transformations can be represented, sometimes, by a formula. That is, you can have a formula for translation, reflection or for a stretch. Can you identify which transformation is in operation from a given formula? Here, we have some examples of formulas that are used for transformations. Are you able to identify which one of those formulas is paired with which transformation? Can you just briefly do it?

Like must I say what this is and what that is?

Yes! All these are transformed from $f(x)$. This is the original function, if we transform this to $y = f(-x)$ what do you think the transformation is? Describe it fully.

Oh sir, I think, oh sir, I'm not sure about that one.

Try to think!

Ah sir, I don't know. It's difficult. Ok sir let me try. This is, you can say, a translation maybe.

Can you write the translating for me? (Writes correct vertical translation) Ok then, this one, what do you think it is?

Ah, I think its translation because $f(x)$ only changes, and then if you say $-f(x)$ it means you are taking this to the other side its actually like down, so that means downwards.

Ok. Any other formula you can identify there?

I think this one. This one means that it's stretching because it's actually moving the thing, because you moving it 2 units to the left through the x-axis and then the y-axis value 3 units

Please elaborate!

It's stretching because you taking your scale, if you have 1cm scale you reducing it to half

Right. This one will be very quick. There we have the diagram that is the original and is $f(x)$, and the other one is the image. Then from there can you identify what transformation involved?

Sir, its stretching, yes because it's becoming Oh its actually reflected because this part comes to this side, the original is this one, it's the big one, oh its compressing vertically, compressing because it was a bit bigger, now it has been compressed becoming a bit smaller or narrower.

What about number 4?

This one is the original, this is reflection, no not a reflection, its translation because they are moving the object.

Which one moved?

The bigger one.

How is the movement?

It's one unit to the left and oh its only one unit to the left and it doesn't go up or down.

And then the other?

The other one didn't go up, didn't compress I've forgotten the word. Like it became smaller. Ok compression.

Right, let's go on to the next one. We are almost done

This one sir it's flipped.

Are we on number 5?

No it was rotated anticlockwise like 90 degrees anticlockwise.

Now the last one?

The last one is stretched. This one is the original? This one, the smaller one. It stretched vertically.

How big is the stretch?

Oh sir, it's stretched by 2.

Thank you very much, have a good day.

Interviewee 6.

Can you briefly describe to me what you understand by a translation?

Translation is moving, the movement of one image onto the other side of the Cartesian plane. It depends whether you are translating it by how many points, up or down, left or right. That's what I understand by a translation.

What about a reflection?

A reflection is a mirror image of one image onto the other part of the Cartesian plane as well depending on reflection either on the x or the y –axis

Right! Then, lastly, the stretch?

Stretch is expanding an –image.

Normally when we do these transformations they may be represented by some formula

Are you able to identify any of those by mere looking at the formula?

I think so, yes!

Right. Here are some formulae, can you identify these. We have the original function $y=f(x)$ then all these are transformed images. Say whether it's reflection or translation or stretch and how big is it, its direction and size and whatever?

Yoh, ah well I think this one is a translation.

How about the translation?

It's enlarged by 3 units and moved to the left.

Write it down. As you write please also talk.

Ok the $y=-f(x)$ is a reflection about the y-axis and $f(x + 3) - 2$ is a translation which is whether the image is expanded by 3 units and moved 2 units to the left.

And this one right?

That's all I know sir.

Right lastly we have the graphs. Now, we have the original graph labeled and the transformed graph is not labeled. Can you tell me the transformation that is involved in mapping the original to the other?

Oh the transformation? This is a glide reflection and it's moved a unit to the left and expanded by 2 units.

That is number 3. What about for number 4.

Oh number 4 is translated a unit to the right, and for 2nd one it's translated 2 units up, that's for number 4.

Then, what about the last 2 diagrams?

Which one is the original one? Ok that one is rotated I think 90 degrees clockwise that's number 5

Ok and number 6?

Which is the original?

The smaller one.

Oh it's expanded by 2 units, it's expanded by the factor of 2

Ok! Thank you very much.

Interviewee 7.

Can you say what you understand by reflection?

Ok. Reflection is the mirror image of an object or a graph about a line, like on a Cartesian plane. You can get an object that is reflected about the y-axis, about the x-axis, about a line which is $y=x$ and about a line which is $y=-x$

What about translation?

Translation is when you shift, the movement of an object from its original space, which it originally occupied. Like you can shift upwards or downwards, to the left or the right.

All right do the 3rd one, stretch?

Yoh stretch, defining it. Ok sir. Stretch, I can't clearly define it but I know what it is.

Ok, you can demonstrate, let's say you got your sine graph and then you are stretching it.

It's going to move, it can either be stretched horizontally or it can be stretched vertically.

Ok! Label the original and the one you stretched?

When you stretch it gets 2 sine, that one is $2 \sin x$.

What about the direction?

I think its horizontal sir.

Then also label. Right, we have a diagram here; can you tell us what transformation has happened here?

It's a compression. Vertical one of the $y=\sin x$.

What do you mean?

Ok sorry sir, this has been compressed horizontally. A horizontal line which is usually landscaped. This is your horizontal line and that's your vertical.

I see, so you mean the horizontal is the x-axis? So instead of saying horizontal, can you say in the direction of the x-axis and vertically, in the direction of the y – axis? You are right. Man you tell your opinion about this topic?

Which one?

The topic of transformations of functions

Personally it's something I enjoyed but it's challenging but with practice you tend to perfect it

Tell me how you came to understand the concepts in it.

When Mr. ...(teacher) teaches and gives us homework every day about what we learnt in class, he is giving us not because he is trying to be funny, but he is giving it so that he can see what we can do on our own.

Does the knowledge of transformations of functions have any effect on other topics?

Like on other topics? Transformations or functions, you pick up some of it in algebra, quadratic equations and what not because your parabola is quadratic, your linear equations so, yes it has an effect on algebra, it has an effect on trigs.

Explain briefly its effects on algebra?

Algebra like linear equations is a straight line graph, which is also a function then your exponential equations is your exponential graph which is also a function then your quadratic equation is your parabola.

Given the original function labeled $y=f(x)$. From these formulae, can you please show us a formula for its reflection in the x-axis?

Reflection in the x-axis? Ok let me just refresh my memory. Reflection, eish, can't recall this one!

What about in the y-axis, still using that function?

Nay.

Can you give us a formula for the stretch?

An upward stretch or a vertical stretch?

Any.

It's like a translation in the negative x direction? It's to the left.

Thank you very much I think I've learnt a lot from you.

Interviewee 8.

May you please define briefly what you understand by a reflection?

A reflection is a mirror image of the original points through the y-axis or the x-axis

What about stretch?

A stretch is when a shape can be either stretched horizontally or vertically

Ok, let's have the function $f(x)$. Can you indicate by means of a formula using the $f(x)$, the image of this function after it has been reflected in the x-axis?

Do you mean the mirror is on the x-axis?

Yes, continue.

The x co-ordinates change. X becomes positive and the y-axis becomes negative.
The y-axis and the y value changes.

Can you effect this by an equation?

I do not know.

What about a translation?

Let's say we move the function 2 places vertically, ok from it if it is from a negative 2 and it moved vertically it will be a negative 10 it will end up by the x-axis, 40 in the y-axis.

Say your function $f(x)$ is this parabola, can you show by drawing an image of the vertical translation?

(Correct idea shown, but not accurate)

All right. Tell me how you feel about the topic transformations of functions?

Ok I find it difficult especially at the parabola especially when I have to work out the equations. That's when I find it difficulties. But with the straight line it's much easier but I believe that if I practice more and get help when I get stuck I can improve better.

So are you practicing?

Yes I practice before exams sir.

Only before exams? Why don't you practice in-between?

In-between, sir, we will be doing a different chapter so I don't want to get confused that's why I practice before the test.

Alright thank you very much.

Pleasure, sir.

Interviewee 9.

Briefly describe to me what you understand by the term translation?

Translation is when a shape moves horizontally or vertically by units to a certain image.

Ok. What about a reflection?

A reflection is a mirror image depending on the line that you put it; it can either be reflected in the x-axis or the y-axis. Ok or any other line which is part of the x-axis or the y-axis

Say something about a stretch?

Stretch is when you enlarge a drawing by stretching it horizontally or vertically making it bigger.

Ok, I believe you have much knowledge on these concepts. How is the topic in relation to other mathematical topics?

It's ok nothing much I can say I know.

All right, let's again focus on a function. Would you prefer mapping it algebraically or graphically?

I prefer graphically.

Let's say you have a function $f(x) = 2x + 5$. Let's say we want to map this function; we want to translate it by 2 units vertically. What will be the resulting function be?

Translating is just moving it, shifting it vertically by 2 units.

Right, what about if you reflect it in the x-axis, how would be the resulting function?

Let's say we have the function $y=2x+5$, it's going to cut the y-axis at 5, then cut the x-axis at 2,5. Let's say we put the mirror line on x-axis, and draw the image diagram, and it will cut x-axis at negative 2 and a half.

You may show by drawing, how it will look like?

This is our $y=2x+5$ so we want to reflect this in the x-axis, then you put a mirror here, our image will be here. (Drawing the correct image). Ok this is the reflection

All right. That's fantastic. Do you have anything else to say about the topic?

Nothing much I could say, it is interesting, its ok sir it's not that difficult at least, I understand more or less.

Ok thank you very much, I wish you succeed in your studies. This is quite an important topic, because it has very positive influences on other mathematical topics.

Yes sir, thank you.

All right, thank you too.

Interviewee 10.

Tell me how you feel about the topic of functions and their transformations?

I find it difficult sir, I am struggling

You struggle with what exactly? What is really the problem? Don't have enough material? Is it the nature of the topic, the words used or, maybe how you interaction with the subject teacher?

I believe it's because of the interaction with the teacher and also my understanding when I read the book, I find it difficult to understand what they are saying, like ah! I don't know how to put it.

Oh! Say it all if you can.

I believe it's not meant to be difficult but when I read it, like the theory and the explanation, I don't know how to do it. When it comes to the questioning and the background I don't know why or how it because I didn't understand the theory given that it dates back on my grade 10 teacher.

Ok, can you just briefly attempt to explain to me what you understand by the term reflection of a function?

I understand that it is on the y and the x-axis, I can make an example sir.

Oh, yes. Do it.

You can say reflect $y=x+2$ on the y-axis, so I begin with writing the transformation of a reflection sir. So I have indicated that I need to change the x value to make it a negative sir.

Ok! So from that one if the y remains the same with x changing, then the reflection will be on which axis?

The y-axis.

Ok right what about the x-axis? What happens, write here.

Y value changes its sign.

Thank you very much. Right. May you attempt the concept of stretch on a function?

Ah! I believe sir is when they say you must move a function on a particular co-ordinate like move this function 5 units to the left. I believe it's like that sir.

We have successfully discussed reflection and stretch. What about translation?

Mmmh, I can't recall.

If you cannot recall it's no problem.

Ah I forgot sir

Ok. Let's move on. Here we are given a function $f(x)$, which maybe any function of any form. It may be linear, it may be quadratic, it may be exponential or can be trigonometric. If it is mapped to $y=f(-x)$, can you identify the transformation which has taken place?

Umh- it's a reflection of y sir.

Ok. Do the same for the remaining ones?

No sir, I believe that I'm wrong here sir.

You can just simply explain in brief what transformations the formulae represent?

Umh.

How is it? Say it. Difficult or easy, is it something enjoyable?

I can't say it's difficult but it's not enjoyable, not always but I believe if I tackle more questions, I can succeed in getting the marks in a test sir.

Ok right, can you say anything about this one? We are given the diagrams.

Oh yes sir.

From the diagrams can you identify the transformation of $f(x)$. Function $f(x)$ is the original. What has happened to $f(x)$ to get to this other function?

I can say this has transformed. Because this one is $y=\sin x$ then from there the unidentified line of the graph is $y=\cos x$, sir.

What about here the $f(x)$ is the original and it transforms to this one and that one. Can you identify what transformation has occurred?

I believe that $f(x)$ has been translated 2 units up sir and also one unit to the right sir.

All right thank you very much. I enjoyed having this conversation with you.

interviewee 11.

Can you tell me in general how you feel about the topic of functions and their transformations?

Yoh sir, to tell you the truth I hate it, I hardly understand it sir.

Ok. What do you hate? What is hard? Is it the book, or how they are explained on you?

I never got time to practice it, maybe that is why.

Can you recall anything about it?

Yes, sir.

Well, briefly explain what you understand by the process of translation of a function?

Ah I can't!

You can start with any functions, on the Cartesian plane. How do you reflect it? Reflect it either in the x or the y – axis. So can you briefly describe what you understand by that?

Ok umh, when you reflect something along the x-axis your y-values, I think, become negative, when you reflect it along the y-axis your x-values become negative.

But some values can be negative to start with.

To clarify, when the value is positive the reflection transforms them to become negative and when they are negative they become positive.

Right what about stretching?

It's difficult.

Ok. Then looking at these functions. All these are transformations coming from the function $y = f(x)$, $f(x)$ can be a linear function, a parabola or trigonometric etc. Can you write what transformation mapped $y=f(x)$ to these ones. You may talk as you write.

No sir, I hardly understand.

Not even a bit?

Nothing sir.

Ok. What about if you are given a function $f(x)$ and then it transforms to that one. So briefly what do you say about the transformation there?

This one is $y=\sin x$, maybe it has been reduced by 2.

Ok. What about here?

The first has been shifted by one unit to the right and the second is shifted vertically by two units upwards.

Thank you very much for your information.

Interviewee 12.

From what you have done so far on the topic of transformations of functions, briefly tell me how you feel about the topic.

The topic is quiet short and for me it's easy by the time I was writing this test. I didn't read a book. I hadn't studied the topic but now I studied. I think I understand better.

Do you enjoy the topic? Are you comfortable with it?

No sir I'm not comfortable because I don't like it at all sir.

Why? Is there a reason for that?

I think it's lack of interest in the topic. Most of the topics like ...ah. This topic is not interesting. It's boring.

What is boring?

It is challenging.

Although challenging, tell me what you understand by the concepts involved in transformations of functions.

Sometimes they can tell you to translate maybe by a formula or reflect on the y-axis. Let's say they have given you something like any function and you translate this function on the x-axis sir. If you say that, you must actually put a minus on the y-axis. Am I right sir? Or usually maybe given two points x and y then they say translate it in the y-axis and then if you translate it it's gonna be minus x to y that's what I understand.

Say it specifying concepts.

Or maybe they say rotate 180° the point maybe clockwise let's say from the 1st quadrant rotate 180° the image is going to be x-y but it's on the other quadrant. And then maybe they say enlarge or reduce the point it is different.

I don't get it quite well. Is reduce not the opposite of enlarge?

Yes sir, it's the opposite. It's opposite but mathematically it's almost the same thing. Reduce with the factor of half or enlarge with the factor of 2 or let's start by enlarging. Its x is to y then they say enlarge x by 2 it's going to be $2x + y$ after enlarging it. If they say reduce point x by half its going to be half $x + y$. oh sorry sir, eish, from x to y it's going to be half $x + y$

You may talk about stretching?

Ah stretching, sir, I find it quite confusing because stretching and enlarging I think it's one and the same thing.

Ok. Here we have a function $f(x)$, from $f(x)$ there is a mapping or transformation onto $y=f(-x)$ or $y=f(x)$ onto $y=-f(x)$. By mere looking at the formula can you deduce what kind of transformation mapped $f(x)$ onto each specific function?

What happened sir?

Maybe meaning they reflected the function, is it a stretch, translation or what?

For me sir it's a translation. Can I ask something sir?

You can ask.

What's the difference between translation and reflecting sir?

To me translation is moving an object in a straight line through a specific distance and then reflection is flipping the object. From its definition, what is a real life example of a reflection?

A mirror.

So if you look at the mirror what do you see?

An image or a reflection.

On that background can you identify any of the transformations of the functions that are given to you?

So sir, must I write the rotation.

Write any that you know.

Sir, oh! There is something. These two are quite tricky, sir, and they are confusing me. Sir, translation one place to the right.

Write it.

Which one?

These ones and those ones.

The $y = \frac{1}{2} f(x)$, $y = f(x) - 3$ and $y = f(x+3) - 2$. Then sir you said moving a point, sir, from one place to another you said its translation, sir, and this point has actually moved 3 units to the right. These ones I'm not familiar with them sir

All right. Let's say you are given a function $f(x)$ graphically and it is transformed to this smaller one. Can you describe what has happened in terms of transformation?

From this graph sir they have reduced the x values by 2 for it is to be smaller, but I don't know which word to use but they have reduced the x value.

Ok. Right what about this next one?

$F(x)$ is the original it's transformed to that one and from here they have moved the graph upwards 2 units and then from here to here moved 1 unit to the left then this one 2 units upwards.

All right thank you very much. How about this one?

This one sir the original graph is this one. So I think this graph is the image of this one. The image after 90 degrees rotation.

Right. Write it there.

Actually they translated the graph on the y-axis.

Ok. What about the last one, number 6?

This is the original graph. The original is this one. Which is cosine and as a result it was enlarged sir. Maybe increased by a factor of 3 from here to there. Stretched but I don't know by how many.

All right let's say by factor 3.

They reflected the x-axis on the y-axis.

Thank you very much.

interviewee 13.

That task exercises you wrote were about the transformations of functions. Can you tell me how you felt about that topic?

I felt normal about it; it's just a subject I need to understand.

Tell me in terms of the level of difficulty, did you enjoy it or whatever?

Its easy sir but some parts are challenging and difficult for me.

Challenges are normal to meet and you can overcome them. May I ask you some questions? What do you understand by the concept of translation?

Translation, it involves the moving of points by certain units sideways, upwards or downwards

Ok, What about reflection?

Reflection involves reflecting a graph in the y-axis or the x-axis.

How does the reflected graph look like?

Upside down if it's parabolic and then if it's an exponential graph it will be on the other quadrant.

Then, say something about a stretch?

I've never heard of a stretch.

It's a transformation of a function. It's a concept you are going to meet it but since you are in grade eleven you should have done it already. Maybe the teacher is still going to introduce it to you. Then from what you know, are you able to identify, from a formula, which transformation has taken place?

Yes sir.

In this question, we have a function $f(x)$ where $f(x)$ can be linear, quadratic, hyperbola or trigonometric. Now if $f(x)$ transforms to $f(-x)$ or $-f(x)$ or $3f(x)$ and so on, can you briefly explain all of these.

All of them? The 1st one, I think, it's a reflection in the x-axis. I'm not sure.

May you write as you talk?

Sir, this is the turning point, the x-axis point and the y-axis point or shifting of a graph. And then this one, I think, probably given a graph you substitute in this bracket and then you get a new set of formula and then new sets of turning points, I think.

I do not get it well. Write it.

Yoh, sir I don't know.

And then what is this one substituting?

Of what is in the brackets. Ah sir I think that's all I can do.

All right thank you. Right now let's say you are given the graphs, which are drawn. Right opposite is the graph for $\sin x$. From there can you identify what transformation has taken place?

No, sir.

Right, here, we have $f(x)$ it moves or transforms to that one, can you explain what transformation it is?

The turning point has been increased between 0 and 2, the 2nd one the graph has shifted 1 unit to the right.

Here we have the original, this one, and then what has happened to it so as to get this one?

I think its! Yoh, sir Im not familiar with this type of transformation.

What about this one where the amplitude has changed, it has increased by 2! So if the amplitude has increased by 2 what do you say?

By the value of a is now 3, not 1, $y=a \cos bx$.

What is a there?

Oh it's there, its 3.

We are simply refreshing what we have done. Ok thank you very much.

interviewee 14.

The exercises you did before were about the transformation of function. Can you briefly tell me how you feel about that topic?

I was confused at first but now as I am practicing everyday I see that it's an easy topic.

How about in terms of your enjoyment? Did you enjoy it?

Partly and the other part I gave up when I study alone and then I ask my other friends.

So, you help each other. Right in relation to that I would like to hear from you what you understand by the concept of reflection?

Talking about reflection means that the shape of the object doesn't change size it only reflects to the other side.

Which other side? Side of what?

The axis, either the x or the y -axis.

Ok, say something about the concept of translation?

Translation is when an object moves from its position depending on the scale given. It moves or shifts?

What about the concept of stretch?

Stretching I don't really know it because we haven't done it at school.

Ok. Thank you very much. All right among those transformations, we have discussed, if those are shown by a formula will you be able to identify them?

I don't know unless I practise them.

At least try to identify them now. Like here, we are given a function $y=f(x)$ and you are to describe fully the transformation which is represented by the formula. We are moving from $f(x)$ up to the given one.

This is reflecting about the y-axis.

Right, may you write it as you talk. You are free to answer them all or those that you feel like. Talk as you write.

This one I don't know it, $y=3f(x)$ and $y= - 3f(2x) + 4$. This one it has been lifted by the factor of $\frac{1}{2}$, $y =\frac{1}{2} f(x)$ and then this one is reflected about the x-axis and then shifted 3 units to the left. This one $f(x - 2) + 3$ must be a translation of 3 units to the left and 2 units up. Ah, even this one I don't really know it.

Can you describe the transformation in number 3, function $f(x)$ transforms to this other one.

I think it's stretching. Stretching to the stretch factor of 2.

Right lets go to the next one.

Function of x which is a parabola. It moves to that one.

What type of movement?

It's a translation shifted 2 units up and the other one 1 unit to the right, this line $y=x$.

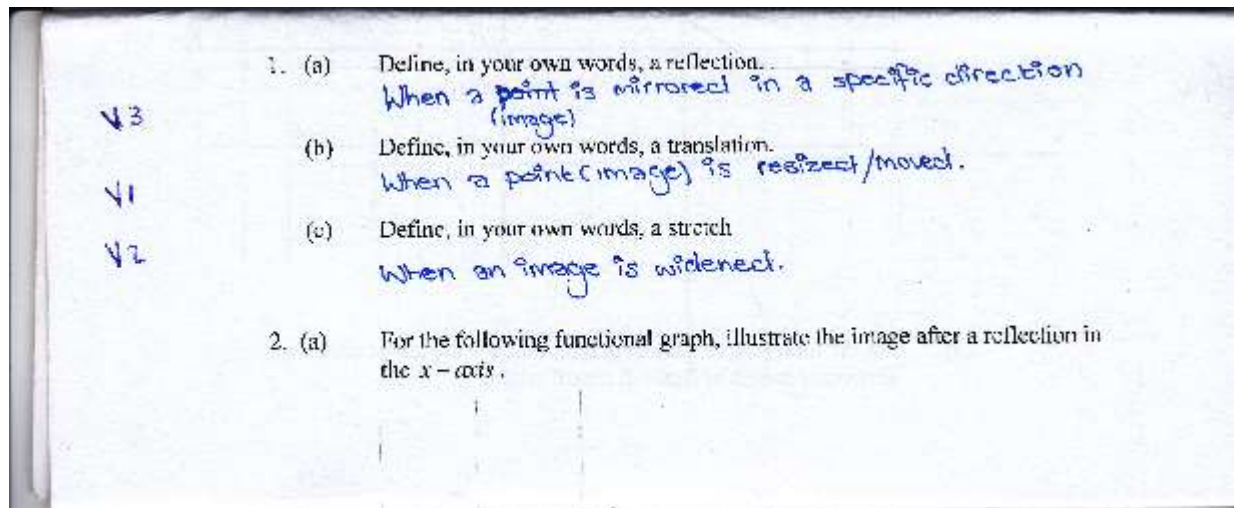
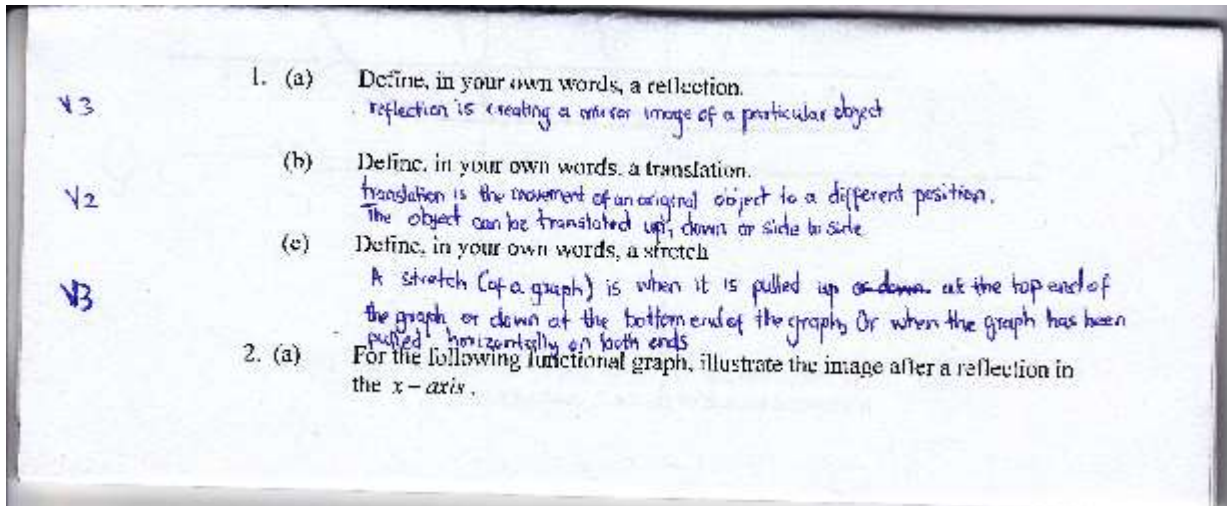
Then the last number.

It has been stretched by the scale factor of 3 downwards because this is the initial one. The original one is this one oh it's that one. The image is the bigger one. Oh, it has been stretched and is bigger than the other one, then the other one to the stretch factor of 3.

All right thank you very much.

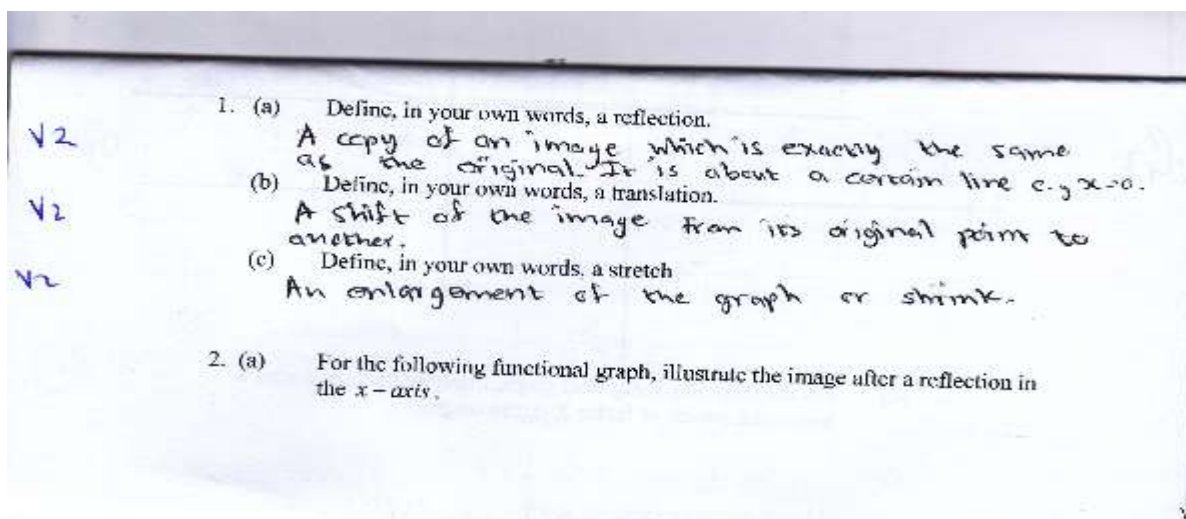
APPENDIX D: SAMPLE OF LEARNERS' VERBAL CONCEPT IMAGES OF REFLECTION, TRANSLATION AND STRETCH OF FUNCTIONS

(a) Mostly correct conceptions

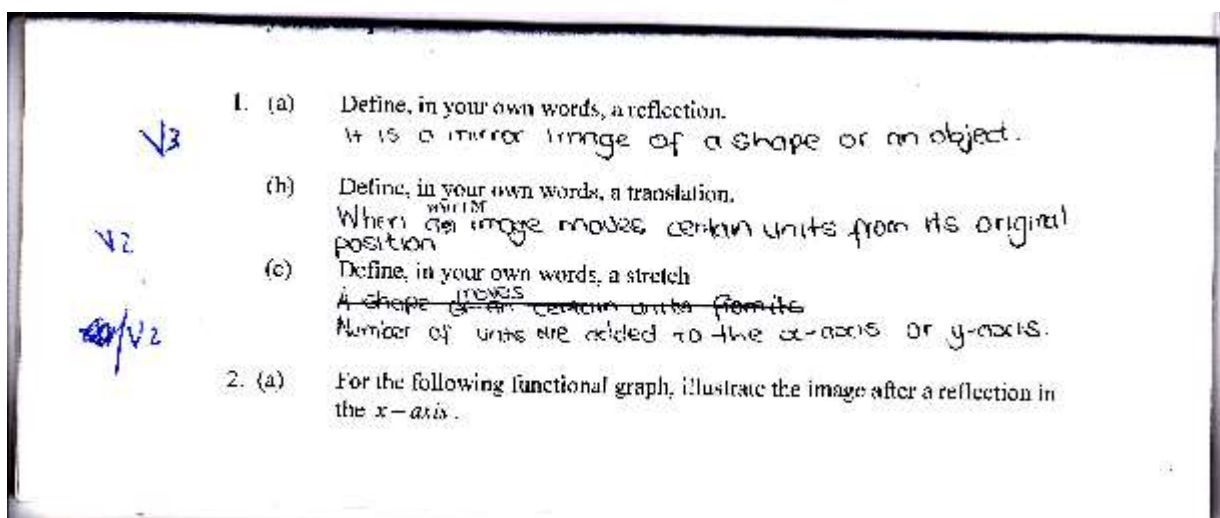


(b) Mostly misconceptions

(i)

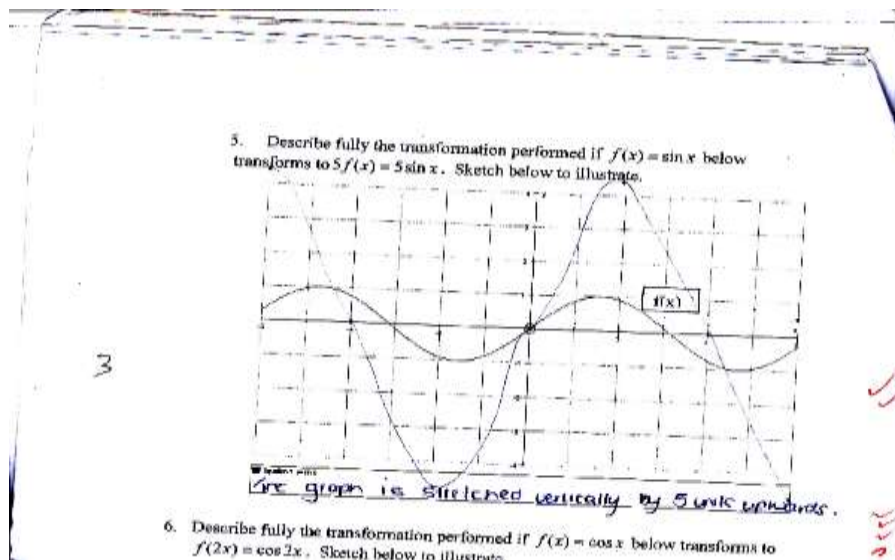


(ii)

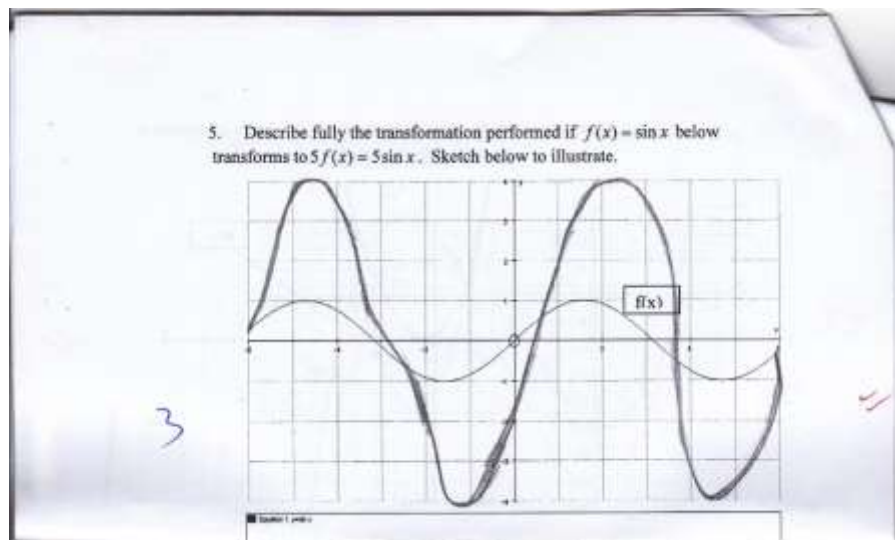


APPENDIX E: SAMPLE OF LEARNERS' GRAPHICAL IMAGES OF REFLECTION, TRANSLATION AND STRETCH OF FUNCTIONS

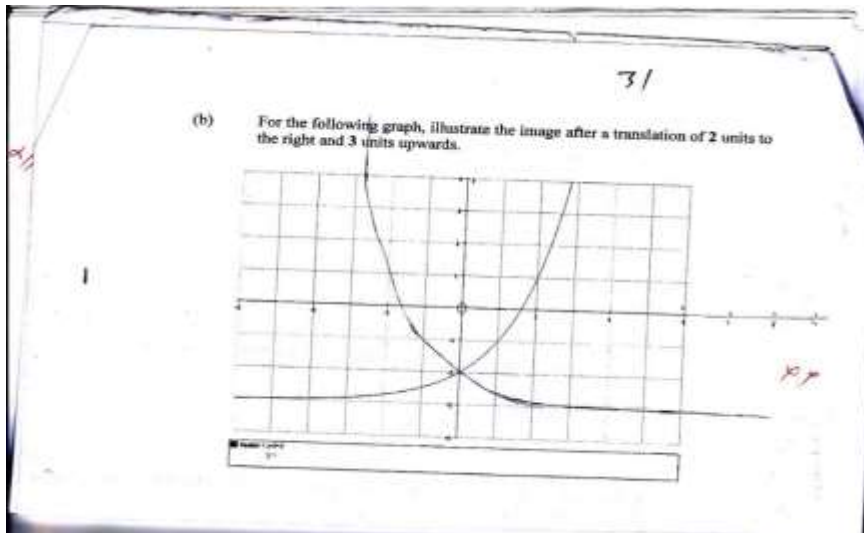
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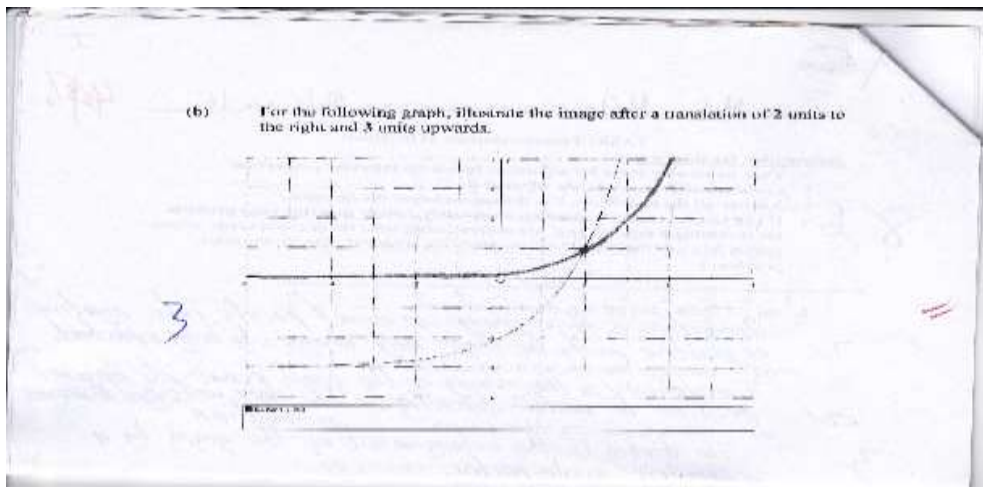
(b)



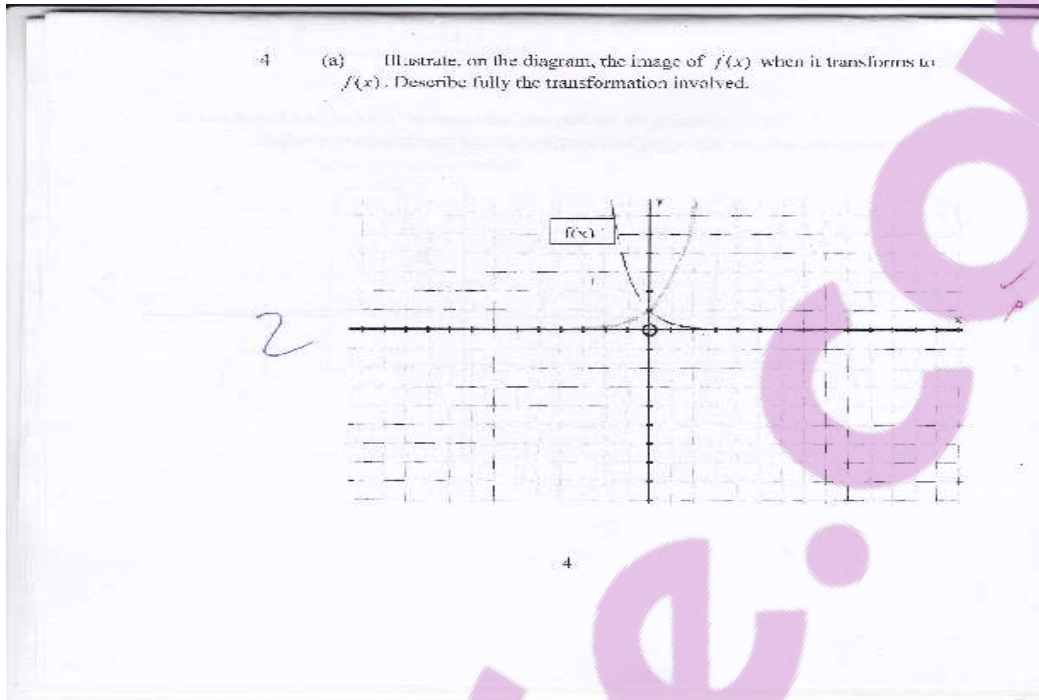
(c)



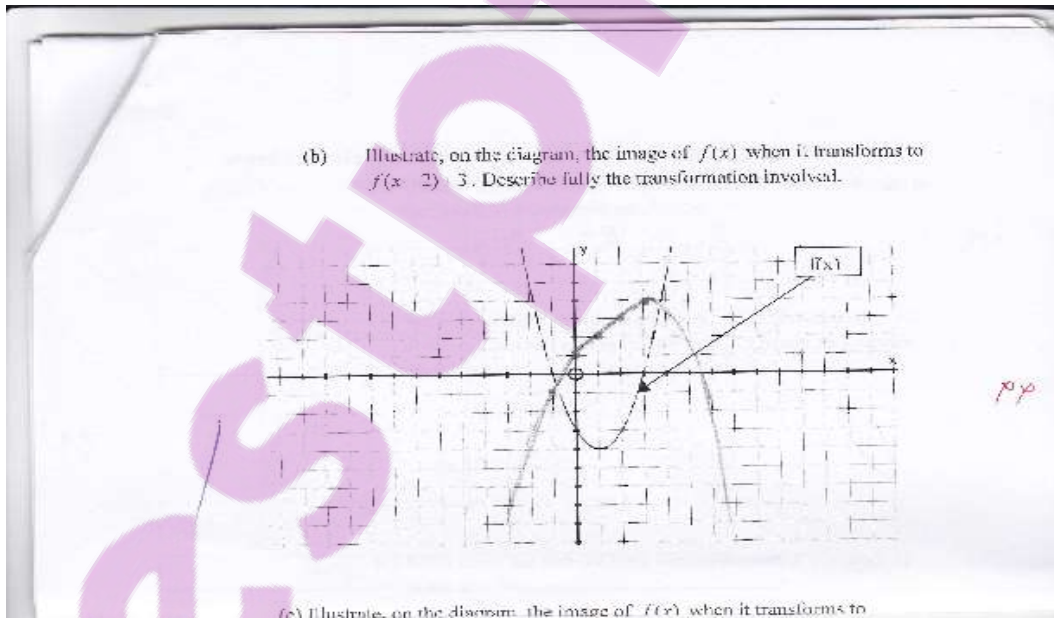
(d)



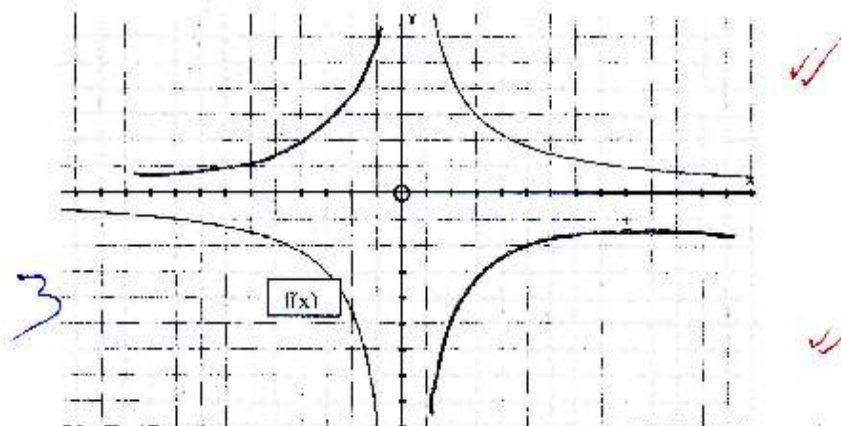
(e)



(f)

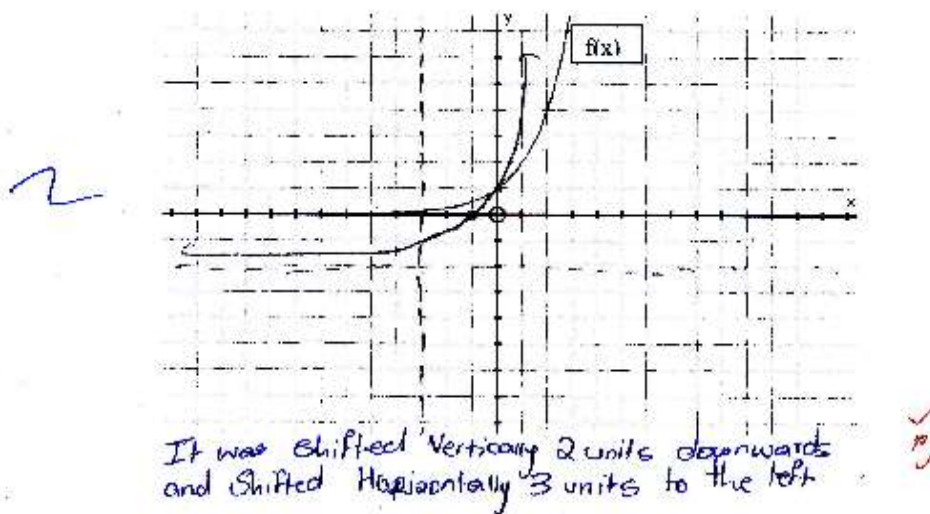


3. (a) Illustrate, on the diagram, the image of $f(x)$ when it transforms to $f(-x)$. Describe fully the transformation involved.



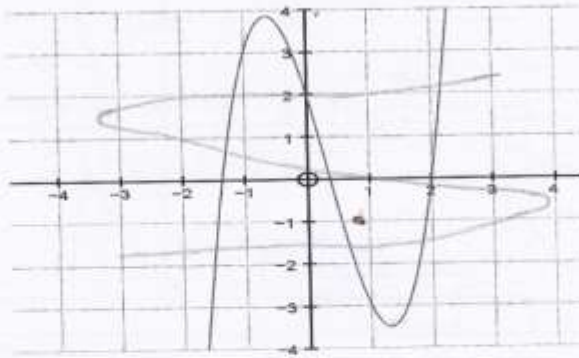
The graph has been translated through the x -axis

- (b) Illustrate, on the diagram, the image of $f(x)$ when it transforms to $f(x-3)-2$. Describe fully the transformation involved.



It was shifted vertically 2 units downwards and shifted horizontally 3 units to the left

(c) For the following graph, illustrate the image after a horizontal stretch of factor $\frac{1}{2}$, y-axis invariant.



2

APPENDIX F: SAMPLE OF LEARNERS' SYMBOLICAL IMAGES OF REFLECTION, TRANSLATION AND STRETCH OF FUNCTIONS

7.

(a) Write down the formula for a translation of $y = \sin x$ 4-units to the right and 3-units upward.

$y = 3 \sin 4x$ ✓

(b) Write down the two separate formulae for reflections of $y = 2^x$ in the x -axis and also in the y -axis. Compare and contrast the results.

1. $y = -2^x$ ✓
 2. $y = 2^{-x}$ ✓
 - they both intercept at (1, 2) ✓

(c) Write down the formula for a vertical stretch of $y = \cos x$ by factor 2.

$y = 2 \cos x$ ✓

8.

(a) Write down the formula for a translation of $y = \cos x$ 4-units to the left and 3-units downward.

$y = -4 \cos -3x$ ✗

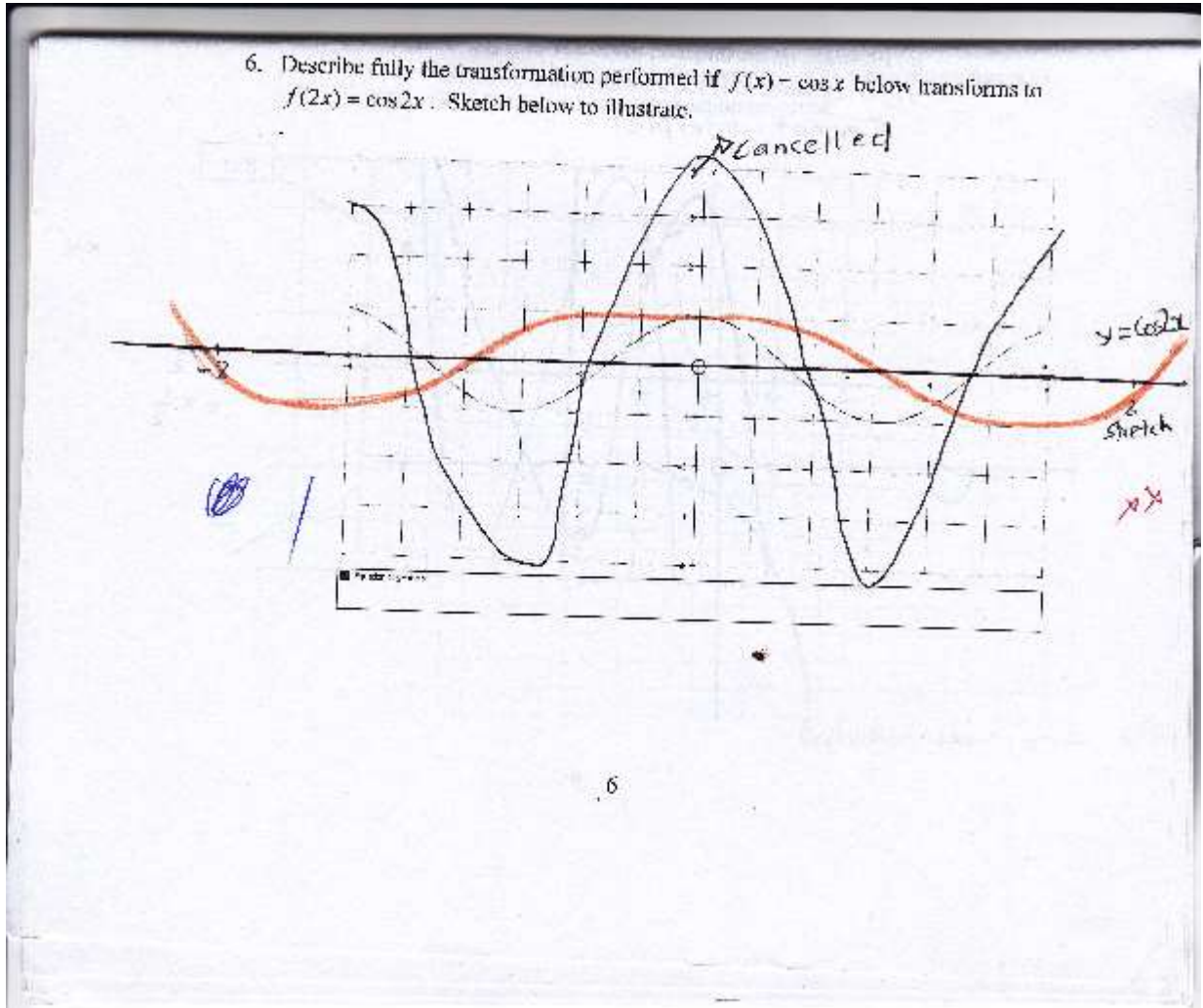
(b) Write down the two separate formulae for reflections of $y = \frac{2}{x}$ in the y -axis and also in the x -axis. Compare and contrast the results.

$y = \frac{2}{-x}$ $y = -\frac{2}{x}$ ✓ ✓
 - they both have an error at 0 ✗

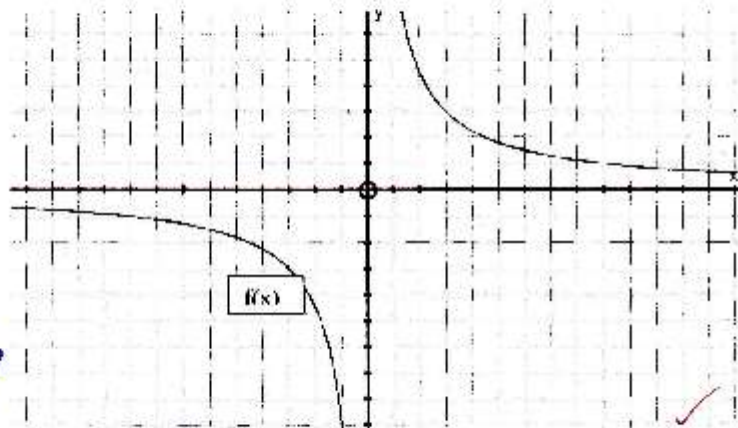
(c) Write down the formula for a horizontal stretch of $y = \sin x$ by factor $\frac{1}{2}$.

$y = \frac{1}{2} \sin x$ ✓

APPENDIX G: SAMPLE OF LEARNERS' IDENTIFICATION OF REFLECTION, TRANSLATION OR STRETCH OF FUNCTIONS

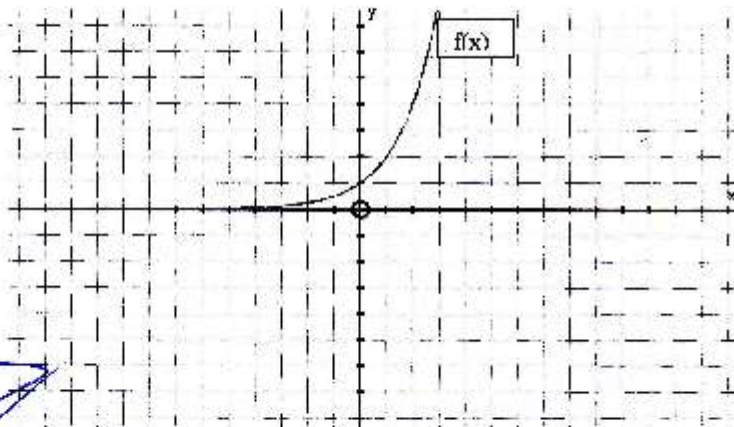


3. (a) Illustrate, on the diagram, the image of $f(x)$ when it transforms to $f(-x)$. Describe fully the transformation involved.



$(x; y) \rightarrow (-x; -y) \rightarrow (x; y)$: The transformation transformed from $(-x; -y)$ into $(x; y)$

- (b) Illustrate, on the diagram, the image of $f(x)$ when it transforms to $f(x+3)-2$. Describe fully the transformation involved.



$(x; y) \rightarrow (-3; 2)$ - The transformation involved is $(x; y)$ transformed to $(-3; 2)$

APPENDIX H: REQUEST LETTER FOR PRINCIPALS FOR PERMISSION TO CONTACT RESEARCH AT THEIR SCHOOLS.

10 March 2011

The Principal

Dear Sir/Madam

REQUEST FOR PERMISSION TO RESEARCH AT YOUR SCHOOL

I hereby request for permission to do research for my PhD studies with learners at your school.

DESCRIPTION OF THE STUDY

The research title is “**GRADE 11 MATHEMATICS LEARNERS’ CONCEPT IMAGES AND MATHEMATICAL REASONING ON TRANSFORMATION OF FUNCTIONS**”. The purpose of this study is to investigate how learners interpret the transformation concepts on functions and their representations, which are outlined by the FET syllabus at Grade 11 level. A group of Grade 11 Mathematics learners would be asked to do some content related activities. These activities and the follow-up clinical interviews would inform the study and would be expected to generate new knowledge for Mathematics Education field of study.

CONFIDENTIALITY

The name of school and learners will be kept completely confidential at all presentations and in the academic writing about the study.

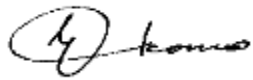
TIME INVOLVEMENT

The activities and the follow-up video and audio-recorded clinical interviews will take place soon after school and will not interfere with any school activity.

SUBJECTS RIGHTS

The learners who will participate will do so voluntarily and after written consent forms have been received from their parents / guardians. Individual privacy will be maintained for all published and written data of the study.

Yours faithfully

A handwritten signature in black ink, appearing to read 'Mukono Shadrick', written in a cursive style.

Mukono Shadrick

APPENDIX I: EXEMPLAR ACCEPTANCE LETTER BY PRINCIPAL TO DO RESEARCH AT THE SCHOOL.



Department of Education
District D9, JHB East

SANDOWN HIGH SCHOOL



P O Box 65130, Benmore, 2010
Phone: 011 884-1423/2/3 Fax: 011 884-1490
Email: admin@sandownhigh.co.za

22 March 2011

Professor L. Molamu
Registrar
Unisa

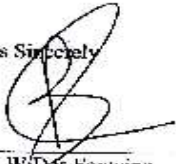
Sir/Madam

RE: Mr S. Mukono Student Number 4768-265-5

This is to confirm that Mr S. Mulono Student Number 4768-265-5 is employed by Sandown High School.

We have given our approval and consent for Mr Mukono to do his thesis at Sandown High School.

Yours Sincerely


Mr B. Wides Fortune
Principal

**APPENDIX J: CONSENT LETTER FOR PERMISSION FROM PARENTS /
GUARDIANS TO CONTACT THE RESEARCH WITH THEIR CHILDREN.**

Mr. S. Mukono

UNISA PhD Thesis

Cell: +2784 5788710

E-mail smukono63@yahoo.com

May 2012

Dear Parent / guardian

INFORMATION AND CONSENT TO PARTICIPATE IN THIS RESEARCH PROJECT

I hereby request for your consent to let your child participate in this research studies.

DESCRIPTION OF THE STUDY

The research is an investigation into how Mathematics learners interpret the concepts involved in Transformation of Functions and represent them, as outlined in the FET syllabus objectives at Grade 11 level. Your child would participate in a group chosen to do some content related activities. These activities and the follow-up clinical interviews would inform the study and would be expected to generate new knowledge for the field of Mathematics Education.

CONFIDENTIALITY

The name of your child will be kept completely confidential at all presentations and in the academic report writing about the study.

BENEFITS AND TIME INVOLVEMENT

The performance in the activities and possible follow-up clinical interviews form part of learning and, as such, would be communicated to your child's Mathematics teacher. Your child and his / her teacher would be active contributors to any facts or theories from the results of this study. The data collection will take place during the best time convenient for

the teacher, be it during learning time or soon after school, and will not prejudice any school activity.

SUBJECTS RIGHTS

All the learners who will participate will do so voluntarily and after parents / guardians have completed the consent forms. Individual privacy will be maintained for all published and written data of the study.

CONSENT

May you please complete, sign the attached consent form and return it with your child. This letter and attached form need to be produced as proof of your consent to recordings for this study only. Please feel free to contact me any time if you have questions or concerns about this research.

Thank you.

Mukono S. (Mr.)



MATHEMATICS TEACHER AND RESEARCHER

[Please fill in the attached concern form]

APPENDIX K: CONSENT FORM FOR PARENTS OR GUARDIANS

Please show your consent by **ticking**.

I give consent for my child to participate in the research study activities outlined above.

_____ Yes

_____ No

I give consent for video- audio recordings to be made during the follow-up clinical interviews for this research study. The camera would focus only on paper and writing hand, not on the face, of the participants.

_____ Yes

_____ No

I give consent for the video-audio recordings to be watched, listened to and transcribed for the research study.

_____ Yes

_____ No

Signature: _____

Date: _____

APPENDIX L: ETHICAL CLEARANCE TO CONTACT RESEARCH STUDY



APPENDIX M: CERTIFICATE OF EDITING

Hazel Cuthbertson

Language Services Practitioner

Member: Professional Editors' Group, South Africa

30 January 2015

Certificate of language editing

Title of PhD thesis: *Gr 11 mathematics learners' concept images & mathematical reasoning ON TRANSFORMATIONS of functions*

Candidate: Shadrick Mukono

Examining university: University of South Africa

To whom it may concern

This is to certify that the manuscript of the above-mentioned PhD thesis was edited by me, Hazel Cuthbertson, in terms of language usage and expression, in my capacity as a professional English-language editor.

I focused on language issues, including grammar, tenses, use of terminology, sentence construction, spelling conventions, and consistency of reference style. I inserted comments and suggestions, for the subsequent attention of the student in consultation with his supervisor.

I edited the thesis in draft form prior to the candidate's production of the final document.

Hazel Cuthbertson

hazel.cuthbertson@gmail.com

(Signature does not appear, for electronic privacy reasons)

APPENDIX N: SUPERVISOR'S APPROVAL STATEMENT

Supervisor's Statement on S. Mukono's Thesis

To whom it may concern

The turnit-in report shows that the overall similarity index is 9%. Except for two sources whose match index is 1%, all sources used and cited have a match index of <1%. I consider the thesis to have an acceptable degree of originality and that all sources consulted have been referenced plausibly and duly acknowledged.

Yours in Mathematics Education



Prof LD Megari
Supervisor

Own Plagiarism

APPENDIX O: TURN – IT – IN REPORT

→
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From PhD Thesis (Mukino)

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