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CHAPTER 1

BACKGROUND AND OVERVIEW OF THE STUDY

1.1 INTRODUCTION

Be it in the first, second or third world context, humans have in their environment objects in three dimensions (spatial objects); it may be a simple match-box, a pyramid or a twenty-storey building. Mathematics, or to be precise geometry, often demands the representation of such objects in two dimensions or vice versa. Onslow (1990:9) states that transferring between the second and third dimensions is important in many occupations such as architecture, drafting and design. The question arises as to the extent to which the school (be it in South Africa or elsewhere) prepares learners, through the teaching of geometry, for the above named occupations.

The National Council of Teachers of Mathematics (NCTM) published the Curriculum and Evaluation Standards for School Mathematics (1989) which stressed the importance of developing spatial sense in students. By spatial sense it meant the intuitive feel of one's environment and objects in it (Sgroi 1990:21). Spatial sense encompasses spatial visualization-perceiving spatial objects which includes having a mental picture of such objects (imagery).

Moses (1990:59) highlights the fact that the spatial development at the primary school was compromised for other disciplines of mathematics. According to her, the NCTM Commission on Standards for School Mathematics (1989) clearly states that teachers should devote less attention to complex paper-and-pencil computations and rote learning of rules. The time currently spent on these topics should instead be devoted to areas such as geometry and problem-solving. Students should be afforded the opportunity to visualize and represent geometric figures with special attention to developing spatial sense and learn to appropriate geometry as a means of describing the physical world. The author laments that elementary school mathematics textbooks contain few activities that deal with spatial sense (Moses 1990:59).

Wheatley (1990:10) concurs with the above assertion that a review of the United States school mathematics shows that rules, procedures, and analytical reasoning dominated the curriculum and that little attention was given to spatial visualization.

According to van Niekerk (1997:1) South Africa inherited its geometry from England at a time when the teaching of geometry in England was more conservative than in any country in the world. In the 1930s Euclid was followed more closely in England than elsewhere. Any informal approach to the teaching of geometry in high school was regarded as a waste of time. Theorems were introduced as early as possible. She states that van Zyl (1942) feels that the earlier introduction of informal geometry is desirable but the fact that primary school teachers are not trained to do this at universities militated against this.

The mathematics syllabi of several universities in South Africa as evidenced in their calendars reflect no geometry UDW (1998), RAU (1999), Vista (1996), Transkei (1997) and North (Qwaqwa campus (2000); an optional module in Euclidean and non-Euclidean geometry UFH (1996); analytical geometry in the first year Venda (1999) and analytical geometry in the first, second and third years Rhodes (2000). One geometry paper at a B.Sc Hons level UPE. University graduates are thus not prepared for the teaching of geometry in the schools. Even in the primary schools, the teaching of geometry especially, informal geometry, is underemphasized. Whilst the Eastern Cape Department of Education’s mathematics syllabus for Grade 1 to Standard 4 (1995) comprises geometric shapes such as triangles, rectangles and angles, there is no mention of three-dimensional shapes and their representation in two-dimensions, how individuals perceive objects in space (vision) and the location of objects in space. Such omissions deny learners further opportunities to be creative thinkers.

Could the high failure rate in matric paper 2, of which geometry forms the bulk of the paper, be a result of the status quo as described above or do teachers emphasize paper 1 at the expense of paper 2 (they handle paper 1 first and run out of time for paper 2)? These issues need to be researched, but are not the scope of the present study.

Freudenthal (1971:421) states that in many countries plane geometry started in the 7th grade, geometry in space in the 10th and 11th grades. Children who were good in plane geometry often failed in space geometry. Their spatial imagination had been killed by too much exclusive plane geometry. This emphasizes the fact that geometry teaching should start from three-dimensional shapes to two-dimensional shapes.

Not only was spatial visualization underemphasized in elementary schools, but the little geometry that learners were exposed to, was taught wrongly. According to Freudenthal (1971:418) geometry failed because it was taught in such a manner that its deductivity could not be reinvented by the learner but only imposed. For Freudenthal, starting with axioms and theorems is a wrong approach to the teaching of geometry (Streefland 1993:122). Indeed, starting with the fine, polished product such as axioms and theorems (the generalizations), denies learners the opportunity of finding out how such theorems or axioms were arrived at. The starting point should be from the child's everyday life experiences of spatial objects (his reality). Geometry should be related to the science of the physical space - of the space in which the child lives and moves, as an organization of the learner's spatial experiences (Freudenthal 1971:418).

Dina van Hiele-Geldof, the wife of P.M. van Hiele, and a student of Freudenthal's experiments emphasize the importance of the reinvention of geometry and not its imposition during instruction. She did not start with definitions of triangles, squares and rectangles but let children discover properties of these and hence be in a position to define them by themselves (re-invention / self-discovery). This is a more meaningful approach to the teaching of geometry. The importance of this approach is re-iterated by Presmeg (1989:22) in that seeing patterns in spatial objects, for example, various geometric shapes and lines of symmetry in the sufi mandala or Shri Yantra (complicated drawings) can empower learners with skills that are essential for solving problems in Euclidean geometry.

Hershkowitz, Parzysz and Van Dormolen (1996:165) state that visual education is important for learners to be able to effectively and correctly interact with shapes. Some learners who are not gifted at symbolic thinking need to visualize a given phenomenon (actually see or have a mental picture) in order to understand it. Visual ways of thinking and reasoning are thus important for such learners. Herschkowitz et al (1996:165) state that visual thinking and reasoning may be acquired through a well planned visual education but lament that visual education is often a neglected area in curricula.

The development of geometric thinking has been studied by a Netherlands husband-and-wife team Dina van Hiele-Geldof and Pierre van Hiele. They identified five levels of understanding: visualization; analysis; informal deduction; formal deduction and rigor. The first three identify thinking within the capability of elementary school learners whilst the last two involve thinking needed in high school and college level geometry (Usnick, Miller, Stonecipher, Speer and Brakies 1993:393). The levels are sequential and hierarchical; each having its own language. Teachers should identify the level at which the learner operate otherwise both might be at different wavelengths during instruction. It is through instruction and not biological maturity that a child progresses from one level to the next. This is accomplished through five instructional phases.

The Gauteng Department of Education (GDE) and Gauteng Institute for Curriculum Development (GCID) Draft Progress Map Document (1999) suggests a basis for a mathematics curriculum for Grades 1 to 9 which emphasizes the teaching of informal geometry as a basis for the learning of formal geometry.

Spatial development manifests itself in different activities. Two such activities are creativity and divergent thinking.

Vervalin (1971:59) defines creative thinking as:

> the process of bringing a problem before one's mind clearly (as by imaging, visualizing, supposing, musing, contemplating, etc.) and then originating or inventing an idea, concept, realization or picture along new or unconventional lines. It involves study and reflection rather than action.

In the context of this study, learners will solve geometrical problems. In this exercise, they are left to their own resources with the researcher acting only as a facilitator as the learners grapple with possible solutions to the problems. They (learners) should thus reflect on the problems, interpret them in various ways, articulate their understandings to others in a group in their own words and together negotiate meaning so as to arrive at solutions. These actions are manifestations of creativity on the learners' part.

Divergent thinking is typified by, inter alia, a flexibility to select and/or construct a large number of possible ideas, associations and implications (Hutchinson 1971:232). It includes critical thinking, analysis, synthesis, induction, deduction, lateral thinking, amongst others. Ennis (1987:12) defines critical thinking as reasonable reflective thinking that is aimed at deciding what to do or believe. When selecting or constructing associations and implications during the problem-solving process, one firstly reflects on the problem, one looks at alternative ways of solving a problem, so as to arrive at such associations and implications. To arrive at associations and implications of a problem condition, one analyzes such conditions, reasons inductively and deductively, has to synthesize the associations and implications so as to arrive at the solution to the problem. Lateral thinking implies not digging deeper but digging elsewhere in solving a problem i.e. searching for other solution pathways or implications of the problem.

This study attempts to establish the extent to which geometry education promotes creativity and divergent thinking in learners. If geometry has a vital role to play in the learners acquiring these skills, then denying learners geometry education deprives them of acquiring the said skills. These skills are transferable i.e. they can/are used outside the realm of the school. For individuals to be self-reliant, they have to be creative and divergent thinkers. Thus the school has an important role to play in empowering learners with these life skills and no opportunity for realizing this should be spared.

1.2 PROBLEM STATEMENT

The teaching of geometry, in the past, especially in the primary school, has not been coupled with activities of creativity and divergent thinking. Learners were merely given theorems and axioms based on geometric shapes and not afforded the opportunity to discover these for themselves, thereby not re-inventing the deductivity of geometry. This has not only negatively impacted on the internalization of geometry concepts but has also fostered in learners and educators a negative attitude towards geometry as geometry was presented as a body of knowledge to be learnt of, in most cases, by rote.

Knowledge should emanate from the solution of problems (Bereiter 1992:337-340). The problem-centred approach to teaching and learning encompasses creativity and divergent thinking. Learning geometry using activities of creativity and divergent thinking should lead to a deeper conceptual understanding. The study attempts to establish the extent to which these activities i.e. providing rich learning experiences in the context of problem solving (the problem-centred approach) promotes the internalization of geometry concepts. In the context of this study creativity and divergent thinking are not ends in themselves but means towards an end viz. conceptual understanding

1.3 AIMS OF THE RESEARCH

1.3.1 Aim of the Study

The aims of the research is to establish the extent to which emphasis on creativity and divergent thinking in the teaching and learning of geometry will enhance the internalization of geometry concepts.

1.3.2 Objectives of the Study

The objectives that are framing the research project are:

- The concepts creativity and divergent thinking will be analysed in the broader educational context.
- The nature of geometry teaching and learning will be investigated through evaluating effective and applicable theories and models.
- The implementation of the concepts creativity and divergent thinking in the

teaching and learning of geometry will be done via the problem centred approach.

- The gathering of appropriate and applicable data in an adequate empirical investigation.
- The description of approximate methods and tools for gathering and presenting these data.
  - Establishing the extent to which emphasis on creativity and divergent thinking enhance the conceptualization of geometry concepts.

1.4 RESEARCH DESIGN

The design is based on a review of the literature to determine the elements of creativity and divergent thinking in geometry education as well as the empirical study which is divided into quantitative and qualitative research.

1.4.1 Literature study

The literature is reviewed in Section 5.2.1

1.4.2 Empirical study

The research was done in the form of a diagnostic teaching experiment of Grade 7 learners at Attwell Madala Junior Secondary School (which merged with Northcrest Senior Secondary School to form Northcrest High School in2001 with the understanding that Grade 7 would be phased out in 2002).

There was a pre-test to find out the students' knowledge of and their skills on the various topics that will be dealt with. The post-tests determined students progress as a result of engaging in various topis to be handled (the quantitative aspect). The researcher was the teacher.

In the qualitative research, the researcher basically introduced concepts and made illustrations of whatever was dealt with in each training episode. There were about fifty episodes of thirty five minutes each. There will be a description of the creativity and divergent thinking as they manifest themselves in student problem-solving. The observation method using direct observation was used during the problem-solving process. The problem-centred approach was used wherein learners solve geometrical problems appearing in worksheets and the researcher only acted as a guide or facilitator. Learners were provided with materials required by the activities in each worksheet. Learners will be encouraged to discuss and write as much as possible. In addition, individual interviews using an audio-tape were conducted to find out learner attitudes towards the subject, what they enjoyed most and the difficulties they experienced with the various activities in the work sheets. Two questionnaires to find out learner attitudes, knowledge after a teaching episode and difficulties were administered.

Formative assessment was done on a daily basis to evaluate the problem-solving process and on that basis plan for the next session.

1.5 SIGNIFICANCE OF THE STUDY

The Outcomes Based Education (OBE) currently being implemented in South African schools aims at developing individuals who can think, are creative and are problem solvers. Thinking (divergent thinking) and creativity are inherent in problem solving. For individuals to have these attributes, they must acquire what Cuoco, Mark and Goldenberg (1996:376-378) refer to as ‘habits of mind’. Some of these ‘habits of mind’ can be classified thus:

Divergent thinking

Learners should be

- conjecturers
- experimenters (e.g. try various heuristics during problem solving)
- describers (e.g. give precise descriptions of the steps in a process)
- tinkerers (e.g. putting ideas apart and putting them back again or differently)
- visualizers (e.g. doing things in one’s head that, in the right situation, could be done with one’s eyes)
- pattern sniffers (from patterns generalizations can be made)

Creativity

Learners should be

- creators
- inventors (invent notation if ordinary language is too cumbersome, invent mathematics for utilitarian purposes and for fun)
- describers (e.g. develop the habit of writing down their thoughts, results, conjectures, arguments, proofs and opinions about the mathematics they do)

Goldenberg, Cuoco and Mark (1997:1-3) state that geometry is the ideal vehicle for building the ‘habits of mind’. In geometry learners can search for invariants (patterns) and use these to arrive at generalizations, experiment, analyze, synthesize, visualize, describe and give proofs for their conjectures. Unfortunately geometry is a neglected field. Curricula tend to present a visually impoverished mathematics (Goldenberg et al 1997:4).

In OBE terminology the above ‘habits of mind’ are collectively referred to as knowledge, skills, values and attitudes. OBE posits that learners must acquire skills, knowledge, and values and their attitudes must be positively influenced through instruction. Through the teaching of informal geometry (this pertains to, inter alia, the use of geometric shapes, for example, three-dimensional objects in the learners’ environment as a starting point for them to discover the properties of such shapes and not to start with theorems and axioms pertaining to such shapes as a point of departure in geometric instruction), the acquired knowledge, the skills to be acquired include the ability to manipulate spatial objects, use polygons in tiling/tessellations and solve puzzles; the learners’ affectivity is developed as he sees the beautiful patterns in a tessellation (aesthetic value) and he tends to appreciate the beauty of geometry and spatial objects (attitude). Informal geometry is a perfect springboard for the implementation of OBE. Goldenberg (1999:193) states that the curriculum should include content that develops the ‘habits of mind’ in learners.
In his article titled ‘Creative and productive thinking in the classroom’, Hutchinson (1971:229) states that all the abilities of all children must be developed fully if education is to do its job. Recent research, however, points to the fact that some important abilities are not being developed by current classroom methods. Thinking and creativity are some of the neglected abilities in the classroom.

The study aims at highlighting the contribution that can be made by using activities of creativity and divergent thinking in the internalization of geometry concepts.

1.6 TERMINOLOGY

1.6.1 Geometry education

According to Freudenthal (1971:413-414) mathematics is an activity of solving problems. Geometry is thus an activity of solving problems concerning shapes, vision and location.

Geometry education concerns itself with philosophies, principles and methodologies in the teaching of geometry. In the school situation the emphasis is on the teaching of geometric concepts.

Shape

Learners recognise different geometric shapes and patterns in the world around us. geometric shapes are embedded in spatial objects and create an opportunity to move from two-dimensional to 3-dimensional perceptions and vice versa.

Vision

Projections of reality from various vantage points are an important part of geometry.

Location

Learners have to be exposed to different systems for determining position and how to use them appropriately.

1.6.2 Creativity

In an attempt to define creativity we look at personality traits of creative individuals. Such individuals are always thinking; pay attention to detail; are always willing to listen to others' opinions; are critical of their work; are analytic; see patterns in phenomena other would take for granted; are original; have adaptive flexibility, spontaneous flexibility, associational fluency, expressional fluency, word fluency, the capacity to be puzzled, the ability to be curious; are motivated, confident, intellectually persistent and have an intense aesthetic and moral commitment to work (Taylor & Holland 1964:19-25).

1.6.3 Divergent thinking

Cangelosi (1996:166) defines divergent thinking as reasoning that practices unanticipated and unusual responses. This includes cognitive processes such as critical thinking, analysis, synthesis, interpretation, investigating, conjecturing, induction, deduction, comparing, classifying, generalizing and evaluation. Such thinking enhances creativity in learners.

1.6.4 Problem solving

In the problem-solving process, one is confronted with a problem of which there is blockage. Problem solving includes strategies such as working backwards, making a drawing, create your own problem, think of a similar problem that was solved successfully in attempting to solve a problem. Problem solving is teacher-centred in the sense that the teacher can direct a learner at the said strategies.

1.6.5 The problem-centred approach

In the problem-centred approach, instruction begins with problems. It is from the solution of problems that the learner acquires knowledge. The problem-centred approach is different from acquiring knowledge and applying it . Problem solving is a means to acquiring knowledge.

By problematizing what he studies, the learner is made to be more independent and responsible for his learning. The learner interprets the problem conditions in the light of his repertoire of experiences (knowledge and strategies previously assimilated). The teacher provides the necessary scaffolding during this process.

1.7 PROGRESS OF THE INVESTIGATION

Chapter 2 deals with creativity and divergent thinking. The objectives for the chapter are to:

- Analyze creativity
- Analyze divergent thinking

Chapter 3 deals with perspectives on geometry education. The objectives of the chapter are to:

- Expose the nature of geometry teaching and learning.

Chapter 4 deals with creativity and divergent thinking in geometry teaching and learning in a problem-centred context. The chapter has the following objectives:

- Describe elements of creativity and divergent thinking in the problem-centred teaching and learning of geometry.

Chapter 5 outlines the methodology to be followed in conducting the research. The objective is to:

- Describe appropriate methods and tools to be used in the gathering and processing of data.

Chapter 6 gives the research findings and an analysis thereof. The objective is to:

- Establish the extent to which creativity and divergent thinking enhance the conceptualization of geometry concepts.

Chapter 7 gives a summary, conclusions and recommendations of the study.

CHAPTER 2

CREATIVITY AND DIVERGENT THINKING

2.1 INTRODUCTION

In this chapter the concepts of creativity and divergent thinking are analyzed. Creativity is analyzed in three ways. Firstly, some definitions from the literature are reviewed. Secondly, the creative process is described. Thirdly, creativity in the problem solving process is described. In analyzing divergent thinking, the cognitive processes such as defining, explaining, analyzing, deducing, investigating, reflecting and comparing are explored and their role in geometrical problem solving explained.

Creativity is a complex concept and as such there are many definitions of creativity. Taylor (1988:118) states that there are some fifty to sixty definitions and that the list is expanding every day. Torrance (1988:43) concurs with this view in that creativity defies precise definition and is almost infinite. In an attempt to make explicit some of these definitions, characteristics of some of the world's renowned creative individuals such as Leonardo da Vinci, Piero di Cosimo, Pasteur and Einstein are mentioned. Personality and motivational traits such as originality, redefinition, adaptive flexibility, motivation, moral commitment in one's work, confidence and resourcefulness are mentioned to elucidate the concept of creativity. Characteristics such as the tendency to strive for more comprehensive answers, looking at a problem from different viewpoints and find different solutions to the problem and intellectual thoroughness are used to shed some light on the concept of creativity.

Creativity is a process and not an event. It occurs in stages such as preparation, incubation, illumination and verification.

The chapter also explores conditions under which creativity takes place. What are the prerequisites for creativity? When is act or product referred to as creative? Is creativity an attribute for the intelligent people only? In a classroom situation when is a learner's performance considered creative? The chapter attempts to answer these questions.

2.2 THE CONCEPT OF CREATIVITY

The concept of creativity is explored under the following main headings viz. definitions of creativity, the creative process and creativity in the problem-solving process.

2.2.1 Definitions of creativity

Csikszentmihalyi (1996:27-28) defines creativity on the basis of a systems model that is based on three components viz. the domain which consists of symbolic rules and procedures, for example, mathematics; the field which includes individuals who act as gatekeepers to the domain. How novel a product is depends on the audience/field prior experiences (Lubart 1994:290), and lastly the person who changes the domain through his innovation. To be creative in mathematics, one must first learn the rules in mathematics and one's ability to have made a difference in mathematics must be recognized by teachers and scholars (in the field).

Shapiro (1968:15-16) defines creativity in a similar way. Creativity is perceived as a potential capacity by means of which an individual (**person**) may produce something original that serves to fill the gap in (changes) a particular field of human endeavour (**domain**). Furthermore creativity is a problem solving process.

Creativity is essentially the development or modification of existing relationships into new ones. Do we see a given concept in a new light? Westcott (1978:360-361) defines creativity in mathematics as simply the production of new relationships and the discovery of new applications of mathematical ideas.

According to Lubart (1994:290-291) the creativity of a product depends, inter alia, on its novelty and appropriateness (the central features) i.e. is it an innovation?, does it address a need?, is it sensible? and is it useful? is the product of quality?, is it important and does it have a production history? (peripheral features) i.e. has the individual worked hard enough and not merely followed rules to produce this qualitative important product? These peripheral features can enhance or diminish the basic creativity of a product.

In the classroom situation, a learner may see the meaning of a given concept in a new light. For such a learner this is a new and unique experience for him. This, according to Parnes (1975:255) is an "aha" experience for the learner - the fresh and relevant association of thoughts, facts and ideas into a new configuration which is pleasing. What the learner has just experienced may not be new to his educator. It is novel to the learner but is already known to other (Lubart 1994:290). This is a case of a learner re-inventing the subject matter. In a mathematics classroom it would be a case of a learner thinking like a mathematician does which is how learners should be taught mathematics and what a curriculum should provide for (Cuoco et al 1996:376).

Hayes (1989:135) states that a creative act must be seen as original or novel and valuable or interesting and must reflect on the mind of the creator.

According to Weisberg (1994:160) creativity does not occur in a vacuum. The creative individual must see some inadequacy in the initial product (a gap in knowledge) and must know how to overcome this inadequacy.

An analysis of creativity should take into account four interrelated factors viz.

- physical reality
- mental reality (one's knowledge / domain)
- social stratification - social groups who are influenced by the discovery, and
- culture (Lamb 1991:9)

These factors influence what can be discovered/created and what is worth discovering.

Lamb (1991:16) lists Briskman's criteria for determining a creative product. According to these criteria, a creative product must be a novel product yet relative to the background knowledge entailed in the previous products - an improvement of previous products; it must solve a problem that could not be addressed by the previous products and it must meet standards which are part of the prevailing background.

There are common features on the above definitions of creativity. They emphasize a recombination of existing ideas into something new. Taylor (1988:118) refers to these as 'Gestalt' or 'Perception' type definitions. The individual expresses himself in a manner which is unique to him as in the manner in which a learner re-invents mathematics. These are 'aesthetic', or 'expressive' definitions and there are 'Solution Thinking' definitions i.e. the mind sees the relationship between two items in such a way as to generate a third item (problem solving).

2.2.1.1 The nature of the creative person

Looking at a few well known creative individuals, personality and motivational traits as well as characteristics of creativity will render us in a better position to understand creativity. It is these qualities that the study aims at identifying in learners during geometric problem-solving.

(a) **Creative individuals**

According to Beck (1979:29-33), Leonardo da Vinci, the famous painter, was always thinking, paid attention to detail, was willing to listen to others' opinions, was critical of his works and analyzed these. A creative individual always sees the potential for inventiveness in a given situation. Piero di Cosimo, a younger contemporary of Leonardo would study a wall where sick persons habitually spat. He would envisage battles of horses, fantastic cities and enormous landscapes (Beck 1979:30). Einstein, the famous physicist of the relativity theory $E = mc^2$, was unable to understand the obvious - what other men accepted as known what was really unknown (Taylor & Holland 1964:29). It is from this unwillingness to accept the obvious that deeper implications of phenomena are unearthed.

(b) **Personality and motivational traits of creativity**

A creative person does not have to be a knowledgeable person. Taylor and Holland (1964:17-18) state that Pasteur, though ignorant about silkworms was able to solve a problem concerning silkworms. Independence, the ability to see patterns and sense problems are qualities an individual must possess in geometric problem solving. Guilford listed the following intellectual characteristics as most likely to be valid measures of creative talent: originality, redefinition, adaptive flexibility, spontaneous flexibility, associational fluency, expressional fluence, word fluence, ideational fluency, elaboration and probably some evaluation factors (Taylor & Holland 1964:19). Other creativity traits include the capacity to be puzzled, the ability to know when one does not know and the ability to be curious. These qualities can lead one to making original contributions.

Being motivated, confident, intellectually persistent and initiative, having an intense aesthetic and moral commitment in one's work and being non-conforming and resourceful are motivational traits to creativity (Taylor & Holland 1964:24-25).

Taylor and Holland (1964:28-29) state that:

> whereas the typical person focuses on adjusting to his environment, the creative individual tries to adjust the environment to him, to improve it in ways that he feels are urgently needed.

(c) **Characteristics of creativity**

In addition to the traits mentioned in S.2.2.1, creative individuals possess the following characteristics:

- the tendency to strive for more comprehensive answers. They look at a problem from different viewpoints and find different solutions to the problem?
- intellectual thoroughness. Every step towards the solution of a problem is justified and there is no ambiguity as to their inferences
- striving for original and more workable products
- tendency to resist idea reduction and its opposite pole, the willingness to reduce ideas. On the one hand, if an idea has no bearing to the problem at hand it must be quickly eliminated. On the other hand, creative individuals are open to several solution pathways to a problem and investigate the merits of each, thereby eliminating the possibility of overlooking important implications of the

problem situation

- the abilities to form and test hunches. Making conjectures and validating such conjectures is important in geometry problem solving (Taylor & Holland 1964:21-23).

Intelligence may be a necessary but not a sufficient condition for creativity. Hayes (1989:143) proposes that creative performance has its origin in the motivation of the creative person and not in innate cognitive abilities. He further states that this motivation leads a person to acquire a lot of knowledge that then becomes critical to creative performance.

Cuoco et al (1996:376) state that a mathematics curriculum should let learners acquire the ‘habits of mind’ i.e. they should think like mathematicians do; they should acquire ways in which mathematicians think about problems. In this way learners become creative in the manner in which they learn mathematics. ‘Habits of mind’ entail learners experiencing the process of creating, inventing, conjecturing, experimenting, being pattern sniffers and describers. Acquiring the habits of mind lets learners experience what goes on before new results are polished and presented (habits of mind give learners the genuine research experience).

In the process of learning mathematics, learners should make conjectures/guesses and experiment to validate their conjectures. They should see patterns. This is particularly relevant to geometry learning where learners are expected to find the relationship between various geometric shapes and within a given shape. The authors also mention that learners should be tinkerers - putting ideas apart and putting them back together or differently, for example, finding the different nets of a cube gives learners the opportunity to visualize a cube from different perspectives. Learners should be inventors (develop the habit of inventing mathematics for utilitarian purposes and for fun). Learners should also be visualizers - do things in their minds which rightly should be done with their eyes; for example, finding the nets of a cube without using the concrete object (cube). Lastly learners should also give precise descriptions of the various steps they have followed during, for example, the problem solving process.

2.2.2 The creative process

Shapiro (1968:15) essentially refers to the creative thinking process as the association of concepts such that new combinations are formed and that it is a problem solving process. Weisberg (1994:150) takes particular problems and shows the role of creative thinking in solving them. In explaining why individuals produced or failed to produce a solution to Duncker's candle problem Weisberg (1994:151) states that all attempts to solve problems are based on past experience, novel solutions to a problem came about in an evolution (process) and that creative thinking is a basic human capacity.

2.2.2.1 The phases of creativity

The product of the creative act i.e. the masterpiece, is a culmination of a lot of work, the tip of an iceberg. This work occurs in phases. According to Fourie et al (1991:149), the creative act has four phases namely, preparation, incubation, illumination and verification. Ford and Usnick (1999:20) state that the four phases are analogous to the four phases of Polya's (1973) model of mathematical problem-solving, viz.

- phase 1 - understand the problem
- phase 2 - devise a plan
- phase 3 - carry out the plan

- phase 4 - look back

These phases have an impact on geometric problem solving.

1. **Understand the problem/preparation**

During this phase, there is a thorough examination of possibilities of the idea. The person accesses information pertinent to the problem. He reads, makes notes, discusses, puts questions, collects information, investigates, tries several solutions and compares them (Fourie et al 1991:149). The individual also analyses the problem, identifies the knowns and the unknowns from the given data and draws a diagram(s) to illustrate the given information.

2. **Devise a plan/incubation**

The conscious preparation in 1 is followed by subconscious activity. Relaxation sets in, wrong ideas and incorrect solutions disappear. During this stage, students are encouraged to think about a solution strategy. Ford and Usnick (1999:21-22) state that in order to encourage students in this vital thinking step, they should be made to write the solution strategy down, but not to solve it yet, they should make inferences from the given data and see patterns in the given geometric shapes on the basis of theorems and axioms learned previously. Weisberg (1994)149-150) highlights the importance of this stage in creativity through Coleridge's and Kekule's experiences. An entire poem appeared full-blown without conscious work on Coleridge's part, also in a dream, the ring-like structure of benzene appeared to Kekule in the form of a snake seizing hold of its own tail.

3. **Carry out the plan/illumination**

This moment is the climax of the act of creativity. It is the moment of discovery. Suddenly the person understands the idea or the solution to the problem (Fourie et al 1991:149). Students now implement their plan to solve the problem.

4. **Look back/verification**

The results of the previous phase are evaluated and verified. Does the solution fit the problem conditions? i.e. does it make sense? Are there other ways of arriving at the same result? Repeating the solution with an alternative approach encourages diverse and flexible thinking (Ford & Usnick 1999:22).

Cangelosi (1996:159-160) presents an alternative model to the above. Problem solving, of which creativity is inherent, depends on deductive reasoning and it is through an application objective that deductive reasoning is attained. This is achieved through a four-stage lesson:

- initial problem confrontation and analysis
- subsequent problem confrontation and analysis
- rule articulate, and
- extension and subsequent lessons (Cangelosi 1996:159).

Stage 1

Learners are given a pair of similar problems except that the content to be applied in the solution of these problems is applicable to the solution of one problem and not the other. Learners have to use deductive reasoning to be able to realise this.

Stage 2

In addition to learners using deductive reasoning to solving the pair of problems, they analyze solutions to additional pairs of problems. They should explain why the content works for one problem and not the other.

Stage 3

Learners formulate rules that are applicable to the solution of the problem. They link these rules to relevant rules they have used in solving previous problems and in this way come to make generalizations of how such problems can be solved.

Stage 4

The content of stage 3 is used as a starting point for subsequent content. In other words, learners are invited to reflect on how high level problems could be resolved or partially solved on the basis of their present results.

From the above, it is evident that creativity and problem solving go hand in hand. Ford and Usnick (1999:20) state that teachers could either develop problem-solving through creativity or creativity through problem solving. The latter approach is used in this study.

2.2.2.2 Conditions of creativity

According to Hennessey and Amabile (1988:27) creativity will be maintained when intrinsic motivation is maintained and it will be undermined when intrinsic motivation is undermined. An individual must have genuine love for what he is doing. In this way he will be willing to spend a lot of time in what he is doing and will thus be knowledgeable in his field and increase the likelihood of him arriving at some novel solution to a problem. Weisberg (1998:172-173) states that one's creativity can be increased if one is motivated, is committed (spends long periods of time working in his field), is knowledgeable about the old (domain) and is willing to take risks.

2.2.2.3 A cognitive model of creativity

Ambruster (1989:177) states that the creative process involves the acquisition of knowledge and skills (domain), the transformation of the knowledge into new forms and the rendering of these forms into a shareable product. Each of these stages involves cognition. Individuals also have to reflect on their thinking during these stages (meta-cognition). This includes the setting of goals or subgoals, planning the next cognitive move, monitoring and evaluating the effectiveness of the cognitive strategies and revising the cognitive strategies.

2.2.2.4 Cognitive/thinking techniques and processes of creativity

Traits such as being independent, confident, self-assertive, radical, non-conforming and imaginative are necessary but not sufficient for creativity on the part of an individual. Individuals have to be trained to possess creative attitudes, abilities and thinking techniques (Davis, 1971:262). For the purpose of this study the researcher concentrated on thinking techniques. Learners should be given opportunities (content) in the teaching-learning situation to acquire these skills.

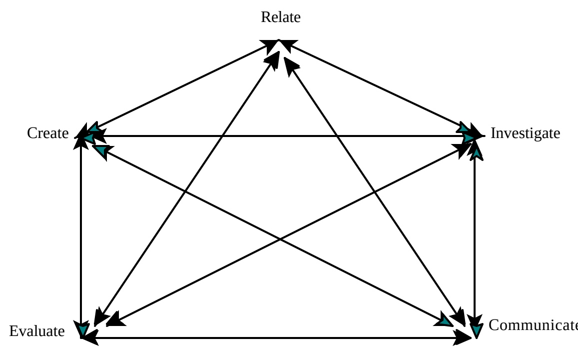
Davis (1971:263) lists:

1. Attribute/listing
2. Morphological-synthesis
3. Checklisting and
4. Synectics as thinking techniques that enhance creativity.

By attribute-listing is meant the listing of properties of an object with a view to changing or improving such properties. This exercise serves a dual purpose: firstly, it sensitizes an individual to the properties of the structure and secondly it provides an opportunity for innovation on the part of the individual (Davis 1971:264).

The Morphological-synthesis technique:

Students first identify two or more important attributes (e.g. colour, shape) of a problem and list specific values (e.g. red, blue, orange, square, triangular, round) for each. They then examine all possible combinations, utilizing one value of each attribute (Davis, 1971:264). Some of the following combinations could emerge: red-triangular, red-round, red-square, blue-triangular, blue-round.



Checklisting

Students consider each item on a prepared list as a possible source of innovation in respect to a given problem (Davis 1971:265).

The synectic approach

This procedure emphasizes the use of analogies in, for example, problem solving. In solving a problem, one could think of a similar problem solved previously. This will be explained in chapter four.

Kokol-Voljv and Sheffield (1999:32) suggest the following as a model that could be creative in exploring the depth and complexities of geometric problems (fig. 1).

Fig. 1 A model for developing mathematical creativity. Each process influences and is influenced by others.

In explaining how the above model works, Sheffield (1999:51) states that in solving a problem, a student might start off by relating ideas about solving that problem to previous problems that have been solved, investigate those ideas i.e. see if they work by solving the problem using such, create new problems to work on (on the basis of the solution of the problem at hand), evaluate solutions (a step that encourages critical thinking) and communicate the results.

2.3 DIVERGENT THINKING

2.3.1 Orientation

As has already been mentioned, divergent thinking includes lateral thinking, deducing, inducing, identifying, synthesizing, analyzing, evaluating, problem solving, critical thinking, differentiating, amongst others.

The above named cognitive processes are essential for effective problem solving. Learners should be empowered with these skills to be good problem solvers. Problem solving using the said skills reveals divergent thinking on the part of learners as per the definition of divergent thinking (as proposed above). The following exposition is a description of some of the elements of divergent thinking.

2.3.2 Teaching for thinking

In his article: Why teach thinking? Nickerson(1987:414-430) answers, inter alia, the following questions:

1. What constitutes good thinking?
2. Is the teaching of thinking necessary?

In addressing the first question, Nickerson's response characterizes attributes of good thinking, including attributes such as using evidence skillfully, i.e. judging, critical thinking, organizing thoughts and articulating them concisely and coherently and distinguishing between logically valid and invalid inference. These actions are manifestations of thinking. In answering the second question, Nickerson states that students do not acquire high-level thinking skills easily or well on their own or through ordinary instruction and hence would benefit from direct instruction/teaching for thinking.

Teaching for thinking as described above means the creation of situations in the geometry classroom such that thinking will prosper. In this endeavour Nickerson (1994:414-415) states that learners should be exposed to domain - specific knowledge. One cannot think deeply about geometry unless one knows something about geometry; normative principles of reasoning - learners should know when to conclude that if A is the case then B (the necessary condition) or A if and only if B (the necessary and sufficient condition); informal principles of thought such as algorithms and heuristics;

metacognitive knowledge (see S. 3.2.6) and to knowledge, skills and strategies required for thinking.

2.3.3 A taxonomy of critical thinking dispositions and abilities

As mentioned earlier, Ennis (1987:10) defines critical thinking as “reasonable reflective thinking that is focused on deciding what to believe or do”. This definition is broad and encompasses thinking skills in general. These skills should contribute towards an individual deciding what to believe or do. In the context of this study this is pertaining to deciding on which skills to use, why those skills and how to use such skills in geometry problem solving. The ability to solve problems, acquired in this way, reveals divergent thinking.

An individual must acquire abilities that are necessary for critical thinking as defined above. Ennis (1987:12-15) gives a list of such abilities. Some of these abilities have been modified to suit the present study.

1. **Focusing** on a question/problem

- this entails **identifying** or formulating a question and criteria for **judging** possible answers

2. **Analyzing** a problem

or **classifying, differentiating, integrating, sorting**

- identifying the knowns and the unknowns/the givens and the required
- seeing similarities and differences
- seeing the distinctive parts of the whole and the role of each part to the total picture

3. **Reflecting** on the situation

- questions such as what? why? and how? go a long way towards reflecting on a situation

4. **Deductive** reasoning

- given a statement/generalisation/theorem, does it suit particular instances? i.e. **application** of theorems or **proving theorems** or **investigating** to validate

5. **Inductive** reasoning

- from several particular instances, can a generalisation be made?

investigating them and arriving at a conclusion?

6. **Defining** terms

- students should define (qualities, characteristics) of geometric figures e.g. rhombus, square, cube, etc. and define concepts e.g. congruence and similarity.

7. Deciding on an action/**problem solving**

- understanding the problem
- devise a plan
- carry out the plan
- look back (Polya, 1973)

8. Interaction with others/**communicating**

- learners share views on solution of problem
- share their strategies towards problem solving

see social constructivism mentioned in S. 3.6

2.3.4 Developing reasoning skills

Quellmalz (1987:94-95) presents a taxonomy of reasoning skills and her HOT (higher-order-thinking) method for teaching them. She proposes that the main aspects of higher order thinking are:

1. The **identification** of the task or problem type.
2. Definition and clarification of essential elements and terms (**analysis**).
3. **Judgement** and connection (**synthesis**) of relevant information, and

procedures for drawing conclusions (**generalising**) and/or **solving problems**.

She shows how these aspects of thinking are used in three subjects - natural science, social science and literature and proposes a process model of how these thinking skills can be combined. Krulik and Rudnick (1993:334-335) posit another model of higher order thinking/reasoning. The model is based on **critical thinking** and **creative thinking**.

Critical thinking examines, relates and evaluates all aspects of a situation or a problem. It encompasses skills such as **focusing** on parts of a problem, **gathering** information pertinent to the problem, **remembering** and **associating** such information to the answer to the present problem thereby determining the reasonableness of the answer.

Creative thinking includes skills such as **synthesizing** ideas, **generating** original and effective ideas and **applying** these so as to produce a complex and workable product.

The model can be diagrammatically represented as follows:

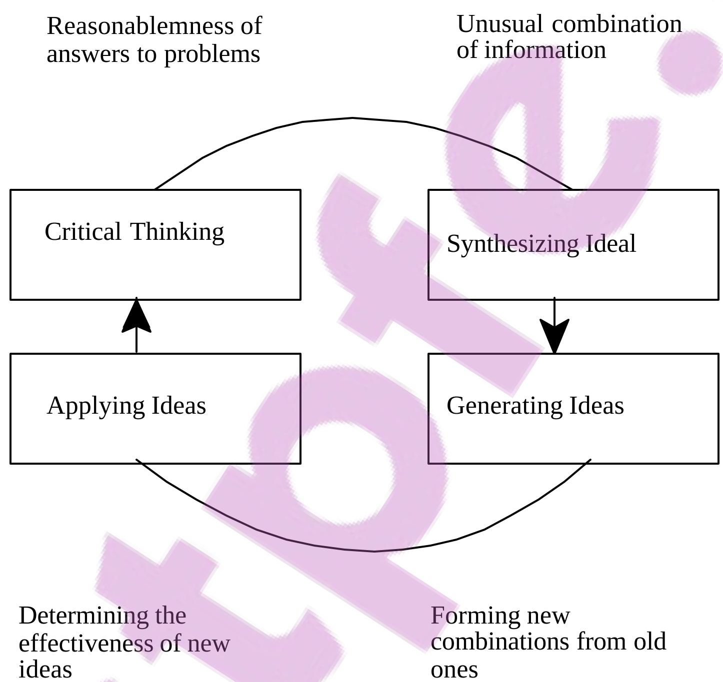


Fig. 2 A model for creative thinking

2.3.5 Cognitive processes

The various cognitive processes to be mentioned hereunder, including those in S. 2.3.3 and 2.3.4 are manifestations of divergent thinking. This is so as a result of the definition of divergent thinking in this study.

- Comparing

  In letting students compare objects, e.g. a collection of pyramids, prisms and cones, etc., they are given the opportunity to think as they observe the similarities and differences that those objects possess (Raths, Wassermann, Jonas & Rothstein, 1986:6). According to the authors, it is interesting to compare the students' comparisons if they were given the same objects. They have the advantage of noting the likenesses and/or differences they overlooked. Comparing involves analysis.

- Observing

  Behind the notion of observing is the act of perceiving involving, amongst other senses, seeing and noting what is seen. In observing geometric structures, their characteristics/properties are noted. One thus discovers information, reflects/thinks on the meaningfulness of the information thus discovered.

- Summarizing

  Summarizing requires putting into condensed form content that has been presented. This requires consciousness without omitting important points. This activity requires thought. In summarizing the properties of geometric figures, one may observe their properties in terms of their likeness and difference (compares) then puts such likenesses or differences concisely. The activities of comparing, observing and summarizing compliment one another.

- Classifying/Sorting

  Classifying or sorting entails putting objects into groups according to some principle (Raths et al, 1986:9). In classifying a collection of polyhedrons, those having features in common (comparison) are put together e.g. spheres are put separate from prisms or pyramids. This classification is arrived at on the basis of shape.

  Raths et al (1986:10) state that classifying involves analysis and synthesis. In analysing, learners look for similarities and differences and those objects having common features are put together (synthesis).

- Interpreting

  Raths et al (1986:10) state that "... interpreting is a process of putting meaning into and taking meaning out of our experiences". This entails making inferences and qualifying such meanings. Students could, for example, be asked the meaning a given three-dimensional drawing has for them. In this exercise, students can draw the top view, front view or side view.

- Hypothesizing/Conjecturing

  A hypothesis is a tentative or provisional answer to a problem. It represents a guess. This imaginative projection to a possible solution to a problem is thought-provoking. Hypotheses need verification/validating.

- Applying Facts and Principles in New Situations

  Principles, rules, generalisations or laws that students have acquired previously are used to solve problems or applied in new situations. Students use theorems in solving riders in geometry. This is the deductive approach to teaching mathematics. In this type of thinking, the individual is being tested to see if facts and principles learned in one context can be used in another.

! Investigating/Designing Projects

A project is a large-scale assignment involving many different activities. Students may be required to design apparatus (improvised) for geometry instruction. Significant projects involve thinking operatons such as comparing, summarizing, observing, interpreting, applying principles, imagining and evaluating.

Krulik and Rudnick (1993:334) are of the opinion that the last step in Polya model of mathematical problems solving i.e. looking back, offers fertile ground for creative and critical thinking and the development of higher order thinking skills in learners. Sheffield (1999:51) shares this view in stating that when the problem is solved, the creativity has just begun. Students are satisfied with getting an answer to a problem and not reflecting on it any further. In this way they miss the excitement of thinking deeply about mathematical ideas and discovering new concepts.

To overcome this shortcoming, thereby allowing for creative and critical thinking in students, Krulik and Rudnick (1993:334-335) state that at least five steps should occur:

1. Test the reasonableness and practicality of the answer - critical thinking.
2. Write a summary paragraph about the problem and its solution (what the student did to solve the problem) - summarizing; metacognition.
3. Find other solutions - lateral thinking.
4. Change the conditions of the problem - ‘what if’ - creativity.
5. Extend the problem to a mathematical formula, concept or generalization

- induction.

2.3.6 Metacognition

Learners should be encouraged to reflect on their own thinking in the didactic situation. In geometrical problem solving they should reflect on how they arrived at the strategies they used for the solution of a problem and report about these. As they discuss these amongst themselves, they can be in a position to see/detect flaws in their reasoning and refine their reasoning. Ambruster (1989:177) states that metacognition includes both **knowledge** and **control** that the individuals have over their own cognitive processes. Metacognitive knowledge includes knowledge about how to monitor, control and evaluate one's performance on cognitive tasks with the aim of revising/refining one's cognitive strategies for effectiveness.

2.3.7 Language and thought

Language plays a critical role in regulating cognitive activity - the vocabulary an individual has at his disposal affects what he thinks about (Ellis & Siegler 1994:343; Stein 1974:91). The more concepts the learner is conversant with the greater the likelihood that he can find the relationships amongst them. In problem solving, a learner's creativity may be blocked if he does not possess a language with words for concepts necessary for the solution of a problem (Stein 1974:91).

2.4 CREATIVITY, DIVERGENT THINKING AND THE PROBLEM-CENTRED APPROACH TO TEACHING AND LEARNING

In the problem-centred approach, instruction begins with the solution of problems, and for this purpose learners have to be creative and divergent thinkers. In solving the problem, they have to use strategies previously acquired, e.g. think of a similar previous problem that was solved successfully and use strategies used in that problem on the present problem (problem solving). Once the problem has been solved, the learner reflects on the meaningfulness of the solution (critical thinking). Problem solving and critical thinking are elements of creativity and divergent thinking. When the problem is solved, the creativity has just begun (Sheffield 1999). From the solution of the problem, the learner looks for new relationships e.g. changes the problem and solves the new problem. These relationships and understandings are what is left after the problem has been solved. They constitute knowledge for the participant (Hiebert, Carpenter, Fennema, Fuson, Human, Murray, Oliver & Wearne 1996).

2.5 CONCLUSION

The definitions of creativity reviewed above seem to indicate that the creativity of a product depends on its novelty and usefulness (Lubart 1994, Hayes 1989). An individual may also improve an already existing product as he has seen some inadequacy in it. In this manner the final product is more appropriate and this also indicates creativity on the individual who has brought about the change. In the first deduction from the definitions of creativity, the creative individual comes with an entirely new product altogether while in the second case an individual improves an existing product. In the latter case the individual must first be knowledgeable about the product in order to improve it (domain) and the improved product should be recognised as creative by experts who are knowledgeable about the domain (field) (Csikszentmihalyi 1996). In the former case the individual need not be knowledgeable about the domain but just comes with an innovation as in the case of Pasteur mentioned in S. 2.2.1.1(b). Intelligence may be a necessary but not a sufficient condition for creativity (Haynes 1989).

Of importance to the present study is creativity as it manifests itself in the classroom. If a learner discovers the meaning of a concept that is already known to the educator for himself, or sees the concept in a new light, that is a new and unique experience for him. That is creativity on his part. This is the case when learners re-invent the subject matter and this is the manner in which the subject matter should be taught.

The creativity model presented by Ford & Usnick (1999) which is Polya's model (1973) is analogous to the one by Fourie et al (1991). Cangelosi (1996) presents a more encompassing model but has elements in common with the above model, for example, confrontation and analysis of the problem is similar to understanding a problem and rule articulation is similar to devising a plan and carrying it out. In looking back, one searchers for alternative solution pathways to the problem and asks oneself if the problem was phrased differently what the solution would be which is tantamount to extension and subsequent lessons. The difference between the models is in Cangelosi pairing problems that cannot be solved in a similar way and letting learners discover this for themselves thereby leading to gaining more insight into and creativity in problem solving.

Creativity is the association of concepts such that new combinations are formed and this is a problem solving process (Shapiro 1968).
Creativity is maintained if a person is intrinsically motivated in what he is doing. He is thus willing to spend long hours at what he is doing and this increases the likelihood of him arriving at a novel product (Weisberg 1988:173).

Relating ideas about solving problems, investigating those ideas, creating new problems from the solutions of the problem at hand, evaluating solutions and communicating results is a creative model to geometrical problem solving. In the classroom setting, these activities occur as learners interact with one another and also with the teacher (social interaction) and on this basis, they, as a collective, negotiate meaning together with individuals adapting their ideas to be in line with the more functional ones to the situation(socio-constructivism). Since learners and the teacher come with their unique experiences to the classroom, these experiences form the basis to learning mathematics (socio-cultural basis).

Divergent thinking is typified by cognitive processes such as comparing, observing, summarizing, classifying, interpreting, hypothesizing, applying facts, investigating, associating, analyzing, synthesizing ideas and generalizing. Learners do not acquire these thinking skills on their own and should be taught these and the learners' attention should be specifically drawn to them during geometrical problem solving. It is from the use of such skills that conceptual understanding is enhanced.

CHAPTER 3

PERSPECTIVES ON GEOMETRY EDUCATION

3.1 INTRODUCTION

Classroom instruction is guided by a philosophy/philosophies of people, learning theories and approaches. Geometry education is likewise grounded in philosophies, learning theories and approaches.

Spatial sense is an intuitive feel of one's surroundings and objects in them (environment). This encompasses, inter alia, geometric experiences of shapes in space - how such shapes are visualized and the spatial abilities inherent in such visualization. Learners need to be afforded the opportunity to identify and classify different geometric shapes and patterns in their environment and to appreciate the dynamic aspects of shapes in space i.e. vision and location. Instructional activities that facilitate the development of spatial sense will be explored.

As has already been mentioned, classroom instruction is based on, inter alia, learning theories. It will therefore be necessary to review some influential theories. Constructivism is a view of knowledge that suggests that the learner does not remember knowledge exactly as it was presented, but that he interprets what is being presented on the basis of what he already knows. Thus the learner does not passively receive knowledge, but actively constructs it.

Cognitive science posits a way of understanding the way in which individuals process information. Three models viz. Anderson's model of cognition, Greeno's model of problem solving and the parallel distributed processing networks are identified (van der Sandt 2000:54-64).

Philosophers, psychologist and didacticians' contributions to the teaching and learning of geometry also need to be explored. The works of Vygotsky and Feuerstein inform us on how learning takes place in general. Geometry learning is not immune to what their viewpoints postulate. The works of Piaget, van Hieles and Freudenthal go a step further and inform geometry learning in particular.

3.2 GEOMETRY EDUCATION

3.2.1 Orientation

An individual's intuitive feel for objects (geometric shapes) in the individual's environment is referred to as spatial sense. A learner manipulates objects (shape), he identifies and classifies them according to their properties; he looks at these objects from various vantage points (vision) and he also needs to locate their positions in space.

According to Wheatley (1990:10), spatial sense should be thought of as imagery/visualization i.e. seeing the actual object and noting its properties or having a mental picture of an object. These actions culminate in knowing the object i.e. in having learnt about the object. Wheatley looks at the three components of imagery viz. construction, representation and transformation of self-generated images as proposed by Kosslyn. Once an image (of an object) has been constructed it does not remain in the consciousness. If needed, it must be recalled by calling it up. Such representation may not take the same form as the originally constructed image (Wheatley 1990:10). Spatial sense often requires an image to be transformed, e.g. transforming a rhombus to a square. Estimation, an important skill in measurement, shows how imagery facilitates geometry learning: mental images serve as cues to facilitate estimation - the mental image of a right angle could be used as a benchmark to estimate accurately an angle measure of 45E (Happs & Mansfeld 1992:42-45).

Visualization is the cornerstone to the learning of geometry. Visualization depends on what is being visualized and the spatial abilities inherent in such visualization.

3.2.2 Spatial perception/visualization

Spatial visualization is an act of perceiving an object or having a mental picture (image) of such an object with a view to knowing the properties of the object. Children's perception of their environment incorporates all the sensory inputs viz. round, right, touch, small and body position to help them discover what their environment is like. Smothergill, Fergus and Timmons (1975:4) state that visualization has an important role to play in spatial development/functioning of children.

3.2.2.1 Visual education

According to Sgroi (1990:21-22), spatial visualization relies heavily on the mathematical language and communication. Learners should know mathematical concepts and communicate them to their educators and peers. They should be given the opportunity to communicate in such a way that they develop a vocabulary that is not only written but is also mental and pictorial. These exercises are meant to enhance the learners' spatial development. In practice, however, this is not the case. Wheatley (1990:10) states that the spatial visualization is underemplasized in the mathematics curriculum in the United States. Herschkowitz et al (1996:161) concur with this view in that unlike numerical or algebraic education, visual education is an often neglected area in the curriculum especially with regards to learners interacting with real shapes in space. Prior to the recommendations of the GDE and the GICD as mentioned in S.3.2.3, the situation was much the same in South African schools.

The teaching of visualization has the following merits:

- visualization is an essential part of human intelligence - visual education contributes to intellectual development,
- it is an alternative way of learning geometry that stresses visual thinking rather than the traditional (Euclidean) approach,
- learners are actively involved in the teaching-learning situation - manipulate objects (Hershkowitz et al 1996:166).

3.2.2.2 Spatial visualization in the mathematics curriculum

Learners differ in capabilities. Mathematics should thus be presented in a manner that takes this into account. In geometry learning gifted students rely on symbolic thinking while those less gifted should visualize the problem in problem solving. Certainly visualization does not harm the gifted learner but if left out of the curriculum, limits the chances of success in geometric problem solving for the less gifted learner. While acknowledging the fact that there are individual differences, Bishop (1989:7-14) feels that visualization is of value to all students in all aspects of the mathematics curriculum.

Furthermore, visualization in the mathematics curriculum has the following positive effects on concept formation:

- the mathematical content can be pushed much further with younger learners (they are not capable of a lot of symbolic thinking at this stage; in fact they rely on visualization in instruction)
    - an alternative approach in presenting mathematics
    - results in rich concept images which form a basis for abstraction (Eisenberg & Dreyfus 1989:4).

Bishop (1989:7) also concurs in that visual representations offer a powerful introduction to complex abstractions in mathematics. He states that there are five types of visualizations (what is being visualised - the visual images) as identified by Pressmeg viz:

- concrete, pictorial imagery
- pattern imagery
- memory images for formulae
- kinaesthetic imagery (fingers manipulating objects), and
- dynamic (moving) imagery

The students do not stay with one of these imageries but use different ones in different situations. For these imageries to be enhanced, learners should be provided with three dimensional objects and be given the opportunities to represent these in two dimensions and vice versa.

The visualization process is individual. Individuals differ in the 'visuality' of their problem solving approaches. There is never a unique way to carry out a particular task - each individual interprets a task in a unique way. What happens depends on an individual's preferences; his unique visualizations and the ability to recall and use them

(Bishop 1989:10). This is in line with the constructivist theory of learning to be discussed in detail in S.3.7.

3.2.2.3 Visualization in multicultural mathematics classrooms

The South African society consists of people of different cultures; we thus have multicultural classes. Mathematics classrooms are no exception.

For some learners in a multicultural classroom, the language of instruction may not be the mother tongue. For these learners the need for actual objects and mental pictures (visualization) is important for understanding as they are disadvantaged by the language of instruction (Presmeg (1989:19).

There are various ways in which visualization may enhance understanding in a multicultural classroom. If, for example, the geometric shape a learner visualizes is some pattern in the learner's culture he will have a deeper understanding of such shape. In this regard, Presmeg (1989:21) states: "Visual imagery which is meaningful in the pupil's frame of reference may lead to enhanced understanding of mathematical concepts at primary and secondary levels". Furthermore, from a complicated drawing from another culture, for example, the Sufi mandala or Shri Yantra, learners may see patterns and thus identify various geometric shapes and lines of symmetry. These skills are essential in solving problems in Euclidean geometry (Presmeg 1989:21). Thus borrowing from another culture, whose members are in the classroom affords all learners the opportunity to enhance their understanding of the said concepts.

3.2.3 Space and shape

Shape, vision and location are some of the themes a learner grapples with in the learning of geometry. The Gauteng Department of Education(GDE) and the Gauteng Institute for Curriculum Development (GICD) curriculum document (1999) recommends that these are the themes to be taught in Grade 1 to 9. Shape deals with the recognition of different geometric shapes and patterns in our environment, vision deals with projections of reality from various vantage points while location deals with finding the location of various objects and places (GDE & GICD 1999:18-19). This recommendation comes as an alternative to the unsatisfactory curriculum for geometry learning of the past which emphasized deduction at this early stage (Grade 1 - 9) and underplayed the role of informal geometry as the scaffolding for learning formal geometry.

Hershkowitz, Parazysz and van Dormolen (1996:162) state that by interacting with real shapes in space is meant:

- the discovering of similarities and difference among objects/the identification and classification of shapes,
- the analyzing of the components of form/properties,
- to recognise shapes in different representations/visualization, and
- to appreciate/understand the dynamic aspects of shapes in space, for example, the relative position of several shapes to each other, the relative position of the observer and the objects that he looks at and the processes of changing shapes (vision and location).

3.2.4 Spatial abilities

Del Grande (1990:14-18) describes spatial perception in terms of seven spatial abilities. These skills are essential for enhanced spatial visualization. Learners should be provided with structured and manipulative materials that can help them create visualizations and thus the visualization process itself. The seven spatial abilities Del Grande mentions are:

- eye-motor co-ordination
- figure-ground perception
- perceptual constancy
- position-in-space perception
- perception of spatial relationships
- visual discrimination, and
- visual memory

as suggested by Hoffer (1977). These abilities have a bearing on the study of mathematics in general and geometry in particular.

Owens (1990:48) brings to the fore the relationship between spatial abilities and spatial visualization. In so doing, he mentions that according to Bishop (1983) and Halpern (1986), spatial abilities comprise two main factors:

- a visualization factor, and
- an orientation factor

The visualization factor focuses on the ability to imagine, rotate, twist or invert an object while the orientation factor focuses on the ability to detect arrangements of elements within a pattern (analysis).

According to Del Grande (1990:19), data collected from teachers indicates that pupils perform geometric tasks very well when such tasks are related to the spatial abilities of the pupils. This means that pupils have to be empowered with skills (spatial abilities) for visualization in the teaching-learning situation.

3.2.5 Informal Geometry as a Basis for Learning Formal Geometry

Del Grande (1990:19) states:

> Geometry has been difficult for pupils due to an emphasis on the deductive aspects of the subject and a neglect of the underlying spatial abilities, acquired by hands-on activities, that are necessary prerequisites for understanding and mastery of geometrical concepts.

Three dimensional shapes which can be handled by the learners should be used as a starting point in learning formal geometry. It is from such objects that concepts such as symmetry, perpendicularity, parallelism, congruency, amongst others, that are necessary for formal geometry, are internalized.

Rowan (1990:24) concurs with the above viewpoint in that students should first meet geometric ideas initially through hand-on experiences with the geometric nature of their surroundings. The ability to name geometric figures should emanate from experiences that lead to the development of the underlying concepts.

The van Hiele level theory states that a student cannot understand material at a certain level without having had experiences that enable him to think intuitively at a preceding level and that if the language of instruction is at a higher level than the student’s thought processes, he will not understand the instruction (Mayberry 1983:59). An analysis of the responses to geometrical problems given to 19 pre-service elementary teachers showed that they were below level 3 (the van Hiele 5 levels are discussed in S.3.7) as they failed to answer correctly questions that require level 3 thinking and hence were not properly prepared to understand formal geometry (Mayberry 1983:60). These teachers require grounding in and understanding of tasks requiring lower thought levels. These lower levels impact on formal geometry.

Furthermore, what emanates from the learner's environment goes a long way to providing the leaner with opportunities for visualization and the acquisition of problem solving skills as the Suli Mandala or Shri Yantra examples mentioned in S.3.2.2.3 suggest.

3.3 PIAGET

3.3.1 Orientation

According to Piaget a child's ability to perform a given cognitive task depends on his level of intellectual development. Depending on the nature of the task, a child cannot perform that task unless he is biologically mature enough to perform such task. This has important implications for instruction which includes geometry learning. If a child cannot perform a task on an abstract level, such a child should first work on a concrete level then as he matures be given tasks on an abstract level. Piaget outlines various geometric tasks, e.g. topological relations, with the intent that these should be geared towards specific levels of a child's intellectual development.

3.3.2 Piaget's stages of intellectual development

According to Piaget, the development of thinking process occurs through consecutive stages (which depend on biological maturity) each characterized by its own distinctive process (Behr 1983:28). These stages are delineated as the sensory-motor, pre-operational, concrete operational thought and formal operational.

3.3.3 Piaget and Inhelder's topological primacy theory

The work of Piaget and Inhelder (1971) shows that there is a definite order in the manner in which geometric ideas are organised by children: initially, topological relations (connectedness or enclosure) are constructed and only later, projective (rectilinearity) and Euclidean (angularity, parallelism and distance) are constructed (van Niekerk 1997:29).

3.3.3.1 Topological primary

Piaget and Inhelder use haptic and drawing evidence when investigating topological primacy.

Haptic evidence

The first experiments required learners to discriminate and recognise shapes by handling them instead of looking at them (Haptic perception).

Van der Sandt (2000:54) states that children of age 2 years 6 months to 4 years recognised and drew closed, rounded shapes and those based on simple relations such as openness and closure. Gruber and Voneche (1977:577-578) also state that according to Piaget, the child conserves only topological properties of spatial objects. He is able to draw squares, circles and rectangles as vague objects sharing the common property of being always closed; he represents rings and crosses with an opening by the same vague open figure. This distinction between open and closed figures can be likened to a distinction between inside and outside. From the age of 4 years to 7 years, learners recognize simple Euclidean shapes while from approximately 7 or 8 years they begin to recognize the connection between shapes.

Drawing as evidence

In this section, learners produce drawings either spontaneously or with the aid of visual models or visual memory (van der Sandt 2000:54).

Learners of up to around 3 years 6 months produced different types of scribbles depending on open or closeness of the figure; those between 3 years and 4 years produced “real drawings” indicating only the topological relationships (real pertains to not producing scribbles); those from 4 years to 6 years 6 months could differentiate Euclidean shapes (first rectangles and squares, then triangles and later a rhombus) and lastly those from 6 years 6 months onwards could draw shapes correctly by recalling a mental image of those shapes (Van der Sandt 2000:55).

Van der Sandt (2000:57) states that for Piaget and Inhelder (1971) the inability to draw a shape accurately reflects the inadequacy of mental tools for spatial development because drawing is an act of representation and not perception.

Skills as mentioned above are important in the manner in which learners interact with geometric shapes in their environment.

3.3.3.2 Projective space

Projective relations involve relations between figure and subject. They are concerned with the co-ordination of objects, for example, a subject sighting an object. In sighting an object, the learner gets to understand that the line of vision is straight (one cannot see around an object). Van der Sandt (2000:58) states that very young learners can perceptually recognize a straight line but cannot present a straight line mentally. Learners from 4 years to 7 years are able to form straight lines parallel to the edge of the table. There is a progressive ability to form/draw lines independent of the edge of the table, while those around 7 years onwards understand what a projective straight line is. They can spontaneously draw lines from different viewpoints.

3.3.3.3. Euclidean space

Piaget and Inhelder's (1971) experiments showed that there is a gradual acquisition and conservation of parallelism, the discovery of proportion and conservation of angles, and the development of a simple co-ordinate system of reference required to construct horizontal and vertical axes (Van der Sandt 2000:62). Young children up to 4 years have no understanding of the planes; for learners between 4 to 7 years, the horizontal and vertical axes remain undiscovered.

Identifying firstly the level of geometric thought the child operates on forms a basis for what the child is able to understand in Euclidean geometry (formal geometry).

3.4 VYGOTSKY

Vygotsky holds a close reciprocal between through and language. A child's development is typified by pre-intellectual speech as well as non-verbal thought. Only with the inter functioning of the two entities, does thought become verbal and speech become intellectual. It is in the act of thinking words without pronouncing them (internalized speech) that the child's thought processes are facilitated; that language becomes a vehicle of thought (Behr 1983:26) and van Niekerk (1997:24). Vygotsky's followers, Curia and Gal'perin are of the opinion that speech plays a major factor in thought, especially in connection with concepts. The visual experiences of a child are supported by an extended use of appropriate language (Behr 1983:26). A child with linguistic competence can articulate his visual experiences in various ways. Language thus plays a crucial role in the spatial development of the child.

Vygotsky argued for the evolution of the higher cognitive processes from social (interpersonal) to the individual (intrapersonal). Language is thus important as it plays a vital role in interacting with other beings. It is equally important in the classroom as a means of communication between the educator and other learners and amongst the learners themselves. Confrey (1995:189) states that Vygotsky's higher cognitive functions included meaning, memory, attention, thinking and perception. In all these cognitive functions, the social dimension is primary in time and fact and the individual dimension is derivative and secondary. It is from the interpersonal relations that an individual acquires knowledge which he then internalizes. For Vygotsky, all knowledge is situated in culture and in time; what becomes personally meaningful has been shaped by our interactions with other human beings.

For Vygotsky, teaching is a form of interaction between a more knowledgeable other and a novice (Confrey 1995:203). It is through this interaction that a learner is schooled into using established socio-cultural distinctions and descriptions appropriately. The learner first experiences active problem solving activities in the presence of others (his teachers and his peers), but gradually performs these processes independently. The teacher's guidance and support is pertinent in this regard. He corrects the learner as he makes mistakes until such time that the learner has internalized the problem solving strategies. This is what teaching entails.

Vygotsky agrees with the van Hieles that instruction and not age or biological maturation plays an important role in the child's mental development and hence spatial development.

According to Rieber and Carton (1987:208-209), psychological research on instruction is usually limited to establishing what the child has and knows today, i.e. the child's actual development. The authors maintain that this is inadequate since the level of development is not only measured by what has matured. Cognisance should also be taken of the zone of proximal development, Vygotsky's brainchild. This is the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under the guidance of the teacher. By this, Vygotsky implied that with appropriate instruction, there may be potential for a child to reach higher conceptual levels than he would be able to achieve naturally (Jaworski 1994:26). Everything else being constant, it can be assumed that a child whose mental age is 8 years can perform tasks for such age. Such a child can, through demonstrations, leading questions and by introducing the initial elements of the task's solution i.e. coaching, perform a task characteristic to a 12 year old. The difference between the child's actual level of development and the level of performance that he achieves in collaboration with the teacher, defines the level of proximal development (Rieber & Carton 1987:208-209). This emphasizes the importance of instruction in the child's level of development which encompasses spatial development.

3.5 FEUERSTEIN

With his Mediated Learning Experience (MLE), Feuerstein (1983:37-70) puts

Vygotsky's ideas and findings in a slightly different manner. The MLE takes as point of departure the fact that the child is deficient in certain, amongst others, mental processes and thus aims at eliminating these. These deficiencies could be as a result of a lack of cultural transmission - the content handled at school is not an integral part of the child's culture. Such content should thus be mediated to the learner. In problem solving, for example, the teacher should empower learners with skills so as to be able to find solutions to problems. The teacher should create an environment, in the classroom, wherein creativity and divergent thinking, as pertinent skills in the problem solving process would prosper.

3.6 CONSTRUCTIVISM AND SOCIO-CONSTRUCTIVISM

Constructivism is a view about knowledge that suggest that learners do not remember content exactly as it was presented, but that they interpret instructional situations in many different ways. New knowledge is interpreted on the basis of pre-existing knowledge (Olivier 1992:13).

Learners make new ideas meaningful when they integrate them into existing networks of knowledge. When a learner hears the word 'triangle', he may, depending on his experiences, think of a tricycle, a tripod or a triangular road sign. In any event, the learner comes to understand that a triangle has a 'three' of something. In support of the above, Yackel, Cobb, Wood, Wheatley and Merkel (1990:13) state that in tackling new mathematical tasks, children use strategies that have already been internalized, to complete such tasks.

According to Orton (1994:37):

> Constructivism is based on the view that, ..., we all have to make sense

> of the world ourselves, and we continue to develop our understanding throughout life, just as we all come to our definition of any particular word and continue to amend it.

No two individuals can have the same meaning of a certain concept at a given time. Moreover, such meaning changes for an individual and so does the manner in which meaning from previous knowledge was constructed.

According to Njisane (1992:27): “Constructivism is Piaget’s brainchild”. Indeed, when one considers the stages of intellectual development in Piaget’s theory and the implications they have for teaching, one sees why the above assertion is true. Let us consider the concrete operational stage and the implications that this stage has for the teaching and learning of mathematics. Such learners are in the primary school and concrete objects should be used in teaching mathematics.

Von Glasserfeld (1989:131) makes a distinction between ‘training’ and ‘teaching’ and in so doing distinguishes between behaviourism and constructivism. In both teaching and training, communication plays a vital role. From a behaviourist perspective, communication between a teacher and a student consists of providing stimuli and reinforcements that will solicit behavioural responses considered appropriate by the teacher. The process is concerned about the acquisition of skills and does not necessarily require understanding on the part of other learner (training). Communication is not possible from the constructivist perspective unless the receiver has a conceptual structure that is compatible with what the speaker has in mind. The receiver’s conceptual structure is the basis with which he interprets and understands what the speaker says. There is negotiation of meaning between the speaker and the receiver. This entails understanding on the part of the receiver so that he does nothing that contravenes the speaker’s expectations. Understanding is essential in teaching.

Without this, the teacher and the student are invariably at different wavelengths and no meaningful teaching and learning takes place.

Radical constructivism is based on two hypotheses viz.

1. Knowledge is actively constructed by the cognizing subject, not passively received from the environment.
2. Coming to know is an adaptive process that organizers one's experiential world; it does not discover an independent, pre-existing world outside the mind of the knower (Lerman 1989:211).

In summary, these hypotheses state that knowledge is actively constructed by a thinking learner and that such construction is an adaptive process that occurs in the mind of the learner.

It has already been mentioned that according to constructivism, knowledge is based on past constructions. Such constructions come about through assimilation and accommodation. Assimilation refers to the use of information already existing in one's personal networks of knowledge to interpret new knowledge. If the new knowledge is contradicted during assimilation, it is accommodated by adapting one's existing concepts.

The constructivist theory states that learning is a process of invention - learners must experience, hypothesize, manipulate objects, ask questions, investigate and negotiate. A learner-centred, problem-centred approach to teaching is thus necessary (to be explained in S.4.4). In this regard, the learner discovers his own knowledge and the teacher is a guide or facilitator.

Conceptual understanding underpins constructivism. This is in line with Simon and

Schifter's (1993:331) view that in the construction of knowledge, understandings are constructed by learners as they attempt to make sense of their experiences. In this exercise, learners bring to bear a web of prior understandings that are used in interpreting new experiences. This shows that knowledge is not received from one person to another in intact form.

Whereas radical constructivism focuses on an individual, social constructivism is based on social interaction. Taylor and Campbell (1993) suggest that knowledge is socially constructed through its negotiation and mediation with others (Jaworski 1994:24)., This emphasizes the socio-cultural context of learning and highlights the importance of language in learning. Knowledge construction in the social constructivist perspective implies that through the use of language and social interchange, individual knowledge can be challenged and new knowledge constructed. In the teaching-learning situation, knowledge is negotiated, shared and acquired between the teacher and the students and amongst the students. Whereas constructivism in general is a view of knowledge, social constructivism is used as a learning theoretical framework. The move from a radical to a social view of knowledge construction can be likened to the move from a Piagetian to a Vygotskian view of learning as shown in section S.3.4 and S.3.5.

In the classroom situation we have learners as well as the teacher. There are individuals who interact with one another, they share their views and experiences and along the way knowledge is constructed. Knowledge is acquired through the sharing of their experiences, it is socially constructed. We thus speak of socio-constructivist view about knowledge.

The socio-constructivist view of knowledge construction is relevant to classroom practice:

- A distinction should be made between educational procedures that aim at generating understanding (teaching) and those that aim at the mere acquisition of skills (training). This is echoed by both Jaworski (1994:26-27) and von Glasserfeld (1989:135). This distinction would encourage teachers to clarify the particular goals they want to achieve. Curricula and in particular, classroom instruction, would be more effective as it would deliberately separate the task of achieving a certain level of performance in a skill from that of generating conceptual understanding within a given problem area.

- Since the learner uses his own conceptual structures to interpret new knowledge, the teacher must have an adequate model of the learner's conceptual network. Without this model, the teacher and the learner may be at different wavelengths with regards a given topic.

- The teacher understands that the learner is trying to make sense in his experiential world. He (teacher) is thus interested in the student's errors. Indeed, without knowing the learner's conceptual network and how this leads him to errors, the teacher will not be able to correct learner misconceptions (the teacher must first find the learner's knowledge base - that which leads him to the misconception and then correct it otherwise the misconception will not be corrected).

- Socio-constructivism implies that there should be communication between the teachers and the learners and among learners i.e. a language centred classroom is necessary. However, this linguistic communication does not serve to transfer knowledge to the student but is a tool in the process of guiding the student's construction (Jaworski 1994:26-27).

3.7 VAN HIELE

According to van Hiele (1986:36), learners progress through levels of thought in geometry. Van Niekerk (1997:34) states that the van Hiele theory has the following characteristics:

- Learning is a discontinuous process implying that there are qualitatively different levels of thinking.
- These levels are sequential and hierarchical. A learner cannot function adequately at one level without having mastered most of the previous level. The progress from one level to the next is more dependent upon instruction than age or biological maturation.
- Concepts that are implicitly understood at one level become understood implicitly at another level.
- Each level has its own language. Thus two people that reason at different levels cannot understand each other. They cannot follow the thought process of the other. Language structure is a critical factor in the movement through the levels.

Van Hiele (1986:34-47) distinguished five levels of geometric thought which can be summarized as follows:

- Level one: visual

  Figures or objects are seen as a whole; individual properties of such are not distinguished.
- Level two: descriptive/analytical

  Figures or objects can be identified by their properties; each property is seen in isolation, i.e. there is no comparison to other figures or objects.

- Level three: abstract/relational

  Figures or objects are still determined by their properties, but the relationship between properties and figures evolve.
- Level four: formal deduction

  Theories and deductive proof can be constructed.
- Level five: rigor/mathematical

  Proofs counter to intuition can be accepted as long as the deductive argument is valid; learners can manipulate geometric statements such as axioms, definitions and theorems.

Teppo (1991:210) states that van Hiele currently characterizes his model in terms of three rather than the five foregoing levels and these can be summarized as follows:

- Level one: visual

  Learners recognize objects globally.
- Level two: descriptive

  Learners recognise objects by their geometric properties
- Level three: theoretical

  Learners use deductive reasoning to prove geometric relationships.

Instruction should be geared toward finding out the level at which a child operates then building up from there, otherwise the child and the teacher may be at different wavelengths and instruction is bound to fail. A child at level N will answer most questions at that level but will not answer questions at level N+1 (Mayberry 1983:58).

There are five phases/stages of learning that facilitate geometric thinking (Teppo 1991:212) and (van Hiele 1986:54). Student progress from one level to the next is a result of passing through the five instructional phases. Van Hiele (1986:50) states that if a child comes to conclude that ‘every square is a rhombus’ that is not as a result of maturity but is a result of the learning process. These instructional phases can be summarized as follows:

! Phase one: information

Learners get reacquainted with material for instruction.

! Phase two: bound orientation

Learner explore the field of inquiry through carefully guided activities.

! Third phase: explication

The learners and the teacher discuss the object of study. Language appropriate to the particular level is stressed.

! Fourth phase: free orientation

Students learn by general tasks to find different types of solutions.

! Fifth phase: integration

Students build an overview of all they have learned of the subject. At this stage, rules may be composed and memorized.

Gutiérrez, Jaime and Fortuny (1991: 238-250) present a method for the evaluation of the acquisition of the van Hiele levels in solid geometry. Findings from their research led them to the following conclusions:

! There is a possibility that a student can develop two consecutive levels of thinking simultaneously. However, the acquisition of a lower level is more complete than a higher level when this happens.

! Depending on the level of the problem some students use several levels at the same time.

! The above do not reject the hierarchical nature of the levels but highlights the complexity of the human reasoning process i.e. people do not behave in a simple, linear manner which the assignment of one single level would seem to suggest.

- A student may operate on a certain level on one task but may be on a different level at a different ask.

The van Hiele theory offers a theoretical framework for the teaching and learning of geometry. Indeed it is essential that the teacher and the learner first find common ground as a basis for learning. Once this has been established instruction can then be taken to higher grounds. The theory points out to the levels of geometric thinking a child goes through and that it is through instruction that a learner will proceed from a lower level to a higher one.

3.8 HANS FREUDENTHAL

Freudenthal is the founder of the so called realistic mathematics education in the Netherlands. In realistic mathematics education, reality does not only serve as an area of applying mathematical concepts but is also the source of learning i.e. the content of mathematics instruction (Treffers 1993:39).

For Freudenthal, mathematics does not only mean mathematizing reality i.e. transforming a problem field in reality into a mathematical problem but also mathematizing mathematics itself i.e. the processing within the mathematical system. The former refers to horizontal mathematizing while the latter refers to vertical mathematizing (Streefland 1993:111). As an illustration of these two concepts, Hershkowitz et al (1996:180-181) gives the example of the position a mouse must not be in, between two walls if it is not to be seen by a cat which is on the other side of the walls. The line of vision is straight, so the mouse must not be in line with the cat's vision (horizontal mathematizing). The cat has many lines of vision in between the two walls. All the cat's lines of vision between the walls form a region within which the cat can see (the right angle). Sight lines have thus been transformed into a single angle (vertical mathematizing).

According to Freudenthal (1971:418) geometry failed because it was taught in such a manner that its deductivity could not be reinvented by the learner but only imposed. For Freudenthal, starting with axioms and theorems is a wrong approach to teaching geometry (Streefland 1993:122). Indeed, starting with the fine polished product such as axioms and theorems, denies the learners the opportunity of finding out how such theorems or axioms are arrived at. The starting point should be from the child's everyday life experiences of spatial objects (from his reality). Geometry should be related to the science of the physical space - of the space in which the child lives and moves, as an organization of the learner's spatial experiences (Freudenthal 1971:418).

Freudenthal (1971:421) states that:

> In many countries plane geometry started in the 7th grade, geometry in space in the 10th and 11th grades. Children who were good in plane geometry, often failed in space geometry. Their spatial imagination had been killed by too much exclusive plane geometry.

This emphasizes that geometry instruction should start form three-dimensional objects (spatial objects) to two dimensions (plane geometry). The GDE and GICD curriculum document (1993) agrees with this viewpoint.

Dina van Hiele-Geldof, the wife of P.M. van Hiele and a student of Freudenthal's experiments emphasize the importance of the reinvention of geometry and not its imposition during instruction. She did not start with definitions of triangles, squares, rectangles, but let children discover properties of these and hence be in a position to define them by themselves (reinvention). This is a more meaningful approach to the teaching of geometry.

3.9 COGNITIVE SCIENCES

The cognitive sciences perspective in students learning of geometry endeavours to integrate research and practical work in the fields of psychology, philosophy, linguistics and artificial intelligence (Clements & Battista 1981:434). The cognitive science perspective aims at understanding the way in which individuals process information. Three models are distinguished within this perspective namely Anderson's model of cognition, Greeno's model of geometric problem solving and the parallel distributed processing networks. These models inform us on how learners internalize geometric concepts. The first two models will be discussed a they have an impact on the study.

3.9.1 Anderson's model of cognition (ACT)

Anderson's Adaptive Control of Thought (ACT) model distinguishes between two kinds of knowledge, namely declarative and procedural. Declarative knowledge is "knowing that" or "knowing why", while procedural knowledge is "knowing how".

Procedural learning occurs in the execution of a skills while declarative is knowing why the skill is executed or is justifiable.

According to the ACT model, learning involves the acquisition of declarative knowledge which is then interpreted by general procedures (procedural knowledge) (Van Niekerk 1992:44). Pythagoras theorem states that in a right angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.

Knowing how to prove this theorem constitutes declarative knowledge while using this theorem to calculate the hypotenuse given two sides of a right angled triangle is procedural knowledge.

These two types of knowledge are used in geometry teaching.

3.9.2 Greeno's model of problem solving

In this model, a computer simulation was designed to solve problems that ninth-grade students could solve. In essence, the simulation involves three domains of knowledge which are required in order to solve problems in geometry.

The domains could be summarized in the following manner:

1. Recall theorem(s) that could be used to solve these problems. Each step in solving a problem consists of an inference in which some new relation is deduced from the information that was given or that had previously been inferred, for example, "if two angles are congruent, then they have equal measure". Thus the solution of a problem consists of a series of "if - then" steps which lead to the goal of the problem.

2. Do we see patterns that correspond to or warrant the use of theorem(s) mentioned in 1? To use the theorem "corresponding angles formed by parallel lines and a transversal are congruent", one needs to see a pattern of corresponding angles formed by parallel lines and a transversal first. If this pattern exists, it can thus be concluded that such angles are equal.

3. What plan are we going to follow, using the theorem(s), to solve the problem. For example, in showing that two angles are equal, one can use relations such as corresponding angles or alternating angles or to prove that triangles containing such angles are congruent.

Greeno gives strategies that can be used in geometry problem solving. These are general guidelines that can be used by a learner in geometric problem solving.

3.10 CONCLUSION

Spatial sense was defined as an intuitive feel of one's environment. This encompasses geometric experiences of shapes in space. The manner in which such shapes are visualized and the spatial abilities inherent in such visualization falls within the ambit of geometry education. In this regard the learner manipulates objects, he identifies and classifies them according to their properties an communicates about such properties (cognitive processes).

To be able to perform the above activities, the learner must be able to visualize shapes. Visual education which manifests itself in spatial abilities is necessary to this end.

The exposition of the nature of geometry education is essential for a deeper understanding of what geometry education entails. This exposition is done, in this study, through the literature review in Chapter 5.

Classroom instruction is based on, inter alia, learning theories. These theories will inform the teaching practice in geometry at Northcrest High School. For this purpose constructivism as a view of knowledge was reviewed. Cognitive science aims at understanding the way in which individuals process information. The three models (Anderson's model of cognition, Greeno's model of geometric problem solving and the parallel distributed networks) inform us on how learners learn geometry. The works of Piaget (a radical constructivist) and Vygotsky also inform us on how learners learn geometry. For van Hiele, learners go through levels of geometric thinking and these levels must first be identified by the educator, lest the educator and the learner operate at different wavelengths during instruction. Freudenthal emphasizes the fact that the teaching of geometry should emanate from the child's environment ie. three-dimensional shapes then to formal geometry of axioms and the theorems and not vice versa. Freudenthal also emphasizes the fact that the learner should be given the opportunity to re-invent geometry for himself (self discovery) - the learner must reflect on how the polished product (theorems and axioms) were arrived at.

Visualization is important in the learning of geometry. Learners in secondary schools should be given the opportunity to visualize geometric shapes - manipulate concrete objects and discover their properties, make conjectures and generalizations pertaining to such shapes and also have a mental picture of the shapes. These activities level the ground for learners being able to see patterns and be able to solve riders in high school.

The researcher also shares the view that knowledge is not transferred directly from one person (teacher) to the other (learner) but that the learner interprets such knowledge on the basis of what he already knows. Finding out the learner's knowledge base that is pertinent to the content to be presented is essential in instruction. This is in line with finding out the level of geometric thinking at which the learner operates (van Hiele levels). In the above process, the teacher interacts with the learner (social interaction) and this is the basis for knowledge construction for both the learner and the teacher (socio-constructivism).

CHAPTER 4

CREATIVITY AND DIVERGENT THINKING IN GEOMETRY TEACHING LEARNING I NA PROBLEM-CENTRED CONTEXT

4.1 INTRODUCTION

In order for individuals to perform their duties effectively, be it in the workplace or in life in general, they need to be creative and be good thinkers. The school has an important duty in developing such skills in learners.

In Chapter 1 mention was made of the fact that emphasizing disciplines of mathematics such as algebra at the expense of geometry and in particular informal geometry at school denies learners further opportunities to be creative and divergent thinkers. This chapter focuses on elements of creativity and divergent thinking and their role in enhancing the internalization of geometry concepts in the problem-centred context.

Divergent thinking includes processes such as lateral thinking, deducing, inducing, identifying, synthesizing, analyzing, evaluating, problem-solving, critical thinking and differentiating. These processes are essential for effective geometric problem-solving. Learners should be empowered with these skills to be good problem solvers . This process can be achieved by teaching for thinking, developing in learners critical thinking disposition and abilities as well as the cognitive processes. The concept creativity has been associated with originality and inventiveness amongst other things. In this study, learners will be expected to compose songs, write poems, posters, analogies, metaphors and essays on geometric concepts. Furthermore, learners construct musical instruments using geometric shapes. They also produce art from tessellations and tangrams.

4.2 THE PROBLEM-CENTRED PERSPECTIVE

The problem-centred approach currently being followed in many schools in South Africa utilizes the constructivist views on learning (Murray, Olivier and Human 1993:193). The following will be an account of the objectives of the problem-centred approach to learning, the role of the teacher in such learning, problem solving as a learning type and the role of social interaction in problem-centred learning.

The problem-centred approach encompasses creativity and divergent thinking. Creativity and divergent thinking are inherent in problem solving. Engaging in activities of creating and divergent thinking in the teaching and learning of geometry facilitates teaching in context i.e. the correct teaching and learning environment. Creativity divergent thinking, problem solving and teaching in context are embedded in the problem-centred approach and this results in better learning. The empirical study in this research uses this approach and the following empirical format should be realized.

Creativity and divergent thinking + problem solving + teaching in context →better learning.

the problem-centred approach

(see also diagram)

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Fig. 28 The research format

4.2.1 Objectives

The problem-centred approach to the development of the number component of mathematics in South Africa has as its main objectives (1) the development of problem solving skills with regard to the routine and non-routine real life situations within the mathematics framework; and (2) the utilizations of problem solving as a learning type in order to solve problems (Van Niekerk 1992:50). The first objective entails the exposition of learners to real life problems requiring the use of mathematics for their solution. This exercise goes a long way in applying mathematics to real life situations. The second objective exposes learners to different strategies of problem solving, understanding of mathematics from different viewpoints and makes them perceive mathematics as problem solving and not just as part of mathematics.

4.2.2 Instruction using the problem-centred approach

Bereiter (1992:337-340) makes a distinction between two types of knowledge viz: referent-based and problem-based knowledge. In the former, knowledge centres around attempting to understand what the text says i.e. acquiring knowledge for its own sake whereas in the latter, knowledge is constructed around the solution of problems, it is situation-based, the knowledge that one possesses is used to solve particular problems and is changed if not suitable (accommodation) and as a result new knowledge is constructed (constructivism). For knowledge to be functional in making sense of the world, it should be attached to persistent problems in the learner's mind otherwise it will only be recalled when cues for such knowledge are given which is then referent-based. Bereiter advocates for problem-based rather than referent-based knowledge in instruction. It is from persistent, high level problems learners are dealing with, that a sophisticated body of world knowledge will be constructed (Bereiter 1992:353).

Hiebert, Carpenter, Fennema, Fuson, Human, Murray, Olivier & Wearne (1996:12) concur with Bereiter (1992) in that learners should problematize the subject i.e. both the curriculum and instruction should begin with problems, dilemmas and questions for learners. Problematizing the subject most likely leads to understanding. Hiebert et al (1996:16-18) distinguish between the understanding during the problem solving process (functional understanding) and the understanding that remains after the activity is over (structural understanding) - the residue. The residue depends on the nature of the problem that was solved (structure), strategies for problem solving used and the other learners' dispositions toward mathematics.

In this study, worksheets, questionnaires, interviews and tests are used during the problem solving process (teaching in context). Furthermore, learners write poems and posters, compose songs and use metaphors and analogies on geometric concepts. It is from these activities that it is envisaged, the learners' knowledge of geometric concepts will emanate (the problem-centred approach)

4.2.2.1 Problem solving as a learning type

Problem solving is a process and it entails the use of creative guidelines (heuristics) that help in the solution of problems. These guidelines are general suggestions and cannot guarantee success. Some of the guidelines/heuristics of problem solving involve: working backwards; making a drawing; creating your own problem and solving it and thinking of a similar problem that was solved successfully.

Through the problem solving process, learners come to perceive mathematics as problem solving. In other words, children learn mathematics through problem solving, for example, in finding the nets of a cube, they represent a three-dimensional object in two dimensions. It is about learning mathematics through problem solving. Learners engage in a variety of problem situations and along the process learn mathematical content (Hiebert et al 1996:19).

In problem solving, learners are afforded an opportunity to apply mathematical knowledge to real life situations. Learners use mathematical knowledge to solve real life problem. It is mathematics for problem solving. Krulik and Rudnick (1989:5) state mathematics instruction aims at, inter alia, the attainment of knowledge and facts and the ability to use information and facts. The latter ability is the essential part of the problem solving process.

The acquisition of problem solving skills (strategies) en route to solving mathematical problems is a learning mechanism in the problem-centred approach.

The are many methods and strategies for problem solving. Amongst others, Nickerson (1994:425-430) suggests:

! problem decomposition or subgoaling
  - breaking a problem into simpler problems; combing solutions to these subproblems may yield a solution to a problem
! hill climbing
  - if there is no strategy, one should walk (metaphor) in such a way that one is always going up; if one is descending one must make a u-turn. However descending also helps as one may pick up what he missed in going up.
! forward chaining
  - one begins with the givens and works directly toward the goal, and
! specialization and generalization
  - taking special cases may help; there may be a pattern in the special cases

considered, as some of the heuristics one may use in problem solving.

Learners should be encouraged to communicate strategies they used in problem solving and to write have a repertoire of problem solving strategies.

In the problem-centred approach the learner uses these strategies as he grapples with the solution of problems and learns mathematical content in the process. The teacher offers these strategies to the learners in guiding them towards the solution of the problem. Problem solving is thus teacher-centred while the problem-centred approach is learner-centred. Problem solving (strategies) is used in the problem-centred approach to aid learners when they cannot solve the problem on their own. Creativity and divergent thinking are essential skills in problem solving.

4.2.2.2 The role of social interaction

The problem-centred classroom is the place of problem posing and problem solving. These processes are achieved through invention, explanation, negotiation, sharing and evaluation - social interaction plays a crucial role in this regard. Social interaction creates the opportunity for children to talk about their thinking and encourages reflection; students learn not only through their constructions but also from one another and they also interact with the teacher.

Communication is important in a classroom that uses the problem-centred approach to learning especially during groupwork. The teacher can encourage classroom communication by creating opportunities for discussion, allowing learners to discuss their problem solving strategies with one another, planning opportunities for general classroom communication in, for example, obtaining feedback from the various groups during problem solving and communicating with the learners.

In South Africa we have multi-cultural classrooms. It is common to find children of four or more different cultural backgrounds in the same classroom. These diversities in culture reinforce the learners problem solving strategies if the teacher asks for different points of view during the problem solving process.

4.2.2.3 The role of the teacher

The teacher has a vital role to play in the learning of geometry in a problem-centred classroom. He must become a facilitator, someone who helps learning to take place through self-discovery.

A change of attitude is necessary on the part of the teacher of geometry using the problem-centred approach to that of traditional teaching which was teacher centred. He should create situations to develop creative thinking; develop a wide range of problem-centred activities and materials to aid problem-solving development in learners; encourage learners to think critically, adapt ideas that make sense to them, invent many different ways to solve problems and to expand and enhance the development of geometrical concepts though problem-solving activities.

The teacher should plan situations that develop thinking on the part of the learner. In this regard, he looks for thought-provoking activities, a wide range of problem-related materials to help learners think about solutions and opportunities for learners to pose problems for others to solve.

To user a problem-based approach successfully, the teacher must let learners struggle together towards the solutions of problems without suggesting strategies to solve the problem yet provide sufficient scaffolding to keep learners interested and on-task (Erickson 1999:519). Scaffolding may involve additional geometry instruction for unknown content when necessary and questioning techniques that help learners to draw out their own thinking are essential for success in this approach. To be able to do this the teacher must be knowledgeable about the subject.

The teacher must select tasks that are challenging to the learners, tasks that link with the learners' experiences and for which learners can see the relevance of ideas and skills they already possess (Hiebert et al 1996:16). In essence the teacher provides the correct teaching and learning environment (methodology).

4.2.2.4 The role of learners

In the problem-centred approach learners should be free to express themselves without fear of reprisal. Mistakes are often as constructive as the correct solution strategies they offer in helping the learners understand the mathematics involved (Erickson 1999:518). Learners should realise that learning means learning from others and must take advantage of other's ideas and results of their investigations. This is social interaction which leads to socio-constructivism (constructing knowledge as a collective).

4.3 CREATIVITY IN GEOMETRY EDUCATION

The study aims at describing creativity in geometry education. For this purpose some creative processes in the learning of geometry are mentioned.

4.3.1 Synectics

Stein (1975:172) defines synectics as the joining together of different elements which at a face value do not appear to have anything in common. Synectics is based on the use of metaphors and analogies in such a manner as to achieve creative results. Cangelosi (1996:168) states that metaphors and analogies are used to stimulate learners to reconstruct old ideas, thereby promoting divergent thinking. In this manner, learners free themselves of convergent thinking. Using metaphors and analogies to describe a concept leads to a better understanding of such concept and stimulates creativity on the individual using the metaphors or analogies. In the context of the study, learners get a deeper understanding and more insight of geometrical concepts.

On the other hand there are problems with which we are so familiar and are too close to us such that we cannot see the total picture. We tend to concentrate on the individual parts and do not see how these parts fit onto the whole. An appropriate analogy or metaphor provides us with the necessary distance so that we get a better view of the problem and move to a creative solution (Stein 1975:172).

4.3.1.1 Metaphors and analogies

Some examples of metaphorical thinking and analogies from the literature include the Wright brothers who based their work on turning and stabilizing the airplane on observations of buzzards keeping their balance in flight, Kekule, imagining a snake swallowing its tail, thought of carbon atoms in a ring rather than in a linear chain and Archimedes formed an analogy of his body submerged in a bath to conclude that a body displaces its own volume.

4.3.2 Geometry and poetry

Thompson (1994:88) states that she encouraged her students to find geometry in Sandburg's poems. This was essentially a project for the learners and had to have both a poem or an except from the poem and an illustration on a poster board. She found that it was best to let one student read a poem and let the others discover the geometry as they looked at the illustration. The project was fun for all and students became aware of the geometry in their lives.

Learners can use poems to appreciate and as a means of internalizing geometric concepts.

4.3.3 Tangrams

Teeguarden (1999:57) states that in most cases learners as well as their educators view mathematics as only computation and skill-based exercises while there are many activities such Tangram Art which enrich the curriculum and provide an opportunity for learners to see that mathematics and creativity really do go hand in hand i.e. there is a correlation between mathematics (and in particular geometry) and art (a manifestation of creativity).

Legend has it that a Chinese Emperor, Tan accidentally broke his mirror into seven pieces. He then required (as a game) people to create designs using all the seven shapes. This is how tangrams originated (Dunkels 1990:38-42).

Tangrams, an ancient 7-piece Chinese puzzle provide children with an opportunity to create numerous pieces and designs. These designs could be letters of the alphabet, all numerals zero to nine, hundreds of human and animal figures (Teeguarden 1999:181). In handling and manipulating the pieces of a tangram in their designs, learners also internalize the properties of the shapes they use.

4.3.4 Tessellations

According to Clauss (1991:52) the word tessellation comes from a Latin word which means to pave with tiles. The tiles are in the form of polygons. A tessellation can be defined as the pattern fitting together of polygons so as to cover a plane in such a manner that there are no spaces between them and no overlapping. Clauss (1991:52) states that investigating whether the fitting together of various polygons to make a tessellation can be enjoyable and exciting mathematics-inquiry activity for elementary and middle school learners.

Investigating tessellations further affords learners the opportunity to familiarize themselves with various polygons such as triangles, trapeziums, parallelogram, pentagon, heptagon and octagon (the learners learn the properties of these shapes) and learners also learn the language of tessellations such as ‘flip’, ‘rotate’ and ‘transform’.

4.3.5 Pentominoes

A pentomino is a two-dimensional shape made from five congruent squares such that each square has at least one of its sides in common with another square (Onslow 1990:5).

Pentominoes give learners the opportunity to examine concepts of congruence, similarity, tessellations, perimeter, area and volume in a problem solving environment. From the twelve pentominoes learners can determine which ones have no symmetry, one, two or four lines of symmetry. Many interesting tessellations can be made by sliding or flipping a pentomino.

4.3.6 Essays, posters, songs and musical instruments

Writing essays and posters, creating music (songs), musical instruments, art products and drama using geometric concepts reflects creativity in learners and enhances understanding. Parnes (1975:235) refers to the above activities as creative experiences. The study focuses on such creative experiences for the learners. These creative experiences help learners to link geometric concepts to their environments.

4.3.7 Creative correlations in geometry

Learners should have a vocabulary list of the words they encounter each day in their learning of geometry. They should write the meanings as well as illustrations of such words. This is critical for mastery of geometry concepts by second language learners of geometry. Writing words such as triangle as tr^{a}angle not only helps learners internalize the meaning of such words but also gives learners the opportunity to be creative (Westcott 1978). In this study learners are required to have a vocabulary list as well as be creative in writing words as in the example of triangle above.

4.3.8 Further scenarios for creativity in geometry

Teeguarden (1999:57) states that many mathematical structures and concepts lead to patterns that can be transformed into geometrical and other artistic designs. Such patterns afford students an opportunity to see that mathematics (in particular geometry) and creativity really go hand in hand. In outlining principles, the art and craft in curriculum design, Goldenberg (1999) puts a case for connected geometry. He states that geometry can connect with diverse interests such as science, mechanical design and play. Connecting geometry with other fields leads to the acquisition of the habits of mind and this promotes creativity in geometry learning. In mechanical design, for example, one's creativity is called to the fore. Furthermore connecting geometry to art enriches the curriculum and make students realize the importance of geometry in the real world. Art as tangrams and tessellations (geometric structures) reveal the partnership between geometry and art. Teeguarden (1999:57) reports that, together with daughter Sheryl Teeguarden Riley they developed and used beautiful math-art projects with 7-9 year olds at workshops. The marh-art projects included tangram art, magic square art, tessellation art, mod math art, fibonacci art, fractical art and flexagoms. These projects highlighted geometric concepts such as symmetry, patterns, magic squares, spatial perception, reflections, translations and rotations (Teeguarden 1999:62).

Usnick and Ford (1999:63) reiterate the above fact that mathematics and art are not on opposite ends of a continuum, but that there are connections between the two domains. The authors mention that the most glaring connection between the two disciplines is geometry. This further emphasizes the importance of geometry in relating mathematics to the real world.

Tessellations, as has already been mentioned, are projects that combine mathematics and art. When polygons are fitted together to cover a plane with no spaces between them and no overlapping, the pattern formed is called a tessellation. According to Giganti and Cittadino (1990:6), students have to be encouraged to develop and explore their own artistic creativity and to exercise their spatial sense within mathematics through the art of tessellation.

Kurina (1999:133-136) discusses an experiment conducted with 30 students aged 14-15 years. The students were given 2 sets of problems aimed at developing creativity in the process of their solutions.

According to the author, geometry is an opportunity for ‘visual creativity’. Questions such as ‘how otherwise could it look like?’ or ‘how could it continue?’ are vital in developing this type of creativity.

On the basis of the solutions to the four problems given to the students, Kurina (1999:136) concluded, amongst other things, that creativity is possible only if individuals are given many opportunities, in the experiment shapes encompassing many symmetries, geometry is opportunity to ‘visual creativity’, the cultivation of creativity requires time and the atmosphere should be conducive for such activity.

In an interesting manner and in an activity where students are active and discover geometric concepts for themselves (in this case finding all the eleven nets of a cube thereby representing a three dimensional object in two dimensions and vice versa), Mueller-Philipp (1999:138-141) describes the creative act of students in finding the nets. They are being original (a task they have not done before) as they roll the cube around, often in their minds only, in order to discover the nets. The exercise is extended to find pentominoes (the nets of an open box).

Meissner (1999:177-178) describes a procedure to be followed in constructing a diversity of solids from different types of polygons and circle sectors (bricks). Manipulating these pieces helps in constructing solids that shows creativity of individuals, develops spatial abilities and enlarges manual skills.

4.3.9 A model for creativity in geometry

The following is an outline of the cognitive processes inherent in the model for mathematical creativity (Fig. 1). In engaging in activities of creativity during geometrical problem solving, for example, in drawing a poster to illustrate a geometry concept(s), one has to generate ideas (lateral thinking), draw, evaluate his product (critical thinking) and communicate about his product (describe).

- Investigate
  - understanding the problem)identify, define, draw) (Polya 1973), attribute listing (Davis 1971) and identification of the problem (Quellmalz 1987).
- Relate
  - compare, sort/classify, matching/associating, apply ideas (Knulik & Rudnick (1993) and synthesis (Quellmalz 1987).
- Create
  - devise a plan (Polya 1973), generate ideas (Knulik & Rudnich 1993), self-discovery and movitation
- Evaluate
  - critical thinking (Ennis 1987), looking back (Polya 1973), judgements and generalizing (Quellmalz 1987) and reflection.
- Communication
  - communicate results (Goldenberg et al 1997), describe, state and explain.

4.4 DIVERGENT THINKING IN GEOMETRY TEACHING AND LEARNING

Divergent thinking in the learning of geometry is evidenced by the learners' flexibility in selecting and constructing a large number of possible ideas, associations and implications for geometry concepts. Learners dig deeply and widely into the implications of geometry concepts during the problem solving process. This, according to Erickson (1999:516) is the situation in the problem-centred approach to the teaching of mathematics. Learners explore problems, make conjectures and generalizations about mathematical concepts and processes. They do not seek one solution to a problem as is the case in convergent thinking. Open ended problems enhance divergent thinking in learners.

4.4.1 Cognitive processes

In the context of this study, the cognitive processes include critical thinking, lateral thinking, comparing, describing, defining, generalizing, evaluating, induction, deduction, visualizing, conjecturing, transforming, reflecting, manipulating, communicating, symbolizing, analyzing, synthesizing, equating, investigating, counting, problem solving, connecting and sorting.

The processes mentioned above are used during the geometrical problem solving process within the context of the problem-centred approach.. Learners have to reflect on the meaningfulness of answers to problems i.e. does the answer reflect reality? and how does the answer relate to the learner's experiences (critical thinking); lateral thinking i.e. not digging at the same spot during the problem solving but digging elsewhere; compare i.e. look for similarities and differences between and amongst geometric shapes; describe i.e. mention the properties of geometric objects in a clear and unambiguous manner, define geometric shapes and concepts; observe patterns and be able to make generalizations from such observations; in group discussion, learners evaluate the appropriateness of each member's contribution to the solution of a problem; use inductive reasoning to arrive at generalizations; make deductions on the basis of their knowledge of geometry concepts; visualize not only by seeing concrete objects but also see these objects in their minds (imagery); make conjectures and also verify such conjectures through experimentation; transform a given shape by changing its orientation through, for example, flipping the shape; reflect on their solutions to problems and communicate these solutions; manipulate geometric shapes, for example, tessellate polygons; analyze, for example, a polyhedron as having faces, vertices and edges; put together parts/properties of a shape so as to arrive at the shape; arrive at relationships between or amongst geometric concepts so as to arrive at an equation describing the relationship of such concepts as in Euler's formula, count and put shapes that belong to the same group as in sorting objects.

4.5 CREATIVITY AND DIVERGENT THINKING IN GEOMETRY TEACHING AND LEARNING IN A PROBLEM-SOLVING CONTEXT

The following model outlines the role of creativity, divergent thinking, problem solving and teaching in context in the problem-centred approach to the learning of geometry.

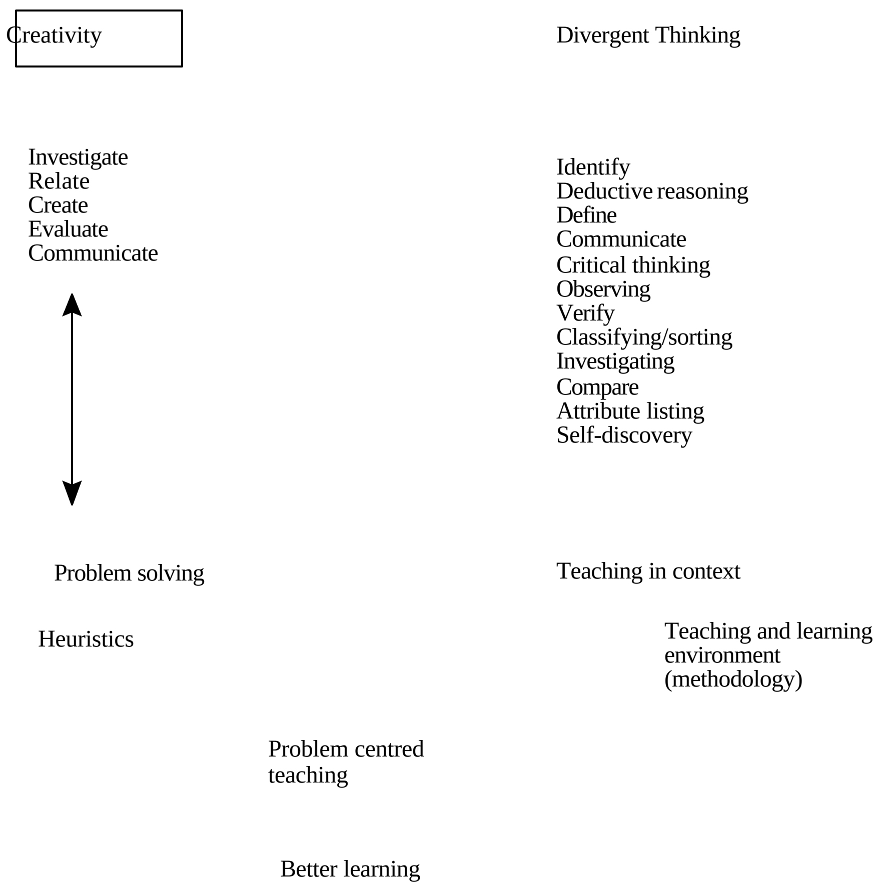


Fig. 29 The problem-centred approach to the teaching and learning of geometry

4.6 CONCLUSION

In line with what constructivism stands for, the problem-centred approach to the learning of geometry points that the learner should be given problems to solve on his own with the educator acting as facilitator. In this exercise, the learner interprets the problem on his own, decides on strategies to follow and evaluates the solution to the problem in the light of whether it makes sense on the basis of his experiences and whether it depicts reality. The problem-centred approach allows learners to think and be creative and not to absorb information without reflection.

Visualization is important in the problem-centred approach to the learning of geometry. The learner solves problems that require him to think and visualize geometric shapes - draw such shapes, discover their properties, make conjectures and prove these and make generalizations pertaining to such shapes. It is from the solution of problems that the learners’ knowledge emerges.

An individual has to be creative and be a divergent thinker in order to solve problems. Activities of creativity and divergent thinking provide rich learning experiences for learners to explore geometry concepts and to be able to solve problems and these, in turn, enhance the conceptual understanding.

The problem-centred approach sets the foundation for creative and divergent thinking activities to influence an be influenced by problem solving and teaching in context (the correct teaching and learning environment) and this culminates in better learning.

Divergent thinking is evidenced by qualities such as the ability to conjecture, construct, model, reflect, calculate, articulate, sort, transform, solve problems, make patterns, generalize, analyze, synthesize, visualize, evaluate, represent, be a critical thinker, deduce, induce, identify and think horizontally and vertically. The empirical study in Chapter 5 aims at creating opportunities during geometric problem solving such that these qualities can prosper. The empirical study in Chapter 5 also gives learners the opportunity to use their culture (songs, poems, musical instruments, analogies, metaphors and drama) to illustrate geometric concepts. Learners are usually creative in this respect.

Creativity and divergent thinking are related skills in the sense that both activities require the use of the same cognitive processes (see diagram). The cognitive process mentioned above are elements of creativity and divergent thinking and will be used in the problem-centred teaching and learning of geometry in the empirical study.

CHAPTER 5

METHOD OF RESEARCH

5.1 INTRODUCTION

The study attempts to identify and highlight situations in a problem-centred geometry classroom wherein creativity and divergent thinking are used in the internalization of geometry concepts. In order to accomplish this, firstly a review of literature will be made to highlight research on creativity and divergent thinking in geometry education and secondly there will be a description of the empirical study that was conducted to identify situations of creativity and divergent thinking during various geometrical problem solving processes with Grade 7 learners using worksheets, questionnaires and interviews, and tests.

In the literature review, mention will be made of characteristics of creativity such as the abilities to form and test hunches, making conjectures and validating them (Taylor and Holland 1964:21-23), phases of creativity such as preparation, incubation, illumination and verification (Fourie et al 1991:149) (as well as scenarios for creativity in geometry such as Tangram Art and Tessellation Art (Teeguarden 1999:181) and finding the nets of a cube (Mueller-Philipp 1999:138-141). The various cognitive processes such as comparing, observing, sorting, imaging, summarizing, interpreting, investigating and reasoning skills which are manifestations of divergent thinking in geometry are also mentioned.

The problem-centred approach (explained in Chapter 4) was used. The worksheets are designed such that the learners work in groups during problem solving. The researcher will, in every session, motivate learners to concentrate on the fine details and the process i.e. how they arrive at their solutions, how they use their informal language to explain the process. It is about the understanding (conceptual understanding) by the learners and not about their success in completing the exercises (training in executing certain skills). The nature of the activities in the worksheets as well as the groupwork during such activities aims at encouraging learners to be creative thinkers who collectively negotiate meaning for common understanding.

The quantitative research aims at finding out what learners know prior to the geometrical problem solving process as well as measure the growth of creativity and divergent thinking during the problem solving process. The PSS is used in the analysis of results in the questionnaires and tests. The qualitative study uses the observation method to report on activities in the worksheets, these being punctuated by interviews and questionnaires to find out learner feelings and progress.

Learners also write essays, poems, metaphors, analogies and posters on geometric concepts. Furthermore, learners design various patterns using pentominoes, colour tessellating shapes thereby highlighting the correlation between art and geometry, make interesting shapes like human beings and animals using tangram pieces and construct musical instruments using geometric shapes.

The sample consisted of twelve Grace 7 learners at Northcrest High School in 2001. There were eight boys and four girls. All the learners wrote the three test (pre-test and two post-test) and the two questionnaires administered. Eight learners (of which five were boys and three were girls) were interviewed. All the twelve learners were engaged in specific tasks of creativity using geometric concepts such as composing songs and writing posters.

5.2 RESEARCH DESIGN

There will be a review of literature to find out the elements of creativity and divergent thinking in geometry education. The empirical study is divided into quantitative and qualitative research. Activities in the worksheets will require the use of cubes, various polygons cut from cartridge paper as well as diagrams appearing on the worksheets. Interviews will be recorded on audio-tape.

5.2.1 Literature study

According to Taylor and Holland (1964:21-23) creative individuals have the tendency to strive for more comprehensive answers, look at the problem from different viewpoints and find different solutions to the problem; are intellectually thorough, every step towards the solution of a problem is justified and there is no ambiguity to their inferences and have a tendency to resist idea reduction and its opposite pole, the willingness to reduce ideas. These characteristics should be manifest in creative acts. The study aims at identifying such acts in geometrical problem solving.

According to Fourie et al (1991:149), the creative act has four phases namely, preparation, incubation, illumination and verification. Ford and Usnick (1999:20) state that the four phases are analogous to the four phases of Polya's (1973) model of mathematical problem solving viz. phase 1 - understand the problem, phase 2 - devise a plan, phase 3 - carry out the plan and phase 4 - look back. These phases offer a general strategy for problem solving and could be used in geometry.

The possession of traits such as being independent, confident, self-assertive, radical, non-conforming and imaginative are necessary but not sufficient for creativity on the part of the individual. Individuals have to be trained to possess creative attitudes, abilities and thinking techniques (Davis 1971:262). For the purposes of this study, the researcher will concentrated on thinking techniques such as attribute listing, morphological-synthesis, checklisitng and synectics (Davis 1971:263).

Teeguarden (1999:57) states that many mathematical structures and concepts lead to patterns that can be transformed into geometrical and other designs. Such patterns afford students an opportunity to see that mathematics (in particular geometry) and creativity really go hand in hand. Furthermore, connecting geometry to art enriches the curriculum and makes learners realise the importance of geometry in the real world. Learners were afforded the opportunity to make shapes from tangram pieces as well as tessellations from polygons.

According to Kurina (1999:133-136) geometry is an opportunity for 'visual creativity'. Questions such as 'how otherwise could it look like?' or 'how could it continue?' are vital in developing this type of creativity.

In an interesting manner and in an activity where students are active and discover geometric concepts for themselves (in this case finding all the eleven nets of a cube thereby representing a three dimensional object in two dimensions and vice versa), Mueller-Philipp (1999:138-141) describes the creative acts of students in finding the nets. They are being original as they roll the cube around, often in their minds only, in order to discover the nets.

Nickerson (1994:414-430) lists judging, critical thinking, organizing thoughts and articulating concisely and coherently and distinguishing between logically valid and invalid inferences as attributes of good thinking. Nickerson states that students do not acquire these skills easily or well on their own or through ordinary instruction and hence would benefit from direct instruction / teaching for thinking. The study aims at creating situations in the geometry classroom such that learners can acquire the above named attributes. In addition to these attributes, the study aims at learners acquiring cognitive processes such as the ability to compare, observe, summarize, classify/sort, interpret, imagine, hypothesize/conjecture, apply facts and principles in new situations and investigate/design projects in geometrical problem solving (Raths et al 1986:6-14).

The study also aims at developing reasoning skills such as the identification of the task or problem type, definition and classification of essential elements and terms (**analysis**) and **judgements** and connection (**synthesis**) of relevant information, and procedures for drawing conclusions (**generalizing**) and/or **solving problems** (Quellmalz 1987:94-95).

5.2.2 Empirical study

According to Fraenkel and Wallen (1990:143), there are two fundamental types of numerical data, viz. quantitative and categorical data. The other type of data is qualitative. The data to be analyzed in this study is both quantitative and qualitative.

5.2.2.1 Quantitative research

Prior to commencing with the activities in the worksheets, the Grade 7 learners wrote a pre-test comprising some of the concepts that are covered in the study. By the end of May 2001, the same group of learners wrote post-test 1 on the material that had been covered in the worksheets, interviews and questionnaires. The learners wrote post-test 2 by the end of August 2001 on material based on the worksheets, interviews, questionnaires and on specific activities that called for creativity on the part of learners, for example, composing songs, writing essays, posters, poems, metaphors and analogies based on geometric concepts. A post-test 2 was written to find out firstly if learners had retained material covered between the pre-test and post-test 1 which had been handled between March and May with contact session between the teacher and the class of four periods of thirty five minutes a week an secondly to find out if the above activities had helped reinforce/consolidate geometric concepts mentioned prior to writing post-test 1. Learners only did the specific activities that called for creativity on their part as mentioned above only in August then wrote post-test 2. Contact sessions as described above were adhered to. Learners had not been engaged in any geometry activity in the classroom between the end of May and the beginning of August. The researcher was the teacher during the teaching episodes as described above.

Statistical methods for the analysis of data such as the mean, mode, median and standard deviation were used to measure the growth in the spatial development of the learners as a result to the exposure and non-exposure to the material that was covered.

A bar graph was used to illustrate learner performance on the three types of tests mentioned. The SPSS was used to determine the reliability and validity of the data collected in the questionnaires and tests.

Another graph (straight lines) was used to show learner performance in the three tests. Each line shows the progress each learner made from the pre-test to the post-test 1 and to post-test 2. Twelve learners wrote the three tests.

5.2.2.2. Qualitative research

There will be a detailed description of the activities and objectives of the worksheets as well as the questionnaires administered and interviews conducted.

a) **Research method**

Stallings and Mohlman (1988:469) state that there are many techniques that are used to observe and record human behaviour and physical environments. These include, amongst others, narrative descriptions of unfolding phenomena during geometrical problem-solving. In this study, observations will be recorded with the audiotape as well as with paper and pencil.

Stallings and Mohlman (1988:473) state that narrative descriptions are advantageous in that the context of the observation can be described in a rich and holistic manner and that the natural sequence of events is preserved.

Structured observation is characterized by three stages viz. the recording of events in a systematic manner as they happen, the coding of such events into prescribed categories and the subsequent analysis of the events to give descriptions of teacher-pupil interaction (Galton 1988:475).

b) **Data collection**

To collect data, worksheets, questionnaires and interviews were used. In addition learners were engaged in activities such as designing patterns using pentominoes and making musical instruments using geometric shapes. Recording the interviews on audio-tape allows for repeated observation thereby increasing the likelihood of inter observer agreement (Galton 1988:475). Formative assessment was used in the study with a view of collecting information on learner performance so as to improve instruction.

(i) Worksheets/Appendix A

During instruction, learners worked through worksheets 1- 10 (Appendix A). In these worksheets learners responded to activities based on polygons, polyhedra, nets of a cube and other polyhedra, pentominoes, tessellations, tangrams and three dimensional puzzles. Learner responses were analysed and depending on such responses, remedial work was done (formative assessment).

(ii) Questionnaire/Appendix B

Wolf (1988:478) defines a questionnaire as a self-report instrument which is used for gathering information about variables of interest to the investigator. It consists of a number of questions on paper that the respondent reads and answers. A questionnaire is based on the assumptions that the respondent understands the questions, has the information to answer the questions and is willing to answer the questions honestly.

In this study, the respondents, Grade 7 learners, answered questions which are based on the work covered in the worksheets handled previously. This serves to measure the growth of spatial development of the learners. A description of learner responses to the items in the questionnaire was made.

(iii) Tests/Appendix C

Learners wrote a pre-test and two post-tests. Learner responses to these tests are analysed in Chapter 6.

The pre-test was written at the beginning of March (05.03.01), post-test 1 on 18.05.01 and post-test 2 on 14.09.01.

(iv) Interviews/Appendix D

Interviews were conducted with individual learners. Observations from such interviews were recorded with an audio-tape as well as with paper and pencil (Stallings & Mohlman 1988:471). Galton (1988:475) states that mechanical recording using an audiotape allows for repeated observation thereby increasing the likelihood of interobserver agreement. Hitchcock and Hughes (1989:80-85) distinguish between structured and semi-structured interviews. In the former, the interviewee answers either yes or no to a question while in the latter, the interviewer asks for the reasons or an explanation for the yes or no. The latter situation asks for insight or divergent thinking on the interviewee and this is the concern of this study i.e. to search for elements of creativity and divergent thinking in geometric problem solving. A description of learner responses to questions during the interview was made.

Interviews were conducted during the period between the pre-test and post-test 1.

(v) Learner portfolios/Appendix E

Learners were required to write songs, posters, poems, essays, metaphors and analogies on geometric concepts; write words they encountered in geometry illustrating the meanings of such words in their writing; colour in their designs in tessellating shapes; make designs using tangrams; tessellate pentominoes thereby arriving at various patterns of such pentominoes and construct musical instruments using geometric shapes.

Samples of the above activities have been included in S.6.2.1.5 – 6.2.1.14.

Learners engaged in these activities in between post-test 1 and 2.

5.3 RELIABILITY AND VALIDITY

Galton (1988:476) states that reliability serves to indicate how free a particular measurement is of error. The SPSS is used to determine the reliability of the collected data.

According to Fraenkel and Wallen (1990:90) the validity of an instrument “... revolves around the defensibility of the inferences researchers make from the data collected through the use of an instrument”. In this study, validity is ensured by the amount and type of evidence there is to support the researcher’s interpretations according to the data collected.

5.4 CONCLUSION

The foregoing has been an outline of methods and tools t obe used in the collection and processing of data.

In the collection of data, worksheets, questionnaires, interviews and tests are used. Furthermore, learners are required to develop portfolios containing, inter alai, poems, songs, posters and musical instruments using geometric shapes.

In processing data, there will be a description of learner responses aimed at identifying elements of creativity and divergent during geometrical problem solving. It is envisaged that as a result of using such elements, learners will have a better understanding of geometry concepts. The SPSS will be used in determining the reliability of questionnaires and tests.

CHAPTER 6

DATA INTERPRETATION

6.1 INTRODUCTION

Learners were engaged in several activities in geometrical problem solving. These included worksheets, questionnaires, interviews, tests and they had to write songs, poems, essays, posters, geometry words in such a manner that the writing depicted what the word means, metaphors and analogies. Furthermore, learners had to tessellate polygons and pentominoes thereby producing various design and coloured these to produce works of art. Using tangram pieces, learners had to produce shapes of human beings, animals or letters of the alphabet using all the seven pieces of the tangram. Learners also constructed musical instruments using geometric shapes.

The first worksheet required learners to describe the properties of plane figures, define a polygon and compare polygons in their own words. These exercises encouraged learners to develop their own informal language to describe what they do and understand during geometrical problem solving. Furthermore, learners had to find the sum of the angles of various polygons.

The second worksheet required learners to classify polyhedra and to use standard language to label such shapes.

The third worksheet required learners to define three dimensional shapes, such as a pyramid, prism, cylinder and a cone, a face, an edge and a vertex in their own words. Furthermore, learners had to derive Euler’s formula.

In the fourth worksheet, learners had to define the net of a cube and find eleven nets of the cube.

The fifth worksheet required learners draw nets of prisms and pyramids.

In the sixth worksheet learners had to define a pentamino and a hexomino and to find all the pentominoes and hexominoes.

The seventh worksheet required learners to tessellate various simple shapes (polygons) and to identify these in everyday life, cultural artifacts and nature. Learners had to describe the properties of the polygons they tessellated. Furthermore, learners had to tessellate pentominoes. Lastly learners had to define a tessellation in their own words.

The eighth worksheet, also on tessellations, required learners to make designs using various polygons, conclude whether all polygons tessellate and to use mathematical language to explain and describe patterns of shapes (using the language of tessellations and transformations viz. translate, rotate and reflect). The latter exercise encouraged learners to work co-operatively (groupwork was used) too, as in other activities, negotiate mathematical language for common understanding and communication.

The ninth worksheet required learners to create designs from tangrams, observe various geometric shapes in their designs and to communicate about their designs.

The tenth worksheet required learners to construct seven puzzle pieces (Worksheet 10) using cubes and to use the seven pieces to construct more complicated structures.

The worksheets described above were punctuated by questionnaires and interviews based on the worksheets that had been covered.

Learners wrote a pre-test prior and two post tests. The study lays a foundation for the application of the problem-centred approach to the teaching and learning of geometry as described in S. 4.2.

6.2 QUALITATIVE RESEARCH

The qualitative study uses the observation method in reporting learners' activities in the worksheets, these being punctuated by interviews and questionnaires to find out learner feelings and progress. The learners' responses to the pre-test and the two post-tests administered are also analysed. There is a description as well as a statistical analysis of the learners' responses.

6.2.1 Divergent thinking and creativity in geometry teaching and learning

Divergent thinking includes cognitive processes such as critical thinking, analysis, synthesis, interpreting, investigating, sorting, conjecturing, induction, deduction, comparing, generalizing and evaluation. The worksheets are based on the use of these cognitive processes in geometrical problem solving. Creativity and divergent thinking are related skills in the sense that the elements of creativity depicted on the model for mathematical creativity (Fig.1, p. 25) are cognitive processes typifying divergent thinking. On the other hand, in creating- bringing about an innovation one has to generate original and effective ideas – lateral thinking, synthesizing ideas and critical thinking (divergent thinking).

The problem-centred approach to the teaching of geometry cannot be adequately administered in the formal setting i.e. using worksheets, interviews, questionnaires and tests. Learners were also required to compose songs, write poems, construct musical instruments, draw posters, have metaphors and analogies on geometric concepts, alternative forms of assessment. Their creativity and divergent thinking is called to the fore in engaging in the activities. Post test 2 was also used to determine the effectiveness of these activities on the internalization of geometry concepts.

6.2.1.1 Worksheets/Appendix A

Twelve learners in Grade 7 at Northcrest High School worked on ten worksheets from March to May 2001. These worksheets appear as Appendix A. The following is the analysis of the learners' responses to the items appearing in those worksheets. The worksheets took an average of three to four periods of thirty five minutes each to complete.

a) **Worksheet 1**

The worksheet is intended to assess the learners' knowledge of geometric concepts. In responding to the questions in the worksheet, they communicate their perceptions in groups and also write these down. For this purpose the worksheet contains eleven geometric shapes: a triangle, six quadrilaterals, a pentagon, a hexagon, a circle, and a four-sided shape consisting of three straight lines and a curve. In addition learners were given these shapes in the form of cartridge paper so as to manipulate them. Firstly learners are required to describe the properties of each shape in their own words and secondly to give examples from their environment that resemble the shape in question.

Triangle

An equilateral triangle, isosceles and scalene triangles and a right-angled triangle had been cut out of cartridge paper. Each group out of the four groups manipulated each of these triangles at a given time.

On mentioning the properties of a triangle, the groups stated that a triangle has three straight lines (sides) and there are three corners. The fact that they mentioned corners instead of angles showed that they did not as yet possess the formal geometry language to describe this property, and that they communicate using their informal language to describe what they understand during geometrical problem solving.

Learners were then required to give examples of shapes in their environment that resemble triangles. The groups mentioned the roofing of a rondavel, a tent and a sledge. In giving these examples learners were actually referring to the faces of these polyhedra that they are triangular in shape. The distinction between two-dimensional and three-dimensional shapes had not yet been made. This is in line with van Hiele (1986:34-47) level one geometric throught-objects are recognized globally. Nonetheless, learners were able to relate triangles to the shapes in their environment.

The learners attention was drawn to the four triangles cut out of cartridge paper they had manipulated. They were asked how these triangles differed from one another. Learners did not know the names of the four triangles at this stage. The groups were unanimous that in an 'equilateral triangle' all sides appear to be equal-amacala ayalingana and they intuitively felt that the corners also appeared to be equal, one group mentioned that in an 'isosceles triangle' two sides appear to be equal and the 'base angles' also appear to be equal, all groups were unanimous that all sides are unequal and that one angle (which was an obtuse angle) is certaining greater than each of the remaining ones and lastly what was striking about the right angled triangle was that one learner mentioned that one angle ifana necorner yendlu - referring to the right angle that it is like the corner of a house. As a result of having looked at the differences of the four triangles, learners mentioned their properties (level 2, van Hiele). This ability is a result of having been specifically directed to find such differences (instruction) – mediated learning experience (Feuerstein1983). To move from level 1 to 2 is a result of instruction and not biological maturity (van Hiele 1986). The learner who mentioned that a right angle is like a corner of a house relates the concept of a right angle to his previous experiences so as to make sense of the concept (constructivism).

The researcher mentioned the names of the four triangles handled and guided the learners in verifying their guesses (intuition) that in an equilateral triangle all sides and angles are equal and in an isosceles triangle the base angles and two sides are equal while in a scalene triangle the sides are unequal and lastly in a right-angled triangle one angle is equal to 90E and is called a right angle and that there will always be a longest side, the hypotenuse by using a protractor and a ruler.

Learners have identified properties of the four triangles and communicated their perceptions/observations in their groups. They have related triangles to shapes in their environment. Learners have also made conjectures about, for example, the angles of an equilateral triangle and verified these – have experimented (Goldenbeg 1999). Tjese processes are essential elements of creativity and divergent thinking.

Quadrilaterals

The learners' attention was directed to all the polygons appearing in the worksheet. They were required to mention the polygons they thought belonged together (sorting) because they had something in common. Since the learners were just mentioning that they had straight lines and angles, the researcher had to direct learners to the number of sides the polygons had. On the basis of this hint they picked up figures 3.2, 3.3, 3.4, 3.5, 3.6 and 3.7 as belonging together since they had four sides. One learner argued that fig 3.10 also had four sides (criticalthinking). The learners were then directed to the type of sides this shape has in comparison to the other figures and it was agreed that the shape did not belong to the rest. The researcher then mentioned that figures 3.2 - 3.7 are called quadrilaterals as they have four sides (quad means 4).

Rectangle

Three groups described a rectangle as a shape with four sides and four corners. One group mentioned that there are two long sides and two short sides. This particular group was asked what they thought was the relationship between the two long sides or the two short sides. The group agreed amongst themselves that the sides appear to be equal. The researcher asked them to verify that by measuring the sides. All the groups were asked what could be concluded about the angles/corners of a rectangle. The same learner who had described a right-angle as like a corner of a house had the same response for the angles of a rectangle. If that was the case then, learners concluded that all the angles of a rectangle are equal and they are 90E each. Learners mentioned some properties of a rectangle – attribute listing, a thinking technique (Davis 1971), level 2 reasoning (van Hiele 1986). Learners gave examples of a chalkboard, desk, plank, door, duster, window, brush, book and house as rectangles. (Level 1 objects are rcognised globally). In giving properties of a rectangle learners are on level 2 while in giving examples of a rectangle, they are on level 1 as they do not refer to the faces of the 3D objects mentioned but the whole object. This is in line with Guitterez et al’s (1991) findings that a learner may operate on a certain level on one task, but may be on a different level on a different task. As in the case of examples of a triangle the distinction between 2D and 3D objects has not as yet been made.

Square

The researcher drew a square on the chalkboard and asked the groups to describe the properties of the shape. The groups quickly pointed to the fact that a square has four sides and four angles which are right angles. Their attention was then specifically drawn to the sides of the shape. They answered that they appear to be equal. Learners were asked to verify this conjecture. Learners gave a box, window, book, chessboard, towel and tiles as examples of a square.

Parallelogram

According to the groups, a parallelogram has four sides and four angles - inamacala atsolo amanye agoso. The researcher introduced the concept of opposite sides and angles. Learners were then asked what can be said about the opposite sides and opposite angles of a parallelogram. The groups intuitively felt that the opposite sides and angles appear to be equal. The researcher asked how else this assertion could be verified without using a protractor and ruler. As the learners could not respond, the researcher mentioned that by cutting the parallelogram along one diagonal such that two triangles are formed and super imposing one triangle on the other (by flipping) . In this way it is proven that one pair of opposite angles and the two pairs of opposite sides are equal. This was demonstrated to the learners. The groups were then required to show that the other pair of opposite angles are also equal (using a similar parallelogram by cutting along the other diagonal).

The concept of parallel lines was introduced as lines that will never meet no matter how far we draw them. The groups were asked if the opposite sides of a parallelogram or a square could ever meet if we produced them. They concluded that the opposite sides of a parallelogram are parallel (hence the name parallelogram).

Rhombus

The groups mentioned that a rhombus has four sides and angles and that the four sides appear to be equal and as in a parallelogram the opposite angles appear equal.

Kite

Learners were familiar with this type of shape. They mentioned that it is a kite and that it has four sides and angles. To them the two short sides and the two long sides appear to be equal. The researcher let them verify this using rulers. The researcher then asked the learners to join two vertices of a kite such that the long diagonal is formed. Learners were asked what could be said about the two triangles that are formed. One group mentioned that they appeared to be equal. The groups were given kites cut out of cartridge paper and asked to prove this. They cut the kite along the long diagonal and placed one triangle on top of the other and concluded that the triangles are equal. They further concluded that the obtuse angles in the two triangles are equal.

In the diagram in their worksheet, the groups were asked to draw the other diagonal of the kite in such a say that four triangles are formed. Learners were asked the type of angle formed where the diagonals meet. They recalled that the angles are right angles, hence we have four right angled triangles with the two smaller ones equal and the two bigger ones also equal.

Trapezium, concave quadrilateral, pentagon and hexagon

Groups mentioned that figure 3.6 has four sides and angles, figure 3.7 has four sides and two groups identified only three angles, figure 3.8 has five sides and five angles and figure 3.9 has six sides and six angles. The researcher mentioned that figure 3.6 is called a trapezium Learners were then asked what can be said about the sides of such a shape. Groups stated they were unequal. The researcher then specifically referred groups to the pair of 'parallel' lines and asked learners if those two lines could ever meet. Groups realized that those were parallel lines and that a trapezium has a pair of parallel lines.

The researcher mentioned that figure 3.7 is a concave quadrilateral, explained the concept and showed learners that the shape has four angles. Lastly figure 3.8 and 3.9 are referred to as a pentagon and hexagon respectively. The concepts were explained to the learners i.e. pent-means 5 and hex means 6.

Learners were able to sort/classify polygons. One learner mentioned the fact that Fig. 3.10 also has four sides and should be included amongst quadrilaterals. This showed reflection/critical thinking on the part of this learner. However, learners realized the figure has a curve as one of its sides and is thus different from other figures. Learners also made conjectures and verified these (experimentation/investigation).

Circle

The groups stated that a circle is round, has no sides and corners. They gave the wall of a rondavel, a ball, coin, wheel, lid, juice container, rim, roll and tyre as examples of circular objects. Again the distinction between 2D and 3D objects has not yet been made.

The worksheets also required learners to mention why figures 3.10 and 3.11 are different from the other figures. The groups mentioned that the two shapes have curve(s) as their side(s) while all the other shapes have straight lines as their sides.

On the basis of the above answer, learners were told that figures 3.1 - 3.9 are polygons (and figures 3.10 and 3.11 are not) and then required to define a polygon. The groups responded that a polygon is a shape that is made up of straight lines. At this stage, the researcher drew an open shape made of straight lines on the chalkboard and asked learners if this shape was the same as those referred to. They realized that a polygon is a closed shape. The researcher then mentioned that a polygon is a plane figure (flat) as opposed to a box which has volume/thickness. The box is an example of a polyhedron. The researcher also highlighted that the other examples of rectangles learners had given are actually polyhedra with faces being rectangles (polygon).

In stating that a wardrobe is a rectangle, the learners could be looking at the total picture and ignoring the thickness of the wardrobe. Such learners are referred to be at level 1 of the van Hiele levels of geometric thinking. Since learners in this study are Grade 7 geometry learners, it can be assumed that their learning of geometry in the lower grades did not emanate from 3D objects in their surroundings but started with 2D shapes. This disadvantage makes the learners to be at level 1 reasoning even though they are at the senior phase of primary schooling. This seems to point to the wrong approach to the teaching of geometry at the lower grades. Geometry teaching should emanate form the child's everyday of spatial objects in his environment (Freudenthal 1971).

The worksheet also requires learners to compare various polygons i.e. mention the similarities and differences between polygons:

Rectangle and square

The groups mentioned that these shapes are similar in that both have four straight lines and four angles and that the angles are right angles in both shapes. The researcher drew the groups' attention to opposite sides in the two shapes and asked if they could ever meet. They then remembered and concluded that in both shapes the opposite sides are parallel.

According to the groups a rectangle differs from a square in that in a rectangle the opposite sides are equal while in a square all the four sides are equal. Learners were able to compare (a cognitive process) a rectangle and a square.

Parallelogram and rhombus

The similarities mentioned were that both shapes have four sides of which the opposite side are parallel. Both shape also have four angles. Learners needed guidance in concluding that in both shapes the opposite angles are equal., The difference was that in a parallelogram only the opposite sides are equal while in a rhombus all sides are equal.

Parallelogram and trapezium

The similarities that were mentioned are that both shapes have four sides and four angles.

The differences were that in a parallelogram two pairs of opposite sides are equal and parallel whereas in a trapezium only one pair of opposite sides is parallel and are not equal. The opposite sides of a parallelogram are equal while in a trapezium the opposite angles are not equal.

Lastly the worksheet required learners to find the sum of the angles of a quadrilateral, pentagon, hexagon and a heptagon and from these deduce a formula for the sum of the angles of a polygon with n-sides.

The researcher informed learners that the sum of the angles of a triangle is 180E. Using the informal proof by tearing the angles of a triangle and placing them adjacent to each other on a straight line, learners were convinced that their sum is 180E since a straight line is 180°.

Armed with this discovery, learners were asked what could be the sum of the angles of a quadrilateral. Since the learners were lost, the researcher suggested that they divide the quadrilateral into two triangles by joining two opposite vertices. So a quadrilateral is made up of two triangles, and so what will be the sum of the angles of a quadrilateral? One group stated that it will be 180° + 180° = 360°.

Using the above trick, learners could calculate the sum of the angles of a pentagon, hexagon and heptagon. However, one learner mentioned that instead of drawing three triangles in a pentagon we could draw five triangles in which case the sum of the angles would be 180° + 180° + 180° + 180° + 180° = 180° x 5 = 900° instead of 180° x 3 = 540°. The researcher stated that the procedure is to draw the minimum number of triangles i.e. do not subdivide an already existing triangle into further triangles. However this showed critical thinking on the part of the learner.

The following pattern emerged from the above calculations:

A polygon with sum of angles

3 sides n = 3 180° → 1 x 180° → (3 - 2) 180° = 180°

4 sides n = 4 360° → 2 x 180° → (4 - 2) 180° = 360°

5 sides n = 5 540° → 3 x 180° → (5 - 2) 180° = 540°

6 sides n = 6 720° → 4 x 180° → (7 - 2) 180° = 540°

7 sides n = 7 900° → 5 x 180°E → (7 - 2) 180° = 540°

:

n sides → (n - 2) 180°

Learners were able to deduce the sum of the angles of various polygons once they could divide a polygon in question into the minimum number o triangles. However, they could not make a generalization as to what the sum of the angles of a polygon with n sides is.

(a) **Worksheet 2**

Learners had been required to bring packaging material from their homes. Of the materials in each group were prisms and cylinders (learners did not know initially that the materials they had brought were either prisms or cylinders). Learners were then required to put together materials they thought belonged together in a group (classify) and give reasons for their specific groups. Of importance they had to point at a feature(s) that made them group the objects together. Learners really enjoyed working with these materials and, there was a lot of discussion in the group.

'Prisms'

Three groups had put together rectangular and triangular prisms together on the basis that there were rectangles in them; one group put these two types of prisms separately on the basis that one group had triangles in them and so they thought it belonged to a group of its own.

'Cylinders'

All the groups put together cylinders together on the basis that they were all round and broad.

'Cones'

The researcher had provided each group with an ice-cream cone and all the groups put the cones separately on the basis that although the cone is round and has a sharp point at the end - itsolo ekuggibeleni; the bottle that were classified under cylinders do not.

'Spheres'

Learners were shown a ball and asked under which of the preceding it could be placed. Three groups felt that it could be placed under cylinders as it was round while the remaining group was not sure. The researcher informed the learners that the ball is not a cylinder as there is something that it does not possess that the cylinders the learners had brought had. They were required to find this property. One learner realised that a ball does not have a flat base.

'Hemisphere'

Learners were informed that a hemisphere is half a sphere. They were then required to give examples of hemispheres. They had difficulty answering this question whereupon the researcher used an example of an orange (sphere) is cut into two equal parts we obtain two hemispheres. Examples such as half an apple and a ball were then mentioned.

'Pyramids'

Using diagrams from textbooks and photographs of pyramids of the Egyptian Pharaohs, the researcher introduced the concept of a pyramid. Learners felt that pyramids were a group of their own as they are pointed at the end and thus cannot be classified as prisms.

Having done the above activities the research introduced the formal geometry language to describe the above groupings. Learners were also referred to illustrations of prisms, pyramids, cylinders, cones , spheres and hemispheres in van Niekerk's Packaging Booklet (2000).

Each of the four groups was given the above booklet and was required to sort the packaging materials on page 3-6 by making use of the SOLIDS page on pages 19-20 and fill in table 2 (see Appendix 2). This activity was done with a lot of enthusiasm and discussion by learners in their groups. Learners even went to the extent of making elongations of objects in order to justify their having put them under cones.

Learners were able to classify/sort packaging material they had brought from their environment on the basis of properties of such material – an essential cognitive process.

Letting learners classify and in the process discover the properties of three dimensional objects (packaging materials) in their environment is in line with Freudenthal's (1971) recommendation that geometry learning should emanate from the child's environment.

(b) **Worksheet 3**

Learners were required to write their own definitions of a pyramid, prism, cylinder and cone. They had to discuss these definitions and make sure that they were clear. This activity was the most problematic one for learners. This was compounded by the fact that learners are second language learners of mathematics. The researcher encouraged learners to express themselves in vernacular. In this endeavour, learners stated that, for example, a prism is a box - iyibhokisi; a cylinder is a bottle, ifana nombhobho - it is like a pipe; a cone is a container of ice cream and a pyramid is a tomb for Pharaohs.

The activity requiring learners to define a face, edge and vertex of a solid was not so problematic as the one above. Amongst the definitions that emerged were that a face ngumphambili - is the front of a solid, it is flat; an edge is a line where faces meet and that a vertex is where the edges meet. These definitions seem to indicate that learners have grasped the essence of what these geometric concepts mean.

Making learners define (a cognitive process) shaps by relating or mentioning the properties of such shapes, even if it is in vernacular, enhances their (learners) understanding of such shapes

The next activity required learners to count the number of faces, edges and vertices of six solids (see worksheet 3) then fill in the table indicating the number of faces, edges and vertices for each solid. This activity was tackled with great enthusiasm and discussion. In some instances some of the groups made errors in their calculations. One group was particularly good and filled in the table correctly. However, learners could not readily see the relationship between the number of faces, edges and vertices (Euler's formula). The researcher made the learners add the number of faces and vertices and then compare the sum to the number of edges in each solid. One learner realised that:

$F + V - 2 = E$ He actually gave this formula!

This formula can be rewritten as $F + V = E + 2$, the usual form of Euler's formula.

Learners have to be guided to see patterns and make deductions if they cannot do these activities on their own. The level at which a learner can perform with guidance/instruction/mediation from the teacher is higher than that at which he performs independently – Bygotsky's brainchild (Jaworski 1994).

The last activity in the worksheet required learners to define a polyhedron using their knowledge of prisms and pyramids as examples of polyhedra. Three groups gave examples of these solids such as box, book, duster, door and tomb of Pharaohs indicating the difficulty they have in defining concepts. One group defined a polyhedron as a solid with faces, vertices and edges.

The learners' work in the above activities shows that definitions are problematic for them. They are more at home in manipulating objects like counting the number of faces, edges and vertices in solids. Curricula should take cognisance of the fact and pay more attention on applications rather than theory especially in the lower grades of schooling.

(c) Worksheet 4

Nets of a cube

On a diagram appearing on the worksheet was a die revealing three faces with one, four and five dots. Learners were asked the number of dots appearing opposite the face with five and four dots. All the learners responded that there are two and three dots respectively. This shows their knowledge of this particular cube.

Another diagram on the worksheet is a net (fold out) of the die above. Learners were required to fill in the correct number of dots on the net to reflect the die. The correct response is:

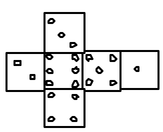
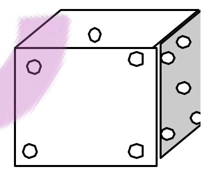


Fig. 15: One net of a cube

The group experienced problems with the task The researcher explained the concept of a foldout and asked the learners where a 4 or 2 would be if the dice rests on a 6 (as was the case in the dice at hand). Thereafter all the learners filled the rest of the dots on the net with enthusiasm and discussion.

The researcher informed the learners that the above net is just one foldout of a cube. There are ten others. Learners had to find the remaining nets and define the net of a cube.

The results of the groups varied from five, six, seven and eight nets. There was a lot of arguments amongst the learners within the groups as to whether, for example, it was not the same net in different orientations - one learner in a particular group realised that we have to jika i.e. rotate the figure until it is the same as the other (spatial abilities); some had problems in jumping faces, i.e. connecting faces which are not connected in the solid and learners did not have a mental picture (imagery) of a net i.e. they could not draw nets without using the concrete object (the cube). If a learner cannot operate on an abstract level, concrete objects should be used – Piaget's levels of intellectual development (Behr 1983).



Learners defined the net of a cube as a foldout of a cube. The researcher added that it is a two-dimensional representation of a cube which is three-dimensional i.e. on paper this is how a cube can be represented.

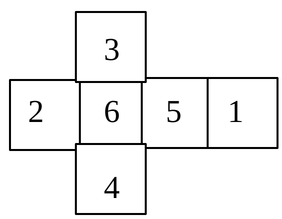
The following are some of the nets the learner drew.

Fig. 16: Nets of a cube drawn by learners

Learners were able to define the net of a cube and they were able to draw nets of a cube even though they had to use concrete objects (dies) and could not imagine such nets.

(e) Worksheet 5

Nets of other polyhedra



Learners were given a rectangular based prism and were required to draw nets of this solid. Learners immediately drew the net of a cube (the one that resembles a cross) – this is as a result of what learners already know (constructivism). The difference between this solid and a cube was pointed out to them i.e. in a cube, all faces are squares whereas in this solid all faces are rectangles of different size. In attempting the nets of this solid for the second time, one group out of three was successful. There were a lot of arguments and discussion during this activity with the other two groups even arguing with the researcher as to the correctness of their nets. In convincing the other two groups that their nets did not represent the solid, the researcher pointed out that in making a 'fold out' of the solid the sequence of the faces was wrong.

The following is a sample of learners ‘ nets showing the correct and the wrong nets.

Fig. 17: Nets of a prism drawn by learners

With regard to the net of a triangular prism, two groups were successful in writing this net while the other group produced the net but its dimensions did not depict the solid. The following are the two types of nets produced.

Fig. 18: Nets of a triangular prism drawn by learners

Learners produced the following net of a square pyramid.

Fig: 19: A net of a square pyramid drawn by a learner

The researcher asked if this the only way in which a square-based pyramid could be represented on paper (two dimensions). Since there were no responses the researcher drew the following shapes on the chalkboard.

Fig. 20: Other nets of a square pyramid

The researcher then asked learners the relationship between these shapes and the net they had drawn. One group realized that in these figures, the ‘triangles’ in their net were shifted to other positions. They identified the triangles that were shifted as a result of a flip. Learners were now learning the formal geometry – the language of tessellations (flipping and rotating) as a result of manipulating geometric shapes. All the groups were then made to arrive at the three shapes above ‘moving’ from their original net. The researcher asked if there were other ways in which a square pyramid could be represented other than the shapes discussed and one group came with the following:

Fig. 21: A net of a square pyramid

Learners were then required to draw nets of a hexagonal prism, hexagonal pyramid and a triangular pyramid. These were done successfully by the group except for one group which missed one hexagon in the hexagonal prism. The learners had the following nets:

Fig. 22: Nets of a triangular and hexagonal pyramid and a hexagonal prism drawn by learners

Learners were able to draw nets of polyhedra. Drawing is an essential activity in creativity and divergent thinking.

(a) **Worksheet 6**

Pentominoes and hexominoes

Learners were given a diagram of figures which are pentominoes and those that are not (see number 1 & 2 Worksheet 6). Pentominoes are arrangement of five squares on a plane in such a manner that there is no overlapping between the squares i.e. a side of one square completely covers the side of an adjacent square. In 3 (same diagram) pentaminoes and shapes that are not pentominoes were given. Learners had to choose which of those are pentominoes. On the basis of these activities, learners were required to define a pentomino. All the learners were able to define a pentomino since its properties had been fully defined in the activities. In this approach learners were able to arrive at the definition of a pentomino by themselves (self discovery). They wrote facts such as - azidibananga kakuhle/ngokupheleleyo. Learners were required to qualify 'kakuhle' and they wrote - no overlapping, corner to corner.

The above would seem to indicate that learners are better prepared to arrive at definitions if the above approach is used i.e. the self-discovery approach/deduction.

The diagram sheet indicated two examples of pentominoes. Learners were required to draw the other ten pentominoes. Initially learners drew the same shapes in different orientations. The researcher indicated such instances to them (told them that the shapes are the same) and asked them why the shapes are the same. There was a lot of arguments and discussion and in some instances one learner in a group would realise that rotating or flipping would result in the other shape they thought was different from the one that was being rotated or flipped.

The following are some of the pentominoes learners drew.

Fig. 23: Pentominoes drawn by learners

Learners discovered the properties of pentominoes by themselves (self-discovery) and were able to define a pentomino.

A hexomino is the arrangement of six squares in a similar fashion. These are some of the hexominoes that learners produced.

Fig. 24: Hexominoes drawn by learners

(a) Worksheet 7

Tessellations

By transforming (flip) $\triangle$ABF where necessary, a 'pattern' is formed in Fig. 4 (see worksheet 7).

Learners were required to name all angles in the figure equal to <ABF. In triangles having the same orientation as $\triangle$ABF, for example $\triangle_s$FGK, KLP, PQU, BCG, GHL and CDH, learners did not experience problems identifying angles equal to <ABF. They mentioned angles such as FGK, KLP, PQI, BCG, GHL and CDH. The researcher directed the 'learners' attention to triangles such as $\triangle_s$, BFG, GKL, LPQ, QUV, CGH, HLM, MQR, RVW and DHI. They were asked if there wasn't an angle in these triangles equal to <ABF. One group realised that $\triangle$BFG is a rotation of $\triangle$ABF, hence B $\hat{F}$FG = A $\hat{B}$ F. Having realised this learners looked for triangles similar/having the same orientation as $\triangle$BFG, thus they found that other angles equal to ABF are GKL, LPQ, QUV, CGH, HLM, MQR, RVW and DHI, amongst others.

All the learners stated that AE is parallel to JF. Asked whether there was no other relationship, for example, in terms of size/magnitude, learners stated FG iphindwe kathathu ukufumana uAE i.e. AE is three times more than FG.

Groups arrived at the same relationship of $\triangle$ABF and $\triangle$UFI though using different solution pathways (divergent thinking). $\triangle$ABF is the same as $\triangle$PQU. One version was that $\triangle$UFI is made up of nine triangles all equal to $\triangle$PQU thus $\triangle$PQU = $\triangle$ABF which is one part out of total of nine i.e. $\triangle$ABF = 1/9 $\triangle$UFI in area. Learners are developing different strategies to problem solving.

In another version the sides of the two triangles were looked at. Starting with $\triangle$PQU, to get to $\triangle$KMU, the sides of $\triangle$PQU are doubled and to get to $\triangle$UFI, the sides of $\triangle$PQU are multiplied by three, thus $\triangle$UFI is three times $\triangle$PQU - $\triangle$ABF.
From the relationship arrived at above, learners concluded that the lengths of the sides of $\triangle$UFI are three times those of $\triangle$ABF.

Groups mentioned the pattern in the figure could be considered to be made of, for example, $\triangle_s$, BFG, AKC, APD, AUE, BLD and parallelograms AFGB, BFGC, PUID and KPHC amongst others. These responses are varied and for learners to have arrived at them showed their creativity and divergent thinking abilities – looking at the same problem from different viewpoints. The learners' ability to correctly pick these triangles and parallelograms seems to indicate that they have internalized properties of such shapes.

The worksheet also comprised of a second tessellation (fig 5). The pattern in the figure could be thought of as being made up of equilateral triangles. Learners were asked the kind of polygon formed by triangles 1 + 2 + 3 + 4 + 5 + 6. The groups stated that it is a hexagon.

Learners identified the parallelogram formed by $\triangle 1 + \triangle 2$ as a rhombus and in a rhombus all sides are equal whereas in parallelogram ABGF the opposite sides are equal.

The polygon formed by $\triangle 1 + \triangle 2 + \triangle 3$ was identified as a trapezium with one pair of parallel lines.

Groups came to an understanding that a tessellation is the fitting together of shapes in such a way that there are no gaps in between them and that there is no overlap.

Learners continued the tessellation of the pentomino in 4 (see also patterns using pentominoes).

(b) Worksheet 8

Tessellations and quadrilaterals

Each of the three groups was given eight polygons all of the same kind cut out of cartridge paper. The polygons handled were two quadrilaterals of different shapes, two pentagons of different shapes, a hexagon and an octagon (see Fig 7 worksheet 8). These polygons were exchanged amongst the groups.

The groups were required to place their polygons (of the same kind) onto cardboard such that there were no spaces in between them and there was no overlapping of the polygons. They argued amongst themselves as to what to do with a shape to make it fit into, for example, a pattern that had already been formed with three or four other shapes of the same kind. The exercise was really challenging to them and in some cases patterns had to be started afresh. Learners were asked to write down whether they were successful in getting given shapes to tessellate. The groups produced beautiful patterns which they appreciated. The researcher informed learners that if the polygons fit each other as described above the pattern produced is referred to as a tessellation. They were then asked where else (or in what form) have they seen tessellations. They responded that this occurred in the form of tiles in their homes and even in the classroom which was tiled.

Learners were then given pentagons (Fig 8/worksheet 8) and the above procedure was repeated.

On being asked whether all polygons tessellate, the response was that some do not as was evidenced by the frustration on learners after having tried every trick they could think of. The most glaring example was that of an octagon where there will always be a square in between octagons.

Learners produced beautiful patterns of geometric shapes in their tessellations. They were specifically asked to mention the properties of the polygons they used in their tessellations, thus these properties are consolidated in learners.

(i) Worksheet 9

Tangrams

Creativity was not forthcoming on the part of the learners. One group constructed a boat out of the seven pieces of the tangram. The other two groups could not manipulate the shapes to get to whatever they had thought. The researcher suggested that they construct figures of animals, human beings in various positions and of letters of the alphabet. Still no creative work was forthcoming. This activity was later picked up (see S.6.2.1.11).

Learners identified the shapes they used in their design from the tangram and also highlighted the properties of such shapes.

(j) Worksheet 10

Three-dimensional puzzles

Using cubes, learners in three groups built the seven pieces of the puzzle as shown in figure10 (see worksheet 10). These seven pieces were used to build other structures (see figure 11).

The seven pieces were built with a lot of enthusiasm. No problems were experienced by all the three groups in this regard. The first group built structures A, B, C and D. The second group built structure G, B and A. Group 3 were successful in building structures D and A. Not all the puzzle pieces were used to build a given structure, thus learners had to experiment until they were successful.

To be able to build these structures, learners had to rotate, slide and flip the puzzle pieces. They assembled some of the structures horizontally then raised the assembled structures to a vertical position. The third group, however, experienced great difficulties and some frustration. With a little help from the researcher in as far as the activities of rotating, sliding and flipping were concerned, the group got to build structures D and A.

Learners could manipulate three-dimensional shapes as they could flip and rotate the structures they had built using figure 10 to build structures in figure 11.

6.2.1.2 Questionnaires/Appendix B

Two questionnaires were administered. The learners' responses to the questionnaires is based on the following Likert scale (as proposed by the researcher):

- comprehensive understanding (correct identification, description and explanation – a variety of methods used)
- good understanding (mostly correct identification, description and explanation - a variety of methods used)
- some understanding (partial identification, description and explanation – limited methods used)
- little understanding (incorrect/inappropriate identification, description and explanation, using no specific methods used)
- No understanding/no attempt.

a) **Questionnaire 1**

Twelve learners answered this questionnaire. Most of the material based on questions in this questionnaire had been covered in some form in the worksheets (worksheets 1-10). Table 1 shows the frequencies and percentages of learners who responded in a particular way.

Item 1

Learners were required to define a polygon. One learner answered that a polygon is a closed shape made up of straight lines. She also gave examples such as a triangle and a parallelogram (comprehensive understanding). Another learner gave an example of a triangle and mentioned the fact that such shape has angles (good understanding). The remaining ten learners could not define a polygon (no understanding).

The fact that most learners could not recall the definition of a polygon would seem to indicate that more work / a different strategy would have to be used to make learners familiar and hence be in a better position to define a polygon.

Item 2

In an attempt to define a polyhedron, learners gave varied definitions. One learner stated that a polyhedron is a shape having length, breadth and height (level 1 – Van Hiele) and gave examples such as a duster and a book. This learner was defining a special type of polyhedron i.e. a prism. Another learner gave an example of a cone in attempting to define a polyhedron. These two learners show a good understanding of a polyhedron. Five learners mentioned the fact that a polyhedron is like a square, a rectangle or a book. These learners were focusing on a property of a polyhedron (prism) i.e. a face or were looking at a prism (book) and were stating that the total shape resembled a two-dimensional shape (a rectangle). These learners reveal the fact that they have not been exposed to the differences between a two dimensional and a three dimensional shape i.e. the former is planar and has area while the latter has area (e.g. in the faces), thickness or volume (some understanding). The other five learners could not define a polyhedron (no understanding).

Item 3

Learners were required to define a tetrahedron. One learner correctly defined a tetrahedron as a solid with four faces (comprehensive understanding). Three learners defined a tetrahedron as a shape with four sides (some understanding). These learners have internalised tetra to mean four but sides could be referring to two dimensional shapes. The remaining eight learners could not answer this question (no understanding).

Item 4

Learners were required to give four examples of quadrilaterals. One learner gave a square, rectangle, dice and door as such examples. Another learner also added a duster and a loaf of bread as examples. These two learners show some understanding of quadrilaterals). From this it can be concluded that some learners have not made the distinction between a two dimensional and a three dimensional shape. Three learners gave correct responses such as a square, parallelogram, rhombus, trapezium and rectangle (comprehensive understanding). The remaining seven learners did not attempt this question (no understanding).

Item 5

Even though the five learners mentioned in item 4 could give examples of quadrilaterals, three of them could only draw one such quadrilateral i.e. a parallelogram (some understanding), one drew a square, rectangle and a rhombus while the other learner drew a rectangle and a square (good understanding). The remaining eight learners could no draw quadrilaterals (no understanding).

Item 5, 6, 7 and 8

Learners were required to define an edge, vertex and a face of a polyhedron. Only four learners attempted these questions. Of these learners, three correctly defined an edge as a portion of a polyhedron where faces meet (comprehensive understanding), two correctly defined a vertex as the portion where the edges meet (comprehensive understanding) and all four learners referred to a face as a side of a polyhedron (good understanding). On the basis of the learners' experiences (they appear to have been exposed to sides of polygons), they intuitively define a face as a 'side' of a polyhedron. This is in line with the constructivist theory mentioned in S. 3.6.

Item 9

All the twelve learners wrote Euler's formula incorrectly or did not recall such formula at all (no understanding).

Item 10

Learners could not answer the question as to what can be said about the angles of a parallelogram (no understanding).

Item 11

In responding to the question as to what can be said about the sides of a rhombus, two learners correctly stated that such sides are equal (good understanding). The other learners did not attempt this question (no understanding).

Item 12

Learners were required to define a net. One learner responded that a net is 'when we open a die' i.e. it is a fold out (some understanding). The other eleven learners could not define a net. Definitions are problematic for learners. They need to be encouraged to write their perceptions in whatever manner they are capable of.

Item 13

Given three diagrams representing the net of a triangular pyramid, a dice and a rectangular prism, learners were required to state the solids represented by such nets. Five learners attempted this question. One out of these learners correctly identified the pyramid, four correctly identified the die (cube) while four learners identified shape C as a box or prism. These five learners show good understanding while the remaining seven show no understanding of the concepts.

Item 14

One learner drew five nets of a cube. Another learner drew four such nets while all the other learners could not draw such nets. These two learners show a good understanding of nets of a cube. The remaining ten reveal no understanding of such concept.

Even through few learners could define and identify (elements of creativity and divergent thinking), geometric shapes, a majority of them could not. This showed no understanding of such concepts. This could be due to the fact that learners had forgotten about such concepts (as questionnaires punctuated the worksheet) or that learners performed better while working in groups (learners worked in groups in answering worksheets while they worked individually in answering the questionnaires).

(b) Questionnaire 2

Twelve learners wrote this questionnaire. Most of the material based on questions appearing in this questionnaire had already been dealt with in some form in the worksheets (Appendix A). Table 3 shows the frequencies and percentages of learners who responded in a particular way.

Item 1

Learners were required to mention the similarities between a rectangle and a square. One learner mentioned that both shapes have four sides. This is indicative of some understanding on the part of the learner. The rest of the learners did not answer this question (no understanding).

Item 2

This item required of learners to give the differences between a rectangle and a square. Two learners stated that in a square all sides are equal whereas in a rectangle the opposite sides are equal (good understanding). Other learners did not attempt the question (no understanding).

Item 3

One learner answered this question which required learners to give similarities between a parallelogram and a rhombus. This learner mentioned that both shapes have four sides (some understanding). The remaining eleven learners did not attempt this question (no understanding).

Item 4

Learners were required to define a rhombus in terms of a parallelogram (class inclusion). All the learners could not do this (no understanding).

Item 5

Learners were required to give two examples (from their environment) of a prism as well as a cylinder. The four learners who answered this question mentioned a box, a loaf of bread, a door and a duster as examples of a prism (some understanding). One learner gave a glass as an example of a cylinder. The remaining eight learners did not attempt this question (no understanding).

Item 6

This item required learners to mention the type of faces in a triangular pyramid, a cube and a rectangular prism. The learners could not answer this question (no understanding).

Item 7

No learner could define a pentomino (no understanding).

Item 8

Learners were given four diagrams and were required to choose those that they thought were pentominoes (Questionnaire 2). A and C are pentominoes. Nine learners identified A as a pentomino (some understanding). Five learners identified C and a pentomino. These learners did not attempt this question (no understanding).

Item 9

Learners were required to define a tessellation. One learner stated that a tessellation is an arrangement of polygons such that there is no overlapping and no spaces between/amongst them (comprehensive understanding). The other learners could not define a tessellation (no understanding).

Item 10

In answering the question as to whether all polygons tessellate, four learners correctly stated that not all polygons tessellate (good understanding). The remaining eight learners did not attempt this question (no understanding).

Item 11

From the polygons that did tessellate, learners were asked what they can conclude about the sum of the angles where the polygons meet. No learner concluded that the sum of the angles is 360° (no understanding).

Item 12

In the language of tessellations, learners were asked what must have been done to triangle A to get triangle B (Appendix B). No learner stated that there was a flip or a rotation (no understanding).

Item 13

Learners were required to define a tangram. All the learners could not define a tangram (no understanding).

In giving similarities between polygons, for example, a rectangle and a square, learners do not give comprehensive answers (they omitted the fact that in both shapes all angles are equal to 90° and that the opposite sides are parallel). Learners are also not conversant with three dimensional shapes. Most of them could not give examples of prisms and cylinders from their environment. Learners still experience problems in defining geometric concepts like a pentomino, tessellation or tangram. They could however, choose pentominoes from shapes that are not pentominoes. This seems to indicate that learners are more at home with applications rather than definitions.

Learners responses to this questionnaire also reveal that they do well in group work and not individually.

6.2.1.3 Tests/Appendix C

a) **Pre-test**

Twelve learners wrote the pre-test on 05.03.01 within a period of forty minutes. They had not been taught geometry in this grade i.e. Grade 7. Table 5 shows the frequencies and percentages of learners responding in a particular way.

Item 1

Given perpendicular lines, learners were asked what angles are called when such lines meet. One learner correctly stated that it is a right angle (some understanding). Two learners stated that a rectangle is formed which is somehow correct given the fact that there are right angles in a rectangle but incorrect in the context of the question while the remaining five learners gave incorrect answers such as a reflex angle, a hexagon, a triangle or did not answer the question. The rest of the learners did not attempt this question (no understanding).

Item 2

Learners were given three triangles and were asked what these triangles appeared to be what kind of triangles. Only one learner stated that they appeared to be isosceles triangles (comprehensive understanding). The other learners gave varied responses such as squares, acute angled, kite or did not answer the question (no understanding).

Item 3

Given a drawing of two parallel lines, learners were required to provide a word that describes this phenomenon. Four learners stated that the two lines are parallel (comprehensive understanding). Responses from the rest of the learner seemed to indicate unfamiliarity with the concept of parallel lines (no understanding).

Item 4

Learners were asked to draw a rectangle. Eight of the learners drew a rectangle (comprehensive understanding), while the remaining four drew triangles (no understanding).

Item 5

Learners were required to draw a right angled triangle. One learner was able to draw a right angled (some understanding), four learners drew a right angled triangle (comprehensive understanding), while the remaining seven learners drew acute angled triangles (no understanding).

Item 6

Learners had to draw a parallelogram. One learner drew a parallelogram clearly indicating the parallel lines (comprehensive understanding). Four learners drew a rectangle (which can be considered a special type of parallelogram with all angles equal to 90°) (some understanding) while the remaining seven learner drew parallel lines (no understanding).

The responses of the learners seem to indicate that they do have some notion of what a parallelogram is (parallel lines and a rectangle being some special case of a parallelogram) but that such knowledge is incomplete.

Item 7

This item required learners to draw a square. Two learner were able to draw squares (comprehensive understanding). The remaining learners either drew a rectangle, triangle or a circle (no understanding).

Item 8

Learners were asked if a right angled triangle always has a longest side and if so, which one? Eight learners answered in the affirmative but did not state which side is the longest (some understanding). Two learners did not attempt this question (no understanding).

Item 9

In this item learners were asked if a right angled triangle always has a largest angle and if so which one? Four learners stated that a right angled triangle always has a largest angle and stated which one (good understanding). Four other learners wrote that such triangle does have a largest angle (some understanding) while the remaining two learners did not attempt the question (no understanding).

Item 10

Learners were required to mention/state something about the sides of an isosceles triangle. Responses to this item showed unfamiliarity with the properties of an isosceles triangle. No learner stated that in an isosceles triangle, two sides are equal (no understanding).

Item 11

Learners were asked about the relationship amongst the angles of an isosceles triangle. As in item 10, learners did not state that the base angles (angles at the base of the equal sides) are equal (no understanding).

Item 12

Learners were asked how they recognize parallel lines i.e. to define parallel lines. All the learners did not attempt this question (no understanding).

Item 13

This item required learners to mention some ways in which squares and rectangles are alike (similarities). The majority of the learners did not attempt this question. Those that did mentioned irrelevant concepts to the question (no understanding).

Learners were also asked if all squares are also rectangles. They did not respond to this question (no understanding).

Item 14

Learners had to state whether the following statements are true or false and to give reasons for their answers.

(i) All isosceles triangles are right angled triangles

(ii) Some right angled triangles are isosceles triangles

Most learners did not attempt these questions; those who did had irrelevant answers to the questions (no understanding).

Items in the above test are based on properties of polygons such as the isosceles and right angled triangle, the rectangle, square and parallelogram. The test items are also based on perpendicular and parallel lines.

The quality of the learners' responses to these items seem to indicate that they have not been well founded in the concepts mentioned in the test in the lower grades. The test was written in March, prior to teaching geometry in Grade 7. It can be argued that the learners had forgotten the geometry they learnt the previous year(s). However, basic concepts (polygons) such as a triangle, rectangle and a square should have been entrenched in their minds as examples of such are abundant in the learners' environment. The same holds for perpendicular and parallel lines.

The items in this test required learners to define, draw, describe and compare geometric shapes. Most learners showed unfamiliarity with these shapes.

(b) Post test 1

Learners had previously engaged in activities of creativity and divergent thinking in a problem solving context that were meant to enhance the internalization of geometry concepts. The effectiveness of such internalization is tested in post test 1 and 2.

The following is a description of learner responses to the items in the test (Appendix C). Twelve learners wrote this test within a period of forty minutes. The test was written on 18.05.01. Table 7 shows the frequencies and percentages of a particular understanding by learners (see Likert scale S. 6.2.1.2).

Item 1

Learners were given diagrams of four types of triangles and were required to identify these.

a) All the twelve learners correctly identified (a) as an isosceles triangle (comprehensive understanding).

b) Eight learners correctly identified (b) as an equilateral triangle (comprehensive understanding) while the remaining four only stated that it is a triangle (some understanding).

c) Four learners stated that the figure in (c) is a scalene triangle (comprehensive understanding). Five learners mentioned the fact that the triangle is obtuse angled (some understanding). The remaining three did not answer the question (no understanding).

d) All the twelve learners correctly identified (d) as a right angled triangle (comprehensive understanding).

Learners seem to have internalized the properties of the four types of triangles mentioned above as the majority of them were able to identify them. Responses to item 1.1 hereunder seem to bear testimony to this.

Item 1.1

The isosceles triangle

Nine learners stated that an isosceles triangle has two equal sides. Of these learners, four also mentioned base angles but did not state that such angles are equal. The remaining three learners mentioned that in an isosceles triangle two sides are equal and the base angles are equal. The property that an isosceles triangle has two equal sides has been internalized by all the learners.

The equilateral triangle

Five learners mentioned that in an equilateral triangle, three sides are equal. Of these learners, one also mentioned the fact that all the angles are equal and each is equal to $60°$. The remaining seven learners could not state the properties of an equilateral triangle even though the majority of them had correctly identified the equilateral triangle in 1. This seems to indicate the importance of exposing learners to different ways of asking the same question.

Scalene triangle

In Item 1, four learners stated that (c) is a scalene triangle while five had mentioned the fact that it is an obtuse angled triangle. In mentioning the properties of a scalene triangle only one learner mentioned the obtuse angle. This seems to indicate the fact that learner have to be encouraged to verbalize their observations and to write these down. This was picked up in the revision of the test.

One learner correctly stated that in a scalene triangle, all the three sides are unequal while the other ten learners either did not attempt this question ro gave wrong properties of the triangle.

Right angled triangle

Seven learners mentioned that in a right angled triangle, one of the angles is 90E. Of these learners, one stated that the other angles are acute angles. Four learners indicated that a right angled triangle has a longest side, the hypotenuse. One learner did not attempt this question.

Item 1.2

Learners were asked if a right angled triangle will always have a longest side and if so which one? Six learners responded in the affirmative to this question (some understanding) with four of these learners stating that such side is the hypotenuse - an improvement to the response to the same question in the pre test (comprehensive understanding). The other six learners did not respond to the question (no understanding).

Item 1.3

In a similar fashion to item 1.2 and item 9 in the pre test, learners were asked if a right angled triangle will always have a largest angle and if so which one. Eight learners responded in the affirmative and stated that such angle is a right angle/90° (comprehensive understanding). Four learners did not attempt this question (no understanding)

Learner responses to this item seem to indicate that hey are on level one – van Hiele (1986),. They mention the properties of a right-angled triangle.

Item 2

Learners were asked to draw a rectangle and a square. All the twelve learners were able to draw these two shapes and in their drawings indicated the properties of the shapes (e.g. in a rectangle the opposite sides are equal and in a square, all sides are equal by marking them as such) (comprehensive understanding).

Learners were also asked about the similarities between a rectangle and a square. The properties mentioned were that both shapes have four sides and four angles, that in both shapes the opposite sides are parallel and that all the angles are equal to 90°.

In stating the differences between a rectangle and a square, learners mentioned the fact that in a rectangle the opposite sides are equal whereas in a square all the sides are equal. It would appear from the properties of a rectangle and a square mentioned that the learners have internalized these concepts.

Item 3

Learners had to draw a rhombus and a parallelogram. All the learners correctly drew a rhombus and a parallelogram and indicated the fact that in both shapes, the opposite sides are parallel and in a parallelogram the opposite sides are equal while in a rhombus all sides are equal and marking them as such in their drawings (comprehensive understanding).

Learners failed to correctly respond to the question requiring them to define a rhombus in terms of a parallelogram. They had to realize that a rhombus is a special kind of parallelogram where all the sides are equal. Instead they gave the properties of both a rhombus and a parallelogram, but did not make a further deduction from such properties as required by the question. Learners have not yet reached the stage of formal proof (van Hiele level 3 according to Teppo (1991)).

Item 4

Learners had to define an edge, a vertex and a face of a solid. Seven learners correctly defined an edge of a solid as where the faces meet (comprehensive understanding) see fig. and a vertex as where the edges meet. Learners experienced considerable difficulty in defining a face of a solid. Two learners stated that it is a side of a solid in the same way as a polygon has sides. One learner mentioned that a face is a polygon on a solid. From these, it would appear that learners intuitively feel that a face of a solid is a polygonal shape on the solid.

Item 5

Learners were required to write down Euler's formula. Five learners correctly wrote Euler's formula as

$$V + F = E + 2$$

The rest of the learners could not recall such formula. All the learner could not manipulate this formula so as to find the number of edges given the number of faces and vertices in 5.1 and the number of vertices given the number of faces and edges in 5.2. In the revision of the test attention was given to this difficulty using different algorithms in algebra. Further exercises in manipulating the formula to solve for the variables in Euler's formula showed that learners had grasped the algorithm. They possess procedural knowledge (Anderson's model of cognition – see S. 3.9.1).

Item 6

Learners had to define a tessellation. Three learners defined a tessellation as a fitting together of planar shapes such that there is no overlapping between them and there are no spaces between them. One learner referred to a tessellation as a tiling. The rest of the learners did not attempt this question.

The responses of the learners to this test show a marked improvement to their responses to similar questions in the pre-test. They seem to have internalized properties of geometric shapes such as an isosceles, equilateral and a right angled triangle, a square, rectangle, parallelogram and a rhombus. They are in a position to define concepts such as a vertex, face and edge of a solid and have an idea of what a tessellation is.

It has also emerged from the test that learners should be exposed to different ways of asking the same question as mentioned under the equilateral triangle in item 1.1. Learners should also be encouraged to verbalize and write down their observations as mentioned under the scalene triangle in item 1.1. These language skills (speaking and writing) enhance performance in mathematical (geometry) problem solving. Difficulties in defining concepts continue to recur (learners are more comfortable with applications) as mentioned in the worksheets. The learners' performance has improved as compared to the pre test. This is as a result of engaging in activities of creativity and divergent thinking during geometrical problem solving (the worksheets). Responses to this test show that learners could define, identify, draw and compare geometric shapes – essential elements of creativity and divergent thinking.

(c) Post Test 2

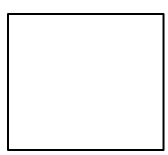
Twelve learners wrote this test on 14.09.01 within a period of forty minutes. Table 9 shows the frequencies and percentages of a particular understanding by learners (see Likert scale S. 6.2.1.2).

Item 1

Learners were asked to explain geometric shapes by making use of a diagram. They were encouraged to show as many properties of the shape as possible in their (learners) diagrams.

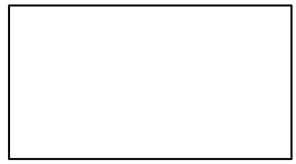
a square

All the learners were able to draw a square and they marked all the sides to be equal and show that all the angles are right angles (comprehensive understanding).



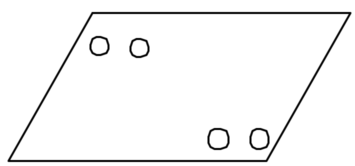
a rectangle

The properties of a rectangle were shown by all the learners according to the diagram shown hereunder.



The opposite sides are equal and all angles are right angles.

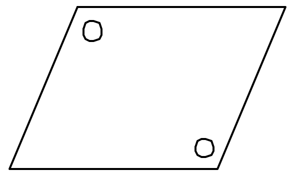
a rhombus



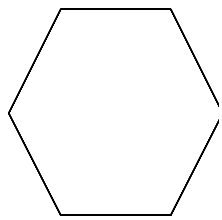
All the learners were abl [illegible] ing the fact that the opposite sides are parallel and that all the sides are equal as shown in the above diagram (good understanding). Two learners also indicated that in a rhombus the opposite angles are equal (comprehensive understanding).

a parallelogram

All the learners were able to draw a parallelogram showing that the opposite sides are equal and parallel. Two students (as in the case of the rhombus) indicated that the opposite angles are equal. One of these learners also indicated that if a line is drawn to join two opposite corners i.e. a diagonal, two triangles are formed with equal areas.

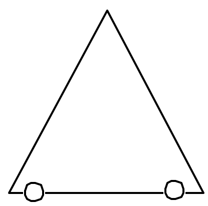


a hexagon



All the learners drew a six-sided figure and they also indicated that such a shape has six angles. Only three learners stated that the sum of the interior angles of a hexagon is 620°.

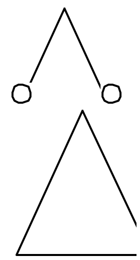
an isosceles triangle



All the learners drew an isosceles triangle indicating the fact that two sides are equal and the base angles are equal.

a kite

Nine learners successfully drew a kite. Of these niner learners, seven indicated the two isosceles triangles showing the equal sides and angles as depicted in the figure hereunder.



The above responses seem to indicate that learners have internalised the properties of these polygons.

Item 2

Learners were required to match five objects on the left to corresponding descriptions on the right.

Eight learners correctly matched a stop sign to an octagonal prism. The other learners incorrectly matched the stop sign to other descriptions.

All the twelve learners correctly matched a beehive to a hexagonal prism. This could have been as a result of the fact that an actual beehive had been brought to class by the learners and such shape thoroughly explored.

All learners correctly matched a tomb of Pharaohs, a box of breakfast cereal and an ice cream container to a square pyramid, a rectangular prism and a cone respectively.

These shapes had been handled during investigations involving the worksheets (Appendix 5).

The four learners who had not matched a stop sign had matched it to descriptions such as triangular pyramid, pentagonal prism and triangular prism.

Item 3

Learners were required to draw four nets of a cube. Six learners drew four nets (comprehensive understanding), four learners drew three nets (good understanding) while two were able to draw two nets - some understanding (the 'cross' and the T-shaped ones which were drawn by all the learners).

Item 4

Given four nets of a triangular pyramid, a rectangular prism, a hexagonal pyramid and an open box where the open portion is a base of a square pyramid, learners were required to state what solids these nets represented.

All the learners identified 4.1 and 4.2 as nets of a triagular pyramid and a rectangular prism respectively.

Four learners successfully identified 4.3 as the net of a hexagonal pyramid.

Two learners described 4.4 as a pyramid on a box while one leaner fully described the net as an open box where the open portion is a base of a square based pyramid.

The rest of the learners did not attempt this question.

On the whole the learners seem to have made considerable progress in handling solids. Learners could explain, draw, match and state propeties of geometric shapes - essential elements of creativity and divergent thinking in the problem solving context. The learners' scores in this test are a marked improvement of their performance in the pre test and post test 1.

6.2.1.4 Interviews/Appendix D

Semi-structured interviews were conducted with the learners. In this type of interview, learners were asked the same questions. Subsequent questions depended on the learner's response to the original question. Most of the questions asked in these interviews were asked after worksheets (Appendix A) had been handled in class.

Learners A, B, C and D were asked questions 1, 2, 3, 4, 12, 13, 14 while learners E, F, G and H were asked questions 5, 6, 7, 8, 9, 10, 11 and 13.

Question 1 and 2

Shape A

Learner A was shown shape A and asked what it is. The response was that it is a rectangle. The researcher then asked for the properties of such shape. The learner responded that two sides are equal whereupon the researcher asked which ones. Having pointed to the two 'short' ones, the researcher wanted to know if those were the only equal sides in the shape. The learner pointed to the 'long' sides. The researcher then mentioned that these sides are opposite to each other thus in a rectangle, the opposite sides are equal. The learner could not respond when asked if something else could be said about a rectangle.

Learner B also identified shape A as a rectangle. The learner mentioned that two pairs (indicating them on the figure) of sides are equal. On being asked what can be said about the angles of the rectangle, the learner responded that they are all right angles.

Having identified shape A as a rectangle, **learner C** mentioned that the opposite sides of a rectangle are equal. The research asked the learner to show him such sides. This was done successfully. The learner also mentioned that the angles of a rectangle are all equal whereupon the researcher asked the magnitude of such angles. The learner responded that they are right angles (90°).

Learner D identified shape A as a rectangle and mentioned that the opposite sides of a rectangle are equal, the learner also mentioned that the angles are also equal and their magnitude is 90°.

Responses by the learners seem to indicate that they have internalised the properties of a rectangle in as far as the sides and angles are concerned.

Shape B

Learners A, B, C and D identified shape B as a square. The learners mentioned that the square has equal sides and all angles equal 90°.

The learners seem to be conversant with the properties of a square.

Shape C

Learner A identified the shape as a parallelogram. On being asked about the properties of a parallelogram, the learner mentioned that it has parallel lines. The researcher then asked the learner to point the parallel pines. This was done successfully from which the learner concluded that the opposite sides are parallel.

Learner B also identified the shape as a parallelogram but could not mention the properties of the shape. The researcher then asked what could be said about the opposite sides of the shape (the researcher pointed to these). The learner responded that they are equal.

Having identified shape C as a parallelogram, **learner C** was asked to mention the properties of shape C. Since there was no response, the researcher asked if the opposite sides of shape C could ever meet. The researcher asked what such lines are called and the learner recalled that they are parallel lines.

Learner D also identified shape C as a parallelogram and stated that the opposite sides are equal and parallel.

Except for some probing here and there learners seem to have internalised the properties of a parallelogram. This probing is essential as it is a means of letting learners communicate their perceptions thereby consolidating their understanding of geometric concepts.

Shape D

All the four learners identified shape D as a rhombus. They stated that in a rhombus all sides are equal and that the opposite sides are parallel.

This success could be due to the fact that the previous questions were based on a parallelogram and that a rhombus is a special type of parallelogram.

Shape E

Learners A, B and D identified shape E as a hexagon and mentioned that it has six sides and angles. **Learner C** had problems identifying the shape and had to be reminded that a shape with six sides is a hexagon. He stated that the shape has six angles.

Question 3

All the four learners mentioned that the sum of the angles of a triangle is 180E.

Question 4

Learners A, B and C could not tell what the sum of the angles of a parallelogram (a quadrilateral) is. The researcher then asked if we draw a straight line to join opposite vertices how many triangles would be formed. Thus the parallelogram would consist of two triangles then the sum of the angles of a parallelogram is 180° x 2 = 360°.

Learner D recalled that the sum of the angles of a parallelogram is 360°.

With regards to the sum of the angles of a hexogon, all four learners could not recall what the sum is and were made to divide the shape into three triangles and then find the sum by multiplying 180° by 3. They obtained 540°.

Learners need some guidance to obtain the sum of the interior angles of polygons.

Question 13

Learners were asked what they enjoyed most in the work that was covered (polygons, polyhedra, Euler's formula, nets, pentominoes, tessellations, tangrams and three-dimensional puzzles).

From the varied responses given by all the eight learners interviewed it emerged that activities that lended themselves to practical work were enjoyed by the learners. These activities included classifying packaging materials (solids), counting the number of faces, vertices and edges of solids, finding nets of a cube (especially manipulating a cube to find these) and nets of other solids, writing different pentominoes, tessellating various polygons and solving three dimensional puzzles.

Question 14

Activities that require learners to define geometric concepts or required them to mention properties of shapes were least enjoyed.

Question 12

Learners A, B, C and D mentioned the tiling in their classroom, in the kitchens at their homes and the tiling on the side walks in town.

Question 5

Learners E, F and H stated that the pointed portion of the solid is the face. **Learner G** had to be reminded that the portion is the face.

Question 6

All the four learners found the number of faces to be five.

Question 7

All the four learners found the number of edges to be nine.

Question 8

All four learners found the number of vertices to be six.

Question 9

All the four learners could not recall Euler's formula. The researcher had to remind each of them that the formula states F + V = E + 2.

Question 10

All four learners had to be guided in the substitution to realise that this solid satisfy Euler's formula

F + V + E + 2

5 + 6 = 9 + 2

Learners seem to experience problems in the manipulation of equations.

Question 11

All the learners experienced difficulty in defining a net. They were guided into seeing that a net is a fold out of a three-dimensional shape (i.e. a two-dimensional representation of a three dimensional shape).

Learners seem to have grasped properties of some polygons like a square, rhombus and hexagon. They can manipulate solids by being able to calculate the number of edges, faces and vertices of such solids. However, they experience problems manipulating equations (substitution).

They (learners) can relate tessellations on paper to various types of tiling in their environment.

Learners still experience a great deal of difficulty defining geometry concepts e.g. a net.

6.2.1.5 Essays

Learners were required to write essays on geometric shapes. They had to use their imagination and creativity in linking the theory that they learn at school to their everyday life situations and make their essays as interesting as possible.

In writing essays on geometric shapes, learners related to objects in their surroundings. One learner whose topic was a square, mentioned that objects such as a desk, book and window panes have a square shape.

Another learner did not confine herself to one geometric shape but related shapes such as a square, rectangle, isosceles triangle and a right angle to objects in her surroundings. According to the learner, the square is found in objects such as a cardboard, TV, radio, picture, and a ceiling; a rectangle in windows, lawn, stoop, wardrobe, bed and mat; a heart takes the shape of an isosceles triangle and a right angle is a corner of a cardboard or a corner of a wall.

The learner above wrote about tessellations or tiling which is essentially a pattern of geometric shapes.

Yet another learner mentioned a square-shaped box (cube) and imagined this box being squeezed from the top left edge downward to the right. From such a process would emerge a shape that looks like a rhombus (cut box iwe icala).

Learners are able to relate geometric shapes learnt at school to objects in their environment. The fact that they are familiar with these objects would seem to suggest that understanding the geometric concepts associated with these objects is enhanced.

6.2.1.6 Songs

One learner used the tune of a popular Brenda Fassie song to highlight the properties of a rectangle like it has four sides and that if a diagonal is drawn there will be two triangles. The learner also mentioned that in buildings, a rectangular shape is used. She ends the song by stating that even if people can look down on the rectangle and say it has not so attractive corners, she will continue being the rectangle's friend.

Another learner defined an acute, a right and an obtuse angle through a tune as follows:

6.2.1.7 Poems

In one poem a learner mentions that in a box are squares (faces) and that to make boxes one must have squares. She mentions that bags and window pans are also square-shaped.

Learners recited their poems in a praise singing fashion. Another learner wrote a poem about a 'circle'. The learner mentioned that such shape is found on footballs, on the roads (traffic circle) and some sweets are circular in shape.

Yet another learner wrote a poem about a right angle. This learner mentions that right angles are found in corners of houses, in window panes, doors, books, duster and table. She highlights the importance of this type of angle in designing these objects.

While learners enjoy themselves in writing poetry thereby putting to the fore their creative talents, they also learn geometric concepts along the way.

6.2.1.8 Analogies and metaphores

Learners were asked the question: ‘what will you eat for breakfast?’ wherein their knowledge of geometric shapes would be called to the fore. What appear hereunder are responses from two learners. Both learners essentially talk about slices of bread which are prisms and faces thereof which are right-angled triangles.

Another learner mentioned that she would eat sandwiches with cheese and polony inside, cheese because it comes in square shaped forms while the polony is in circular form. These sandwiches are cut diagonally to produce right-angled triangles to make the sandwiches even more appealing for her.

In addition she would eat wors because it is shaped like a cylinder. She would then wash everything down with coffee.

6.2.1.9 Tessellations

Learners were given a tessellation and were required to colour (produce some artwork). Their colouring of the shape being tessellated was such that it would be in line with what a tessellation is, i.e. the same drawing/colouring should appear in the exact position in all the shapes. Learners produced the following tessellations.

6.2.1.10 Pentominoes

Having drawn various pentominoes (see Worksheet 6/Appendix 6) learnesr cut similar pentominoes out of cartridge paper. Using these pentominoes they designed and coloured the following patterns (artwork). These patterns of pentominoes show the correlation between geometry and art while sensitizing learners to the properties of squares and pentominoes. Learners had the following patterns for pentominoes

6.2.1.11 Tangrams

Given seven pieces of a tangram viz. five right angled triangles of different sizes, a square and a parallelogram, learners were required to use all the seven pieces to design any shape (it could be an animal, person or a letter of the alphabet). Learners produced the following shapes.

Learners were also asked to identify the seven pieces of the tangram and to mention the properties of such shapes. They experienced no problems in this regard seeming to indicate that they had internalised the properties of the shapes.

6.2.1.12 Creative correlations in geometry

Learners had been required to, during the discussions of the various geometric concepts (in the worksheets) have a vocabulary list of all the concepts they encounter each day and to write the meaning of such concepts in their vocabulary book. The following is an example of such list.

- angle
  - acute, right, obtuse, reflex and a complete revolution
- parallel lines
- perpendicular lines
- triangle
  - right angled, isosceles, equilateral and scalene
- rectangle
- square
- parallelogram
- rhombus
- quadrilateral
- pentagon
- hexagon
- heptagon
- octagon
- convex quadrilateral
- concave quadrilateral

- polygon
- circle
- polyhedron
- prism
  - rectangular, triangular and hexagonal
- pyramid
  - square based, hexagonal and triangular
- cone
- cylinder
- face of a polyhedron
- vertex of a polyhedron
- edge of a polyhedron
- net of a polyhedron
- pentominoes
- hexaminoes
- tessellation
- tangram

Using their vocabulary list, learners were required to write geometric concepts in such a manner that they depicted the concept at hand. In this way, their creativity was called to the fore e.g. triangle can be written as Tr△angle. Instead of writing the letter ‘i’ in triangle, a triangle is inserted. This consolidates the concept of a triangle in learners and also affords learners the opportunity to be creative in writing such concepts. The following are some of the learners’ portfolios in this regard.

6.2.1.13 Musical instruments

Learners made various types of musical instruments using cartridge paper. Parts of the musical instruments are made up of geometric shapes such as rectangles, squares and circles.

A popular instrument is the guitar. Six learners constructed this type of instrument. All these learners made the portion with strings using a rectangle. The broad portions consisted of a square in two such instruments; a circle in two other instruments; a triangle in a bass guitar and a shape that is not a polygon as it has three straight sides with the fourth side a curve. The learners creativity is called to the fore as they came with these interesting designs. Some of the learners' designs appear hereunder.

Fig 12. Musical instruments

Another popular instrument is the drum or rubira. Basically the instrument is made of a cylinder which is covered with leather on top. Three learners produced this type of instrument.

Yet another instrument that was produced is the keyboard. This instrument came in various shapes and colours. Three learners made these instruments. One learner cut the black and white buttons from the actual cartridge paper that was the instrument while another learner cut these buttons from a separate cartridge paper then pasted these on the instrument while the last learner drew these buttons on his three-dimensional keyboard in red and black. Different learners interpreted and designed the keyboard differently thus allowing for diversity.

One learner constructed a tambourine. Essentially the instrument consists of two cylindrical rings joined at one point though one ring is on top of the other. Attached to the top ring are smaller rings which go right round this top ring. These are the rings which produce sound on beating the instrument against one's hand.

A harp, an instrument consisting of a cylindrical ring with 'strings' attached on top. The strings are made of small rectangles. One learner produced this instrument.

Lastly one learner produced an instrument that is made of three cylinders which are blown and the resulting sound is controlled by tapping on a rectangular shape which is on top of these cylinders.

Learners came up with different original ideas to construct various musical instruments. Of importance they used geometric shapes in producing these. This helped them relate geometric shapes (polygons and polyhedra) to their environment. The fact that they produced different shapes of the same instrument, for example, different types of guitars, showed originality and hence creativity on their part.

6.2.1.14 Posters

Learners were given another opportunity to show their creative talents by making posters to illustrate geometric concepts learned. The following is a sample of such posters.

Fig. 13 Posters

6.3 QUANTITATIVE RESEARCH

The following is the statistical analysis of the learners' performance in the questionnaires and the tests.

The learners responses to items in the questionnaires as well as the tests are qualified in terms of the Likert scale of comprehensive, good, some, little and no understanding as described in S. 6.2.1.2. From this, the frequencies and percentage frequencies of learner responses to items are calculated and tabulated and this is followed by a reliability analysis. The mean, median, mode, standard deviation and range of the responses are also tabulated. There is also a comparison of the three tests. Lastly, there is a line and bar graphs to indicate individual learner progress.

QUESTIONNAIRE 1

		no understanding	little understanding	some understanding	good understanding	comprehensive understanding
1	Count	10.0			1.0	1.0
	%	83.3			8.3	8.3
2	Count	5.0		5.0	2.0	
	%	41.7		41.7	16.7	
3	Count	8.0		3.0		1.0
	%	8.3		25.0		
4	Count	7.0		2.0		3.0
	%	58.3		16.7		25.0
9	Count	12.0				
	%	100.0				

10	Count	12.0				
	%	100.0				
11	Count	10.0			2.0	
	%	83.3			16.7	
12	Count	11.0		1.0		
	%	91.7		8.3		
13	Count	7.0			5.0	
	%	58.3			41.7	
14	Count	10.0	2.0			
	%	83.3	16.7			

Table 1: An interpretation of learner responses to items in questionnaire 1

DESCRIPTIVE STATISTICS

The mode, which is the most frequent value/understanding value, the median, which is the middle most value/understanding value and the mean, which is the average of all the understanding values per question was calculated and the results were as follows:

Statistics

	Mean	Median	Mode	Std. Deviation	Range
1	1.5833	1.0000	1.00	1.37895	4.00
2	2.3333	3.0000	1.00	1.23091	3.00
3	1.8333	1.0000	1.00	1.33712	4.00'

4	2.3333	1.0000	1.00	1.77525	4.00
9	1.0000	1.0000	1.00	.00000	.00
10	1.0000	1.0000	1.00	.00000	.00
11	1.5000	1.0000	1.00	1.16775	3.00
12	1.1667	1.0000	1.00	.57735	2.00
13	2.2500	1.0000	1.00	1.54479	3.00
14	1.1667	1.0000	1.00	.38925	1.00

Table 2: The mean, median, mode, standard deviation and range understanding values to items in the questionnaire.

It is interesting to note that the modal understanding value for all of the questions that were posed to the learners was" 1" which translated from the scale is "no understanding".

It was only for question 3 that the median value is 3 whilst for the rest of the questions it is "I" which translated from the scale is "no understanding". The standard deviations appear to be small numerically.

RELIABILITY ANALYSIS

Cronbachs Alpha

Cronbach's alpha was also calculated as part of the reliability test to assess how valid the results were and will we get similar results to generalise if we increased the sample size. A value of 0.7 or higher is a very good value that can lead us to say that we will get the same results if we carried out this survey with a larger sample of students. The

Cronbach's alpha was calculated for the questions in questionnaire 1 and the results are as follows:

****** Method 1 (space saver) will be used for this analysis ******

R E L I A B I L I T Y A N A L Y S I S - S C A L E (A L P H A)

Item-total Statistics

	Scale Mean if Item Deleted	Scale Variance if Item Deleted	Connected Item-Total Correlation	Alpha if Item Deleted
1	14.5833	49.1742	.8359	.8804
2	13.8333	51. 9697	.7752	.8852
3	14.3333	48.6061	.9037	.8750
4	13.8333	43.2424	.8930	.8794
9	15.1667	67.2424	.0000	.9143
10	15.1667	67.2424	.0000	.9143
11	14.6667	51.8788	.8323	.8815
12	15.0000	61.0909	.6447	.8993
13	13.9167	48.2652	.7730	.8871
14	15.0000	61. 8182	.8614	.8986

Reliability Coefficients

N of Cases = 12.0 N of Items = 10

Alpha = .9031

The overall alpha in this case is 0.9031 which is an excellent reliability coefficient

QUESTIONNAIRE 2

		no understanding	some understanding	Good understanding	comprehensive understanding
1	Count	11.0	1.0		
	%	91.7	8.3		
2	Count	10.0		2.0	
	%	83.3		16.7	
3	Count	11.0	1.0		
	%	91.7	8.3		
4	Count	12.0			
	%	100.0			
5	Count	8.0	4.0		
	%	66.7	33.3		
6	Count	12.0			
	%	100.0			
7	Count	12.0			
	%	100.0			
8	Count	3.0	9.0		
	%	25.0	75.0		
9	Count	11.0			1.0
	%	91.7			8.3
10	Count	8.0		4.0	
	%	66.7		33.3	
11	Count	12.0			
	%	100.0			

12	Count	12.0			
	%	100.0			
13	Count				
	%	100.0			

Table 3: An interpretation of learner responses to questionnaire 2

DESCRIPTIVE STATISTICS

The mode, which is the most frequent value/understanding value, the median, which is tht middle most value/understanding value and the mean, which is the average of all th(understanding values per question was calculated and the results were as follows:

Statistics

	Mean	Median	Mode	Std. Deviation	Range
1	1 .1667	1.0000	1.00	.57735	2.00
2	1.5000	1.0000	1.00	1.16775	3.00
3	1.1667	1.0000	1.00	.57735	2.00
4	1.0000	1.0000	1.00	.00000	.00
5	1.6667	1.0000	1.00	.98473	2.00
6	1.0000	1.0000	1.00	.00000	.00
7	1.0000	1.0000	1.00	.00000	.00
8	2.5000	3.0000	3.00	.90453	2.00
9	1.3333	1 .0000	1.00	1.15470	4.00
10	2.0000	1.0000	1.00	1.47710	3.00
11	1.0000	1.0000	1.00	.00000	.00
12	1.0000	1 .0000	1.00	.00000	.00
13	1.0000	1.0000	1.00	.00000	.00

Table 4: The mean, median, mode, standard deviation and range understanding values to items in questionnaire 2.

It is interesting to note that the modal understanding value for all of the questions that were posed to the learners was “1" which translated from the scale is “no understanding”. It was only for question 8 that the median value is 3 whilst for the rest of the questions it is “1" which translated from the scale is “no understanding”. The standard deviations appear to be small numerically.

RELIABILITY ANALYSIS

Cronbachs Alpha

Cronbach's alpha was also calculated as part of the reliability test to assess how valid the results were and will we get similar results to generalise if we increased the sample size.

A value of 0.7 or higher is a very good value that can lead us to say that we will get the same results if we carried out this survey with a larger sample of students. The Cronbach's alpha was calculated for the questions in questionnaire 2 and the results are as

****** Method 1 (space saver) will be used for this analysis ******

R E L I A B I LIT Y   A N A L Y S I S - S C ALE (A L P H A)

* * * Warning * * * Zero variance items

Item-total Statistics

	Scale Mean if Item Deleted	Scale Variance if Item Deleted	Corrected Item-Total Correlation	Alpha if Item Deleted
1	16.1667	24.1515	.7583	.7747
2	15.8333	19.6061	.7560	.7587
3	16.1667	24.1515	.7583	.7747
4	16.3333	28.7879	.0000	.8130
5	15.6667	20.7879	.7829	.7570
6	16.3333	28.7879	.0000	.8130
7	16.3333	28.7879	.0000	.8130
8	14.8333	24.8788	.3425	.8052
9	16.0000	20.1818	.7010	. 7666
10	15.3333	17.5152	.7353	.7701
11	16.3333	28.7879	.0000	.8130
12	16.3333	28.7879	.0000	.8130
13	16.3333	28.7879	.0000	.8130

Reliability Coefficients

N of Cases = 12.0 N of Items = 13

Alpha = .8074

The overall value of 0.8074 is a very good reliability coefficient.

Learners wrote a pre test and two post tests. The following are their scores.

Learner	Pre test total marks 28	Post test 1 Total marks 34	Post test 2 Total marks 40
A	2	11	26
B	1	14	25
C	5	20	32
D	4	15	29
E	4	15	28
F	1	15	26
G	4	14	25
H	4	14	26
I	3	23	34
J	3	12	24
K	3	13	27
L	2	21	30

PRE-TEST

		no understanding	some understanding	good understanding	comprehensive understanding
1	Count	11.0	1.0		
	%	91.7	8.3		
2	Count	11.0			1.0
	%	91.7			8.3
3	Count	8.0			4.0
	%	66.7			33.3
4	Count	4.0			8.0
	%	33.3			66.7
5	Count	7.0	1.0		4.0
	%	58.3	8.3		33.3
6	Count	7.0	4.0		1.0
	%	58.3	33.3		8.3
7	Count	10.0			2.0
	%	83.3			16.7
8	Count	2.0	10.0		
	%	16.7	83.3		
9	Count	2.0	4.0	6.0	
	%	16.7	33.3	50.0	
10	Count	12.0			
	%	100.0			
11	Count	12.0			
	%	100.0			
12	Count	12.0			
	%	100.0			
13	Count	12.0			
	%	100.0			
14	Count	12.0			
	%	100.0			

Table 5: An interpretation of leaner responses to questions in the pre-test

DESCRIPTIVE STATISTICS

The mode, which is the most frequent value/understanding value, the median, which is middle most value/understanding value and the mean, which is the average of all understanding values per question was calculated and the results were as follows:

Statistics

	Mean	Median	Mode	Std. Deviation	Range
1	1.1667	1.0000	1.00	.57735	2.00
2	1.3333	1.0000	1.00	1.15470	4.00
3	2.3333	1.0000	1.00	1.96946	4.00
4	3.6667	5.0000	5.00	1.96946	4.00
5	2.5000	1.0000	1.00	1.93061	4.00
6	2.0000	1.0000	1.00	1.34840	4.00
7	1.6667	1.0000	1.00	1.55700	4.00
8	2.6667	3.0000	3.00	.77850	2.00
9	3.1667	3.5000	4.00	1.11464	3.00
10	1.0000	1.0000	1.00	.00000	.00
11	1.0000	1.0000	1.00	.00000	.00
12	1.0000	1.0000	1.00	.00000	.00
13	1.0000	1.0000	1.00	.00000	.00
14	1.0000	1.0000	1.00	.00000	.00

Table 6: The mean, median, mode, standard deviation and range understanding values to questions in the pre-test.

It is interesting to note that the modal understanding value for all of the questions that were posed to the learners was “1" which translated from the scale is “no understanding”. It was only for questions 4, 8 and 9 that the median value is 3 whilst for the rest of the questions it is “1" which translated from the scale is “no understanding”. The standard deviations appear to be small numerically.

RELIABILITY ANALYSIS

Cronbachs Alpha

Cronbach's alpha was also calculated as part of the reliability test to assess how valid the results were and will we get similar results to generalise if we increased the sample size. A value of 0.7 or higher is a very good value that can lead us to say that we will get the same results if we carried out this survey with a larger sample of students. The Cronbach’s alpha was calculated for the questions in questionnaire A and the results are as follows:

****** Method 1 (space saver) will be used for this analysis ******

L I A B I L I T Y A N A L Y S I S - S C ALE (A L P H A)

Item-total Statistics

	Scale Mean if Item Deleted	Scale Variance if Item Deleted	Corrected Item-Total Correlation	Alpha if Item Deleted
1	24.3333	76.0606	.6018	.8140
2	24.1667	70.3333	.5570	.8075
3	23.1667	55.7879	.7746	.7850
4	21.8333	60.1515	.6031	.8066
5	23.0000	55.0909	.8247	.7782
6	23.5000	61.7273	.8924	.7775
7	23.8333	63.7879	.6531	.7977
8	22.8333	74.6970	.5314	.8130
9	22.3333	78.7879	.1225	.8358
10	24.5000	82.4545	.0000	.8305
11	24.5000	82.4545	.0000	.8305
12	24.5000	82.4545	.0000	.8305
13	24.5000	82.4545	.0000	.8305
14	24.5000	82.4545	.0000	.8305

Reliability Coefficients

N of Cases = 12.0 N of Items = 14

Alpha = .8256

This is an excellent reliability coefficient implying that we are receiving a true representation of responses from the learner.

POST-TEST 1

posttest1				
		no understanding	some understanding	comprehensive understanding
1.a	Count			12
	%			100
1.b	Count		4	8
	%		33.3	66.7
1.c	Count	3	5	4
	%	25	41.7	33.3
1.d	Count			12
	%			100
1.2	Count	6	2	4
	%	50	16.7	33.3
1.3	Count	4		8
	%	33.3		66.7
2	Count			12
	%			100
3	Count			12
	%			100
4.1	Count	5		7
	%	42		58

Table 7: An interpretation of learner responses to questions in post-test 1

DESCRIPTIVE STATISTICS

The mode, which is the most frequent value/understanding value, the median, which is the middle most value/understanding value and the mean, which is the average of all the understanding values per question was calculated and the results were as follows:

Statistics

	Mean	Median	Mode	Std. Deviation	Range

1a	5.0000	5.0000	5.00	.00000	.00
1b	4.3333	5.0000	5.00	.98473	2.00
1c	3.1667	3.0000	3.00	1.58592	4.00
1d	5.0000	5.0000	5.00	.00000	.00
1.2	2.6667	2.0000	1.00	1.87487	4.00
1.3	3.6667	5.0000	5.00	1.96946	4.00
2	5.0000	5.0000	5.00	.00000	.00
3	5.0000	5.0000	5.00	.00000	.00
4.1	3.3333	5.0000	5.00	2.05971	4.00

Table 8: The mean, median, mode, standard deviation and range of understanding values to questions in post-test 1.

It is interesting to note that the modal understanding value for all of the questions that were posed to the learners was “5" which translated from the scale is “comprehensive understanding”. It was only for question lc and question 1.2 that the median values were 3 and I which translated from the scale is “some understanding” and “no understanding”. The standard deviations appear to be small numerically.

RELIABILITY ANALYSIS

Cronbachs Alpha

Cronbach's alpha was also calculated as part of the reliability test to assess how valid the results were and will we get similar results to generalise if we increased the sample size. A value of 0.7 or higher is a very good value that can lead us to say that we will get the same results if we carried out this survey with a larger sample of students. The Cronbach's alpha was calculated for the questions in post-test I and the results are as follows:

****** Method 1 (space saver) will be used for this analysis ******

R E L I A B I L I T Y   N A L Y S I S - S C A L E (A L P H A)

Item-total Statistics

	Scale Mean if Item Deleted	Scale Variance if Item Deleted	Corrected Item-Total Correlation	Alpha if Item Deleted
1a	32.1667	59.9697	.0000	.8547
1b	32.8333	46.8788	.8989	.7979
1c	34.0000	40.0000	.8701	.7827
1d	32.1667	59.9697	.0000	.8547
1.2	34.5000	37.9091	.8033	.7930
1.3	33.5000	35.7273	.8649	.7832
2	32.1667	59.9697	.0000	.8547
3	32.1667	59.9697	.0000	.8547
4.1	33.8333	34.5152	.8765	.7826

Reliability Coefficients

N of Cases = 12.0   N of Items = 9

Alpha = .8413

This is an excellent reliability coefficient implying that we are receiving a true representation of responses from the learner.

POST-TEST 2

Posttest 2					
		no understanding	some understanding	good understanding	comprehensive understanding

1.1	Count				12
	%				100
1.4	Count			10	2
	%			83.3	16.7
1.5	Count			10	2
	%			83.3	16.7
1.6	Count				12
	%				100
2.1	Count	4			8
	%	33.3			66.7
2.2	Count				12
	%				100
2.3	Count				12
	%				100
2.5	Count				12
	%				100
3	Count		2	4	6
	%		16.7	33.3	50
4.1	Count				12
	%				100
4.2	Count				12
	%				100

Table 9: An interpretation of learner responses to questions in post-test 2

DESCRIPTIVE STATISTICS

The mode, which is the most frequent value/understanding value, the median, which is the middle most value/understanding value and the mean, which is the average of all the understanding values per question was calculated and the results were as follows:

Statistics

	Mean	Median	Mode	Std. Deviation	Range
1.1	5.0000	5.0000	5.00	.00000	.00
1.4	4.1667	4.0000	4.00	.38925	1.00
1.5	4.1667	4.0000	4.00	.38925	1.00
1.6	5.0000	5.0000	5.00	.00000	.00
2.1	3.6667	5.0000	5.00	1.96946	4.00
2.2	5.0000	5.0000	5.00	.00000	.00
2.3	5.0000	5.0000	5.00	.00000	.00
2.5	5.0000	5.0000	5.00	.00000	.00
3	4.3333	4.5000	5.00	.77850	2.00
4.1	5.0000	5.0000	5.00	.00000	.00
4.2	5.0000	5.0000	5.00	.00000	.00

Table 10: The mean, median, mode, standard deviation and range understanding values to questions in post-test 2.

It is interesting to note that the modal understanding value for all of the questions that were posed to the learners was “5" which translated from the scale is “comprehensive understanding”. It was only for question 1.4 and question 1.5 that the median values were 4 which translated from the scale is “good understanding”. The standard deviations appear to be small numerically.

RELIABILITY ANALYSIS

Cronbachs Alpha

Cronbach's alpha was also calculated as part of the reliability test to assess how valid the results were and will we get similar results to generalise if we increased the sample size.

A value of 0.7 or higher is a very good value that can lead us to say that we will get the same results if we carried out this survey with a larger sample of students. The Cronbach's alpha was calculated for the questions in questionnaire A and the results are as follows:

****** Method 1 (space saver) will be used for this analysis ******

R E L I A B I LIT Y   A N A L Y S I S - S C ALE (A L P H A)

Item-total Statistics

	Scale Mean if Item Deleted	Scale Variance if Item Deleted	Corrected Item-Total Correlation	Alpha if Item Deleted
1.1	46.3333	8.9697	.0000	.5180
1.4	47.1667	7.7879	.4742	.4496
1.5	47.1667	7.7879	.4742	.4496
1.6	46.3333	8.9697	.0000	.5180
2.1	47.6667	1. 6970	.6614	.5159
2.2	46.3333	8.9697	.0000	.5180
2.3	46.3333	8.9697	.0000	.5180
2.5	46.3333	8.9697	.0000	.5180
3	47.0000	5.4545	.8000	.2593
4.1	46.3333	8.9697	.0000	.5180
4.2	46.3333	8.9697	.0000	.5180

Reliability Coefficients

N of Cases = 12.0   N of Items = 11

Alpha =.5128

This alpha value is extremely poor. The overall alpha in this case is 0.5128 which is not high. From the last column above, we have the value of the alpha if a particular question was removed. For example if we removed question 1.1 then the value of the alpha is 0.5180 which increased the overall alpha of 0.5128 As soon as a variable that is removed and it causes the overall alpha to decrease, then that variable is important and reliable, but if a variable is removed and it's removal caused the alpha to increase, then that variable is NOT a reliable variable and is problematic in the questionnaire. The following variables/questions were removed: questions 1.1, 1.6,2.1,2.22.3,2.5,4.1,4.2 and a new Cronbach's alpha was calculated as follows:

****** Method 1 (space saver) will be used for this analysis

R ELI A B I LIT Y A N A L Y S I S S C ALE (A L P H A)

Reliability Coefficients

N of Cases = 12.0 N of Items = 4

Alpha = .6216

This alpha value is reasonable. It is clear then that the **questions 1.1, 1.6, 2.2, 2.3, 2.5, 4.1, 4.2** are not a reliable reflection in this test. The content of the questions are not the issue but simply the way in which these questions have been answered by the students.

COMPARISONS BETWEEN THE PRE-TEST, POST-TEST 1 AND POST-TEST 2

The data was tested and found to follow a normal distribution i.e. it was parametric. Hence the paired sample t-test was then used to test for differences amongst the learners in their scores they achieved with respect to the pre-test, post-test 1 and post test 2.

COMPARISON OF THE PRE-TEST AND POST-TEST 1

H_0: there is no difference in the mean scores achieved by the learners in the pre-test and the post-test 1.

H_1: there is a difference in the mean scores achieved by the learners in the pre-test and the post-test 1.

The results are summarized below:

		Paired Differences		t	df	Sig. (2-tailed) p-value
		Mean	Std. Deviation			
Pair 1	PRETEST-POSTTST1	-12.5833	3.75278	-11.615	11	.000

Table 11: A comparison of the pre-test and post-test 1.

At the significance level of a = 0.05 we will reject H_0 because the p-value is given as 0.000 which is less than a. Hence we conclude that there is a difference in the mean scores achieved by the learners in the pre-test and the post-test 1.

COMPARISON OF THE PRE-TEST AND POST-TEST 2

H_0: there is no difference in the mean scores achieved by the learners in the pre-test and the post-test 2.

H_1: there is a difference in the mean scores achieved by the learners in the pre-test and the post-test 2.

The results are summarized below:

Paired Samples Test

		Paired Differences		t	df	Sig. (2-tailed) D-value
		Mean	Std. Deviation			
Pair 1	PRETEST-POSTTST2	-25.5000	3.94277	-22.404	11	.000

Table 12: A comparison of the pre-test and post-test 2

At the significance level of a = 0.05 we will reject H_0 because the p-value is given as 0.000 which is less than a. Hence we conclude that there is a difference in the mean scores achieved by the learners in the pre-test and the post-test 2.

COMPARISON OF THE POST-TEST 1 AND POST-TEST 2

H_0: there is no difference in the mean scores achieved by the learners in the post-test I and the post-test 2.

H_1: there is a difference in the mean scores achieved by the learners in the post-test! and the post-test 2.

The results are summarized below:

		Paired Differences		t	df	Sig. (2-tailed) D-value
		Mean	Std.			

			Deviation			
Pair 1	POSTTST1-POSTTST2	-12.9167	3.02890	-14.773	11	.000

Table 13: A comparison of post-test 1 and post-test 2.

At the significance level of a = 0.05 we will reject H_0 because the p-value is given as 0.000 which is less than a. Hence we conclude that there is a difference in the mean scores achieved by the learners in the post-test I and the post-test 2.

The frequency tables for the 3 tests are summarized below:

PRETEST

		Frequency	Percent	Cumulative Percent
Valid	1.00	2	16.7	16.7
	2.00	2	16.7	33.3
	3.00	3	25.0	58.3
	4.00	4	33.3	91.7
	5.00	1	8.3	100.0
	Total	12	100.0	

POSTTEST1

		Frequency	Percent	Cumulative

				Percent
Valid	11.00	1	8.3	8.3
	12.00	1	8.3	16.7
	13.00	1	8.3	25.0
	14.00	3	25.0	50.0
	15.00	3	25.0	75.0
	20.00	1	8.3	83.3
	21.00	1	8.3	91.7
	23.00	1	8.3	100.0
	Total	12	100.0	

POSTTEST2

		Frequency	Percent	Cumulative Percent
Valid	24.00	1	8.3	8.3
	25.00	1	8.3	16.7
	26.00	3	25.0	41.7
	27.00	1	8.3	50.0
	28.00	1	8.3	58.3
	29.00	1	8.3	66.7
	30.00	1	8.3	75.0
	32.00	1	8.3	83.3
	34.00	1	8.3	91.7
	35.00	1	8.3	100.0

	Total	12	100.0	

Table 14: A summary of the frequencies of the three tests.

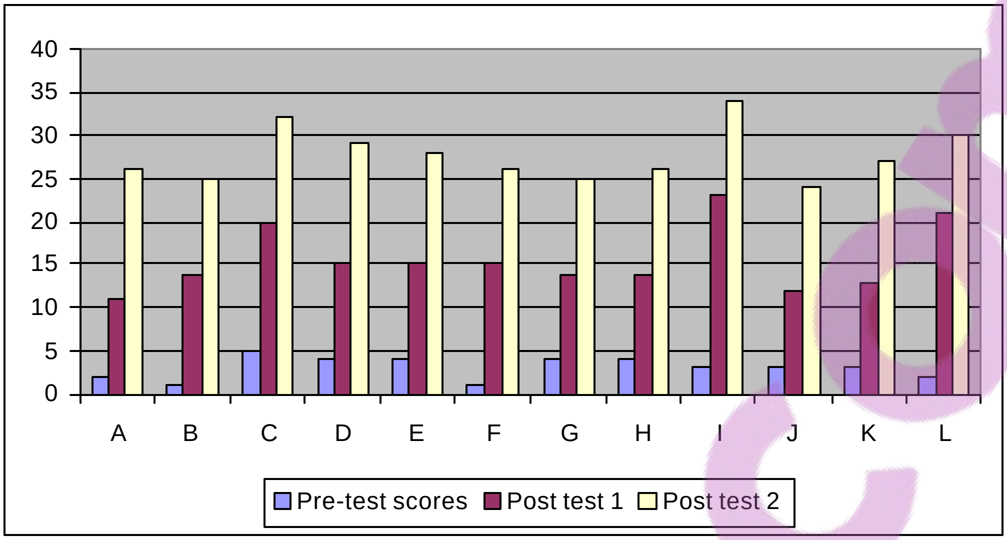
The descriptive statistics for the three tests are summarized below:

Statistics

		PRETEST	POST TEST1	POST TEST2
N	Valid	12	12	12
Mean		3.0000	15.5833	28.5000
Median		3.0000	14.5000	27.5000
Mode		4.00	14.00	26.00
Std. Deviation		1.27920	3.72847	3.58025
RanQe		4.00	12.00	11.00
Percentiles	25	2.0000	13.2500	26.0000
	50	3.0000	14.5000	27.5000
	75	4.0000	18.7500	31.5000

Table 15: A comparison of the means, medians, modes, standard deviations, ranges and percentiles of the three tests

These results can also be illustrated graphically in the following manner.



Key 1cm = 5 marks

Fig. 26 A bar graph illustrating learner performance in the pre-test and post-test 1 & 2.

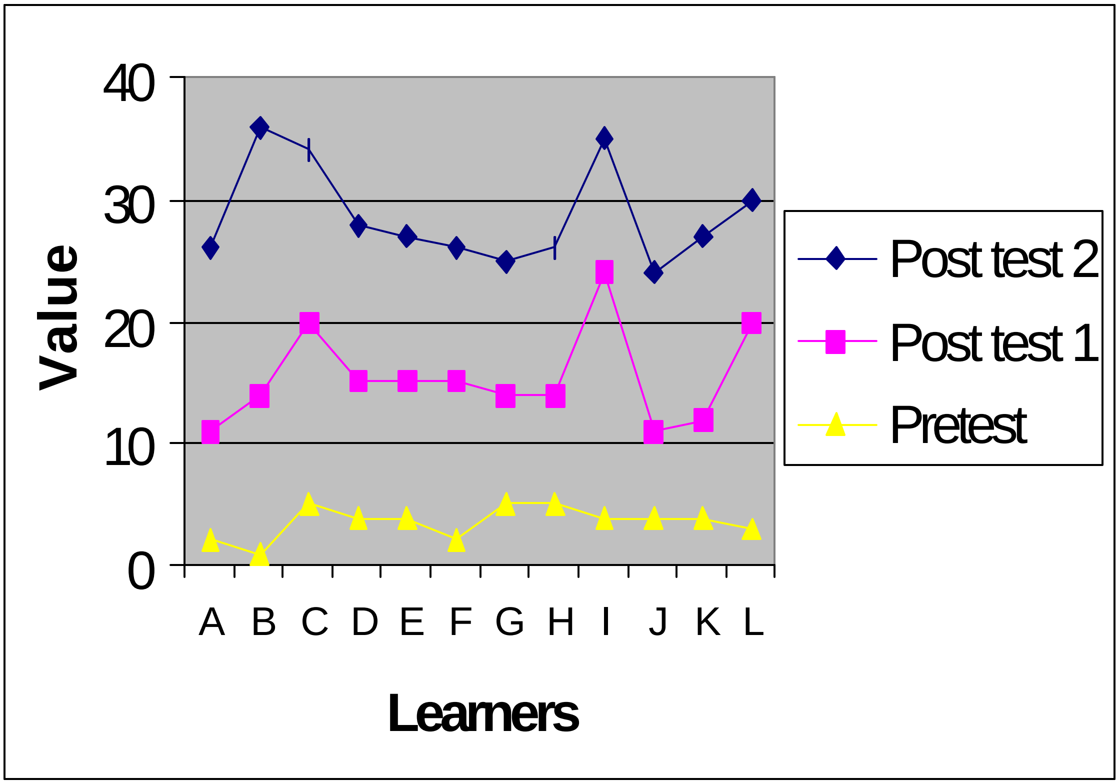


Fig. 27 Line graph indicating individual learner progress

The sample (12 learners) that was used in the study is small, hence a generalization cannot be fairly made. It is thus imperative that individual learner progress of some ofthe learners be made os as to establish whether activities of creativity and divergent thinking in a problem-based context tend to enhance the internalization of geometry concepts.

Elements of creativity and divergent thinking in these individual learners are traced in the worksheets, interviews, questionnaires and activities mentioned in S. 6.2.1.5 - 6.2.1.14 to
finally culminate in learner perfonnance in the post tests. These elements are indicated in
the model for the problem-based approach to the teaching and learning of geometry (see Fig. 29).

Worksheets and questionnaires were administered, interviews conducted during teaching episodes between the pre-test and post-test 1. Activities in S. 6.2.1.5 - 6.2.1.14 were undertaken between post test 1 and 2. These tasks serve a dual purpose: firstly learners have to be creative and secondly their understanding of geometry concepts is tested. The correctness of th geometry concepts as depicted in the learners' products (e.g. the musical
instruments) indicates whether the concepts have been retained (these activities were undertaken between post test 1 and 2 and no teaching of geometry took place during this period). These activities thus afford learners the opportunity to reveal their understanding of geometry concepts and also offer a rich teaching and learning environment for geometry.

Learner A

This learner obtained 2 marks out of28 in the pretest. The leamer's responses to questions
in the test showed a lack of understanding of geometry concepts such as right angle, parallel lines, isosceles triangle and rectangle.

Worksheet 1 (see S.6.2.1.1) required learners to describe the properties of each of the geometric shapes drawn in their own words and to give examples from their environment
that resemble the shape in question. On mentioning the properties of a right angled triangle (in his group), this learner stated that it is like the comer of a house. He related the concept of a right angle to his previous experiences. The learner gave the same response on describing the angles of a rectangle. the learner could describe and relate - essentail elements of creativity and divergent thinking.

This worksheet also requires learners to find the sum of the angles of some polygons and hence deduce a formula for the sum of the angles of a polygon with n-sides. Having been
given the trick that to find the sum of th angles of a given polygon, divide it into triangles and knowing that the sum of the angles ofa triangle is 180°°, the sum of the angles of such polygon is given by 180°° multiplied by the number of triangles. Leamer A had argued that
a pentagon could be divided into five triangles thus the sum of the angles is 900°.

The researcher had then mentioned that it (pentagon) should be divided into a minimum number of triangles so that the sum would be 180° x 3 = 540°. This had, nonetheless showed critical thinking on the part of the learner.

Worksheet 4 required learners to draw the nets of a cube. In most cases, learners were producing the same net in different orientations. Leamer A realised that in rotating certain shapes, we end up with a previous one (a spatial ability). this learner produced most nets as shown in Fig. 98.

In the interviews that were conducted, learners A could describe some properties of a rectangle - opposite sides are equal, identify a square and mention that a square has equal sides and all angles are equal to 900, identify a parallelogram and mention that the opposite sides are parallel.

Learner A's performance in the post-tests improved. The learner correctly defined most geometric shapes and was able to identify shapes in post-test 1. The learner correctly matched the five objects to corresponding descriptions in post-test 2. There was a marked improvement as compared to the pre-test. Learners were also required to write analogies and metaphors based on geometric concepts.

Leamer A's choice of sandwiches with cheese and polony inside and wors for breakfast illustrates his understanding of a square, circles, right angles and cylinders.

Learner B

This learner obtained I mark out of28 in the pretest showing a lack of understanding of the geometry concepts tested. In worksheet 1, learner B argued that Fig. 3.10 is a quadrilateral since it has four sides (one of the sides is a curve). It was pointed to this learner that one of this shape's sides is a curve and thus is not a polygon since a polygon is made up of straight lines. Her argument showed critical thinking on her part.

In sorting packaging materials (3 -dimensional shapes) worksheet 2, learner B realised that a ball is not a cylinder as the other learners claimed since it does not possess a flat base.

The concept of a hexomino (worksheet 6) as an arrangement of six squares on a plane much that there is no overlapping between the squares i.e. a side of one square completely covers
the side of an adjacent square was clearly demonstrated by this learner in the seven hexominoes drawn on diagram

In the interviews that were conducted (see S. 6.2.1.4) learner B correctly identified a rectangle and stated that the opposite sides are equal and that all angles are right angles. The learner also correctly identified a square and mentioend that all sidres are equal and the
angles equal 90°.

In the activities mentioned in S. 6.2.1.5 learner B chose to write an essay that would depict her understanding of geometry concepts such as a square, rectangle, rhombus, isosceles
triangle and a right angle. She related these concepts to objects in her surroundings. These concepts were correctly applied to these objects. This highlights the partnership between
geometry and creativity and cements the leamer's understanding of geometry concepts.

Furthermore, learners were required to write geometric concepts in such a manner that they depicted the concept at hand. The learner wrote concepts such as triangle, parallel, pentagon, octagon, square, rectangle, rhombus, parallelogram, trapezium, kite, hexagon and prism (see S. 6.2.1.12) in such a manner that the meaning of these concepts is shown.

The learner answered most questions in post-test I and 2 correctly showing an improvement as compared to the pretest.

Learner C

This learner obtained the highest score of 5 marks our of 28 in the pretest.

In worksheet 3 learners were required to use their inductive reasoning to derive Euler's formula after having counted the number of faces, edges and vertices of six three dimensional shapes. Learners could not readily see the relationship between the number of these three concepts. Finally learner C realised the relationship and wrote the formula:

$F+V-2=E$

which can be rewritten as $F + V = E + 2$, the usual form of Euler's formula.

Leamer C responses to questions in worksheet 7 showed that he had internalized the concept fo Tessellations as he could identify angles FGK, KLP, PQI, BCG, GAL and CDH

as corresponding to LABF and also realised that the pattern in the figure 4 could be made up of triangles similar to triangles BFG, AKC, APID and KPHC in different orientations. To arrive at this conclusion, the learner had to first identify the corresponding angles and

sides in these triangles and parallelograms. This entrenches the properties of triangles and parrallelograms and congruence of shapes. Tessellations playa crucial part in this regard.

Leamer C's responses to some questions in the questionnaires reveal comprehensive understanding of the concepts involved (se S. 6.2.1.2). The learner (in questionnaire 1) defines a tetrahedron as a solid with four faces, correctly defines an edge as a portion of a polyhedron where faces meet and a vertex as the portion where the edges meet.

In questionnaire 2, the learner defines a tessellation as an arrangement of polygons in such a manner that there is no overlapping and no spaces between/amongst them.
Worksheet 10 tested the learners' ability to manipulate 3-dimensional shapes. They had to build complicated structures using some of the 7-puzzles they had built using cubes. Leamer C was leading his group in building structures A, B. C and D. Indeed he was called to assist group 3 who were struggling. To be able to build these structures, learners had to flip, rotate and slide the puzzle pieces so as to fit into each other and moreover, they had
to decide which of the puzzle would be suitable to build a given structure. This activity developed the eye-motor co-ordination (Del Grande 1990) and enhance their (learners) spatial invisualization.

Leamer B' s performance in the post tests improved even further as compared to the pretest. His answers to questions in post test 1 showed comprehensive understanding of concepts
such as an equilateral, scalene and right-angled triangle. He gave a comprehensive description of these triangles. He also mentioned that in a right-angled there will always be a longest side i.e. the hypotenuse and a largest angle, the right angle - an improvement to his response to the same questions in the pretest.

Learner F

This leamer's responses to questions in the pretest revealed little or no understanding of the geometric concepts involved.

In the interviews conducted (see Appendix D), learner F correctly gave the number offaces, edges and vertices of a given object and mentioned properties of shapes such as a square,
rhombus and hexagon when such shapes were shown to him as was the case in worksheets during the teaching episodes.

Leamer F produced the first tessellation (top left) - see S. 6.2.1.9. this activity, whilst affording the learner the opportunity to showcase his artistic abilities, highlighted a tessellation as a repeated pattern of similar shapes such that there is no overlapping or spaces between them. His drawing are the same in each repeated shape.

The same learner produced the pattern of pen tomi noes (top right) - see S. 6.2.1.10 in the form of a L. This activity entrenches the properties of the squares (all sides equal and all angles right angles) and what a pentomino is (all the sides and angles of the squares being used are equal as they fit onto one another without any overlapping or spaces between them). To produce a pattern requires the learner to be creative.

The learner drew a van showing a triangle, rectangles, trapezium, square and parallelogram correctly (see S. 6.2.1.14). this seems to indicate that he has internalized the properties of
such shapes and can highlight them in a creative way.

Learner F’s performance in the post tests improved. He was able to identify and define the above named shapes which was an improvement to his performance in the pretest.

Learner L

As with the learners mentioned above, learner L's responses to questions in the pretest showed little or no understanding fo the geometric concepts involved.

The learner could draw the net of a square-based pyramid (see worksheet 5) - fig 19. Learner L led his group to arrive at yet another net of a square-based pyramid (see fig 20
this seems to point to his ability to represent pyramids in two dimensions.

Learners were required to make musical instruments out of cartridge paper with the parts of such instruments made up of geometric shapes. Learner L produced a guitar which was
made up of a triangle and a rectangle (see fig. 12). In addition to his, the learner also produced a tambourine consisting of cylindrical rings. The learner correctly stated the properties of the geometric shapes being used in these instruments.

Leamer L was in the group that produced a man crossing, using the seven puzzle pieces of a tangram consisting of five right-angled triangles, a square and a parallelogram. In a similar fashion as above, learners described the properties of the shapes that were used.

Leamer L' s poster consisted of a bus whose parts consisted of, inter alia, triangle, rectangle, circle, square and parallelogram. The fact that these were correctly shown seems to indicate
that the learner has internalized the properties of the said shapes.

The leamer's perforn1ance in the post tests revealed comprehensive understanding of the above named concepts as compared to his performance in the pretest.

Creativity and divergent thinking were qualified in terms of the elements or activities (of creativity and divergent thinking) mentioned in diagram under S. 4.5. Learners engaged in these activities during geometrical problem solving in a problem-based context using worksheets, questionnaires, interviews, tests and activities in S. 6.2.1.5 - 6.2.1.14 as the teaching and learning environment. These elements/activities were identified in the various problem solving exercises learners were engaged in during the study. It would seem reasonable to suggest that the learners' performance in the post tests improved as compared to the pretest as a result of exposure to such activities. Learners gave comprehensive answers to most questions in these tests. It seems that these activities of creativity and divergent thinking during geometrical problem solving contributed to the learners' internalization of geometry concepts.

In between post test 1 and 2 there was no teaching of geometry except to let learners engage inactivities mentioned in S. 6.2.1.5 - 6.2.1.14. These activities afforded learners the opportunity to reveal their understanding of geometry concepts in a creative way. The fact that learners correctly reflected geometric concepts in these activities would seem to indicate that such concepts have been retained. This also further highlights the importance of creativity and divergent thinking in the learning of geometry - these two concepts seem

to be the fertile ground for entrenching geometry concepts in learners.

6.4 CONCLUSION

The activities conducted in the worksheets have an impact on teaching practice and curriculum design. Learner, especially those in the lower grade like Grade 7 have problems in giving precise definitions of geometric concepts. Learners should be given the latitude to express themselves in vernacular. Given this opportunity learners do describe geometric shapes or concepts. Their definitions have then to be put in English. On returning to such definitions after a few days have elapsed, the learners have forgotten. Definitions are generally problematic for learners, they are more at home in manipulating shapes such as three dimensional shapes, for example, finding nets of solids, calculating the number of edges, faces and vertices of solids, tessellating shapes and working with three dimensional structures to build other complicated structures. This would seem to indicate that geometry at these lower levels must emanate from the three dimensional objects that learners are familiar with and this should gradually be extended to formal geometry in the upper grades. There should be more questions on application or manipulation of geometric shapes rather than on definitions in these grades. The mathematics curriculum should also emphasize the above approach to the teaching of geometry.

Approaches that emphasize learning through self discovery should be encouraged and pursued in the teaching and learning of geometry as was the case with learners arriving at a definition of a pentomino described in this study.

Responses to the questionnaires reveal some progress in the internalization of geometry concepts.

Using the semi-structured interviews, addressed some of the misconceptions that some interviewees had. In this manner the interview played a formative type of assessment and corrected learner misconceptions. Learners revealed that they enjoyed working with concrete objects such as finding nets of solids, classifying solids, working with three dimensional puzzles and tessellating polygons. The interviews also revealed that they enjoyed defining geometric concepts the least.

Learner performance in both post tests was a marked improvement from the pre test. This was evidenced by individual performance in all the tests as well as the difference between the means of the three tests. The fact that learners' performance in post test 2 is slightly higher than in post test 1 would seem to indicate that learners have internalized the geometric concepts handled.

The following elements of creativity and divergent thinking were identified in the learner responses to the worksheets, questionnaires, interviews and tests: communicate, describe, manipulate, relate, verify, conjecture, define, compare, deduce, critical thinking, classify/sort, self-discovery, identify, draw, state/explain and match.

Learner performance in the two post tests shows that learners have internalized geometry concepts as a result of engaging inactivities of creativity and divergent thinking as compared to their performance in the pretest - see the means, modes, medians, frequencies, standard deviations and the alpha values for reliability in 6.3. The comparison of the pretest and post test 1, pretest and post test 2 and post test 1 and post test 2 also reflect an improvement in learner performance as a result of engaging in these activities. The results in the post test reflect the extent to which activities of creativity and divergent thinking enhance the internalization of geometry concepts.

The various activities in th empirical study encouraged learners to solve problems, for example in the worksheets, they had to be creative and divergent thinkers in the process (the cognitive processes were used). Furthermore, learners were creative in, inter alia, composing songs and writing poems and posters based on geometry concepts.

For geometry to be taught in a meaningful way, a rich learning environment should be provided for by the teacher. In the context of this study, learners had to solve problems and from the solution of such problems and further problems emanating from the solution of previous ones, the learners acquire knowledge. Learners' attention should be specifically directed towards activities of creativity and divergent thinking during geometrical problem solving. In solving problems, learners should be made to be, inter alia, conjecturers, experimenters and verify their initial guesses. This should relate what they learn in the classroom to their everyday life situations, for example, write metaphors and analogies, compose songs and write posters on geometry concepts. These activities enhanced their conceptual understanding of geometry.

CHAPTER 7

SUMMARY, CONCLUSION AND RECOMMENDATIONS

7.1 SUMMARY OF THE RESEARCH

The research aimed at establishing the extent to which creativity and divergent thinking enhance the internalization of geometry concepts in a problem-solving context and in encouraging the development of learners who are creative, divergent thinkers and problem solvers.

The above aims are realised through the achievement of the objectives as mentioned in S. 1.3.1.

7.1.1 The Concept of creativity

The creativity of a product depends on its novelty and usefulness (appropriateness). Such a product could be a new invention altogether or could be an improvement of an already existing product due to some inadequacy in it. In the improvement of an existing product the individual must be knowledgeable about the subject matter (domain) in which the initial product was invented. A product must be judged to be creative by experts in the domain i.e. the field (Csikszentmihalyi 1996:27-28).

Intelligence may be a necessary but not a sufficient condition for creativity. If an individual is intrinsically motivated, he is willing to spend long hours in what he is doing, he becomes as much knowledgeable about the old as possible and this increases the likelihood of him arriving at a novel product (Weisberg 1988:173).

Of importance to this study is the manner in which creativity manifests itself in the classroom. If a learner discovers the meaning of a concept (the meaning is already known to the educator) for himself (self-discovery) or sees the concept in a new light, that is an AHA! experience for the learner (Parnes 1975:225) and that is creativity on his part. This is the situation when mathematics learners think like the mathematicians do, they re-invent mathematics i.e. the process before the product is arrived at (Cuoco et al 1996:376-378).

In geometrical problem solving, learners should relate ideas about solving problems, investigate those ideas, create new problems from the solutions of problems at hand, evaluate solutions and communicate results (Kokol-Voljƒ and Sheffield 1999:32).

7.1.2 Divergent Thinking and Geometry Education

In the study divergent thinking included the following cognitive process:

- focusing on a question - identifying a question and judging possible answers
- analyzing a problem
- reflecting on the situation as well as metacognition
- deductive and inductive reasoning
- defining terms
- problem solving
- communicating
- comparing
- observing
- summarizing

- classifying/sorting
- interpreting
- imagining
- conjecturing
- applying facts to new situations
- critical thinking, and
- experimenting.

Learners used these processes in geometric problem solving and hence the internalization of geometry concepts.

Geometric shapes in the child's environment (polygons and polyhedra) were a starting point in the teaching and learning of geometry in a Grade 7 classroom. It is from the exploration of the properties of such shapes that various relationships (which are summarized in axioms and theorems) can be arrived at. This is in line with Freudenthal's recommendation that the starting point in geometry should be from the child's everyday life experiences of spatial objects (Streefland 1993:122). This affords the learner the opportunity to re-invent geometry (Freudenthal 1971:418).

An individual's intuitive feel of objects (geometric shapes) in the individual's environment is referred to as spatial sense. According to Wheatley (1990:10), spatial sense should be thought of as imagery/visualization i.e. seeing the actual object and noting its properties or having a mental picture of an object.

Del Grande (1990:14-18) describes spatial perception in terms of seven spatial abilities (see 3.3.4). These skills are essential for enhanced spatial visualization.

The works of Piaget, Vygotsky, Freudenthal, Feuerstein, van Hiele and the cognitive sciences provide us with insight on how geometry is learned.

7.1.3 Creativity and divergent thinking in geometry education

Creativity in geometry education entails, inter alia

- **Synectics**
  - the joining together of two different elements which at a face value do not have anything in common. It is based on the use of metaphors and analogies in such a manner as to achieve creative results.

- **Poetry, essays, posters, songs and musical instruments**

  - these can be used to make learners become aware of the geometry in their lives.

- **Tangrams and tessellations**

  - these can be used to change the mindset that mathematics is only concerned with computations. In manipulating tangram pieces to make designs and in tessellating polygons, learners familiarize themselves with and get to know the properties of the shapes they are manipulating. Furthermore, tangrams and tessellations bring to the fore the correlation between mathematics and in particular geometry and art.

- **Pentominoes**

  - these give learners the opportunity to examine concepts of congruence, similarity, tessellations, perimeter, area and volume in a problem solving environment. Tessellating pentominoes also produces interesting patterns (artwork).

- **Creative correlations in geometry**

  - writing words such as triangle as tr▲angle not only gives learners the opportunity to internalize the meaning of such words but also an opportunity to be creative in the writing of such words (Westcott 1978).

Divergent thinking in geometry education involved learners being engaged in the activities as outlined in 7.1.2.

Informal geometry forms a basis for the learning of formal geometry. Geometry instruction should start with the three dimensional shapes then proceed to their representation in two dimensions. From this order should come axioms and theorems (formal geometry). This is in line with the traditional deductive principles of

- from the concrete to the abstract and
- from the known to the unknown.

The problem-centred approach to the teaching of geometry helps produce individuals who can think, are creative and are problem solvers. Learners are left to their own resources with the educator acting as facilitators. They look for strategies to solve problems and communicate such strategies to their peers and negotiate meaning for common understanding in groupwork during the problem solving process.

7.**1.4 Findings from the research**

The qualitative research was divided into activities in the worksheets, questionnaires, interviews and descriptions of learner responses to the pre test and two post tests.

The pre test revealed that learners did not know of much of the geometric concepts that were to be handled i.e. properties of polygons, polyhedra, nets of solids, tessellations and tangrams. In relating polygons to shapes in their environment, learners gave examples of a door, book and duster as rectangles. They were actually referring to faces in these solids as rectangles. The distinction between 3-D and 2-D shapes was made at a later stage. Even though learners could compare various polygons, their comparisons were not comprehensive, they had to be guided towards this end.

Learners could classify packaging material as to whether such material were prisms, pyramids, cylinders or cones. The derivation of Euler's formula was problematic but one learner finally arrived at the formula. Defining geometric concepts is problematic for the learners, they enjoy working with concrete objects - finding nets of solids and calculating the number of faces, edges and vertices of solids. Learners could tessellate polygons and were able to arrive at conclusions such as that not all polygons will tessellate, the sum of the angles of a quadrilateral is 360E and that from the quadrilaterals handled, all quadrilaterals will tessellate.

Even though they could produce some designs from the tangram pieces, these were not varied and learners were not creative enough in this regard.

Learners enjoyed working with three dimensional structures and could build complicated structures by combining simpler ones together.

The responses to the two questionnaires administered were answered in a similar fashion to the responses in the worksheets. From the interviews, it became apparent that learners enjoyed working with concrete shapes such as cubes and tessellating shapes.

The responses to the test items in the two post tests shows a growth in spatial development compared to their responses to the pre test.

The quantitative research revealed a growth in spatial development of learners as shown by the high test scores in the two post tests as compared to the pre test. This has been illustrated using two graphs (see 6.3). Scores in post test 2 are a little higher than scores in post test 1 which was written on more or less the same material three months ago. This seems to reflect that learners internalized concepts handled before writing post test 1.

7.2 CONCLUSIONS

The following conclusions were arrived at in the study of creativity and divergent thinking in geometry education.

7.2.1 Curriculum

Learners in the senior phase of the primary school could not distinguish between 3-D and 2-D shapes. They mentioned a door, book and duster as examples of rectangles and a ball and orange as examples of circles. According to the van Hiele theory, this reflects level 1 reasoning. Learners were actually referring to the faces in these solids or deliberately ignoring the thickness of the solids. The fact that the above distinction could not be made seems to indicate that geometry instruction in the lower levels did not start with 3-D shapes. These shapes were not represented in two dimensions and vice-versa so as to make the distinction between the two possible. According to Freudenthal (1971:418) this is a wrong approach to the teaching of geometry. Geometry should emanate from the child's everyday experience of geometric shapes (which are 3-D objects).

7.2.2 Informal Geometry

Informal geometry forms the basis for the learning of formal geometry. Informal geometry is the bridge between what the child has been exposed to prior to coming to school (3-D shapes in his environment) and the formal geometry he learns at school.

Learners enjoy working with real life objects such as packaging materials, cubes and tessellations. These are not foreign to them. These objects should be the starting point in geometry education.

Informal proofs between geometric concepts such as tearing the three angles (corners) of a triangle cut out of paper and putting these next to each other on a straight to prove that the sum of the angles of a triangle is 180E serves as a starting point for a formal proof of the theorem.

7.2.3 Definitions

Definitions (van Hiele level 3)are problematic for learners at this stage. They have difficulty recalling these. Learners are more comfortable with working with concrete objects such as calculating the number of faces, vertices and edges rather than defining these. Geometry instruction should take cognisance of this fact and even though definitions are important, emphasis should not be put to recall of such definitions at this stage.

7.3 RECOMMENDATIONS

The following recommendations can be made from the study.

7.3.1 Language

Learners should be given the opportunity to verbalize their perceptions of geometric shapes and concepts in vernacular. Their earlier perceptions of such shapes is grounded in their mother tongue. They are more prone to giving comprehensive descriptions of such shapes if they are given the opportunity to do so in their mother tongue. Their descriptions are then phrased in the medium of instruction which in this case is English.

7.3.2 Vocabulary List

Learners should be encouraged to have a vocabulary list of the words they encounter in the learning of geometry each day. This is especially significant for second language learners of geometry.

7.3.3 Re-inventing Geometry

Learners should not be told what the properties of a given shape are or of the relationship between geometric concepts, for example, a square is a rectangle with equal sides. They should be made to discover these for themselves (self discovery). Through probing and making use of illustrations the teacher should guide learners to re-invent geometry. The problem is that the teacher may be impatient or may feel that this guidance and probing is time consuming. What a child has discovered for himself is likely to be retained than what he is merely told.

7.3.4 Heuristics

Learners should be encouraged to have their own heuristics to problem solving. They should list the steps in problem solving and seek alternative solutions during problem solving. They should communicate about their problem solving strategies and have a repertoire of these.

7.3.5 Patterns

Learners have to be trained to look for patterns/invariants in a given shape be it in three or two dimensions. The search for patterns is a necessary step towards the solution of geometrical problems (riders) in high school.

7.3.6 Correlation of mathematics with other subjects/learning areas

Knowledge is one; it has been compartmentalized into ‘subjects’ for the sake of convenience. Every opportunity should be utilized to integrate the learning of mathematics to the learning of other subjects. Tessellations correlate mathematics to art. Moreover, the topic also contributes to the aesthetic development of a child. Such topics should not be neglected in geometry instruction.

7.3.7 Creativity in a geometry classroom

Learners should be made to write essays, songs, poems, posters, analogies and metaphors on geometric concepts, and words in geometry with illustrations of the meaning of such words. They should construct musical instruments using geometric shapes. They should also tessellate polygons and pentominoes and colour these to produce works of art. This not only helps learners to internalize geometric concepts but is also fertile ground for learners to be creative.

7.3.8 The teaching of solids

In most schools for African learners, three dimensional shapes are not taught (see 7.2.1). Learners have no idea of how such shapes can be represented in two dimensions. This denies learners the opportunity to be trained in seeing patterns. Is the omission of the above topic the reason why African learners perform poorly in Mathematics Paper II in Grade 12 and which consists mostly of geometry? Do teachers emphasize paper 1 at the expense of paper 2 (they handle paper 1 first and run out of time for paper 2)? These issues need to be researched.

7.3.9 Praise singing

Africans do praise singing in honour of their political leaders. This could be on geometry shapes as well. This helps learners internalize such concepts.

7.3.10 Creativity and divergent thinking in the learning of geometry

Activities of creativity and divergent thinking should be used in the teaching and learning of geometry using the problem-centred approach. These activities enhance conceptual understanding and groupwork should be used during such activities.

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