Contents

List of Figures

Chapter 0

Preface

In this thesis we study representations of Cayley graphs and transitive graphs with automata. This study had been inspired by the notion of Cayley automatic groups introduced by Kharlampovich, Khoussainov and Miasnikov [1]. The notion of Cayley automatic groups emerged from the area of automatic structures and the area of automatic groups. Automatic groups were introduced by Thurston as a tool for computations on fundamental groups of 3–manifolds. Thurston noted that the fundamental groups of many 3–manifolds can be computed by finite automata, and that motivated him to connect group theory, theory of 3–manifolds and formal language theory. The theory of automatic groups is now one of the cutting edge research topics in group theory [2]. Khoussainov and Nerode initiated a development of the theory of automatic structures in the mid of the 90s as a systematic way to represent algebraic structures with finite state machines such as finite automata. The area of automatic structures had become an essential part of the theory of computable structures. This theory also provides a theoretic framework for extending the theory of finite models [3].

Both automatic groups and automatic structures use the same computational model – multitape synchronous automata. Cayley automatic groups have the following key properties.

- The definition does not depend on the choice of generators.
- Synchronous automata are used in the definition of Cayley automatic groups just like in case of automatic groups.
- Cayley automatic groups properly extend the class of automatic groups. For instance, the Heisenberg groups and the Baumslag–Solitar groups are Cayley automatic.
- The word problem is decidable in quadratic time.

The last property, in some respects, provides a universal algorithm for solving word problem in Cayley automatic groups.

In this thesis we study representations with automata of three important families of structures. The first one is the class of the Baumslag–Solitar groups. The second one is the family of the wreath products of groups. Wreath product is an important algebraic construct in group theory. This algebraic construct can naturally be extended to the class of transitive graphs. One of our goals is to study automata–theoretic properties of this wreath product operation. For instance, we study if the wreath product of two groups A and B is Cayley automatic in case both A and B are Cayley automatic. The third one is the family of transitive non–Cayley graphs including the Diestel–Leader graph.

In a more general setting, our goal is to give characterizations of Cayley automatic groups. This is a very challenging problem and, perhaps, does not have a general satisfactory solution. However, our results show that for some classes of finitely generated groups and transitive graphs the problem can be solved positively. In order to address the characterization problem, we introduce and then study some numerical characteristics of Turing transducers for which all heads move synchronously first forth and then back.

The Baumslag–Solitar groups play an important role in combinatorial and geometric group theory. The Baumslag–Solitar group $BS(m, n)$ has two generators a and t with the single relation $t^{-1}a^m t = a^n$. These groups often provide examples distinguishing different classes of groups. For instance, this family of groups contains residually finite groups, non–Hopfian groups, and Hopfian groups that are not residually finite. The Baumslag–Solitar group $BS(m, n)$ is not automatic unless $m = 0$, $n = 0$ or $|m| = |n|$. Kharlampovich, Khoussainov and Miasnikov proved that some of the Baumslag–Solitar groups are Cayley automatic [1]. But, they could not prove that all Baumslag–Solitar groups are Cayley automatic. Therefore, they posed a question whether all Baumslag–Solitar groups are Cayley automatic [1]. In this thesis we construct Cayley automatic representations for all Baumslag–Solitar groups, and then study these representations in relation with some metric properties of these groups.

Cayley graphs (obtained from finitely generated groups) are directed graphs whose labels are some group generators. If we remove the labels and the directions of edges of Cayley graphs, then we obtain transitive graphs. For the time being we call these graphs undirected Cayley graphs. Transitive graphs are natural algebraic objects. They are, by definition, connected graphs such that any two vertices of a graph can be mapped to each other via an automorphism of the graph. The famous random graph is an example of transitive graph and so are undirected Cayley graphs. An important difference between the random graph and undirected Cayley graphs is that the degree of every vertex of the random graph is infinite, while the degree of every vertex of an undirected Cayley graph is constant. It is natural to ask whether any automatic transitive graph can be, in some respect, close to an automatic undirected Cayley graph. Gromov introduced the notion of closeness between groups via the notion of quasi–isometry [44]. Informally (but which is sufficient for the preamble) two graphs are quasi–isometric if there exists a bilipschitz map between the graphs which has a constant distortion. In light of this, we ask if every automatic transitive graph can be quasi–isometric to an automatic undirected Cayley graph. This question was formulated by Miasnikov and Kharlampovich in personal communication. In this thesis we show that the family of all automatic transitive graphs is substantially wider than the class of automatic undirected Cayley graphs. We show that the class of automatic transitive graphs includes an infinite family of automatic non–Cayley transitive graphs. The limit of this family is the Diestel–Leader graph which is known to be not quasi–isometric to any Cayley graph [45]. In this thesis we show that the Deiestel–Leader graph is automatic.

The construction of wreath products of groups plays a significant role in combinatorial and geometric group theory. The wreath product $A \wr B$ is not finitely presented and, therefore, not automatic unless B is finite or

A is the trivial group. In this thesis we construct Cayley automatic representations for the wreath products of the form $G \wr \mathbb{Z}$ and then study these representations in relation with some metric properties of these groups and some subgroups of $G \wr \mathbb{Z}$. We use a deterministic pushdown automaton and a nested stack automaton to construct representations for Cayley graphs of the wreath products of the form $G \wr F_n$ and $G \wr \mathbb{Z}^2$, respectively. Then we study these representations in relation with some metric properties of these groups and some of their subgroups. It is still an open question whether the wreath products $G \wr F_n$ and $G \wr \mathbb{Z}^2$ are Cayley automatic for any nontrivial group G. Any solution to this problem will provide a significant advance in understanding automaticity of graphs, in particular transitive graphs and Cayley graphs of groups.

Finally, in this thesis we investigate the problem of characterization of Cayley automatic groups. In order to address this problem we define and then study three numerical characteristics of a special class of Turing transducers. We first show that automatic representations of Cayley graphs can be expressed in terms of the Turing transducers for which all heads move synchronously first forth and then back. Then we study admissible asymptotic behaviour for these three numerical characteristics of Turing transducers of this special class. As we already indicated, this thesis is one attempt (and, perhaps, the first attempt) towards the grand task of characterization of automatic transitive graphs. We hope that much more work will follow and shed more light on our understanding of interconnections between automata, groups, and transitive graphs.

Chapter 1

General Introduction

This thesis contributes to the field of automatic structures [4, 5, 6] and contains new results on representations of Cayley graphs and transitive graphs by automata. The study of automatic representations of Cayley graphs was initiated by Kharlampovich, Khoussainov and Miasnikov by introducing the notion of Cayley automatic groups [1]. Cayley automatic groups are also referred to as Cayley graph automatic or graph automatic groups in the literature.

The set of Cayley automatic groups contains all automatic groups in the sense of Thurston [2]. However, the set of Cayley automatic groups is considerably wider than the set of automatic groups. For example, the set of Cayley automatic groups contains all finitely generated nilpotent groups of nilpotency class at most two. Cayley automatic and Cayley biautomatic groups retain key algorithmic properties which hold for automatic and biautomatic groups, respectively: the word problem for Cayley automatic groups is decidable in quadratic time and the conjugacy problem for Cayley biautomatic groups is decidable. The first order theory for Cayley graphs of Cayley automatic groups is decidable.

There have been three longstanding open problems for automatic groups asking if they are biautomatic, if they have decidable conjugacy problem and whether or not the isomorphism problem is decidable within the set of automatic groups. Miasnikov and Šunić showed that the answers for such problems for the set of Cayley automatic groups are negative [7]. Based on the results from [8, 9, 10], they showed examples of Cayley automatic groups which are not Cayley biautomatic and have undecidable conjugacy problem, and they proved that the isomorphism problem is undecidable within the set of Cayley automatic groups. These results confirm that Cayley automatic groups sound as a suitable generalization of automatic groups.

The set of automatic groups has a geometric characterization called the fellow traveller property [2, Theorem 2.3.5]. In particular, the fellow traveller property implies that all automatic groups are finitely presented and their isoperimetric functions are at most quadratic [2, Theorem 2.3.12]. These two properties do not hold, in general, for Cayley automatic groups. For example, the Heisenberg group H_3 is Cayley automatic [1, Example 6.6], but it is not automatic since its isoperimetric function is at least cubic [2, § 8.1]. The lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ is Cayley automatic [1, Theorem 10.6], but it is not automatic since it is not finitely presented [11]. There are many other examples of Cayley automatic groups which are not automatic. See [1] and Chapters 3–4 of the present work for more examples.

We emphasize that study of automatic representations of Cayley graphs is not a trivial issue. Furthermore, there exist automatic representations of Cayley graphs which have unexpected and counterintuitive properties. For example, there exists an automatic representation of the Cayley graph $\Gamma(\mathbb{Z}, S)$, where $S = \{1\}$, for which the set of words representing the set $\{z \in \mathbb{Z} | z > n\}$ is not regular for every $n \in \mathbb{Z}$ [12, Corollary 4.4]. There exists a FA–representation of the group $\mathbb{Z}^2$ for which every nontrivial cyclic subgroup is not FA–recognizable [13]. Therefore, we obtain an automatic representation of the Cayley graph $\Gamma(\mathbb{Z}^2, S)$, where $S = \{(1,0),(0,1)\}$, for which the set of words representing a nontrivial cyclic subgroup of $\mathbb{Z}^2$ is not regular for every such a subgroup. So, in Chapter 4 we pay particular attention to some properties of the obtained representations of Cayley graphs.

Given a Cayley graph of a Cayley automatic group with respect to some finite set of generators, by removing labels and edges orientation we obtain an automatic transitive graph. However, not every automatic transitive graph of finite degree can be obtained from a Cayley graph by removing labels and edges orientation. Chapter 5 of the present work shows examples of such automatic transitive graphs. The reason why we focus only on transitive graphs of finite degree is partly explained by the following theorem proved by Peter Cameron.

Theorem. *[14, Theorem 4.1] Let G be a countable group which cannot be expressed as the union of finite number of translates of non–principal square root sets and a finite set, where a non principal square root is a set $\sqrt{a} = \{g \in G | g^2 = a\}$ for $a \neq 1$ and a translate is a set $\sqrt{a}h = \{gh | g \in \sqrt{a}\}$ for $h \in G$. Then the set of Cayley graphs for G which are isomorphic to the Rado graph is residual.*

The hypothesis of this theorem is satisfied for many groups. In particular, the hypothesis holds for every group G for which there exists a homomorphism from G onto the infinite cyclic group. Informally speaking, this theorem means that "almost all" Cayley graphs $\Gamma(G, S)$ for infinite sets $S \subset G$ with labels and edges orientation removed are isomorphic to the Rado graph. But, the Rado graph does not admit an automatic representation [15, 16].

Oliver and Thomas proved that a finitely generated group has an automatic representation iff it is virtually abelian [17]. This result is based partly on the celebrated Gromov's theorem for groups of polynomial growth. But, the problem of finding characterizations for Cayley automatic groups is more complicated, and it seems to require new approaches. In Chapter 6 we approach this problem by studying asymptotic behavior of the numerical characteristics of Turing transducers which are associated to automatic Cayley graphs.

The outline of the thesis is as follows. In Chapter 2 we give an introduction to automatic structures and Cayley automatic groups. In Chapter 3 we study automatic representations of Cayley graphs of the Baumslag–Solitar groups. In Chapter 4 we study representations of Cayley graphs of wreath products of groups by finite automata, pushdown automata and nested stack automata. In Chapter 5 we study automatic representations of non–Cayley transitive graphs. In Chapter 6 we introduce and then study the numerical characteristics of Turing transducers which are associated to automatic Cayley graphs. Chapter 7 concludes the thesis and contains some open questions.

We provide more details for Chapters 2–6 below.

Chapter 2 – Cayley graphs as automatic structures

This chapter contains an introduction to automatic structures and Cayley automatic groups. Section 2.1 starts by introducing synchronous finite automata and automatic structures, see Definitions 2.1.1, 2.1.2, 2.1.3, 2.1.4 and 2.1.5. Then Section 2.1 continues by recalling some fundamental properties of automatic structures. Theorems 2.1.1 and 2.1.2 show decidability of the first order theory of an automatic structure. Theorem 2.1.4, generalizing Theorem 2.1.1, shows decidability of the first order theory of an automatic structures for sentences containing the quantifiers $\exists^{\infty}$ *there exists infinitely many* and $\exists^{(k,m)}$ *there exists k modulo m*. Theorem 2.1.3 shows that if a structure is first order interpretable in an automatic structure, then it is automatic. Theorem 2.1.5 shows that any FA–recognizable relation can be obtained as a finite union of concatenations of FA–recognizable relations of the special form. Theorem 2.1.6 gives the characterization of FA–recognizable relations $R \subset \Sigma^{*n}$, $|\Sigma| \geqslant 2$ in terms of their definability in the structure $\mathcal{W}_k = (\Sigma^*, (\sigma_a)_{a \in \Sigma}, \preceq_p, e\ell)$ for any, equivalently all, $k \geqslant 2$. Theorem 2.1.7 gives the characterization of FA–recognizable relations over the unary alphabet in terms of their definability in the structure $(\mathbb{N}, \leqslant, (\equiv_p)_{p \in \mathbb{N}})$. Theorem 2.1.8 gives the characterization of automatic structures in terms of their interpretability in the structure $\mathcal{W}_k$ for any, equivalently all, $k \geqslant 2$. The Myhill–Nerode type theorem for FA–recognizable relations is stated in Theorem 2.1.9. Theorem 2.1.10, informally speaking, shows that the study of automatic structures of finite signature boils down to the study of automatic graphs.

Section 2.2 starts by introducing Cayley automatic groups and Cayley biautomatic groups, see Definitions 2.2.1 and 2.2.2. Then Section 2.2 continues by recalling some algorithmic properties for Cayley automatic groups and Cayley biautomatic groups. Theorem 2.2.1 shows that the word problem in a Cayley automatic group is decidable in quadratic time. Theorem 2.2.2 shows that the conjugacy problem in a Cayley biautomatic group is decidable. After that Section 2.2 continues by showing that Cayley automatic groups and Cayley biautomatic groups are closed with respect to some algebraic operations. Theorem 2.2.3 shows that Cayley automatic groups and Cayley biautomatic groups are closed under direct product. Theorem 2.2.4 shows that if a group have a subgroup of finite index which is Cayley automatic, then the group is Cayley automatic. Theorem 2.2.5 shows that under certain conditions a semidirect product of Cayley automatic groups is a Cayley automatic group. In Theorem 2.2.6 we recall the auxiliary fact – the normal form theorem for amalgamated free products. Theorem 2.2.7 shows that the free product of two Cayley automatic groups is a Cayley automatic group. Theorems 2.2.8 and 2.2.9 show that under certain conditions an amalgamated free product of two Cayley automatic groups is a Cayley automatic group.

Chapter 3 – The Baumslag–Solitar groups

In this chapter we study Cayley automatic representations for the Baumslag–Solitar groups $BS(m,n) = \langle a, t \,|\, t^{-1}a^m t = a^n \rangle$. The Baumslag–Solitar groups play an essential role in group theory as examples and test–cases for theories and techniques. Representing the group $BS(1,n), n \in \mathbb{N}$ as the certain set of linear function acting on the real line, it was shown that $BS(1,n), n \in \mathbb{N}$ is Cayley automatic [1, Theorem 13.1]. It had been an open question whether or not all Baumslag–Solitar groups are Cayley automatic. In this chapter we show that all Baumslag–Solitar groups are Cayley automatic.

Section 3.1 starts by recalling the definitions of Baumslag–Solitar groups and HNN extensions, see Definitions 3.1.1 and 3.1.2. Then Section 3.1 continues by recalling the normal form theorems for HNN extensions, see Theorems 3.1.1 and 3.1.2. In the end of Section 3.1 we recall that all Baumslag–Solitar groups can be obtained as HNN extensions.

In Section 3.2 we show that all Baumslag–Solitar groups are Cayley automatic and discuss some properties of the constructed Cayley automatic representations. For given positive integers m and n, in Theorem 3.2.1 we construct a Cayley automatic representation of the Baumslag–Solitar group $BS(m,n)$. The construction uses the normal form theorem for HNN extensions (see Theorem 3.1.2).

Let m and n be positive integers. In Proposition 3.2.1 we show that for the Cayley automatic representation $\psi : L \to BS(m,n)$ constructed in Theorem 3.2.1 the inequalities $\lambda|g|+\xi \leqslant |w| \leqslant \mu|g|+\delta$ hold for all $g \in BS(m,n)$, where $|g|$ is the length of g in the group $BS(m,n)$ with respect to the generators a and t and $|w|$ is the length of the word $w = \psi^{-1}(g)$, and $\lambda > 0$, $\mu > 0$, ξ and δ are some constants. Proposition 3.2.1 follows from the metric estimates for the Baumslag–Solitar groups obtained by Elder and Burillo [18].

The main results of Chapter 3 are published in [19].

Chapter 4 – Wreath products of groups

In this chapter we study representations of Cayley graphs of wreath products of groups with respect to finite automata, pushdown automata and nested stack automata. Wreath products of groups appear in many situations in group theory and provide a way of constructing many interesting examples. For wreath products of groups there is an abundance of results on quantitative characteristics such as growth rate, isoperimetric profiles and drift of simple random walks. This makes studying wreath products of groups relevant to seeking connections between characteristics of groups and the computational power of automata which is sufficient for representing their Cayley graphs.

In Section 4.1 we give a short introduction to the notion of wreath products of groups. In Section 4.2 we study Cayley automatic representations of the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ and their properties. In Theorem 4.2.1 we show that the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ is a Cayley biautomatic group by constructing a certain Cayley biautomatic representation for $\mathbb{Z}_2 \wr \mathbb{Z}$. In Lemma 4.2.1 we show the formula for the length of the word w representing an element $g \in \mathbb{Z}_2 \wr \mathbb{Z}$ with respect to the Cayley biautomatic representation of $\mathbb{Z}_2 \wr \mathbb{Z}$ constructed in Theorem 4.2.1. In Lemma 4.2.2 we show the formula for the length of an element $g \in \mathbb{Z}_2 \wr \mathbb{Z}$ in the Cayley graph of $\mathbb{Z}_2 \wr \mathbb{Z}$. Using Lemmas 4.2.1 and 4.2.2, in Proposition 4.2.1 we show that the inequalities $\frac{1}{3}|g| + \frac{2}{3} \leqslant |w| \leqslant |g| + 1$ are satisfied for all $g \in \mathbb{Z}_2 \wr \mathbb{Z}$.

In Proposition 4.2.2 we show that the sets of representatives of the elements of the normal subgroup $\mathbb{Z}_2^{(\mathbb{Z})} \trianglelefteq \mathbb{Z}_2 \wr \mathbb{Z}$ and the subgroup $\mathbb{Z} \leqslant \mathbb{Z}_2 \wr \mathbb{Z}$ with respect to the Cayley biautomatic representation constructed in Theorem 4.2.1 are recognized by finite automata. Proposition 4.2.3 shows that there are Cayley automatic representations of $\mathbb{Z}_2 \wr \mathbb{Z}$ for which the set of representatives of the elements of the normal subgroup $\mathbb{Z}_2^{(\mathbb{Z})} \trianglelefteq \mathbb{Z}_2 \wr \mathbb{Z}$ is not recognizable by a finite automaton. Similarly, Proposition 4.2.4 shows that there are Cayley automatic representations of $\mathbb{Z}_2 \wr \mathbb{Z}$ for which it is not recognizable by a finite automaton whether the lamp at a certain position is lit.

In Section 4.3 we study Cayley automatic representations of the wreath products of groups $G \wr \mathbb{Z}$ and their properties. In Theorem 4.3.1 we show that $G \wr \mathbb{Z}$ is Cayley automatic for a Cayley automatic group G by constructing a certain Cayley automatic representation for $G \wr \mathbb{Z}$. In Theorem 4.3.2 we show that $G \wr \mathbb{Z}$ is Cayley biautomatic for a Cayley biautomatic group G, using the Cayley automatic representation constructed in Theorem 4.3.1. In Lemma 4.3.1 we show a lower bound for the length of the word w representing an element $g \in G \wr \mathbb{Z}$ with respect to the Cayley automatic representation constructed in Theorem 4.3.1. In Lemma 4.3.2 we show the formula for the length of an element $g \in G \wr \mathbb{Z}$ in the Cayley graph of $G \wr \mathbb{Z}$. Using Lemmas 4.3.1 and 4.3.2, in Proposition 4.3.1 we show that under certain conditions the inequalities of the form $\lambda|g| + \xi \leqslant |w| \leqslant \mu|g| + \delta$ are satisfied for all $g \in G \wr \mathbb{Z}$, where $\lambda > 0$, $\mu > 0$, ξ and δ are some constants.

In Section 4.4 we study representations of some Cayley graphs of the groups $\mathbb{Z}_2 \wr F_n$ with deterministic pushdown automata and their properties. Section 4.4 starts by recalling some necessary definitions. In Definition 4.4.1 we recall the notion of pushdown automata. In Proposition 4.4.1 we recall briefly some basic properties of context–free languages. In Definition 4.4.2 we recall the notion of deterministic pushdown automata, and then in Definition 4.4.3 we define the class $\mathcal{P}_1$ as the class of all languages recognized by deterministic pushdown automata. In Definitions 4.4.4 and 4.4.5 we recall the notion of parallel poly–pushdown and poly–context–free languages, and then define the classes $\mathcal{P}$ and $\mathcal{P}^c$ as the classes of all parallel poly–pushdown and poly–contex–free languages, respectively. Thus, we obtain a hierarchy of classes of languages: $\mathcal{P}_1 \subset \mathcal{P} \subset \mathcal{P}^c$. In Proposition 4.4.2 we recall some basic properties which hold for parallel poly–pushdown and poly–context–free languages. In Definition 4.4.6 we define C–Cayley automatic groups, where C is some class of languages. The notion of C–Cayley automatic groups was introduced by Elder and Taback as a generalization of the notion of Cayley automatic groups. In Proposition 4.4.3 we show that the definition of $\mathcal{P}^c$–Cayley automatic groups does not depend on the choice of generators.

Section 4.4 continues by showing that the groups $\mathbb{Z}_2 \wr F_n$ are $\mathcal{P}_1$–Cayley automatic and studying some properties of the constructed $\mathcal{P}_1$–Cayley automatic representations of these groups. In Theorem 4.4.1 we construct a $\mathcal{P}_1$–Cayley automatic representation of the group $\mathbb{Z}_2 \wr F_2$. In Proposition 4.4.4 we show that for the $\mathcal{P}_1$–Cayley automatic representation constructed in Theorem 4.4.1 the inequalities $\frac{1}{3}|g| + \frac{2}{3} \leqslant |w| \leqslant 3|g| + 1$ are satisfied for all $g \in \mathbb{Z}_2 \wr F_2$, where $|g|$ is the length of g in the Cayley graph of $\mathbb{Z}_2 \wr F_2$ and $|w|$ is the length of the word w representing g. In Propositions 4.4.5 we show that for the $\mathcal{P}_1$–Cayley automatic representation constructed in Theorem 4.4.1 the sets of representatives of the subgroups $\mathbb{Z}_2^{(F_2)} \trianglelefteq \mathbb{Z}_2 \wr F_2$ and $F_2 \leqslant \mathbb{Z}_2 \wr F_2$ are recognized by deterministic pushdown automata. In Proposition 4.4.6 we show that the sets of representatives of the two–generated subgroups $H_1 \leqslant \mathbb{Z}_2 \wr F_2$ and $H_2 \leqslant \mathbb{Z}_2 \wr F_2$ are regular languages, but the set of representatives of the two–generated subgroup H_3 is recognized by a deterministic pushdown automaton. In Theorem 4.4.2 we show how to construct $\mathcal{P}_1$–Cayley automatic representations for the groups $\mathbb{Z}_2 \wr F_n$, $n \geqslant 3$. In Proposition 4.4.7 we show that for the $\mathcal{P}_1$–Cayley automatic representation constructed in Theorem 4.4.2 the inequalities $\frac{1}{3}|g| + \frac{2}{3} \leqslant |w| \leqslant (2n-1)|g| + 1$ are satisfied for all $g \in \mathbb{Z}_2 \wr F_n$, where $|g|$ is the length of g in the Cayley graph of $\mathbb{Z}_2 \wr F_n$ and $|w|$ is the length of the word w representing g.

In Section 4.5 we study $\mathcal{P}_1$– and context–free–Cayley automatic representations of groups $G \wr F_n$. In Theorem 4.5.1 we show that a group $G \wr F_n$ is $\mathcal{P}_1$–Cayley automatic if G is Cayley automatic. In Theorem 4.5.2 we show that, under certain conditions, a group $G \wr F_n$ is context–free–Cayley automatic if G is context–free–Cayley automatic. In Proposition 4.5.1 we show that, under certain conditions, for the context–free–Cayley automatic representation of $G \wr F_n$ constructed in Theorem 4.5.2 the inequalities of the form $\lambda|g| + \xi \leqslant |w| \leqslant \mu|g| + \delta$ are satisfied for all $g \in G \wr F_n$, where $|g|$ is the length of g with respect certain generators of $G \wr F_n$, $|w|$ is the length of the word representing g, $\lambda > 0$, $\mu > 0$, ξ and δ are some constants.

We conclude Section 4.5 by giving two definitions. In Definition 4.5.1 we introduce the notion of geodesic representations of groups. In particular, the $\mathcal{P}_1$–Cayley automatic representations of the groups $\mathbb{Z}_2 \wr F_n$ constructed in Theorems 4.4.1 and 4.4.2 are geodesic. In Definition 4.5.2 we recall the notion of quasi–geodesic representations of groups introduced by Elder and Taback. In particular, all Cayley automatic representations are quasi–geodesic. If a representation $\psi : L \to G$ is geodesic, then it is quasi–geodesic.

In Section 4.6 we study representations of a Cayley graph of the group $\mathbb{Z}_2 \wr \mathbb{Z}^2$ with nested stack automata. Section 4.6 starts by recalling the definitions related to the notion of nested stack automata and indexed languages, see Definitions 4.6.1–4.6.5. In Theorem 4.6.1 we construct an indexed–Cayley automatic representation $\psi : L \to \mathbb{Z}_2 \wr \mathbb{Z}^2$ of the group $\mathbb{Z}_2 \wr \mathbb{Z}^2$ for which L is a regular language. The indexed–Cayley automatic representation of the group $\mathbb{Z}_2 \wr \mathbb{Z}^2$ constructed in Theorem 4.6.1 is not quasi–geodesic, and therefore, it is not geodesic.

The main results of Chapter 4 are published in [20].

Chapter 5 – Non–Cayley automatic transitive graphs

In this chapter we obtain examples of automatic non–Cayley transitive graphs. Furthermore, we construct an automatic representation of the Diestel–Leader graph which is known not to be quasi–isometric to any Cayley graph. Two graphs are quasi–isometric as metric spaces if there is a mapping between them which is coarsely Lipschitz and coarsely surjective. Informally speaking, quasi–isometry provides an equivalence relation on graphs that ignores their local details. The examples obtained in this chapter show that the class of automatic transitive graphs is essentially wider than the class of all automatic undirected Cayley graphs.

In Section 5.1 we show that the non–Cayley transitive graphs $H_{n,m}$ constructed by Thomassen and Watkins are automatic. We start Section 5.1 by constructing the graph $H_{2,3}$. In Proposition 5.1.1 we show that $H_{2,3}$ is a non–Cayley transitive graph. In Definition 5.1.1 we define the notion of line graph, and then in Lemma 5.1.1 we show that for an automatic digraph its line graph is automatic. Using Lemma 5.1.1, in Proposition 5.1.2 we show that the graph $H_{2,3}$ is automatic. In Proposition 5.1.3 we show that the graph $H_{n,m}$ is an automatic non–Cayley transitive graph for every pair of integers n and m such that $n \geqslant 2$, $m \geqslant 3$ and $n \neq m$.

In Section 5.2 we show that the Diestel–Leader graph is automatic. The Diestel–Leader graph is obtained as the limit of a sequence of transitive graphs each of which is quasi–isometric to the 5–regular tree T_5. In Proposition 5.2.1 we show an auxiliary fact which implies that every transitive graph in this sequence is automatic. Then we give a description of the Diestel–Leader graph which is used then in Theorem 5.2.1 to show that it is automatic.

The main results of Chapter 5 are published in [19].

Chapter 6 – On characterizations of Cayley automatic groups

In this chapter we address the problem of finding characterizations of Cayley automatic groups by studying asymptotic behavior of the numerical characteristics of Turing transducers associated to automatic Cayley graphs.

In Section 6.1 we define the special class of multi–tape Turing transducers $\mathcal{T}$ for which the heads move synchronously, first forth and then back. In Lemma 6.1.1 we show that Turing transducers of the class $\mathcal{T}$ can be presented in terms of multi–tape synchronous finite automata. Then we explain how to construct the labeled directed graph Γ_T for a given Turing transducer $T \in \mathcal{T}$. Let Γ be a labeled directed graph which has a constant number of outgoing edges labeled by different labels in every vertex; for example, it can be a Cayley graph of a finitely generated group. Lemma 6.1.2 shows that if Γ is automatic there exists a Turing transducer $T \in \mathcal{T}$ for which $\Gamma_T \cong \Gamma$. Lemmas 6.1.1 and 6.1.2 imply Theorem 6.1.1 showing that Γ is automatic iff there exists $T \in \mathcal{T}$ for which $\Gamma_T \cong \Gamma$. We say that Γ is presented by T if $\Gamma_T \cong \Gamma$.

In Section 6.2 we introduce the numerical characteristics for Turing transducers of the class $\mathcal{T}$ – growth functions, Følner functions and average length growth functions. These three numerical characteristics are the analogs of growth functions, Følner functions and drifts of simple random walks for Cayley graphs of groups.

In Section 6.3 we consider asymptotic behavior of the growth function $b_n, n = 0, \ldots, \infty$ and the Følner function $f_n, n = 1, \ldots, \infty$ for Turing transducers of the class $\mathcal{T}$. In Claim 6.3.1 we show that if a Cayley graph is presented by $T \in \mathcal{T}$, then the growth function of T coincides with that of the Cayley graph. Using Claim 6.3.1, in Examples 6.3.1 we show that the growth series $\sum b_n z^n$ for Turing transducers of the class $\mathcal{T}$ may not be rational. Then in Example 6.3.2 we show that the growth function for Turing transducers of the class $\mathcal{T}$ may have intermediate growth. Then we discuss asymptotic behavior of Følner functions for Turing transducers of the class $\mathcal{T}$. In Claim 6.3.2 we show that if a Cayley graph is presented by $T \in \mathcal{T}$, then the Følner function of T coincides with that of the Cayley graph. Using Claim 6.3.2, in Theorem 6.3.1 we show that for every integer $i \geqslant 1$ there exists a Turing transducer of the class $\mathcal{T}$ for which $f_n \sim n^{(n^i)}$.

In Section 6.4 we study asymptotic behavior of the average length growth function $\ell_n, n = 1, \ldots, \infty$ for Turing transducers of the class $\mathcal{T}$. We first recall some necessary definitions related to the notion of random walks on graphs. In Claim 6.4.1 we show that if a Cayley graph $\Gamma(G, S)$ is presented by $T \in \mathcal{T}$, then $\ell_n = E_{\mu^{*n}}[|w|]$ for every $n = 1, \ldots, \infty$, where, for any $g \in G$, $|w|$ is the length of the word w representing g with respect to T and $\mu^{*n}(g)$ is the probability, defined on the ball of radius n of the graph $\Gamma(G, S)$, that a n–step simple symmetric random walk on $\Gamma(G, S)$, which starts at the identity $e \in G$, ends up at g.

It is easy to construct Turing transducers of the class $\mathcal{T}$ for which $\ell_n \asymp$

$\sqrt{n}$ and the growth function b_n is polynomial. We describe such Turing transducers in Example 6.4.1. A more complicated technique is required to construct a Turing transducer of the class $\mathcal{T}$ for which $\ell_n \asymp \sqrt{n}$ and the growth function b_n is exponential. We construct such a Turing transducer in Lemma 6.4.1. It is easy to construct Turing transducers of the class $\mathcal{T}$ for which $\ell_n \asymp n$ and the growth function b_n is exponential. We describe such Turing transducers in Example 6.4.2. Is there a Turing transducer of the class $\mathcal{T}$ for which ℓ_n grows between $\sqrt{n}$ and n? We answer this question positively in Theorem 6.4.1 by showing that for every $\alpha < 1$ there exists a Turing transducer $T \in \mathcal{T}$ for which $\ell_n \asymp n^\beta$ for some β such that $\alpha < \beta < 1$ and the growth function b_n is exponential.

The results of Chapter 6 are published in [21].

Chapter 2

Cayley graphs as automatic structures

In this chapter we give an introduction to automatic structures and Cayley automatic groups.

2.1 Finite automata and automatic structures

Recall briefly the definitions of finite automata and regular languages. For an introduction to finite automata and regular languages see, e.g., [22].

A nondeterministic finite automaton $\mathcal{M}$ over an alphabet Σ is a tuple (Q, δ, q_0, F), where Q is a finite set of states, $q_0 \in Q$ is an initial state, $\delta \subset Q \times \Sigma \times Q$ is a transition table, $F \subset Q$ is a subset of final states. Let $w \in \Sigma^*$ be a word $w = \sigma_1 \sigma_2 \dots \sigma_n$, $\sigma_i \in \Sigma$ for $i = 1, \dots, n$. We say that the automaton $\mathcal{M}$ accepts the word w if there is a sequence of states $q_0, q_1, \dots, q_n$, $q_i \in Q$ for $i = 1, \dots, n$ such that $(q_{i-1}, \sigma_i, q_i) \in \delta$ for $i = 1, \dots, n$, and $q_n \in F$.

The language recognized by $\mathcal{M}$ is the set of all words accepted by $\mathcal{M}$. A language $L \subset \Sigma^*$ is called regular if it is recognized by a finite automaton over Σ.

An automaton $\mathcal{M} = (Q, \delta, q_0, F)$ over an alphabet Σ is called deterministic if for every $q \in Q$ and $\sigma \in \Sigma$ there exists at most one q' such that $(q, \sigma, q') \in \delta$. Deterministic finite automata are equivalent to nondeterministic ones in the sense of computing power, i.e., deterministic and nondeterministic finite automata recognize the same class of regular languages.

A finite automaton can be thought as a one–way read–only Turing machine. Recall that a two–way finite automaton is a read–only Turing machine which uses a constant amount of space on their work tape. Since a constant amount of space can be converted into a finite number of states of a Turing machine, then a two–way finite automaton can be equivalently defined as a read–only one–tape Turing machine.

A two–way finite automaton is called deterministic if no more than one instruction is allowed for every configuration of the automaton. It appears that the classes of two–way finite automata, two–way deterministic finite automata and finite automata are equivalent in the sense of computing power [23, Theorem VIII.1.5], i.e., all these classes recognize the same class of regular languages.

A read–only synchronous n–tape Turing machine is a read–only n–tape Turing machine for which all n heads move synchronously either to the left or to the right. We suppose that an input for a n–tape Turing machine is given as a n–tuple of strings written on n tapes. If $n > 1$, then the class of read–only synchronous n–tape Turing machines is clearly weaker than the class of read–only n–tape Turing machines in the sense of computing power.

A read–only one–way synchronous n–tape Turing machine is a read–only synchronous n–tape Turing machine for which the heads are allowed to move only to the right. By [23, Theorem VIII.1.5] we have that the class of read–only synchronous n–tape Turing machines is equivalent to the class of read–only one–way synchronous n–tape Turing machines in the sense of computing power.

For a given finite alphabet Σ, we denote by $\Sigma_\diamond$ the alphabet $\Sigma_\diamond = \Sigma \cup \{\diamond\}$, where $\diamond \notin \Sigma$. A finite automaton over the alphabet $\Sigma_\diamond^n \setminus \{(\diamond, \ldots, \diamond)\}$ is called a synchronous n–tape automaton. Due to [24] a synchronous n–tape automaton is interpreted as a read–only one–way synchronous n–tape Turing machine. The following definition is originated from [25, 26].

Definition 2.1.1. *Let $(w_1, \ldots, w_n) \in \Sigma^{*n}$ be a n–tuple of strings. The convolution of this tuple $\otimes(w_1, \ldots, w_n)$ is the string of length $\max\{|w_i|, i = 1, \ldots, n\}$ over the alphabet $\Sigma_\diamond^n \setminus \{(\diamond, \ldots, \diamond)\}$ such that the kth symbol is $(\sigma_1, \ldots, \sigma_n)$, where σ_i is the kth symbol of w_i if $k \leqslant |w_i|$ and $\diamond$ if $k > |w_i|$. The convolution $\otimes R$ of a n–ary relation $R \subset \Sigma^{*n}$ is defined as the set of convolutions of all n–tuples from R.*

We also denote a convolution $\otimes(w_1, \ldots, w_n)$ as $w_1 \otimes \cdots \otimes w_n$. The set of finite automata recognizable (FA–recognizable) relations is defined as follows.

Definition 2.1.2. *We say that a n–ary relation $R \subset \Sigma^{*n}$ is FA–recognizable if the convolution $\otimes R$ is recognizable by a synchronous n–tape automaton.*

Recall some basic definitions related to automatic structures. We refer the reader to [27, Chapters B and C] for more details.

A structure $\mathcal{A} = (A; R_1^{n_1}, \ldots, R_k^{n_k}, f_1^{\ell_1}, \ldots, f_m^{\ell_m})$ is defined by a domain A, atomic relations $R_1^{n_1}, \ldots, R_k^{n_k}$ and functions $f_1^{\ell_1}, \ldots, f_m^{\ell_m}$ on A, where each $R_i^{n_i}, i = 1, \ldots, k$ is a n_i–ary relation and $f_j^{\ell_j}, j = 1, \ldots, m$ is a ℓ_j–ary function. We write an upper index in R_i^A to emphasize that it is a relation on A. A structure $\mathcal{A}$ can be uniquely associated with the relational structure $\mathcal{A}_R$ by replacing each function $f_j^{\ell_j}, j = 1, \ldots, m$ with its graph $Graph(f_j^{\ell_j}) := \{(x_1, \ldots, x_{\ell_j}, y) \,|\, f_j^{\ell_j}(x_1, \ldots, x_{\ell_j}) = y\}$.

Definition 2.1.3. *We say that a relational structure $\mathcal{A} = (A; R_1^{n_1}, \ldots, R_k^{n_k})$ is automatic over a finite alphabet Σ if the domain $A \subset \Sigma^*$ and the atomic relations $R_i^{n_i} \subset \Sigma^{*n_i}, i = 1, \ldots, k$ are FA–recognizable.*

An $\mathcal{A}$–formula $\Phi(x_1, \ldots, x_k)$ is a formula for which all non–logical symbols belong to the signature $\mathcal{A}$. A relation R of arity n is first order definable in a structure $\mathcal{A}$ if there exists a $\mathcal{A}$–formula $\Phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and m elements $c_1, \ldots, c_m \in A$ such that $(x_1, \ldots, x_n) \in R$ iff $\mathcal{A} \models \Phi(x_1, \ldots, x_n, c_1, \ldots, c_m)$. The formula $\Phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is called a first order definition in $\mathcal{A}$ of the relation R. The following theorem is due to [4, 28].

Theorem 2.1.1. *If $\mathcal{A}$ is an automatic structure over Σ then there is the algorithm that for a relation R defined by a first order formula $\Phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ in $\mathcal{A}$ with parameters $c_1, \ldots, c_m$ constructs a n–tape synchronous automaton recognizing $\otimes R$.*

Definition 2.1.4. *We say that two relational structures $\mathcal{A} = (A; R_1^A, \ldots, R_k^A)$ and $\mathcal{B} = (B; R_1^B, \ldots, R_k^B)$ of the same signature are isomorphic if there exists* *a bijection* $\varphi : A \to B$ *such that for every* $i = 1, \dots, k$, $(x_1, \dots, x_n) \in R_i^A$ *iff* $(\varphi(x_1), \dots, \varphi(x_n)) \in R_i^B$, *where* n *is the arity of* R_i.

The isomorphism type of a structure $\mathcal{A}$ is the equivalence class of all structures that are isomorphic to $\mathcal{A}$. If one admits that $\varphi : A \to B$ is an embedding such that for every atomic relation R of some arity n: $(x_1, \dots, x_n) \in R^A$ iff $(\varphi(x_1), \dots, \varphi(x_n)) \in R^B$ then $\mathcal{A}$ is called a substructure of $\mathcal{B}$.

Definition 2.1.5. *We say that a structure* $\mathcal{A} = (A; R_1^{n_1}, \dots, R_k^{n_k}, f_1^{\ell_1}, \dots, f_m^{\ell_m})$ *is automatic (or FA–presentable) if the relational structure* $\mathcal{A}_R = (A; R_1^{n_1}, \dots, R_k^{n_k}, Graph(f_1^{\ell_1}), \dots, Graph(f_m^{\ell_m}))$ *is isomorphic to a structure* $\mathcal{B}$ *which is automatic over some finite alphabet* Σ.

The first order theory of a structure $\mathcal{A}$ is the set of all first order sentences that are true in $\mathcal{A}$. A first order theory is called decidable if there is an algorithm that decides whether or not a given first order sentence belongs to the theory. As a corollary of Theorem 2.1.1 we obtain the following theorem.

Theorem 2.1.2. *The first order theory of an automatic structure* $\mathcal{A}$ *is decidable.*

We say that an equivalence relation $\xi \subset A \times A$ on a structure $\mathcal{A} = (A, R_1^A, \dots, R_n^A)$ is a congruence relation if for every atomic relation R^A of some arity m the following holds: if $(x_i, y_i) \in \xi$ for all $i = 1, \dots, m$, then $(x_1, \dots, x_m) \in R^A$ iff $(y_1, \dots, y_m) \in R^A$. A congruence relation ξ on a structure $\mathcal{A}$ defines the quotient structure $\mathcal{A}/\xi$ of the same signature.

Let $\Delta(x_1, \dots, x_k)$ and $\Phi_i(y_1^1, \dots, y_k^1, \dots, y_1^{r_i}, \dots, y_k^{r_i}), i = 1, \dots, n$ be $\mathcal{A}$–formulas that define the structure $(B, R_{r_1}^B, \dots, R_{r_n}^B)$, where $B =$

$\{(x_1, \ldots, x_k) | x_i \in A, \mathcal{A} \models \Delta(x_1, \ldots, x_k)\}$ and $R^B_{r_i}$ are defined by Φ_i for $i = 1, \ldots, n$, respectively. If then there is a congruence relation ξ on the structure $(B, R^B_{r_1}, \ldots, R^B_{r_n})$ defined by a $\mathcal{A}$–formula $\Psi(x_1, \ldots, x_k, y_1, \ldots, y_k)$, then we say that the structure $(B, R^B_{r_1}, \ldots, R^B_{r_n})/\xi$ is first order definable in $\mathcal{A}$. We say that a structure $\mathcal{C}$ is first order interpretable in $\mathcal{A}$ if it is isomorphic to a structure which is first order definable in $\mathcal{A}$. As a corollary of Theorem 2.1.1 we obtain the following theorem (cf. [4, 29]).

Theorem 2.1.3. *If a structure $\mathcal{A}$ is automatic and $\mathcal{B}$ is a first order interpretable structure in $\mathcal{A}$, then $\mathcal{B}$ is automatic.*

As a corollary of Theorem 2.1.3 we obtain the following proposition.

Proposition 2.1.1. *Let $\mathcal{A}$ be an automatic structure. Then a substructure of $\mathcal{A}$ with a first order definable domain is automatic. The structure $\mathcal{A}/\xi$ defined by a first order definable congruence ξ relation is automatic.*

By Theorem 2.1.3 we obtain the following proposition (cf. [4, 29]).

Proposition 2.1.2. *Let $\mathcal{A} = (A, R_1, \ldots, R_n)$ be a structure. Let r_j be the arity of R_j for $j = 1, \ldots, n$. Suppose that there exist a regular language $L \subset \Sigma^*$, a surjective map $\nu : L \to A$ and FA–recognizable relations $L_\xi \subset L^2$, $L_j \subset L^{r_j}$, $j = 1, \ldots, n$ such that the following properties hold:*

- $(x_1, x_2) \in L_\xi$ *iff* $\nu(x_1) = \nu(x_2)$;
- $(x_1, \ldots, x_{r_j}) \in L_j$ *iff* $(\nu(x_1), \ldots, \nu(x_{r_j})) \in R_j$ *for* $j = 1, \ldots, n$.

Then the structure $\mathcal{A}$ is automatic.

On the other hand, if a structure $\mathcal{A} = (A, R_1, \ldots, R_n)$ is automatic then the condition of Proposition 2.1.2 is true. Therefore, this condition gives an equivalent definition of automatic structures (cf. [4]). The map $\nu : L \to A$ and the regular languages $L, L_1, \ldots, L_n$ together with a collection of automata recognizing them constitute an automatic representation of the structure $\mathcal{A}$.

We use $\exists^\infty$ and $\exists^{(k,m)}$ to denote the existential quantifiers interpreted as *there exist infinitely many* and *there exist k modulo m many*, respectively. Let us give two simple examples of using the quantifiers $\exists^\infty$ and $\exists^{(k,m)}$. Consider the structure $(\mathbb{N}, <)$. The first order definition $\exists^{(0,2)} y(y < x)$ describes the unary relation $R(x) \subset \mathbb{N}$ which contains all even natural numbers. Consider a graph $\Gamma(V, E)$. The first order definition $\exists^\infty y(E(x, y))$ describes the unary relation $R(x) \subset V$ that contains all vertices of $\Gamma(V, E)$ for which the number of adjacent vertices is infinite.

Theorem 2.1.1 can be generalized as follows (cf. [12, 29]).

Theorem 2.1.4. *Let $\mathcal{A}$ be an automatic structure over Σ. For a relation R given by a first order definition $\Phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ in $\mathcal{A}$ with parameters $c_1, \ldots, c_m$ and, probably, containing the quantifiers $\exists^\infty$ and $\exists^{(k,m)}$ there exists the algorithm which constructs a n–tape synchronous automaton recognizing $\otimes R$.*

Recall some characterization theorems for FA–recognizable relations and automatic structures. For a given non–empty set $A \subset \{1, \ldots, n\}$ we denote by Σ_A the subset $\Sigma_A = \{(\tau_1, \ldots, \tau_n) | \tau_i \neq \diamond \,\text{iff}\, i \in A\} \subset \Sigma_\diamond^n$. There is the following decomposition theorem [24].

Theorem 2.1.5. *A relation $R \subset \Sigma^{*n}$ is FA–recognizable iff R is a finite union of products of the form: $R_0 \dots R_k$, where each $R_i \subset \Sigma^{*n}_{A_i}$ for $i = 0, \dots, k$ is FA–recognizable and $A_k \subset \dots \subset A_0 \subset \{1, \dots, n\}$.*

For $\Sigma = \{0, \dots, k-1\}$ let $\mathcal{W}_k$ be the structure $\mathcal{W}_k = (\Sigma^*, (\sigma_a)_{a \in \Sigma}, \preceq_p, e\ell)$, where $\sigma_a(u) = ua$, $u \preceq_p v$ if u is a prefix of v, $e\ell(u, v)$ if u and v have the same length. The structure $\mathcal{W}_k$ is automatic. The theorem below provides the characterization of FA–recognizable n–ary relations $R \subset \Sigma^{*n}$, $|\Sigma| \geqslant 2$ in terms of their definability in $\mathcal{W}_k$ (cf. [25, 26]).

Theorem 2.1.6. *For k–ary alphabet Σ with $k \geqslant 2$ a relation $R \subset \Sigma^{*n}$ is FA–recognizable iff it is first order definable in the structure $\mathcal{W}_k$.*

Let Σ be a unary alphabet, $|\Sigma| = 1$. Let us identify Σ^* and $\mathbb{N}$ in a natural way. We denote by $\equiv_p$ the congruence relation modulo p on $\mathbb{N}$. The theorem below provides the characterization of FA–recognizable n–ary relations $R \subset \Sigma^{*n}$ in terms of their definability in the structure $(\mathbb{N}, \leqslant, (\equiv_p)_{p \in \mathbb{N}})$ (cf. [29, 30]).

Theorem 2.1.7. *A relation $R \subset \Sigma^{*n}$, $|\Sigma| = 1$, is FA–recognizable iff it is first order definable in the structure $(\mathbb{N}, \leqslant, (\equiv_p)_{p \in \mathbb{N}})$.*

The theorem below provides the characterization for automatic structures in terms of their interpretability in $\mathcal{W}_k$ (cf. [31]).

Theorem 2.1.8. *A structure $\mathcal{A}$ is automatic iff it is first order interpretable in $\mathcal{W}_k$ for some, equivalently all, $k \geqslant 2$.*

Recall the Myhill–Nerode type theorem for the characterization of FA–recognizable n–ary relations in Σ^{*n} (cf. [4]). For a given element $w \in \Sigma^{*n}$, the convolution $\otimes w$ can be represented as the concatenation $r_0 \dots r_k$, where

$r_i \in \Sigma_{A_i}^{*n}$, $i = 0, \ldots, k$ such that $A_k \subset \cdots \subset A_0 \subset \{1, \ldots, n\}$. We define the sort of w as the set A_k. Given $L \subset \Sigma^{*n}$, define $\eta_L \subset \Sigma^{*n} \times \Sigma^{*n}$ as the equivalence relation on Σ^{*n} satisfying the following properties:

- if $(u, v) \in \eta_L$ then u and v are of the same sort, say B;
- for every $w \in \Sigma^{*n}$ corresponding to some sequence $A_k \subset \cdots \subset A_0$ such that $A_0 \subset B$, $u \cdot w \in L$ iff $v \cdot w \in L$.

The Myhill–Nerode type theorem for FA–recognizable relations is as follows.

Theorem 2.1.9. *A relation $R \subset \Sigma^{*n}$ is FA–recognizable iff the number of equivalence classes of η_R is finite.*

If $n = 1$, then Theorem 2.1.9 has the form of the classical Myhill–Nerode theorem for regular languages.

For a given structure $\mathcal{A}$ of finite signature there is a way to construct the graph $\mathcal{G}(\mathcal{A})$ such that $\mathcal{G}(\mathcal{A})$ is interpretable in $\mathcal{A}$ and $\mathcal{A}$ is interpretable in $\mathcal{G}(\mathcal{A})$ [27, 32]. This, informally speaking, shows that to study automatic structures of finite signature it is enough to study automatic graphs. We have the following theorem.

Theorem 2.1.10. *For every structure $\mathcal{A}$ of finite signature there is a graph $\mathcal{G}(\mathcal{A})$ such that $\mathcal{A}$ is automatic iff $\mathcal{G}(\mathcal{A})$ is automatic.*

2.2 Cayley automatic groups

Let G be a group generated by a finite set $S = \{s_1, \ldots, s_n\}$. Recall that the Cayley graph $\Gamma(G, S)$ is the labeled directed graph with the set of vertices identified with G and two vertices $u, v \in G$ are connected by the edge (u, v) labeled by $s \in S$ if $us = v$. The Cayley graph $\Gamma(G, S)$ can be viewed as the structure:

$$\mathcal{A}_{G,S} = (G; E_{s_1}, \ldots, E_{s_n}), \tag{2.1}$$

where $E_{s_i} = \{(u, v) | u, v \in G, us_i = v\}$, $i = 1, \ldots, n$. The following definition first appeared in [1].

Definition 2.2.1. *We say that G is a Cayley automatic group if the structure $\mathcal{A}_{G,S}$ given by* (2.1) *is automatic.*

It follows from Theorem 2.1.1 that if $\mathcal{A}_{G,S}$ is an automatic structure for some finite set of generators $S \subset G$, then $\mathcal{A}_{G,S'}$ is an automatic structure for every finite set of generators $S' \subset G$. Therefore, Definition 2.2.1 does not depend on the choice of generators.

Let $\mathcal{B}_{G,S}$ be the following structure:

$$\mathcal{B}_{G,S} = (G; E_{s_1}, \ldots, E_{s_n}, E'_{s_1}, \ldots, E'_{s_n}), \tag{2.2}$$

where $E'_{s_i} = \{(u, v) | u, v \in G, s_i u = v\}$, $i = 1, \ldots, n$. Cayley biautomatic groups were introduced in [1] as follows.

Definition 2.2.2. *We say that G is a Cayley biautomatic group if the structure $\mathcal{B}_{G,S}$ given by* (2.2) *is automatic.*

Similarly, Definition 2.2.2 does not depend on the choice of generators.

Let G be a Cayley automatic group and $\Gamma(G, S)$ be the Cayley graph with respect to some finite set of generators $S = \{s_1, \ldots, s_n\} \subset G$. By Definition 2.2.1, there exists an automatic representation for the Cayley graph $\Gamma(G, S)$:

a regular language $L \subset \Sigma^*$, binary relations $L_{s_1}, \dots, L_{s_n} \subset L^2$ recognizable by two–tape synchronous finite automata $\mathcal{M}_{s_1}, \dots, \mathcal{M}_{s_n}$, respectively, and a bijection $\psi : L \to G$ such that $(w_1, w_2) \in L_{s_i}$ iff $\psi(w_1)s_i = \psi(w_2)$. In the theorem below we address the word problem for Cayley automatic groups (see also [1, Theorem 8.1]).

Theorem 2.2.1. *The word problem in the group G is decidable in quadratic time.*

Proof: Let us be given $w_1 \in L$ and $s \in S$. We first describe the algorithm for constructing the word $w_2 \in L$ such that $\psi(w_1)s = \psi(w_2)$ (see also [2, Theorem 2.3.10]). Let Ω be a subset of states of the automaton $\mathcal{M}_s$ and $\sigma \in \Sigma$. Put $T_{\Omega,\sigma}$ to be the set of arrows in the diagram of $\mathcal{M}_s$ labeled by the symbols (σ, η), $\eta \in \Sigma_\diamond$ which start in the states of Ω. We denote by $Q_{\Omega,\sigma}$ the subset of states of $\mathcal{M}_s$ which are ends of the arrows from $T_{\Omega,\sigma}$. Let q_0 be the initial state of $\mathcal{M}_s$. Put $\Omega_0 = \{q_0\}$.

Let $w_1 = \sigma_1 \dots \sigma_k$, $\sigma_i \in \Sigma$ for $i = 1, \dots, k$. Let us describe the Turing machine T_s such that if w_1 is fed as an input then T_s returns w_2 as an output. The Turing machine T_s works as follows.

- T_s reads the first symbol σ_1 of w_1, writes a symbol encoding the set T_{Ω_0,σ_1} and moves the head right to the next cell. Put $\Omega_1 = Q_{\Omega_0,\sigma_1}$.
- T_s reads the second symbol σ_2 of w_1, writes a symbol encoding the set T_{Ω_1,σ_2} and moves the head right to the next cell. Put $\Omega_2 = Q_{\Omega_1,\sigma_2}$.
- T_s proceeds as above until it reads σ_k and writes a symbol encoding the set $T_{\Omega_{k-1},\sigma_k}$. Then there are two possibilities: either the set $\Omega_k =$

$Q_{\Omega_{k-1},\sigma_k}$ contains an accepting state of the automaton $\mathcal{M}_s$ or it does not contain one. If it does not contain an accepting state of $\mathcal{M}_s$, then there is exactly one path in $\mathcal{M}_s$ from one of the states of Ω_k to an accepting state labeled by the symbols from the set $\{(\diamond, \eta) | \eta \in \Sigma\}$. A length of such a path is less than or equal to the number of states in $\mathcal{M}_s$. Let this path be labeled by a sequence $(\diamond, \eta_1), \ldots, (\diamond, \eta_m)$. The machine T_s then writes the word $\eta_1 \ldots \eta_m$ and moves the head back to the kth cell. If Ω_k contains an accepting state of $\mathcal{M}_s$, then the machine T_s does nothing at this step.

- Working backwards, the machine T_s reconstructs the rest of the word w_2.

Summarizing the description above we see that the Turing machine T_s moves the head to the right until the end of an input, makes a finite number of moves bounded from above by the number of states of $\mathcal{M}_s$, and moves the head back to the first cell. This algorithm takes $O(|w_1|)$ time to get the output w_2, where $|w_1| = k$ is the length of the input w_1.

Let $g = s_{i_1}^{j_1} \ldots s_{i_m}^{j_m}$ be a representation of $g \in G$ in terms of the generators $s_1, \ldots, s_n$, where $i_\ell = 1, \ldots, n$ and $j_\ell \in \{+1, -1\}$ for $\ell = 1, \ldots, m$. By the pumping lemma we know that if $\psi(w_2) = \psi(w_1)s$ for $s \in S$ then $||w_2| - |w_1|| \leqslant C_s$, where C_s is the number of states in $\mathcal{M}_s$. We denote by e the identity of the group G. Put $C = \max\{C_s | s \in S\}$ and $D = |\psi^{-1}(e)|$, where $|\psi^{-1}(e)|$ is the length of the word $\psi^{-1}(e)$. For every $k = 1, \ldots, m$ we have that $|\psi^{-1}(s_{i_1}^{j_1} \ldots s_{i_k}^{j_k})| \leqslant Ck + D$.

For a given word $s_{i_1}^{j_1} \ldots s_{i_m}^{j_m}$ in the group G, the algorithm deciding whether or not $s_{i_1}^{j_1} \ldots s_{i_m}^{j_m} = e$ in the group G is as follows. We start with the word

$\psi^{-1}(e)$ as an input and get the output $\psi^{-1}(s_{i_1}^{j_1})$ by the algorithm above. Then we take $\psi^{-1}(s_{i_1}^{j_1})$ as an input and get the output $\psi^{-1}(s_{i_1}^{j_1}s_{i_2}^{j_2})$ by the algorithm above. Repeating this procedure we get the word $\psi^{-1}(g) = \psi^{-1}(s_{i_1}^{j_1} \dots s_{i_m}^{j_m})$. Thus, it takes $O(m^2)$ time to get the word $\psi^{-1}(g)$. Then, we verify whether or not $\psi^{-1}(g)$ coincides with $\psi^{-1}(e)$, which takes only a constant amount of time. □

As regards the conjugacy problem we have the following theorem. See also [1, Theorem 8.5]

Theorem 2.2.2. *The conjugacy problem in a Cayley biautomatic group is decidable.*

Proof: Let G be a Cayley biautomatic group and $S \subset G$ be a finite set of generators of G. By Definition 2.2.2, there exists an automatic representation of the structure $\mathcal{B}_{G,S}$ given by (2.2). Choose any automatic representation of the structure $\mathcal{B}_{G,S}$. Let a bijection $\psi : L \to G$ be a part of the chosen automatic representation. Let us be given $g_1, g_2 \in G$. It follows from Theorem 2.1.1 that there exist a finite automaton recognizing the language $L_{g_1,g_2} = \{w | g_2\psi(w) = \psi(w)g_1\} \subset L$. Therefore, the emptiness problem for the set L_{g_1,g_2} is decidable. If the set L_{g_1,g_2} is not empty, then g_1 and g_2 are conjugate. □

The direct product of two automatic groups is automatic [2, Theorem 4.1.1]. Similarly, for Cayley automatic groups we have the following theorem. See also [1, Corollary 10.4].

Theorem 2.2.3. *The direct product of two Cayley automatic groups is a Cayley automatic group. The direct product of two Cayley biautomatic groups is a Cayley biautomatic group.*

Proof: Let G_1 and G_2 be Cayley automatic groups. Let $S_1 \subset G_1$ and $S_2 \subset G_2$ be some finite sets of generators of G_1 and G_2, respectively. Since G_1 and G_2 are Cayley automatic groups, there exist bijections $\psi_1 : L_1 \to G_1$ and $\psi_2 : L_2 \to G_2$ which provide automatic representations for the Cayley graphs $\Gamma(G_1, S_1)$ and $\Gamma(G_2, S_2)$, respectively. We assume that $L_1 \subset \Sigma_1^*$ and $L_2 \subset \Sigma_2^*$ are regular languages such that $\Sigma_1 \cap \Sigma_2 = \varnothing$. Put $L = L_1 L_2$. Let $\psi : L \to G_1 \times G_2$ be the bijection from L to the direct product $G_1 \times G_2$ such that $\psi(w_1 w_2) = (\psi_1(w_1), \psi_2(w_2)) \in G_1 \times G_2$, where $w_1 \in L_1$ and $w_2 \in L_2$. It is easy to verify that $\psi : L \to G_1 \times G_2$ provides an automatic representation for the Cayley graph $\Gamma(G_1 \times G_2, S_1 \cup S_2)$. Similarly, the theorem holds for Cayley biautomatic groups. $\square$

If $H \leqslant G$ is a subgroup of finite index in G then H is automatic iff G is automatic [2, Theorem 4.1.4]. This theorem cannot be generalized straightforwardly for Cayley automatic groups. It is proved that if $H \trianglelefteq G$ is a normal subgroup of finite index in G and H is a Cayley automatic group, then G is a Cayley automatic group [1, Theorem 10.1]. This theorem can be easily generalized as follows.

Theorem 2.2.4. *Let $H \leqslant G$ be a subgroup of finite index in G. If H is a Cayley automatic group then G is a Cayley automatic group.*

Proof: Let $S = \{h_1, \dots, h_n\}$ be a set of generators of H. Since H is a Cayley automatic group, there exist a bijection $\psi : L \to H$ between a regular language $L \subset \Sigma^*$ and the group H which provides an automatic representation for the Cayley graph $\Gamma(H, S)$. Let $K = \{k_1, \dots, k_m\} \subset G$ be a set of representatives of the right cosets Hg, $g \notin H$. We use the language $L_0 = \{\mathbf{k_1}, \dots, \mathbf{k_m}\}$ for representing the elements $k_1, \dots, k_m$. We assume that

$\mathbf{k_i} \notin \Sigma$, $i = 1, \ldots, m$. Put $L' = \{w\sigma | w \in L \wedge \sigma \in L_0 \cup \{\epsilon\}\}$, where ϵ is the empty word. The language L' is clearly regular. Define the map $\psi' : L' \to G$ as follows. If $w' = w\mathbf{k_i}$ for $w \in L$, then put $\psi'(w') = \psi(w)k_i$. If $w' = w$ for $w \in L$, then put $\psi'(w') = \psi(w)$.

Let $g = hk_i$ for $h \in H$ and $i = 1, \ldots, m$. Then $\psi'^{-1}(g) = \psi^{-1}(h)\mathbf{k_i}$. For a given $j = 1, \ldots, m$ we have $k_i k_j = h_{i_1}^{\gamma_1} \ldots h_{i_r}^{\gamma_r} k_\ell$, where $h_{i_1}, \ldots, h_{i_r} \in S$ and $\gamma_1, \ldots, \gamma_r \in \{-1, +1\}$. Then we obtain that $gk_j = hh_{i_1}^{\gamma_1} \ldots h_{i_r}^{\gamma_r} k_\ell$. Therefore, $\psi'^{-1}(gk_j) = \psi^{-1}(hh_{i_1}^{\gamma_1} \ldots h_{i_r}^{\gamma_r})\mathbf{k}_\ell$. Thus, the right multiplication by k_j is FA–recognizable.

For a given $p = 1, \ldots, n$ we have $k_i h_p = h_{j_1}^{\delta_1} \ldots h_{j_s}^{\delta_s} k_q$, where $h_{j_1}, \ldots, h_{j_s} \in S$ and $\delta_1, \ldots, \delta_s \in \{-1, +1\}$. Then we obtain that $gh_p = hh_{j_1}^{\delta_1} \ldots h_{j_s}^{\delta_s} k_q$. Therefore, $\psi'^{-1}(gh_p) = \psi^{-1}(hh_{j_1}^{\delta_1} \ldots h_{j_s}^{\delta_s})\mathbf{k_q}$. Thus, the right multiplication by h_p is FA–recognizable. Therefore, the bijection $\psi' : L' \to G$ provides an automatic representation for the Cayley graph $\Gamma(G, S \cup K)$. $\square$

Let us be given groups A, B and a homomorphism $\varphi : B \to Aut(A)$ from B to the group of automorphisms $Aut(A)$. The semidirect product $A \rtimes_\varphi B$ is the group that can be identified with the product $B \times A = \{(b, a) | b \in B, a \in A\}$ with the group operation given by the formula $(b_1, a_1)(b_2, a_2) = (b_1 b_2, a_1^{b_2} a_2)$, where $a_1^{b_2} = \varphi(b_2)(a_1)$.

Let A and B be Cayley automatic groups. Let $Y \subset B$ be a finite set of generators of B. We fix a bijection $\psi_A : L_A \to A$ which provides an automatic representation for a Cayley graph of A. Suppose that for every $y \in Y$ the automorphism $\varphi(y) : A \to A$ is FA–recognizable with respect to ψ_A. We have the following theorem. See also [1, Theorem 10.3].

Theorem 2.2.5. *The semidirect product $A \rtimes_\varphi B$ is a Cayley automatic group.*

Proof: Let $X \subset A$ be a finite set of generators of A. Let $\psi_A : L_A \to A$ and $\psi_B : L_B \to B$ be some bijections which provide automatic representations for the Cayley graphs $\Gamma(A, X)$ and $\Gamma(B, Y)$, respectively, where $L_A \subset \Sigma_A^*$ and $L_B \subset \Sigma_B^*$. We obtain the automatic representation of $\Gamma(A \rtimes_\varphi B, X \cup Y)$ as follows. We represent every pair $(u, v) \in L_B \times L_A$ as the convolution $u \otimes v$. Put $L = \{u \otimes v | u \in L_B, v \in L_A\}$. Let $\psi : L \to B \times A$ be the bijection such that $\psi : u \otimes v \mapsto (b, a)$, where $b = \psi_B(u)$ and $a = \psi_A(v)$.

Let us show that the bijection $\psi : L \to A \rtimes_\varphi B$ provides an automatic representation of the Cayley graph $\Gamma(A \rtimes_\varphi B, X \cup Y)$. Let $x \in X$. Then $(b, a)(e, x) = (b, a^e x) = (b, ax)$. Therefore, the right multiplication by the element (e, x) is FA–recognizable. Let $y \in Y$. Then $(b, a)(y, e) = (by, a^y) = (by, \varphi(y)(a))$. We assumed that $\varphi(y) : A \to A$ is a FA–recognizable automorphism with respect to $\psi_A : L_A \to A$ for every $y \in Y$. Therefore, the right multiplication by the element (y, e) is FA–recognizable. $\square$

Theorem 2.2.5 enables to construct examples of groups which are Cayley automatic, but not Cayley biautomatic [7]. There exist semidirect products $\mathbb{Z}^d \rtimes_\tau F_n$, $\tau : F_n \to GL_d(\mathbb{Z})$ such that the conjugacy problem in these groups is undecidable [8, 9]. By Theorem 2.2.5, the semidirect product $\mathbb{Z}^d \rtimes_\tau F_n$ is Cayley automatic for every $\tau : F_n \to GL_d(\mathbb{Z})$. Recall that, by Theorem 2.2.2, the conjugacy problem for a Cayley biautomatic group is decidable. Therefore, the Cayley automatic semidirect products $\mathbb{Z}^d \rtimes_\tau F_n$, for which the conjugacy problem is undecidable, are not Cayley biautomatic.

Consider now amalgamated free products. Let us be given two groups $G = \langle S_1; R_1 \rangle$ and $H = \langle S_2; R_2 \rangle$, their subgroups $A \leqslant G$ and $B \leqslant H$, and an isomorphism $\varphi : A \to B$ between them. The amalgamated free product of

A and B with respect to $\varphi : A \to B$ is the group $P := \langle S_1, S_2; R_1, R_2, \{a = \varphi(a) : a \in A\}\rangle$. The groups G and H are naturally embedded into P. If $A = B = \{e\}$, then P is the free product $G * H$.

Let us choose a set of representatives of left cosets $gA, g \in G$ and $hB, h \in H$, putting the identity e to be the representative of the left cosets eA and eB. Let $g_i, i = 0, 1, \ldots, n$ be elements of the set $G \cup H$. We say that the sequence $g_n, \ldots, g_1, g_0$ is a normal form if the following holds.

- The elements $g_n, \ldots, g_1$ are from the chosen set of representatives.
- Successive elements g_i and g_{i-1}, $1 < i \leqslant n$ come from different groups G and H.
- If $n > 0$, then no $g_i, i = 1, \ldots, n$ is equal to e.
- It is assumed that $g_0 \in A$.

The normal form theorem for amalgamated free products is as follows (see, eg., [33]).

Theorem 2.2.6. *For every element $g \in P$ there is a unique representation $g = g_n \ldots g_1 g_0$ such that the sequence $g_n, \ldots, g_1, g_0$ is a normal form.*

We have the following theorem. See also [1, Theorem 10.8].

Theorem 2.2.7. *Let G and H be Cayley automatic groups. Then $G * H$ is a Cayley automatic group.*

Proof: Let $\psi_1 : L_1 \to G$ and $\psi_2 : L_2 \to H$ be some bijections which provide Cayley automatic representations of G and H, respectively, where $L_1 \subset \Sigma_1^*$ and $L_2 \subset \Sigma_2^*$ are regular languages. It is assumed that $\Sigma_1 \cap \Sigma_2 = \varnothing$

and the empty word ϵ does not correspond to any nontrivial element of G or H. It follows from Theorem 2.2.6 that every nontrivial element $g \in G * H$ has a unique representation as $g = g_n \dots g_1$ such that the following holds.

- None of $g_i, i = 1, \dots, n$ is equal to the identity e.
- Successive elements g_i and g_{i-1}, $1 < i \leqslant n$ come from different groups G and H.

For a given nontrivial element $g = g_n \dots g_1 \in G * H$ let $w_g \in (\Sigma_1 \cup \Sigma_2)^*$ be the concatenation of the words corresponding to $g_n, \dots, g_1$ with respect to ψ_1 and ψ_2. Put $L = \{w_g | g \in G * H \wedge g \neq e\} \cup \{\epsilon\}$. It is easy to see that $L \subset (\Sigma_1 \cup \Sigma_2)^*$ is a regular language. Let $\psi : L \to G * H$ be the bijection such that $\psi(w_g) = g$ for every nontrivial $g \in G * H$ and $\psi(\epsilon) = e$. It can be verified that ψ provides a Cayley automatic representation of $G * H$. $\square$

Theorem 2.2.7 can be generalized as follows. See also [1, Theorem 10.9 (2)].

Theorem 2.2.8. *Let G and H be Cayley automatic groups. Let $A \leqslant G$ and $B \leqslant H$ be finite subgroups and $\varphi : A \to B$ be an isomorphism between them. Then the amalgamated free product P is a Cayley automatic group.*

Proof: Let $\psi_1 : L_1 \to G$ and $\psi_2 : L_2 \to H$ be some bijections which provide Cayley automatic representations of G and H respectively, where $L_1 \subset \Sigma_1^*$ and $L_2 \subset \Sigma_2^*$ are regular languages. It is assumed that $\Sigma_1 \cap \Sigma_2 = \varnothing$ and the empty word ϵ does not correspond to any nontrivial element of G or H.

We denote by $\preceq_1$ some length–lexicographical order on L_1. Let $\Phi_1(w)$ be the following first order formula defined on the regular domain L_1:

$$\forall u(\psi_1(u) \in A \Rightarrow \forall v(\psi_1(v)\psi_1(u) = \psi_1(w) \Rightarrow w \preceq_1 v)) \wedge \neg(\psi_1(w) \in A).$$

Put $L_1' \subset L_1$ to be the language of the words w for which $\Phi_1(w)$ is true. The language L_1' is regular. It can be seen that $\psi_1(L_1')$ is the set of representatives of the left cosets $gA, g \in G \setminus A$. Similarly, let us be given a length–lexicographical order $\preceq_2$ on L_2. Let $\Phi_2(w)$ be the following first order formula defined on the regular domain L_2:

$$\forall u(\psi_2(u) \in B \Rightarrow \forall v(\psi_2(v)\psi_2(u) = \psi_2(w) \Rightarrow w \preceq_2 v)) \wedge \neg(\psi_2(w) \in B).$$

Put $L_2' \subset L_2$ to be the regular language of the words w for which $\Phi_2(w)$ is true. It can be seen that $\psi_2(L_2')$ is the set of representatives of the left cosets $hB, h \in H \setminus B$.

Let us fix a correspondence between the set of nontrivial elements of A and some finite alphabet Σ_A. It is assumed that $\Sigma_A \cap (\Sigma_1 \cup \Sigma_2) = \varnothing$. It follows from Theorem 2.2.6 that every element $g \in P$ has a unique representation as $g = g_n \dots g_1 g_0$, where $g_n, \dots, g_1, g_0$ is a normal form such that $g_n, \dots, g_1 \in \psi_1(L_1') \cup \psi_2(L_2')$. For a given nontrivial element $g = g_n \dots g_1 g_0 \in P$ let $w_g \in (\Sigma_1 \cup \Sigma_2 \cup \Sigma_A)^*$ be the concatenation of the words corresponding to $g_n, \dots, g_1$ with respect to ψ_1 and ψ_2 which is then concatenated with the symbol from Σ_A corresponding to the element $g_0 \in A$, if $g_0 \neq e$. Put $L = \{w_g | g \in P \wedge g \neq e\} \cup \{\epsilon\}$. It is easy to see that $L \subset (\Sigma_1 \cup \Sigma_2 \cup \Sigma_A)^*$ is a regular language. Let $\psi : L \to P$ be the bijection such that $\psi(w_g) = g$ for every nontrivial $g \in P$ and $\psi(\epsilon) = e$. It can be verified that ψ gives a Cayley automatic representation of P. $\square$

Another generalization of Theorem 2.2.7 is as follows.

Theorem 2.2.9. *Let G and H be Cayley biautomatic groups. Let $A \leqslant G$ and $B \leqslant H$ be subgroups of finite index and $\varphi : A \to B$ be an isomorphism. Suppose that there exist Cayley biautomatic representations $\psi_1 : L_1 \to G$ and $\psi_2 : L_2 \to H$ such that $\psi_1^{-1}(A) \subset L_1$ and $\psi_2^{-1}(B) \subset L_2$ are regular languages and the isomorphism $\varphi : A \to B$ is FA–recognizable with respect to ψ_1 and ψ_2. Then the amalgamated free product P is a Cayley automatic group.*

Proof: Let $R = \{r_1, \dots, r_m\} \subset G$ be a set of representatives of the left cosets gA, $g \notin A$ and $S = \{s_1, \dots, s_k\} \subset H$ be a set of representatives of the left cosets hB, $h \notin B$. It follows from Theorem 2.2.6 that every element $g \in P$ has a unique representation as $g = g_n \dots g_1 g_0$, where $g_n, \dots, g_1, g_0$ is a normal form such that $g_i \in R \cup S, i = 1, \dots, n$. Put $L_R = \{\mathbf{r_1}, \dots, \mathbf{r_m}\}$ and $L_S = \{\mathbf{s_1}, \dots, \mathbf{s_k}\}$, where $\mathbf{r_1}, \dots, \mathbf{r_m}$ and $\mathbf{s_1}, \dots, \mathbf{s_k}$ correspond to the elements $r_1, \dots, r_m$ and $s_1, \dots, s_k$, respectively. We denote by L_A the regular language $\psi_1^{-1}(A)$. Put

$$L = L_A \cup L_R L_A \cup L_S L_A \cup L_R L_S L_A \cup L_S L_R L_A \cup L_R L_S L_R L_A \cup L_S L_R L_S L_A \cup \dots .$$

The language L is regular. We define $\psi : L \to P$ as the bijection constructed naturally from the normal form $g_n, \dots, g_1, g_0$ and the bijection $\psi_1|_{L_A} : L_A \to A$.

Let $T \subset G$ be a finite set generating G and $Q \subset H$ be a finite set generating H. Let us be given $g = g_n \dots g_1 g_0$, where $g_n, \dots, g_1, g_0$ is the normal form. Let $t \in T$. Assume first that $g_1 \in R$. The map $g_0 \mapsto g_1 g_0 t$ is FA–recognizable because $\psi_1 : L_1 \to G$ is a Cayley biautomatic representation. If $g_1 g_0 t \notin A$ then to obtain the normal form of gt we need to represent $g_1 g_0 t$ as $r g_0'$, where

$r \in R$ and $g_0' \in A$. But this is FA–recognizable as well because R is finite, L_A is regular and $\psi_1 : L_1 \to G$ is a Cayley biautomatic representation.

Assume now that $g_1 \in S$. The map $g_0 \mapsto g_0 t$ is FA–recognizable because $\psi_1 : L_1 \to G$ is a Cayley automatic representation. If $g_0 t \notin A$ then to obtain the normal of gt we need to represent $g_0 t$ as $r g_0'$, where $r \in R$ and $g_0' \in A$. This is FA–recognizable as well because R is finite, L_A is regular and $\psi_1 : L_1 \to G$ is a Cayley biautomatic representation. Therefore, for the representation $\psi : L \to P$ the right multiplication by $t \in T$ is FA–recognizable.

Let $q \in Q$. Assume first that $g_1 \in R$. The map $g_0 \mapsto \varphi(g_0)q$ is FA–recognizable because $\psi_2 : L_2 \to H$ is a Cayley automatic representation and $\varphi : A \to B$ is FA–recognizable with respect to ψ_1 and ψ_2. If $\varphi(g_0)q \notin B$ then to obtain the normal of gq we need to represent $\varphi(g_0)q$ as $s\varphi(g_0')$, where $s \in S$ and $g_0' \in A$. This is FA–recognizable.

Assume now that $g_1 \in S$. The map $g_0 \mapsto g_1\varphi(g_0)q$ is FA–recognizable because $\psi_2 : L_2 \to H$ is a Cayley biautomatic representation and $\varphi : A \to B$ is FA–recognizable with respect to ψ_1 and ψ_2. If $g_1\varphi(g_0)q \notin B$ then to obtain the normal form of gq we need to represent $g_1\varphi(g_0)q$ as $s\varphi(g_0')$, where $s \in S$ and $g_0' \in A$. This is FA–recognizable. Therefore, for the representation $\psi : L \to P$ the right multiplication by $q \in Q$ is FA–recognizable. Thus, ψ provides a Cayley automatic representation of P. $\square$

Chapter 3

The Baumslag–Solitar groups

In this chapter we prove that the Baumslag–Solitar groups $BS(m,n)$ are Cayley automatic groups and show some properties of the obtained Cayley automatic representations of these groups. The main results of this chapter are published in [19].

3.1 The Baumslag–Solitar groups and HNN extensions

The Baumslag–Solitar groups were introduced by Baumslag and Solitar to show examples of non–Hopfian groups [34].

Definition 3.1.1. *For given nonnegative integers m and n the Baumslag–Solitar group $BS(m,n)$ is a two–generator one–relator group defined as $BS(m,n) = \langle a, t \,|\, t^{-1}a^m t = a^n \rangle$.*

If $m = 0$ or $n = 0$, then $BS(m,n)$ is isomorphic to the free product of a cyclic group and $\mathbb{Z}$. We further suppose that m and n are nonzero.

Let us recall the definition of the HNN extension and the normal form theorem for the HNN extension. For more details see, e.g., [33, Chapter IV, § 2].

Definition 3.1.2. *The HNN extension of a group G relative to subgroups $A, B \leqslant G$ and an isomorphism $\varphi : A \to B$ is the group $G^* = \langle G, t; t^{-1}at = \varphi(a), a \in A\rangle$, where the generator t is called a stable letter.*

The normal form theorem for the HNN extension is as follows. For the proof see, e.g., [33, Theorem 2.1.(II)].

Theorem 3.1.1. *Let us fix representatives of right cosets $Ag, g \in G$ and $Bg, g \in G$, putting the identity $e \in G$ to be the representative of the right cosets Ae and Be. Then every element $g \in G^*$ has a unique representation as $g = g_0 t^{\epsilon_1} g_1 \cdots t^{\epsilon_\ell} g_\ell$ such that the sequence $g_0, t^{\epsilon_1}, g_1, \cdots, t^{\epsilon_\ell}, g_\ell$ satisfies the following properties.*

- *g_0 is an arbitrary element of G and $\epsilon_i \in \{-1, +1\}, i = 1, \ldots, \ell$.*
- *If $\epsilon_i = -1$, then g_i is a representative of a right coset of A in G.*
- *If $\epsilon_i = +1$, then g_i is a representative of a right coset of B in G.*
- *There is no consecutive subsequence $t^{\epsilon}, e, t^{-\epsilon}$.* □

The alternative form of Theorem 3.1.1 is as follows.

Theorem 3.1.2. *Let us fix representatives of left cosets $gA, g \in G$ and $gB, g \in G$, putting the identity $e \in G$ to be the representative of the left cosets eA and eB. Then every element $g \in G^*$ has a unique representation as $g = g_\ell t^{\epsilon_\ell} \cdots g_1 t^{\epsilon_1} g_0$ such that the sequence $g_\ell, t^{\epsilon_\ell}, \cdots, g_1, t^{\epsilon_1}, g_0$ satisfies the following properties.*

- *g_0 is an arbitrary element of G and $\epsilon_i \in \{-1, +1\}, i = 1, \ldots, \ell$.*
- *If $\epsilon_i = -1$, then g_i is a representative of a left coset of B in G.*
- *If $\epsilon_i = +1$, then g_i is a representative of a left coset of A in G.*
- *There is no consecutive subsequence $t^{\epsilon}, e, t^{-\epsilon}$.* □

For given positive integers m and n the Baumslag–Solitar group $BS(m, n)$ can be obtained as the HNN extension of the infinite cyclic group $\mathbb{Z}$ relative to the subgroups $m\mathbb{Z}$ and $n\mathbb{Z}$ and the isomorphism $\varphi : m\mathbb{Z} \to n\mathbb{Z}$. Therefore, Theorems 3.1.1 and 3.1.2 are applicable to $BS(m, n)$.

3.2 The Baumslag–Solitar groups are Cayley automatic

Recall that all Baumslag–Solitar groups are asynchronously automatic and for positive integers m and n the group $BS(m, n)$ is automatic iff $m = n$ [2, § 7.4]. The group $BS(1, n), n \in \mathbb{N}$ is Cayley automatic due to [1, Theorem 13.1]. The following theorem shows that all Baumslag–Solitar groups are Cayley automatic.

Theorem 3.2.1. *For given positive integers m and n the group $BS(m, n)$ is Cayley automatic.*

Proof: As we mentioned in Section 3.1 the group $BS(m, n)$ can be obtained as the HNN extension of the group $\mathbb{Z}$ relative to the subgroups $m\mathbb{Z}$ and $n\mathbb{Z}$ and the isomorphism $\varphi : m\mathbb{Z} \to n\mathbb{Z}$. We denote by a the generator of $\mathbb{Z}$. The isomorphism φ maps a^m to a^n.

Put $e, a, \dots, a^{m-1}$ and $e, a, \dots, a^{n-1}$ to be the representatives of left cosets of the subgroups $m\mathbb{Z}$ and $n\mathbb{Z}$ in $\mathbb{Z}$, respectively. By Theorem 3.1.2 every element $g \in BS(m,n)$ has a unique representation as

$$g = g_\ell t^{\epsilon_\ell} \cdots g_1 t^{\epsilon_1} g_0 \tag{3.1}$$

such that $g_0 = a^k$ for some $k \in \mathbb{Z}$, $\epsilon_i \in \{-1, +1\}$ for $i = 1, \dots, \ell$, if $\epsilon_i = -1$ then $g_i \in \{e, a, \dots, a^{n-1}\}$, if $\epsilon_i = +1$ then $g_i \in \{e, a, \dots, a^{m-1}\}$, and there is no consecutive subsequence $t^\epsilon, e, t^{-\epsilon}$ in the sequence $g_\ell, t^{\epsilon_\ell}, \cdots, g_1, t^{\epsilon_1}, g_0$.

The right–multiplication of an element g by the generator a transforms the representation (3.1) as follows:

$$g_\ell t^{\epsilon_\ell} \cdots g_1 t^{\epsilon_1} a^k \xrightarrow{\times a} g_\ell t^{\epsilon_\ell} \cdots g_1 t^{\epsilon_1} a^{k+1}. \tag{3.2}$$

Let $k = mp + r$, where $p \in \mathbb{Z}$ and $r = 0, \dots, m-1$. The right–multiplication of g by the generator t transforms the representation (3.1) as follows.

- If $r \neq 0$ then

$$g_\ell t^{\epsilon_\ell} \cdots g_1 t^{\epsilon_1} a^k \xrightarrow{\times t} g_\ell t^{\epsilon_\ell} \cdots g_1 t^{\epsilon_1} a^r t a^{np}. \tag{3.3}$$

- If $r = 0$, $\ell \geqslant 1$ and $\epsilon_1 = -1$ then

$$g_\ell t^{\epsilon_\ell} \cdots g_2 t^{\epsilon_2} g_1 t^{-1} a^k \xrightarrow{\times t} g_\ell t^{\epsilon_\ell} \cdots g_2 t^{\epsilon_2} g_1 a^{np}. \tag{3.4}$$

- If $r = 0$, $\ell \geqslant 1$ and $\epsilon_1 = 1$ then

$$g_\ell t^{\epsilon_\ell} \cdots g_1 t a^k \xrightarrow{\times t} g_\ell t^{\epsilon_\ell} \cdots g_1 t\, e\, t a^{np}. \tag{3.5}$$

- If $r = 0$ and $\ell = 0$ then

$$a^k \xrightarrow{\times t} a^r t a^{np}. \tag{3.6}$$

Let q be the function $q : \mathbb{Z} \to \mathbb{Z}$ such that $q(k) = np$, where $k = mp + r$. If $m \geqslant 2$, for the m–ary representation of integers the function $q : mp + r \mapsto np$ is FA–recognizable. If $m = 1$, then the function q is FA–recognizable for any b–ary representation of integers for $b \geqslant 2$.

Put $\Sigma = \{0, \ldots, c, \mathbf{0}, \ldots, \mathbf{d}, t^{+1}, t^{-1}\}$, where c is the symbol denoting $\max\{m, n\} - 1$ and $\mathbf{d}$ is the bold symbol denoting $d = \max\{m - 1, 1\}$. Let $L_d \subset \{\mathbf{0}, \ldots, \mathbf{d}\}^*$ be the language of $(d + 1)$–ary representations of all integers. Put L' to be the set of all words $x_\ell t^{\epsilon_\ell} \ldots x_1 t^{\epsilon_1} \mathbf{y}$ such that the following holds.

- $x_i \in \{0, \ldots, c\}$ and $t^{\epsilon_i} \in \{t^{+1}, t^{-1}\}$, $i = 1, \ldots, \ell$.
- $x_i \in \{0, \ldots, n - 1\}$ if $\epsilon_i = -1$, $i = 1, \ldots, \ell$.
- $x_i \in \{0, \ldots, m - 1\}$ if $\epsilon_i = +1$, $i = 1, \ldots, \ell$.
- There is no consecutive subsequence $t^\epsilon, 0, t^{-\epsilon}$ in the sequence $x_\ell, t^{\epsilon_\ell}, \ldots, x_1, t^{\epsilon_1}$.
- $\mathbf{y} \in L_d$.

Put $L = L' \cup L_d$. It can be seen that L is a regular language. By Theorem 3.1.2 we get the bijection $\psi : L \to BS(m, n)$. It follows from (3.2)–(3.6) and the fact that the function q is FA–recognizable for $(d+1)$–ary representations of integers that the bijection $\psi : L \to BS(m,n)$ provides an automatic representation for the Cayley graph of $BS(m,n)$ with respect to the generators a and t. $\square$

Remark 3.2.1. *If $m = n$ then the function $q : k = mp + r \mapsto mp$ is FA–recognizable for the unary representation of integers. Modifying accordingly the construction of the Cayley automatic representation of $BS(m,n)$ in Theorem 3.2.1 we obtain that $BS(m,n)$ is automatic if $m = n$.*

Remark 3.2.2. *The function $q : k = mp + r \mapsto np$ is recognizable by an asynchronous automaton for the unary representation of integers. Modifying accordingly the proof of Theorem 3.2.1 it can be obtained that $BS(m,n)$ is asynchronously automatic.*

Remark 3.2.3. *Elder and Taback constructed the representation of the Cayley graph of $BS(m,n)$ with respect to the generators a and t using counter automata [35, Proposition 18].*

Let m and n be positive integers. For a given element $g \in BS(m,n)$ we denote by $|g|$ the length of g in the group $BS(m,n)$ with respect to the generators a and t. Let us consider the Cayley automatic representation $\psi : L \to BS(m,n)$ constructed in Theorem 3.2.1. We denote by w the word of the language L such that $\psi(w) = g$. Put $|w|$ to be the length of the word w. We have the following proposition.

Proposition 3.2.1. *There exist constants $\lambda > 0$, $\mu > 0$, ξ and δ such that the following inequalities hold for all $g \in BS(m,n)$:*

$$\lambda|g| + \xi \leqslant |w| \leqslant \mu|g| + \delta. \tag{3.7}$$

Proof: For a given $g \in BS(m,n)$ the normal form $g_\ell t^{\epsilon_\ell} \cdots g_1 t^{\epsilon_1} g_0$ can be obtained as the concatenation of the word $u = g_\ell t^{\epsilon_\ell} \cdots g_1 t^{\epsilon_1}$ and $g_0 = a^k$, see the equation (3.1). Burillo and Elder showed that, if $m \neq n$, then there exist constants $C_1, C_2, D_1, D_2 > 0$ such that the inequalities $C_1(|u| + \log(|k|+1)) - D_1 \leqslant |g| \leqslant C_2(|u| + \log(|k|+1)) + D_2$ hold for all $g \in BS(m,n)$, where $|u|$ is the length of the word u, see [18, Theorem 3.2]. If $m = n$, then there exists a constant C_1 such that the inequalities $C_1(|u| + |k|) \leqslant |g| \leqslant |u| + |k|$ hold for all $g \in BS(m,m)$, see [18, Lemma 3.3]. Therefore, from the construction of the Cayley automatic representation $\psi : L \to BS(m,n)$ we obtain that for some constants $\lambda > 0$, $\mu > 0$, ξ and δ the inequalities (3.7) hold for all $g \in BS(m,n)$. $\square$

Remark 3.2.4. *The inequality $|w| \leqslant \mu|g| + \delta$ can be alternatively obtained as follows. Put $\delta = |\psi^{-1}(e)|$. Since $\psi : L \to BS(m,n)$ provides an automatic representation of the Cayley graph of $BS(m,n)$ with respect to the generators a and t, there exists a constant μ such that* $\max\{||\psi^{-1}(ga)| - |\psi^{-1}(g)||, ||\psi^{-1}(gt)| - |\psi^{-1}(g)||\} \leqslant \mu$ *for all $g \in BS(m,n)$. This implies that the inequality $|w| \leqslant \mu|g| + \delta$ holds for all $g \in BS(m,n)$.*

Chapter 4

Wreath products of groups

In this chapter we study representations of Cayley graphs of wreath products of groups with finite automata, pushdown automata and nested stack automata. The main results of this chapter are published in [20].

4.1 Wreath products: short introduction

In this section we recall the definition of the restricted wreath product of two groups. For more details on wreath products see, e.g., [36, § 6.2]. Let A and B be groups. For a given function $f : B \to A$ we say that f has finite support if the set $\{x \in B | f(x) \neq e\}$ is finite. We denote by $A^{(B)}$ the group of all functions $f : B \to A$ having finite support with the usual multiplication.

For a given $f \in A^{(B)}$ we denote by $f^b \in A^{(B)}$ the function such that $f^b(x) = f(bx)$ for all $x \in B$. Let $\tau : B \to \mathrm{Aut}(A^{(B)})$ be the homomorphism such that for every $b \in B$ the automorphism $\tau(b) : A^{(B)} \to A^{(B)}$ maps f to f^b for all $f \in A^{(B)}$.

The restricted wreath product $A \wr B$ is defined as the semidirect product $A^{(B)} \rtimes_\tau B$. Therefore, the wreath product $A \wr B$ can be obtained as the product $B \times A^{(B)}$ with the multiplication given by $(b, f) \cdot (b', f') = (bb', f^{b'} f')$. Similarly, $A \wr B$ can be obtained as the product $A^{(B)} \times B$ with the multiplication given by $(f, b) \cdot (f', b') = (f f'^{b^{-1}}, bb')$. In this chapter we mostly use the latter way to represent elements of wreath products of groups.

Baumslag proved that the wreath product $A \wr B$ of two finitely presented groups A and B is finitely presented if and only if either A is the trivial group or B is finite [11]. Therefore, all wreath products of groups which we consider in this chapter are not finitely presented, and, therefore, they are not automatic.

We denote by $\mathfrak{i}_A$ the embedding $\mathfrak{i}_A : A \to A \wr B$ for which $\mathfrak{i}_A : a \mapsto (f_a, e)$, where e is the identity of the group B and $f_a \in A^{(B)}$ is the function $f_a : B \to A$ such that $f_a(e) = a$ and $f_a(x)$ is the identity of the group A for every $x \neq e$. We denote by $\mathfrak{i}_B$ the embedding $\mathfrak{i}_B : B \to A \wr B$ for which $\mathfrak{i}_B : b \mapsto (\mathbf{e}, b)$, where $\mathbf{e}$ is the identity of the group $A^{(B)}$; in other words, $\mathbf{e}$ is the function which maps all elements of B to the identity of the group A. For the sake of convenience we will identify A and B with the subgroups $\mathfrak{i}_A(A) \leqslant A \wr B$ and $\mathfrak{i}_B(B) \leqslant A \wr B$, respectively.

Let $S_A = \{a_1, \dots, a_n\}$ and $S_B = \{t_1, \dots, t_m\}$ be sets of generators of the groups A and B, respectively. Put $S = S_A \cup S_B$. The Cayley graph $\Gamma(A \wr B, S)$ is obtained as follows. The vertices of $\Gamma(A \wr B, S)$ are all pairs (f, b) such that $f \in A^{(B)}$ and $b \in B$. Let $v_1 = (f_1, b_1)$ and $v_2 = (f_2, b_2)$ be vertices of the Cayley graph $\Gamma(A \wr B, S)$. There is a directed edge (v_1, v_2) labeled by $a_i, i = 1, \dots, n$ iff $b_1 = b_2 = b$ and $f_1(x) = f_2(x)$ for all $x \neq b$, and

$f_2(b) = f_1(b)a_i$. There is a directed edge (v_1, v_2) labeled by $t_j, j = 1, \ldots, m$ iff $f_1 = f_2$ and $b_1 t_j = b_2$. The notion of the wreath product can be easily generalized for graphs, see, e.g., [37, Definition 2.1].

4.2 The lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$

The lamplighter group is the wreath product $\mathbb{Z}_2 \wr \mathbb{Z}$. The elements of the lamplighter group are all pairs $(f, z) \in \mathbb{Z}_2^{(\mathbb{Z})} \times \mathbb{Z}$ such that $f : \mathbb{Z} \to \mathbb{Z}_2$ is a function having finite support and $z \in \mathbb{Z}$. The multiplication in the group $\mathbb{Z}_2 \wr \mathbb{Z}$ is given by the rule

$$(f_1, z_1)(f_2, z_2) = (f_1 f_2^{-z_1}, z_1 + z_2), \tag{4.1}$$

where $f_2^{-z_1}(z) = f_2(z - z_1)$ for all $z \in \mathbb{Z}$.

We denote by h_0 the function $h_0 : \mathbb{Z} \to \mathbb{Z}_2$ such that $h_0(z) = 0$ for all $z \in \mathbb{Z}$. That is, h_0 is the identity of the group $\mathbb{Z}_2^{(\mathbb{Z})}$. We denote by h_1 the function $h_1 : \mathbb{Z} \to \mathbb{Z}_2$ such that $h_1(z) = 0$ for all $z \neq 0$ and $h_1(0) = 1$. Put $t = (h_0, 1)$ and $a = (h_1, 0)$. It can be seen that a and t generate the group $\mathbb{Z}_2 \wr \mathbb{Z}$. The lamplighter group can be obtained as the group $\langle a, t \,|\, [t^i a t^{-i}, t^j a t^{-j}], a^2 \rangle$.

The lamplighter group has the following geometric interpretation. Every element of the lamplighter group corresponds to a bi–infinite string of lamps indexed by integers each of which has only two states 0 and 1 such that only a finite number of lamps are in the state 1, and the lamplighter pointing at the current lamp. We say that a lamp is lit if it is in the state 1. Similarly, we say that a lamp is unlit if it is in the state 0. The identity of the lamplighter group corresponds to the configuration when all lamps are unlit, and the lamplighter points at the lamp positioned at the origin $z = 0$. The right multiplication by a changes the state of the current lamp. The right multiplication by t (or, t^{-1}) moves the lamplighter by one step to the right $z \mapsto z+1$ (or, to the left $z \mapsto z-1$).

In [1, Theorem 10.6] the authors showed that the lamplighter group is Cayley automatic. Theorem 4.2.1 below shows that the lamplighter group is Cayley biautomatic. The Cayley automatic representation in Theorem 4.2.1 is different from the representation in [1, Theorem 10.6].

Theorem 4.2.1. *The lamplighter group is Cayley biautomatic.*

Proof: Let us be given an element $g = (f, z) \in \mathbb{Z}_2 \wr \mathbb{Z}$. In order to present $f(i) \in \mathbb{Z}_2$ we use the symbols 0 and 1: 0 means that the lamp at the position $z = i$ is unlit, 1 means that the lamp is lit. In order to show the position of the origin $z = 0$, we use the symbols A_0 and A_1 if the lamp at the origin is unlit and lit, respectively. In order to show the position of the lamplighter, we use the symbols C_0 and C_1 if the lamp at the position of the lamplighter is unlit and lit, respectively. If the lamplighter is at the origin, we use the symbols B_0 and B_1.

Let m be the smallest $i \in \mathbb{Z}$ for which $f(i) = 1$; if $f(i) = 0$ for all $i \in \mathbb{Z}$, then put $m = 0$. Put $\ell = \min\{m, z, 0\}$. Let n be the largest j for which $f(j) = 1$; if $f(j) = 0$ for all $j \in \mathbb{Z}$, then put $n = 0$. Put $r = \max\{n, z, 0\}$. We represent the element (f, z) by the word:

$$f(\ell)f(\ell+1)\dots f(-1)A_{f(0)}f(1)\dots f(z-1)C_{f(z)}f(z+1)\dots f(r-1)f(r), \quad (4.2)$$

where $A_{f(0)} = A_0$ and $A_{f(0)} = A_1$ if $f(0) = 0$ and $f(0) = 1$, respectively; also, $C_{f(z)} = C_0$ and $C_{f(z)} = C_1$ if $f(z) = 0$ and $f(z) = 1$, respectively. If

$z = 0$, then the element (f, z) is represented by the word:

$$f(\ell)f(\ell+1)\dots f(-1)B_{f(0)}f(1)\dots f(r-1)f(r), \tag{4.3}$$

where $B_{f(0)} = B_0$ and $B_{f(0)} = B_1$ if $f(0) = 0$ and $f(0) = 1$, respectively.

It can be seen that the language of the words representing all elements of the lamplighter group is regular. If $z \neq 0, -1$, then writing the words representing $g = (f, z)$ and $gt = (f, z+1)$ one under another we have:

$$\begin{array}{ccccccccc} f(\ell) & \dots & A_{f(0)} & \dots & f(z-1) & C_{f(z)} & f(z+1) & \dots & f(r) \\ f(\ell) & \dots & A_{f(0)} & \dots & f(z-1) & f(z) & C_{f(z+1)} & \dots & f(r) \end{array}. \tag{4.4}$$

If $z = 0$, then writing the words representing $g = (f, z)$ and $gt = (f, 1)$ one under another we have:

$$\begin{array}{cccccccc} f(\ell) & \dots & f(-1) & B_{f(0)} & f(1) & f(2) & \dots & f(r) \\ f(\ell) & \dots & f(-1) & A_{f(0)} & C_{f(1)} & f(2) & \dots & f(r) \end{array}. \tag{4.5}$$

The other cases are considered in a similar way. From (4.4) and (4.5) it can be seen that the relation $\langle g, gt \rangle$ is recognized by a synchronous two–tape finite automaton.

If $z \neq 0$, then writing the words representing g and $ga = (fh_1, z)$ one under another we have:

$$\begin{array}{ccccccccc} f(\ell) & \dots & A_{f(0)} & \dots & f(z-1) & C_{f(z)} & f(z+1) & \dots & f(r) \\ f(\ell) & \dots & A_{f(0)} & \dots & f(z-1) & C_{\overline{f(z)}} & f(z+1) & \dots & f(r) \end{array}, \tag{4.6}$$

where $\overline{f(z)} = 1$ and $\overline{f(z)} = 0$ if $f(z) = 0$ and $f(z) = 1$, respectively.

If $z = 0$, then writing the words representing g and $ga = (fh_1, 0)$ one under another we have:

$$\begin{array}{ccccccc} f(\ell) & \dots & f(-1) & B_{f(0)} & f(1) & \dots & f(r) \\ f(\ell) & \dots & f(-1) & B_{\overline{f(0)}} & f(1) & \dots & f(r) \end{array}, \tag{4.7}$$

where $\overline{f(0)} = f(0) + 1 \mod 2$. From (4.6) and (4.7) it can be seen that the relation $\langle g, ga \rangle$ is recognized by a synchronous two–tape finite automaton.

Let us now show that the left multiplications by t and a are recognized by synchronous two–tape finite automata. The element tg equals $(h_0, 1)(f, z) = (h_0 f^{-1}, z+1) = (f^{-1}, z+1)$. Recall that $f^{-1}(z) = f(z-1)$; the shifted function f^{-1} should not be confused with the inverse of f. If $z \neq -1$, the element $(f^{-1}, z+1)$ is represented by the word

$$f^{-1}(\ell+1) \dots A_{f^{-1}(0)} \dots C_{f^{-1}(z+1)} \dots f^{-1}(r+1), \tag{4.8}$$

which is equal to

$$f(\ell) \dots A_{f(-1)} \dots C_{f(z)} \dots f(r). \tag{4.9}$$

If $z \neq 0, -1$, then writing the words representing g and tg one under another we have:

$$\begin{array}{cccccccc} f(\ell) & \dots & f(-1) & A_{f(0)} & \dots & C_{f(z)} & \dots & f(r) \\ f(\ell) & \dots & A_{f(-1)} & f(0) & \dots & C_{f(z)} & \dots & f(r) \end{array}. \tag{4.10}$$

If $z = 0$, then writing the words representing g and $tg = (f^{-1}, 1)$ one under another we have:

$$\begin{array}{ccccccc} f(\ell) & \dots & f(-1) & B_{f(0)} & f(1) & \dots & f(r) \\ f(\ell) & \dots & A_{f(-1)} & C_{f(0)} & f(1) & \dots & f(r) \end{array}. \tag{4.11}$$

The other cases are considered in a similar way. From (4.10) and (4.11) it can be seen that the relation $\langle g, tg \rangle$ is recognized by a synchronous two–tape finite automaton.

The element ag equals $(h_1, 0)(f, z) = (h_1 f, z)$. If $z \neq 0$, then writing the words representing g and ag one under another we have:

$$\begin{array}{ccccccc} f(\ell) & \ldots & A_{f(0)} & \ldots & C_{f(z)} & \ldots & f(r) \\ f(\ell) & \ldots & A_{\overline{f(0)}} & \ldots & C_{f(z)} & \ldots & f(r) \end{array}. \tag{4.12}$$

If $z = 0$, then writing the words representing g and $ag = (h_1 f, 0)$ one under another we have:

$$\begin{array}{ccccc} f(\ell) & \ldots & B_{f(0)} & \ldots & f(r) \\ f(\ell) & \ldots & B_{\overline{f(0)}} & \ldots & f(r) \end{array}. \tag{4.13}$$

From (4.12) and (4.13) it can be seen that the relation $\langle g, ag \rangle$ is recognized by a synchronous two–tape finite automaton.

Thus, we constructed the bijection $\psi : L \to \mathbb{Z}_2 \wr \mathbb{Z}$ from some regular language $L \subset \{0, 1, A_0, A_1, B_0, B_1, C_0, C_1\}^*$ to the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$. We showed that with respect to ψ the right and left multiplications by a and t are recognized by two–tape synchronous finite automata. Therefore, the lamplighter group is Cayley biautomatic. $\square$

For the Cayley biautomatic representation of the lamplighter group described in Theorem 4.2.1 the length of the word w representing an element $g = (f, z) \in \mathbb{Z}_2 \wr \mathbb{Z}$ is obtained in the following lemma.

Lemma 4.2.1. *The length of the word w representing an element $g = (f, z) \in \mathbb{Z}_2 \wr \mathbb{Z}$ equals $|w| = \max\{|n-m|, |n|, |m|, |n-z|, |m-z|, |z|\} + 1$, where m is the smallest $i \in \mathbb{Z}$ for which $f(i) = 1$ and $m = 0$ if $f(i) = 0$ for all $i \in \mathbb{Z}$, and n is the largest j for which $f(j) = 1$ and $n = 0$ if $f(j) = 0$ for all $j \in \mathbb{Z}$.*

Proof: The word representing an element (f, z) is either of the form (4.2), if $z \neq 0$, or of the form (4.3), if $z = 0$. Therefore,

$$\begin{aligned} |w| = |r - \ell| + 1 = |\max\{n, z, 0\} - \min\{m, z, 0\}| + 1 = \\ = \max\{|n - m|, |n|, |m|, |n - z|, |m - z|, |z|\} + 1, \end{aligned} \tag{4.14}$$

where $\ell = \min\{m, z, 0\}$ and $r = \max\{n, z, 0\}$. □

For a given element $g = (f, z) \in \mathbb{Z}_2 \wr \mathbb{Z}$ we denote by $\#\mathrm{supp}\, f$ the cardinality of the set $\mathrm{supp}\, f = \{j \mid f(j) = 1\}$. The word length of g in the lamplighter group with respect to the generators a and t is shown in the following lemma.

Lemma 4.2.2. *The word length of an element $g = (f, z) \in \mathbb{Z}_2 \wr \mathbb{Z}$ in the lamplighter group with respect to the generators a and t is as follows:*

$$\begin{aligned} |g| = \#\mathrm{supp}\, f + \min\{2\max\{-m, 0\} + \max\{n, 0\} + |z - \max\{n, 0\}|, \\ 2\max\{n, 0\} + \max\{-m, 0\} + |z + \max\{-m, 0\}|\}. \end{aligned} \tag{4.15}$$

Proof: Recall that the left–first and the right–first normal forms of an element $g = (f, z) \in \mathbb{Z}_2 \wr \mathbb{Z}$ are defined as

$$g = a_{-j_1} \dots a_{-j_q} a_{i_1} \dots a_{i_p} t^z,$$

$$g = a_{i_1} \dots a_{i_p} a_{-j_1} \dots a_{-j_q} t^z,$$

respectively, where $i_p > \dots > i_1 \geqslant 0$, $j_q > \dots > j_1 > 0$ and $a_k = t^k a t^{-k}$ (see, e.g., [38]). Cleary and Taback proved (see [38, Proposition 3.6]) that the word length of $g = (f, z) \in \mathbb{Z}_2 \wr \mathbb{Z}$ in the lamplighter group with respect to the generators a and t is provided either by the left–first normal form or the right–first normal form. Therefore,

$$|g| = p + q + \min\{2j_q + i_p + |z - i_p|, 2i_p + j_q + |z + j_q|\}.$$

Let us express $|g|$ in terms of m and n for the following three different cases:

- if $m \leqslant -1$ and $n \geqslant 0$ then $|g| = p + q + \min\{-2m + n + |z - n|, 2n - m + |z - m|\}$,
- if $m \geqslant 0$ then $|g| = p + q + n + |z - n|$,
- if $n \leqslant -1$ then $|g| = p + q - m + |z - m|$.

It is easy to see that $\#\mathrm{supp}\, f = p + q$. Therefore, we obtain (4.15). $\square$

By Lemmas 4.2.1 and 4.2.2 we obtain the following proposition.

Proposition 4.2.1. *The following inequalities hold for all $g \in \mathbb{Z}_2 \wr \mathbb{Z}$:*

$$|w| - 1 \leqslant |g| \leqslant 3|w| - 2 \tag{4.16}$$

– or, equivalently,

$$\frac{1}{3}|g| + \frac{2}{3} \leqslant |w| \leqslant |g| + 1. \tag{4.17}$$

Furthermore, the equalities $|w| - 1 = |g|$ and $|g| = 3|w| - 2$ are achieved for an infinite number of elements $g \in \mathbb{Z}_2 \wr \mathbb{Z}$.

Proof: We first prove the inequality $|g| \leqslant 3|w| - 2$. It follows from (4.14) that $|w| \geqslant n - m + 1$. Therefore, $|w| \geqslant \#\mathrm{supp}\, f$. Let us consider separately each of the three following cases: $m \leqslant -1 < 0 \leqslant n$, $n \leqslant -1$ and $0 \leqslant m$.

- Suppose that $m \leqslant -1 < 0 \leqslant n$. If $z \geqslant n$, then $-2m + n + |z - n| = -2m + z \leqslant 2(z - m)$. If $z \leqslant m$, then $2n - m + |z - m| = 2n - m + m - z \leqslant 2(n - z)$. If $m < z < n$, then $-2m + n + |z - n| = 2(n - m) - z$ and $2n - m + |z - m| = 2(n - m) + z$. Therefore, by (4.14), we obtain that $\min\{-2m + n + |z - n|, 2n - m + |z - m|\} \leqslant 2(|w| - 1)$. Therefore, $|g| \leqslant 3|w| - 2$.

- Suppose that $m \geqslant 0$. By (4.14) we obtain that $n + |z - n| \leqslant 2(|w| - 1)$. Therefore, $|g| \leqslant 3|w| - 2$.

- Suppose that $n \leqslant -1$. By (4.14) we obtain that $-m + |z - m| \leqslant 2(|w| - 1)$. Therefore, $|g| \leqslant 3|w| - 2$.

It is easy to provide an infinite sequence of elements $\mathbb{Z}_2 \wr \mathbb{Z}$ for which the equality $|g| = 3|w| - 2$ holds. Let us consider the infinite sequence of elements whose representatives are $B_1, 1B_11, 11B_111$ and so on. By Lemma 4.2.2, the lengths of these elements in the group $\mathbb{Z}_2 \wr \mathbb{Z}$ with respect to the generators a, t are $1, 7, 13$, respectively, and so on. Therefore, the equality $|g| = 3|w| - 2$ holds for all these elements.

Let us show the inequality $|w| - 1 \leqslant |g|$. The identity $e \in \mathbb{Z}_2 \wr \mathbb{Z}$ is represented by the word B_0. Therefore, the inequality holds for $g = e$. Suppose that the inequality holds for some $g \in \mathbb{Z}_2 \wr \mathbb{Z}$. It follows from the construction of the Cayley biautomatic representation (see Theorem 4.2.1) that the length of the word representing ga equals $|w|$, and the lengths of the words representing gt and gt^{-1} are equal to either $|w|$, $|w|+1$ or $|w|-1$. This implies that the inequality holds for the elements ga, gt and gt^{-1}. Therefore, the inequality holds for all $g \in \mathbb{Z}_2 \wr \mathbb{Z}$.

It is easy to provide an infinite sequence of elements $\mathbb{Z}_2 \wr \mathbb{Z}$ for which the equality $|w| - 1 = |g|$ holds for all $g \in \mathbb{Z}_2 \wr \mathbb{Z}$. Let us consider the infinite sequence of elements which representatives are $B_0, A_0C_0, A_00C_0, A_000C_0$ and etc. The lengths of these elements in the group $\mathbb{Z}_2 \wr \mathbb{Z}$ with respect to the generators a, t are $0, 1, 2, 3$ and etc. Therefore, the equality $|w| - 1 = |g|$ holds for all these elements. $\square$

Another property of the Cayley biautomatic representation constructed in Theorem 4.2.1 is as follows.

Proposition 4.2.2. *The sets of representatives of the elements of the normal subgroup* $\mathbb{Z}_2^{(\mathbb{Z})} \trianglelefteq \mathbb{Z}_2 \wr \mathbb{Z}$ *and the subgroup* $\mathbb{Z} \leqslant \mathbb{Z}_2 \wr \mathbb{Z}$ *are regular languages.*

Proof: The elements of the subgroup $\mathbb{Z}_2^{(\mathbb{Z})} \trianglelefteq \mathbb{Z}_2 \wr \mathbb{Z}$ are the elements (f, z) for which $z = 0$. The representatives of these elements have the form (4.3), i.e., they contain either the symbol B_0 or B_1. The language of such representatives is regular. The representatives of the elements of the subgroup $\mathbb{Z} \leqslant \mathbb{Z}_2 \wr \mathbb{Z}$ are the words of the form (4.2) and (4.3) which do not contain the symbols $1, A_1, B_1$ and C_1. Therefore, the language of such representatives is regular. □

It is easy to provide an example of a Cayley automatic representation of the lamplighter group for which the set of representatives of the subgroup $\mathbb{Z}_2^{(\mathbb{Z})} \trianglelefteq \mathbb{Z}_2 \wr \mathbb{Z}$ is not a regular language.

Example 4.2.1. *For a given element* $(f, z) \in \mathbb{Z}_2 \wr \mathbb{Z}$ *let* s *and* t *be the minimum and maximum elements of the set* $\{j \,|\, f(j) = 1\} \cup \{z\}$*, respectively. Let* $\varphi : L \to \mathbb{Z}$ *be some automatic representation of the structure* $(\mathbb{Z}, S)$*, where* $S(x) = x + 1$*, i.e.,* φ *is a bijection between a regular language* $L \subset \Sigma^*$ *and the infinite cyclic group* $\mathbb{Z}$ *such that the binary relation* $\{\langle \varphi^{-1}(x), \varphi^{-1}(x+1)\rangle | x \in \mathbb{Z}\} \subset \Sigma^* \times \Sigma^*$ *is recognized by a synchronous two–tape finite automaton. Let us represent an element* $(f, z) \in \mathbb{Z}_2 \wr \mathbb{Z}$ *as the word*

$$w\#f(s)\dots C_{f(z)}\dots f(t), \tag{4.18}$$

where $\varphi(w) = s$ *and the symbol* $\#$ *is used as a separator. Note that the* *representation* (4.18) *does not contain the symbols* A_0, A_1, B_0 *and* B_1. *It can be verified that the representation* (4.18) *provides a Cayley automatic representation of the lamplighter group* $\mathbb{Z}_2 \wr \mathbb{Z}$. *Let* L *be the language of all words* (4.18) *representing the elements of* $\mathbb{Z}_2 \wr \mathbb{Z}$. *We denote by* $L_0 \subset L$ *the language of representatives of the elements of the normal subgroup* $\mathbb{Z}_2^{(\mathbb{Z})} \trianglelefteq \mathbb{Z}_2 \wr \mathbb{Z}$. *In Proposition 4.2.3 we show that* L_0 *is not regular.*

Proposition 4.2.3. *The language* L_0 *is not regular.*

Proof: Suppose that L_0 is a regular language. Then, by the pumping lemma, for some word $u = w\#f(s)\dots f(p)\dots f(q)\dots C_{f(0)}\dots f(t) \in L_0$ the words $u_n = w\#f(s)\dots f(p-1)v^n f(q+1)\dots C_{f(0)}\dots f(t) \in L_0$ for all $n \geqslant 0$, where $v = f(p)\dots f(q)$. For the element which corresponds to a word u_n the lamplighter is at the position $z = (n-1)(q-p+1)$. This implies that the element is not in the subgroup $\mathbb{Z}_2^{(\mathbb{Z})}$ unless $n = 1$. We get a contradiction. Therefore, L_0 is not a regular language. □

Consider now two subsets of $\mathbb{Z}_2 \wr \mathbb{Z}$: $T_0 = \{(f,z) \mid f(0) = 0\}$ and $T_1 = \{(f,z) \mid f(0) = 1\}$. It is easy to see that $T_0 \cup T_1 = \mathbb{Z}_2 \wr \mathbb{Z}$ and $T_0 \cap T_1 = \varnothing$. It follows from (4.2) and (4.3) that for the Cayley biautomatic representation described in Theorem 4.2.1 the set of words representing the elements of T_0 is a regular language, because an element $g = (f,z)$ belongs to T_0 iff the word representing it contains the symbols A_0 or B_0, and g belongs to T_1 iff the word representing it contains the symbols A_1 or B_1. Let us consider the Cayley automatic representation of the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ described in Example 4.2.1. Let M_0 be the language of representatives of the elements of T_0 with respect to this representation. By the pumping lemma we obtain the following proposition.

Proposition 4.2.4. *The language M_0 is not regular.*

Proof: Suppose that the language M_0 is accepted some finite automaton $\mathcal{M}$. Let us consider an element $g = (f, 0)$ such that $f(j) = 1$ for $j \in [-2t, -t-1]$, where t is greater than the number of states of $\mathcal{M}$, and $f(j) = 0$ for $j \notin [-2t, -t-1]$. Let $u = w\#1\ldots10\ldots0C_0$ be the word given by (4.18) representing $(f, 0)$. The word $u \in M_0$ because $f(0) = 0$. By the pumping lemma, for every nonnegative integer n there exists the word $u_n = w\#1\ldots10\ldots0C_0 \in M_0$ which satisfies the following properties. The word u_n has exactly $t + k(n-1)$ consecutive symbols 1 for some positive integer k (where k is less or equal than the number of states of $\mathcal{M}$), t consecutive symbols 0, and the symbol C_0 in the end. For a given nonnegative integer n we denote by (f_n, z_n) the element of $\mathbb{Z}_2 \wr \mathbb{Z}$ which correspond to the word u_n. If $n = 1$, then $u_n = u$ and $(f_n, z_n) = (f, 0)$. It is easy to see that for a sufficiently large n, $f_n(0) = 1$ and, therefore, $u_n \notin M_0$. Therefore, we get a contradiction. $\square$

4.3 Wreath products $G \wr \mathbb{Z}$

In this section we consider wreath products $G \wr \mathbb{Z}$. For a given group G the elements of the group $G \wr \mathbb{Z}$ are all pairs of the form (f, z), where $z \in \mathbb{Z}$ and $f : \mathbb{Z} \to G$ is a function having finite support. The multiplication in the group $G \wr \mathbb{Z}$ is given by the rule $(f_1, z_1)(f_2, z_2) = (f_1 f_2^{-z_1}, z_1 + z_2)$, where $f_2^{-z_1}(z) = f_2(z - z_1)$ for all $z \in \mathbb{Z}$. Suppose that $G = \langle g_1, \ldots, g_n; R_1, \ldots, R_m \rangle$. Then the wreath product $G \wr \mathbb{Z}$ can be obtained as the group $G \wr \mathbb{Z} = \langle g_1, \ldots, g_n, t; R_1, \ldots, R_m, [t^i g_k t^{-i}, t^j g_k t^{-j}] \rangle$.

It is easy to see that all results of Section 4.2 can be easily modified for a wreath product $G \wr \mathbb{Z}$ if G is a finite group. For an infinite Cayley automatic group G we obtain the following theorem.

Theorem 4.3.1. *For a Cayley automatic group G the group $G \wr \mathbb{Z}$ is Cayley automatic.*

Proof: Let $S = \{g_1, \ldots, g_n\}$ be a set generating the group G. Since the group G is Cayley automatic, there exists a bijection between a regular language L and the group G such that the directed edges of the Cayley graph $\Gamma(G, S)$ labeled by $g_1, \ldots, g_n$ are recognized by synchronous two–tape finite automata $\mathcal{M}_{g_1}, \ldots, \mathcal{M}_{g_n}$, respectively. Without loss of generality we assume that $L \subset \{0, 1\}^*$ (see [29]). Furthermore, we suppose that the empty word $\epsilon \notin L$.

We construct a Cayley automatic representation of the wreath product $G \wr \mathbb{Z}$ as follows. Let us introduce two counterparts of the symbols 0 and 1: bold **0** and bold **1**, in order to specify the beginning of the words of the language L. In order to specify the position of the origin $z = 0$, we use A_0 and A_1 as the first symbol if the word of L representing the element $f(0) \in G$ begins with 0 and 1, respectively. Similarly, in order to specify the position of the lamplighter z, we use C_0 and C_1 as the first symbol if the word of L representing the element $f(z) \in G$ begins with 0 and 1, respectively. The symbols B_0 and B_1 are used in the case when the lamplighter is at the origin $z = 0$.

We show now two simple examples. Let $(f, 1)$ be an element of $G \wr \mathbb{Z}$ such that $f(j) = e$ for $j \notin [-1, 2]$ and let $f(-1) \neq e$, $f(0)$, $f(1)$, $f(2) \neq e$ be represented by the words 011, 1001, 01 and 111, respectively. Then the element $(f, 1)$ should be represented as follows:

$$\mathbf{0}11A_1001C_01\mathbf{1}11.$$

Let $(f, 0)$ be an element of $G \wr \mathbb{Z}$ such that $f(j) = e$ for $j \notin [-1, 1]$ and let $f(-1) \neq e$, $f(0)$ and $f(1) \neq e$ be represented by the words 111, 000 and 01, respectively. Then the element $(f, 0)$ should be represented as follows:

$$\mathbf{1}11B_000\mathbf{0}1.$$

With a slight abuse of notation we will use $f(j)$ to denote the following modification of the word of L representing the element $f(j) \in G$: the first symbol of the corresponding word should be changed to the bold one. We denote by $A_{f(0)}$ and $B_{f(0)}$ the following modifications of the words representing the element $f(0) \in G$: the first symbol of the corresponding word should be changed to A_0 or A_1, and B_0 or B_1 respectively. We denote by $C_{f(z)}$ the following modification of the word representing the element $f(z) \in G$: the first symbol of the corresponding word should be changed to C_0 or C_1.

Let (f, z) be an element of $G \wr \mathbb{Z}$. Let m be the smallest $i \in \mathbb{Z}$ such that $f(i) \neq e$; if $f(i) = e$ for all $i \in \mathbb{Z}$, then put $m = 0$. Put $\ell = \min\{m, z, 0\}$. Let n be the largest j such that $f(j) \neq e$; if $f(j) = e$ for all $j \in \mathbb{Z}$, then put $n = 0$. Put $r = \max\{n, z, 0\}$. If $z \neq 0$, we represent the element (f, z) as follows:

$$f(\ell) \ldots A_{f(0)} \ldots C_{f(z)} \ldots f(r). \tag{4.19}$$

In the case $z = 0$, we represent the element (f, z) as follows:

$$f(\ell) \ldots B_{f(0)} \ldots f(r). \tag{4.20}$$

For a given element $g = (f, z) \in G \wr \mathbb{Z}$ the words representing g and gt written one under another are either of the form (4.4) or (4.5). Therefore, the binary relation $\langle g, gt \rangle$ is recognized by a two–tape synchronous finite automaton. The words representing g and gg_j for some $j = 1, \dots, n$ written one under another are either of the form:

$$\begin{array}{cccccccc} f(\ell) & \dots & A_{f(0)} & \dots & f(z-1) & C_{f(z)} & f(z+1) & \dots & f(r) \\ f(\ell) & \dots & A_{f(0)} & \dots & f(z-1) & C_{f(z)g_j} & f(z+1) & \dots & f(r) \end{array}, \quad (4.21)$$

or of the form:

$$\begin{array}{ccccccc} f(\ell) & \dots & f(-1) & B_{f(0)} & f(1) & \dots & f(r) \\ f(\ell) & \dots & f(-1) & B_{f(0)g_j} & f(1) & \dots & f(r) \end{array}. \quad (4.22)$$

Since the group G is Cayley automatic, the differences of lengths of words $||C_{f(z)}| - |C_{f(z)g_j}||$ and $||B_{f(0)}| - |B_{f(0)g_j}||$ are bounded by a constant from above. By (4.21) and (4.22) this implies that the relation $\langle g, gg_j \rangle$ is recognized by a two–tape synchronous finite automaton. □

Suppose now that G is a Cayley biautomatic group. Then we obtain the following theorem.

Theorem 4.3.2. *For a Cayley biautomatic group G the wreath product $G \wr \mathbb{Z}$ is Cayley biautomatic.*

Proof: Let us show that for the Cayley automatic representation of $G \wr \mathbb{Z}$ constructed in Theorem 4.3.1 the left multiplications by $g_1, \dots, g_n$ and t are recognized by two–tape synchronous finite automata.

Let $g = (f, z)$ be an element of $G \wr \mathbb{Z}$. It is represented either by the word of the form (4.19) or (4.20). The element tg equals $(h_0, 1)(f, z) = (h_0 f^{-1}, z+1) = (f^{-1}, z+1)$, where $f^{-1}(z) = f(z-1)$ and $h_0 : \mathbb{Z} \to G$ is the function such that $h_0(z) = e$ for all $z \in \mathbb{Z}$. The words representing g and tg written one under another have either the form (4.10) or (4.11). It can be seen that the relation $\langle g, tg \rangle$ is recognized be a two–tape synchronous finite automaton.

Consider now the element $g_j g, j = 1, \ldots, n$. It is equal to $g_j g = (h_j, 0)(f, z) = (h_j f, z)$, where h_j is the function $h_j : \mathbb{Z} \to G$ such that $h_j(0) = g_j$ and $h_j(i) = e$ for all $i \neq 0$. The words representing g and $g_j g$ written one under another have either the form (4.12) or (4.13). Since G is a Cayley biautomatic group, the relation $\langle g, g_j g \rangle$ is recognized by a two–tape synchronous finite automaton. $\square$

Let G be a Cayley automatic group. For the Cayley automatic representation of the group $G \wr \mathbb{Z}$ constructed in Theorem 4.3.1 the length of the word w representing an element $g = (f, z) \in \mathbb{Z}_2 \wr \mathbb{Z}$ satisfies the inequality shown in the following lemma.

Lemma 4.3.1. *Let w be the word representing an element $g = (f, z) \in G \wr \mathbb{Z}$. Then the following inequality holds:*

$$|w| \geqslant \max\{|n - m|, |n|, |m|, |n - z|, |m - z|, |z|\} + 1, \qquad (4.23)$$

where m is the smallest $i \in \mathbb{Z}$ for which $f(i) \neq e$ and $m = 0$ if $f(i) = e$ for all $i \in \mathbb{Z}$, and n is the largest j for which $f(j) \neq e$ and $n = 0$ if $f(j) = e$ for all $j \in \mathbb{Z}$.

Proof: In the proof of Theorem 4.3.1 we assumed that a language L representing the elements of G does not contain the empty word ϵ. Therefore, for every word u representing an element of G the length $|u| \geqslant 1$. From the construction of the Cayley automatic representation of $G \wr \mathbb{Z}$ in Theorem 4.3.1 we obtain the inequality (4.23). $\square$

The analog of Lemma 4.2.2 for a wreath product $G \wr \mathbb{Z}$ is almost straightforward. For a function $f : \mathbb{Z} \to G$ having finite support we denote by $|\mathrm{supp} f|$ the sum $\sum_j |f(j)|$, where $|f(j)|$ is the length of the element $f(j)$ in the group G with respect to the generators $g_1, \dots, g_n$. We denote by $\#\mathrm{supp} f$ the number of elements $j \in \mathbb{Z}$ for which $f(j) \in G$ is nontrivial. Then we obtain the following lemma.

Lemma 4.3.2. *The word length of an element $g = (f, z)$ in the group $G \wr \mathbb{Z}$ with respect to generators $g_1, \dots g_n$ and t is as follows:*

$$|g| = |\mathrm{supp}\, f| + \min\{2\max\{-m, 0\} + \max\{n, 0\} + |z - \max\{n, 0\}|, \\ 2\max\{n, 0\} + \max\{-m, 0\} + |z + \max\{-m, 0\}|\}. \tag{4.24}$$

Also, considering the following three different cases for m and n, we obtain:

- *if $m \leqslant -1$ and $n \geqslant 0$ then $|g| = |\mathrm{supp} f| + \min\{-2m + n + |z - n|, 2n - m + |z - m|\}$,*
- *if $m \geqslant 0$ then $|g| = |\mathrm{supp} f| + n + |z - n|$,*
- *if $n \leqslant -1$ then $|g| = |\mathrm{supp} f| - m + |z - m|$.*

Proof: The proof is the same as in Lemma 4.2.2. $\square$

Let G be a Cayley automatic group. Let us choose some Cayley automatic representation of G, $\varphi : L \to G$. Suppose that the following inequality holds for all $g \in G$:

$$|g| \leqslant C|u| + D, \tag{4.25}$$

where $C > 0$ and $D \geqslant 0$ are some constants, and $|u|$ is the length of the word representing $g \in G$ with respect to the chosen Cayley automatic representation of G, and $|g|$ is the word length of g with respect to some generators $g_1, \dots, g_n$. Let w be the word representing an element $g \in G \wr \mathbb{Z}$ with respect to the Cayley automatic representation of $G \wr \mathbb{Z}$ constructed in Theorem 4.3.1. The analog of Proposition 4.2.1 for the wreath product $G \wr \mathbb{Z}$ is as follows.

Proposition 4.3.1. *The following inequality holds for all $g \in G \wr \mathbb{Z}$:*

$$|w| \leqslant K|g| + K_0, \tag{4.26}$$

where K and K_0 are some constants which depend only on the chosen Cayley automatic representation of G. In addition, the following inequality is satisfied for all $g \in G \wr \mathbb{Z}$:

$$|g| \leqslant M|w| - 2, \tag{4.27}$$

where $M = C + D + 2$. Unifying the inequalities (4.26) *and* (4.27) *we obtain:*

$$\frac{1}{K}|w| - \frac{K_0}{K} \leqslant |g| \leqslant M|w| - 2, \tag{4.28}$$

– or, equivalently,

$$\frac{1}{M}|g| + \frac{2}{M} \leqslant |w| \leqslant K|g| + K_0. \tag{4.29}$$

Proof: We first prove the inequality (4.26). Let K_0 be the length of the word which represents the identity $e \in G$ with respect to the chosen Cayley automatic representation of G. Therefore, the inequality holds for $g = e$. For each $j = 1, \dots, n$ put d_j to be the maximum number of the padding symbols

$\Diamond$ in the convolutions $\varphi^{-1}(g) \otimes \varphi^{-1}(gg_j)$; by the pumping lemma, d_j always exists. Put $K = \max\{K_0, d_j \mid j = 1, \ldots, n\}$.

Suppose that the inequality (4.26) holds for some $g \in G \wr \mathbb{Z}$. It follows from the construction of the Cayley automatic representation of $G \wr \mathbb{Z}$ in Theorem 4.3.1 that the words representing gg_j and gg_j^{-1} have the lengths at most $|w| + \max\{d_j \mid j = 1, \ldots, n\}$. The lengths of the words representing gt and gt^{-1} are equal to either $|w|$, $|w| + K_0$ or $|w| - K_0$. This implies that the inequality (4.26) holds for the elements gg_j, gg_j^{-1}, gt and gt^{-1} as well.

Let us prove the inequality (4.27). By (4.25), we obtain that:

$$|\mathrm{supp} f| \leqslant C|w| + D\#\mathrm{supp} f.$$

By Lemma 4.3.1 we obtain that $\#\mathrm{supp} f \leqslant |w|$. Therefore,

$$|\mathrm{supp} f| \leqslant (C + D)|w|. \tag{4.30}$$

As in the proof of Proposition 4.2.1, by the inequality (4.23), we obtain the following upper bound for the second summand of the right–hand side of (4.24):

$$\begin{aligned} 2(|w| - 1) \geqslant \min\{& 2\max\{-m, 0\} + \max\{n, 0\} + |z - \max\{n, 0\}|, \\ & 2\max\{n, 0\} + \max\{-m, 0\} + |z + \max\{-m, 0\}|\}. \end{aligned}$$

Therefore,

$$|g| \leqslant (C + D + 2)|w| - 2.$$

Put $M = C + D + 2$. Then we obtain (4.27). Combining (4.26) and (4.27), we obtain (4.28) and (4.29). $\square$

4.4 The wreath products $\mathbb{Z}_2 \wr F_n$

In this section we consider the wreath products $\mathbb{Z}_2 \wr F_n$, where F_n is the free group of n generators. We will show how to represent the Cayley graphs of the wreath products $\mathbb{Z}_2 \wr F_n$ by deterministic pushdown automata. The idea to use pushdown automata for representing Cayley graphs was motivated by the notion of parallel poly–pushdown groups introduced by Baumslag, Shapiro and Short [39] and the notion of C–graph automatic groups introduced by Elder and Taback [35].

Recall first the definition of a pushdown automaton (see, e.g., [22]).

Definition 4.4.1. *A pushdown automaton is a tuple* $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$*, where*

- Q *is a finite set of states;*
- Σ *is a finite input alphabet;*
- Γ *is a finite stack alphabet;*
- δ *is a map from a subset of* $Q \times \Sigma_\epsilon \times \Gamma$ *to* $Q \times \Gamma^*$*;*
- $q_0 \in Q$ *is the start state;*
- $Z_0 \in \Gamma$ *is the start symbol;*
- $F \subset Q$ *is the set of accepting states.*

The pushdown automaton $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ starts reading an input word $w \in \Sigma^*$ being in the initial state q_0 and the single symbol Z_0 in the stack. At any point in a computation, the instantaneous description is a triple

(q, w, γ), where $q \in Q$ is the current state, $w \in \Sigma^*$ is the remaining input, and $\gamma \in \Gamma^*$ is the stack contents. We say that the pushdown automaton P makes a transition $(q_1, aw, X\gamma) \xrightarrow{P} (q_2, w, \beta\gamma)$, where $q_1, q_2 \in Q, a \in \Sigma_\epsilon, w \in \Sigma^*, X \in \Gamma$ and $\beta, \gamma \in \Gamma^*$ if $\delta(q_1, a, X) = (q_2, \beta)$. In other words, the pushdown automaton P reads off the symbol $a \in \Sigma$ (or, nothing, if $a = \epsilon$), pops the symbol X from the stack, and pushes the word β into the stack.

We say that a word w is accepted by a pushdown automaton P if there is a finite sequence of transitions $(q_0, w, Z_0) \xrightarrow{P} \ldots \xrightarrow{P} (q, \epsilon, \gamma)$ such that $q \in F$. We denote by $L(P)$ the language of the words which are accepted by P. We say that a language L is context–free if $L = L(P)$ for some pushdown automaton P. Recall briefly some properties of context–free languages.

Proposition 4.4.1. *(see, e.g., [40, Theorem 5.2]) Context–free languages are closed under homomorphism, inverse homomorphism, and intersection with regular languages. A homomorphism here means a homomorphic mapping from one finitely generated free monoid to another.*

Definition 4.4.2. *A pushdown automaton* $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ *is deterministic if:*

- $\delta(q, a, X)$ *has at most one element for any* $q \in Q$, $a \in \Sigma$ *or* $a = \epsilon$, $X \in \Gamma$*;*
- *if* $\delta(q, a, X)$ *is nonempty for some* $a \in \Sigma$ *then* $\delta(q, \epsilon, X)$ *must be empty.*

Definition 4.4.3. *A language* $L \subset \Sigma^*$ *is said to be in the class* $\mathcal{P}_1$ *if it is recognizable by a deterministic pushdown automaton.*

Definition 4.4.4. *(see [39, Definition]) A language* $L \subset \Sigma^*$ *is said to be in the class* $\mathcal{P}$ *(the class of parallel poly–pushdown languages) if there are* *finitely many languages $L_i, i = 1, \dots, k$ recognizable by deterministic push-down automata such that $L = \bigcap_{i=1}^{k} L_i$.*

Proposition 4.4.2. *([39, Proposition 3.1]) Suppose that L and M are in $\mathcal{P}$, and that R is regular then the languages $L \cap M$, $L \cup R$ and $L \setminus R$ are in $\mathcal{P}$.*

Definition 4.4.5. *A language $L \subset \Sigma^*$ is said to be in the class $\mathcal{P}^c$ (the class of poly–context–free languages) if there are finitely many languages $L_i, i = 1, \dots, k$ recognizable by pushdown automata such that $L = \bigcap_{i=1}^{k} L_i$.*

The analog of Proposition 4.4.2 clearly holds for languages in the class $\mathcal{P}^c$. It is clear that $\mathcal{P}_1 \subset \mathcal{P} \subset \mathcal{P}^c$. Let C be a class of languages, for example, $\mathcal{P}_1$, $\mathcal{P}$ or $\mathcal{P}^c$.

Definition 4.4.6. *(see [35, Definition 8]) We say that a finitely generated group G is C–Cayley automatic if there exists a bijection $\psi : L \to G$ between a language L in the class C and the group G such that for some set of group generators $S = \{g_1, \dots, g_n\}$ for each $g_j \in S$ the language $L_j = \{w_1 \otimes w_2 | w_1, w_2 \in L, \psi(w_1) g_j = \psi(w_2)\}$ is in C.*

Proposition 4.4.3. *If a group G is $\mathcal{P}^c$–Cayley automatic with respect to some finite generating set $S \subset G$ then it is $\mathcal{P}^c$–Cayley automatic with respect to all finite generating sets of G.*

Proof: By Proposition 4.4.1, the class $\mathcal{P}^c$ is closed under homomorphism and inverse homomorphism. Similarly to Proposition 4.4.2, the class $\mathcal{P}^c$, by design, is closed under intersection. These two facts imply that the definition of $\mathcal{P}^c$–Cayley automatic groups does not depend on the choice of generators (see, e.g., [35, Lemma 14] for the proof). □

Remark 4.4.1. *In the original definition of parallel poly–pushdown groups [39, Definition] the alphabet* Σ *of a language* L *is identified with a finite set of semigroup generators of a group* G, $\psi : L \to G$ *is obtained from the semigroup homomorphism* $\pi : \Sigma^* \to G$, *i.e.,* $\psi(w) = \pi(w)$, *and for each* $\sigma \in \Sigma \subset G$ *the set* $\{(u, v)|u, v \in L, \pi(v) = \pi(u)\sigma\}$ *should be the intersection of finite number of languages recognizable by two–tape deterministic asynchronous pushdown automata.*

Remark 4.4.2. *According to the terminology adopted in [35], the group* G *in Definition 4.4.6 should be called* C*–graph automatic groups. In particular,* $\mathcal{P}^c$*–Cayley automatic groups should be called poly–context–free–graph automatic groups.*

Remark 4.4.3. *In [35] the authors, consider* $\mathcal{DCS}$*–Cayley automatic groups, where* $\mathcal{DCS}$ *is the class of languages recognizable by deterministic linear bound automata. It can be seen that every* $\mathcal{P}$*–Cayley automatic group is* $\mathcal{DCS}$*–Cayley automatic.*

Remark 4.4.4. *The definition of* $\mathcal{P}^c$*–Cayley automatic groups does not depend on the choice of generators. However, the definitions of* $\mathcal{P}_1$ *and* $\mathcal{P}$*–Cayley automatic groups depend on the choice of generators.*

We denote by a and b the generators of the free group $F_2 = \langle a, b \rangle$, and by h the nontrivial element of $\mathbb{Z}_2$. We consider the Cayley graph of the wreath product $\mathbb{Z}_2 \wr F_2$ with respect to the generators a, b and h. Recall that an element of $\mathbb{Z}_2 \wr F_2$ is a pair (f, z), where f is a function $f : F_2 \to \mathbb{Z}_2$ that has finite support and $z \in F_2$ is the position of the lamplighter. We obtain the following theorem.

Theorem 4.4.1. *The group $\mathbb{Z}_2 \wr F_2$ is $\mathcal{P}_1$–Cayley automatic.*

Proof: In order to construct a $\mathcal{P}_1$–Cayley automatic representation of $\mathbb{Z}_2 \wr F_2$ we extend the Cayley automatic representation of $\mathbb{Z}_2 \wr \mathbb{Z}$ obtained in Theorem 4.2.1. In the Cayley automatic representation of $\mathbb{Z}_2 \wr \mathbb{Z}$ we use the symbols $0, 1, A_0, A_1, B_0, B_1, C_0, C_1$. In the $\mathcal{P}_1$–Cayley automatic representation of $\mathbb{Z}_2 \wr F_2$ we use the brackets (,) and [,]. Along with the symbols 0 and 1 we use D_0, E_0 and D_1, E_1. Along with the symbols $A_0, A_1, B_0, B_1, C_0, C_1$ we use $D_0^A, D_1^A, D_0^B, D_1^B, D_0^C, D_1^C, E_0^C, E_1^C$. The symbols A_0, A_1, D_0^A, D_1^A are used to show the position of the origin $e \in F_2$. The symbols $C_0, C_1, D_0^C, D_1^C, E_0^C, E_1^C$ are used to show the position of the lamplighter $z \in F_2$. The symbols B_0, B_1, D_0^B, D_1^B are used if the lamplighter is at the origin. See also the meaning of the symbols $A_0, A_1, B_0, B_1, C_0, C_1$ in Theorem 4.2.1.

We say that a symbol is an A–, B–, C–, D– and E–symbol if it belongs to the set $\{A_0, A_1, D_0^A, D_1^A\}$, $\{B_0, B_1, D_0^B, D_1^B\}$, $\{C_0, C_1, D_0^C, D_1^C, E_0^C, E_1^C\}$, $\{D_0, D_1, D_0^A, D_1^A, D_0^B, D_1^B, D_0^C, D_1^C\}$ and $\{E_0, E_1, E_0^C, E_1^C\}$, respectively. We say that a symbol is basic if it belongs to the set $\{0, 1, A_0, A_1, B_0, B_1, C_0, C_1\}$.

For a given $s \in F_2$, denote by $r(s) \in \{a, a^{-1}, b, b^{-1}\}^*$ the reduced word representing s. We denote by F_a the set of all group elements $s \in F_2$ for which $r(s) = aw$ or $r(s) = a^{-1}w$, $w \in \{a, a^{-1}, b, b^{-1}\}^*$. We denote by F_b the set of all group elements $s \in F_2$ for which $r(s) = bw$ or $r(s) = b^{-1}w$, $w \in \{a, a^{-1}, b, b^{-1}\}^*$. It is clear that $F_2 = F_a \cup F_b \cup \{e\}$. We denote by H the subgroup of $\mathbb{Z}_2 \wr F_2$ generated by a and h. It is clear that H is isomorphic to $\mathbb{Z}_2 \wr \mathbb{Z}$.

For a given $(f, z) \in \mathbb{Z}_2 \wr F_2$, depict it in a way shown Fig. 4.1: a white box means that the value of a function $f : F_2 \to \mathbb{Z}_2$ is $e \in \mathbb{Z}_2$, a black box means that it is $h \in \mathbb{Z}_2$, a black disk specifies the position of the lamplighter z and tells us that the value of f is h. Let us consider the horizontal line going through the identity $e \in F_2$. For a vertex $s \in F_2$ on this line, put $V_s = \{p \in F_b \,|\, f(sp) = h \vee z = sp\}$. Scan this line from the left to the right. If $V_s = \varnothing$, then write the corresponding basic symbol (see Theorem 4.2.1). If $V_s \neq \varnothing$, then write the corresponding D–symbol. For the element in Fig. 4.1 (left) we get $11D_0^A D_0 1$. For the element in Fig. 4.1 (right) we get $D_0 A_1 D_0 1$. If $(f, z) \in H$, then we obtain the same representative as in Theorem 4.2.1. If $(f, z) \notin H$, then D–symbols occur. In this case we continue as follows.

Take any occurrence of D–symbol. This occurrence corresponds to some vertex $s \in F_2$. Let us consider a vertical line going through s. For a vertex $t \in F_2$, $t \neq s$ on this line, put $H_t = \{p \in F_a \,|\, f(tp) = h \vee z = tp\}$. Insert the brackets (and) around the occurrence of a D–symbol. Scan, omitting s, this line from the bottom to the top. If $H_t = \varnothing$, then write the corresponding basic symbol inside the brackets. If $H_t \neq \varnothing$, then write the corresponding E–symbol inside the brackets. Do it for every occurrence of a D–symbol. For the element in Fig. 4.1 (left) we get $11(1E_0 D_0^A E_0)(E_0 D_0 E_1)1$. For the element in Fig. 4.1 (right) we get $(D_0 1)A_1(E_0 D_0 E_1^C)1$. If no E–symbols occur then we stop. If E–symbols occur, we insert the brackets [and] around each occurrence and repeat the step above for horizontal lines. We continue this procedure until no new D– or E–symbols occur. After the procedure is finished, the result is the representative w of $g = (f, z)$.

Let us consider two elements of $\mathbb{Z}_2 \wr F_2$ in Fig. 4.1. For the element in Fig. 4.1 (left) the procedure of constructing the representative is $11D_0^A D_0 1 \to$

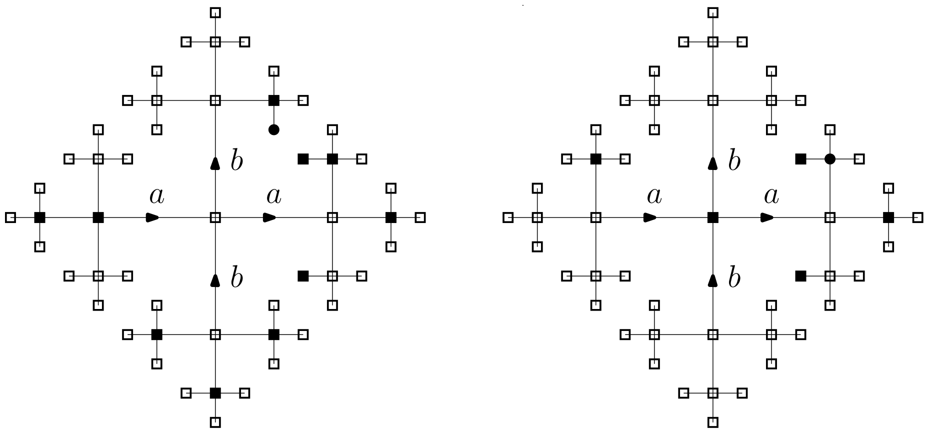

Figure 4.1: Two elements of $\mathbb{Z}_2 \wr F_2$

$11(1E_0D_0^AE_0)(E_0D_0E_1)1 \rightarrow 11(1[1E_01]D_0^A[E_0D_1])([1E_0]D_0[1E_1])1 \rightarrow 11(1[1E_01]D_0^A[E_0(C_1D_1)])([1E_0]D_0[1E_1])1$. For the element in Fig. 4.1 (right) it is $D_0A_1D_01 \rightarrow (D_01)A_1(E_0D_0E_1^C)1 \rightarrow (D_01)A_1([1E_0]D_0[1E_1^C])1$.

Put $\Sigma = \{0, 1, D_0, D_1, E_0, E_1, (,), [,], A_0, A_1, B_0, B_1, C_0, C_1, D_0^A, D_1^A, D_0^B, D_1^B, D_0^C, D_1^C, E_0^C, E_1^C\}$. Let $L \subset \Sigma^*$ be the language of representatives w of all elements $g \in \mathbb{Z}_2 \wr F_2$. It can be seen that the language L consists of the words satisfying the following properties.

- The configuration of brackets (,), [,] is balanced and, moreover, generated by the context–free grammar $S \rightarrow SS \,|\, (T) \,|\, \varepsilon$, $T \rightarrow TT \,|\, [S] \,|\, \varepsilon$ with the axiom S.
- Each pair of matched brackets (and) is associated with a D–symbol which is placed inside these brackets, but not inside any other pair of matched brackets between them. That is, the configuration of the subword between any two matched brackets (and) is $(p[\ldots]q\ldots r[\ldots]s\,\sigma\,t[\ldots]u\ldots v[\ldots]w)$, where σ is a D–symbol and

$p, q, r, s, t, u, v, w \in \{0, 1, C_0, C_1\}^*$.

- The D–symbols $D_0^A, D_1^A, D_0^B, D_1^B$ are allowed to be associated only with a matched pair of brackets (and) of the first level.

- Each pair of matched brackets [and] is associated with an E–symbol which is placed inside these brackets but not inside any other pair of matched brackets between them. That is, the configuration of the subword between any two matched brackets [and] is $[p(\dots)q \dots r(\dots)s\,\sigma\,t(\dots)u \dots v(\dots)w]$, where σ is an E–symbol and $p, q, r, s, t, u, v, w \in \{0, 1, C_0, C_1\}^*$.

- Each pair of matched brackets is separated by at least two symbols.

- The subwords (0, 0), [0 and 0] are not allowed.

- The symbol 0 is not allowed to be the first or the last one of a word.

- Among the symbols A_0,A_1,B_0,B_1,C_0,C_1,D_0^A,D_1^A,D_0^B,D_1^B,D_0^C,D_1^C,E_0^C,E_1^C each word of L contains either exactly one occurrence of a B–symbol and no A–symbols and C–symbols, or exactly one occurrence of an A–symbol and of a C–symbol, and no B–symbols.

It can be seen that L is recognized by a deterministic pushdown automaton. We denote by $\psi : L \to \mathbb{Z}_2 \wr F_2$ the representation of the group $\mathbb{Z}_2 \wr F_2$ described above. The right multiplication by h either interchanges C_0 and C_1, D_0^B and D_1^B, D_0^C and D_1^C, or E_0^C and E_1^C. Therefore, the language $L_h = \{u \otimes v | u, v \in L, \psi(v) = \psi(u)h\}$ is recognized by a deterministic pushdown automaton. The right multiplication by a (or, b) moves the lamplighter by one step to the right (or, up). It is can be verified that each of the languages $L_a = \{u \otimes v | u, v \in L, \psi(v) = \psi(u)a\}$ and $L_b = \{u \otimes v | u, v \in L, \psi(v) = \psi(u)b\}$ is recognized by a deterministic push-down automaton. Therefore, the group $\mathbb{Z}_2 \wr F_2$ is $\mathcal{P}_1$–Cayley automatic. □

Remark 4.4.5. *It is proved [39, Theorem 5.4] that the set of parallel poly–pushdown groups in the sense of [39, Definition] is closed under wreath products. We remark that the proof of [39, Theorem 5.4] does not work for the case of $\mathcal{P}_1$–Cayley automatic groups.*

We denote by w the word representing an element $g = (f, z) \in \mathbb{Z}_2 \wr F_2$ with respect to the $\mathcal{P}_1$–Cayley automatic representation of the group $\mathbb{Z}_2 \wr F_2$ constructed in Theorem 4.4.1. We obtain the following proposition.

Proposition 4.4.4. *The following inequality holds for all $g \in \mathbb{Z}_2 \wr F_2$:*

$$|w| \leqslant 3|g| + 1. \tag{4.31}$$

In addition, the following inequality holds for all $g \in \mathbb{Z}_2 \wr F_2$:

$$|g| \leqslant 3|w| - 2. \tag{4.32}$$

Both inequalities (4.31) *and* (4.32) *imply that for all $g \in \mathbb{Z}_2 \wr F_2$:*

$$\frac{1}{3}|w| - \frac{1}{3} \leqslant |g| \leqslant 3|w| - 2, \tag{4.33}$$

or, equivalently,

$$\frac{1}{3}|g| + \frac{2}{3} \leqslant |w| \leqslant 3|g| + 1. \tag{4.34}$$

Furthermore, the equalities $|w| = 3|g| + 1$ and $|g| = 3|w| - 2$ are achieved for an infinite number of elements $g \in \mathbb{Z}_2 \wr F_2$.

Proof: We first prove the inequality (4.31). The representative of $e \in \mathbb{Z}_2 \wr F_2$ is the word B_0 of length 1, so the inequality holds for $g = e$. For every $g \in \mathbb{Z}_2 \wr F_2$ the lengths of the representatives for g and gh are the same. It can be seen that for every $g \in \mathbb{Z}_2 \wr F_2$ the lengths of the representatives for g and ga (or gb) differ by at most 3. For example, the representative for hb is $(D_1^A C_0)$ and the representative for hba is $(D_1^A [E_0 C_0])$. Therefore, the inequality (4.31) holds for all $g \in \mathbb{Z}_2 \wr F_2$.

Let us prove the inequality (4.32). Consider the subgroup $\mathbb{Z}_2 \wr \mathbb{Z} \leqslant \mathbb{Z}_2 \wr F_2$. For the elements of this subgroup the representatives are the same as in Theorem 4.2.1. Therefore, by Proposition 4.2.1, we obtain that for the elements of the subgroup $\mathbb{Z}_2 \wr \mathbb{Z} \leqslant \mathbb{Z}_2 \wr F_2$ the inequality (4.32) is satisfied. For a given element $g \in \mathbb{Z}_2 \wr F_2$ the representative w has the form $w = v_0(w_1)v_1(w_2)v_2 \ldots v_{n-1}(w_n)v_n$, where the words $v_0, v_1, \ldots, v_n$ do not contain brackets, and every word $w_i, i = 1, \ldots, n$ has the form $w_i = v'_0[w'_1]v'_1[w'_2]v'_2 \ldots v'_{m_i-1}[w'_{m_i}]v'_{m_i}$, where the words $v'_0, v'_1, \ldots, v'_{m_i}$ do not contain brackets, and etc. Therefore, by induction, we obtain that $|g| \leqslant \sum_{i=1}^{n}(3|w_i| - 2) + 3(|v_0 \ldots v_n| + n) - 2 \leqslant 3|w| - 2$ which proves the inequality (4.32). The inequalities (4.33) and (4.34) are obtained from (4.31) and (4.32).

It is easy to construct an infinite sequence of elements of $\mathbb{Z}_2 \wr F_2$ for which the identity $|w| = 3|g| + 1$ holds. Consider the infinite sequence of elements $e, b, ba, bab, baba$ and etc. For these elements the representatives are $A_0, (D_0^A C_0), (D_0^A [E_0 C_0]), (D_0^A [E_0(D_0 C_0)]), (D_0^A [E_0(D_0[E_0 C_0])])$ and etc. Therefore, for each element g of this sequence the identity $|w| = 3|g| + 1$ holds. An infinite sequence of elements of $\mathbb{Z}_2 \wr \mathbb{Z} \leqslant \mathbb{Z}_2 \wr F_2$, for which the identity $|g| = 3|w| - 2$ holds, is constructed in Proposition 4.2.1. $\square$

Another property of the $\mathcal{P}_1$–Cayley automatic representation constructed in Theorem 4.4.1 is as follows.

Proposition 4.4.5. *The set of the representatives of the elements of the normal subgroup $\mathbb{Z}_2^{(F_2)} \trianglelefteq \mathbb{Z}_2 \wr F_2$ is recognized by a deterministic pushdown automaton. The set of the representatives of the elements of the subgroup $F_2 \leqslant \mathbb{Z}_2 \wr F_2$ is recognized by a deterministic pushdown automaton.*

Proof: The representatives of the elements of the normal subgroup $\mathbb{Z}_2^{(F_2)} \trianglelefteq \mathbb{Z}_2 \wr F_2$ are all representatives which contain a B–symbol. Therefore, there exists a finite automaton which for a given representative $w \in L$ recognizes whether it represents an element of the normal subgroup $\mathbb{Z}_2^{(F_2)} \trianglelefteq \mathbb{Z}_2 \wr F_2$. Since the class $\mathcal{P}_1$ is closed under intersection with a regular language, we obtain that the set of representatives of the elements of the normal subgroup $\mathbb{Z}_2^{(F_2)} \trianglelefteq \mathbb{Z}_2 \wr F_2$ is recognized by a deterministic pushdown automaton. The second statement of the proposition can be shown in a similar way. $\square$

Consider now the subgroups H_1, H_2 and H_3 of the group $\mathbb{Z}_2 \wr F_2$ which are generated by the generators a and h, b and h, and ab and h, respectively. It is easy to see that the groups H_1, H_2 and H_3 are isomorphic to the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$. We obtain the following proposition.

Proposition 4.4.6. *The sets of representatives of the elements of the subgroup $H_1 \leqslant \mathbb{Z}_2 \wr F_2$ and the subgroup $H_2 \leqslant \mathbb{Z}_2 \wr F_2$ are regular languages. The set of representatives of the elements of the subgroup $H_3 \leqslant \mathbb{Z}_2 \wr F_2$ is not a regular language, but it is recognized by a deterministic pushdown automaton.*

Proof: The proposition directly follows from the construction of the $\mathcal{P}_1$–Cayley automatic representation of $\mathbb{Z}_2 \wr F_2$ in Theorem 4.4.1. $\square$

The construction of the $\mathcal{P}_1$–Cayley automatic representation of the group $\mathbb{Z}_2 \wr F_2$ in Theorem 4.4.1 can be generalized for the groups $\mathbb{Z}_2 \wr F_n$, $n \geqslant 3$. We obtain the following theorem.

Theorem 4.4.2. *For a given integer $n \geqslant 3$, the group $\mathbb{Z}_2 \wr F_n$ is $\mathcal{P}_1$–Cayley automatic.*

Proof: Let us show how to construct a $\mathcal{P}_1$–Cayley automatic representation of $\mathbb{Z}_2 \wr F_3$. In a similar way one can obtain a $\mathcal{P}_1$–Cayley automatic representation of $\mathbb{Z}_2 \wr F_n$ for any integer $n > 3$. We denote by a, b and c the generators of the free group $F_3 = \langle a, b, c \rangle$, and by h the nontrivial element of $\mathbb{Z}_2$. We consider the Cayley graph of the group $\mathbb{Z}_2 \wr F_3$ with respect to the generators a, b, c and h. Recall that an element of $\mathbb{Z}_2 \wr F_3$ is a pair (f, z), where f is a function $f : F_3 \to \mathbb{Z}_2$ that has finite support and $z \in F_3$ is the position of the lamplighter.

The $\mathcal{P}_1$–Cayley automatic representation of $\mathbb{Z}_2 \wr F_3$ to be described is a modification of that of $\mathbb{Z}_2 \wr F_2$ constructed in Theorem 4.4.1. We recall that in the $\mathcal{P}_1$–Cayley automatic representation of the group $\mathbb{Z}_2 \wr F_2$ constructed in Theorem 4.4.1 the symbols $0, 1, A_0, A_1, B_0, B_1, C_0, C_1$, the brackets (,) and [,], the symbols D_0, D_1, E_0, E_1, and the symbols $D_0^A, D_1^A, D_0^B, D_1^B, D_0^C, D_1^C, E_0^C, E_1^C$ are used. In this theorem we call these symbols basic. For constructing the $\mathcal{P}_1$–Cayley automatic representation of $\mathbb{Z}_2 \wr F_3$ we add the symbols $F_0, F_1, F_{D_0}, F_{D_1}, F_{E_0}, F_{E_1}$, $G_0, G_1, G_{D_0}, G_{D_1}$, the brackets $\langle, \rangle$ and $\{, \}$, and the symbols $F_0^A, F_1^A, F_{D_0}^A, F_{D_1}^A, F_0^B, F_1^B, F_{D_0}^B, F_{D_1}^B$, $F_0^C, F_1^C, F_{D_0}^C, F_{D_1}^C, F_{E_0}^C, F_{E_1}^C$, $G_0^C, G_1^C, G_{D_0}^C, G_{D_1}^C$, which are used to show the position of the origin $e \in F_3$ and the position of the lamplighter $z \in F_3$. We say that a symbol is a F– and G–symbol if it belongs to the set $\{F_0, F_1, F_{D_0}, F_{D_1}, F_{E_0}, F_{E_1}, F_0^A, F_1^A, F_{D_0}^A, F_{D_1}^A, F_0^B, F_1^B, F_{D_0}^B, F_{D_1}^B, F_0^C, F_1^C, F_{D_0}^C, F_{D_1}^C, F_{E_0}^C, F_{E_1}^C\}$ and $\{G_0, G_1, G_{D_0}, G_{D_1}, G_0^C, G_1^C, G_{D_0}^C, G_{D_1}^C\}$, respectively.

For a given $s \in F_3$, denote by $r(s) \in \{a, a^{-1}, b, b^{-1}, c, c^{-1}\}^*$ the reduced word representing s. We denote by F_{ab} the set of all group elements $s \in F_3$ for which $r(s) = aw$, $r(s) = a^{-1}w$, $r(s) = bw$ or $r(s) = b^{-1}w$, where $w \in \{a, a^{-1}, b, b^{-1}, c, c^{-1}\}^*$. We denote by F_c the set of all group elements $s \in F_3$ for which $r(s) = cw$ or $r(s) = c^{-1}w$, where $w \in \{a, a^{-1}, b, b^{-1}, c, c^{-1}\}^*$. It is clear that $F_3 = F_{ab} \cup F_c \cup \{e\}$. We denote by H the subgroup of $\mathbb{Z}_2 \wr F_3$ generated by a, b and h. It is clear that H is isomorphic to $\mathbb{Z}_2 \wr F_2$.

For a given $(f, z) \in \mathbb{Z}_2 \wr F_3$, depict it in a way shown Fig. 4.2: a white box means that the value of a function $f : F_3 \to \mathbb{Z}_2$ is e, a black box means that it is h, a black disk specifies the position of the lamplighter and tells us that the value of f is h. Let us consider the horizontal plane going through the identity $e \in F_3$. For a vertex $s \in F_3$ on this plane, put $V_s = \{p \in F_c \,|\, f(sp) = h \vee z = sp\}$. Scan this plane as it is described in Theorem 4.4.1. If $V_s = \varnothing$, then write the corresponding basic symbol. If $V_s \neq \varnothing$, then write the corresponding F–symbol. For the element in Fig. 4.2 (left) we get $1([1E_0]F_{D_0}^A F_0)1$. For the element in Fig. 4.2 (right) we get $F_0([C_1E_0]F_{D_0}^A[E_0F_0])1$. If $(f, z) \in H$, then we obtain the same representative as in Theorem 4.4.1. If $(f, z) \notin H$, then F–symbols occur. In this case we continue as follows.

Take any occurrence of F–symbol. This occurrence corresponds to some vertex $s \in F_3$. Let us consider a vertical line going through s. For a vertex

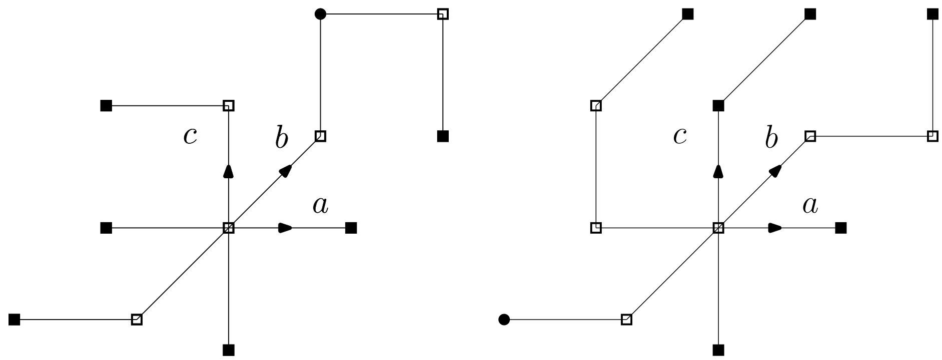

Figure 4.2: Two elements of $\mathbb{Z}_2 \wr F_3$

$t \in F_3$, $t \neq s$ on this line, put $H_t = \{p \in F_{ab} \mid f(tp) = h \vee z = tp\}$. Insert the brackets $\langle$ and $\rangle$ around the occurrence of a F–symbol. Scan, omitting s, this line from the bottom to the top. If $H_t = \varnothing$, then write the corresponding basic symbol inside the brackets. If $H_t \neq \varnothing$, then write the corresponding G–symbol inside the brackets. Do it for every occurrence of a F–symbol. For the element in Fig. 4.1 (left) we get $1([1E_0]\langle 1F^A_{D_0}G_0\rangle\langle F_0G^C_1\rangle)1$. For the element in Fig. 4.1 (right) we get $\langle F_0G_0\rangle([C_1E_0]\langle 1F^A_{D_0}G_1\rangle[E_0\langle F_01\rangle])1$. If no G–symbols occur then we stop. If G–symbols occur, we insert the brackets $\{$ and $\}$ around each occurrence and repeat the step above for a horizontal plane. We continue this procedure until no new F– or G–symbols occur. After the procedure is finished, the result is the representative w of $g = (f, z)$.

Let us consider two elements of $\mathbb{Z}_2 \wr F_3$ in Fig. 4.2. For the element in Fig.4.2 (left) the procedure of constructing the representative is $1([1E_0]F^A_{D_0}F_0)1 \to 1([1E_0]\langle 1F^A_{D_0}G_0\rangle\langle F_0G^C_1\rangle)1 \to 1([1E_0]\langle 1F^A_{D_0}\{1G_0\}\rangle\langle F_0\{G^C_1F_0\}\rangle)1 \to 1([1E_0]\langle 1F^A_{D_0}\{1G_0\}\rangle\langle F_0\{G^C_1\langle 1F_0\rangle\}\rangle)1$. For the element in Fig.4.2 (right) it is $F_0([C_1E_0]F^A_{D_0}[E_0F_0])1 \to \langle F_0G_0\rangle([C_1E_0]\langle 1F^A_{D_0}G_1\rangle[E_0\langle F_01\rangle])1 \to \langle F_0\{(G_{D_0}1)\}\rangle([C_1E_0]\langle 1F^A_{D_0}\{(G_{D_1}1)\}\rangle[E_0\langle F_01\rangle])1$.

Let L be the language of all words representing the elements of the group $\mathbb{Z}_2 \wr F_3$ according to the representation described above. It can be verified that L is recognizable by a deterministic pushdown automaton. We denote by $\psi : L \to \mathbb{Z}_2 \wr F_3$ the representation of the group $\mathbb{Z}_2 \wr F_3$ described above. The right multiplication by h either interchanges C_0 and C_1, D^B_0 and D^B_1, D^C_0 and D^C_1, E^C_0 and E^C_1, F^B_0 and F^B_1, $F^B_{D_0}$ and $F^B_{D_1}$, F^C_0 and F^C_1, $F^C_{D_0}$ and $F^C_{D_1}$, $F^C_{E_0}$ and $F^C_{E_1}$, G^C_0 and G^C_1, or $G^C_{D_0}$ and $G^C_{D_1}$. Therefore, the language $L_h = \{u \otimes v | u, v \in L, \psi(v) = \psi(u)h\}$ is recognizable by a deterministic pushdown automaton. Also, it is easily verified that the languages $L_a = \{u \otimes v | u, v \in L, \psi(v) = \psi(u)a\}$, $L_b = \{u \otimes v | u, v \in L, \psi(v) = \psi(u)b\}$ and $L_c = \{u \otimes v | u, v \in L, \psi(v) = \psi(u)c\}$ are recognizable by deterministic pushdown automata. $\square$

Let us be given an integer $n \geqslant 3$. We denote by w the word representing an element $g = (f, z) \in \mathbb{Z}_2 \wr F_n$ with respect to the $\mathcal{P}_1$–Cayley automatic representation of the group $\mathbb{Z}_2 \wr F_n$ constructed in Theorem 4.4.2. We obtain the following proposition.

Proposition 4.4.7. *The following inequality holds for all* $g \in \mathbb{Z}_2 \wr F_n$*:*

$$|w| \leqslant (2n-1)|g| + 1. \tag{4.35}$$

In addition, the following inequality holds for all $g \in \mathbb{Z}_2 \wr F_n$*:*

$$|g| \leqslant 3|w| - 2. \tag{4.36}$$

Both inequalities (4.35) *and* (4.36) *imply that for all* $g \in \mathbb{Z}_2 \wr F_n$*:*

$$\frac{1}{2n-1}|w| - \frac{1}{2n-1} \leqslant |g| \leqslant 3|w| - 2, \tag{4.37}$$

or, equivalently,

$$\frac{1}{3}|g| + \frac{2}{3} \leqslant |w| \leqslant (2n-1)|g| + 1. \tag{4.38}$$

Proof: Let us show the inequalities (4.35) and (4.36) for $n = 3$. In a similar way one can obtain them for any integer $n > 3$. We first prove the inequality (4.35). Let $\psi : L \to \mathbb{Z}_2 \wr F_3$ be the $\mathcal{P}_1$–Cayley automatic representation of $\mathbb{Z}_2 \wr F_3$ constructed in Theorem 4.4.2. It can be seen that the right multiplication by h does not change the length of a representative: if $\psi(u)h = \psi(v)$ for some $u, v \in L$, then $|u| = |v|$.

It is easily verified that the right multiplications by a, b or c increase the length of a representative by at most 5. For example, the representative of hc is $\langle F_1^A C_0 \rangle$ and the representative of hcb is $\langle F_1^A \{(G_0 C_0)\} \rangle$. The representative of the identity $e \in \mathbb{Z}_2 \wr F_3$ is the word B_0 of length 1. Therefore, we obtain (4.35). The inequality (4.36) can be shown similarly to Proposition 4.4.4. □

4.5 Wreath products $G \wr F_n$

In this section we consider groups $G \wr F_n$. It is easy to see that all constructions and proofs presented in Section 4.4 for the groups $\mathbb{Z}_2 \wr F_n$ can be straightforwardly modified for the groups $G \wr F_n$, if G is a finite group. For an infinite Cayley automatic group G we obtain the following theorem.

Theorem 4.5.1. *Let G be a Cayley automatic group. Then the group $G \wr F_n$ is $\mathcal{P}_1$–Cayley automatic.*

Proof: Let us be given a Cayley automatic representation $\psi_G : L_G \to G$ such that the empty word $\epsilon \notin L_G$. Let n be an integer greater or equal than 2. Recall that in Theorem 4.3.1 we constructed the Cayley automatic representation of the group $G \wr \mathbb{Z}$. In exactly the same way, using the $\mathcal{P}_1$–Cayley automatic representation of the group $\mathbb{Z}_2 \wr F_n$ constructed in Theorems 4.4.1 and 4.4.2, we obtain the $\mathcal{P}_1$–Cayley automatic representation of the group $G \wr F_n$. $\square$

We denote by $\mathcal{P}_1^c$ the class of context–free languages. Let $\psi_G : L_G \to G$ be a $\mathcal{P}_1^c$–Cayley automatic representation of G for which $\epsilon \notin L_G$ and there exists a finite set of generators $S = \{g_1, \ldots, g_m\}$ and a constant N such that for every $u, v \in L_G$, for which $\psi_G(u)g = \psi_G(v)$ for some $g \in S$, the inequality $||u| - |v|| \leqslant N$ holds. We obtain the following theorem.

Theorem 4.5.2. *For a given integer $n \geqslant 2$, the group $G \wr F_n$ is $\mathcal{P}_1^c$–Cayley automatic.*

Proof: The $\mathcal{P}_1^c$–Cayley automatic representation of the group $G \wr F_n$ is obtained in exactly the same way as the representation in Theorem 4.5.1. $\square$

Suppose now that the following inequality holds for some constants C and D:

$$|g| \leqslant C|u| + D, \tag{4.39}$$

where $|u|$ is the length of the word $u = \psi_G^{-1}(g)$, $|g|$ is the length of g in the Cayley graph $\Gamma(G, S)$, and $C > 0$ and $D \geqslant 0$ are some constants. Let n be an integer greater or equal than 2. Let us consider the $\mathcal{P}_1^c$–Cayley automatic representation $\psi : L \to G \wr F_n$ of the group $G \wr F_n$ constructed in Theorem 4.5.2. Put $w = \psi^{-1}(g)$ to be the representative of an element $g \in G \wr F_n$. Similarly to Proposition 4.3.1, we obtain the following proposition.

Proposition 4.5.1. *The following inequality holds all $g \in G \wr F_n$:*

$$|w| \leqslant K|g| + K_0, \tag{4.40}$$

where K and K_0 are some constants which depend only on the chosen $\mathcal{P}_1^C$–Cayley automatic representation of G. In addition, the following inequality is satisfied for all $g \in G \wr F_n$:

$$|g| \leqslant M|w| - 2, \tag{4.41}$$

where $M = C + D + 2$. Unifying the inequalities (4.40) *and* (4.41) *we obtain:*

$$\frac{1}{K}|w| - \frac{K_0}{K} \leqslant |g| \leqslant M|w| - 2, \tag{4.42}$$

or, equivalently,

$$\frac{1}{M}|g| + \frac{2}{M} \leqslant |w| \leqslant K|g| + K_0. \tag{4.43}$$

Proof: We first prove the inequality (4.40). Let K_0 be the length of the word representing the identity $e \in G \wr F_n$ for the chosen $\mathcal{P}_1^c$–Cayley automatic representation $\psi_G : L_G \to G$. We assumed that $\epsilon \notin L_G$. Therefore, $K_0 \geqslant 1$ and the inequality (4.40) holds for $g = e$.

Put $K = \max\{K_0 + 2(n-1), N\}$. Suppose that the inequality (4.40) holds for some $g \in G \wr F_n$. It can be seen that the lengths of the representatives of the words gg_j, gg_j^{-1}, gt and gt^{-1} are at most $|w| + K$. This proves that the inequality (4.40) holds for all $g \in G \wr F_n$.

Let us prove the inequality (4.41). Let (f, z) be an element of $G \wr F_n$. Similarly to Proposition 4.3.1, from (4.39) we obtain that:

$$|\text{supp} f| \leqslant (C + D)|w|. \tag{4.44}$$

Let ℓ be defined by the identity $|g| = |\mathrm{supp} f| + \ell$. Therefore, ℓ is the length of some route which the lamplighter covers before arriving at $z \in F_n$. Similarly to Proposition 4.3.1 we obtain that $\ell \leqslant 2(|w| - 1)$. Put $M = C + D + 2$. Therefore, we obtain that

$$|g| \leqslant (C + D + 2)|w| - 2. \tag{4.45}$$

Combining (4.40) and (4.41), we obtain (4.42) and (4.43). □

Definition 4.5.1. *Let $\psi : L \to G$ be a bijection between some language L and a finitely generated group G. We call a representation $\psi : L \to G$ geodesic if there exists a constant $\lambda > 0$ for which the inequalities*

$$\frac{1}{\lambda}|g| - \lambda \leqslant |w| \leqslant \lambda|g| + \lambda \tag{4.46}$$

hold for all $g \in G$, where $|g|$ is the length of $g \in G$ with respect to some finite set of generators of G and w is the representative of an element $g \in G$, $w = \psi^{-1}(g)$, and $|w|$ is the length of the word w.

The representations of the Baumslag–Solitar groups $BS(m, n)$, the groups $\mathbb{Z}_2 \wr \mathbb{Z}$ and $\mathbb{Z}_2 \wr F_n$, $n \geqslant 2$ constructed in Theorems 3.2.1, 4.2.1, 4.4.1 and 4.4.2 are geodesic. Proposition 4.5.1 shows that if a group G satisfies certain conditions then the representation constructed in Theorem 4.5.2 is geodesic.

Definition 4.5.2. *[35, Definition 4] We say that a representation $\psi : L \to G$ is quasi–geodesic if there exists a constant $\lambda > 0$ for which the inequality*

$$|w| \leqslant \lambda|g| + \lambda$$

holds for all $g \in G$, where $|g|$ is the length of $g \in G$ with respect to some finite set of generators of G, w is the representative of an element $g \in G$, $w = \psi^{-1}(g)$, and $|w|$ is the length of the word w.

Clearly, if a representation $\psi : L \to G$ is geodesic then it is quasi–geodesic. All Cayley automatic representations are quasi–geodesic.

4.6 The wreath product $\mathbb{Z}_2 \wr \mathbb{Z}^2$

In this section we consider the group $\mathbb{Z}_2 \wr \mathbb{Z}^2$. We will show how to represent the Cayley graph of $\mathbb{Z}_2 \wr \mathbb{Z}^2$ by nested stack automata.

Recall that a nested stack automaton uses a memory tree. We suppose that a memory tree is ordered by depth–first search and by taking the leftmost possible outedge at each opportunity. It is allowed to move on up and down on the rightmost branch of a memory tree. Nested stack automata generalize pushdown automata for which a linear stack is used. For an introduction to nested stack automata see, e.g., [40, 41]. Recall some necessary definitions.

Definition 4.6.1. *[41, Definition 3.1] For a finite set Ξ the set $\mathcal{T}$ of memory trees consists of all finite trees T ordered by depth–first search (by taking the leftmost possible outedge at each opportunity) with*

1. *Root vertex v_0;*
2. *Edges labeled by letters from Ξ;*
3. *All edges directed away from the root;*
4. *One distinguished vertex on the path from v_0 to the latest vertex of T, i.e., on the rightmost path.*

A monoid of partial maps on the set of memory trees $\mathcal{T}$ is defined as follows.

Definition 4.6.2. *[41, Definition 3.2] The monoid M_{nsa} is a monoid generated under composition by certain partial maps from $\mathcal{T}$ to itself. Pick $T \in \mathcal{T}$ with distinguished vertex v and let y be the label of the inegde to v. If $v = v_0$, then $y = \epsilon$. The effect of the partial maps D_x, U_x, P_x and Q_x on T is described as follows.*

- *$D_x(T)$, $x \in \Xi$: if $x = y$ and $v \neq v_0$, then $D_x(T)$ is obtained by changing the distinguished vertex of T to the parent of v;*

- *$U_x(T)$, $x \in \Xi \cup \{\epsilon\}$: if $x = y$ and v is not a leaf, make the latest child of v the new distinguished vertex.*

- *$P_x(T)$, $x \in \Xi$: add to T a new edge with source v, label x, and target a new vertex v_1. Make v_1 the latest vertex of T and the new distinguished vertex.*

- *$Q_x(T)$, $x \in \Xi$. If $x = y$ and v is a leaf with parent v_1, delete v and its inedge. Make v_1 the distinguished vertex.*

In each case if T does not satisfy the condition given, the map is not defined at T.

The monoid M_{nsa} acts on $\mathcal{T}$ by partial injective maps. The monoid M_{nsa} has an identity element which we denote by 1: for this element $1(T) = T$ for every $T \in \mathcal{T}$. The definition of nested stack automata is as follows.

Definition 4.6.3. *[41, Definition 3.3] Let Σ be a finite alphabet. A nested stack automaton $\mathcal{M}$ over Σ is a finite directed graph with a designated initial vertex, designated final vertices, and with edges labeled by pairs (m, a) where*

$a \in \Sigma \cup \{\epsilon\}$, and either $m = 1$ or m is one of the generators D_x, U_x, P_x and Q_x of a monoid M_{nsa} in Definition 4.6.2.

A computation of a nested stack automaton is a directed path which starts at the initial vertex of this automaton. A deterministic nested stack automaton is one such that every computation of length n can be continued in at most one way to a computation of length $n+1$. A more formal definition of deterministic nested stack automata is as follows.

Definition 4.6.4. *[41, Definition 3.5] A nested stack automaton over Σ is deterministic if each combination of a vertex, a memory tree T, and a letter $a \in \Sigma$ admits at most one outedge with label (m, a) or (m, ϵ) such that $m(T)$ is defined.*

Let $\mathcal{M}$ be a nested stack automaton. Given a computation of $\mathcal{M}$, the label of this computation $(m, w) \in M_{nsa} \times \Sigma^*$ is defined as follows. Put $m \in M_{nsa}$ to be the result of multiplying of the first components of edge labels in order given by the path. Put w to be the word obtained by reading off the second components of edge labels in order given by the path. For the path of a length 0, put the label to be $(1, \epsilon)$.

Definition 4.6.5. *[41, Definition 3.4] A word $w \in \Sigma^*$ is accepted by $\mathcal{M}$ if there is a computation with the label $(1, w)$ ending at a final state. The set of all accepted words is the language accepted by $\mathcal{M}$.*

An indexed language is one accepted by a nested stack automaton. We denote by $\mathcal{I}$ the class of all indexed languages. Along with Cayley automatic groups, $\mathcal{P}_1$– and $\mathcal{P}_1^c$–Cayley automatic groups one can study $\mathcal{I}$–Cayley automatic groups.

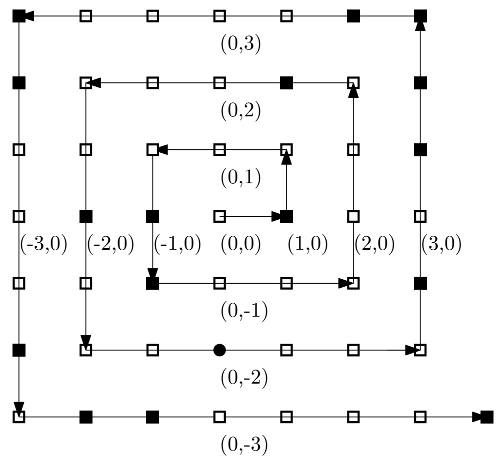

Figure 4.3: The map $t : \mathbb{N} \to \mathbb{Z}^2$ and an element of $\mathbb{Z}_2 \wr \mathbb{Z}^2$

Let us consider the group $\mathbb{Z}_2 \wr \mathbb{Z}^2$. Recall that an element of $\mathbb{Z}_2 \wr \mathbb{Z}^2$ is a pair (f, z), where f is a function $f : \mathbb{Z}^2 \to \mathbb{Z}_2$ that has finite support and $z \in \mathbb{Z}^2$ is the position of the lamplighter. We denote by x and y the standard generators of $\mathbb{Z}^2$, and by h the nontrivial element of $\mathbb{Z}_2$. Let us consider the Cayley graph of the wreath product $\mathbb{Z}_2 \wr \mathbb{Z}^2$ with respect to the generators x, y and h. We obtain the following theorem.

Theorem 4.6.1. *There exists an $\mathcal{I}$–Cayley automatic representation $\psi : L \to \mathbb{Z}_2 \wr \mathbb{Z}^2$ of the group $\mathbb{Z}_2 \wr \mathbb{Z}^2$ such that L is a regular language.*

Proof: Put $\Sigma = \{0, 1, C_0, C_1\}$. Let us consider the map $t : \mathbb{N} \to \mathbb{Z}^2$ such that: $t(1) = (0, 0)$, $t(2) = (1, 0)$, $t(3) = (1, 1)$, $t(4) = (0, 1)$, $t(5) = (-1, 1)$, $t(6) = (-1, 0)$, $t(7) = (-1, -1)$, $t(8) = (0, -1)$ and et cetera. The map $t : \mathbb{N} \to \mathbb{Z}^2$ is shown in Fig. 4.3. For a given element $(f, z) \in \mathbb{Z}_2 \wr \mathbb{Z}^2$, represent it as the word for which the kth symbol is 0 if $f(t(k)) = e$, 1 if $f(t(k)) = h$, C_0 if $f(t(k)) = e$ and $z = t(k)$, and C_1 if $f(t(k)) = h$ and

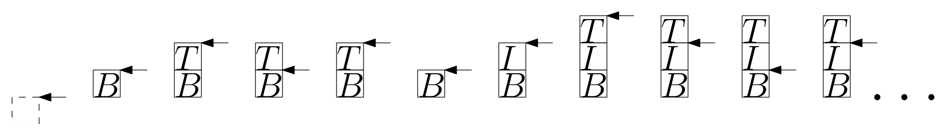

Figure 4.4: The first several iterations of $\mathcal{M}_x$

$z = t(k)$. The last symbol of a word is not allowed to be 0, i.e., it should be either 1, C_0 or C_1. An example of an element of $\mathbb{Z}_2 \wr \mathbb{Z}^2$ is shown in Fig. 4.3: *A white box means that the value of a function $f : \mathbb{Z}^2 \to \mathbb{Z}_2$ is $e \in \mathbb{Z}_2$, a black box means that it is $h \in \mathbb{Z}_2$, a black disk at the point $(0, -2)$ specifies the position of the lamplighter and tells us that the value of f is h. For the element of $\mathbb{Z}_2 \wr \mathbb{Z}^2$ shown in this figure the word representing it is* $010001100000001000001000C_1000101111000011000101100001$.

It can be seen that the language $L \subset \Sigma^*$ of representatives of all elements $g \in \mathbb{Z}_2 \wr \mathbb{Z}^2$ is regular. Also, the language $L_h = \{w_1 \otimes w_2 | w_1, w_2 \in L, \psi(w_1)h = \psi(w_2)\}$ is regular. Let us consider the languages $L_x = \{w_1 \otimes w_2 | w_1, w_2 \in L, \psi(w_1)x = \psi(w_2)\}$ and $L_y = \{w_1 \otimes w_2 | w_1, w_2 \in L, \psi(w_1)y = \psi(w_2)\}$. We will show that L_x and L_y are indexed languages.

Put the stack alphabet $\Xi = \{I, B, T\}$. The symbols B and T denote the bottom and the top of the stack, respectively. The symbol I is used for all intermediate positions.

Let $w \in L_x$. Below we give a description of a nested stack automata verifying that $w \in L_x$. We say that a symbol of the alphabet Σ is a C–symbol if it is either C_0 or C_1. Consider a nested stack automaton $\mathcal{M}_x$ that works as follows until it meets for the first time a letter which contains a C–symbol.

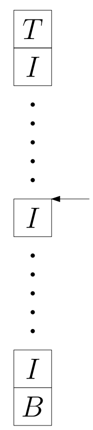

Figure 4.5: A content of a stack of the nested stack automaton $\mathcal{M}_x$

- Initially the stack is empty.
- $\mathcal{M}_x$ reads off the first letter of w and pushes B onto the stack.
- $\mathcal{M}_x$ reads off the second letter of w and pushes T onto the stack.
- $\mathcal{M}_x$ reads off the third letter of w and goes one step down.
- $\mathcal{M}_x$ reads off the fourth letter of w and goes one step up.
- $\mathcal{M}_x$ makes two silent moves popping T and pushing I.
- $\mathcal{M}_x$ reads off the fifth letter of w and pushes T onto the stack.
- $\mathcal{M}_x$ reads off the sixth letter of w and goes one step down.
- $\mathcal{M}_x$ reads off the seventh letter of w and goes one step down.
- $\mathcal{M}_x$ reads off the eights letter of w and goes one step up.

- The process continues by going up and down along the stack between B and T. Each time the top is reached, it is raised up by one. This process is shown in Fig. 4.4: *The content of the stack is shown for the first several iterations of* $\mathcal{M}_x$. In general situation, a content of a stack is shown in Fig. 4.5.

It is easy to see the following. If a letter of a word $w \in L_x$ being read at some position m contains a C–symbol for the first time, then either the next letter at the position $m+1$ or the letter at the position $m+(4n+1)$ contains a proper C–symbol, where n is the current height of the stack. In order to verify the latter case, the automaton $\mathcal{M}_x$, after meeting the letter containing a C–symbol for the first time, works as follows.

- $\mathcal{M}_x$ makes silent moves going up until it reaches the top of the stack.

- Then the automaton $\mathcal{M}_x$ pops a symbol out of stack each time it reads off four consecutive letters.

- After the stack is emptied the automaton reads off the next letter and checks whether it contains a proper C–symbol.

- After meeting a letter containing a C–symbol for the second time, the automaton works without using a stack verifying that $w \in L_x$.

It is easy to see that the automaton $\mathcal{M}_x$ recognizes the language L_x. In a similar way one can obtain the nested stack automaton $\mathcal{M}_y$ that recognizes the language L_y. It can be seen that the nested stack automata $\mathcal{M}_x$ and $\mathcal{M}_y$ are deterministic. □

Remark 4.6.1. *It can be verified that for the $\mathcal{I}$–Cayley automatic representation $\psi : L \to \mathbb{Z}_2 \wr \mathbb{Z}^2$ constructed in Theorem 4.6.1 the following inequality holds:*

$$|g| \leqslant 2|w| - 1,$$

where w is the representative of $g \in \mathbb{Z}_2 \wr \mathbb{Z}^2$, $|w|$ is the length of w and $|g|$ is the length of g in the group $\mathbb{Z}_2 \wr \mathbb{Z}^2$ with respect to the generators x, y and h. Also, it can be easily verified that for an arbitrary constant λ the inequality

$$|w| \leqslant \lambda|g| + \lambda$$

does not hold for all $g \in \mathbb{Z}_2 \wr \mathbb{Z}^2$. So, the $\mathcal{I}$–Cayley automatic representation of the group $\mathbb{Z}_2 \wr \mathbb{Z}^2$ constructed in Theorem 4.6.1 is not quasi–geodesic, and, therefore, it is not geodesic.

Chapter 5

Non–Cayley automatic transitive graphs

In this chapter we study automatic representations of infinite non–Cayley transitive graphs. The main results of this chapter are published in [19].

5.1 Examples of automatic non–Cayley transitive graphs

By Definition 2.1.5, a graph $\Gamma(V, E)$ is called automatic if the structure $(V; E)$ is automatic. Recall that a graph $\Gamma(V, E)$ is transitive if for every pair of vertices $u, v \in V$ there exists an automorphism of the graph which maps u into v. We say that a transitive graph is non–Cayley if it is not an unlabeled and undirected Cayley graph of any finitely generated group (in this chapter all Cayley graphs are supposed to be unlabeled and undirected). We will provide examples of transitive graphs which showcase that the class

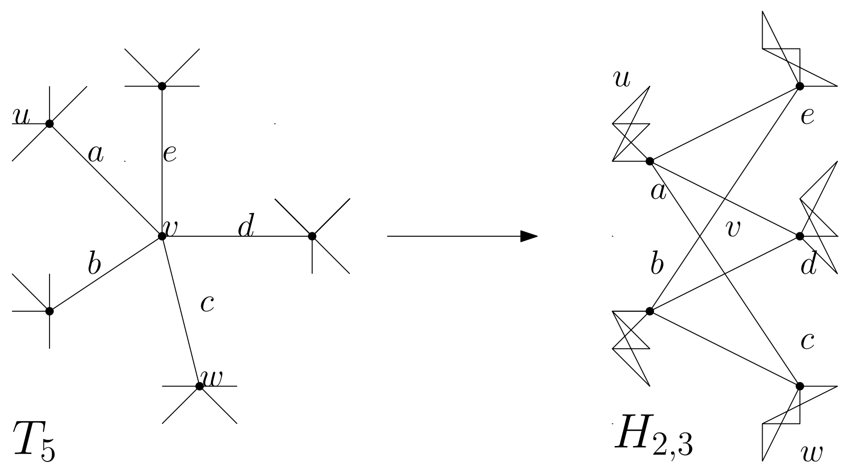

Figure 5.1: Constructing the non–Cayley graph $H_{2,3}$ from the graph T_5

of automatic transitive graphs properly contains the class of all Cayley graphs of Cayley automatic groups.

Thomassen and Watkins constructed an example of an infinite non–Cayley transitive graph [42]. Below we construct this example following [43, § 2]. Let us consider the infinite 5–regular tree T_5 and the bipartite graph $K_{2,3}$. We define the graph $H_{2,3}$ as follows. First, replace vertices of T_5 by disjoint copies of $K_{2,3}$. Second, for each edge (u, v) of T_5, identify a vertex of the $K_{2,3}$ corresponding to u with a vertex of the $K_{2,3}$ corresponding to v such that no point in any $K_{2,3}$ is identified more than once and a vertex in the class of size 2 is always identified with a vertex in the class of size 3, and vice versa. This construction is shown in Fig. 5.1 (see also [43, Figure 1]). For the graph $H_{2,3}$ we obtain the following proposition.

Proposition 5.1.1. *[43, § 2] The graph $H_{2,3}$ is a non–Cayley transitive graph.*

Proof: It is easy to see that $H_{2,3}$ is a transitive graph. Let us show that $H_{2,3}$ cannot be a Cayley graph of any finitely generated group. We say that a group S acts transitively on a graph H if for every pair u, v of vertices of H there exists an action of S which maps u to v. We say that S acts freely on H if every nontrivial action of S moves every vertex of H. Recall that a graph H is a Cayley graph of some group iff there exists a subgroup S of the group of automorphisms of H which acts freely and transitively on H.

Suppose that there exists a subgroup S of the group of automorphisms of $H_{2,3}$ which acts freely and transitively on $H_{2,3}$. Let K be one of the $K_{2,3}$ making up $H_{2,3}$, and let $\{a, b\}$ and $\{c, d, e\}$ be the vertices of K in the classes of size 2 and 3, respectively. Let us consider an automorphism that maps an element of $\{c, d, e\}$ back into $\{c, d, e\}$. Such an automorphism must map the set $\{a, b\}$ to itself because this is the only pair of vertices which has three common neighbors and has a common neighbor in the set $\{c, d, e\}$. Therefore, such an automorphism must fix K.

Put $\theta \in S$ to be an automorphism such that $\theta(c) = d$. Then θ must swap a and b. Put $\theta' \in S$ to be an automorphism such that $\theta'(c) = e$. Then θ' must swap a and b. Therefore, $\theta'\theta^{-1} \in S$ maps d to e, a to a, and b to b. This gives us a contradiction. Hence $H_{2,3}$ cannot be a Cayley graph for any finitely generated group. Therefore, $H_{2,3}$ is a non–Cayley transitive graph. □

In this chapter we use the notion of line graph. For a given digraph D the line graph D' is defined as follows.

Definition 5.1.1. *The line graph of a digraph $D = (V, E)$ is the digraph $D' = (V', E')$ with $V' = E$ and $E' = \{((u, v), (u', v')) \mid (u, v), (u', v') \in$*

$V' \ \wedge \ v = u'\}$.

Before showing that $H_{2,3}$ is automatic let us prove the following auxiliary lemma.

Lemma 5.1.1. *Let D be an automatic digraph. Then its line graph D' is automatic.*

Proof: Let $\psi : L \to V$ be a bijection which provides an automatic representation for the digraph D. Put $L' = \{w_1 \otimes w_2 | w_1, w_2 \in L \wedge (\psi(w_1), \psi(w_2)) \in E\}$. Since ψ provides an automatic representation of D, there exists a two–tape synchronous finite automaton which accepts the language L'. Therefore, L' is a regular language.

Since V' is identified with E we obtain the bijection $\psi' : L' \to V'$ defined as $\psi'(w_1 \otimes w_2) = (\psi(w_1), \psi(w_2))$. Consider the language:

$$L'' = \{w_1' \otimes w_2' | w_1' = u_1 \otimes v_1 \in L' \wedge w_2' = u_2 \otimes v_2 \in L' \wedge \psi(v_1) = \psi(u_2)\}.$$

A synchronous automaton that accepts L'' works as follows. For a given input $w_1' \otimes w_2'$ it verifies that both w_1' and w_2' belong to L', and that v_1 coincides with u_2. Therefore L'' is a regular language. Thus, the bijection $\psi' : L' \to V'$ provides an automatic representation of D'. $\square$

Let us show now that $H_{2,3}$ is automatic.

Proposition 5.1.2. *The graph $H_{2,3}$ is automatic.*

Proof: Let us consider the following representation of T_5. Put $\Sigma = \{a, b, c, d, e\}$. Put L to be the subset of Σ^* such that in every word $w \in L$ none of the subwords *aa*, *bb*, *cc*, *dd* and *ee* occur. Depict the graph T_5 on a plane as it is shown in Fig. 5.1 (left). Let us choose a labeling of edges of T_5 as follows. For some vertex v of T_5 label incident edges by a, b, c, d and e counterclockwise, see Fig. 5.1. Consider the vertex u incident to v for which (v, u) is labeled by a and label the remaining edges incident to u by b, c, d and e counterclockwise. Then consider the vertex w incident to v for which (v, w) is labeled by c and label the remaining edges incident to w by d, e, a, b counterclockwise. Continue this procedure until all edges of T_5 are labeled. It can be seen that, after labeling edges of T_5, for every vertex of T_5 all five edges incident to it have different labels a, b, c, d and e.

Define a representation $\psi : L \to V(T_5)$ as follows. Put $\psi(\epsilon) = v$, where ϵ is the empty word and v is a vertex of T_5. For a given $w \in L$, put the vertex $\psi(w)$ to be the vertex v_n which is the end of the path $v_0, v_1, v_2, \ldots, v_n$, where $v_0 = v$, for every $i = 1, \ldots, n$ the edge (v_{i-1}, v_i) is labeled by ith letter of w and $n = |w|$. It can be seen that $\psi : L \to V(T_5)$ provides an automatic representation of T_5. Let us fix orientations of edges of T_5 such that each vertex has 2 ingoing edges and 3 outgoing edges. We denote the obtained digraph by D_0. It can be verified that $\psi : L \to V(T_5)$ provides an automatic representation of D_0. Let D_0' be the line graph of D_0. It can be verified that the graph $H_{2,3}$ can be obtained from D_0' by removing orientations of edges. In Lemma 5.1.1 we showed that if some digraph is automatic then its line graph is automatic. Therefore, the graph $H_{2,3}$ is automatic. □

Similarly to the graph $H_{2,3}$, for given integers n and m such that $n \geqslant 2$, $m \geqslant 3$ and $n \neq m$ one can obtain the automatic non–Cayley transitive graph $H_{n,m}$. We obtain the following proposition.

Proposition 5.1.3. *Let n and m be integers such that $n \geqslant 2$, $m \geqslant 3$ and*

$n \neq m$. The graph $H_{n,m}$ is an automatic non–Cayley transitive graph.

Proof: The proof is the same as for Propositions 5.1.1 and 5.1.2. □

5.2 The Diestel–Leader graph is automatic

Let D_i, $i = 0, \ldots, \infty$ be the sequence of digraphs such that the digraph D_0 is the 5–regular tree for which every vertex has exactly two ingoing edges and three outgoing edges, and D_{i+1} is the line graph of D_i for every $i \geqslant 0$. Recall the following proposition.

Proposition 5.2.1. *[43, Proposition 3] For a given integer $n \geqslant 1$, the digraph D_n is isomorphic to the digraph whose vertices are the directed paths of length n in D_0, with an edge from $x_1x_2\ldots x_{n+1}$ to $y_1y_2\ldots y_{n+1}$ if $y_i = x_{i+1}$ for every $1 \leqslant i \leqslant n$.*

Put G_n to be the graph obtained from D_n by removing the orientations from the edges. By Lemma 5.1.1, we obtain that G_i, $i = 0, \ldots, \infty$ is a sequence of automatic transitive graphs. It can be seen that $G_0 = T_5$ and $G_1 = H_{2,3}$. Moreover, every graph $G_i, i = 0, \ldots, \infty$ is quasi–isometric to the 5–regular tree T_5. Recall the following definition due to Gromov [44].

Definition 5.2.1. *We say that two graphs G_1 and G_2 are quasi–isometric if there exist a map $\theta : V(G_1) \to V(G_2)$ and some $\lambda \geqslant 1$ such that*

$$\frac{1}{\lambda} d_{G_1}(x, y) - \lambda \leqslant d_{G_2}(\theta(x), \theta(y)) \leqslant \lambda d_{G_1}(x, y) + \lambda$$

for all $x, y \in V(G_1)$, and for any point $y \in V(G_2)$ there is some $x \in V(G_1)$ such that $d_{G_2}(\theta(x), y) \leqslant \lambda$; where $d_{G_1}(x, y)$ and $d_{G_2}(\theta(x), \theta(y))$ are the graph distances between x, y and $\theta(x), \theta(y)$ in G_1 and G_2, respectively.

Diestel and Leader showed that the sequence G_i, $i = 0, \ldots, \infty$ converges to a limit which is a transitive graph [43]. The limit is the Diestel–Leader graph which we denote by G^*. It was conjectured that $G^\star$ cannot be quasi–isometric to any Cayley graph [43]. The conjecture was proved by Eskin, Fisher and Whyte [45].

The graph $G^\star$ can be described as follows, see [43, § 3]. Let X be a 3–regular tree in which each vertex has 2 ingoing edges and 1 outgoing edge. Let Y be a 4–regular tree in which each vertex has 1 ingoing edge and 3 outgoing edges. The digraphs X and Y are shown in Fig. 5.2. All edges are directed from the left to the right, see also [43, Figure 4].

Fix a vertex $O_1 \in V(X)$ and a vertex $O_2 \in V(Y)$. For each $x \in X$, set $r(x)$ to be the signed distance from O_1 to x, i.e., if the unique undirected path from O_1 to x in X has s forward edges and t backward edges then $r(x) = s - t$. Define $r(y)$ similarly for each $y \in Y$. Define the digraph D^* as follows. The set of vertices of D^* is the set $\{(x, y) \in X \times Y : r(x) = r(y)\}$, and D^* has an edge from (x, y) to (x', y') if $(x, x') \in E(X)$ and $(y, y') \in E(Y)$. Let G^* be obtained from D^* by removing the orientations from the edges.

Theorem 5.2.1. *The graph $G^\star$ is an automatic transitive graph such that no Cayley graph is quasi–isometric to it.*

Proof: We only need to prove that G^* is automatic. Put $L_0 = \{\epsilon\}$, $L_1 = \{\#\} \cdot \{\#\}^*$, $L_2 = \{u\} \cdot \{u\}^*$, $L_3 = L_1 \cdot L_2$, $L_4 = \{0\} \cdot \{0, 1\}^* \cup \{1\} \cdot \{0, 1\}^*$, $L_5 = \{d\} \cdot \{0, 1\}^*$ and $L_6 = L_1 \cdot L_5$. Put the language L' to be the set of words $u \otimes v$ such that $|u| = |v|$ and one of the following holds:

- $u, v \in L_0$,

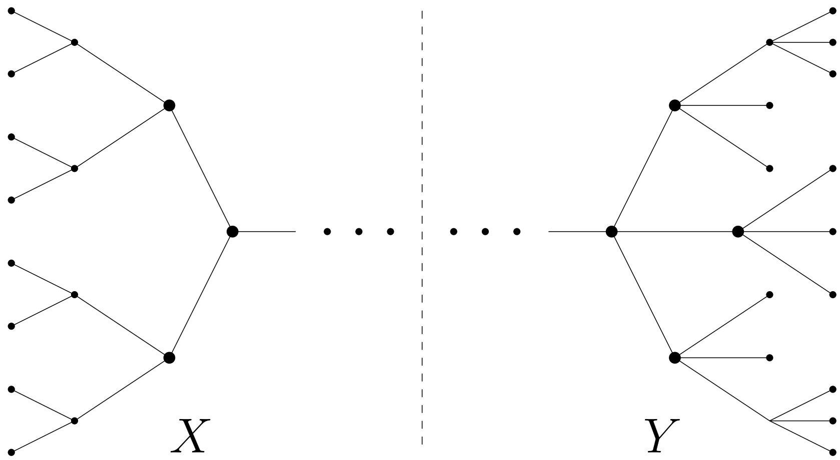

Figure 5.2: The digraphs X and Y

- $u \in L_1$ and $v \in L_4$,
- $u \in L_2$ and $v \in L_5 \cup L_6$,
- $u \in L_3$ and $v \in L_5$.

We construct the map $\psi' : L' \to V(X)$ as follows. For a given $w \in L'$ from the vertex $O_1 \in V(X)$ we make as many steps forward along outgoing edges as we have symbols u in the first row of w. Then, if there is a symbol d in the second row of w, we make one step backward to the vertex we have not visited before: there is only one way to choose this vertex. After that we make as many steps backward as we have symbols 0 and 1 in the second row of w: we go up if a symbol being read in the second row is 0 and go down if it is 1.

The resulting vertex defines $\psi'(w)$. It can be seen that $|r(\psi'(w))|$ is the number of the hash symbols # in w. Hash symbols # appear in the first row of w iff $r(\psi'(w)) < 0$. Similarly, hash symbols # appear in the second row of

w iff $r(\psi'(w)) > 0$.

For example, consider the following word of the language L':

$$\begin{matrix} u & u & u & u & u & u \\ \# & \# & \# & d & 0 & 1 \end{matrix}.$$

In order to obtain the vertex corresponding to this word we should make six steps forward from O_1, then three steps backward such that doing the second step we go up and doing the third step we go down.

Put $L_7 = \{0\} \cdot \{0,1,2\}^* \cup \{1\} \cdot \{0,1,2\}^* \cup \{2\} \cdot \{0,1,2\}$, $L_8 = \{d_1, d_2\} \cdot \{0,1,2\}^*$ and $L_9 = L_1 \cdot L_8$. Put the language L'' to be the set of words $u \otimes v$ such that $|u| = |v|$ and one of the following hold:

- $u, v \in L_0$,
- $u \in L_1$ and $v \in L_7$,
- $u \in L_2$ and $v \in L_8 \cup L_9$,
- $u \in L_3$ and $v \in L_8$.

We construct the map $\psi'' : L'' \to V(Y)$ as follows. For a given $w \in L''$ from the vertex $O_2 \in V(Y)$ we make as many steps forward along outgoing edges as we have symbols u in the first row of w. Then if there is one of the symbols d_1 or d_2 in the second row of w we make one step backward to the vertex we have not visited before: this vertex is uniquely defined depending on whether we have d_1 or d_2 in the second row of w. After that we make as many steps backward as we have symbols $0, 1$ and 2 in the second row of w: we go up if a symbol being read in the second row is 0, we go straight if it is 1, and go down if it is 2.

The resulting vertex defines $\psi''(w)$. It can be seen that $|r(\psi''(w))|$ is the number of the hash symbols # in w. Hash symbols # appear in the first row of w iff $r(\psi''(w)) < 0$. Similarly, hash symbols # appear in the second row of w iff $r(\psi''(w)) > 0$.

For example, let us consider the following word in L'':

$$\begin{array}{cccccc} u & u & u & u & u & u \\ \# & \# & d_1 & 0 & 1 & 2 \end{array}.$$

In order to obtain the vertex corresponding to this word we should make six steps forward from O_2, then four steps backward such that doing the second step we go up, doing the third step we go straight and doing the fourth step we go down.

Put $L = \{w' \otimes w'' | w' \in L', w'' \in L'' \wedge r(\psi'(w')) = r(\psi''(w''))\}$. It can be seen that L is a regular language. We define the map $\psi : L \to \{(x, y) \in X \times Y : r(x) = r(y)\}$ as $\psi(w' \otimes w'') = (\psi'(w'), \psi''(w''))$. It can be seen that ψ provides an automatic representation of the digraph D^*. Therefore, ψ provides an automatic representation of the graph G^*. $\square$

Chapter 6

On characterizations of Cayley automatic groups

In this chapter we address the problem of finding characterizations of Cayley automatic groups. Our approach is to define and then study three numerical characteristics of Turing transducers of the special class $\mathcal{T}$ which is obtained from automatic representations of labeled directed graphs. In Section 6.1 we define the class of Turing transducers $\mathcal{T}$. Then we show that automatic representations of Cayley graphs of groups can be expressed in terms of Turing transducers of the class $\mathcal{T}$. This explains why the study of admissible asymptotic behavior for some numerical characteristics of Turing transducers of the class $\mathcal{T}$ is relevant to the problem of finding characterizations for Cayley automatic groups. In Section 6.2 we introduce three numerical characteristics for Turing transducers of the class $\mathcal{T}$. Asymptotic behavior of the numerical characteristics of Turing transducers of the class $\mathcal{T}$ is discussed in Sections 6.3 and 6.4. The results of this chapter are published in [21].

6.1 Turing transducers of the class $\mathcal{T}$

Recall that a $(k+1)$–tape Turing transducer T for $k \geqslant 1$ is a multi–tape Turing machine which has one input tape and k output tapes. See, e.g., [46, § 10] for the definition of Turing transducers. The special class of Turing transducers $\mathcal{T}$ that we consider in this chapter is described as follows. Let us be given a $(k+1)$–tape Turing transducer $T \in \mathcal{T}$ and an input word $x \in \Sigma^*$. Initially, the input word x appears on the input tape, the output tapes are completely blank and all heads are over the leftmost cells. First the heads of T move synchronously from the left to the right until the end of the input x. Then the heads make a finite number of steps (probably no steps) further to the right, where this number of steps is bounded from above by some constant which depends on T. After that, the heads of T move synchronously from the right to the left until it enters a final state with all heads over the leftmost cells.

We say that T accepts x if T enters an accepting state; otherwise, T rejects x. Let $L \subseteq \Sigma^*$ be the set of inputs accepted by T. We say that T translates $x \in L$ into the outputs $y_1, \ldots, y_k$ if for the word x fed to T as an input, T returns the word y_i on the ith output tape of T for every $i = 1, \ldots, k$. It is assumed that for every input $x \in L$, the output $y_i \in L$ for every $i = 1, \ldots, k$. Let $L' \subseteq L^k$ be the set of all k–tuples of outputs $(y_1, \ldots, y_k)$. We say that T translates L into L'.

Let $T \in \mathcal{T}$. Lemma 6.1.1 below shows a connection between Turing transducers of the class $\mathcal{T}$ and multi–tape synchronous finite automata.

Lemma 6.1.1. *There exists a $(k+1)$–tape synchronous finite automaton $\mathcal{M}$*

such that a convolution $x \otimes y_1 \otimes \cdots \otimes y_k \in \Sigma_\diamond^{(k+1)*}$ *is accepted by* $\mathcal{M}$ *iff* T *translates the input* x *into the outputs* $y_1, \ldots, y_k$*.*

Proof: The lemma is obtained straightforwardly from the following two well known facts. The first fact is that the class of regular languages is closed under reversal. The second fact is as follows. Let the convolutions $\otimes R_1$ and $\otimes R_2$ of two relations $R_1 = \{(x, y)|x, y \in \Sigma^*\}$ and $R_2 = \{(y, z)|y, z \in \Sigma^*\}$ be accepted by two–tape synchronous finite automata. Then the convolution $\otimes R$ of the relation $R = \{(x, z)|\exists y[(x, y) \in R_1 \wedge (y, z) \in R_2]\}$ is accepted by a two–tape synchronous finite automaton. $\square$

In other words, one can say that multi–tape synchronous finite automata simulate Turing transducers of the class $\mathcal{T}$. In a different context, the notion of simulation for finite automata appeared, e.g., in [47, 48].

For a given k, put $\Sigma_k = \{1, \ldots, k\}$. Let $T \in \mathcal{T}$ be a $(k+1)$–tape Turing transducer translating a language L into $L' \subseteq L^k$. We construct a labeled directed graph Γ_T with the labels from Σ_k as follows. The set of vertices $V(\Gamma_T)$ is identified with L. For given $u, v \in L$ there is an oriented edge (u, v) labeled by $j \in \Sigma_k$ if T translates u into some outputs $w_1, \ldots, w_k$ such that $w_j = v$. It is easy to see that each vertex of the graph Γ_T has k outgoing edges labeled by $1, \ldots, k$.

Let Γ be a labeled directed graph for which every vertex has k outgoing edges labeled by $1, \ldots, k$. Recall that, by Definition 2.1.5, Γ is called automatic if there exists a bijection between a regular language and the set of vertices $V(\Gamma)$ such that for every $j \in \Sigma_k$ the set of oriented edges labeled by j is accepted by a synchronous two–tape finite automaton. From Lemma 6.1.1 we obtain that Γ_T is automatic. Suppose that Γ is automatic. Lemma

6.1.2 below shows that Γ can be obtained as Γ_T for some $(k+1)$–tape Turing transducer $T \in \mathcal{T}$.

Lemma 6.1.2. *There exists a $(k+1)$–tape Turing transducer $T \in \mathcal{T}$ for which $\Gamma_T \cong \Gamma$.*

Proof: The lemma can be obtained from the following fact. Let $R = \{(x, y) | x, y \in L\}$ be a binary relation such that $\otimes R$ is recognized by a two–tape synchronous finite automaton, where L is a regular language. Suppose that for every $x \in L$ there exists exactly one $y \in L$ such that $(x, y) \in R$. Then there exists a two–tape Turing transducer $T_R \in \mathcal{T}$ for which T_R translates x into y iff $(x, y) \in R$ and T_R rejects x iff $x \notin L$. The construction of the Turing transducer T_R is shown in Theorem 2.2.1 (see also [2, Theorem 2.3.10]). The resulting $(k+1)$–tape Turing transducer $T \in \mathcal{T}$ is obtained as the combination of k two–tape Turing transducers $T_{R_1}, \ldots, T_{R_k}$, where $R_1, \ldots, R_k$ are the binary relations defined by the directed edges of Γ labeled by $1, \ldots, k$, respectively. $\square$

Lemmas 6.1.1 and 6.1.2 together imply the following theorem.

Theorem 6.1.1. *A labeled directed graph Γ is automatic iff there exists a Turing transducer $T \in \mathcal{T}$ for which $\Gamma \cong \Gamma_T$.*

Let $\Gamma(G, S)$ be a Cayley graph for some set of generators $S = \{s_1, \ldots, s_k\}$. Let us fix an order of the elements in S as $s_1, \ldots, s_k$. We say that the Cayley graph $\Gamma(G, S)$ is presented by $T \in \mathcal{T}$ if, after changing labels from j to s_j for every $j \in \Sigma_k$ in Γ_T, $\Gamma_T \cong \Gamma(G, S)$. The isomorphism $\Gamma_T \cong \Gamma(G, S)$ defines the bijection $\psi : L \to G$ up to the choice of the word of L corresponding to the identity $e \in G$. By Theorem 6.1.1 we obtain that if $\Gamma(G, S)$ is presented by $T \in \mathcal{T}$, then G is a Cayley automatic group and T provides an automatic representation for the Cayley graph $\Gamma(G, S)$. Moreover, for each automatic representation of $\Gamma(G, S)$ there is a corresponding Turing transducer $T \in \mathcal{T}$ which presents $\Gamma(G, S)$.

6.2 Numerical characteristics of Turing transducers

We now introduce three numerical characteristics for Turing transducers of the class $\mathcal{T}$. Let $T \in \mathcal{T}$ be a $(k+1)$–tape Turing transducer translating a language L into $L' \subseteq L^k$. Given a word $w \in L$, feed w to T. Let $w_1, \ldots, w_k \in L$ be the outputs of T for w. We denote by $T(w)$ the set $T(w) = \{w_1, \ldots, w_k\}$. Given a set $W \subseteq L$, we denote by $T(W)$ the set $T(W) = \bigcup_{w \in W} T(w)$. Let us choose a word $w_0 \in L$. Put $W_0 = \{w_0\}$, $W_1 = T(W_0)$ and, for $i > 1$, put $W_{i+1} = T(W_i)$. Let $V_n = \bigcup_{i=0}^{n} W_i$, $n \geqslant 0$. Put $b_n = \#V_n$.

- We call the sequence $b_n, n = 0, \ldots, \infty$ the growth function of the pair (T, w_0).

For a given finite set $W \subseteq L$ put

$$\partial W = \{w \in W | T(w) \nsubseteq W\}.$$

In other words, ∂W is the set of words $w \in W$ for which at least one of the outputs of T for w is not in W. Define the function $F\o l(\varepsilon) : (0, 1) \to \mathbb{N}$ as

$$F\o l(\varepsilon) = \min\{\#W | \#\partial W < \varepsilon \#W\}.$$

It is assumed that the function $F\o l(\varepsilon)$ is defined on the whole interval $(0,1)$, i.e., for every $\varepsilon \in (0,1)$ the set $\{W \subseteq L | \#\partial W < \varepsilon \# W\}$ is not empty.

- We call the sequence $f_n = F\o l(\frac{1}{n}), n = 1, \dots, \infty$ the Følner function of T.

Let M be a finite multiset of words of L. We denote by $T(M)$ the multiset obtained as follows. Initially, $T(M)$ is set to be empty. Then, for every word w in M add the outputs of T for w to $T(M)$. If w has the multiplicity m in M, then this procedure must be repeated m times. Let M_0 be the multiset consisting of the word w_0 with the multiplicity one. Put $M_1 = T(M_0)$ and, for $i > 1$, put $M_{i+1} = T(M_i)$. The total number of elements (multiplicities are taken into account) in the multiset M_n is k^n. Put ℓ_n to be

$$\ell_n = \frac{\sum\limits_{w \in M_n} m_w |w|}{k^n}, \tag{6.1}$$

where m_w is the multiplicity of a word w in M_n and $|w|$ is the length of w. In other words, ℓ_n is the average length of the words in the multiset M_n.

- We call the sequence $\ell_n, n = 1, \dots, \infty$ the average length growth function of the pair (T, w_0).

6.3 Asymptotic behavior of growth and Følner functions

In this section we discuss asymptotic behavior of growth functions and Følner functions of Turing transducers of the class $\mathcal{T}$. We first consider behavior of growth function $b_n, n = 0, \dots, \infty$.

Let G be a group with a finite set of generators $Q \subseteq G$. Put $S = Q \cup Q^{-1}$. For a given $g \in G$ we denote by $\ell_S(g)$ the minimal length of a word representing g in terms of S. We denote by B_n the ball of the radius n, $B_n = \{g \in G | \ell_S(g) \leqslant n\}$. Recall that the growth function of the pair (G, Q) is the function $\#B_n, n = 0, \ldots, \infty$, where $\#B_n$ is the number of elements in the ball B_n. Let $T \in \mathcal{T}$ be a Turing transducer translating a language L into $L' \subseteq L^k$, where $k = \#S$. Choose any word $w_0 \in L$. The following claim is straightforward.

Claim 6.3.1. *Suppose that the Cayley graph $\Gamma(G, S)$ is presented by T. Then the growth function b_n of the pair (T, w_0) coincides with the growth function of the pair (G, Q).*

One of the important questions in the group theory is whether or not for a given pair (G, Q) the growth series is rational. A similar question naturally arises for a pair (T, w_0). It is easy to show an example of a pair $(T, w_0), T \in \mathcal{T}$ for which the growth series is not rational.

Example 6.3.1. *Stoll proved that the growth series of the Heisenberg group H_5 with respect to the standard set of generators is not rational [49]. The Cayley graph of H_5 is automatic [1, Example 6.7]. Therefore, we obtain that there exists a pair $(T, w_0), T \in \mathcal{T}$ for which the growth series $\sum b_n z^n$ is not rational.*

Moreover, a Turing transducer of the class $\mathcal{T}$ may have a function $b_n, n = 0, \ldots, \infty$ of intermediate growth.

Example 6.3.2. *Miasnikov and Savchuk constructed an example of a 4–regular automatic graph which has intermediate growth [50]. Therefore, we* *obtain that there exists a pair $(T, w_0), T \in \mathcal{T}$ for which the function $b_n, n = 0, \ldots, \infty$ has intermediate growth.*

We now consider the behavior of Følner function $f_n, n = 1, \ldots, \infty$ for Turing transducers of the class $\mathcal{T}$. Følner functions were first considered by A. Vershik for Cayley graphs of amenable groups [51]. Recall first some necessary definitions regarding Følner functions [52].

Let G be an amenable group with a finite set of generators $Q \subseteq G$. Put $S = Q \cup Q^{-1}$. Let E be the set of directed edges of $\Gamma(G, S)$. For a given finite set $U \subseteq G$ the boundary ∂U is defined as

$$\partial U = \{u \in U | \exists v \in G[(u, v) \in E \wedge v \notin U]\}.$$

The function $F\o l_{G,Q} : (0, 1) \to \mathbb{N}$ is defined as

$$F\o l_{G,Q}(\varepsilon) = \min\{\#U | \#\partial U < \varepsilon \#U\}.$$

The Følner function $\mathrm{F\o l}_{G,Q} : \mathbb{N} \to \mathbb{N}$ is defined as $\mathrm{F\o l}_{G,Q}(n) = F\o l_{G,Q}(\frac{1}{n})$. The following claim is straightforward.

Claim 6.3.2. *Suppose that the Cayley graph $\Gamma(G, S)$ is presented by a Turing transducer $T \in \mathcal{T}$. Then for the Følner function f_n of T, $f_n = \mathrm{F\o l}_{G,Q}(n)$.*

In this section we say that $f_1(n) \sim f_2(n)$ if there exists $K \in \mathbb{N}$ such that $f_1(Kn) \geqslant \frac{1}{K} f_2(n)$ and $f_2(Kn) \geqslant \frac{1}{K} f_1(n)$, i.e., $f_1(n)$ and $f_2(n)$ are equivalent up to a quasi–isometry. Let $Q' \subseteq G$ be another set generating G. Then $\mathrm{F\o l}_{G,Q}(n) \sim \mathrm{F\o l}_{G,Q'}(n)$. In this section Følner functions are considered up to quasi–isometries. So, instead of $\mathrm{F\o l}_{G,Q}(n)$, we will write $\mathrm{F\o l}_{\mathrm{G}}(n)$.

Let $G_1 = \mathbb{Z} \wr \mathbb{Z}$. Put $G_{i+1} = G_i \wr \mathbb{Z}, i \geqslant 1$. It is shown [52, Example 3] that $\mathrm{F\o l}_{G_i}(n) \sim n^{(n^i)}$. It follows from Theorem 4.3.1 that for every integer

$i \geqslant 1$ there exists a Turing transducer $T_i \in \mathcal{T}$ for which a Cayley graph of G_i is presented by T_i. The following theorem shows that the logarithm of Følner functions for Turing transducers of the class $\mathcal{T}$ can grow faster than any given polynomial.

Theorem 6.3.1. *For every integer $i \geqslant 1$ there exists a Turing transducer of the class $\mathcal{T}$ for which $f_n \sim n^{(n^i)}$.*

Remark 6.3.1. *Consider the group $\mathbb{Z}\wr(\mathbb{Z}\wr\mathbb{Z})$. It is shown [52, Example 4] that* $\text{Føl}_{\mathbb{Z}\wr(\mathbb{Z}\wr\mathbb{Z})}(n) \sim n^{(n^n)}$. *In particular,* $\text{Føl}_{\mathbb{Z}\wr(\mathbb{Z}\wr\mathbb{Z})}(n)$ *grows faster than* $\text{Føl}_{G_i}(n)$ *for every $i \geqslant 1$. However, it is not known whether or not there exists a Turing transducer $T \in \mathcal{T}$ for which a Cayley graph of $\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})$ is presented by T.*

6.4 Random walk and average length growth functions

In this section we discuss asymptotic behavior of average length growth functions of Turing transducers of the class $\mathcal{T}$ and its relation to random walks on graphs.

Recall first some necessary definitions [53]. Let G be an infinite group with a set of generators $Q = \{s_1, \dots, s_m\} \subseteq G$. Put $S = Q \cup Q^{-1} = \{s_1, \dots, s_m, s_1^{-1}, \dots, s_m^{-1}\}$. Let μ be a symmetric measure defined on S, i.e., $\mu(s) = \mu(s^{-1})$ for all $s \in S$. The convolution $\mu^{*n}(g)$ on B_n is defined as

$$\mu^{*n}(g) = \sum_{g=g_1 \dots g_n} \prod_{i=1,\dots,n} \mu(g_i),$$

where $g_i \in S$, $i = 1, \ldots, n$.

Let $c_n(g)$ be the number of words of length n over the alphabet S representing the element $g \in G$. If μ is the uniform measure on S, then $\mu^{*n}(g) = \frac{c_n(g)}{(2m)^n}$. Therefore, $\mu^{*n}(g)$ is the probability that a n–step simple symmetric random walk on the Cayley graph $\Gamma(G, S)$, which starts at the identity $e \in G$, ends up at the vertex $g \in G$. In this section we consider only uniform measures μ. We denote by $E_{\mu^{*n}}[\ell_S]$ the average value of the functional ℓ_S on the ball B_n with respect to the measure μ^{*n}. For some Cayley graphs of wreath products of groups we will show asymptotic behavior of $E_{\mu^{*n}}[\ell_S]$ of the form $E_{\mu^{*n}}[\ell_S] \asymp f(n)$, where $g(n) \asymp f(n)$ means that $\delta_1 f(n) \leqslant g(n) \leqslant \delta_2 f(n)$ for some constants $\delta_2 \geqslant \delta_1 > 0$.

Let $T \in \mathcal{T}$ be a Turing transducer translating a language L into L'. Suppose that the Cayley graph $\Gamma(G, S)$ is presented by T. Let us choose any word $w_0 \in L$. The Turing transducer T provides the bijection $\psi : L \to G$ such that $\psi^{-1}(e) = w_0$. Therefore, we can consider the average of the functional $|w|$ on the ball B_n with respect to the measure μ^{*n}, where $|w|$ is the length of a word $w \in L$. The following claim is straightforward.

Claim 6.4.1. *For a n–step symmetric simple random walk on the Cayley graph $\Gamma(G, S)$, $E_{\mu^{*n}}[|w|] = \ell_n$, where ℓ_n is the nth element of the average length growth function of the pair (T, w_0).*

The following proposition relates ℓ_n and $E_{\mu^{*n}}[\ell_S]$.

Proposition 6.4.1. *There exist constants C_1 and C_2 such that $\ell_n \leqslant C_1 E_{\mu^{*n}}[\ell_S] + C_2$ for all n.*

Proof: Recall that, by definition, there exists a constant c such that for every input $x \in L$ and an output $y_j \in L, j = 1, \ldots, 2m$, $|y_j| \leqslant |x| + c$. Put $C_1 = c$ and $C_2 = |w_0|$. Therefore, we obtain that the inequality $\ell_n \leqslant C_1 E_{\mu^{*n}}[\ell_S] + C_2$ holds for all n. $\square$

It is easy to give examples of Turing transducers of the class $\mathcal{T}$ for which $\ell_n \asymp \sqrt{n}$ and the growth function b_n is polynomial using a unary–like representation of integers. See Example 6.4.1 below.

Example 6.4.1. *Let $Q = \{s_1, \ldots, s_m\}$ be the standard set of generators of the group $\mathbb{Z}^m$, where $s_i = (\delta_i^1, \ldots, \delta_i^m)$ and $\delta_i^j = 1$ if $i = j$, $\delta_i^j = 0$ if $i \neq j$. Put $S = Q \cup Q^{-1}$. It can be seen that there exists a $(2m+1)$–tape Turing transducer $T \in \mathcal{T}$ translating a language L into a language $L' \subseteq L^{2m}$ for which $\Gamma(\mathbb{Z}^m, S)$ is presented by T. It is easy to see that a language L and an isomorphism between Γ_T and $\Gamma(\mathbb{Z}^m, S)$ can be chosen in a way that $\ell_S(g) = |w|$, where $g \in \mathbb{Z}^m$ and $w \in L$ is the word corresponding to g. In particular, put the empty word ϵ to be the representative of the identity $(0, \ldots, 0) \in \mathbb{Z}^m$. Therefore, for such a Turing transducer T, $\ell_n = E_{\mu^{*n}}[\ell_S]$. For a symmetric simple random walk on the m–dimensional grid, $E_{\mu^{*n}}[\ell_S] \asymp \sqrt{n}$. For the proof see, e.g., [54]. So, for the pair (T, ϵ), $\ell_n \asymp \sqrt{n}$. The growth function b_n of (T, ϵ) is polynomial. Thus, we obtain $(2m+1)$–tape Turing transducers $T_m, m = 1, \ldots, \infty$ for which $\ell_n \asymp \sqrt{n}$ and the growth function b_n is polynomial.*

A more complicated technique is required in order to show an example of a Turing transducer of the class $\mathcal{T}$ for which $\ell_n \asymp \sqrt{n}$ and the growth function b_n is exponential. We will construct such a Turing transducer in Lemma 6.4.1.

Let H be a group with a set of generators $S_H = \{t_1, \ldots, t_k\}$. Consider the group $\mathbb{Z}_2 \wr H$. Let $h \in \mathbb{Z}_2^{(H)}$ be the function $h : H \to \mathbb{Z}_2$ such that $h(g) = e$ if $g \neq e$ and $h(e) = a$, where a is the nontrivial element of $\mathbb{Z}_2$. Let $Q = \{t, th, ht, hth | t \in S_H\}$ be the set of generators of the group $\mathbb{Z}_2 \wr H$. Put $S = Q \cup Q^{-1}$. Consider a symmetric simple random walk on the Cayley graph $\Gamma(\mathbb{Z}_2 \wr H, S)$. It is easy to see that a n–step random walk on $\Gamma(\mathbb{Z}_2 \wr H, S)$ corresponds to a n–step random walk on H. Put $P = S_H \cup S_H^{-1}$. Let R_n be the number of different vertices visited after walking n steps on $\Gamma(H, P)$. We call R_n the range of a n–step random walk on $\Gamma(H, P)$. In the following proposition the asymptotic behavior of $E_{\mu^{*n}}[\ell_S]$ is expressed in terms of $E_{\mu^{*n}}[R_n]$ – the average range for a n–step random walk on $\Gamma(H, P)$.

Proposition 6.4.2. *Let H and S be as above. For a symmetric simple random walk on $\Gamma(\mathbb{Z}_2 \wr H, S)$, $E_{\mu^{*n}}[\ell_S] \asymp E_{\mu^{*n}}[R_n]$.*

Proof: For the proof see [55, Lemma 2]. □

Lemma 6.4.1. *There exists a set of generators S_1 of the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ for which the following statements hold.*

(a) For a simple symmetric random walk on $\Gamma(\mathbb{Z}_2 \wr \mathbb{Z}, S_1)$, $E_{\mu^{*n}}[\ell_{S_1}] \asymp \sqrt{n}$.

(b) There exists a Turing transducer $T_1 \in \mathcal{T}$ such that $\Gamma(\mathbb{Z}_2 \wr \mathbb{Z}, S_1)$ is presented by T_1 and $\ell_n \asymp \sqrt{n}$.

Proof: Let us consider the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$. Let t be a generator of the subgroup $\mathbb{Z} \leqslant \mathbb{Z}_2 \wr \mathbb{Z}$ and $h \in \mathbb{Z}_2^{(\mathbb{Z})} \leqslant \mathbb{Z}_2 \wr \mathbb{Z}$ be the function $h : \mathbb{Z} \to \mathbb{Z}_2$ such that $h(z) = e$ if $z \neq 0$ and $h(0) = a$. Let $Q_1 = \{t, th, ht, hth\}$ be the set of generators of $\mathbb{Z}_2 \wr \mathbb{Z}$ and $S_1 = Q_1 \cup Q_1^{-1}$. For a simple symmetric random walk on $\Gamma(\mathbb{Z}, \{t, t^{-1}\})$, $E_{\mu^{*n}}[R_n] \sim \sqrt{n}$, where $\sim$ here means asymptotic equivalence. For the proof see, e.g., [54]. Therefore, from Proposition 6.4.2 we obtain that for a simple symmetric random walk on $\Gamma(\mathbb{Z}_2 \wr \mathbb{Z}, S_1)$, $E_{\mu^{*n}}[\ell_{S_1}] \asymp \sqrt{n}$.

Let $Q_1' = \{t, h\}$ be a set of generators of $\mathbb{Z}_2 \wr \mathbb{Z}$. Put $S_1' = Q_1' \cup Q_1'^{-1} = \{t, t^{-1}, h\}$. By Theorem 4.2.1 and Proposition 4.2.1, there is an automatic representation of the Cayley graph $\Gamma(\mathbb{Z}_2 \wr \mathbb{Z}, S_1')$, the bijection $\psi_1 : L_1 \to \mathbb{Z}_2 \wr \mathbb{Z}$, for which the inequalities $\frac{1}{3}\ell_{S_1'}(g) + \frac{2}{3} \leqslant |w| \leqslant \ell_{S_1'}(g) + 1$ hold for all $g \in \mathbb{Z}_2 \wr \mathbb{Z}$, where L_1 is a regular language, $w = \psi_1^{-1}(g) \in L_1$ is the word corresponding to g and $|w|$ is the length of w. It is easy to see that $\frac{1}{2}\ell_{S_1}(g) \leqslant \ell_{S_1'}(g) \leqslant 3\ell_{S_1}(g)$. Therefore, we obtain that $\frac{1}{6}\ell_{S_1}(g) + \frac{2}{3} \leqslant |w| \leqslant 3\ell_{S_1}(g) + 1$ for all $g \in \mathbb{Z}_2 \wr \mathbb{Z}$. This implies that $\frac{1}{6}E_{\mu^{*n}}[\ell_{S_1}] + \frac{2}{3} \leqslant E_{\mu^{*n}}[|w|] \leqslant 3E_{\mu^{*n}}[\ell_{S_1}] + 1$. The bijection $\psi_1 : L_1 \to \mathbb{Z}_2 \wr \mathbb{Z}$ provides an automatic representation for the Cayley graph $\Gamma(\mathbb{Z}_2 \wr \mathbb{Z}, S_1)$. By Lemma 6.1.2, we obtain that there exists a 9–tape Turing transducer $T_1 \in \mathcal{T}$ translating the language L_1 into some language $L_1' \subseteq L_1^8$ for which $\Gamma(\mathbb{Z}_2 \wr \mathbb{Z}, S_1)$ is presented by T_1. Therefore, we obtain that for T_1, $\ell_n \asymp \sqrt{n}$. Since the growth function of the group $\mathbb{Z}_2 \wr \mathbb{Z}$ is exponentinal, the growth function b_n of T_1 is exponential. $\square$

It is easy to give examples of Turing transducers of the class $\mathcal{T}$ for which $\ell_n \asymp n$ and the growth function b_n is exponential. See Example 6.4.2 below.

Example 6.4.2. *Let F_m be the free group over m generators $s_1, \dots, s_m$. Put $Q = \{s_1, \dots, s_m\}$ and $S = Q \cup Q^{-1}$. There exists a natural automatic representation of the Cayley graph $\Gamma(F_m, S)$, the bijection $\psi : L \to F_m$, for which L is the language of all reduced words over the alphabet S. In particular, the empty word ϵ represents the identity $e \in F_m$. The bijection ψ maps a word $w \in L$ into the corresponding group element of F_m. It is clear* *that* $\ell_S(g) = |w|$*, where* $w = \psi^{-1}(g)$*. For a symmetric simple random walk on* $\Gamma(F_m, S)$*,* $E_{\mu^{*n}}[\ell_S] \asymp n$*. Therefore,* $E_{\mu^{*n}}[|w|] \asymp n$*. Therefore, for each* $m > 1$ *we obtain the pair* $(T, \epsilon), T \in \mathcal{T}$ *for which* $\ell_n \asymp n$*. Since the growth function of the free group* F_m *is exponential, the growth function* b_n *of the pair* (T, ϵ) *is exponential.*

Is there a Turing transducer of the class $\mathcal{T}$ for which ℓ_n grows between $\sqrt{n}$ and n? We will answer this question positively in Theorem 6.4.1 which follows from Proposition 6.4.3 below.

Let G be a group with a set of generators $S_G = \{g_1, \ldots, g_m\}$. Put $P = S_G \cup S_G^{-1}$. Assume that for a symmetric simple random walk on $\Gamma(G, P)$, $\ell_n(\mu) \asymp n^\alpha$ for some $0 < \alpha \leqslant 1$. Consider the wreath product $G \wr \mathbb{Z}$. Let t be a generator of the subgroup $\mathbb{Z} \leqslant G \wr \mathbb{Z}$. Let $h_i \in G^{(\mathbb{Z})} \leqslant G \wr \mathbb{Z}$, $i = 1, \ldots, m$ be the functions $h_i : \mathbb{Z} \to G$ such that $h_i(z) = e$ if $z \neq 0$ and $h_i(0) = g_i$. Put $Q = \{h_i^p t h_j^q \,|\, i, j = 1, \ldots, m; p, q = -1, 0, 1\}$ to be the set of generators of the group $G \wr \mathbb{Z}$ and $S = Q \cup Q^{-1}$. Consider a n–step random walk on $\Gamma(G \wr \mathbb{Z}, S)$. The following proposition shows asymptotic behavior of $E_{\mu^{*n}}[\ell_S]$.

Proposition 6.4.3. *Let* G*,* S *and* α *be as above. For a symmetric simple random walk on* $\Gamma(G \wr \mathbb{Z}, S)$*,* $E_{\mu^{*n}}[\ell_S] \asymp n^{\frac{1+\alpha}{2}}$*.*

Proof: For the proof see [56, Lemma 3]. □

Theorem 6.4.1. *For every* $\alpha < 1$ *there exists a Turing transducer* $T \in \mathcal{T}$ *for which* $\ell_n \asymp n^\beta$ *for some* β *such that* $\alpha < \beta < 1$ *and the growth function* b_n *is exponential.*

Proof: Let us consider the sequence of wreath products $G_m, m = 1, \ldots, \infty$ such that $G_1 = \mathbb{Z}_2 \wr \mathbb{Z}$ and $G_{m+1} = G_m \wr \mathbb{Z}$, $m \geqslant 1$. From

Lemma 6.4.1 (a) and Proposition 6.4.3 we obtain that for every $m > 1$ there exists a proper set of generators $Q_m \subseteq G_m$ such that for a symmetric simple random walk on the Cayley graph $\Gamma(G_m, S_m)$, $E_{\mu^{*n}}[\ell_{S_m}] \asymp n^{1-\frac{1}{2^m}}$, where $S_m = Q_m \cup Q_m^{-1}$. By Theorem 4.3.1 and Proposition 4.3.1, for every $m > 1$ there is an automatic representation of the Cayley graph $\Gamma(G_m, S'_m)$, the bijection $\psi_m : L_m \to G_m$, for which the inequalities $\delta'_1 \ell_{S'_m}(g) + \lambda'_1 \leqslant |w| \leqslant \delta'_2 \ell_{S'_m}(g) + \lambda'_2$ hold for all $g \in G_m$ for some constants $\delta'_2 > \delta'_1 > 0, \lambda'_1, \lambda'_2$, where L_m is a regular language and $S'_m = Q'_m \cup {Q'_m}^{-1}$ for some proper set of generators $Q'_m \subseteq G_m$, and $w = \psi_m^{-1}(g)$ is the word representing g. Therefore, the inequalities $\delta_1 \ell_{S_m}(g) + \lambda_1 \leqslant |w| \leqslant \delta_2 \ell_{S_m}(g) + \lambda_2$ hold for all $g \in G_m$ for some constants $\delta_2 > \delta_1 > 0, \lambda_1, \lambda_2$. This implies that $\delta_1 E_{\mu^{*n}}[\ell_{S_m}] + \lambda_1 \leqslant E_{\mu^{*n}}[|w|] \leqslant \delta_2 E_{\mu^{*n}}[\ell_{S_m}] + \lambda_2$. Therefore, $E_{\mu^{*n}}[|w|] \asymp n^{1-\frac{1}{2^m}}$.

For every $m > 1$ the bijection $\psi_m : L_m \to G_m$ provides an automatic representation of the Cayley graph $\Gamma(G_m, S_m)$. It follows from Lemma 6.1.2 that there is a $(k_m + 1)$–tape Turing transducer $T_m \in \mathcal{T}$ translating the language L_m into $L'_m \subseteq L_m^{k_m}$ for which, after proper relabeling, $\Gamma_{T_m} \cong \Gamma(G_m, S_m)$. The numbers $k_m, m = 1, \dots, \infty$ can be obtained recurrently as follows. It is easy to see that $k_{m+1} = 2(k_m + 1)^2$ for $m \geqslant 1$ and $k_1 = 8$, which is simply the number of elements in S_1 (see Lemma 6.4.1). So, we obtain that for $T_m, m > 1$, $\ell_n \asymp n^{1-\frac{1}{2^m}}$. For every $m > 1$, since the growth function of the group G_m is exponential, the growth function b_n of T_m is exponential. $\square$

Chapter 7

Conclusion and open problems

In this thesis we studied representations of three important families of structures with automata. The first one is the class of the Baumslag–Solitar groups and it is discussed in Chapter 3. The key results of Chapter 3 are Theorem 3.2.1 and Proposition 3.2.1. The second one is the family of the wreath products of groups and it is discussed in Chapter 4. The key results of Chapter 4 are as follows. In Section 4.2 these results are Theorem 4.2.1 and Proposition 4.2.1. In Section 4.3: Theorems 4.3.1 and 4.3.2, and Proposition 4.3.1. In Section 4.4: Theorems 4.4.1 and 4.4.2, and Propositions 4.4.4and 4.4.7. In Section 4.5: Theorems 4.5.1 and 4.5.2, and Proposition 4.5.1. In Section 4.6: Theorem 4.6.1. The third one is the family of transitive non–Cayley graphs including the Diestel–Leader graph and it is discussed in Chapter 5. The key results of this chapter are Proposition 5.1.3 and Theorem 5.2.1. Furthermore, in this thesis we investigated the characterization problem of Cayley automatic groups. In order to address this problem, in Chapter 6 we introduce and then study three numerical characteristics for

Turing transducers from the special class. The key results of this chapter are Theorems 6.3.1 and 6.4.1.

The open questions below suggest possible directions for future work.

Let us consider some problems that are apparent from the results of this thesis. In Theorem 4.2.1 we show that $\mathbb{Z}_2 \wr \mathbb{Z}$ is Cayley biautomatic and, therefore, it is Cayley automatic. Theorem 4.4.1 shows that $\mathbb{Z}_2 \wr F_2$ is $\mathcal{P}_1$–Cayley automatic.

Question 7.0.1. *Is the group $\mathbb{Z}_2 \wr F_2$ Cayley automatic?*

Similarly, Theorem 4.6.1 shows that there exists an $\mathcal{I}$–Cayley automatic representation $\psi : L \to \mathbb{Z}_2 \wr \mathbb{Z}^2$ for which the domain L is regular.

Question 7.0.2. *Is the group $\mathbb{Z}_2 \wr \mathbb{Z}^2$ Cayley automatic?*

A positive answer to one of the questions 7.0.1–7.0.2 would give a new non–trivial construction for Cayley automatic representations of the wreath products of groups. On the other hand, a negative answer to one of these questions would give a method to prove non–Cayley automaticity of groups. For given Cayley automatic groups A and B, the reader may ask whether the group $A \wr B$ is Cayley automatic. One of our future goals is to find all Cayley automatic wreath products of groups. We assume that a complete classification of Cayley automatic wreath products of groups is a very important step in characterization of Cayley automatic groups.

An alternative approach to investigate the characterization problem of Cayley automatic groups is to study numerical characteristics of Turing transducers from the class $\mathcal{T}$. Theorem 6.3.1 shows that for every integer $i \geqslant 1$ there exists a Turing transducer of the class $\mathcal{T}$ for which $f_n \sim n^{(n^i)}$.

Question 7.0.3. *Is there a Turing transducer $T \in \mathcal{T}$ for which the Følner function grows faster than $n^{(n^i)}$ for all $i \geqslant 1$?*

Theorem 6.4.1 claims that for every $\alpha < 1$ there exists a Turing transducer $T \in \mathcal{T}$ for which $\ell_n \asymp n^{\beta}$ for some β such that $\alpha < \beta < 1$.

Question 7.0.4. *Is there a Turing transducer $T \in \mathcal{T}$ for which ℓ_n grows faster than n^{α} for every $\alpha < 1$ but slower than n?*

One of our future goals is to get new examples of asymptotic behavior of the numerical characteristics for Turing transducers of the class $\mathcal{T}$.

One of the important classes of Cayley automatic representations is the class of geodesic Cayley automatic representations (see Definition 4.5.1). One of our future goals is to give a characterization of geodesic Cayley automatic representations. The $\mathcal{P}_1$–Cayley automatic representation of the group $\mathbb{Z}_2 \wr F_2$ constructed in Theorem 4.4.1 is geodesic. On the other hand, the $\mathcal{I}$–Cayley automatic representation of the group $\mathbb{Z}_2 \wr \mathbb{Z}^2$ constructed in Theorem 4.6.1 is not geodesic.

Question 7.0.5. *Is there a geodesic $\mathcal{P}_1$–, $\mathcal{P}_1^c$– or $\mathcal{I}$–Cayley automatic representation of the group $\mathbb{Z}_2 \wr \mathbb{Z}^2$?*

Answering the question 7.0.5 would be a good advancement in understanding geodesic Cayley automatic representations.

The last two questions 7.0.6 and 7.0.7 below are not relevant to the problem of characterization of Cayley automatic groups. However, answering one of these questions would show some limitations for Cayley automatic representations of groups. Let us fix some integer $z_0 \in \mathbb{Z}$. Example 4.2.1 together with Propositions 4.2.3 and 4.2.4 show that there are Cayley automatic representations of the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ for which the following requests are not decidable by a finite automaton:

- Is the lamplighter at the position $z = z_0$?
- Is the lamp at the position $z = z_0$ lit?

The subgroup $\mathbb{Z} \leqslant \mathbb{Z}_2 \wr \mathbb{Z}$ is the set of elements of $\mathbb{Z}_2 \wr \mathbb{Z}$ for which all lamps are unlit. It can be seen that for both representations (the one described in Theorem 4.2.1 and the one in Example 4.2.1) the languages of the words representing the elements of the subgroup $\mathbb{Z} \leqslant \mathbb{Z}_2 \wr \mathbb{Z}$ are regular.

Question 7.0.6. *Is there a Cayley automatic representation of the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ for which the language of the words representing the elements of the subgroup $\mathbb{Z} \leqslant \mathbb{Z}_2 \wr \mathbb{Z}$ is not regular?*

Let us consider now two subsets of $\mathbb{Z}_2 \wr \mathbb{Z}$: $S_0 = \{(f,z) \mid f(z) = 0\}$ and $S_1 = \{(f,z) \mid f(z) = 1\}$. It is easy to see that $S_0 \cup S_1 = \mathbb{Z}_2 \wr \mathbb{Z}$ and $S_0 \cap S_1 = \varnothing$. It can be seen that for both representations (the one described in Theorem 4.2.1 and the one in Example 4.2.1) the languages of the words representing the elements of S_0 are regular.

Question 7.0.7. *Is there a Cayley automatic representation of the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ for which the language of the words representing the elements of S_0 is not regular?*

Bibliography

[1] O. Kharlampovich, B. Khoussainov, A. Miasnikov, From automatic structures to automatic groups, Groups, Geometry, and Dynamics 8 (1) (2014) 157–198.
URL `arxiv.org/abs/1107.3645v2`

[2] D. B. A. Epstein, J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson, W. P. Thurston, Word Processing in Groups, Jones and Barlett Publishers. Boston, MA, 1992.

[3] L. Libkin, Elements of finite model theory, Springer, 2012.

[4] B. Khoussainov, A. Nerode, Automatic presentations of structures, in: D. Leivant (Ed.), Logic and Computational Complexity, Vol. 960 of Lecture Notes in Computer Science, Springer Berlin Heidelberg, 1995, pp. 367–392.

[5] B. Khoussainov, M. Minnes, Three lectures on automatic structures, Proceedings of Logic Colloquium (2007) 132–176.

[6] B. Khoussainov, A. Nerode, Open questions in the theory of automatic structures., Bulletin of the EATCS 94 (2008) 181–204.

[7] A. Miasnikov, Z. Šunić, On automatic transitive graphs, in: A.-H. Dediu, C. Martín-Vide (Eds.), Language and Automata Theory and Applications 2012, Vol. 7183 of Lecture Notes in Computer Science, Springer Berlin Heidelberg, 2012, pp. 401–407.

[8] O. Bogopolski, A. Martino, E. Ventura, Orbit decidability and the conjugacy problem for some extensions of groups, Trans. Amer. Math. Soc. 362 (4) (2010) 2003–2036.

[9] Z. Šunić, E. Ventura, The conjugacy problem is not solvable in automaton groups, Journal of Algebra 364 (2012) 148–154.

[10] G. Levitt, Unsolvability of the isomorphism problem for [free abelian]–by–free groups, arXiv:0810.0935 [math.GR].

[11] G. Baumslag, Wreath products and finitely presented groups, Mathematische Zeitschrift 75 (1) (1961) 22–28.

[12] B. Khoussainov, S. Rubin, F. Stephan, Definability and regularity in automatic structures, in: STACS 2004, Springer, 2004, pp. 440–451.

[13] A. Nies, P. Semukhin, Finite automata presentable abelian groups, Annals of Pure and Applied Logic 161 (3) (2009) 458–467.

[14] P. J. Cameron, Homogeneous cayley objects, Europ. J. Combinatorics 21 (2000) 745–760.

[15] B. Khoussainov, A. Nies, S. Rubin, F. Stephan, Automatic structures: richness and limitations, Logical Method in Computer Science 3 (2:2) (2007) 1–18.

[16] C. Delhommé, Automaticité des ordinaux et des graphes homogènes., Comptes Rendus Mathematique 339 (1) (2004) 5–10.

[17] G. P. Oliver, R. M. Thomas, Automatic presentations for finitely generated groups, in: V. Diekert, B. Durand (Eds.), STACS 2005, Vol. 3404 of Lecture Notes in Computer Science, Springer Berlin Heidelberg, 2005, pp. 693–704.

[18] J. Burillo, M. Elder, Metric properties of Baumslag–Solitar groups, International Journal of Algebra and Computation 25 (5) (2015) 799–811. URL arxiv.org/abs/1402.3859

[19] D. Berdinsky, B. Khoussainov, On automatic transitive graphs, in: A. Shur, M. Volkov (Eds.), Developments in Language Theory 2014, Vol. 8633 of Lecture Notes in Computer Science, Springer International Publishing, 2014, pp. 1–12.

[20] D. Berdinsky, B. Khoussainov, Cayley automatic representations of wreath products, International Journal of Foundations of Computer Sceince 27 (2) (2016) 147 – 159.
URL http://arxiv.org/abs/1511.01630

[21] D. Berdinsky, Cayley automatic groups and numerical characteristics of Turing transducers, in: S. Brlek, C. Reutenauer (Eds.), Developments in Language Theory 2016, Vol. 9840 of Lecture Notes in Computer Science, Springer–Verlag Berlin Heidelberg, 2016, pp. 26 – 37.
URL http://arxiv.org/abs/1606.08357

[22] J. Hopcroft, R. Motwani, J. D. Ullman, Introduction to Automata Theory, Languages, and Computation, Addison–Wesley, 2001.

[23] P. Odifreddi, Classical recursion theory. Volume II, North–Holland, 1999.

[24] S. Eilenberg, C. C. Elgot, J. C. Shepherdson, Sets recognized by n–tape automata, Journal of Algebra 13 (4) (1969) 447–464.

[25] R. Büchi, Weak second–order arithmetic and finite automata, Z. Math. Logik Grundlagen Math. 6 (1960) 66–92.

[26] C. C. Elgot, Decision problems of finite automata design and related arithmetics, Trans. Amer. Math. Soc. 98 (1961) 21–51.

[27] S. Rubin, Automatic Structures, PhD Thesis, The University of Auckland, 2004.

[28] B. R. Hodgson, Théories décidables par automate fini, PhD Thesis, University of Montréal, 1976.

[29] A. Blumensath, Automatic Structures, Diploma Thesis, RWTH, 1999.

[30] A. A. Nabebin, Multitape automata in a unary alphabet, Trudy Moskov. Orden. Lenin. Energet. Inst. 292 (1976) 7–11.

[31] A. Blumensath, E. Grädel, Automatic structures, In 15th Symposium on Logic in Computer Science LICS (2000) 51–62.

[32] W. A. Hodges, Model Theory, Cambridge University Press, 1993.

[33] R. C. Lyndon, P. E. Schupp, Combinatorial group theory, Springer–Verlag Berlin Heidelberg New York, 1977.

[34] G. Baumslag, D. Solitar, Some two–generator one–relator non–Hopfian groups, Bulletin of American Mathematical Society 68 (1962) 199–201.

[35] M. Elder, J. Taback, c–graph automatic groups, Journal of Algebra 413 (2014) 289–319.

[36] M. I. Kargapolov, J. I. Merzljakov, Fundamentals of the theory of groups, Springer–Verlag New York Inc., 1979.

[37] A. Erschler, Generalized wreath products, International Mathematics Research Notices Article ID 57835 (2006) 1–14.

[38] S. Cleary, J. Taback, Dead end words in lamplighter groups and other wreath products, The Quarterly Journal of Mathematics 56 (2) (2005) 165–178.

[39] G. Baumslag, M. Shapiro, H. Short, Parallel poly–pushdown groups, Journal of Pure and Applied Algebra 140 (3) (1999) 209–227.

[40] R. H. Gilman, Formal languages and their application to combinatorial group theory, in: A. V. Borovik (Ed.), Groups, Languages, Algorithms, Vol. 378 of Contemporary Mathematics, American Mathematical Society, 2005, pp. 1–36.

[41] R. Gilman, M. Shapiro, On groups whose word problem is solved by a nested stack automaton, arXiv:math/9812028.

[42] C. Thomassen, M. E. Watkins, Infinite vertex–transitive, edge–transitive, non–1–transitive graphs, Proceedings of American Mathematical Society 105 (1) (1989) 258–261.

[43] R. Deistel, I. Leader, A conjecture concerning a limit of non–Cayley graphs, Journal of Algebraic Combinatorics 14 (1) (2001) 17–25.

[44] M. Gromov, Geometric group theory, Vol. 2: Asymptotic invariants of infinite groups, edited by A. Niblo and Martin A. Roller, London Math. Soc. Lecture Note Ser., V. 182, Cambridge Univ. Press, Cambridge, 1993.

[45] A. Eskin, D. Fisher, K. Whyte, Coarse differentiation of quasi–isometries I: spaces not quasi-isometric to Cayley graphs, Ann. of Math 176 (1) (2012) 221–260.

[46] A. Meduna, Automata and Languages. Theory and Applications, Springer–Verlag London Ltd., 2000.

[47] C. Calude, E. Calude, B. Khoussainov, Deterministic automata simulation, universality and minimality, Annals of Pure and Applied Logic 90 (1) (1997) 263–276.

[48] C. S. Calude, E. Calude, B. Khoussainov, Finite nondeterministic automata: Simulation and minimality, Theoretical Computer Science 242 (1) (2000) 219–235.

[49] M. Stoll, Rational and transcendental growth series for the higher Heisenberg groups, Invent. Math. 126 (1996) 85–109.

[50] A. Miasnikov, D. Savchuk, An example of an automatic graph of intermediate growth, Ann. Pure Appl. Logic 166 (10) (2015) 1037–1048.

[51] A. Vershik, Countable groups that are close to finite ones, Appendix in F. P. Greenleaf, Invariant Means on Topological Groups and their Applications, Moscow, Mir, 1973 (in Russian), a revised English translation: Amenability and approximation of infinite groups, Selecta Math. 2 (4) (1982) 311–330.

[52] A. Erschler, On Isoperimetric Profiles of Finitely Generated Groups, Geometriae Dedicata 100 (1) (2003) 157–171.

[53] A. Vershik, Numerical characteristics of groups and corresponding relations, Journal of Mathematical Sciences 107 (5) (2001) 4147–4156.

[54] F. Spitzer, Principles of random walk, Van Nostrand, Princeton, 1964.

[55] A. Dyubina, An example of the rate of growth for a random walk on a group, Russian Mathematical Surveys 54 (5) (1999) 1023–1024.

[56] A. Erschler, On the asymptotics of drift, Journal of Mathematical Sciences 121 (3) (2004) 2437–2440.