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Chapter 1

Introduction

This thesis is in the mathematical field of combinatorics. It considers topological properties of graphs by topological methods. The terminology and notation used will mostly be consistent with Jonathan L. Gross and Thomas W. Tucker's *Topological Graph Theory* [14].

1.1 Preliminaries

A ***graph*** $G = (V, E)$ is a set V of ***vertices*** and a set E of *edges*. ***Edges*** are unordered pairs u, v of vertices, denoted uv. Two edges of a graph are ***adjacent*** if they share a common vertex. Two vertices are ***adjacent*** if they share a common edge. An edge and a vertex on that edge are ***incident***. The ***neighbours*** of a vertex u are all the vertices adjacent to u. The ***valency*** (or ***degree***) of a vertex in a graph is the number of edges incident to the vertex, with loops counted twice. A vertex with valency n is called an n***-valent*** vertex. An n***-regular*** graph has vertices of valency n only. The maximum valency of a graph G is denoted by $\Delta(G)$.

(a) A loop B_1 (b) Multiple edges D_2

Figure 1.1: A loop on a single vertex is a bouquet B_1. Double edges on a pair of vertices is a dipole graph D_2.

A ***path*** in a graph is a sequence of vertices such that from each of its vertices (except the last vertex in the sequence) there is an edge to the next vertex in the sequence. A closed path is called a ***cycle***. Two vertices u and v are ***connected*** if G contains a path between u and v. A graph is ***connected*** if every pair of vertices in the graph is connected. A ***loop*** is an edge that connects a vertex to itself (see Figure 1.1a). If there are more than one edge between a pair of vertices, we call them ***multiple edges*** (see Figure 1.1b). The graphs we talk about in this thesis allow loops and multiple edges.

A graph of two vertices and n multiple edges joining them is called a ***dipole*** graph, denoted by D_n. Figure 1.1b is D_2. A graph of n loops joining at a single vertex is called a ***bouquet of circles***, or simply a bouquet, denoted by B_n. Figure 1.1a is B_1.

All graphs in this thesis are connected graphs (except where otherwise noted), allowing loops and double edges.

If the vertices and edges of a graph G' form subsets of the vertices and edges of a given graph G, we call G' a ***subgraph*** of G. We can also call a subgraph G' a ***root*** of G. G can be also called a G'***-rooted*** graph.

To ***delete a vertex*** u from a graph G (denoted by $G - u$) is to remove vertex u and all edges incident to u from G. To ***delete an edge*** $e = uv$ from a graph G (denoted by $G - e$) is to remove edge e from G. To ***contract an edge*** $e = uv$ in a graph G (denoted by $G \cdot e$) is to remove edge e from G and replace vertices u and v by a new vertex w, and replace any edges ux incident to u and vy incident to v by wx and wy respectively.

A ***cut-vertex*** is a vertex of a connected graph G, such that its removal results in disconnecting G. Similarly, if the removal of a subgraph of G results in disconnection, the subgraph is called a ***cut-subgraph***.

For a vertex set U and two non-empty subsets of U, U_1 and U_2, if $U_1 \cap U_2$ is empty and $U_1 \cup U_2 = U$, we call U_1 and U_2 a ***bipartition*** of U.

A ***complete graph*** K_n on n vertices is a graph with edge set $E = \{uv \mid u, v \in V \text{ and } u \neq v\}$. In Figure 1.2, the left graph is K_5. A ***bipartite graph*** $G = \{V \cup V', E\}$ is a graph whose vertices can be partitioned into two sets V and V' such that there are no edges between vertices in the same set. A ***complete bipartite graph*** $K_{m,n}$ on $m+n$ vertices is a bipartite graph $G = \{V \cup V', E\}$ such that $|V| = m$ and $|V'| = n$ and $E = \{uv \mid u \in V \ v \in V'\}$. The graph $K_{m,n}$ has mn edges. In Figure 1.2, the graph on the right is $K_{3,3}$.

A graph also admits a natural topology, called the ***graph topology***, by identifying every edge $\{v_i, v_i\}$ with the unit interval $I = [0, 1]$ and gluing them together at coincident vertices. A ***surface*** is a 2-dimensional compact manifold without boundary. Given a graph G and a surface S, if there is a homeomorphism $\phi : G \to S$ such that each connected component of $S - \phi(G)$ is homeomorphic to an open disc, then $\phi(G)$ is an ***embedding*** on S. This means a graph G can be embedded on surface S if it can be drawn on the surface without any edge crossings. The ***genus*** of a connected orientable

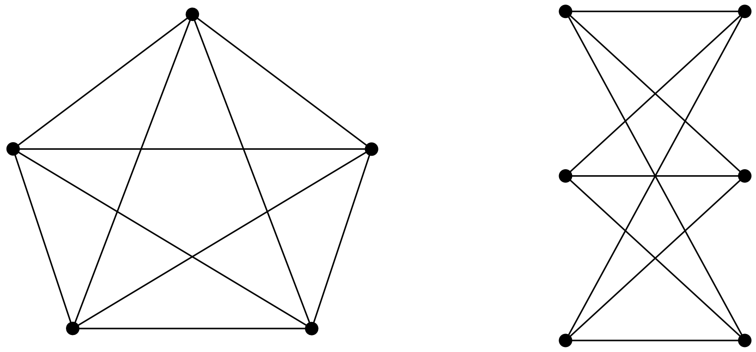


Figure 1.2: The complete graph on five vertices K_5 and the complete bipartite graph on six vertices $K_{3,3}$

surface is an integer representing the maximum number of cuts along non-intersecting closed simple curves without rendering the resulting manifold disconnected. It equals the number of handles on the surface. The orientable surface of genus zero is the ***plane*** (or equivalently the ***sphere***). The orientable surface of genus one is the ***torus*** which adds a handle on sphere.

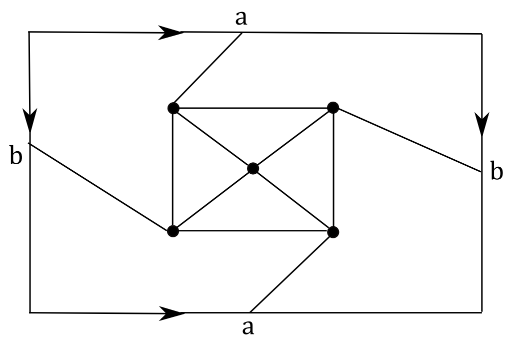
Note that a sphere is compact, but a plane is not compact in general topology. In this thesis, we see the plane the same as the sphere.

In this thesis, we only discuss embeddings of graphs on *orientable* surfaces.

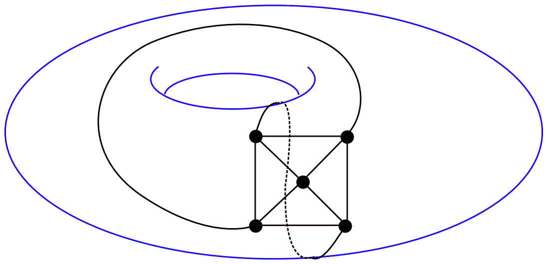
The well known Euler's formula gives the relation between an embedding of a graph and the orientable surface it is embedded on:

$$2 - 2g = v - e + f$$

where v, e and f are respectively the number of vertices, edges and faces of an embedding, and g is the genus of the surface. Figures 1.3a and 1.3b show an embedding of K_5 on a surface of genus 1 with $v = 5$, $e = 10$ and $f = 5$.



(a) An embedding of K_5 on a torus in rectangle shape



(b) An embedding of K_5 on a torus

Figure 1.3: An embedding of K_5 on torus which is a surface of genus 1. The figure on the left is a 2-dimensional rectangle, which is cut from the torus vertically first and then horizontally.

To check whether a graph can be embedded on the plane gives the question of ***planarity test***. There are many well known algorithms for planarity testing. Kuratowski published the first paper on characterizing the planarity of a graph.

A ***subdivision*** of a graph results from inserting some 2-valent vertices into edges.

Theorem 1.1.1. *(Kuratowski's Theorem) [19] A finite graph is planar if and only if it does not contain a subgraph that is a subdivision of K_5 or $K_{3,3}$.*

The above theorem can be used to test the planarity of small graphs by hand. However, its computational complexity is very high. It can not be used to design any algorithms practically. In recent years, linear algorithms have been well developed in different ways and utilized by some mathematical software. Magma uses Boyer and Myrvold's edge addition method in [6]. Fraysseix and Mendez introduce Trémaux trees to do the test in [9].

Most questions in the field of genus embedding can not be answered as easily as the question of planarity.

Let us review some basic definitions first before the questions.

An embedded graph uniquely defines cyclic orders of edges incident to the same vertex. The set of all these cyclic orders is called a rotation system which will be explained at page 7. Embeddings with the same rotation system are considered to be ***equivalent***. In this thesis, we use an embedding of a graph to represent an equivalent class of embeddings. We use $g_h(G)$ to denote the number of embeddings of graph G on an orientable surface of genus h.

The ***genus of a graph*** $\gamma_{min}(G) = min\{h : g_h(G) > 0\}$ is the minimum genus of a surface that the graph can be embedded on. Every graph also has its ***maximum genus*** $\gamma_{max}(G) = max\{h : g_h(G) > 0\}$, which is the maximum genus of a surface that the graph can be embedded on.

Theorem 1.1.2. *[10] A graph can be embedded on an orientable surface of any genus between its minimum and maximum genus.*

For a given graph, how many distinct embeddings does it have on each surface of genus h gives the question of ***genus distribution*** which is represented by a polynomial.

$$g(G) = g_0(G) + g_1(G)x + g_2(G)x^2 + g_3(G)x^3 + \cdots = \sum_{n=0}^{\infty} g_n(G)x^n$$

There are two fundamental ways to calculate the genus distribution of a graph. The first is the ***face-tracing method*** [14]. It is a well-used topological method in this field, and will be used in this thesis. The second is the ***joint tree method*** introduced by Liu [26]. Any graph's genus distribution can be calculated by both of the two ways with the same amount of calculations which is at least

$$\Pi_{i\geq 2}((i-1)!)^{n_i}$$

where n_i is the number of vertices with valency i in G [26].

Rotation systems are important for applying the face-tracing method. Let $G = (V, E)$ be a graph. A ***rotation*** at $v \in V$ is

$$\rho(v) = (e_1(v), e_2(v), \ldots, e_{\rho(v)}(v))$$

which is a cyclic ordering of the edges at v. A ***rotation system*** of G is $\rho(G) = \{\rho(v)|\forall v \in V\}$, which is a collection of clockwise rotations for **all** vertices of the graph.

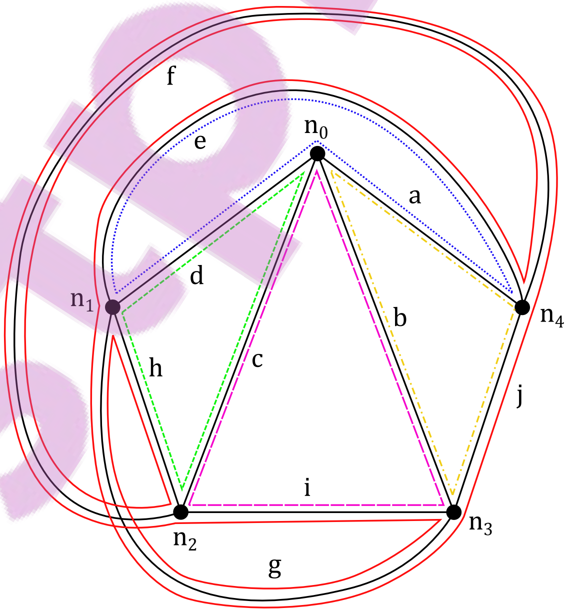


Figure 1.4: A rotation system of K_5 with five face walks

For the given rotation system of K_5 in Figure 1.4, the rotation system is:

$$\begin{array}{lllll} n_0: & \{a & b & c & d\} \\ n_1: & \{d & h & g & e\} \\ n_2: & \{h & c & i & f\} \\ n_3: & \{b & j & g & i\} \\ n_4: & \{a & e & f & j\} \end{array}$$

Theorem 1.1.3. *[14] Every rotation system for a graph G induces a unique embedding of G into an orientable surface. Conversely, every embedding of a graph G into an orientable surface induces a unique rotation system for G.*

This means for a graph G there is a bijection between its rotation systems and embeddings. There are $\Pi_{i\geq 2}((i-1)!)^{n_i}$ rotation systems for a graph G. So there are $\Pi_{i\geq 2}((i-1)!)^{n_i}$ different orientable embeddings of the graph.

Face-tracing method

For a given rotation system of a graph G, choose an initial vertex v_0 of G and a first edge e_1 incident on v_0. Let v_1 be the other vertex incident to e_1. The second edge e_2 in the face walk is the edge after e_1 according to the rotation ρ_{v_1} at vertex v_1. If the edge e_1 is a loop, then e_2 is the edge after the other occurrence of e_1 according to the rotation ρ_{v_0} at vertex v_0. The face walk is finished at edge e_n if the next two edges in the walk would be e_1 and e_2 again. Two edges, which are consecutive in a rotation at a vertex, make a ***corner*** (e.g. e_1e_2). To start a different boundary walk, begin at the second edge of any corner that does not appear in any previously traced faces. If there are no unused corners left in the rotation system, then all faces have been traced. The number of face walks can be used to calculate the genus of the embedding using Euler's formula.

For an embedding of a graph, each edge appears exactly twice. Sometimes, the two appearances of an edge are involved in one face walk. The two appearances of edge f in Figure 1.4 are involved in the red face walk $hfegjfig$. Sometimes, the two appearances of an edge are involved in two different faces. The two appearances of edge a in Figure 1.4 are involved in the blue face walk aed and the yellow face walk abj. For all the figures in this thesis, edges of graphs are black lines. Face walks are colored lines (sometimes dashed lines).

There are five faces in the embedding of K_5 in Figure 1.4: aed, abj, bci, cdh, $hfegjfig$. By Euler's formula, the genus of this embedding of K_5 is $\frac{2-(v-e+f)}{2} = \frac{2-(5-10+5)}{2} = 1$. The graph K_5 is not a planar graph, so the embedding in Figure 1.4 is a minimum embedding. According to Theorem 1.1.2, we need to perform face-tracing on $((4-1)!)^5 = 6^5 = 7776$ embeddings or rotation systems to calculate the genus distribution of K_5. Readers might try to do face-tracing on the 16 rotation systems of the graph K_4. Its genus distribution is $g(K_4) = 2 + 14x$.

1.2 Literature review on graph embedding

To determine the genus of a graph is NP-complete [37]. It is also NP-complete to calculate the genus distribution of a graph. But this difficulty has not stopped mathematicians working on it, especially on some interesting families of graphs.

Gross and Furst introduced the genus embedding distribution of graphs in [13]. From then on, there have been many studies on different types of graphs by different methods.

One thing that interests us is how to find an efficient algorithm to derive a graph's embedding genus distribution from some graphs whose genus distributions are given or relatively easy to get. Many techniques have been developed. The simplest case is a family of graphs which have one graph in each generation. A graph in one generation is developed from the graph in the last generation by adding some vertices and/or edges. This family has infinitely many generations which grow recursively. Ladder type graphs and cross type graphs [41] are families of graphs of this kind.

A ***ladder graph*** L_n is defined by connecting two paths of length $n-1$ by $n+2$ edges including double edges on both ends. It is a 3-regular graph. Figure 1.5 shows how to construct a ladder graph L_3 from its predecessor L_2. The first step is adding two vertices of valency 2 on the right end edge. The second step is connecting the two new vertices by a new edge. L_1 has $((3-1)!)^2 = 4$ different embeddings. We can get the genus distribution of L_1 by hand drawing and get $g(L_1) = 2 + 2x$. But L_n has $((3-1)!)^{2n} = 4^n$ different embeddings. We can get the genus distribution of L_n from the genus distribution of L_1 through a formula derived from the topological operation in [43].

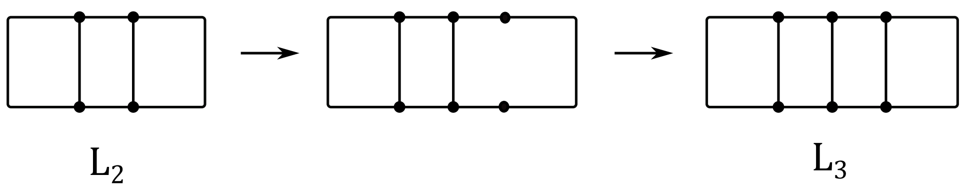


Figure 1.5: The topological operation from L_2 to L_3 by adding an edge

Gross introduced a partial genus distribution to the face-tracing method. With these techniques, we can get the *partial genus distribution* of a graph which is derived from some topological operations of another graph with a known partial genus distribution. A lot of results have been published on different topological operations. The papers published by Gross and his fellow researchers [20], [15], [16], [17], [31], [32] inspire the second chapter.

In [15], Gross discusses topological operations including adding an edge, deleting an edge, contracting an edge and splitting a vertex. These are fundamental operations which can be used to construct more complex topological operations.

The splitting operation is used in [15] by Gross. Let w be a vertex of a graph G, and let U and V be the sets of a bipartition of the neighbours of w. In the graph $G - w$, let every vertex of U be joined to a new vertex u and let every vertex of V be joined to a new vertex v, and join the vertices u and v. This operation is called ***splitting*** the graph G at vertex w.

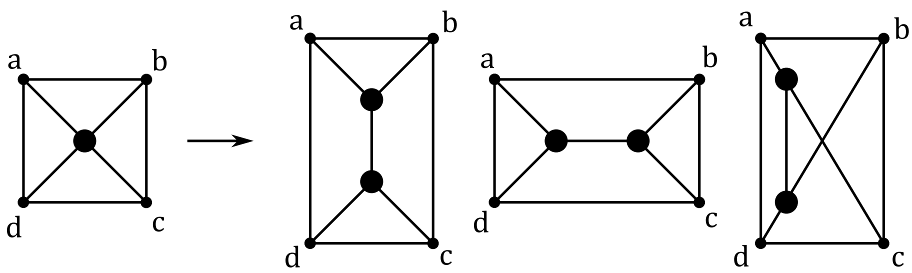


Figure 1.6: Split W_4 [15]

In Figure 1.6, splitting the central vertex of wheel graph W_4 results in two copies of $K_2 \times C_3$ and a $K_{3,3}$, depending on how the split is made. The splitting is from Gross' paper [15].

In Chapter 2 of this thesis, face-contractions are designed based on Gross' work in [15]. We extend Gross' work on vertex 2-splitting, and generalize it to i-splitting.

Topological operations applied on vertices are discussed in [17], [16] and [20].

In [17] and [16], Gross, Khan and Poshni demonstrate how to perform amalgamations and self-amalgamations on 2-valent vertices. Any vertex in a graph can be seen as a root vertex. The operation in [17] allows pasting any two vertices of valency 2 from two

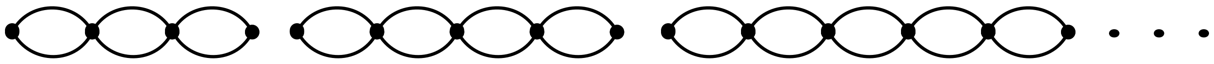


Figure 1.7: The doubled paths DP_3, DP_4, DP_5

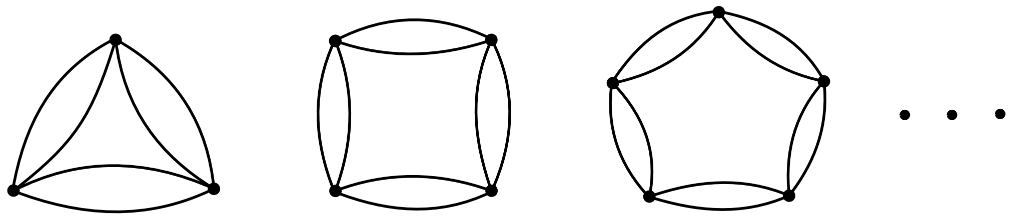


Figure 1.8: The doubled cycles DC_3, DC_4, DC_5 [16]

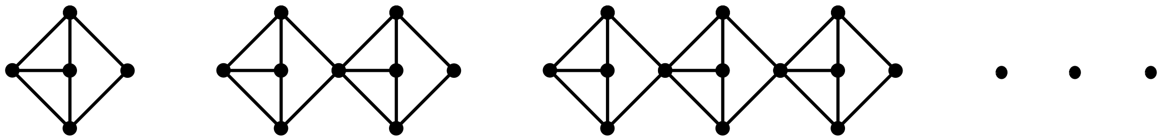


Figure 1.9: Open chains of copies of K_4 [20]

separate graphs, so an iterated amalgamation of arbitrarily many copies of any graph can be achieved. The graph family of doubled paths in Figure 1.7 is an application of vertex amalgamation. The operation in [16] allows pasting any two vertices of valency 2 from one graph. The graph family of doubled cycles in Figure 1.8 is an application of self-amalgamation of vertices. Another paper [20] extended the amalgamation method by allowing **one** of the two roots to have arbitrarily high valency. The open chain of copies of K_4 in Figure 1.9 is an application of this kind operation.

The pearl-making method in Section 2.4 and bouquet-making method in Sections 2.4 and 2.5 are topological operations of merging two vertices of a graph.

Topological operations applied on edges are discussed in [31] and [32].

Figure 1.10: Open chains of copies of $K_{3,3}$ [31]

In [31], the authors demonstrate how to calculate the genus distribution of an arbitrary chain of copies of one or more graphs, that results from the iterative amalgamation along their root-edges on the condition that their root-edges have 2-valent vertices only. This operation can be used to calculate genus distributions for various infinite families of 3-regular graphs include open chains of copies of a specific graph. Closed end ladders in Figure 1.5 and open chains of copies of $K_{3,3}$ in Figure 1.10 are applications of this operation.

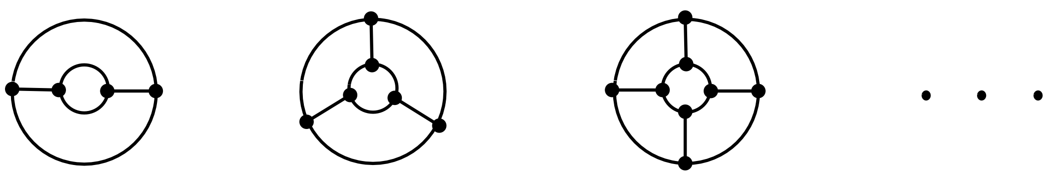


Figure 1.11: Circular ladders CL_2, CL_3, CL_4 [32]

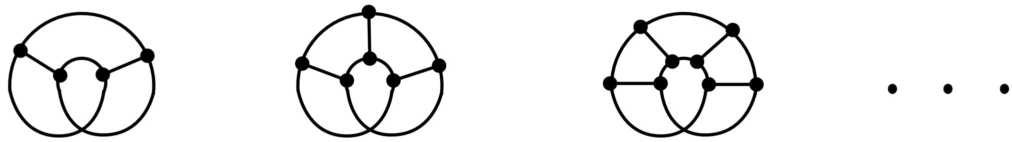


Figure 1.12: Mobiüs ladders ML_2, ML_3, ML_4 [32]

In [32], Gross, Khan and Poshni demonstrate how to calculate the genus distribution of the graph obtained by self-edge-amalgamation of its two root-edges with their root-edges have 2-valent vertices only. Circular ladders in Figure 1.11 and Mobiüs ladders in Figure 1.12 are good applications of self-edge-amalgamation.

All the results in the second chapter are in the category of partial genus distribution named by different topological operations. Some results are achieved in this category in this thesis.

Some important papers on genus embedding distributions of graphs on orientable or non-orientable surfaces include the following.

Furst, Gross and Statman in [11] calculate the genus distribution of closed-end ladders and cobblestone paths. It is the first time that partial genus distribution is introduced.

Gross in [18] finds the genus distribution for bouquets of circles.

Tesar in [36] find the orientable genus distribution of Ringel ladders. It can also be achieved by applying self-amalgamation on the two end edges of closed-end ladders.

Kwak and Lee in [22] compute the genus distribution for dipoles, and Kim and Lee in [21] compute the genus distribution for bouquets of dipoles by using a method concerning the cycle structure of permutations in the symmetric group.

Wan, Liu, Feng and Wang in [39] and [41] develop a surface generating technique based on the joint tree method to determine the genus distribution of ladder type graphs and cross type ladders.

Other authors define variations on genus distributions. Chen in [7] uses overlap matrices to calculate total embedding distributions of necklace, cobblestone path and closed end ladder. Kwak and Shim in [24] compute the total genus distribution of bouquets of circles by an edge-attaching technique. Chen, Liu and Tao Wang in [8] calculate the total embedding distributions of the cacti and necklaces by using overlap matrices. We will not consider these variations in this thesis.

For ***network graphs***, which are of interest to computer scientists, their embedding genus distributions are much harder to achieve, but their minimum genus and maximum genus are well discussed. Network graphs include hypercubes Q_n and star graphs S_n.

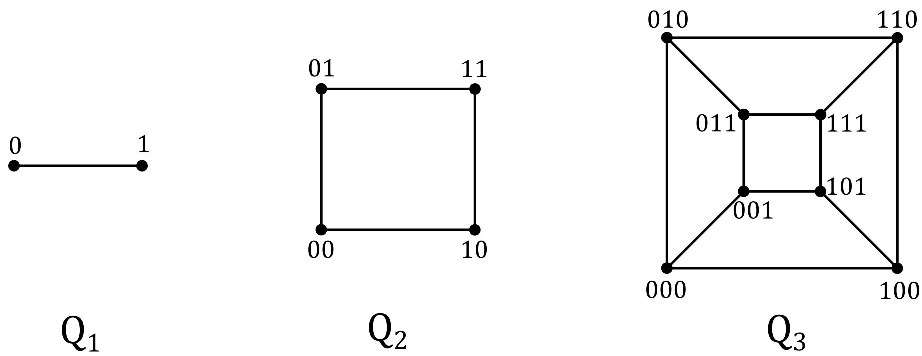


Figure 1.13: Hypercubes Q_1, Q_2 and Q_3 with their vertices labeled by binary numbers

A ***hypercube*** graph Q_n (in Figure 1.13) of order n consists of 2^n vertices labeled by a binary sequence $a_1 a_2 \cdots a_n$ of length n, where $a_i = 0$ or 1. There is an edge between two vertices if and only if their binary labels differ in exactly one place [33]. From the definition, it is an n-regular graph with $n \cdot 2^{n-1}$ edges.

Beineke and Harary in [4] find the minimum genus of Q_n to be $(n-4)2^{n-3}+1$. The paper introduces a method to construct minimum embedding of Q_n with only faces of size 4. We calculate the minimum genus of cube-connected cycles CCC_n using a face-expansion operation in Chapter 2. Please refer to Chapter 2 for the definition.

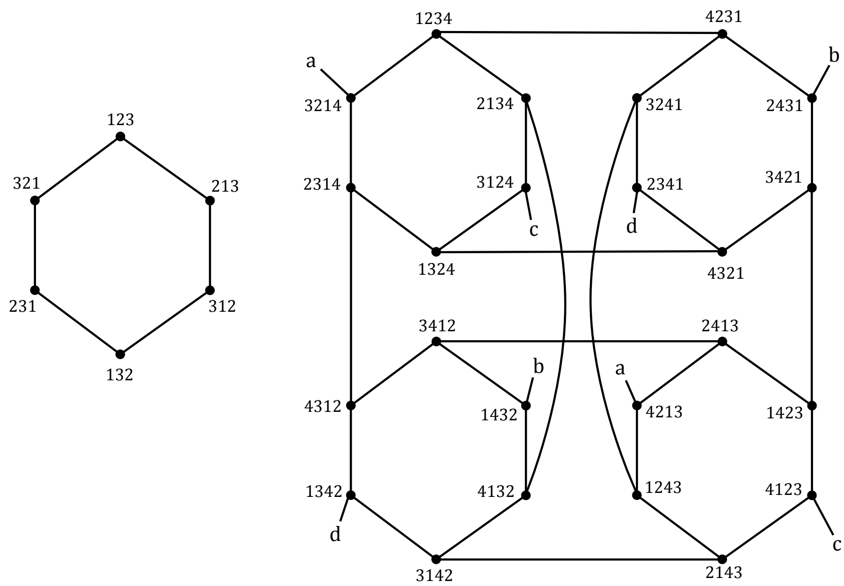


Figure 1.14: Star graphs S_3 and S_4. Connecting the lines with the same letter will make S_4. We keep the edges a, b, c and d unconnected to make the figure clear.

A ***star*** graph S_n (in Figure 1.14) of order n has $n!$ vertices labeled with a unique permutation on $\{1, \ldots, n\}$. There is an edge between any two vertices if and only if their corresponding permutations differ exactly in the first and one other position. S_n is an $(n-1)$-regular graph.

Abbasi in [1] calculates the minimum genus of S_n to be $n!(n-4)/6+1$. He introduces a way to construct minimum embedding of S_n with only faces of size 6. We calculate the minimum genus of star-connected cycles SCC_n by face-expansion operation in Chapter

2.

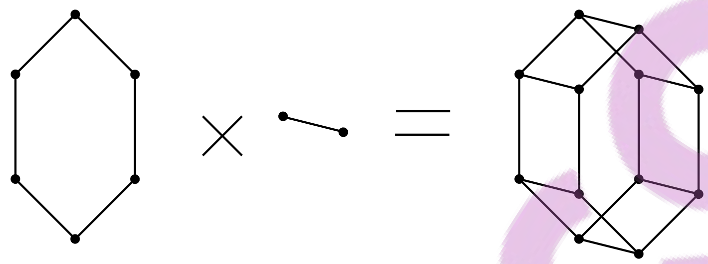


Figure 1.15: The Cartesian product of S_3 with P_2

Given a graph G_i, with vertex set $V(G_i)$ and edge set $E(G_i)$, $(i = 1, 2)$ the ***Cartesian product*** $G_1 \times G_2$ has for its vertex set

$$V(G_1 \times G_2) = \{(u_1, u_2) : u_1 \in V(G_1), u_2 \in V(G_2)\}$$

and for its edge set $E(G_1 \times G_2) =$

$$\{[(u_1, u_2), (v_1, v_2)] : u_1 = v_1 \text{ and } [u_2, v_2] \in E(G_2) \text{ or } u_2 = v_2 \text{ and } [u_1, v_1] \in E(G_1)\}$$

White in [42] calculates the minimum genus of the Cartesian product of the complete bipartite graph $K_{2m,2m}$ with itself is $1 + 8m^2(m - 1)$.

Here $G(n, d)$ denotes any connected regular bipartite graph on $2n$ vertices and of valency d. Pisanski in [29] proves that any Cartesian product $G(n, d) \times G_1(n_1, d_1) \times G_2(n_2, d_2) \times \cdots \times G_m(n_m, d_m)$, such that $\max\{d_1, d_2, \ldots, d_m\} \leq d \leq d_1 + d_2 + \cdots + d_m$ has a quadrilateral embedding, thereby establishing its minimum genus. Pisanski extended the result to $G \times Q_n$ in [30]. In an quadrilateral embedding of an graph, all the faces are length 4, that means this embedding achieves the maximum number of faces, which is corresponding to the graph's minimum genus by Euler's formula.

Bonnington and Pisanski in [5] find the minimum genus embeddings of the Cartesian product of a complete regular tripartite graph with a even cycle $K_{m,m,m} \times C_{2n}$.

A substitution technique is developed in Chapter 3 to construct minimum genus embedding of the Cartesian product of a star graph and another graph which contains hexagon and quadrilateral faces only.

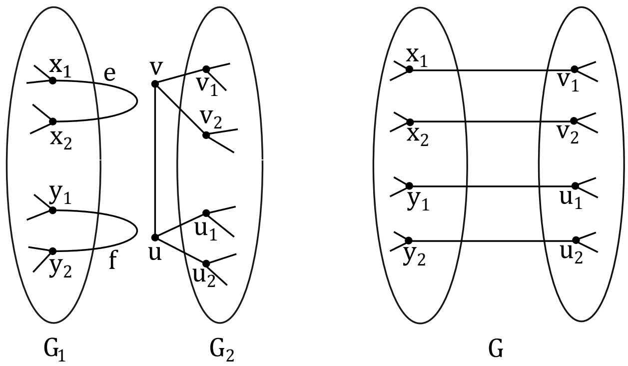


Figure 1.16: Applying dot product on cubic graphs G_1 and G_2 with the resulting cubic graph G (modified from [27])

A ***snark*** is a 3-regular graph whose cycles are length at least 5 and whose edges cannot be colored by only three colors without two edges of the same color meeting at a vertex. The smallest snark is the Petersen graph. Isaacs defines a dot product to join two snarks to construct a new snark in [12].

The dot product is defined in [27] as follows. Let G_1 be a cubic graph and $e = x_1x_2$ and $f = y_1y_2$ be two non-adjacent edges in G_1. Let G_2 be a cubic graph and $uv \in E(G_2)$ be an edge of G_2. Denote the neighbors of u distinct from v by u_1 and u_2, and the neighbors of v distinct from u by v_1 and v_2. Remove the edges e and f from G_1 and remove the vertices u and v from G_2 and denoted by G_1' and G_2' the resulting graphs respectively. Construct a graph G by adding edges x_1v_1, x_2v_2, y_1u_1 and y_2u_2. The added edges are called the ***product edges***. Graph G is called a ***dot product*** (in Figure 1.16) of graphs G_1 and G_2 (denoted by $G = G_1 \cdot G_2$) [27]. According to the definition of dot product, $G_1 \cdot G_2 \neq G_2 \cdot G_1$.

If graphs G_1 and G_2 are snarks, then their dot product is also a snark [12]. Applying the dot product on two Petersen graphs (in Figure 1.17) results in two different snarks of P^2 (in Figure 1.18), depending on the edges chosen.

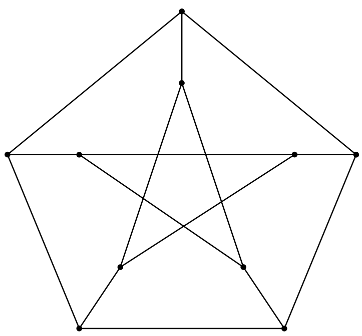


Figure 1.17: Petersen graph

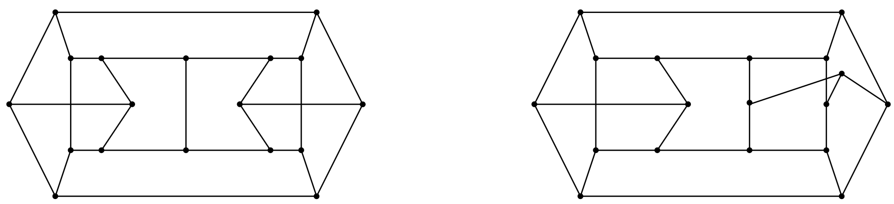


Figure 1.18: Petersen power P^2 contains two different graphs. (modified from [27])

Mohar and Vodopivec in [27] construct a dot product of n copies of the Petersen graph whose minimum genus is precisely k (for $\forall k, 1 \leq k \leq n$). It is a minimum genus embedding construction of P^n.

Based on the result in [27], we construct embeddings of P^n of any genus between 1 and $2n+1$ in Chapter 4.

We extend the results on dot product and introduce the extended dot product to join two 4-regular graphs in Chapter 5.

Chapter 2

Partial Genus Distribution

This chapter uses partial genus distributions to discuss the embedding changes resulting from topological operations. The topological operations we consider are face-contraction, vertex-splitting, vertex-augment, pearl-making, bouquet-making, and face-expansion.

2.1 Face-contraction

In this section, we derive the embedding genus distribution of a graph obtained by face-contraction on another graph whose partial genus distribution is known. Contractions of faces of size 3 and 4 will be discussed here. The contractions of larger faces can be achieved in a similar way.

If a graph G contains a cycle of size 3, we call it a 3-***face*** graph (Figure 2.1). The three vertices on the cycle are called ***end vertices***. We can also define 4-***face*** graph, and n-***face*** graph ($n \geq 3$) in the same way. We discuss faces with end vertices' valencies at least 3. Valency 2 vertices can be removed without changing embeddings.

A ***star rooted*** graph contains a root vertex of valency n. We call it an n-***star*** graph.

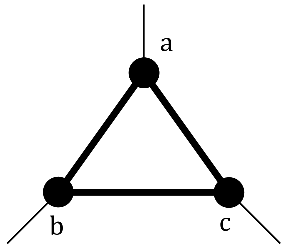


Figure 2.1: A 3-face graph with end vertices a, b and c all valency 3

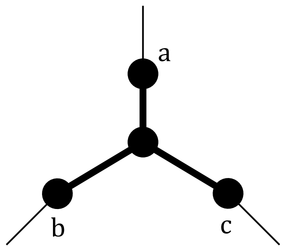


Figure 2.2: A 3-star graph with central vertex valency 3

Figure 2.2 is a 3-star graph with three neighbours valency 2.

2.1.1 3-Face contraction

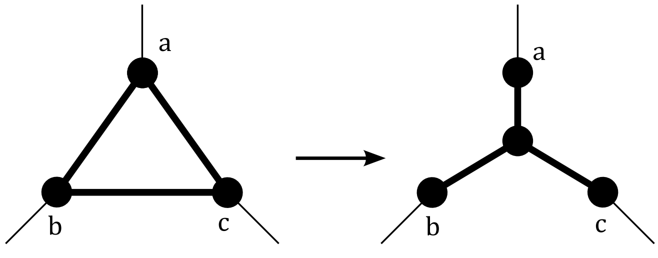


Figure 2.3: Contraction from a 3-face to a 3-star

Let graph (G, abc) be a 3-face graph with end vertices a, b, c of valency l, m, n at least 3. Face-contraction 'shrinks' a 3-face graph (G, abc) to a 3-star graph (G/abc) with the central vertex valency 3 and the three end vertices valency $l-1$, $m-1$ and $n-1$ respectively. This operation is called 3***-face contraction***. Figure 2.3 shows a 3-face graph (G, abc) with end vertices valency 3 contracted to a 3-star graph (G/abc) with end vertices valency 2.

The order of end vertices in G is important. This is because different vertices have different valencies, which is an important concern for our generalized contraction operation.

2.1.1.1 The contraction of a 3-face with 3-valent end vertices.

There are two steps to perform the contraction.

First step of contraction

Delete edge bc from the 3-face root (Figure 2.4).

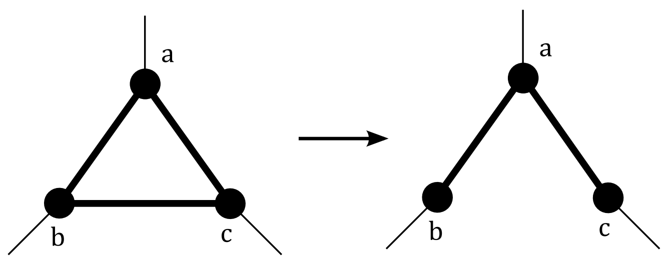


Figure 2.4: First step in the 3-face contraction by deleting edge bc

Jonathan Gross' work on the genus distribution of graphs obtained by deleting an edge uses the following definitions and results [15]:

Let $d_i(G, e)$ be the number of embeddings of G onto the surface S_i in which two different face walks are incident on edge e.

Let $s_i(G, e)$ be the number of embeddings of G onto the surface S_i in which the same face walk is incident on edge e.

The numbers $d_i(G, e)$ and $s_i(G, e)$ are called ***partials***. The sequences $\{d_i(G, e)|i \geq 0\}$ and $\{s_i(G, e)|i \geq 0\}$ are called ***partial genus distributions*** [15].

From the definition, we have $g_i(G) = d_i(G, e) + s_i(G, e)$ and

$$g(G) = \sum_{i=0}^{\infty}(d_i(G, e) + s_i(G, e))x^i = \sum_{i=0}^{\infty} d_i(G, e)x^i + \sum_{i=0}^{\infty} s_i(G, e)x^i$$

The partial genus distribution shows more detailed information on face-tracing than normal genus distribution.

Note that different kind of partials can be designed and used for different genus distribution problems. For instance, it is sufficient to use the partials of a single edge when deleting an edge. For double edge partials, it is sometime necessary to distinguish partials according to the characteristics of faces passing through the double edges [15].

Let u and v be two vertices with valency 2 in graph G. We connect vertices u and v with a new edge e. There are four different ways to insert an edge which are shown in Figure 2.5.

Deleting an edge $e = uv$ inverts the operation of joining vertices u and v. Deleting an edge root of a graph G, we have the lemma below. Note that all embeddings are open 2-cell in this thesis.

Lemma 2.1.1. *[15] In an embedding of single-edge-rooted graph, if two distinct face walks are incident on the root edge, the two faces are merged into one and the genus stays the same. If a single face walk is twice incident on the root edge, then that face is split into*

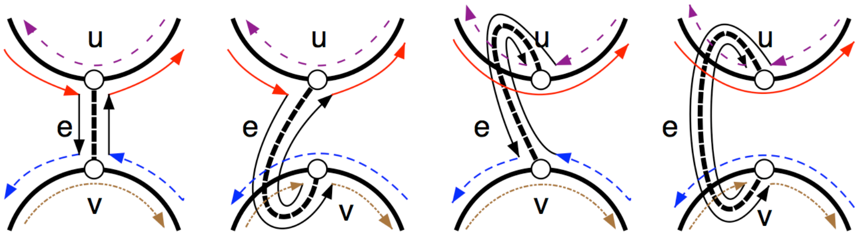


Figure 2.5: Insert an edge e to connect vertices u and v [15]

two faces, and the genus drops by one.

Proof. Let g_0, v_0, e_0, f_0 be the genus, and the number of vertices, edges and faces, respectively, of graph (G, e). Let g_1, v_1, e_1, f_1 be the genus, and the number of vertices, edges and faces, respectively, of graph $(G - e)$.

For graph (G, e), we have $2 - 2g_0 = v_0 - e_0 + f_0$. For graph $G - e$, we have $2 - 2g_1 = v_1 - e_1 + f_1$.

- When two distinct face walks are incident on edge e, by deleting e, we have $v_1 = v_0$, $e_1 = e_0 - 1$ and $f_1 = f_0 - 1$. So $g_1 = g_0$.

- When a single face walk is twice incident on edge e, by deleting e, we have $v_1 = v_0$, $e_1 = e_0 - 1$ and $f_1 = f_0 + 1$. So $g_1 = g_0 - 1$.

$\square$

Production rules describe the relationship between the partial genus distribution of a graph G and the partial genus distribution of another graph which is derived from G by some topological operations.

By using the lemma above, we have the following production rule between (G, e) and $(G - e)$.

Lemma 2.1.2. *[15] Let (G, e) be a single-rooted graph with two 3-valent end vertices. The following production rule describes the relationship between partial genus distribution of (G, e) and the genus distribution of $(G - e)$:*

$$\begin{cases} d_i(G,e) & \rightarrow & \frac{1}{4}g_i(G-e) \\ s_i(G,e) & \rightarrow & \frac{1}{4}g_{i-1}(G-e) \end{cases}$$

Note that the production rule above is not an equation, but a mapping. In all the production rules in this thesis, we will use arrows to make them consistent with Jonathan L. Gross' results.

From the production rule above, the following result is immediate.

Theorem 2.1.3. *[15] Let (G, e) be a single-edge-rooted graph with two 3-valent end vertices. Then the genus distribution of $(G - e)$ is derived by using the following equation:*

$$g_i(G-e) = \frac{1}{4}d_i(G,e) + \frac{1}{4}s_{i+1}(G,e)$$

Second step of contraction

Add a new vertex of valency 2 on the edge which is incident with vertex a. Relabel vertex a by o, the new vertex by a. (see Figure 2.6)

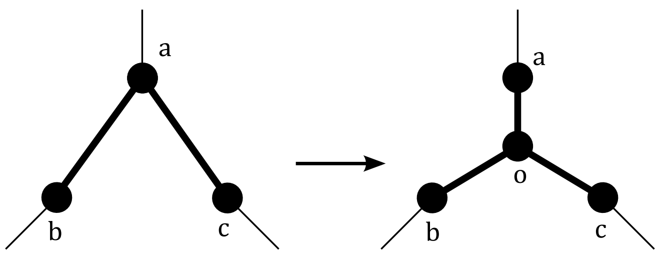


Figure 2.6: Second step in the 3-face contraction by adding a vertex

Lemma 2.1.4. *[13] Adding a new vertex of valency 2 on an edge of a graph embedding, the genus of the graph embedding does not change.*

Proof. It is straightforward by Euler's formula. □

Together the two steps contract a 3-face to a 3-star.

Theorem 2.1.5. *The genus distribution of (G/abc) can be derived by using the following equation.*

$$g_i(G/abc) = \frac{1}{4}d_i(G,bc) + \frac{1}{4}s_{i+1}(G,bc)$$

Proof. This is an immediate consequence of the above two steps. □

There is also a linear relationship between the total number of embeddings of graph G and graph G/abc.

Theorem 2.1.6. *The total number of embeddings of (G/abc) is a quarter of the total number of embeddings of (G, abc).*

$$\sum_{i=0}^{\infty} g_i(G/abc) = \frac{1}{4}\sum_{i=0}^{\infty} g_i(G,abc)$$

Proof. According to the property of planar embedding, the inside and outside of the triangle belong to different open discs. So $s_0(G,abc) = 0$. Then we have

$$\sum_{i=0}^{\infty} s_{i+1}(G,bc) = \sum_{i=0}^{\infty} s_i(G,bc)$$

so

$$\begin{aligned}\sum_{i=0}^{\infty} g_i(G/abc) &= \frac{1}{4}\sum_{i=0}^{\infty} d_i(G,bc) + \frac{1}{4}\sum_{i=0}^{\infty} s_{i+1}(G,bc)\\ &= \frac{1}{4}\sum_{i=0}^{\infty} d_i(G,bc) + \frac{1}{4}\sum_{i=0}^{\infty} s_i(G,bc)\\ &= \frac{1}{4}\sum_{i=0}^{\infty} g_i(G,abc)\end{aligned}$$

□

2.1.1.2 The contraction of a 3-face abc with valencies 3, 4 and 4

We use Gross' result to delete an edge as the first step. Let (G, e) be a single-edge-rooted graph with two 4-valent end vertices.

Theorem 2.1.7. *[15] The genus distribution of $(G - e)$ is derived by the following equation.*

$$g_i(G-e) = \frac{1}{9}d_i(G,e) + \frac{1}{9}s_{i+1}(G,e)$$

Proof. Similar method to Theorem 2.1.3 by deleting an edge of two 3-valent end vertices. □

Second step does not apply. We have the following result.

Theorem 2.1.8. *The genus distribution of (G/abc) is derived by the following equation.*

$$g_i(G/abc) = \frac{1}{9}d_i(G,bc) + \frac{1}{9}s_{i+1}(G,bc)$$

2.1.1.3 The contraction of a 3-face abc with valencies 3, m and n

We generalize Gross' result on deleting an edge. Let (G, e) be a single-edge-rooted graph with end vertices of valency m and n.

Theorem 2.1.9. *The genus distribution of $(G - e)$ is derived by the following equation.*

$$g_i(G-e) = \frac{d_i(G,e)}{(m-1)(n-1)} + \frac{s_{i+1}(G,e)}{(m-1)(n-1)}$$

Adding a vertex of valency 2, we have the following result.

Theorem 2.1.10. *The genus distribution of (G/abc) is derived by the following equation.*

$$g_i(G/abc) = \frac{d_i(G,bc)}{(m-1)(n-1)} + \frac{s_{i+1}(G,bc)}{(m-1)(n-1)}$$

2.1.1.4 When there is no vertex in the 3-face abc with valency 3

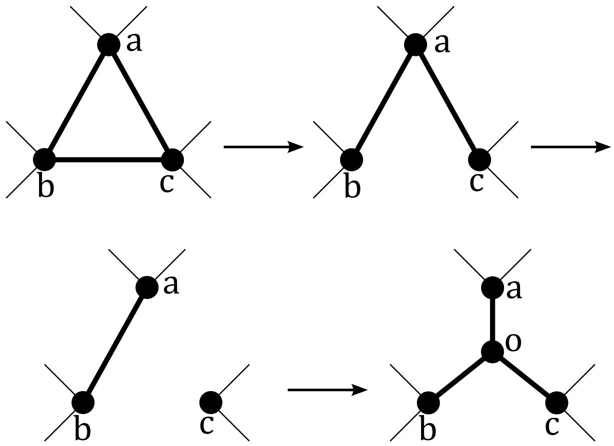


Figure 2.7: Contraction of a 3-face abc with all end vertices valency 4

The procedure above does not work when there is no vertex in the 3-face abc of valency 3. The procedure of contraction of an easy case is illustrated in Figure 2.7, and outlined below.

- First, delete edge bc from the 3-face.
- Second, delete edge ac.
- Third, add a 2-valent vertex o on edge ab and connect o and c.

When the edge ac is not a cut-edge in $G - bc$, we can use the formula above to delete edge ac at second step. If vertex c is 4-valent in G, then we can use Gross's result to connect vertices o and c. If vertex c has a higher valency value than 4 in G, a more complex equation is required to achieve this. The corresponding partial genus distribution could be very complex depending on the valency of vertex c.

When the edge ac is a cut-edge of $G - bc$, after deleting edge ac, we have two graphs G_1 which includes edge ab, and G_2 which includes vertex c. We can use the following of Gross' results to connect two vertices from two different graphs in the second and third step. ***Bar-amalgamation*** is a topological operation which connects two disjoint rooted graphs (A, u) and (B, v) by adding an edge uv (called the ***bar***). The resulting graph is denoted $(A, u)|(B, v)$. Figure 2.8 shows a bar-amalgamation to two copies of $K_4 - e$.

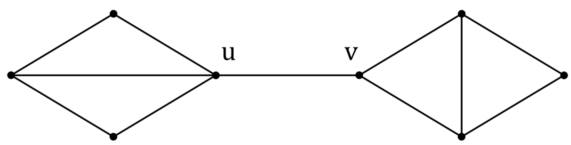


Figure 2.8: Bar-amalgamation [13]

Theorem 2.1.11. *[13] Let (A, u) and (B, v) be rooted graphs. The genus distribution of the $bar-amalgamation$ $(A, u)|(B, v)$ is obtained by multiplying the convolution of the genus distributions of A and B by the product of the valencies of vertices u and v in the graphs A and B, respectively.*

$$g((A, u)|(B, v)) = d(u)d(v)g(A, u)g(B, v)$$

Let the valency of the three end vertices of face abc in graph G be l, m, n. The following theorem applies to contractions of any 3-face which has end vertices valency at least 3 on the condition that ac is a cut-edge of $G - bc$. Let G_1 and G_2 denote the two graphs resulting from deleting ac on $G - bc$.

Theorem 2.1.12. *The genus distribution of (G/abc) is derived by using the following* *equation.*

$$g_i(G/abc) = \frac{2(d_i(G,bc) + s_{i+1}(G,bc))}{(l-1)(m-1)(n-1)}$$

Proof. According to Theorem 2.1.9, we have

$$g_i(G-bc) = \frac{d_i(G,bc)}{(m-1)(n-1)} + \frac{s_{i+1}(G,bc)}{(m-1)(n-1)}.$$

From Theorem 2.1.11, we have the equation below after deleting edge ac on the second step.

$$g(G-bc) = (l-1)\cdot(n-2)g(G_1)g(G_2)$$

On the third step, insert a new vertex o of valency 2 on edge ab, and connect vertex o of graph G_1 and vertex c of graph G_2. We have

$$g(G/abc) = 2\cdot(n-2)\cdot g(G_1)\cdot g(G_2) = \frac{2}{(l-1)}\cdot g(G-bc).$$

So

$$\begin{aligned} g_i(G/abc) &= \frac{2}{(l-1)}\cdot g_i(G-bc) \\ &= \frac{2}{(l-1)}\cdot\left(\frac{d_i(G,bc)}{(m-1)(n-1)} + \frac{s_{i+1}(G,bc)}{(m-1)(n-1)}\right) \\ &= \frac{2d_i(G,bc)}{(l-1)(m-1)(n-1)} + \frac{2s_{i+1}(G,bc)}{(l-1)(m-1)(n-1)}. \end{aligned}$$

□

Example 2.1.13. *For a complete graph K_4, after contracting a 3-face abc to a 3-star, the resulting graph is noted by K_4/abc (Figure 2.9). We use face-tracing method and Euler's formula to calculate the partial genus distribution.*

k	0	1
$d_k(K_4)$	2	8
$s_k(K_4)$	0	6
$g_k(K_4)$	2	14

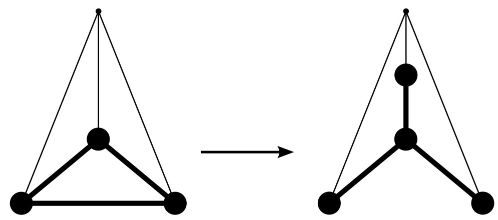


Figure 2.9: An example of face-contraction

$$
\begin{aligned}
g_0(K_4/abc) &= \tfrac{1}{4}d_0(K_4, bc) + \tfrac{1}{4}s_1(K_4, bc)\\
&= \tfrac{1}{4}\cdot 2 + \tfrac{1}{4}\cdot 6\\
&= 2\\
g_1(K_4/abc) &= \tfrac{1}{4}d_1(K_4, bc)\\
&= \tfrac{1}{4}\cdot 8\\
&= 2
\end{aligned}
$$

2.1.2 4-Face contraction

2.1.2.1 When the end vertices of the 4-face are all 3-valent

The vertices v_1, v_2, v_3 and v_4, edges e_1, e_2 and e are described as shown in Figure 2.10. There are three steps for the 4-face contraction.

- First. Delete edge e_1 with two end vertices each 3-valent. We have

$$g_i(G - e_1) = \frac{1}{4}d_i(G, e_1) + \frac{1}{4}s_{i+1}(G, e_1).$$

- Second. Delete edge e_2 adjacent to e_1. We have

$$g_i(G - e_1 - e_2) = \frac{1}{2}d_i(G - e_1, e_2) + \frac{1}{2}s_{i+1}(G - e_1, e_2).$$

  After deleting edge e_2, we assume the resulting graph is connected.

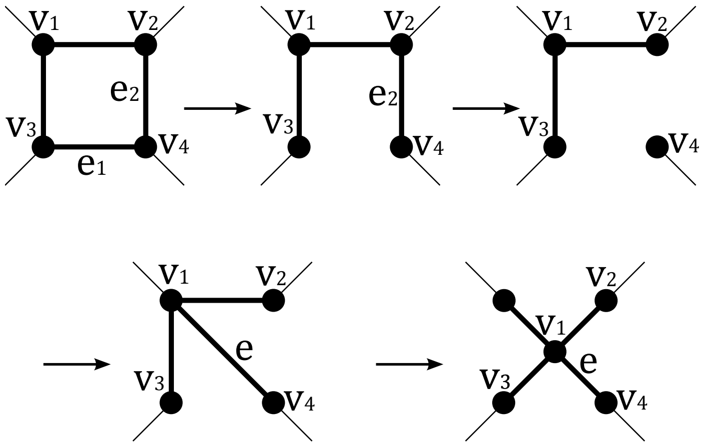


Figure 2.10: Contraction of a 4-face with all end vertices valency 3

- Third. Add an edge e to connect two vertices v_1 and v_3 and insert a vertex of valency 2 on the corresponding edge incident to v_1.

We now explain in detail how to connect two vertices of valency 3 and 1 in a graph G (Figure 2.11).

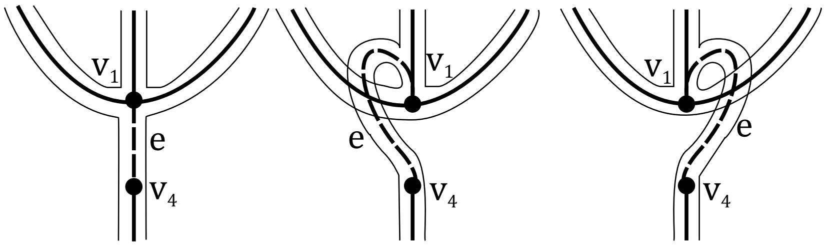


Figure 2.11: Join two vertices of valency 3 and 1 in a graph.

Let (G, v_1, v_4) be a two-vertex-rooted graph with root vertices v_1 and v_4 of valencies 3 and 1 respectively. The graph resulting from joining roots v_1 and v_4 by an edge e is denoted by $G + e$. We calculate the genus distribution of $(G + e)$ from the partial genus distribution of (G, v_1, v_4).

The genus distribution of (G, v_1, v_4) is partitioned according to different combinations of the three face-walks that pass through vertex v_1 and the one face-walk that passes through vertex v_4. There are seven partials, which cover all the embeddings of graph (G, v_1, v_4). In each partial, the first three letters represent the three corresponding face-walks pass through vertex v_1. The fourth letter represents the face-walk pass through vertex v_4.

- Let $abcd_i(G, v_1, v_4)$ denote the number of embeddings of G on the surface S_i in which the three face-walks pass through root v_1 are distinct from each other, and different from the face-walk pass through root v_4.

- Let $abca_i(G, v_1, v_4)$ denote the number of embeddings of G on the surface S_i in which the three face-walks pass through root v_1 are distinct from each other, and one of the three coincides with the face-walk passing through root v_4.

- Let $aacd_i(G, v_1, v_4)$ denote the number of embeddings of G on the surface S_i in which just two of the three face-walks pass through root v_1 are the same, and all of them are different from the face-walk passing through root v_4.

- Let $aacc_i(G, v_1, v_4)$ denote the number of embeddings of G on the surface S_i in which just two of the three face-walks pass through root v_1 are the same, and the other one of the three coincides with the face-walk passing through root v_4.

- Let $aaca_i(G, v_1, v_4)$ denote the number of embeddings of G on the surface S_i in which just two of the three face-walks pass through root v_1 are the same, and the face-walk passing through root v_4 coincides with these two.

- Let $aaad_i(G, v_1, v_4)$ denote the number of embeddings of G on the surface S_i in which the three face-walks pass through root v_1 are the same, and different from the face-walk passing through root v_4.

- Let $aaaa_i(G, v_1, v_4)$ denote the number of embeddings of G on the surface S_i in which the three face-walks pass through root v_1 are the same, and coincide with the face-walk passing through root v_4.

When adding an edge e in each of the seven cases, we have the following production rules.

Lemma 2.1.14. *Let (G, v_1, v_4) be a two-vertex-rooted graph with 3-valent and 1-valent root vertices respectively. Then the following production rules show the relationship between partial genus distribution of G and the genus distribution of $G + e$ obtained by joining roots v_1 and v_4 by an edge e.*

$$
\begin{array}{llcrr}
(1) & abcd_i(G, v_1, v_4) & \longrightarrow & & 3g_{i+1}(G+e) \\
(2) & abca_i(G, v_1, v_4) & \longrightarrow & g_i(G+e)+ & 2g_{i+1}(G+e) \\
(3) & aacd_i(G, v_1, v_4) & \longrightarrow & & 3g_{i+1}(G+e) \\
(4) & aacc_i(G, v_1, v_4) & \longrightarrow & g_i(G+e)+ & 2g_{i+1}(G+e) \\
(5) & aaca_i(G, v_1, v_4) & \longrightarrow & 2g_i(G+e)+ & g_{i+1}(G+e) \\
(6) & aaad_i(G, v_1, v_4) & \longrightarrow & & 3g_{i+1}(G+e) \\
(7) & aaaa_i(G, v_1, v_4) & \longrightarrow & 3g_i(G+e) &
\end{array}
$$

Proof. It is straightforward by Euler's formula and face-tracing. □

By the production rule above, we have the following theorem.

Theorem 2.1.15. *Let (G, v_1, v_4) be a two-vertex-rooted graph with 3-valent and 1-valent root vertices. Then the genus distribution of $G + e$ is:*

$$\begin{aligned} g_i(G+e) &= 3abcd_{i-1}(G, v_1, v_4) + 2abca_{i-1}(G, v_1, v_4) + 3aacd_{i-1}(G, v_1, v_4) \\ &\quad +2aacc_{i-1}(G, v_1, v_4) + aaca_{i-1}(G, v_1, v_4) + 3aaad_{i-1}(G, v_1, v_4) \\ &\quad +abca_i(G, v_1, v_4) + aacc_i(G, v_1, v_4) + 2aaca_i(G, v_1, v_4) + 3aaaa_i(G, v_1, v_4) \end{aligned}$$

Proof. We discuss the seven production rules one by one.

- For production rule (1), there are three embeddings of $G+e$ that result from adding an edge e to an embedding $abcd_{i-1}(G)$ of G.

- For production rule (2), there are two embeddings of $G+e$ that result from adding an edge e to an embedding $abca_{i-1}(G)$ of G. There is one embedding of $G+e$ that results from adding an edge e to an embedding $abca_i(G)$ of G.

- For production rule (3), there are three embeddings of $G+e$ that result from adding an edge e to an embedding $aacd_{i-1}(G)$ of G.

- For production rule (4), there are two embeddings of $G+e$ that result from adding an edge e to an embedding $aacc_{i-1}(G)$ of G. There is one embedding of $G+e$ that result from adding an edge e to an embedding $aacc_i(G)$ of G.

- For production rule (5), there is one embedding of $G+e$ that results from adding an edge e to an embedding $aaca_{i-1}(G)$ of G. There are two embeddings of $G+e$ that result from adding an edge e to an embedding $aaca_i(G)$ of G.

- For production rule (6), there are three embeddings of $G+e$ that result from adding an edge e to an embedding $aaad_{i-1}(G)$ of G.

- For production rule (7), there are three embeddings of $G+e$ that result from adding an edge e to an embedding $aaaa_i(G)$ of G.

The equation follows by combining the seven cases.

□

Other genus distribution equations which are derived from production rules can be proved in the similar way.

Note that when deleting edge e_2 gives two disconnected graph, we need to use a bar-amalgamation of Theorem 2.1.11. Let G_1, which contains V_2, and G_2, which contains V_4, be the two disconnected graphs after deleting e_2. By bar-amalgamation in Theorem 2.1.11, we have:

$$g(G - e_1) = 2 \times 1 \times g(G_1) \times g(G_2)$$

When connecting V_1 and V_4 with e, we can use bar-amalgamation again and have:

$$g(g - e_1 - e_2 + e) = 3 \times 1 \times g(G_1) \times g(G_2) = \frac{3}{2} g(G - e_1).$$

Example 2.1.16. 4*-face contraction of a small graph*

For the graph G in Figure 2.12, after shrinking a 4-face $v_1v_2v_3v_4$ to a 4-star, the resulting graph is denoted by $(G/v_1v_2v_3v_4)$. We use face-tracing and Euler's formula to calculate the partial genus distributions in each step.

- *Step* 1

  Delete e_1. $d_k(G)$ is the number of embeddings of G onto the surface S_k in which two different face walks are incident on edge e_1. Let $s_k(G)$ be the number of embeddings of G onto the surface S_k in which the same face walk is incident on edge e_1.

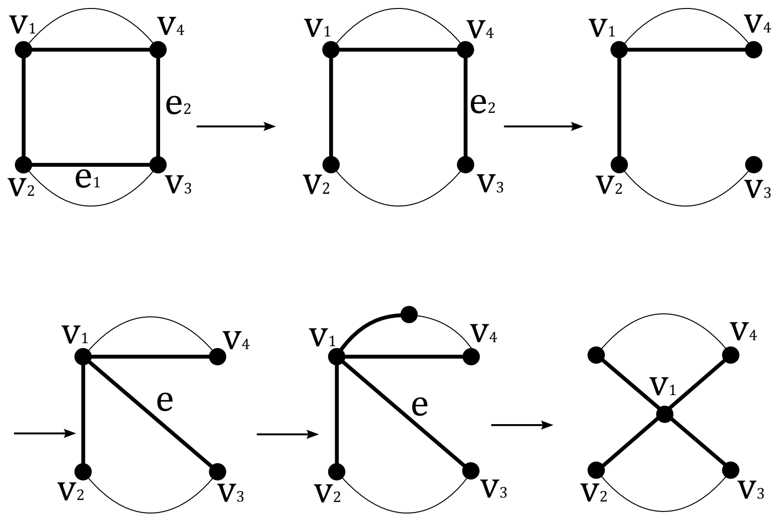


Figure 2.12: An example of 4-face contraction

k	0	1
$d_k(G)$	4	8
$s_k(G)$	0	4
$g_k(G)$	4	12

$$\begin{aligned} g_0(G-e_1) &= \tfrac{1}{4}d_0(G,e_1)+\tfrac{1}{4}s_1(G,e_1) \\ &= \tfrac{1}{4}\cdot 4+\tfrac{1}{4}\cdot 4 \\ &= 2 \\ g_1(G-e_1) &= \tfrac{1}{4}d_1(G,e_1) \\ &= \tfrac{1}{4}\cdot 8 \\ &= 2 \end{aligned}$$

- *Step* 2

Delete e_2. $d_k(G-e_1)$ *is the number of embeddings of* $G-e_1$ *onto the surface* S_k *in* *which two different face walks are incident on edge* e_2*. Let* $s_k(G-e_1)$ *be the number of embeddings of* $G-e_1$ *onto the surface* S_k *in which the same face walk is incident on edge* e_2*.*

k	0	1
$d_k(G-e_1)$	2	0
$s_k(G-e_1)$	0	2
$g_k(G-e_1)$	2	2

$$\begin{aligned} g_0(G-e_1-e_2) &= \tfrac{1}{2}d_0(G-e_1)+\tfrac{1}{2}s_1(G-e_1) \\ &= \tfrac{1}{2}\cdot 2+\tfrac{1}{2}\cdot 2 \\ &= 2 \\ g_1(G-e_1-e_2) &= 0 \end{aligned}$$

- *Step* 3

Connect v_1 *and* v_3 *with an edge.*

k	0
$abcd_k$	0
$abca_k$	0
$aacd_k$	0
$aacc_k$	0
$aaca_k$	2
$aaad_k$	0
$aaaa_k$	0

$$\begin{aligned} g_0(G - e_1 - e_2 + e) &= 2aaca_0(G - e_1 - e_2) \\ &= 2 \cdot 2 \\ &= 4 \\ g_1(G - e_1 - e_2 + e) &= aaca_0(G - e_1 - e_2) \\ &= 2 \cdot 1 \\ &= 2 \end{aligned}$$

The genus distribution of the resulting graph after 4*-face contraction is:* $4 + 2x$.

2.1.2.2 When the 4-face graph's four root vertices v_1, v_2, v_3 and v_4 are of valency 3, m, 3 and n $(m, n \geq 3)$ respectively.

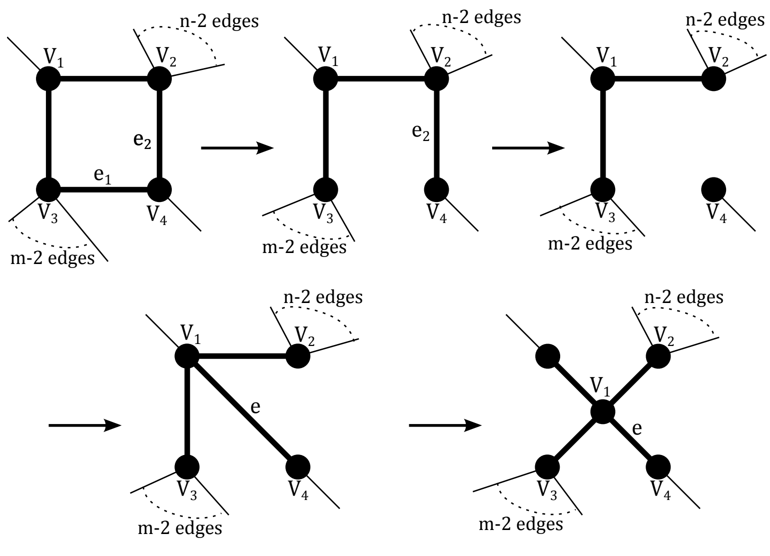


Figure 2.13: Contraction of a 4-face with end vertices v_1, v_2, v_3 and v_4 of valency 3, m, 3 and n $(m, n \geq 3)$ respectively.

The first two steps to calculate the genus distribution change are a little different, because of the higher valencies of vertices of v_2 and v_3. The third step stays the same and is omitted here. The operation procedure is shown in Figure 2.13.

- Step 1: Delete edge e_1 with two end vertices v_2 and v_3 of valency m and 3 respectively.

$$g_i(G - e_1) = \frac{1}{2(m-1)} d_i(G, e_1) + \frac{1}{2(m-1)} s_{i+1}(G, e_1)$$

- Step 2: Delete another edge e_2 with two end vertices v_3 and v_4 of valency 2 and n respectively.

$$g_i(G - e_1 - e_2) = \frac{1}{(n-1)} d_i(G - e_1, e_2) + \frac{1}{(n-1)} s_{i+1}(G - e_1, e_2)$$

  Note that we assume that after deletion of the second edge, the resulting graph is still connected. If it is not, we must consider this separately by using bar-amalgamation.

2.1.2.3 When the 4-face graph's four root vertices are valency h, m, l, n respectively ($m, n \geq 3$ and $h, l > 3$).

The first two steps to calculate the genus distribution change are a little different. The third step is complicated when connecting two vertices v_1 and v_4 of high valencies with an edge. We require a new step to split the vertex v_1 of valency $h+1$ (the third step make the valency grow by 1) to two new vertices of valency $h-1$ and 4, respectively. Step 4 can be achieved by the results in Section 2.2.

- Step 1: Delete edge e_1 with two end vertices of valencies m and l respectively.

$$g_i(G - e_1) = \frac{1}{(m-1)(l-1)} d_i(G, e_1) + \frac{1}{(m-1)(l-1)} s_{i+1}(G, e_1)$$

- Step 2: Delete edge e_2 with two end vertices of valencies $(l-1)$ and n respectively. We assume that after deletion of the second edge, the resulting graph is still connected.

If it is not, we must consider this separately.

$$g_i(G - e_1 - e_2) = \frac{1}{(l-2)(n-1)} d_i(G - e_1, e_2) + \frac{1}{(l-2)(n-1)} s_{i+1}(G - e_1, e_2)$$

Note that this topological method is designed for a single graph. If the left side of the resulting equations can be changed to the same variables on the right side, then this method will have a potential to be used in graph families to obtain iteration functions.

2.2 Vertex-splitting

2.2.1 Introduction

Jonathan L. Gross gave an initial work on vertex splitting [15]. We call it 2-splitting here.

Let w be a vertex in graph G, and let U and V be a bipartition of neighbours of w. Remove vertex w, and let every vertex of U be joined to a new vertex u, and every vertex of V be joined to a new vertex v, and join u and v by a new edge. This operation is called a 2-***splitting*** of the vertex w on the graph G (Figure 2.14).

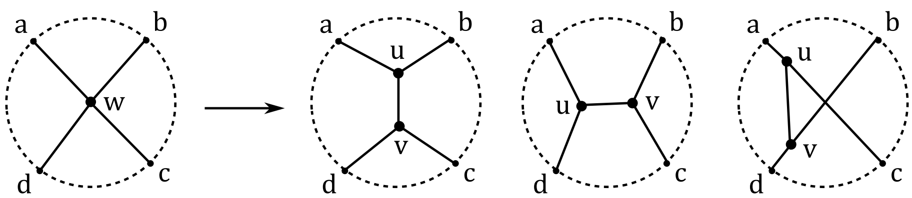


Figure 2.14: 2-splitting a vertex of valency 4 into two vertices of valency 2. The resulting three splits are illustrated. The dashed cycle represent the rest of graph G.

Let w be an n-valent vertex in a graph G, and let r and s be integers at least 2, with

$r + s = n + 2$. Then the number of ways to split graph G at vertex w, such that the end vertices of the new edge have valency r and s is denoted by $N(r, s)$.

Proposition 2.2.1. *[15] For a 2-split, we have*

$$N(r,s) = \begin{cases} \frac{n!}{(r-1)!\cdot(n-r+1)!} = \binom{n}{r-1} & if\ r \neq s \\ \frac{n!}{2\cdot(r-1)!\cdot(n-r+1)!} = \frac{1}{2}\binom{n}{r-1} & if\ r = s \end{cases}.$$

Let U be a vertex subset of a graph G, and ρ_U be an assignment of rotations to every vertex of U. Let $g_i^{\rho_U}(G)$ be the number of embeddings of G on a surface of genus i such that the rotation at every vertex $u \in U$ is $\rho_U(u)$. The sequence

$$g^{\rho_U}(G) = \{g_i^{\rho_U}(G)\}_0^\infty$$

is called the ***relative genus distribution*** of G with respect to ρ_U following [15].

The set of all rotation assignments for U is denoted by R_U. The following results of Gross are straightforward, and will be widely used later.

Proposition 2.2.2. *[15] Let G be a graph and let U be a subset of its vertex set. Then for all $i \geq 0$*

$$g_i(G) = \sum_{\rho \in R_U} g_i^\rho(G).$$

Corollary 2.2.3. *[15] Let G be a graph, let U be a subset of its vertex set. Then*

$$g(G) = \sum_{\rho \in R_U} g^\rho(G).$$

As we will see in the following theorem, there is a linear relationship between the genus distribution of a graph G and the genus distributions of its three split graphs shown in Figure 2.14.

Theorem 2.2.4. *[15] Let G be a graph and w a 4-valent vertex of G. Let H_1, H_2 and H_3 be the three graphs into which G can be split at w, such that the two new vertices of each split are 3-valent (see Figure 2.14). Then*

$$g(G) = \frac{1}{2}g(H_1) + \frac{1}{2}g(H_2) + \frac{1}{2}g(H_3).$$

Based on Gross' work, we will construct equations for 2-splittings for higher values of n. We will also consider 3-splits and uneven splits.

2.2.2 2-split a vertex of valency n when $r = s$

Let us discuss a small example and perform 2-split on a vertex of valency 6 (when $n = 6$ and $r = s = 4$).

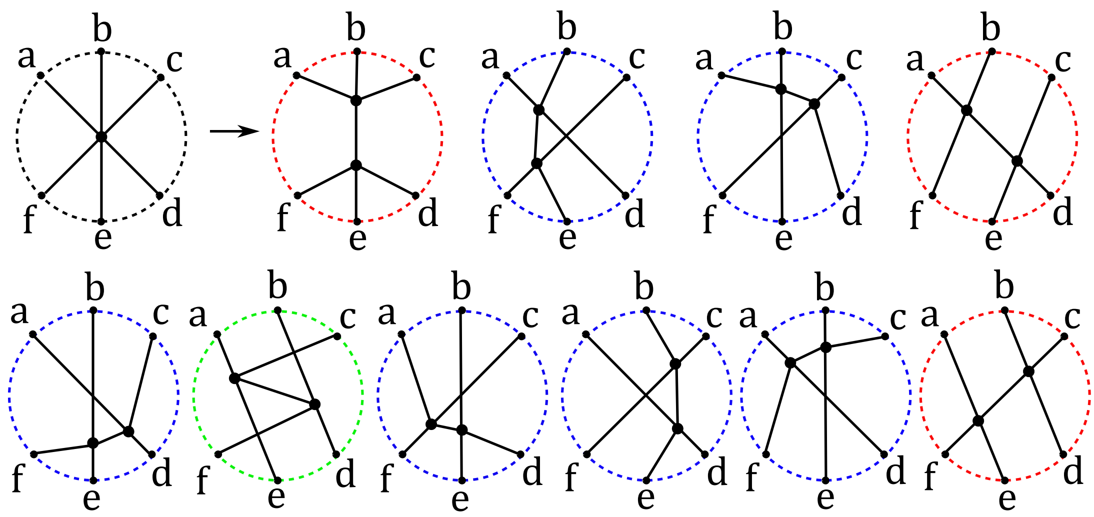


Figure 2.15: Split a vertex of valency 6 into two vertices of valency 4. The 10 splits are illustrated.

Let G be a graph and w a 6-valent vertex of G (Figure 2.15). According to Proposition 2.2.1, there are ten graphs into which G can be split at w.

Proposition 2.2.5. *Let G be a graph and w a 6-valent vertex of G. Let $H_1, \ldots, H_{10}$ be the 10 graphs into which G can be split at w, so that the two new vertices of each split are 4-valent. Then*

$$g(G) = \frac{1}{3}\sum_{i=1}^{10} g(H_i).$$

The proof below uses a similar procedure to Gross' 2-split in Theorem 5.4 in [15].

Proof. In the graph H_i, for $i = 1, 2, \ldots, 10$, let u_i and v_i be the vertices into which vertex w of G splits, and $U_i = \{u_i, v_i\}$. In G, let $U = \{w\}$. We have:

$$g(H_i) = \sum_{\rho \in R_{U_i}} g^\rho(H_i) \text{ for } i = 1, 2, \ldots, 10$$

and

$$g(G) = \sum_{\rho \in R_U} g^\rho(G).$$

It is straightforward to see that the embeddings of $H_1, \ldots, H_{10}$ are mutually disjoint. We can see from Figure 2.15 that the ten split graphs are different when the rest of the graph is not symmetric. If we see the dashed lines as edges, the three red split graphs are the same. That is why we use the word 'mutually disjoint' here.

We make two assertions on the operation of contracting edge u_iv_i.

(1) It induces a 3-to-1 correspondence from the union of the sets of embeddings of these ten graphs onto the set of embeddings of G.

(2) It preserves the genus of the surface.

For an arbitrary embedding $\iota : G \to S$, in which the rotation at vertex w is taken to be $abcdef$. Each of the ten graphs $H_1, \ldots, H_{10}$ has exactly 36 embeddings that coincide with the embedding ι on all the vertices of $V_G - \{w\}$, so there are 360 embeddings altogether in the union of the embeddings of these ten graphs. There are exactly 120 embeddings of G that have the same rotations as ι on the vertices of $V_G - \{w\}$.

The contraction operation decreases the number of vertices and edges, each by 1, and preserves the number of faces. By Euler's formula, the genus of the embedding surface is unchanged. So the equation of the above theorem follows. □

Note that the splitting operation are designed to apply on graphs, rather than embeddings. When we are drawing split graphs like Figure 2.15, it is easy to be confused by the rotations.

Let G be a graph and let w a n-valent vertex of G. We refer to page 42 for the definition of r. By Proposition 2.2.1, there are $\frac{1}{2}\binom{n}{r-1}$ graphs into which G can be split at w. The proof of the following theorem uses a similar procedure to Gross' 2-split [15].

Theorem 2.2.6. *Let G be a graph and w be a n-valent vertex of G. Let $H_1, H_2, \ldots, H_{\frac{1}{2}\binom{n}{r-1}}$ be the $\frac{1}{2}\binom{n}{r-1}$ graphs into which G can be split at vertex w, so that the two new vertices of each split are $(\frac{n}{2}+1)-valent$. Then*

$$g(G) = \frac{2}{n} \cdot \sum_{i=1}^{\frac{1}{2}\binom{n}{r-1}} g(H_i).$$

Proof. We have

$$g(H_i) = \sum_{\rho \in R_{U_i}} g^{\rho}(H_i) \text{ for } i = 1, 2, \ldots, \frac{1}{2}\binom{n}{r-1}$$

and

$$g(G) = \sum_{\rho \in R_U} g^{\rho}(G).$$

We make two assertions on the operation of contracting the edge $u_i v_i$.

(1) It induces an $\frac{n}{2}$-to-1 correspondence from the union of the sets of embeddings of these ten graphs onto the set of embeddings of the graph G.

(2) It preserves the genus of the surface.

For each of the $\frac{1}{2}\binom{n}{r-1}$ split graphs, there are $(r-1)!(s-1)!$ embeddings that coincide with a given embedding ι on all the vertices of $V_G - \{w\}$. There are exactly $(n-1)!$

embeddings of G that have the same rotations as ι on the vertices of $V_G - \{w\}$. It is straightforward to find the $\frac{n}{2}$-to-1 correspondence here.

The contraction operation decreases the number of vertices and edges, each by 1, and preserves the number of faces. By Euler's formula, the genus of the embedding surface is unchanged. So the equation of the above theorem follows. $\square$

These 2-splits discussed above are even splits and have $r = s$. We can extend the result to uneven splits with $r \neq s$.

2.2.3 2-split a vertex of valency n into two vertices of valency r and s, $r \neq s$

We consider an easy case and 2-split a vertex of valency 5 into vertices of valency 3 and 4.

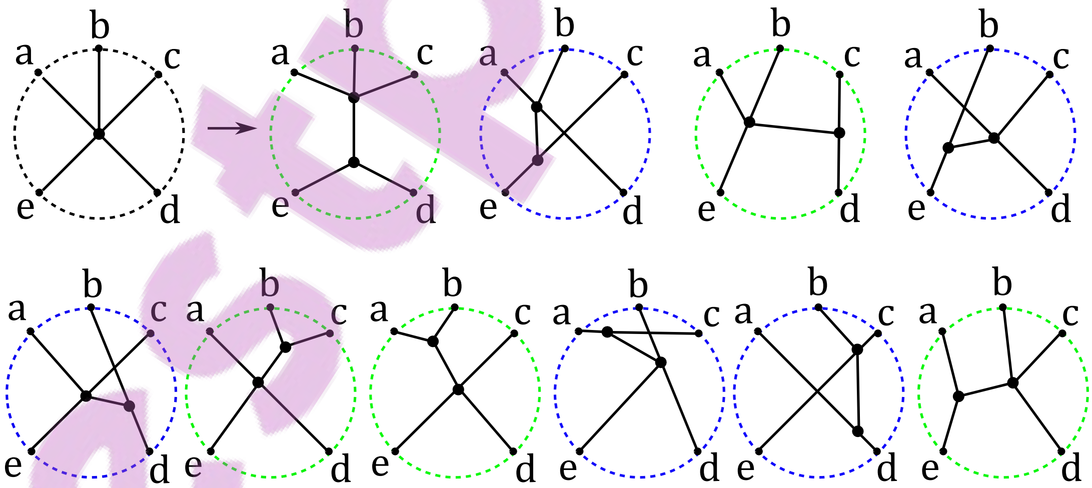


Figure 2.16: Split a vertex of valency 5 into two vertices of valencies 3 and 4. The 10 splits are illustrated.

Let G be a graph and w a 5-valent vertex of G (Figure 2.16). By Proposition 2.2.1, there are ten graphs into which G can be split at w. The proof procedure for the following proposition is similar to Theorem 2.2.6. In this proof and all other splitting cases that follow, we only list the two assertions which are the key part in the proofs.

Proposition 2.2.7. *Let G be a graph and w be a 5-valent vertex of G. Let $H_1, \ldots, H_{10}$ be the ten graphs into which G can be split at w, so that the two new vertices after splitting are of valency 4 and 3 respectively. Then*

$$g(G) = \frac{1}{5} \sum_{i=1}^{10} g(H_i).$$

Proof. We make two assertions on the operation of contracting the edge $u_i v_i$.

(1) It induces a 5-to-1 correspondence from the union of the sets of embeddings of these ten split graphs onto the set of embeddings of the graph G.

(2) It preserves the genus of the surface. □

Now, we extend this result to any uneven 2-split.

Let G be a graph and w a n-valent vertex of G. By Proposition 2.2.1, there are $\binom{n}{r-1}$ graphs into which G can be split at w.

Theorem 2.2.8. *Let G be a graph and w a n-valent vertex of G. Let $H_1, H_2, \ldots, H_{\binom{n}{r-1}}$ be the $\binom{n}{r-1}$ graphs into which G can be 2-split at w, so that the two new vertices of each split are valency r and s, $r \neq s$. Then*

$$g(G) = \frac{1}{n} \cdot \sum_{i=1}^{\binom{n}{r-1}} g(H_i).$$

Proof. There are $\binom{n}{r-1}$ different split graphs. For the splitting operation, we have two assertions:

(1) It induces a n-to-1 correspondence from the union of the sets of embeddings of these $\binom{n}{r-1}$ split graphs onto the set of embeddings of the graph G.

(2) It preserves the genus of the surface. □

2.2.4 3-split a vertex of valency n into 3 vertices of valency t

Let w be a vertex of a graph G, and let $U_1, U_2, \ldots, U_i$ be the subsets of a subdivision of the neighbours of w. In graph $G - w$, let every vertex of $U_j, j = 1, 2, \ldots, i$ be joined to a new vertex u_j and join vertices u_j and w. This operation is called ***i-split*** of G at vertex w. We discuss the 3-split first, then generalize the results to i-splits.

Let w be an n-valent vertex of a graph G, and let r, s and t be integers at least 2, with $r + s + t = n + 3$. Then the number of ways to perform 3-split on graph G at vertex w so that the three new vertices have valency r, s and t is denoted $N(r, s, t)$.

Proposition 2.2.9. *For a* 3*-split, we have*

$$N(r,s,t) = \begin{cases} \binom{r+s+t-3}{t-1}\binom{r+s-2}{s-1} & \text{if} \quad r,s,t \ \text{distinct} \\ \frac{1}{2}\binom{2s+t-3}{t-1}\binom{2s-2}{s-1} & \text{if} \quad r = s \neq t \\ \frac{1}{6}\binom{3t-3}{t-1}\binom{2t-2}{t-1} & \text{if} \quad r = s = t \end{cases}.$$

Proof. This is elementary counting. □

Let G be a graph and w a 6-valent vertex of G (Figure 2.17). By Proposition 2.2.9, there are 15 graphs into which G can be split at w.

Proposition 2.2.10. *Let G be a graph and w be a* 6*-valent vertex of G. Let $H_1, \ldots, H_{15}$ be the* 15 *graphs into which G can be* 3*-split at w, so that the three new vertices are all*

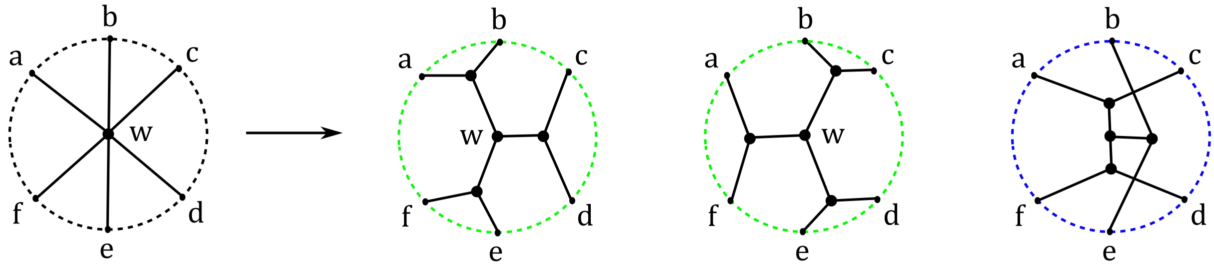


Figure 2.17: Split a vertex of valency 6 into three vertices of valency 2. Three examples of the 15 splits are illustrated.

valency 3. *Then*

$$g(G) = \frac{1}{2}\sum_{i=1}^{15} g(H_i).$$

Proof. There are $N(3,3,3) = 15$ different split graphs. For the contraction operation, we have two assertions:

(1) It induces a 2-to-1 correspondence from the union of the sets of embeddings of these 15 split graphs onto the set of embeddings of the graph G.

(2) It preserves the genus of the surface. $\square$

Let G be a graph and w a n-valent vertex of G, where $n = 3t - 3$. By Proposition 2.2.9, there will be $\frac{1}{6}\binom{3t-3}{t-1}\binom{2t-2}{t-1}$ graphs into which G can be 3-split at w evenly.

Theorem 2.2.11. *Let G be a graph and w be a n-valent vertex of G. Let H_1, $\ldots, H_{\frac{1}{6}\binom{3t-3}{t-1}\binom{2t-2}{t-1}}$ be the $\frac{1}{6}\binom{3t-3}{t-1}\binom{2t-2}{t-1}$ graphs into which G can be 3-split at w, so that the three new vertices are all of valency $\frac{n}{3}+1$. Then*

$$g(G) = \frac{3}{n} \cdot \sum_{i=1}^{\frac{1}{6}\binom{3t-3}{t-1}\binom{2t-2}{t-1}} g(H_i).$$

Proof. There are $N(t,t,t) = \frac{1}{6}\binom{3t-3}{t-1}\binom{2t-2}{t-1}$ different split graphs. For the contraction operation, we have two assertions:

(1) It induces a $\frac{n}{3}$-to-1 correspondence from the union of the sets of embeddings of these $\frac{1}{6}\binom{3t-3}{t-1}\binom{2t-2}{t-1}$ split graphs onto the set of embeddings of the graph G.

(2) It preserves the genus of the surface. $\square$

2.2.5 3-split a vertex of valency n into three vertices of valency s, s and t

We perform a 3-split on a vertex of valency 5 into three vertices of valency 2, 3 and 3 respectively.

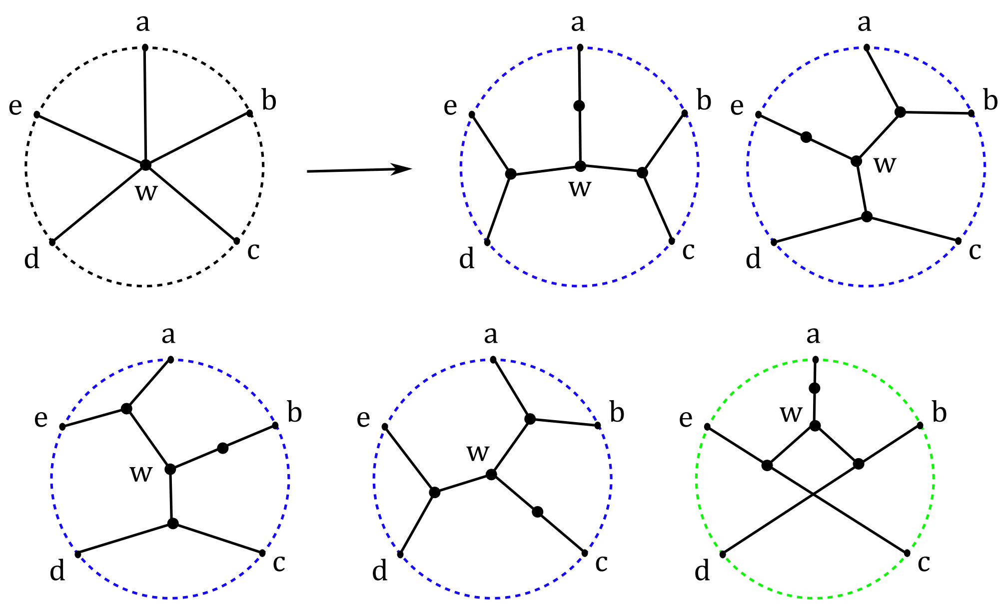


Figure 2.18: Split a vertex of valency 5 into three vertices of valency 1, 2 and 2 respectively. Five examples of the 15 splits are illustrated.

Proposition 2.2.12. *Let G (Figure 2.18) be a graph and w be a $5-$valent vertex of G. Let $H_1, \ldots, H_{15}$ be the $N(3,3,2) = 15$ graphs into which G can be 3-split at w, so that the* *three new vertices are of valency* 3, 3, 2 *respectively. Then*

$$g(G) = \frac{1}{5}\sum_{i=1}^{15} g(H_i).$$

Proof. There are 15 different split graphs. For the contracting operation, we have two assertions:

(1) It induces a 5-to-1 correspondence from the union of the sets of embeddings of these 15 split graphs onto the set of embeddings of the graph G.

(2) It preserves the genus of the surface. □

Theorem 2.2.13. *Let G be a graph and w be a n-valent vertex of G. Let H_1, $\ldots, H_{\frac{1}{2}\binom{2s+t-3}{t-1}\binom{2s-2}{s-1}}$ be the $\frac{1}{2}\binom{2s+t-3}{t-1}\binom{2s-2}{s-1}$ graphs into which G can be* 3*-split at w, so that the three new vertices are of valency $\frac{n-t+3}{2}, \frac{n-t+3}{2}, t$ respectively. Then*

$$g(G) = \frac{1}{n}\cdot\sum_{i=1}^{\frac{1}{2}\binom{2s+t-3}{t-1}\binom{2s-2}{s-1}} g(H_i).$$

Proof. There are $N(s,s,t) = \frac{1}{2}\binom{2s+t-3}{t-1}\binom{2s-2}{s-1}$ different split graphs. For the contraction operation, we have two assertions:

(1) It induces a n-to-1 correspondence from the union of the sets of embeddings of these $\frac{1}{2}\binom{2s+t-3}{t-1}\binom{2s-2}{s-1}$ split graphs onto the set of embeddings of the graph G.

(2) It preserves the genus of the surface. □

2.2.6 3-split a vertex of valency n into three vertices of distinct valencies r, s and t

We discuss an example and 3-split a vertex of valency 6 into three vertices of valency 2, 3 and 4 respectively.

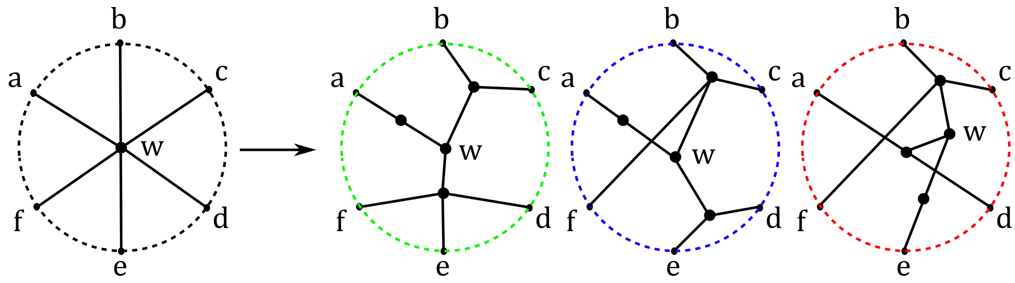


Figure 2.19: Split a vertex of valency 6 into three vertices of valency 1, 2 and 3. Twelve examples of the 60 splits are illustrated.

Theorem 2.2.14. *Let G be a graph and w be a 6-valent vertex of G (Figure 2.19). Let $H_1, \dots, H_{60}$ be the $N(2,3,4) = 60$ graphs into which G can be 3-split at w, so that the three new vertices are of valencies 2, 3, 4 respectively. Then*

$$g(G) = \frac{1}{12} \sum_{i=1}^{60} g(H_i).$$

Proof. There are $N(2,3,4) = 60$ different split graphs. For the contraction operation, we have two assertions:

(1) It induces a 12-to-1 correspondence from the union of the sets of embeddings of these 60 split graphs onto the set of embeddings of the graph G.

(2) It preserves the genus of the surface. □

Theorem 2.2.15. *Let G be a graph and w be a n-valent vertex of G. Let $H_1, \dots, H_{\binom{r+s+t-3}{t-1}\binom{r+s-2}{s-1}}$ be the $\binom{r+s+t-3}{t-1}\binom{r+s-2}{s-1}$ graphs into which G can be 3-split at w, so that the three new vertices are of distinct valencies r, s and t. Then*

$$g(G) = \frac{1}{2n} \cdot \sum_{i=1}^{\binom{r+s+t-3}{t-1}\binom{r+s-2}{s-1}} g(H_i).$$

Proof. There are $N(r,s,t) = \binom{r+s+t-3}{t-1}\binom{r+s-2}{s-1}$ different split graphs. For the contraction operation, we have two assertions:

(1) It induces a $2n$-to-1 correspondence from the union of the sets of embeddings of these $\binom{r+s+t-3}{t-1}\binom{r+s-2}{s-1}$ split graphs onto the set of embeddings of the graph G.

(2) It preserves the genus of the surface. $\square$

2.2.7 Even i-split

Let w be a vertex of a graph G, n be the number of neighbours of w, and let $U_1, U_2, \ldots, U_i$, $(i \geq 3)$ be subsets of the neighbors of w with k vertices in each subset such that $n = k \cdot i$. We call the corresponding operation ***even i-split*** of G at vertex w.

Proposition 2.2.16. *The number of ways to perform even i-split on G at a vertex of valency $k \cdot i$ is*

$$SP_{k \cdot i}(G) = \frac{n!}{i!(k!)^i}.$$

Proof. This is elementary counting. $\square$

Theorem 2.2.17. *Let G be a graph and w be a n-valent vertex of G. Let $H_1, \ldots, H_{SP_{k \cdot i}(G)}$ be the $SP_{k \cdot i}(G)$ graphs into which G can be i-split at w, so that the i new vertices are all of valency $k+1$ such that $n = k \cdot i$. Then*

$$g(G) = \frac{i}{n} \cdot \sum_{j=1}^{SP_{k \cdot i}(G)} g(H_j).$$

Proof. There are $SP_{k \cdot i}(G)$ different split graphs. For the contraction operation, we have two assertions:

(1) It induces a $\frac{n}{i}$-to-1 correspondence from the union of the sets of embeddings of these $SP_{k \cdot i}(G)$ split graphs onto the set of embeddings of the graph G.

(2) It preserves the genus of the surface. $\square$

Equations for uneven i-split could be achieved in the same way. Each case needs to be discussed separately.

2.3 Vertex-augment operation

A vertex w of valency n in a graph G is substituted by a path of $n-1$ edges in a way that each edge which was adjacent to w joins to a vertex on the path. Only the end vertices of the path are valency 2; all other vertices are valency 3. The resulting graphs are called ***augments*** of graph G on the vertex w.

The vertex-augment operation is an extension of vertex splitting. The augment is a rearrangement of a rotation of a vertex along a path, without changing face numbers to keep the genus unchanged. This operation can also apply for a vertex augment on a tree.

2.3.1 Vertex-augment

Proposition 2.3.1. *Let w be an n-valent vertex of a graph G, then the number of ways to augment G at vertex w is $n!/2$.*

Proof. This is elementary counting. □

Example 2.3.2. *When we vertex-augment a vertex of the 4-dipole D_4, it has twelve corresponding augments, as shown in Figure 2.20.*

We can see that contracting a graph along one of its paths inverts the vertex augment on the condition that all the vertices within the path are valency 3 (the two valency 2 end vertices does not make any topological difference).

Theorem 2.3.3. *Let G be a graph and w a 4-valent vertex of G. Let $H_1, H_2, \ldots, H_{12}$ be the 12 graphs into which G can be augmented at w. Then*

$$g(G) = \frac{1}{8}\sum_{i=1}^{12} g(H_i).$$

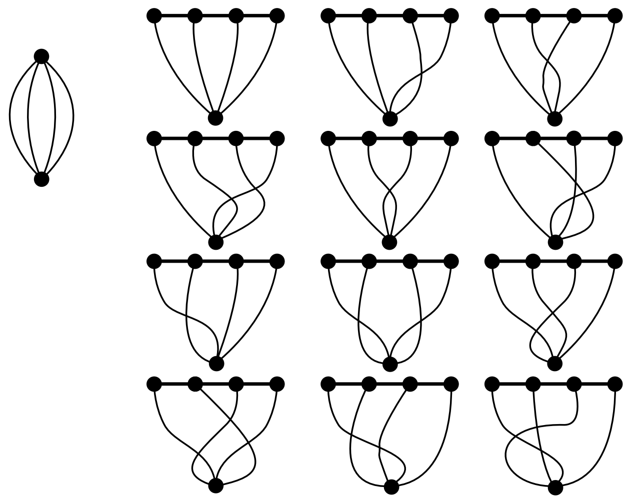


Figure 2.20: Applying vertex-augment on an end vertex of D_4 and resulting 12 augments.

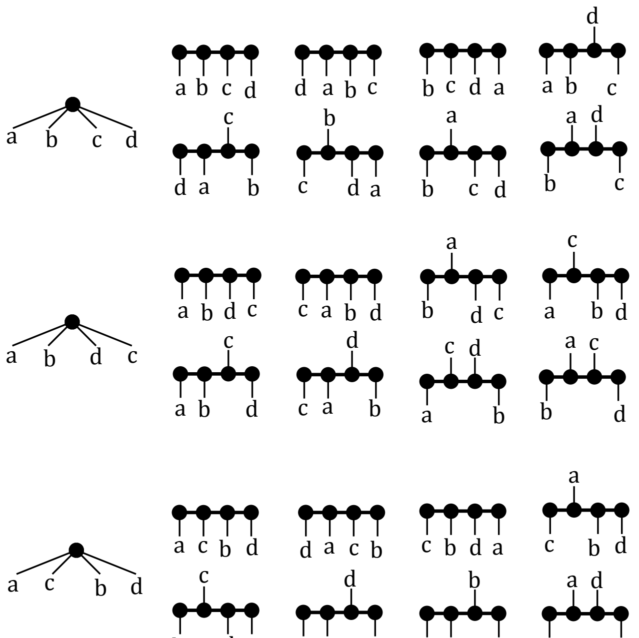


Figure 2.21: Apply vertex-augment on an end vertex of valency 4 resulting in 8-to-1 correspondences. Continued in Figure 2.22.

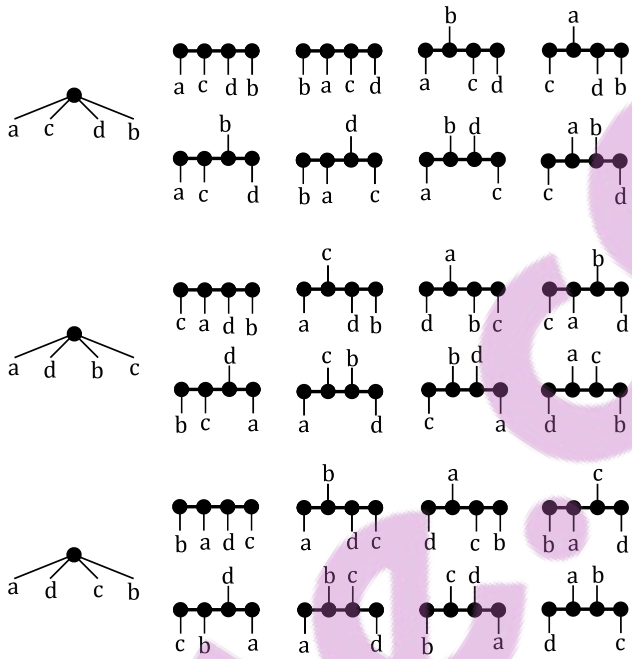


Figure 2.22: Apply vertex-augment on an end vertex of valency 4 resulting in 8-to-1 correspondences. See also Figure 2.21.

Proof. In the graph H_i, for $i = 1, 2, \ldots, 12$, let u_i, v_i, m_i and n_i be the vertices into which vertex w of graph G augments, and $U_i = \{u_i, v_i, m_i, n_i\}$. Also let $U = \{w\}$. We have

$$g(H_i) = \sum_{\rho \in R_{U_i}} g^{\rho}(H_i) \text{ for } i = 1, 2, \ldots, 12$$

and

$$g(G) = \sum_{\rho \in R_U} g^{\rho}(G).$$

It is straightforward to see that the embeddings of $H_1, \ldots, H_{12}$ are distinct.

The two assertions below are on the operation of contracting the edges $u_i v_i,\ v_i m_i$ and $m_i n_i$.

(1) It induces a 8-to-1 correspondence (Figures 2.21 and 2.22) from the union of the sets of embeddings of these 12 graphs onto the set of embeddings of the graph G.

(2) It preserves the genus of the surface.

Consider an arbitrary embedding $\iota : G \to S$, in which the rotation at vertex w is assumed to be *abcd*. Each of the 12 graphs $H_1, \ldots, H_{12}$ has exactly 4 embeddings that coincide with the embedding ι on all the vertices of $V_G - \{w\}$, so there are 48 embeddings altogether in the union of the embeddings of these 12 graphs. There are exactly 6 embeddings of G that have the same rotations as ι on the vertices of $V_G - \{w\}$. Figures 2.21 and 2.22 are an illustration of the 8 correspondences.

The contraction operation decreases the number of vertices and edges by 3, and preserves the number of faces. By Euler's formula, the genus of the embedding surface is unchanged. So the equation follows.

□

We generalize the results and apply vertex-augment on an n-valent vertex of a graph.

Theorem 2.3.4. *Let G be a graph and w an n-valent vertex of G. Let $H_1, H_2, \ldots, H_{n!/2}$ be the $n!/2$ graphs into which G can be augmented at w. Then we can have the linear relationship between the genus distribution of G and the genus distribution of the augmented graphs.*

$$g(G) = \frac{1}{n \cdot 2^{n-3}} \cdot \sum_{i=1}^{n!/2} g(H_i).$$

Proof. In the graph H_i, for $i = 1, 2, \ldots, n!/2$, let $u_{i,1}$, $u_{i,2}$, $\ldots$, $u_{i,n}$ be the vertices into which vertex w of graph G augments, and $U_i = \{u_{i,1}, u_{i,2}, \ldots, u_{i,n}\}$. Let $U = \{w\}$. We have

$$g(H_i) = \sum_{\rho \in R_{U_i}} g^\rho(H_i) \text{ for } i = 1, 2, \ldots, n!/2$$

and

$$g(G) = \sum_{\rho \in R_U} g^\rho(G).$$

It is straightforward to see that the embeddings of $H_1, \ldots, H_{n!/2}$ are mutually disjoint.

Regarding the operation of contracting the edges $u_{i,1}u_{i,2}$, $u_{i,2}u_{i,3}$, $\ldots$, $u_{i,n-1}u_{i,n}$.

(1) It induces a $n \cdot 2^{n-3}$-to-1 correspondence from the union of the sets of embeddings of these $n!/2$ graphs onto the set of embeddings of the graph G.

(2) It preserves the genus of the surface.

For an arbitrary embedding $\iota : G \to S$, each of the $n!/2$ graphs $H_1, \ldots, H_{n!/2}$ has exactly 2^{n-2} embeddings that coincide with the embedding ι on all the vertices of $V_G - \{w\}$, so there are $n!2^{n-3}$ embeddings altogether in the union of the embeddings of these $n!/2$ graphs. There are exactly $(n-1)!$ embeddings of G that have the same rotations as ι on the vertices of $V_G - \{w\}$.

The contraction operation decreases the number of vertices and edges by $n-1$, and preserves the number of faces. By Euler's formula, the genus of the embedding surface is unchanged. So the equation of the above theorem follows.

□

2.3.2 Application of the vertex-augment

It is difficult to calculate the genus distribution of wheel graphs. We find the genus distribution of an easier related graph, the ***fan graph*** F_n, which consists n vertices on a $n-path$ (no repeated vertices) with an extra vertex called a ***hub*** joined to the n vertices by n edges called ***spokes***. If we apply vertex augment on an end vertex of a dipole graph D_n, we will find that all the resulting augment graphs are F_n. Then we have the following results.

Theorem 2.3.5. *There is a linear relationship between the genus distribution of F_n and* *the genus distribution of D_n.*

$$g(F_n) = \frac{2^{n-2}}{(n-1)!} g(D_n)$$

Proof. After performing vertex augment on a vertex of dipole graph D_n, we can find easily that $H_1, H_2, \ldots, H_{n!/2}$ are topologically equivalent to F_n. Then

$$n \cdot 2^{n-3} \cdot g(D_n) = \sum_{i=1}^{n!/2} g(H_i) = n!/2 \cdot g(F_n).$$

The resulting equation is straightforward. □

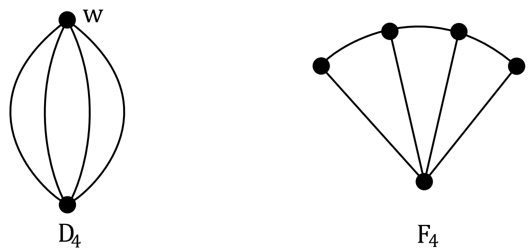


Figure 2.23: Perform vertex augment on an end vertex of D_4, we have F_4.

Example 2.3.6. *The fan F_4.*

Performing vertex augment on a vertex of D_4, we have F_4 (Figure 2.23).

The genus distribution of D_4 is $g(D_4) = 6 + 30x$ [21]. So we can have the genus distribution of F_4 by the theorem above.

$$g(F_4) = \frac{2^{4-2}}{(4-1)!} g(D_4) = 4 + 20x$$

If we apply a vertex augment operation on the hub of F_n, we do not simply find a ladder graph, but a family of cross type ladder graphs [41].

We can define a vertex augment by the rearrangement of adjacent edges of a vertex on a cycle which contains vertices of valency 3 only. It is a reverse operation of face-contraction.

We can also define a vertex augment by the rearrangement of adjacent edges of a vertex upon a tree and preserve the face numbers to keep the genus unchanged. A linear relationship between the genus distribution of the original graph and the genus distributions of augment graphs could be achieved.

The result of vertex augment also applies to vertices which have loops adjacent to them. For example, augmenting on the central vertex of bouquet of circles along a path of size one (an edge) will results in graphs which can be formed by putting some loops on both sides of a dipole graph. Gross' 2-split theory also applies here.

According to these results of vertex augments, the genus distribution of any graph has a linear relationship with the genus distribution of some cubic graphs. The cubic graphs are the results of applying a vertex augment on every vertex whose valency is bigger than 3.

2.4 Pearl-making method

For a given graph G, this method supplies a way to get the genus distribution of the graph by adding a finite number of loops on a root edge with two end vertices of valency 2.

Following the procedure outlined below, we can add any number of loops. Figure 2.24 shows how to add 2 loops.

Procedure:

- 1. Choose any edge a with two end vertices valency 2 as the root edge of a given

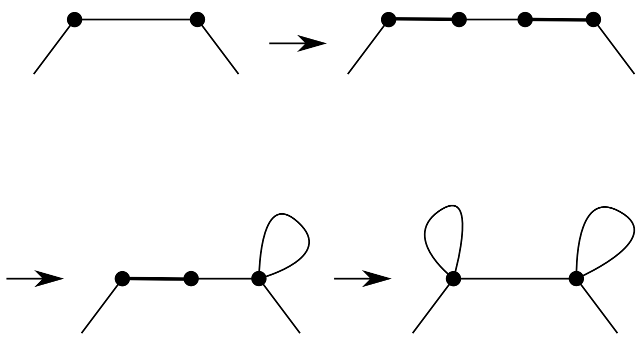


Figure 2.24: Apply pearl-making method on an edge of a graph to add two loops.

graph.

- 2. Divide edge a to get a path of size $2n-1$ with only vertices of valency 2.
- 3. Merging the two end vertices of each edge and get n loops connected by $n-1$ edges.

We consider the production rule by applying the pearl-making method once.

In a graph G_0, choose an edge root a_0 which has two end vertices of valency 2. Then divide a_0 into three new edges c_1, b_1 and a_1 by inserting two new vertices of valency 2. Merge the two end vertices of edge c_1, and get the graph G_1. Note that a_1 is the reserved edge root to make more loops.

Recall that $d_i(G, e)$ is the number of embeddings of graph G on surface S_i in which two different face walks are incident on edge e, and that $s_i(G, e)$ is the number of embeddings of graph G on surface S_i in which the same face walk is incident on edge e.

Lemma 2.4.1. *The production rule can be derived by applying the pearl-making method*

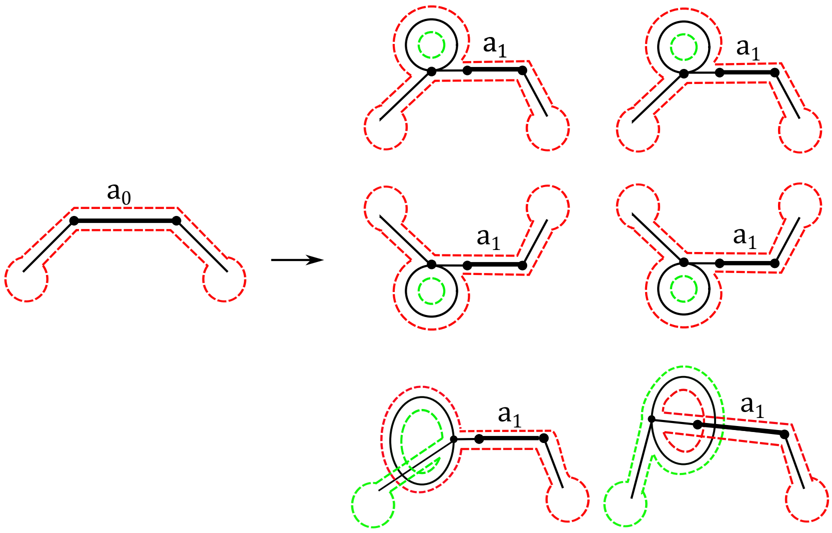


Figure 2.25: In an embedding of a graph G_0, when the two appearances of the chosen root edge a_0 are involved in one face, there are six corresponding embeddings of graph G_1 after applying pearl-making method once on a_0, and all of them have the two appearances of a_1 involved in the same face.

once on edge a_0 of graph G_0.

$$\begin{cases} d_i(G_0, a_0) & \rightarrow & 4d_i(G_1, a_1) + 2s_{i+1}(G_1, a_1) \\ s_i(G_0, a_0) & \rightarrow & 6s_i(G_1, a_1) \end{cases}$$

Note that the production rule is not expressed in equation form, but in mapping form by arrows.

Proof. Let g_0, v_0, e_0 and f_0 be the genus, number of vertices, edges and faces of a given graph embedding of G_0. Let g_1, v_1, e_1 and f_1 be the genus, number of vertices, edges and faces of the resulting graph embedding of G_1. By Euler's formula, we have $2-2g_0 = v_0 - e_0 + f_0$ and $2-2g_1 = v_1 - e_1 + f_1$.

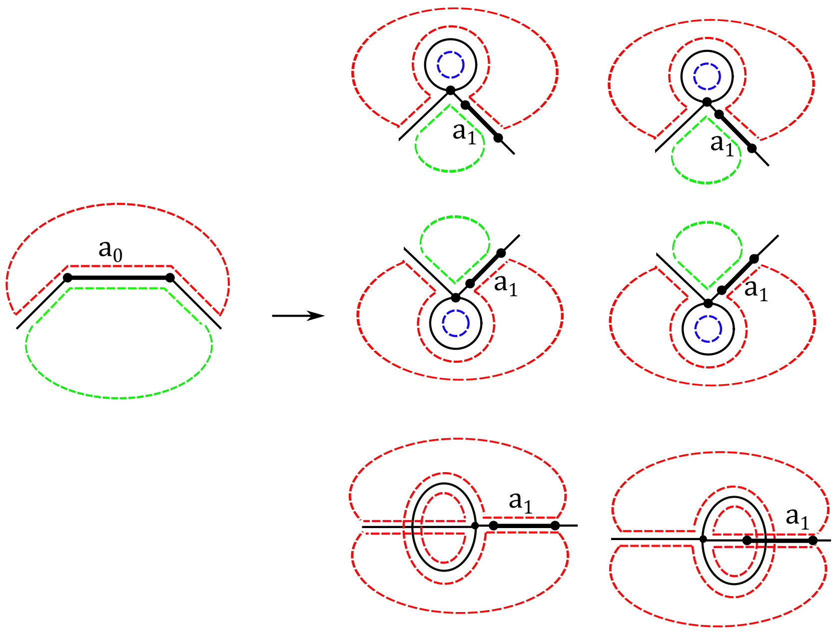


Figure 2.26: In an embedding of a graph G_0, when the two appearances of the chosen root edge a_0 are involved in two different faces, there are six corresponding embeddings of graph G_1 after applying pearl-making method once on edge a_0. Four of them have the two appearances of a_1 involved in different faces, and two of them have the two appearances of a_1 involved in one face.

When the two occurrences of the root edge are on the same face (Figure 2.25), $v_1 = v_0 + 1$, $e_1 = e_0{+}2$, $f_1 = f_0 + 1$, so $g_1 = g_0$.

When the two occurrences of the root edge are on the different faces (Figure 2.26), $v_1 = v_0 + 1$, $e_1 = e_0 + 2$, $f_1 = f_0 + 1$ in the first four cases and $f_1 = f_0 - 1$ in the last two cases, so $g_1 = g_0$ in the first four cases and $g_1 = g_0 + 1$ in the last two cases. □

Following the production rule in Lemma 2.4.1, we get G_1 from G_0. If we apply the pearl-making method along the same edge $n - 1$ times, we get G_{n-1}.

Lemma 2.4.2. *Following the procedure, we can get the production rule from* (G_{n-1}, a_{n-1}) *to* (G_n, a_n).

$$\begin{cases} d_i(G_{n-1}, a_{n-1}) & \rightarrow & 4d_i(G_n, a_n) + 2s_{i+1}(G_n, a_n) \\ s_i(G_{n-1}, a_{n-1}) & \rightarrow & 6s_i(G_n, a_n) \end{cases}$$

Note that the production rule is not expressed in equation form, but in mapping form by arrows.

Theorem 2.4.3. *From the production rule above, we can get the following partial genus distribution of* (G_n, a_n).

$$\begin{cases} d_i(G_n) & = & 4^n d_i(G_0) \\ s_i(G_n) & = & (6^n - 4^n) d_{i-1}(G_0) + 6^n s_i(G_0) \end{cases}$$

Proof. From the production rule in Lemma 2.4.2, we have

$$d_i(G_n) = 4d_i(G_{n-1}) = 4^2 d_i(G_{n-1}) = \cdots = 4^n d_i(G_0)$$

and

$$\begin{aligned} s_i(G_n) &= 2d_{i-1}(G_{n-1}) + 6s_i(G_{n-1}) \\ &= 2(6d_{i-1}(G_{n-2}) + 4d_{i-1}(G_{n-2})) + 6^2 s_i(G_{n-2}) \\ &= 2(4^{n-1} + 6 \cdot 4^{n-2} + 6^2 \cdot 4^{n-3} + \cdots + 6^{n-2} \cdot 4 + 6^{n-1}) d_{i-1}(G_0) + 6^n s_i(G_0) \\ &= 2(4^{n-1} \cdot (1 + \tfrac{6}{4} \cdot \tfrac{(\frac{6}{4})^{n-1} - 1}{\frac{1}{2}})) d_{i-1}(G_0) + 6^n s_i(G_0) \\ &= (6^n - 4^n) d_{i-1}(G_0) + 6^n s_i(G_0). \end{aligned}$$

□

The two occurrences of the root edge can not be as shown in Figure 2.27. By merging the two ends of the root edge, the number of vertices drops by one. The number of edges does not change. The number of faces does not change either, which contradicts Euler's formula.

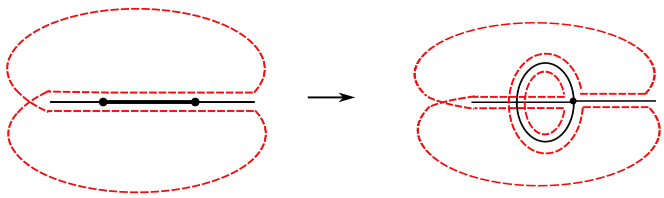


Figure 2.27: This kind of embedding of graph G_0 with the two appearances of edge a_0 belong to one face and has a face walk crossing does not exist. Pearl-making on this kind of embedding cannot be performed.

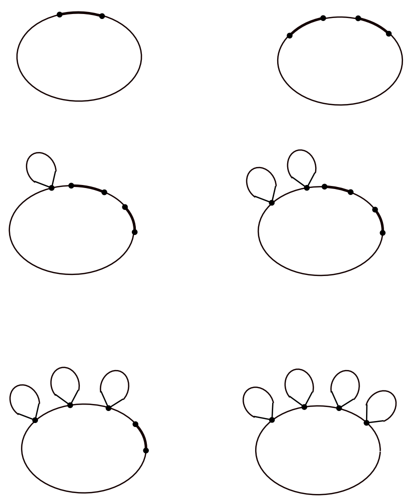


Figure 2.28: Apply pearl-making method four times on a cycle G_0 and get G_1,G_2, G_3 and G_4.

Example 2.4.4. *The example shows how the pearl-making method applies on a simple cycle (Figure 2.28). Let G_0 be a simple cycle. By adding one loop, we get G_1. With more* *loops added, we get G_n for any number of pearls n.*

$$
\begin{aligned}
d_0(G_1) &= 4d_0(G_0) &&= 4\\
s_1(G_1) &= 2d_0(G_0)+6s_1(G_0) &&= 2\\
d_0(G_2) &= 4d_0(G_1) &&= 16\\
s_1(G_2) &= 2d_0(G_1)+6s_1(G_1) &&= 20\\
d_0(G_3) &= 4d_0(G_2) &&= 64\\
s_1(G_3) &= 2d_0(G_2)+6s_1(G_2) &&= 152\\
d_0(G_4) &= 4d_0(G_3) &&= 256\\
s_1(G_4) &= 2d_0(G_3)+6s_1(G_3) &&= 1040
\end{aligned}
$$

$$
\begin{aligned}
g(G_1) &= 4+2x\\
g(G_2) &= 16+20x\\
g(G_3) &= 64+152x\\
g(G_4) &= 256+1040x
\end{aligned}
$$

$$
\begin{aligned}
g_0(G_n) &= 4^n\\
g_1(G_n) &= 6^n-4^n\\
g(G_n) &= 4^n+(6^n-4^n)x
\end{aligned}
$$

2.5 First bouquet-making method—merging root vertices

2.5.1 Introduction

In this section, we present a method to get the genus distribution of graphs by merging two adjacent root vertices. The information we need is not the genus distribution of the original graph (we call it a dipole-rooted graph), but the partial genus distribution on the two side edges connected to the dipole-root.

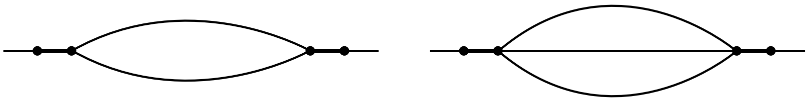


Figure 2.29: A D_2-rooted graph and a D_3-rooted graph

If a dipole graph D_n, together with another two adjacent edges whose end vertices are all valency 2 and $n+1$ respectively, are a subgraph of a graph G, then G is called a D_n***-rooted graph***. Figure 2.29 shows a D_2-rooted graph and a D_3-rooted graph.

Figure 2.30: A B_2-rooted graph and a B_3-rooted graph

If a bouquet of circles B_n, together with another two adjacent edges whose end vertices are all valency 2 and $2n+2$ respectively, is a subgraph of a graph G, then G is called a B_n***-rooted graph***. Figure 2.30 shows a B_2-rooted graph and a B_3-rooted graph.

2.5.2 D_2-Rooted graphs

In this subsection, we apply the bouquet-making method to D_2-rooted graphs.

2.5.2.1 D_2-Root is a cut-subgraph

Figure 2.31 shows two embedding cases of a D_2-rooted graph where D_2-root is a cut-subgraph.

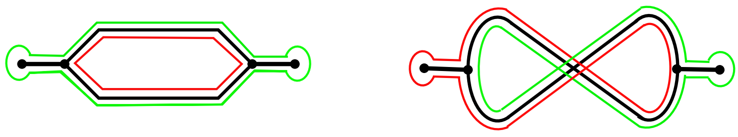


Figure 2.31: Two embedding cases of a D_2-rooted graph with D_2-root as a cut-subgraph

We use two letters to represent the face tracing of two side edges. When the dipole root is a cut-subgraph of G_0, let ss_i^0 be the number of embeddings such that the two appearances of both of the two side edges are involved in the same face. Let ss_i^2 be the number of embeddings such that the two appearances of one side edge belong to one face and the two appearances of the other side edge belong to another face. The partial s_i^0 combines cases of ss_i^0 and ss_i^2 together such that the two appearances of each side edge belong to one face walk. The index i is the genus of the embedding. Note that there are no *sd* or *dd* cases.

Proposition 2.5.1. *The production rule of applying the bouquet-making method on a D_2-rooted graph G_0, where the D_2-root is a cut-subgraph, is*

$$s_i^0(G_0) \to 18s_i^0(G_1) + 12s_{i+1}^0(G_1).$$

Proof. Let G_0 and G_1 denote the graphs before and after applying the bouquet-making method. The sum of all the coefficients on the right side of each production rule is $(6-1)!$, which is the number of rotations on the bouquet vertex.

$$\begin{cases} ss_i^0(G_0) & \rightarrow & 8ss_i^0(G_1) + 16ss_i^2(G_1) + 16ss_i^2(G_1) + 16ss_{i+1}^0(G_1) \\ & & +16ss_{i+1}^0(G_1) + 16ss_i^0(G_1) + 16ss_i^2(G_1) + 16ss_{i+1}^0(G_1) \\ ss_i^2(G_0) & \rightarrow & 8ss_i^0(G_1) + 16ss_i^2(G_1) + 16ss_i^2(G_1) + 16ss_{i+1}^0(G_1) \\ & & +16ss_{i+1}^0(G_1) + 16ss_i^0(G_1) + 16ss_i^2(G_1) + 16ss_{i+1}^0(G_1) \end{cases}$$

The resulting embeddings of $ss_i^0(G_0)$ and $ss_i^2(G_0)$ are the same. The production rule above can be simplified by combining ss_i^0 and ss_i^2 together. We use s_i^0 instead of ss_i^0 and ss_i^2. We have

$$4s_i^0(G_0) \rightarrow 72s_i^0(G_1) + 48s_{i+1}^0(G_1).$$

After simplification, we have the equation. □

Theorem 2.5.2. *Let (G_0, u, v) be a D_2-rooted graph, where the D_2-root is a cut-subgraph of G_0. After applying the bouquet-making method, the partial genus distribution of the resulting B_2-rooted graph G_1 is*

$$s_i^0(G_1) = 18s_i^0(G_0) + 12s_{i-1}^0(G_0).$$

Proof. It is straightforward by the proposition above. □

2.5.2.2 D_2-Root is not a cut-subgraph

Figure 2.32 shows two embedding cases of a D_2-rooted graph where D_2-root is not a cut-subgraph.

We use two letters to represent the face tracing of two side edges. When the dipole root is not a cut-subgraph of G_0, let ss_i^1 be the number of embeddings such that the two appearances of the two side edges are involved in the same face. Let dd_i^0 be the number

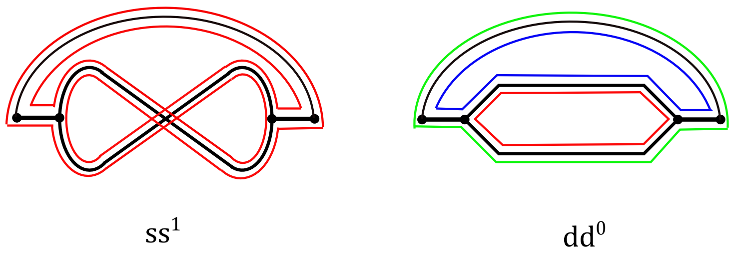


Figure 2.32: Two embedding cases of a D_2-rooted graph when the D_2-root is not a cut-subgraph

of embeddings such that the two appearances of the two side edges are involved in two different faces. The index i is the genus of the embedding. Note that there is no sd case.

Lemma 2.5.3. *The production rule of applying the bouquet-making method on a D_2-rooted graph G_0, where the D_2-root is **not** a cut-subgraph, is*

$$\begin{cases} ss_i^1(G_0) & \rightarrow & 8dd_{i-1}^0(G_1) + 16ss_i^1(G_1) + 16ss_i^1(G_1) + 16dd_i^0(G_1) \\ & & +16dd_i^0(G_1) + 16dd_{i-1}^0(G_1) + 16ss_i^1(G_1) + 16dd_i^0(G_1) \\ dd_i^0(G_0) & \rightarrow & 8dd_i^0(G_1) + 16ss_{i+1}^1(G_1) + 16ss_{i+1}^1(G_1) + 16dd_{i+1}^0(G_1) \\ & & +16dd_{i+1}^0(G_1) + 16dd_i^0(G_1) + 16ss_{i+1}^1(G_1) + 16dd_{i+1}^0(G_1) \end{cases}.$$

The resulting embeddings of $ss_{i+1}^1(G_0)$ and $dd_i^0(G_0)$ are the same according to the production rules above. We use s_i^1 and d_i instead of ss_i^1 and dd_i^0. We have

$$2d_{i-1}(G_0) + 2s_i^1(G_0) \rightarrow 24d_{i-1}(G_1) + 48d_i(G_1) + 48s_i^1(G_1).$$

Theorem 2.5.4. *Let (G_0, u, v) be a D_2-rooted graph, where the D_2-root is not a cut-subgraph of G_0. After applying the bouquet-making method, the partial genus distribution* *of the resulting B_2-rooted graph G_1 is*

$$\begin{cases} s_i^1(G_1) & = & 24s_i^1(G_0) & = & 24d_{i-1}(G_0) \\ d_i(G_1) & = & 24s_i^1(G_0) & + & 12s_{i+1}^1(G_0) \\ & = & 24d_{i-1}(G_0) & + & 12d_i(G_0) \end{cases} .$$

All the embedding cases in each production rule were analyzed by hand face tracing, which include two parts: one part with original embeddings of cases ss^0 and ss^2, one part with original embeddings of cases ss^1 and dd^0.

2.5.3 D_3-rooted graphs

In this part, we apply the bouquet-making method on D_3-rooted graphs.

2.5.3.1 D_3-Root is a cut-subgraph

Figure 2.33 shows one of the $((4-1)!)^2 = 36$ embedding cases of a D_3-rooted graph, where D_3-root is a cut-subgraph.

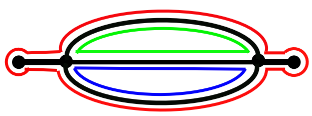


Figure 2.33: An embedding of a D_3-rooted graph with D_3-root being a cut-subgraph

When the dipole root is a cut-subgraph of G, let s_i be the number of embeddings of G such that the two appearances of both of the two side edges are involved in the same face. The two appearances of the two side edges can not be involved in different faces.The index i is the genus of the embedding.

Lemma 2.5.5. *The production rule of applying the bouquet-making method on a D_3-rooted graph, where the D_3-root is a cut-subgraph, is*

$$18s_i(G_0) + 18s_{i+1}(G_0) \to 1680s_i(G_1) + 3360s_{i+1}(G_1).$$

G_0 and G_1 denote the graphs before and after the applying of the bouquet-making method.

Theorem 2.5.6. *Let (G, u, v) be a D_3-rooted Graph with two 4-valent root vertices, where the D_3-root is a cut-subgraph. Then the partial genus distribution of the graph by merging the two root vertices is*

$$s_i(G_1) = \frac{280}{3}s_i(G_0) + \frac{560}{3}s_{i-1}(G_0).$$

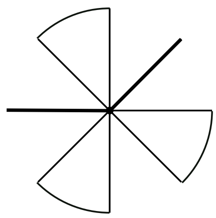


Figure 2.34: The smallest B_3-rooted graph with B_3-root being a cut-subgraph is B_3'

The research of the embedding distribution of B_3-rooted graph and its classifications is based on a graph B_3', which is the smallest B_3-rooted graph as shown in Figure 2.34. The genus distribution of B_3' and the genus distribution of B_3 have a 42 to 1 correspondence. We get $g(B_3) = 40 + 80x$ through calculation by Magma. When we insert the first edge to B_3, it has 6 places to choose from. When we insert the second edge, it has 7 places to choose from. So the genus distribution of B_3' is

$$g(B_3') = 1680 + 3360x.$$

2.5.3.2 D_3-Root is not a cut-subgraph

Figure 2.35 shows two embedding cases of a D_3-rooted graph with the D_3-root not being a cut-subgraph.

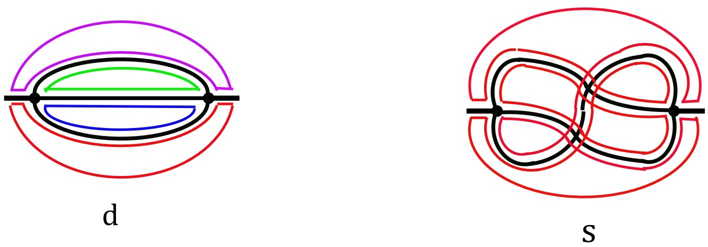


Figure 2.35: The two embedding cases of a D_3-rooted graph with the D_3-root being not a cut-subgraph.

When the dipole root is not a cut-subgraph of G, let s_i be the number of embeddings of G, such that the two appearances of both of the two side edges are involved in the same face. Let d_i be the number of embeddings of G, such that the two appearances of the two side edges are involved in two separate faces. The index i is the genus of the embedding.

Lemma 2.5.7. *The production rule of applying bouquet-making method on a D_3-rooted graph, with the D_3-root not being a cut-subgraph, is*

$$6d_i(G_0)+18d_{i+1}(G_0)+12s_{i+1}(G_0) \rightarrow 672d_i(G_1)+2352d_{i+1}(G_1)+1008s_{i+1}(G_1)+1008s_{i+2}(G_1).$$

Theorem 2.5.8. *Let (G, u, v) be a D_3-rooted graph with two 4-valent root vertices. Then the partial genus distribution of the graph by merging the two root vertices are as follows.*

$$\begin{cases} s_i(G_1) &= 84s_i(G_0) + 84s_{i-1}(G_0) \\ d_i(G_1) &= 112d_i(G_0) + 392d_{i-1}(G_0) \end{cases}$$

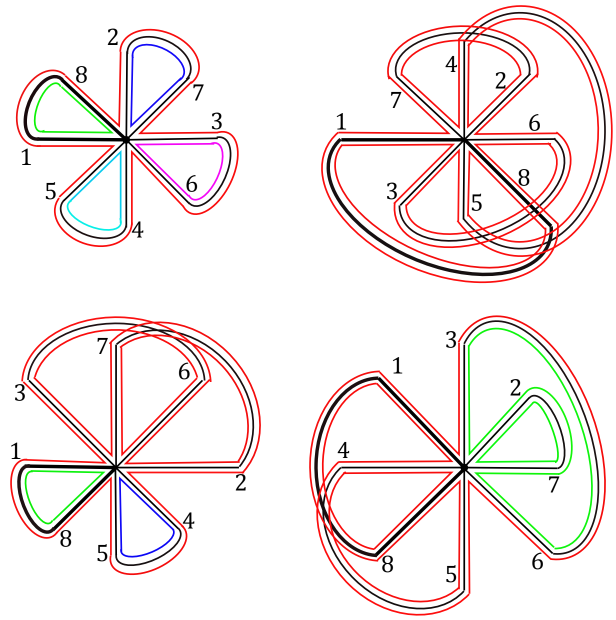


Figure 2.36: A B_3-rooted graph with the B_3-root not being a cut-subgraph can be represented by B_4 topologically. The two edges denoted by number 1 and 8 represent the rest of the graph except the bouquet root. Four embeddings of the 5040 cases are illustrated.

The research of the embedding distribution of B_3-rooted graphs and its classifications (shown in Figure 2.36) are based on graph B_4, which can represent B_3-rooted graphs with the B_3-root not being a cut-subgraph.One loop (labeled 1 and 8) stands in place of the rest of the graph.

2.6 Second bouquet-making method—merging root vertices

This operation produces a method to construct the genus distribution of graphs by merging two adjacent root vertices of a graph called a $B_{m,n}$-rooted graph. The resulting graph is a B_{m+n+1}-rooted graph. The information we need is not the genus distribution of $B_{m,n}$ rooted graph, but the partial genus distribution on the two side edges connected to the $B_{m,n}$ root.

A $B_{m,n}$***-rooted graph*** (Figure 2.37) is a connected graph, which has a root of two bouquets of loops B_m and B_n connected by an edge. Each bouquet of loops has a distinct edge connecting it to the rest of the graph called **side edges**. In Figure 2.37, a and b are the two side edges.

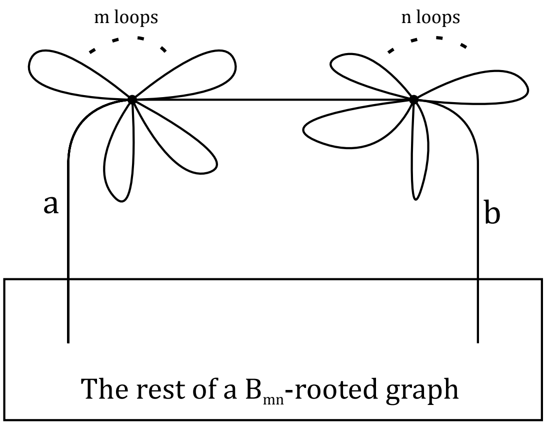


Figure 2.37: A $B_{m,n}$-rooted graph has two root vertices with m and n loops connected to each of them respectively.

2.6.1 When $B_{m,n}$-root is a cut-subgraph

When $B_{m,n}$ is a cut-subgraph of a graph G, let s_i^0 be the number of embeddings of G such that each side edge has its two appearances involved in one face. The index i is the genus of the embedding.

2.6.1.1 $B_{1,1}$-rooted graph

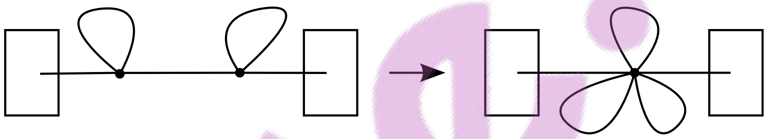


Figure 2.38: The left side is a $B_{1,1}$-rooted graph. After the bouquet-making method, it changes to the B_3-rooted graph on the right.

Lemma 2.6.1. *The production rule of applying the bouquet-making method on a $B_{1,1}$-rooted graph, with $B_{1,1}$-root being a cut-subgraph (Figure 2.38), is*

$$36s_i^0(G_0) \to 1680s_i^0(G_1) + 3360s_{i+1}^0(G_1).$$

The coefficient on the left side of the production rule is $((4-1)!)^2$. The sum of the coefficients on the right side of the production rule is $(8-1)!$.

Theorem 2.6.2. *Let G_0 be a $B_{1,1}$-rooted graph; $B_{1,1}$-root is a cut-subgraph of G_0. Then the partial genus distribution of the graph by applying the bouquet-making method is*

$$s_i^0(G_1) = \frac{140}{3}s_i^0(G_0) + \frac{280}{3}s_{i-1}^0(G_0).$$

2.6.1.2 $B_{2,1}$-rooted graph

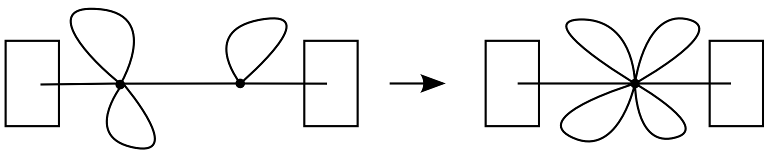


Figure 2.39: The left side is a $B_{2,1}$-rooted graph. After the bouquet-making method, it changes to the B_4-rooted graph on the right.

Lemma 2.6.3. *The production rule of applying the bouquet-making method on a $B_{2,1}$-rooted graph, with $B_{2,1}$-root being a cut-subgraph (Figure 2.39), is*

$$480s_i^0(G_0) + 240s_{i+1}^0(G_0) \to 48384s_i^0(G_1) + 241920s_{i+1}^0(G_1) + 72576s_{i+2}^0(G_1).$$

The sum of the coefficients on the left side of the production rule is $(6-1)! \cdot (4-1)!$. The sum of the coefficients on the right side of the production rule is $(10-1)!$.

Theorem 2.6.4. *Let G_0 be a $B_{2,1}$-rooted graph, and let $B_{2,1}$-root be a cut-subgraph of G_0. Then the partial genus distribution of the graph by applying bouquet-making method is*

$$s_i^0(G_1) = \frac{504}{5}s_i^0(G_0) + 504s_{i-1}^0(G_0) + \frac{756}{5}s_{i-2}^0(G_0).$$

2.6.1.3 $B_{m,n}$-rooted graph

For a $B_{m,n}$-rooted graph, it is not easy to derive a common production rule. The sum of the coefficients on the left side is $(2m+1)! \cdot (2n+1)!$. The sum of the coefficients on the

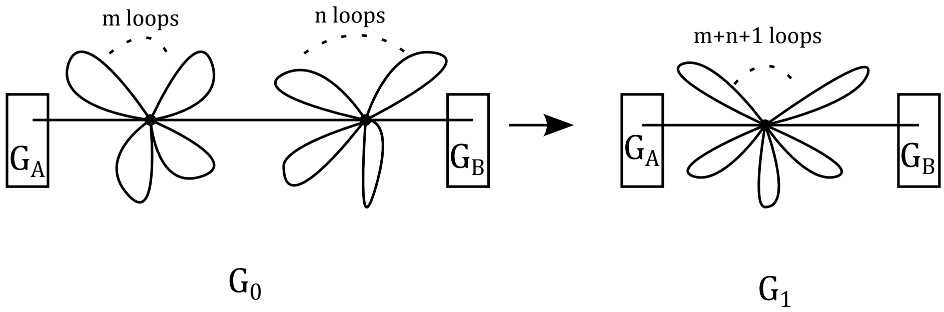


Figure 2.40: The left side is a $B_{m,n}$-rooted graph. After the bouquet-making method, it changes to the B_{m+n+1}-rooted graph on the right.

right side is $(2m+2n+3)!$. It is difficult to distribute these numbers on the left side and the right side of the production rule.

We use another way to express the production rule here.

Lemma 2.6.5. *The production rule of applying bouquet-making method on a $B_{m,n}$-rooted graph, with $B_{m,n}$-root a cut-subgraph (Figure 2.40), is*

$$2mn(2m+1)(2n+1)g(B_m)g(B_n) \to (m+n+1)(2(m+n+1)+1)g(B_{m+n+1}).$$

Proof. Let G_A and G_B be the two subgraphs of G_0, which are connected to rest of the graph by the two side edges. The valency of the two vertices which connect G_A and G_B to the side edges are $x+1$ and $y+1$ respectively. According to Theorem 2.1.13, we have

$$g(G_0) = g(G_A)xg(B_m)2m(2m+1)g(B_n)2n(2n+1)yg(G_B)$$

and

$$g(G_1) = g(G_A)xg(B_{m+n+1})2(m+n+1)(2(m+n+1)+1)yg(G_B).$$

So the mapping

$$g(G_0) \to g(G_1)$$

can be simplified to the production rule. □

It is hard to derive a partial genus distribution equation from the production rule in Lemma 2.6.5. This remains an open problem.

When $B_{m,n}$-root is a cut-subgraph, the genus distribution of G_0 and G_1 are the same as their partial genus distributions, because only s^0 is involved in the face-tracing of two side edges. So the coefficient of x^i is s_i^0.

2.6.2 When $B_{m,n}$-root is not a cut-subgraph

2.6.2.1 $B_{1,1}$-rooted graph

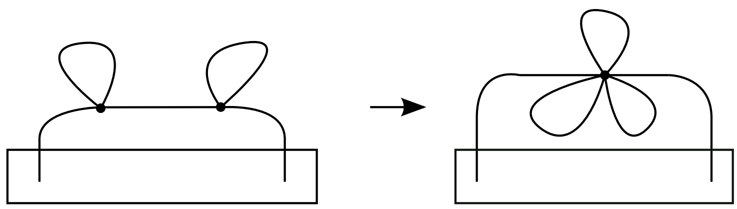


Figure 2.41: The left side is a $B_{1,1}$-rooted graph. By applying the bouquet-making method, it changes to the B_3-rooted graph.

Lemma 2.6.6. *The production rule of applying bouquet-making method on a $B_{1,1}$-rooted graph, with $B_{1,1}$-root being not a cut-subgraph (Figure 2.41), is*

$$16d_i(G_0) + 20s_{i+1}(G_0) \to 672d_i(G_1) + 2352d_{i+1}(G_1) + 1008s_{i+1}(G_1) + 1008s_{i+2}(G_1).$$

Theorem 2.6.7. *Let G_0 be a $B_{1,1}$-rooted graph, where the $B_{1,1}$-root is not a cut-subgraph of the G_0. Then the partial genus distribution of the graph by applying bouquet-making* *method is*

$$\begin{cases} d_i(G_1) &= 42d_i(G_0) &+ 147d_{i-1}(G_0) \\ s_i(G_1) &= \frac{252}{5}s_i(G_0) &+ \frac{252}{5}s_{i-1}(G_0) \end{cases}.$$

2.6.2.2 $B_{2,1}$-rooted graph

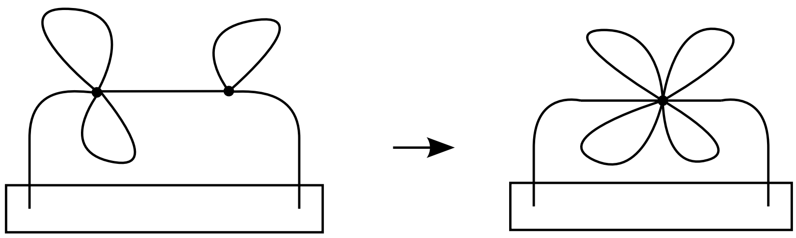


Figure 2.42: The left side is a $B_{2,1}$-rooted graph. After the bouquet-making method, it changes to the B_4-rooted graph.

Lemma 2.6.8. *The production rule of applying the bouquet-making method on a $B_{2,1}$-rooted graph, with the $B_{2,1}$-root being not a cut-subgraph (Figure 2.42), is*

$$\begin{gathered} 96d_i(G_0) + 192d_{i+1}(G_0) + 336s_{i+1}(G_0) + 96s_{i+2}(G_0) \longrightarrow \\ 16128d_i(G_1) + 129024d_{i+1}(G_1) + 72576d_{i+2}(G_1) + 32256s_{i+1}(G_1) + 112896s_{i+2}(G_1) \end{gathered}.$$

Theorem 2.6.9. *Let G_0 be a $B_{2,1}$-rooted graph, where the $B_{2,1}$-root is not a cut-subgraph of the G_0. Then the partial genus distribution of the graph by applying the bouquet making method is*

$$\begin{cases} d_i(G_1) &= 168d_i(G_0) &+ 1344d_{i-1}(G_0) &+ 756d_{i-2}(G_0) \\ s_i(G_1) &= 96s_i(G_0) &+ 1176s_{i-1}(G_0) & \end{cases}.$$

2.7 Face-expansion

The minimum genus embedding of interconnected network graphs are discussed here, including cube-connected cycles (CCC) and star-connected cycles (SCC). The topological operation designed here is face-expansion, which is an inverse operation to face contraction.

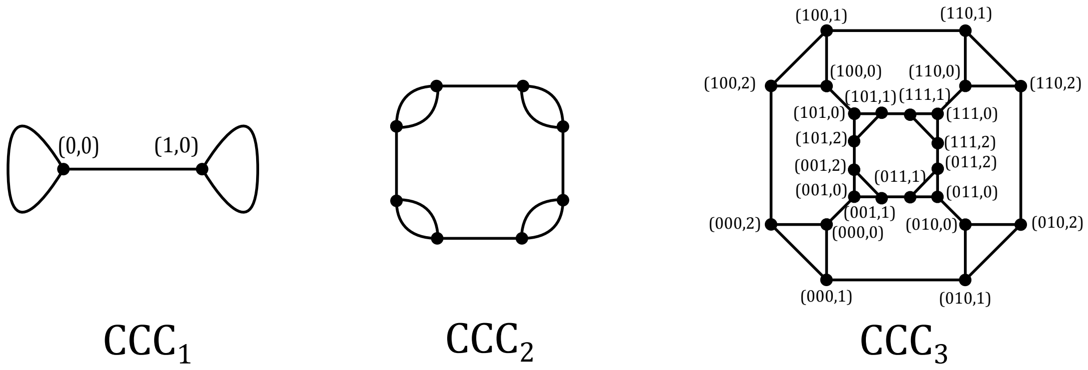


Figure 2.43: Cube-connected cycles CCC_1, CCC_2 and CCC_3 with their vertices labeled by pairs of numbers.

Replacing each vertex in a hypercube graph Q_n by a cycle of n vertices of valency 3 as shown in Figure 2.43, we can get a ***cube-connected cycles*** graph CCC_n. It is defined on a set of $n \cdot 2^n$ vertices, indexed by pairs of numbers (x, y), where x labels from 0 to $2^n - 1$ in binary form and $y \in Z_n$, such that each vertex (x, y) is connected to three vertices: $(x, (y+1)\ mod\ n)$, $(x, (y-1)\ mod\ n)$ and $(x \oplus 2^y, y)$. Here $\oplus$ denotes the bitwise exclusive or operation on binary numbers.

From the definition, CCC_n is a $3-regular$ graph with $n \cdot 2^n$ vertices and $3n \cdot 2^{n-1}$ edges.

A ***star-connected cycles*** graph SCC_n ($n \geq 4$) is obtained by substituting each

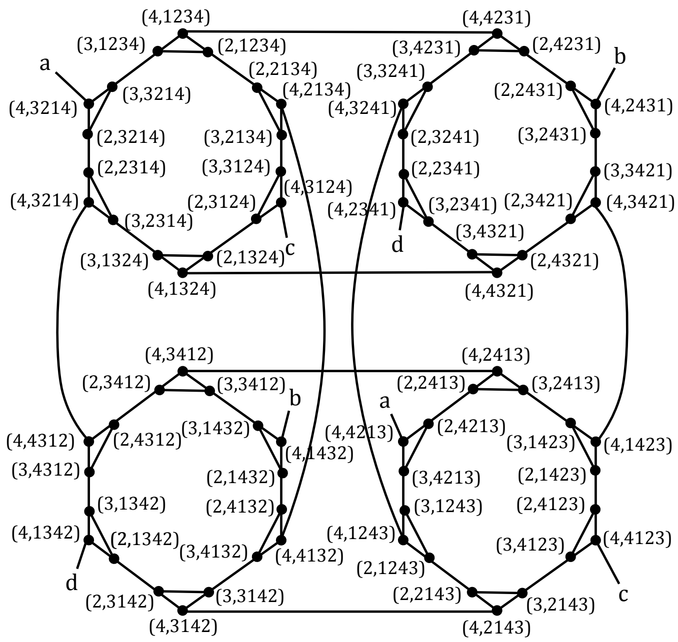


Figure 2.44: A graph of star-connected cycles SCC_4 obtained by replacing each vertex of S_4 with a 3 cycle.

vertex of an star graph S_n with a cycle of $n-1$ vertices of valency 3 [3, 25]. Vertices in SCC_n are denoted by (i, π), where $2 \leq i \leq n$ and π is a permutation of n symbols. An edge exists between two vertices (i, π) and (i', π') if and only if either $\pi = \pi'$ and $min(|i-i'|, n-1-|i-i'|) = 1$, or $i = i'$ and π differs from π' only in the first and the i-th number, such that $\pi(1) = \pi'(i)$ and $\pi(i) = \pi'(1)$. SCC_n is an 3-regular graph with $(n-1) \cdot n!$ vertices and $\frac{3(n-1) \cdot n!}{2}$ edges. An SCC graph is defined upon a star graph in similar way that a CCC graph is defined upon a hypercube. The definition of SCC follows [2]. Figure 2.44 shows SCC_4.

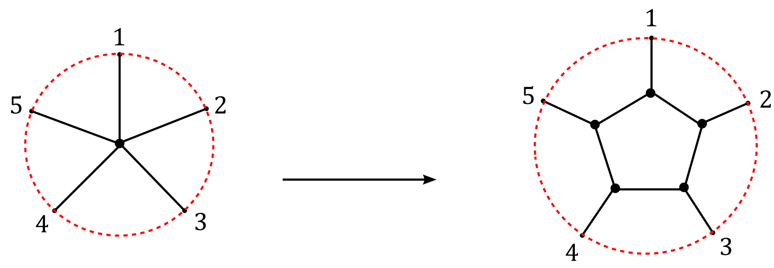


Figure 2.45: Face-expansion on a vertex of valency 5 and result a cycle of 5 vertices of valency 3.

2.7.1 Face-expansion operation

Face-expansion (Figure 2.45) is a topological operation which substitutes a rotation of a vertex v of valency n on an embedding of a graph with a face of length m and $m \leq n$, such that the n edges adjacent to v are connected to the m new vertices in an order which is not against the rotation on v before the operation.

Depending on how the adjacent edges of a vertex are grouped, different kinds of face-expansions can be defined as long as the rotation of the vertex is retained on the cycle.

Theorem 2.7.1. *When applying face-expansion on any rotation of a vertex with valency n of a graph embedding, such that the resulting cycle C_n contains n vertices with valency 3, the resulting embedding's genus does not change.*

Proof. The substitution on a vertex of valency n with a C_n add one face to the embedding. This operation can preserve the embedding's $2-cell$ property (any region is an open disc) by adding one region. So the genus of the resulting embedding will not change by Euler's formula. □

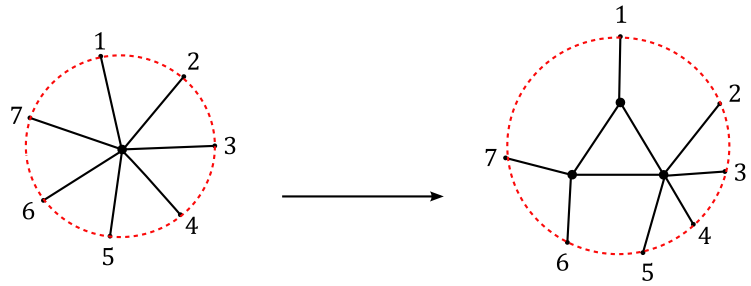


Figure 2.46: Face-expansion on an vertex of valency 7 resulting in a cycle of 3 vertices with valencies 3, 4 and 6.

Theorem 2.7.2. *When applying face-expansion on any rotation of a vertex with valency n of a graph embedding, such that the resulting cycle C_m $(m \leq n)$ contains m vertices with different valencies, the resulting embedding's genus does not change. (Figure 2.46)*

Proof. It is straightforward by using Euler's formula. □

The inverse of face-expansion is face contraction. The results below are straightforward.

Corollary 2.7.3. *When applying face contraction on a face of n vertices with valency* 3 *of a graph embedding, such that the resulting vertex is valency n, the resulting embedding's genus does not change.*

Corollary 2.7.4. *When applying face contraction on a face of m vertices with the sum of valencies equal to $n + 2m$ of a graph embedding, such that the resulting vertex is valency n, the resulting embedding's genus does not change.*

Applying face-expansion on the minimum genus embedding of cube-connected cycles and the minimum genus embedding of star-connected cycles, we have the following results.

Theorem 2.7.5. *The minimum genus of CCC_n is $(n-4) \cdot 2^{n-3} + 1$.*

Proof. The minimum genus of Q_n is $(n-4) \cdot 2^{n-3} + 1$ from [4]. It corresponds to an embedding which has the maximum number of faces (all size four). Applying face-expansion on a minimum embedding, the resulting embedding has the maximum number of faces regarding to the resulting graph CCC_n. □

Theorem 2.7.6. *The minimum genus of SCC_n is $n! \cdot \frac{(n-4)}{6} + 1$.*

Proof. The minimum genus of S_n is $n! \cdot \frac{(n-4)}{6} + 1$ from [1]. It corresponds to an embedding which has the maximum number of faces (all size six). Applying face-expansion on a minimum embedding, the resulting embedding has the maximum number of faces regarding to the resulting graph SCC_n. □

Conjecture 2.7.7. *When applying face-expansion on an minimum embedding of a graph, the resulting embedding is a minimum embedding of the resulting graph.*

The maximum genus of the resulting graph by applying face-expansion on an embedding of a graph is not straightforward at all.

We apply face-expansion on different embeddings of a graph, resulting different graphs, which is shown in Figure 2.47. When applying face-expansion on other embeddings of Q_n (except the minimum embeddings), you will not get graph CCC_n. If applying face-expansion on other embeddings of S_n (except the minimum embeddings), you will not get graph SCC_n.

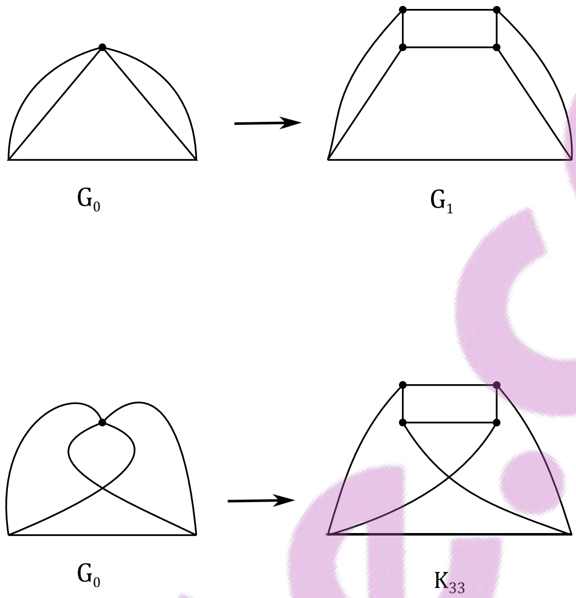


Figure 2.47: Apply face-expansion on different embeddings of G_0 resulting different graphs G_1 which is planar and $K_{3,3}$ with minimum genus 1.

2.8 Conclusion

Partial genus distribution and face tracing methods are extensively used in this chapter to form topological operations. Face-contraction is a one-off operation designed to act on a single graph. Vertex-splitting can be applied on any vertex of any graph as long as the valency of the vertex is bigger than 3. We can, alternatively, calculate the genus distribution of some 3-regular graphs instead of calculating the genus distribution of one graph with high valency vertices. The 3-regular graphs can be chosen flexibly by vertex-splitting and vertex-augment.

Our pearl-making method can be easily used as many times as needed. However, bouquet-making methods are discussed as an one-off operation. Nevertheless, they have the potential to be used recursively along a cut-subgraph or a cycle of a graph.

Face-expansion is the inverse operation of face-contraction. We put face-expansion at the end of Chapter 2, because the network graphs constructed with it will also discussed in Chapter 3.

Chapter 3

Minimum Genus of Cartesian Product

3.1 Introduction

In this chapter, we discuss the minimum genus of the Cartesian product of a star graph S_n with a series of other graphs: a path P_m; a cycle C_m; itself S_n and a different star graph S_m $(m \neq n)$. The results of this chapter will add motivation for studying star graphs.

A classic example of the Cartesian product of graphs is the hypercube Q_n. It can be defined as a repeated Cartesian product: let $Q_1 = K_2$, the complete graph on two vertices, and recursively define $Q_n = Q_{n-1} \times K_2$ for $n \geq 2$. [42].

A ***quadrilateral embedding*** of a graph G is an embedding such that every face is a C_4. Arthur T. White gave a very useful result below.

Theorem 3.1.1. *[42] If a bipartite graph G with v vertices and e edges has a quadrilateral embedding, then that embedding is minimal, and $g(G) = 1 + e/4 - v/2$.*

The construction employed by White [42] to produce quadrilateral embeddings of $G_1 \times G_2$ begins with $|V_2|$ copies of minimum embeddings of G_1 (where G_2 has $|V_2|$ vertices). Some tubes are added between the surfaces, with all other edges (which connect corresponding vertices between the embeddings of G_1,) being arranged on the tubes in a specific way.

The ***addition of a tube*** between two surfaces M_1 and M_2 is performed as follows. Let C_1 be a simple closed curve on M_1, and C_2 be a simple closed curve on M_2, such that the interiors of C_1 and C_2 are homeomorphic to open discs. Remove the two open discs. Adding a tube between M_1, and M_2 is to adjoin a cylinder K with bases C_1 and C_2, such that $K \cap M_1 = C_1$ and $K \cap M_2 = C_2$. Note that adding a tube, which is the same as a handle, creates another orientable surface. That means the genus of the surface grows by one.

The minimum genus of the star graph (refer to page 15 the its definition) is $n!(n-4)/6+1$ [1]. Abbasi constructed the minimum embedding of the star graph [1] by adding tubes between surfaces.

Lemma 3.1.2. *[1] Let $M_1, \ldots, M_n$ be orientable surfaces, all of genus g. If we add $k \geq n-1$ tubes between them to make a connected orientable surface M, then the genus of M is $ng+k-n+1$.*

In the minimum embedding of the star graph, all faces are 6-cycles. The minimum genus corresponds to the maximum number of faces of the embedding according to Euler's formula. We use the same method to construct the minimum embedding of the Cartesian product of some graphs. The technique for adding tubes between surfaces is the key part is this chapter. All of the minimum embeddings in this chapter contain only faces of C_4 and C_6.

3.2 Construction of a minimum embedding of $S_n \times P_2$

First, we discuss the easiest case, $S_4 \times P_2$.

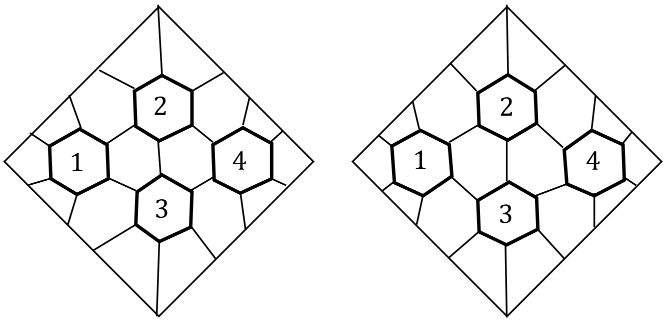


Figure 3.1: The construction of minimum embedding of $S_4 \times P_2$ (adapted from [1]).

Theorem 3.2.1. *The minimum genus of $S_4 \times P_2$ (Figure 3.1) is* 5.

Proof. The proof is also the construction of the minimum embedding of $S_4 \times P_2$.

Let us start with two copies of minimum embeddings of S_4 on two tori. Each of them has only faces of C_6. Four faces labelled $1, 2, 3, 4$ are removed from each torus.[1] Four tubes are added between the cycles sharing the same number. The 24 edges connecting the 24 pairs of vertices from the two embeddings are placed on the four tubes. Faces of size 4 can exist only when two of the four edges are new edges, which connect a pair of vertices from two S_4's, and the other two are old edges, which belong to the two S_4's. So the maximum number of C_4 faces can be counted. There are four tubes added. Each tube has six C_4 faces embedded on it. So there are totally 24 C_4 faces. This embedding construction has the maximum number of C_4 faces, so it has the maximum number of

[1]Later we will refer to these faces as a *team*.

total faces.

This embedding of $S_4 \times P_2$ has 48 vertices, 96 edges, 16 faces of C_6 and 24 faces of C_4. The total number of faces is $F = 16 + 24 = 40$. So the genus of $S_4 \times P_2$ is 5 by using Euler's formula.

$\square$

Similarly, for a minimum embedding of $S_5 \times P_2$, there are $4 \times 5 = 20$ tubes added between two copies of minimum embeddings of S_5, 120 faces of C_4, 120 faces of C_6.

For a minimum embedding of $S_6 \times P_2$, there will be 120 tubes added between 2 minimum embeddings of S_6, 720 faces of C_4, 320 faces of C_6.

Theorem 3.2.2. *The minimum genus of $S_n \times P_2$ is*

$$g(S_n \times P_2) = \frac{n!(2n-7)}{6} + 1.$$

Proof. S_n has $n!$ vertices, $n!(n-1)/2$ edges. $S_n \times P_2$ has $2n!$ vertices, $n \cdot n!$ edges.

For minimum embedding of $S_n \times P_2$, there will be $n!/6$ tubes added between two copies of minimum embeddings of S_n, $n!$ faces of C_4, $\frac{\frac{n!(n-1)}{2}\times 2\times 2+n!\times 2-n!\times 4}{6} = \frac{n!(n-2)}{3}$ faces of C_6, so it has the maximum number of total faces. $\square$

The two vertices of P_2 are both valency one, which make the tube adding an easy operation. When the object of the Cartesian product contains any vertices of valency higher than one, we need to design 'teams' to make the operation applicable, which will be discussed in the following section.

3.3 The selection of teams

For a minimum embedding of a star graph S_n, all the faces are C_6. There are $n!/6$ faces of C_6 which cover all vertices of S_n exactly once and no edges in common. We call the $n!/6$ faces a ***team***, which can be used to construct minimum embeddings of Cartesian product between S_n and another graph. A ***group of teams*** of a star graph S_n contains $n-1$ teams of face C_6, such that each team covers all vertices of S_n, and each edge appears exactly twice in the $n-1$ teams.

The six vertices of a face C_6 in a minimum embedding of a star graph S_n are demonstrated below in permutation forms.

vertex 1	a	$\cdots$	b	$\cdots$	c	$\cdots$
vertex 2	c	$\cdots$	b	$\cdots$	a	$\cdots$
vertex 3	b	$\cdots$	c	$\cdots$	a	$\cdots$
vertex 4	a	$\cdots$	c	$\cdots$	b	$\cdots$
vertex 5	c	$\cdots$	a	$\cdots$	b	$\cdots$
vertex 6	b	$\cdots$	a	$\cdots$	c	$\cdots$

In every face of C_6, the three numbers a, b and c permute in three fixed positions. All the other $n-3$ digit numbers permute in the other $n-3$ positions and make all the faces in a team. No two faces in a team have a common vertex.

For S_n, its vertices are all permutations of digits of length n. There are potentially $\begin{pmatrix} n-1 \\ 2 \end{pmatrix} = (n-1)(n-2)/2$ possible teams. With a properly selected group of $n-1$ teams, a corresponding minimum embedding of S_n can be constructed.

Let us have a look at an example of S_6. Below is a list of ten possible teams. Each team has its first position and other two positions fixed. Each position except the first one appears exactly four times in the ten teams.

team	All possible teams of S_6					
team 1	$\times$	$\times$	$\times$			
team 2	$\times$	$\times$		$\times$		
team 3	$\times$	$\times$			$\times$	
team 4	$\times$	$\times$				$\times$
team 5	$\times$		$\times$	$\times$		
team 6	$\times$		$\times$		$\times$	
team 7	$\times$		$\times$			$\times$
team 8	$\times$			$\times$	$\times$	
team 9	$\times$			$\times$		$\times$
team 10	$\times$				$\times$	$\times$

Each group contains a properly selected five teams from the table above. All groups of properly selected teams are listed in the table below.

group	*teams*
group 1	1, 2, 6, 9, 10
group 2	1, 2, 7, 8, 10
group 3	1, 3, 5, 9, 10
group 4	1, 3, 7, 8, 9
group 5	1, 4, 5, 8, 10
group 6	1, 4, 6, 8, 9
group 7	2, 3, 5, 9, 10
group 8	2, 3, 9, 6, 7
group 9	2, 4, 5, 6, 10
group 10	2, 4, 8, 6, 7
group 11	3, 4, 6, 5, 9
group 12	3, 4, 8, 5, 7

The method to select a group of $n-1$ teams follows.

For S_n, draw an $\begin{pmatrix} n-1 \\ 2 \end{pmatrix} \times n$ table, where each row represents a possible team. In each row, the first position and another two positions are fixed. All the faces of a team are permutations of three digits in the three positions, but no two teams have the same fixed positions. In a properly selected $n-1$ teams, each of the last $n-1$ positions is fixed twice. It is straightforward to see that S_6 has 12 different minimum embeddings, each corresponding to a group of properly selected $n-1$ teams.

Theorem 3.3.1. *The number of minimum embeddings of S_n is equal to the number of groups of properly selected $n-1$ teams of face C_6.*

We begin with a detailed investigation on a properly selected group of $n-1$ teams.

All edges in S_n are included in a group of properly selected $n-1$ teams. Every edge is a transposition of the first digit with another digit of a fixed position. It will appear in any properly selected group of $n-1$ teams.

Each edge of S_n appears exactly twice in a properly selected group of $n-1$ teams.

For each edge, its first appearance has the form:

$$\begin{array}{|c|c|c|c|c|c|}\hline a & \cdots & b & \cdots & c & \cdots d \cdots \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|c|c|c|c|}\hline b & \cdots & a & \cdots & c & \cdots d \cdots \\ \hline\end{array}$$

Its second appearance has the form:

$$\begin{array}{|c|c|c|c|c|c|}\hline a & \cdots & b & \cdots c \cdots & d & \cdots \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|c|c|c|c|}\hline b & \cdots & a & \cdots c \cdots & d & \cdots \\ \hline\end{array}$$

The four positions with digits a, b, c, d are the same. For the first appearance of the edge, the three fixed positions are occupied by a, b, c. For the second appearance of the edge, the three fixed positions are occupied by a, b, d. The edge will not appear in other faces. Because in each properly selected $n-1$ teams, each position can only be fixed twice.

An algorithm is needed to select the groups from the list of possible teams. It is hard to find a formula to calculate the number of groups for S_n.

3.4 Construction of a minimum embedding of $S_n \times P_m$

For each minimum embedding of S_4, there are twelves faces of C_6, which can be divided into 3 teams.

$$\left\{ \begin{array}{lllll} \text{team A:} & A_1 & A_2 & A_3 & A_4 \\ \text{team B:} & B_1 & B_2 & B_3 & B_4 \\ \text{team C:} & C_1 & C_2 & C_3 & C_4 \end{array} \right.$$

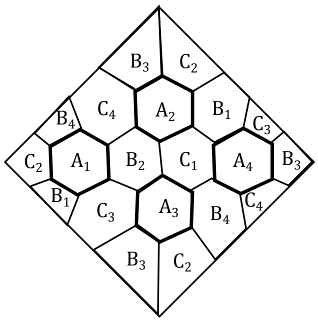


Figure 3.2: A group of teams for the construction of minimum embedding of $S_4 \times P_3$

For each team, the 4 faces of C_6 will cover all the vertices of S_4 and will not have a common vertex. Each edge appears twice in two different teams; the teams are shown in Figure 3.2.

We only need two teams (team A and team B) to construct a minimum embedding of $S_4 \times P_3$.

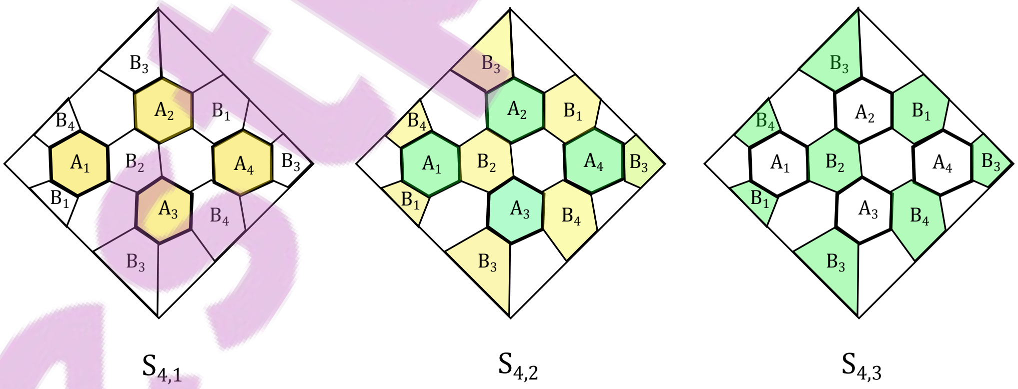


Figure 3.3: The construction of a minimum embedding of $S_4 \times P_3$.

Theorem 3.4.1. *The minimum genus of $S_4 \times P_3$ (Figure 3.3) is* 9.

Proof. We begin with three copies of minimum embeddings of S_4, denoted by $S_{4,1}$, $S_{4,2}$ and $S_{4,3}$. Note that the embedding construction in Theorem 3.2.1. does not work here, because after removing each face from a minimum embedding, we can only add one tube on the cycle. We need to design a way to arrange the location of the new tubes on $S_{4,2}$. This is where we use a group of teams. We add a tube between A_i in $S_{4,1}$ and A_i in $S_{4,2}$, B_i in $S_{4,2}$ and B_i in $S_{4,3}$ for $i = 1, 2, 3, 4$, which are illustrated in the different colors (yellow and green) as shown in Figure 3.3. The minimum genus of $S_4 \times P_3$ is 9 according to Lemma 3.1.2, because it has the maximum number of total faces. □

Based on the definition of team and group of teams at page 92, we let $T(S_n)$ denote the number of groups of teams of size-6 faces in a minimum embedding of S_n, such that each team can cover all the vertices of S_n and do not have a common vertex, and each edge will appear twice in two different teams.

Lemma 3.4.2. *For a star graph S_n, we have $T(S_n) = n - 1$.*

Proof. We know that S_n has $n!$ vertices, $n!(n-1)/2$ edges. A minimum embedding of S_n has $n!(n-1)/6$ faces of C_6 and no faces of other sizes. A team needs $n!/6$ faces to cover all vertices. There are $n-1$ different teams in the embedding. □

Theorem 3.4.3. *The minimum genus of $S_n \times P_m$ is $n!(mn - 3m - 1)/6 + 1$.*

Proof. There will be $(m-1) \times n!/6$ tubes between the m copies of a minimum embedding of S_n. By using the same kind of construction as $S_4 \times P_3$, the genus of $S_n \times P_m$ is

$$g(S_n \times P_m) = m \times (n!(n-4)/6 + 1) + (m-1) \times n!/6 - m + 1 = \frac{n!}{6}(mn - 3m - 1) + 1.$$

□

3.5 Construction of a minimum embedding of $S_n \times T_m$

We can expand the results of $S_n \times P_m$ and construct a minimum embedding of $S_n \times T_m$, where T_m is any tree with m vertices, $m-1$ edges, and maximum valency $\Delta(T_m) \leq (n-1)$. Note that the minimum embedding construction works here because the tree has no cycles.

Theorem 3.5.1. *For a tree T_m, if $\Delta(T_m) \leq n-1$, then minimum genus of $S_n \times P_m$ is $n!(mn+3m-1)/6 - m + 1$.*

Proof. For a minimum embedding of S_n, it has $n!$ vertices, so there are $\frac{n!}{6}$ faces of C_6 will be used to adding tubes. For each edge of T_m, there will be $\frac{n!}{6}$ tubes added. Since T_m has $m-1$ edges, so there are totally $\frac{n!}{6}(m-1)$ tubes added. By Lemma 3.1.2, we have

$$g(S_n \times T_m) = m(n! \times (n-4)/6+1) + n!/6 \times (m-1) - m + 1 = n!(mn+3m-1)/6 - m + 1.$$

□

3.6 Construction of a minimum embedding of $S_n \times C_m$

Based on the embedding construction of $S_4 \times P_3$ (Figure 3.3), we add a tube between the corresponding white-colored faces of in $S_{4,3}$ and $S_{4,1}$. We then place the additional 24 edges on the four tubes to connect the 24 pairs of vertices from $S_{4,3}$ and $S_{4,1}$. There are 12 tubes all together. The genus of the constructed embedding of $S_4 \times C_3$ is 13 by Lemma 3.1.2. The embedding construction above includes 72 faces of size 4 and 12 faces of size 6.

However, we can not say this is a minimum embedding, yet, because a face of size 3 could exist in some embeddings of $S_4 \times C_3$.

More calculations needed to prove it is the minimum embedding.

Lemma 3.6.1. *The minimum genus of* $S_4 \times C_3$ *is* 13.

Proof. $S_4 \times C_3$ has 72 vertices, 180 edges. According to Euler's formula, we have $g = 55 - F/2$. Let F denote the number of faces.

The tube adding construction above is an embedding of $S_4 \times C_3$, but it may not be the minimum genus. If the minimum genus is 13, then $F = 84$. For any embedding, the following three statements are true:

1. Faces of size 3 can exist only when the three edges are all ***new edges***, which connect a pair of vertices from two S_4's.
2. Every face of size 4 contains two new edges and two ***old edges***, which belong to two of the S_4's.
3. Every faces of size 5 contains three new edges and two old edges.

Figure 3.3 is helpful to understand the three statements. The girth of S_4 is 6. We cannot find faces of size 3, 4 or 5 in any embedding of S_4. Faces of size 3, 4 and 5 in any embedding of $S_4 \times C_3$ must contain some new edges, which connect vertices between the three copies of S_4. Through Figure 3.3, it is easy to understand that faces of size 3 can exist only when all the three edges are new edges, that faces of size 4 contains two new edges and two old edges, that faces of size 5 contains three new edges and two old edges.

The Cartesian product $S_4 \times C_3$ has 72 new edges, so it follows that an embedding of $S_4 \times C_3$ has at most 24 faces of size 3, has at most 72 faces of size 4, has at most 24 faces of size 5. All other kind of faces are size 6 or bigger ones. Let F_i represents the number of faces of size i for $i \geq 3$.

$$F = \sum_{i=3}^{\infty} F_i = F_3 + F_4 + F_5 + F_6 + \cdots$$

$$\sum_{i=3}^{\infty} iF_i = 3 \times F_3 + 4 \times F_4 + 5 \times F_5 + 6 \times F_6 + \cdots = 180 \times 2$$

Let $max(F) = F_3 + F_4 + F_5 + F_6 + F_R$. We use F_R to represent the number of faces of size 7 or bigger. To make calculation simple, we treat R as average face size bigger than 6. We assume that each of the faces of size R has on average a new edges and $R - a$ old edges.

We have 72 new edges. So

$$3 \times F_3 + 2 \times F_4 + 3 \times F_5 + a \times F_R \leq 72 \times 2 \Rightarrow F_3 + F_4 + F_5 \leq \frac{72 \times 2 + F_4 - a \times F_R}{3}. \textit{ EQ1.}$$

We have 108 old edges. So

$$2 \times F_4 + 2 \times F_5 + 6 \times F_6 + (R - a) \times F_R \leq 108 \times 2$$

$$\Rightarrow F_4 + 3 \times F_6 \leq 108 - F_5 - \frac{R - a}{2} \times F_R. \textit{ EQ2.}$$

By EQ1, we have

$$3 \times F \leq 72 \times 2 + F_4 + 3 \times F_6 - a \times F_R + 3 \times F_R$$

By EQ2, we have the above value

$$\leq 72 \times 2 + (3 - a)F_R + 108 - F_5 - \frac{R - a}{2} \times F_R = 252 - F_5 - \frac{R + a - 6}{2} F_R$$

$$F \leq 84 - F_5 - \frac{R + a - 6}{2} F_R \leq 84$$

So the maximum value of F is 84, when F_5 and F_R both equal to zero. The constructed embedding above has 84 faces, which means it is a minimum embedding. $\square$

We now expand this result to $S_n \times C_3$.

Theorem 3.6.2. *The minimum genus of* $S_n \times C_3$ *is* $n!(n-3)/2+1$.

Proof. The procedure to construct a minimum embedding of $S_n \times C_3$ is similar to $S_4 \times C_3$ in Lemma 3.5.1.

The star graph S_n has $n!$ vertices, $n!(n-1)/2$ edges and $n!(n-1)/6$ faces of size 6 in its minimum embedding on a surface of genus $n!(n-4)/6+1$ [1].

The Cartesian product $S_n \times C_3$ has $3n!$ vertices, $3(n+1)!/2$ edges. By using an embedding construction as the one in Lemma 3.5.1, there will be $n!/2$ new tubes added between 3 copies of minimum embeddings of S_n. The genus of the resulting embedding is $n!(n-3)/2+1$. The corresponding number of faces is $n!(n+3)/2$ by Euler's formula, including $3n!$ faces of size 4 and $n!(n-3)/2$ faces of size 6.

In any embedding of $S_n \times C_3$, we can have: $F_3 \leq n!$, $F_4 \leq 3n!$ and $F_5 \leq n!$.

There are $6n!$ new edges. So

$$3 \times F_3 + 2 \times F_4 + 3 \times F_5 \leq 6n!$$

$$\Rightarrow F_3 + F_4 + F_5 \leq \frac{6n! + F_4}{3}.$$

There are $3n!(n-1)$ old edges. So

$$2 \times F_4 + 2 \times F_5 + 6 \times F_6 \leq 3n!(n-1)$$

$$\Rightarrow F_4 + 3F_6 \leq \frac{3}{2}n!(n-1) - F_5.$$

We can get the maximum value of F only when all edges appearances are used to construct faces of size 3, 4, 5 and 6 respectively. So

$$F \leq F_3 + F_4 + F_5 + F_6.$$

Then

$$3F \leq 6n! + F_4 + 3F_6 \leq 6n! + \frac{3}{2}n!(n-1) - F_5 \Rightarrow F \leq \frac{n!(n+3)}{2} - \frac{F_5}{3}.$$

Then because $0 \leq F_5 \leq n!$, we have $F \leq n!(n+3)/2$.

The constructed embedding above has $F = n!(n+3)/2$. So it is a minimum embedding construction. $F_3 = 0$, $F_4 = 3n!$, $F_5 = 0$, $F_6 = n!(n-3)/2$.

□

We use C_m to represent a face of size m.

Theorem 3.6.3. *The minimum genus of $S_n \times C_m$ is*

$$n! \cdot m \cdot \left(\frac{n}{6} - \frac{1}{2}\right) + 1.$$

Proof. The proof procedure is similar as $S_n \times C_3$ by using the same kind of embedding construction. Since none of the faces can be size 3, the embedding constructed is minimal.

□

In fact, when calculating the minimum genus of $S_n \times G$, we can use the same kind of embedding construction designed in $S_4 \times P_3$, as long as G does not have subgraph C_3 and satisfies the following theorem.

Theorem 3.6.4. *The tube addition technique can be used to construct embeddings of $S_n \times G$, as long as G does not contain any subgraph of C_3 and $\Delta(G) \leq (n-1)$.*

This theorem can check whether there is enough space to arrange the tubes between the minimum embeddings of S_n. The three teams A, B and C in the example of $S_4 \times C_3$ allows the tube gluing successful. For graphs which contain subgraph C_3, the tube addition technique can be used to construct embeddings of $S_n \times G$, but its minimum needs to be considered separately.

We now discuss more complexed Cartesian products.

3.7 Construction of a minimum embedding of $S_n \times S_n$

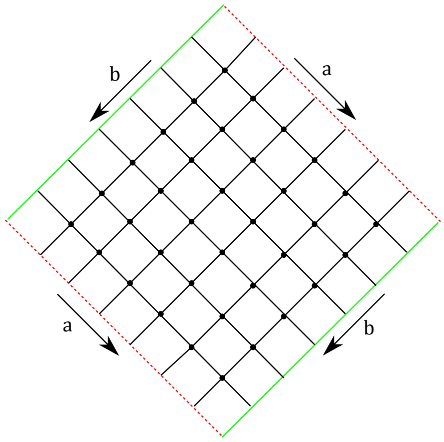


Figure 3.4: A minimum embedding of $S_3 \times S_3$ on a torus: gluing parallel edges

Let us start with the simple case $S_3 \times S_3$ (Figure 3.4). The Cartesian product $S_3 \times S_3$ has $(3!)^2 = 36$ vertices and embeds on a torus.

A minimum embedding of Cartesian product $S_4 \times S_4$ has $(4!)^2 = 576$ vertices and is embedded on a surface of genus 145. It is not practical to draw the figure as it contains 24 copies of Figure 3.2.

Theorem 3.7.1. *The minimum genus of $S_n \times S_n$ is*

$$(n!)^2 \cdot \frac{n-3}{4} + 1.$$

Proof. According to Theorem 3.6.4., we can construct a minimum embedding of $S_n \times S_n$. Because $T(S_n) = n - 1 = \Delta(S_n)$, the genus of $S_n \times S_n$ is

$$n! \cdot (n! \cdot (\frac{n-4}{6}) + 1) + \frac{(n!)^2(n-1)}{12} - n! + 1 = (n!)^2 \cdot \frac{n-3}{4} + 1.$$

□

3.8 Construction of a minimum embedding of $S_n \times S_m$

Theorem 3.8.1. *The minimum genus of $S_n \times S_m$ ($n \geq m$) is*

$$m! \cdot n! \cdot \frac{2n+m-9}{12} + 1.$$

Proof. For a given minimum embedding of S_n, There are $n-1$ different teams. The valency of any vertex in S_m is $m-1$. We have $n-1 \geq m-1$. So the genus of $S_n \times S_m$ ($n \geq m$) is

$$m! \cdot (n! \cdot (\frac{n-4}{6}) + 1) + \frac{m! \cdot n! \cdot (m-1)}{12} - m! + 1 = m! \cdot n! \cdot \frac{2n+m-9}{12} + 1.$$

□

When we construct a minimum embedding of $S_n \times S_m$ when $n \leq m$, there are $n-1$ different teams in a minimum embedding of S_n, which is no more than the valency of S_m. We can not construct an embedding of $S_n \times S_m$ by connecting $m!$ minimum embeddings of S_n, but by the definition of the Cartesian product, $S_n \times S_m = S_m \times S_n$, so we connect $n!$ minimum embeddings of S_m.

Chapter 4

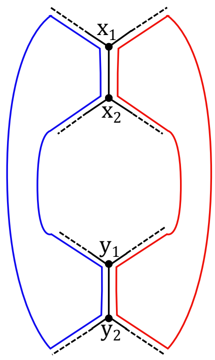
Dot Products of Graphs

4.1 Introduction

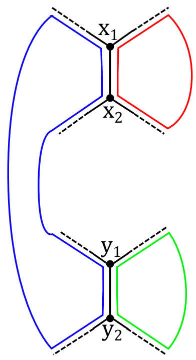
We extend a result of Bojan Mohar and Andrej Vodopivec [27]. In this paper, the authors construct a dot product of n copies of the Petersen graph whose minimum genus is precisely k, where $1 \leq k \leq n$. The author also showed that these are all possible values for the **minimum** genus of P^n.

In this chapter, we extend this result. For each k, $1 \leq k \leq 2n+1$, we construct a dot product of n copies of the Petersen graph with an orientable genus precisely k. The genus here means the genus of any embedding, not necessarily the minimum embedding. We prove the maximum genus of P^n is $2n+1$ and the minimum genus is 1.

Note that the definition of dot product is designed to apply on <u>an embedding</u> of G_1 and <u>an embedding</u> of G_2, and result in <u>an embedding</u> of G. Different embeddings of G_1 and G_2 result in different embeddings of different G; the resulting graph G is not unique. The construction of Petersen powers demonstrates this. As shown in Figure 1.18, P^2 represent 2 graphs. This is why we can not use Duke's result of Theorem 1.1.2, which is



(a) 2-face edges



(b) 3-face edges

Figure 4.1: Edges $e = x_1x_2$ and $f = y_1y_2$ in two different embeddings of a cubic graph G.

for a single graph.

4.2 The genus of Petersen powers

A graph is called a ***Petersen power*** P^n if it can be constructed as a repeated dot product of n copies of the Petersen graph recursively as $P^n = P^{n-1} \cdot P$. The graph $P \cdot (P \cdot P)$ is not a Petersen power according to the definition of dot product [27]. Dot product is a snark preserving operation. We will show that it also has good embedding properties in this chapter.

We call a pair of non-adjacent edges e and f **2*-face edges*** (Figure 4.1a) in an embedding of a cubic graph G, if there exist two distinct face walks, each of which contains exactly one appearance of edges e and f.

We call a pair of non-adjacent edges e and f **3*-face edges*** (Figure 4.1b) in an embedding of a cubic graph G, if there is exactly one face walk that contains both appearances of edges e and f. Three faces in the embedding cover the two appearances of edges e and f.

For a pair of non-adjacent edges of a cubic graph, we could define 1-face edges and 4-face edges as well, but these are not required in the following results.

We now define four dot product operations denoted by 'dot product i', where i is the **increase in genus** from G to $G \cdot P$.

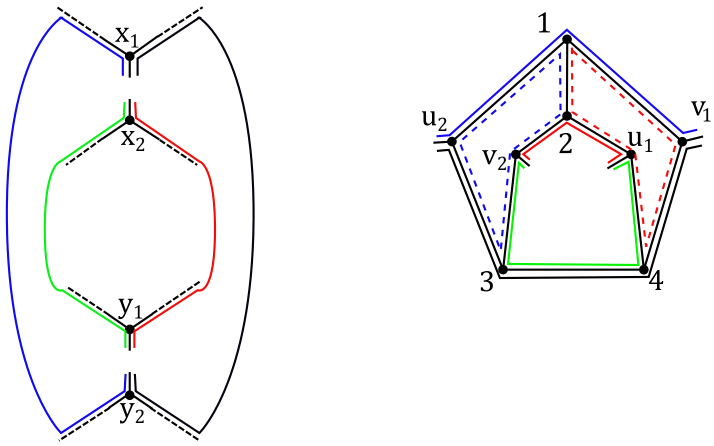


Figure 4.2: Dot product 0

Let G be a cubic graph, and e and f be a pair of 2-face edges in an embedding of G. Then a dot product between an embedding of G and a given embedding of $P - u - v$ (Figure 4.2) exists and we denote it by a ***dot product*** 0.

Lemma 4.2.1 (Lemma 2.3 in [27])**.** *For dot product* 0*, we have* $g(G \cdot P) = g(G)$*. Furthermore, the resulting embedding of graph* $G \cdot P$ *contains a pair of* 2*-face edges.*

Proof. By face-tracing in Figure 4.2 and using Euler's formula, we have $v(G \cdot P) = v(G) + 8$, $e(G \cdot P) = e(G) + 12$, $f(G \cdot P) = f(G) + 4$ and $g(G \cdot P) = g(G)$. Furthermore, we can find a pair of 2-face edges x_1v_1 and y_2u_2. $\square$

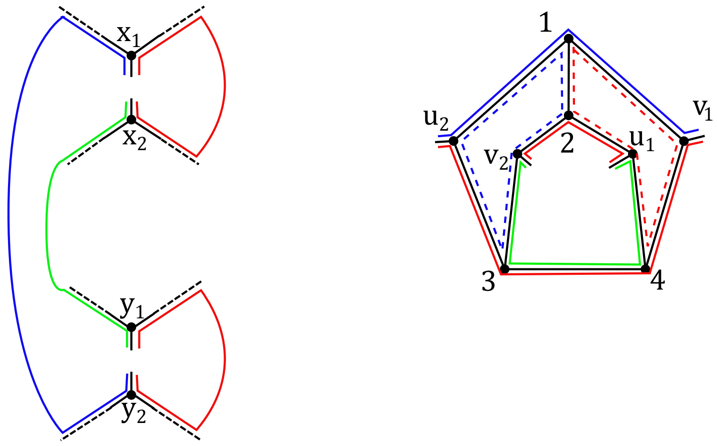


Figure 4.3: Dot product 1

Let G be a cubic graph, and e and f be a pair of 3-face edges in an embedding of G. Then a dot product between G and a given embedding of $P - u - v$ (Figure 4.3) exists and we denote it by a ***dot product*** 1.

Lemma 4.2.2 (Lemma 2.4 in [27])**.** *For dot product* 1*, we have* $g(G \cdot P) = g(G) + 1$*. Furthermore, the resulting embedding of graph* $G \cdot P$ *contains a pair of* 2*-face edges and a pair of* 3*-face edges.*

Proof. By face-tracing in Figure 4.3 and using Euler's formula, we have $v(G \cdot P) = v(G) + 8$, $e(G \cdot P) = e(G) + 12$, $f(G \cdot P) = f(G) + 2$ and $g(G \cdot P) = g(G) + 1$. Furthermore, we can find a pair of 2-face edges x_1v_1 and y_2u_2, and a pair of 3-face edges x_1v_1 and y_1u_1. $\square$

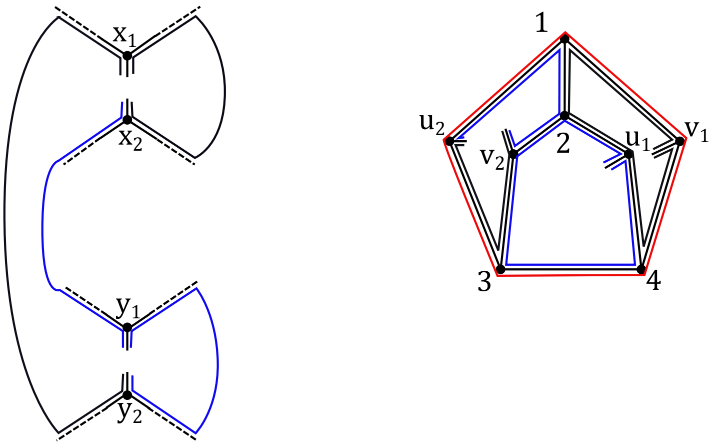


Figure 4.4: Dot product 2

Let G be a cubic graph, and e and f be a pair of 3-face edges in an embedding of G. Then a dot product between G and a given embedding of $P - u - v$ (Figure 4.4) exists and we denote it by a ***dot product*** 2.

Lemma 4.2.3. *For dot product* 2*, we have* $g(G \cdot P) = g(G) + 2$*. The resulting embedding of graph* $G \cdot P$ *contains a pair of* 2*-face edges and a pair of* 3*-face edges.*

Proof. By face-tracing in Figure 4.4 and using Euler's formula, we have $v(G \cdot P) = v(G) + 8$, $e(G \cdot P) = e(G) + 12$, $f(G \cdot P) = f(G)$ and $g(G \cdot P) = g(G) + 2$. Furthermore, we can find a pair of 2-face edges x_2v_2 and y_2u_2, and a pair of 3-face edges x_1v_1 and $1v_1$. □

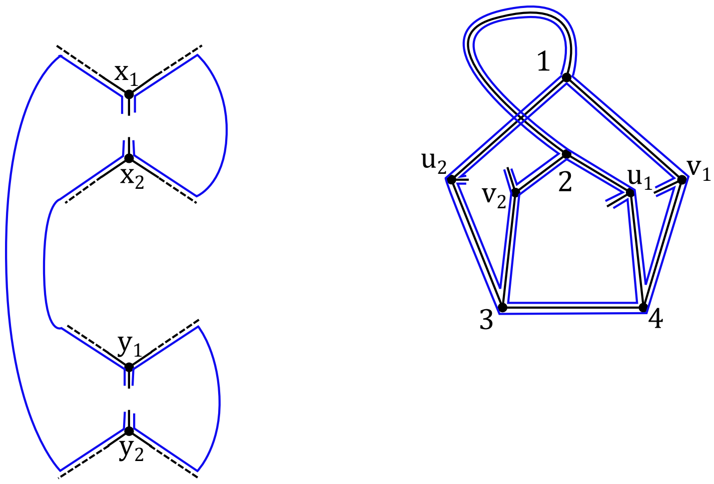


Figure 4.5: Dot product 3

Let G be a cubic graph, and e and f be a pair of 3-face edges in an embedding of G. Then a dot product between G and a given embedding of $P - u - v$ (Figure 4.5) exists and we denote it by a ***dot product*** 3.

Lemma 4.2.4. *For dot product* 3*, we have* $g(G \cdot P) = g(G) + 3$.

Proof. By face-tracing in Figure 4.5 and using Euler's formula, we have $v(G{\cdot}P) = v(G){+}8$, $e(G \cdot P) = e(G) + 12$, $f(G \cdot P) = f(G) - 2$ and $g(G \cdot P) = g(G) + 3$.

Note that dot product 3 is a one-off product. We can not guarantee that the resulting embedding of dot product 3 has any 2-face edges or 3-face edges. □

The results from Lemma 4.2.1 through to Lemma 4.2.4 are important for the main result. In particular, we need to keep track of whether further occurrences of 2-face edges and 3-face edges appear in the resulting product at each step so that the dot product can be applied repeatedly except the last step.

In [27], the results are based on a dot product between a minimum genus embedding of a cubic graph G and a given embedding of $P - u - v$. The results of that paper are based on a dot product between a genus embedding of a cubic graph G (which does not have to be a minimum embedding) with other embeddings of $P - u - v$. We get the full range of the genus of Petersen Powers.

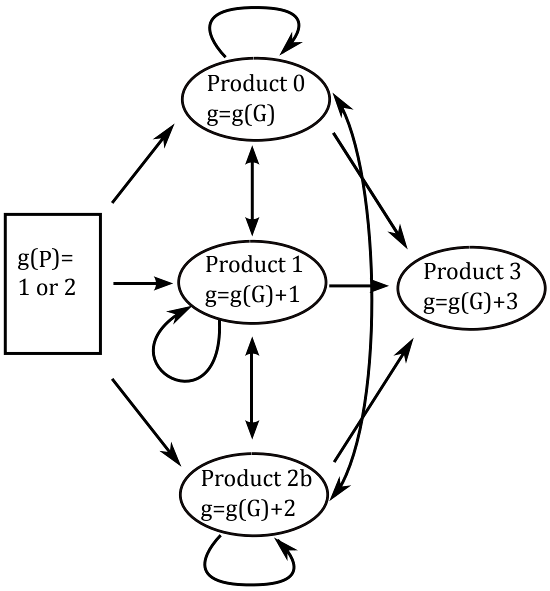


Figure 4.6: The flowchart of embedding construction of P^n with any genus between 1 and $2n + 1$.

Theorem 4.2.5. *For each k, $1 \leq k \leq 2n+1$, there exists a dot product of n copies of the Petersen graph with orientable genus precisely k. The maximum genus of P^n is $2n + 1$. The minimum genus of P^n is 1.*

Proof. A Petersen graph can be embedded on surfaces of genus 1 and genus 2. With the flowchart above, it is straightforward to construct an embedding of P^n of genus k, $1 \leq k \leq 2n + 1$. We can begin with a embedding of Petersen graph of genus 1 or 2. Then perform as many dot product 0, dot product 1 or dot product 2 as needed. The double-oriented arrows between dot product 0, dot product 1 and dot product 2 make the embedding construction very flexible.

Product 3 can be used to construct an embedding of genus $2n+1$: Start with $g(P) = 2$. Then perform dot product 2 by $n - 2$ times, and followed by a dot product 3. An embedding of P^n of genus $2n+1$ has exactly one face walking according to Euler's formula. That means this is an maximum genus embedding of P^n.

An embedding of P^n of genus 1 can be constructed by starting with $g(P) = 1$. Then perform $n - 1$ of dot product 0. $\square$

With different embeddings of $P - u - v$, there are many different possible dot product 0, dot product 1, dot product 2 and dot product 3, which can be constructed. We only need four of them to construct our flowchart and achieve our results.

A Petersen power P^n contains a number of different graphs depending on their construction. All graphs of P^n are cubic graphs, and have the same number of vertices, the same number of edges, the same number of total orientable embeddings. We do not know how their total orientable embeddings are distributed in the range of their genus. There are some graphs of P^n whose minimum genus is 1 and maximum genus is $2n + 1$. There are also some graphs of P^n which have different minimum and maximum genus.

For an embedding of Petersen graph with genus 2 (Figure 4.7), we can find a pair of 2-face edges e_0 and f_0, and a pair of 3-face edges e_0 and f_1. Applying dot product 2 and dot product 3, we will get embeddings of P^2 with genus 4 (Figure 4.8) and 5 (Figure 4.9).

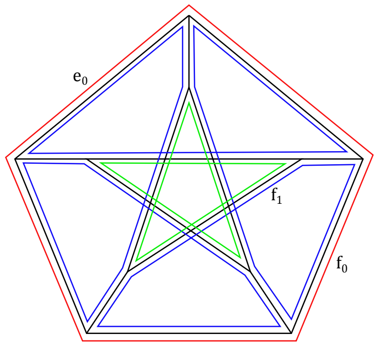


Figure 4.7: An embedding of Petersen graph with three faces and genus 2

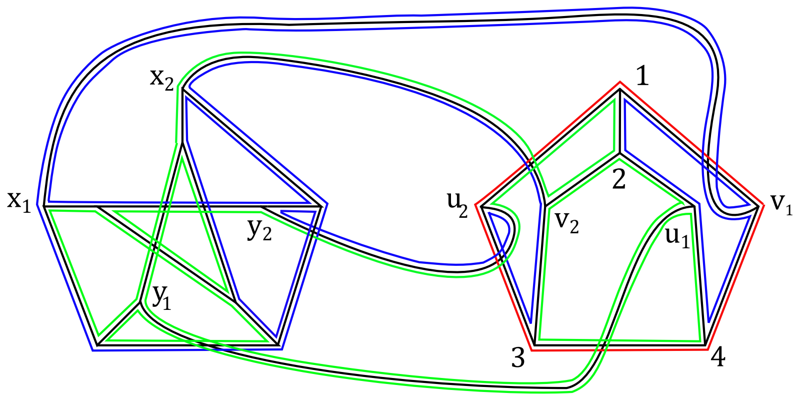


Figure 4.8: Apply dot product 2 and get an embedding of P^2 with three faces and genus 4.

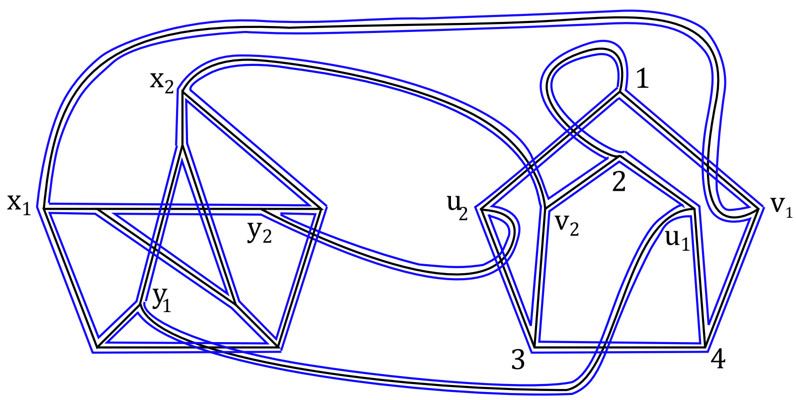


Figure 4.9: Apply dot product 3 and get an embedding of P^2 with one face and genus 5.

Chapter 5

Extended Dot Product of Graphs

This chapter introduces an extended dot product which can be applied on 4-regular graphs. A family of new graphs, $K_{4,4}$ powers, are designed in this chapter to demonstrate how it works. For each k, $n + 3 \leqslant k \leqslant 3n + 1$, we construct an embedding of extended dot product of n copies of $K_{4,4}$ whose orientable genus is precisely k.

Let G_1 be a 4-regular graph and $e = x_1x_2$, $f = y_1y_2$ and $g = z_1z_2$ be three non-adjacent edges in G_1. Let G_2 be a 4-regular graph with an edge uv. As shown in Figure 5.1, we denote the neighbours of u distinct from v by u_1, u_2 and u_3, and denote the neighbours of v distinct from u by v_1, v_2 and v_3. Remove the edges e, f and g from G_1 to get G_1'. Remove the vertices u and v from G_2 to get G_2'. Construct graph G by adding edges x_1v_1, x_2u_1, y_1v_2, y_2u_2, z_1v_3 and z_2u_3 between G_1' and G_2'. The added edges are called ***product edges*** and the graph G is called an ***extended dot product*** of graphs G_1 and G_2, denoted by $G = G_1 \odot G_2$.

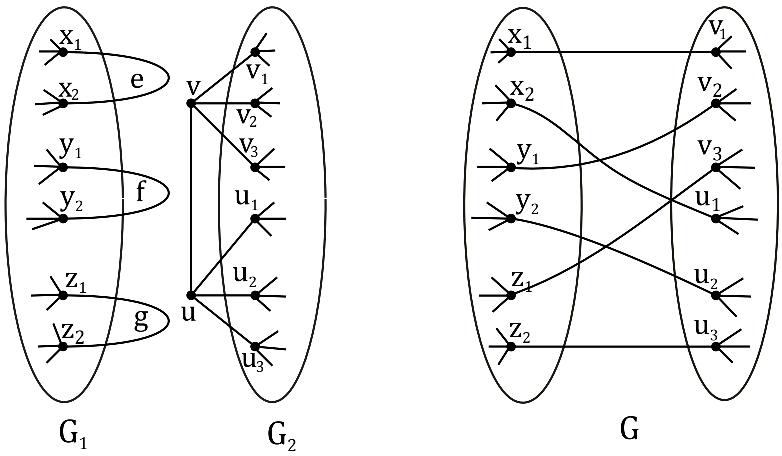


Figure 5.1: Extended dot product of two 4-regular graphs

5.1 Embedding construction of $K_{4,4}^n$

A graph is called a $K_{4,4}$ ***power*** if it can be constructed by the extended dot product of n copies of $K_{4,4}$ recursively as $K_{4,4}^n = K_{4,4}^{n-1} \odot K_{4,4}$. By this definition, $K_{4,4}^n$ is a 4-regular bipartite graph.

Note that $K_{4,4}^n$ is not uniquely defined. There is more than one graph for $K_{4,4}^n$.

Note that $K_{4,4}^3 = (K_{4,4} \odot K_{4,4}) \odot K_{4,4} \neq K_{4,4} \odot (K_{4,4} \odot K_{4,4})$ by definition.

For edges e, f, g in G_1, each has two appearances in an embedding of a 4-regular graph. There are many different types of embeddings of edges e, f, g. We will use five of them, which are shown below.

1. We call non-adjacent edges e, f and g ***type*** 1 ***edges*** (see Figure 5.2) in an embedding of a 4-regular graph, if both appearances of e and g, and one appearance of f belong to one face walk (red). The other appearance of f belongs to another face walk (black).

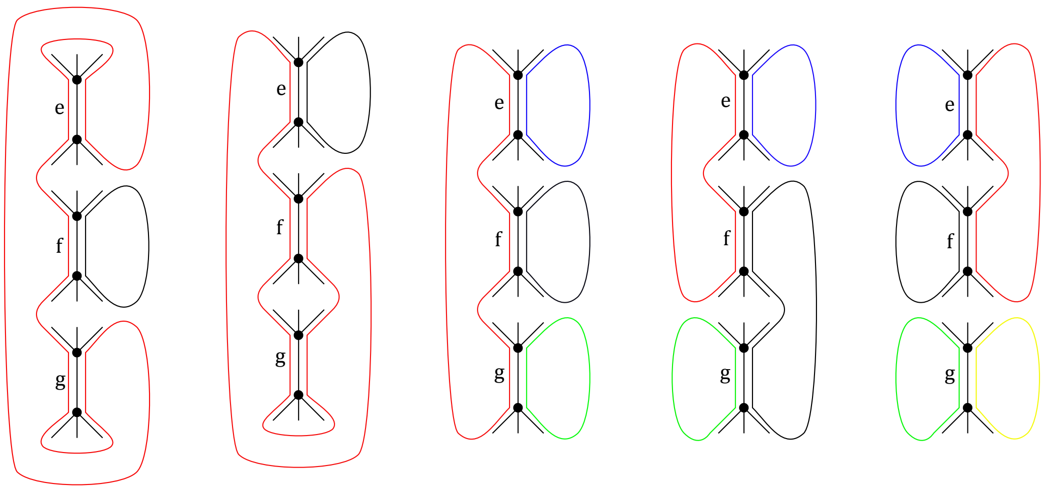


Figure 5.2: For a 4-regular graph G, non-adjacent edges e, f and g are denoted by type i edges ($i = 1, 2, 3, 4, 5$ from left to right), in different embeddings of G.

2. We call non-adjacent edges e, f and g ***type*** 2 ***edges*** (Figure 5.2) in an embedding of a 4-regular graph, if both appearances of f and g, and one appearance of e belong to one face walk (red). The other appearance of e belong to another face walk (black).

3. We call non-adjacent edges e, f and g ***type*** 3 ***edges*** (Figure 5.2) in an embedding of a 4-regular graph, if one side appearances of e, f and g belong to one face walk (red). The other appearances of e, f, g belong to three distinct face walks (blue, black and green).

4. We call non-adjacent edges e, f and g ***type*** 4 ***edges*** (Figure 5.2) in an embedding of a 4-regular graph, if one side appearances of e, f belong to one face walk (red). One side appearance of g and the other appearance of f belong to one face walk (black). The other appearances of e and g belong to two distinct face walks (blue and green).

5. We call non-adjacent edges e, f and g ***type*** 5 ***edges*** (Figure 5.2) in an embedding of a 4-regular graph, if one side appearances of e and f belong to one face walk (red). The other appearances of e and f, and both appearances of g belong to four distinct face walks (blue, black, green and yellow).

For three non-adjacent edges in a 4-regular graph, we can define many types of edges according to the graph's embedding. The number of types could be calculated according to the number of faces involved and their combinations, but these are not required.

We now define six extended dot product operations denoted by 'product i', where i is the increase in genus from G to $G \odot K_{4,4}$. Note that there are two distinct product with $i = 2$, and type 5 will be utilized twice in product 4 and 5.

According to the definition of extended dot product, vertices u and v can not be in the same vertex set in graph $K_{4,4}$, because there are no edges between vertices in one vertex set in a bipartite graph.

After applying extended dot product between an embedding of a 4-regular graph with a given embedding of $K_{4,4} - u - v$, we can get many different products. Six useful products are given, necessary and sufficient for our results.

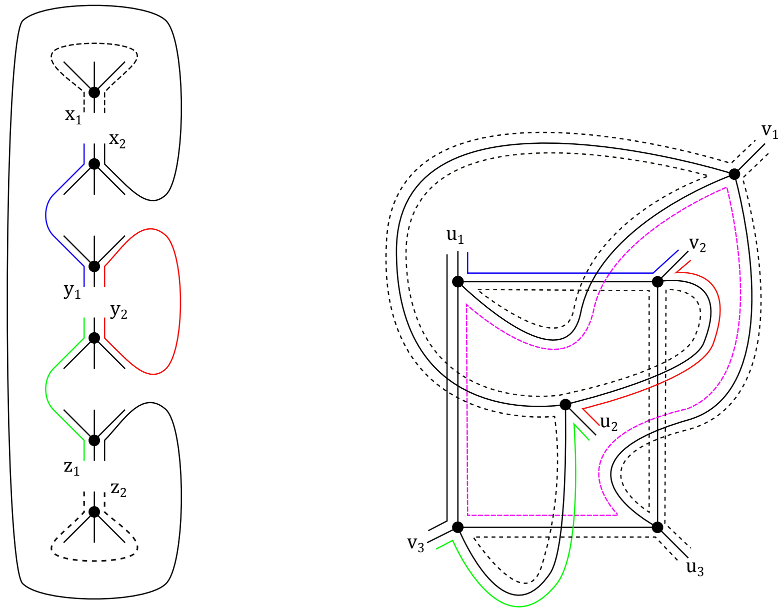


Figure 5.3: Product 1 has six faces involved in the extended dot product.

Let G be a 4-regular graph, and e, f and g be type 1 edges in an embedding of G. Then an extended dot product between an embedding of G and a given embedding of $K_{4,4} - u - v$ exists and we denote it by a ***product*** 1 (Figure 5.3).

Lemma 5.1.1. *For a product 1, we have $g(G \odot K_{4,4}) = g(G) + 1$. Furthermore, the resulting embedding of graph $G \odot K_{4,4}$ contains type 1 edges, type 2 edges and type 4 edges.*

Proof. According to Figure 5.3 and Euler's formula, we have $v(G \odot K_{4,4}) = v(G) + 6$, $e(G \odot K_{4,4}) = e(G) + 12$, $f(G \odot K_{4,4}) = f(G) + 4$ and $g(G \odot K_{4,4}) = g(G) + 1$.

The three type 1 edges are x_1v_1, u_2v_3 and u_3z_2.

The three type 2 edges are u_1v_2, u_2v_1 and u_3z_2.

The three type 4 edges are v_1u_1, u_2v_3 and v_2u_2. $\square$

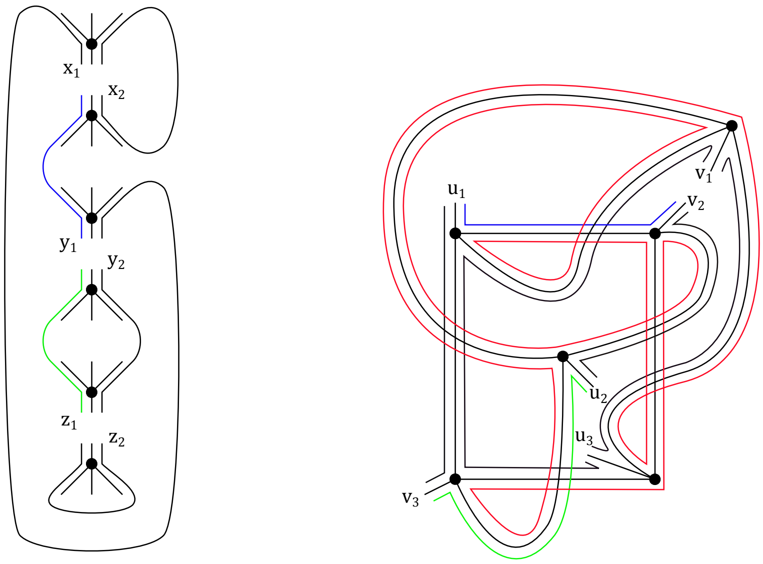


Figure 5.4: Product $2A$ has four faces involved in the extended dot product.

Let G be a 4-regular graph, and e, f and g be type 2 edges in an embedding of G. Then an extended dot product between an embedding of G and a given embedding of $K_{4,4} - u - v$ exists and we denote it by a ***product*** $2A$ (Figure 5.4).

Lemma 5.1.2. *For a product $2A$, we have $g(G \odot K_{4,4}) = g(G) + 2$. Furthermore, the resulting embedding of graph $G \odot K_{4,4}$ contains type 2 edges and type 4 edges.*

Proof. According to Figure 5.4 and Euler's formula, we have $v(G \odot K_{4,4}) = v(G) + 6$, $e(G \odot K_{4,4}) = e(G) + 12$, $f(G \odot K_{4,4}) = f(G) + 2$ and $g(G \odot K_{4,4}) = g(G) + 2$.

The three type 2 edges are y_1v_2, x_1v_1 and u_3z_2.

The three type 4 edges are v_2y_1, v_1u_3 and v_3u_2.

□

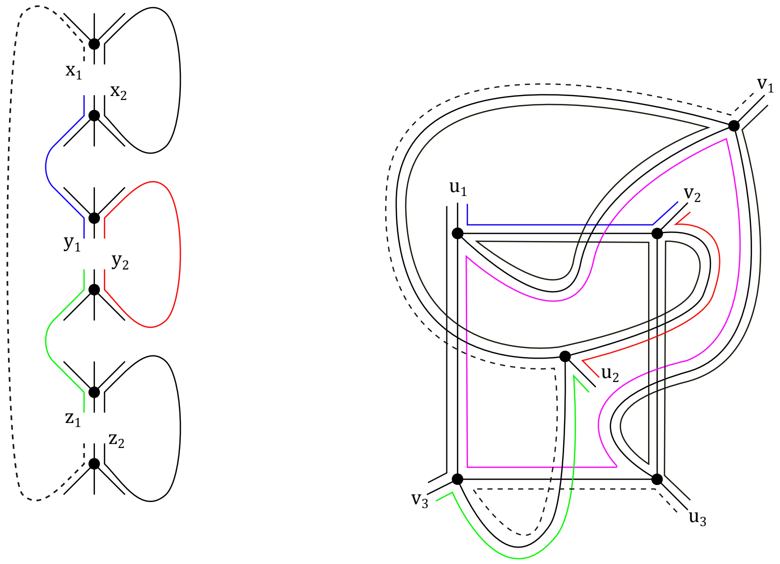


Figure 5.5: Product $2B$ has six faces involved in the extended dot product.

Let G be a 4-regular graph, and e, f and g be type 3 edges in an embedding of G. Then an extended dot product between an embedding of G and a given embedding of $K_{4,4} - u - v$ exists and we denote it by a ***product*** $2B$ (Figure 5.5).

Lemma 5.1.3. *For a product $2B$, we have $g(G \odot K_{4,4}) = g(G) + 2$. Furthermore, the resulting embedding of graph $G \odot K_{4,4}$ contains type 3 edges and type 5 edges.*

Proof. According to Figure 5.5 and Euler's formula, we have $v(G \odot K_{4,4}) = v(G) + 6$, $e(G \odot K_{4,4}) = e(G) + 12$, $f(G \odot K_{4,4}) = f(G) + 2$ and $g(G \odot K_{4,4}) = g(G) + 2$.

The three type 3 edges are x_1v_1, v_2u_1 and z_1v_3.

The three type 5 edges are x_1v_1, v_2u_1 and u_2y_2. $\square$

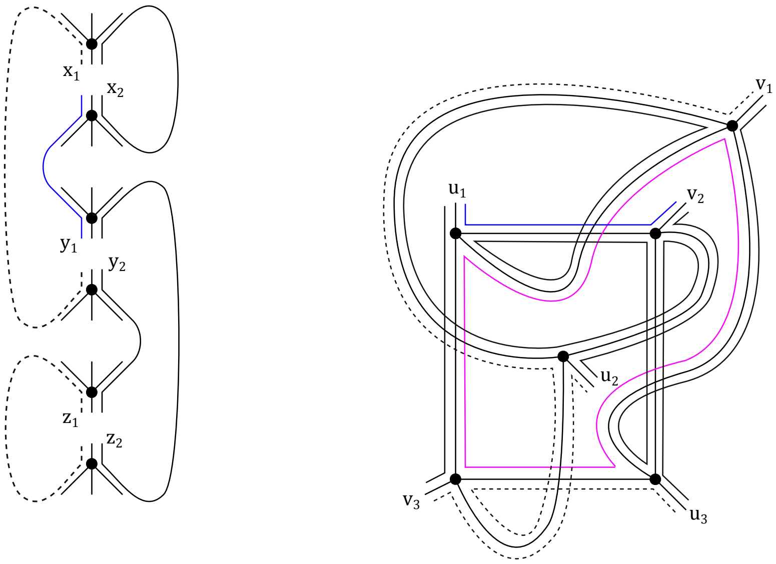


Figure 5.6: Product 3 has four faces involved in the extended dot product.

Let G be a 4-regular graph, and e, f and g be type 4 edges in an embedding of G. Then an extended dot product between an embedding of G and a given embedding of $K_{4,4} - u - v$ exists and we denote it by a ***product*** 3 (Figure 5.6).

Lemma 5.1.4. *For a product* 3*, we have* $g(G \odot K_{4,4}) = g(G) + 3$*. Furthermore, the resulting embedding of graph* $G \odot K_{4,4}$ *contains type* 3 *edges and type* 4 *edges.*

Proof. According to Figure 5.6 and Euler's formula, we have $v(G \odot K_{4,4}) = v(G) + 6$, $e(G \odot K_{4,4}) = e(G) + 12$, $f(G \odot K_{4,4}) = f(G)$ and $g(G \odot K_{4,4}) = g(G) + 3$.

The three type 3 edges are v_1u_3, u_1v_2 and y_2u_2.

The three type 4 edges are v_2u_1, v_1u_2 and v_3u_3.

□

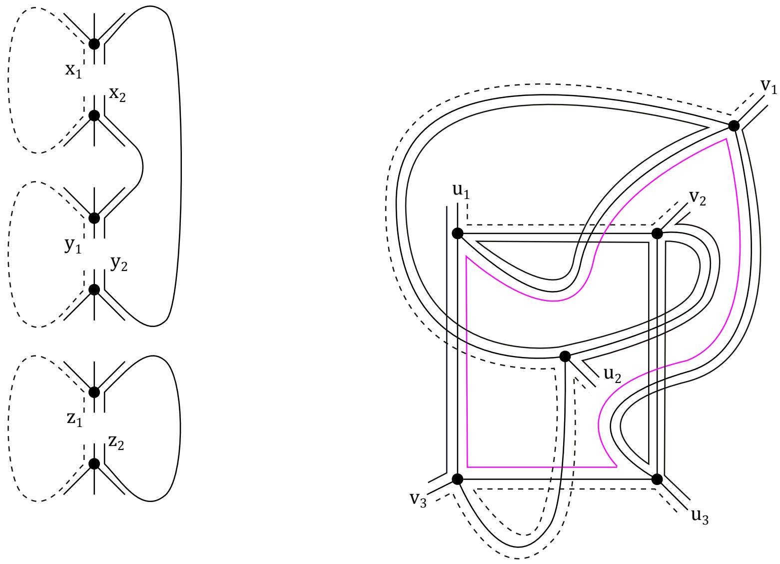


Figure 5.7: Product 4 has three faces involved in the extended dot product.

Let G be a 4-regular graph, and e, f and g be type 5 edges in an embedding of G. Then an extended dot product between an embedding of G and a given embedding of $K_{4,4} - u - v$ exists and we denote it by a ***product*** 4 (Figure 5.7).

Lemma 5.1.5. *For a product 4, we have $g(G \odot K_{4,4}) = g(G) + 4$.*

Proof. According to Figure 5.7 and Euler's formula, we have $v(G \odot K_{4,4}) = v(G) + 6$, $e(G \odot K_{4,4}) = e(G) + 12$, $f(G \odot K_{4,4}) = f(G) - 2$ and $g(G \odot K_{4,4}) = g(G) + 4$. □

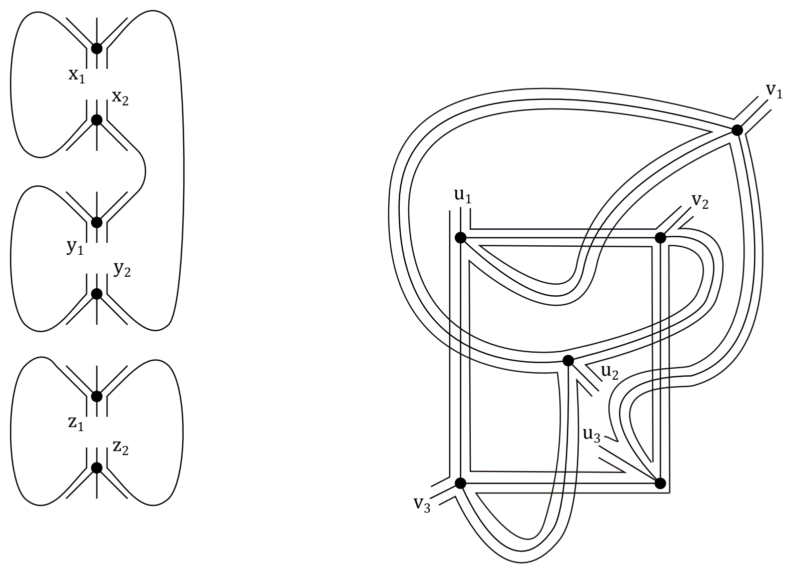


Figure 5.8: Product 5 has one face involved in the extended dot product.

Let G be a 4-regular graph, and e, f and g be type 5 edges in an embedding of G. Then an extended dot product between an embedding of G and a given embedding of $K_{4,4} - u - v$ exists and we denote it by a ***product*** 5 (Figure 5.8). Note that only one face involved in the extended dot product.

Lemma 5.1.6. *For a product* 5*, we have* $g(G \odot K_{4,4}) = g(G) + 5$.

Proof. According to Figure 5.8 and Euler's formula, we have $v(G \odot K_{4,4}) = v(G) + 6$, $e(G \odot K_{4,4}) = e(G) + 12$, $f(G \odot K_{4,4}) = f(G) - 4$ and $g(G \odot K_{4,4}) = g(G) + 5$. □

Figure 5.9 shows the relationships between the 6 product operations.

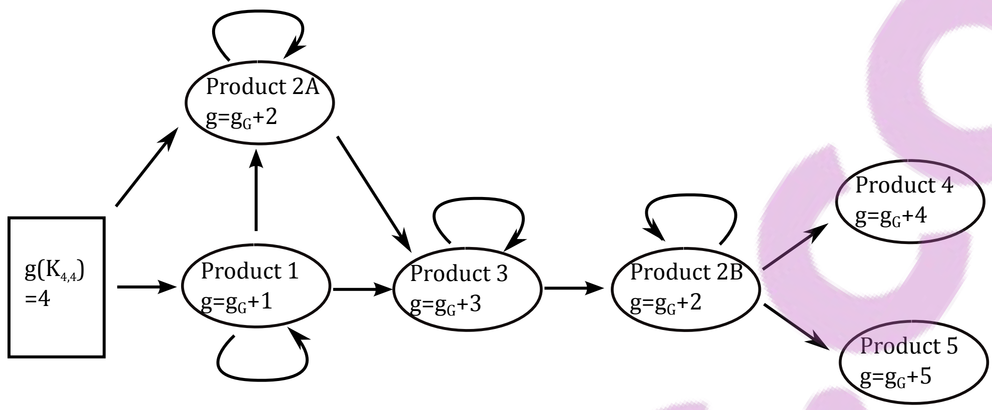


Figure 5.9: The flowchart of embedding construction of $K_{4,4}^n$

Based on the five types of embeddings for e, f, g, there are $5 \cdot (3!)^6$ different product operations with different embeddings of $K_{4,4} - u - v$. We only picked six of them, which are enough to construct embeddings of $K_{4,4}^n$ among the genus range $n + 3 \leqslant k \leqslant 3n + 1$.

Theorem 5.1.7. *For each k, $n+3 \leqslant k \leqslant 3n+1$, there exists an extended dot product of n copies of $K_{4,4}$ whose orientable genus is precisely k.*

Proof. If $n+3 \leq k \leq 2n+2$: We start from an embedding of $K_{4,4}$ with genus equal to 4. Then perform product 1 h times followed by product $2A$ l times, where $0 \leq h \leq n-1$, $0 \leq l \leq n-1$, $h+l+1 = n$. We have $4 + h + 2l = k$.

If $2n+2 \leq k \leq 3n-1$: We start from an embedding of $K_{4,4}$ with genus equal to 4. Then perform product 1, followed by product 3 h times and product $2B$ l times, where $1 \leq h \leq n-2$, $0 \leq l \leq n-3$, $h+l+2 = n$. We have $4+1+3h+2l = k$.

If $k = 3n$: We start from an embedding of $K_{4,4}$ with genus equal to 4. Then perform product $2A$, followed by product 3 $n-2$ times. We have $4+2+3(n-2) = k$.

If $k = 3n+1$: We start from an embedding of $K_{4,4}$ with genus equal to 4. Then perform product $2A$, followed by product 3 $n-4$ times, product $2B$ once and product 5 once.

$\square$

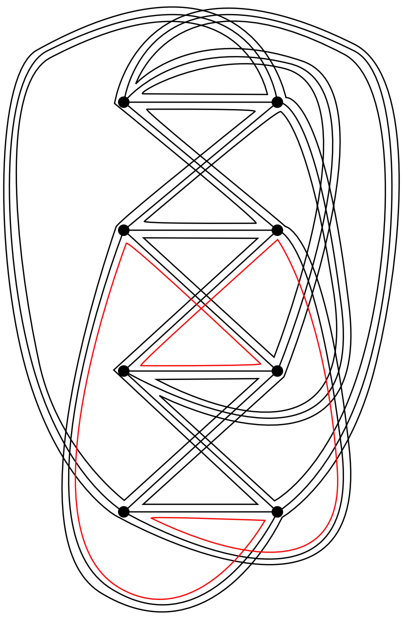


Figure 5.10: An embedding of $K_{4,4}$ with two faces and genus 4

We need to discuss the initial cases separately. The complete bipartite graph $K_{4,4}$ embedds on surfaces of genus 1 ,2, 3 and 4, with corresponding face numbers 8, 6, 4 and 2. An embedding of genus 4 is given in Figure 5.10.

For $K^2_{4,4}$, We can start from an embedding of $K_{4,4}$ with genus equal to 1. Then apply product 3 and get $g(K^2_{4,4}) = 4$, which is smaller than $n+3 = 2+3 = 5$.

Theorem 5.1.8. *The maximum genus of $K^n_{4,4}$ is $3n+1$.*

Proof. $K^n_{4,4}$ has $6n+2$ vertices, $12n+4$ edges. If it can be embedded on a surface of genus

$3n+1$, the embedding will have 2 faces by using Euler's formula. The face number has to be an even number, and the smallest is 2 by Euler's formula. So $3n+1$ is the maximum genus.

□

Theorem 5.1.9. *For any 4-regular graph G, we have $1 \leq g(G \odot K_{4,4}) - g(G) \leq 5$.*

Proof. For an extended dot product $G \odot K_{4,4}$, there are 24 corners in $G - e - f - g$ and $K_{4,4} - u - v$, which are involved in the product operation. A smallest face walk involved in a product needs at least 4 corners, 2 in the embedding of $G - e - f - g$, 2 in the embedding of $K_{4,4} - u - v$, which are straightforward from the 6 figures of productions. So we can make no more than 6 walk faces out of the 24 corners.

On an embedding of G, the two appearances of the three selected edges e, f, g will be involved in m faces. After an extended dot product, $G \odot K_{4,4}$, there are h faces involved. It is straightforward to see that $1 \leq m, h \leq 6$. For example, in product 1, we have $m = 2$ and $h = 6$. In product 5, we have $m = 5$ and $h = 1$.

$$
\begin{array}{rccccc}
 & 2 - 2g(G \cdot K_{4,4}) & & & & \\
= & v(G \odot K_{4,4}) & - & e(G \odot K_{4,4}) & + & f(G \odot K_{4,4}) \\
= & (v(G) + 6) & - & (e(G) - 3 + 6 + 9) & + & (f(G) - m + h) \\
= & v(G) & - & e(G) & + & f(G) + (h - m - 6)
\end{array}
$$

$$g(G \odot K_{4,4}) = g(G) + \frac{6 + m - h}{2}$$

Because $-5 \leq m - h \leq 5$, then $1 \leq (m - h)/2 + 3 \leq 5$.

So we have $1 \leq g(G \odot K_{4,4}) - g(G) \leq 5$.

□

From constructing the genus embedding of $K^n_{4,4}$, we can find that the extended dot product has its limitations to reach the minimum genus, but not the maximum genus.

5.2 Conclusion

This chapter introduced an extended dot product, which can be applied on 4-regular graphs. In fact, we can design 6! different extended dot products for 4-regular graphs according to the 6! different connections between vertices $x_1,\ x_2,\ y_1,\ y_2,\ z_1,\ z_2$ from G_1 and vertices $v_1,\ v_2,\ v_3,\ u_1,\ u_2,\ u_3$ from G_2. Figure 5.11 illustrates another extended dot product with edges x_1v_1, x_2v_2, y_1v_3, y_2u_1, z_1u_2 and z_2u_3 between G_1' and G_2'.

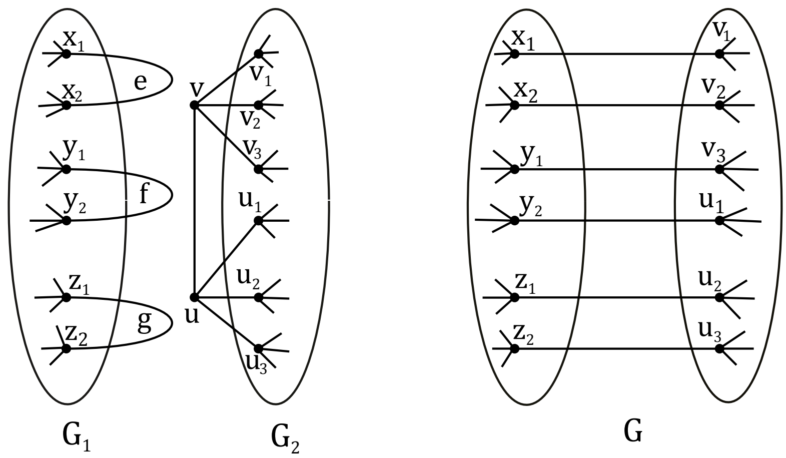


Figure 5.11: Extended dot product of two 4-regular graphs

The extended dot product in Figure 5.11 looks more tidy than the one we discussed in Section 5.2; but the results on genus range are not as good. We have found products which make the genus grow by 2, 3, 4 and 5. But we have not found any product which makes the genus grow by 1. Further research is needed on this topic.

We can also design 4! different dot products for 3-regular graphs according to the 4!

different connections between vertices $x_1,\ x_2,\ y_1,\ y_2$ from G_1 and vertices $v_1,\ v_2,\ u_1,\ u_2$ from G_2. But whether the other 23 dot products are snark-preserving requires consideration.

We could generalize these results further by designing $(2(n-1))!$ different dot products for n-regular graphs ($n \geq 3$).

Chapter 6

Conclusions and Future Research

6.1 Conclusion of results

In Chapter 2 of this thesis, we have presented some topological operations — face-contraction, vertex-splitting, vertex-augment, pearl-making, bouquet-making and face-expansion. We can calculate the genus distributions of some families of graphs which are constructed by these topological operations with known genus distributions.

With the Cartesian product, we can construct the minimum genus embedding of $S_n \times G$, where G does not contain any subgraph C_3 and $\Delta \leq (n-1)$. The team selection technique is designed to make sure that there are only faces of C_6 and C_4 involved in the embedding construction. We have demonstrated the constructions of the minimum embedding of $S_n \times P_m$, $S_n \times T_m$, $S_n \times C_m$ and $S_n \times S_m$. The construction of minimum embeddings of other Cartesian products could be achieved in a similar way. It has been proved that the minimum embedding of $S_n \times C_3$ can be constructed this way as well.

The topological operations in Chapter 2 and 3 are designed to construct graphs. The topological operations discussed in Chapter 4 and 5 are designed to construct embeddings directly. The resulting graphs are by-products.

The dot product supplies a way to construct a family of graphs to allow a certain number of different graphs being contained in each generation. All of them are snarks. We gave an embedding construction of Petersen power P^n, so that in each generation, there exists at least one graph whose minimum genus is 1 and at least one graph whose maximum genus is $2n+1$. We note that this does **not** mean every graph in any generation has these properties.

We used dot product to produce $K_{3,3}$ powers as well, but failed to find any results on minimum or maximum genus.

The extended dot product is designed in this thesis to construct families of 4-regular graphs. It is a generalization of the dot product in Chapter 4. We proved by embedding construction that in each generation of $K_{4,4}^n$, there exists a graph whose maximum genus is $3n + 1$. Note that we do not know whether every graph in generation of $K_{4,4}^n$ has the same maximum genus, but this seems unlikely.

6.2 Future research

The topological operations in Chapter 2 are designed to work on an one-off manner. Further study could be done to consider how to use these results in iterated operations.

The genus distribution of wheel graph W_n can be respected as the diamond of the crown in the field of graph embedding. Theoretically, by adding an edge between the two end vertices of the path of fan graph F_n, we can get a wheel graph W_n. Practically, we need to get the partial genus distribution of F_n regarding the face-tracing of its two end spokes, which can not be achieved directly from the genus distribution of F_n. It is one step away, but sky high to reach.

We can also define a vertex augment by the rearrangement of adjacent edges of a vertex upon a tree and preserve the face numbers to keep the genus unchanged. A linear relationship between the genus distribution of the original graph and the genus distributions of augment graphs could be achieved.

According the results of vertex augmenting, the genus distribution of any graph has a linear relationship with the genus distributions of some cubic graphs. The cubic graphs are results of applying vertex augment on every vertex with valency bigger than 3. Note that this linear relationship is not unique; because the vertex augment can be designed in many ways.

For the Cartesian product, when G contains a subgraph C_3, the construction of minimum embedding of $S_n \times G$ deserves further study.

For Petersen powers P^n, it seems quite possible to get a range for its maximum genus corresponding to the range of minimum genus in [27]. It would also be interesting to consider the range of minimum and maximum genus of $K_{4,4}^n$.

A lot of research have been done on snarks, which is a kind of 3-regular graph (refer to page 17 for the definition). We might try to define a kind of 4-regular graph similar to snarks regarding their cycle size and edge coloring, and find further research topics upon it.

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