TABLE OF CONTENTS

CHAPTER I: INTRODUCTION AND ORIENTATION

Chapter II: Historical development of the derivative and its teaching

CHAPTER III: MODELING AND CONTEXTUALIZATION IN *SKETCHPAD*

CHAPTER IV: RESEARCH METHODOLOGY AND THE DESIGN OF INSTRUMENTS

CHAPTER V: DATA AND DISCUSSION OF DATA

CHAPTER VI: SUMMARY AND CONCLUSIONS

Bibliography 209

Appendices 216

List of Figures

List of Tables

CHAPTER I

INTRODUCTION AND ORIENTATION

1.1   Introduction

The discovery and invention of the calculus by Newton and Leibniz has been acknowledged to be one of the most important developments in the history of mathematical and scientific thought. Tall (1997:319) actually regards the discovery and invention to have been one of the most significant events in the evolution of civilization. The fact that it took many centuries of effort to discover the calculus, points to the cognitive difficulties of communicating its concepts in ways students can understand easily.

The derivative as the instantaneous rate of change and rate of change (gradient/slope) function is an elusive concept. This elusiveness is borne out by the fact that its exact meaning was not unanimously agreed upon for some time even within the mathematics community. The difficulty of the concept is further borne out by the inherent notion of the limit concept and its embodiment of the potential infinite. The basic ideas of Newton's calculus are reckoned to have emanated from considerations of motion (Strauss, 2001:29). Abiding by Zeno's paradox of the Arrow, finding velocity at an instant, implies dividing an infinitely small distance (a zero magnitude) traveled by an infinitely small time interval (a zero instant) which is undefined, something contrary to the common sense physical experience of motion. An understanding of velocity at an instant helps to explain the extent of damage that can be sustained at the point of contact in a collision. It also helps us to understand that two cyclists can complete a race in a dead heat (this means they finish at exactly the same time) at different instantaneous velocities but identical average velocities. In support, Gonzalez-Lopez (2001:129) states that Zeno's paradoxes constitute a paradigmatic example, reflecting the difficulty of reconciling our perceptions with the foundations of physical and scientific theories.

The important role played by the derivative in quantifying the rate of change transcends the ideas of change of distance with change in time or change in velocity with change of

time (acceleration) in the genetic context of motion. Rates of change occurring between any two physical quantities in various contexts can also be handled using the derivative. For example the growth rate at a given time (in economics, ecosystems, pollution rates, birth rates, density, marginal cost, etc). In this connection Kaput (1994) observes that the rapid increase of mathematical applications requires that all citizens be fluent in modeling continuously changing phenomena, especially phenomena of dynamic situations which are at the heart of calculus. He, however, points out that conventional curricula have not been successful in promoting these modeling abilities in undergraduate students yet success in post university careers is more closely linked to success in modeling than in calculus. The challenge to mathematics educators is, therefore, to make calculus more accessible to the majority of students for them to contribute more meaningfully to the socio-economic well being of their societies, even in the context of such a developing country as Zimbabwe.

The proliferation of computer technology legitimately challenges mathematics educators to rethink or broaden the tools with which calculus teaching and learning can be approached to scaffold as many students as possible to higher levels of understanding than otherwise afforded by the traditional pencil-and-paper curricula in which the teacher and the textbook were the arbiters of mathematical truth. Cuoco (2002:294) cautiously points out that the proliferation opens up a whole new set of mathematical possibilities for students and a whole new set of interesting questions for educators, more so when a new tool is designed to serve one field (in this case mathematics) but used in another (education). Weignand and Weller (2001:87) concur with the challenges brought on by new technologies citing, for instance, that computer algebra systems (CAS), have led to new approaches to the function concept. New teaching methods where students can switch between numerical, graphical and symbolical representations at the press of the button, and to new working styles such as working experimentally and making conjectures about solutions through systematic search processes are considered possible in CAS instructional designs. In other words a new genre of classroom practices is opened by the new tools.

The overarching characteristic of the traditional curriculum is its emphasis on learning by rote in which mathematics is reduced to a set of algorithmic procedures to be memorized and regurgitated at examination time but forgotten as quickly as conveniently possible thereafter when minimum understandings have been achieved. Wu (1999:3) laments that it does not add to our comfort to realize that many calculus courses in college lend themselves to learning by rote so that students often graduate from such a course equating the 'derivative' with the "the thing that changes x^n into nx^{n-1}" and nothing more. Tall (1997:289) similarly observes that traditional calculus used to be a mixture of manipulative symbolism and qualitative visualization with possible deductive elements from analysis. In this guise, calculus concepts have not been meaningfully understood. The advent of the computer allows not only a numeric-quantitative approach but also graphical representations to be enactively manipulated at the will of the user offering a possible conceptual approach based on real-time visualization and perceptual control.

However, Artigue (2002:245) cautions that these sophisticated new tools do not become immediately efficient mathematical instruments for the user. If anything, their complexity does not make it easier to master and fully benefit from their potential at the outset. In other words, there is a cost to learning how to use such software effectively and fluently. In respect of dynamic geometry environments, Lagrange (2005:165) adds that even when students acquire some 'enactive' knowledge about functional properties, this knowledge is often far from the algebraic conception of functions in their everyday mathematics. He conversely observes that 'good epistemology' is not a guarantee in itself of consistent practices either (ibid. p 143).

It is further pessimistically noted that merely placing computers in the hands of teachers and students is no panacea if this technology is used as a 'flashcard' or as a substitute arbiter of pure reason (Cuoco & Goldenberg, 1996, Wu, 2006). For instance, the traditional emphasis on the 'how' of mathematics should extend to the 'why' of it in order to fully engage students in meaning negotiation or sense making to form, justify and compress their understanding into rich concept images. Tall (1987:2) already observed the need to develop a curriculum where dynamic software facilities are used to

their fullest advantage, without compromising the human need to have a curriculum which sheds insight into the processes involved. Ruthven and Hennessy (2002:47) express the view that to be successful, technological innovation has to offer new opportunities for current curricular concerns and to be compatible with viable classroom practices. Tall and Ramos (2004) specifically note that the computer environment changes the didactic triad of teacher-student-textbook to a didactic tetrahedron of teacher-student-mathematics and the computer. In a sense, the computer introduces a new factor into the classroom didactic relationships between the students, the educator and the subject matter in that it enables aspects of mathematics to be externalized and manipulated enactively by the user independently of the teacher or the textbook. The didactic tetrahedron however may still require further extension to incorporate collaborative learning as a fifth dimension, and a redefinition of the teacher's role. Figure 1.1 is a conjecture of possible metamorphoses in the didactic relationships, from the semiotic triangle and the didactic triad to the didactic tetrahedron and the didactic pyramid.

a) Semiotic Triangle
b) Didactic Triangle
c) Didactic Tetrahedron
d) Didactic Pyramid
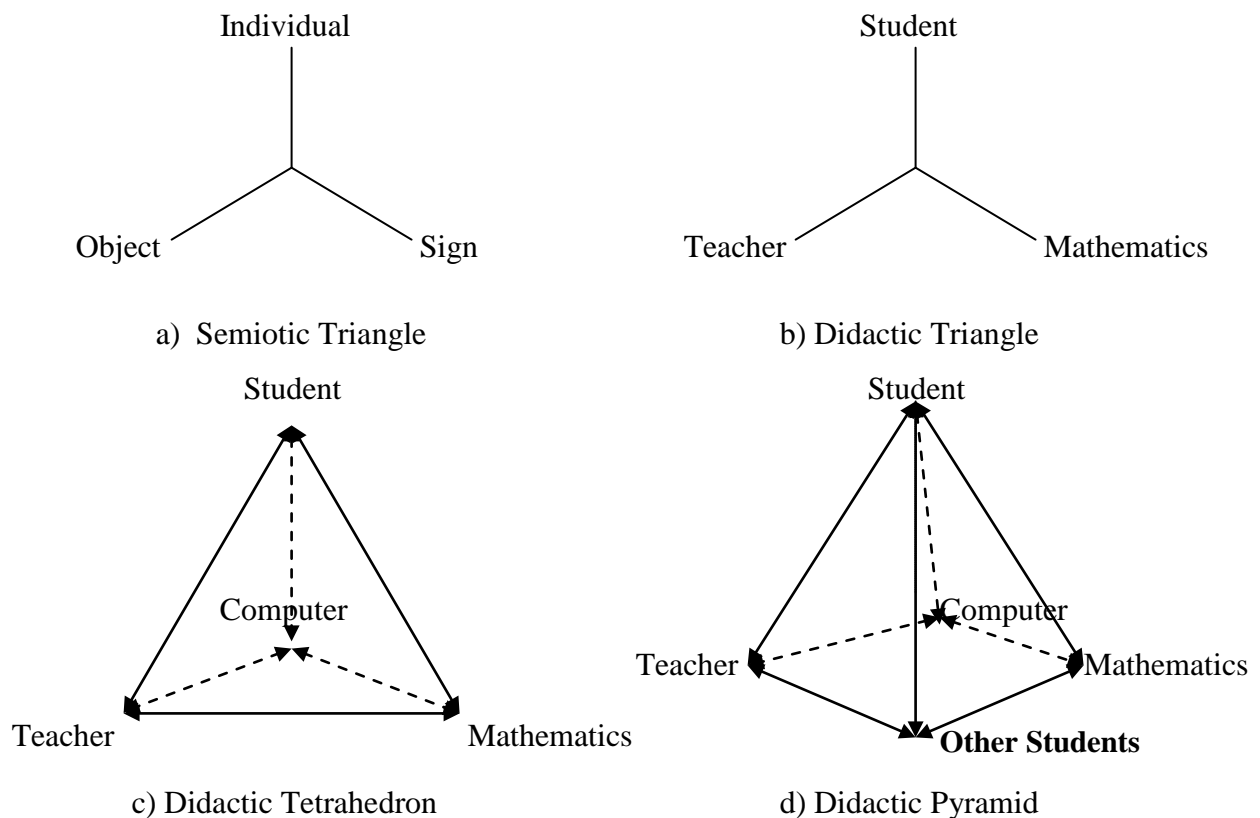
Figure 1.1: The Dynamics of Didactic Relationships

On a similar note, Mariotti (2001:25) argues that the role of dynamic software should be used to introduce students to theoretical thinking through social construction of knowledge and the semiotic mediation offered by the software's graphing interface and interactivity. That is, the software should be used as a tool with which to model not just mathematical concepts as products but also to enactively model the mathematical thinking and reasoning processes involved. Software limitations themselves should also be factored in and harnessed in order to reduce on the narrowing effect of computational representations observed by Hunter, Monaghan and Roper (1993) and Drijvers (2000), among others. That is, the pedagogical role of those limitations should be taken advantage of and utilized to enrich students' concept images as argued by Giraldo and Carvalho (2002).

1.2   Statement of the Problem and Purpose of the Study

The primary goal of this study is to contribute to the understanding of how to support (non-Mathematics major) undergraduate students' efforts in making sense of the concept of derivative in a *Sketchpad* dynamic software environment, at a university in Zimbabwe. The purpose of the study is thus to critically investigate the potential of a dynamic mathematics software environment in developing or modeling sophisticated understandings of the historically difficult meaning of the derivative – both as the instantaneous rate of change and as the rate of change function (or slope function). This is against a backdrop of a high undergraduate failure rate of an introductory university mathematics course offered to science students at the university. The study thus attempts to be another installment towards the developing theory of computationally mediated mathematical learning, which Hoyles, Noss and Kent (2004:309) acknowledge to be a problem that still requires considerably more theoretical elaboration and empirical analysis.

This study builds on previous research that has recognized the power and limitations of using dynamic graphic software to represent calculus concepts. The notion of derivative is recognized not only as an important idea in the calculus, but more importantly as a

central and complex one that is difficult for many students to comprehend especially when approached via the formal (abstract) limit definition route.

The study therefore serves to advance the research agenda by framing the concept of derivative in a dynamic mathematics modeling context. This is achieved through computer aided instruction in mathematics. As a consequence, the study addresses not only the limitations of the concept images that students can develop in a traditional environment but also strives to go beyond to consider the innovative utilization of the static-numeric, dynamic-numeric, static-graphic and dynamic-graphic interfaces, and symbolic algebra interfaces offered by *Sketchpad* in one package.

The study seeks to answer the following questions and sub-questions:
a) Is there a significant difference in the conceptual understanding and modeling of the derivative between the pretest and posttest scores of the experimental and the control groups, at the 5% level? More specifically:
   i) What prerequisite knowledge and skills do students bring to the study and how can it be capitalized upon for students to benefit meaningfully from the availability of dynamic software environments?
   ii) How do the experimental group students' post teaching experiment concept images of the derivative and the associated concepts of function, variability, slope, limit, etc, compare with their pretest understandings?
   iii) How do experimental students' concept images of the derivative and the associated concepts compare with those of control group students at the end of the study?
b) Is there a developmental sequence in the understanding of the derivative that can be mapped and followed in a dynamic software environment? More precisely:
   i) How has the concept of derivative developed historically and what pedagogical lessons can we derive and apply in a dynamic mathematics software environment?
   ii) How effective is the use of dynamic mathematics software as a tool to model the concept of derivative and associated concepts of functions,

variability, limit, continuity, differentiability, and what constraints or limitations are found?

iii) How effective is the chosen instructional model and didactical relationships designed with dynamic mathematics software to develop students' conceptual understanding of the derivative and associated concepts?

c) Is there a change in the conceptual understanding of the concept of the derivative and associated concepts after exposure to the mediation of a dynamic mathematics software environment? More specifically:

i) What is the quality and types of concept images of the derivative that students are able to form with the mediation of *Sketchpad*?

ii) What connections are students able to make between different *Sketchpad* dynamic mathematics representations of the derivative?

iii) What advantages or obstacles do students experience as they attempt to make sense of the derivative with the aid of *Sketchpad* dynamic mathematics software?

1.3   Research methodology adopted in this study

The research methodology adopted in this study has the general form of a classroom design experiment or curriculum/instructional design research as elaborated upon by Cobb, Confrey, diSieza, Lehrer and Schauble (2003), and Gravemeijer (1994) among others. It therefore draws some inspiration from the developmental research proposed by Realistic Mathematics Education philosophy of the Freudenthal Institute in the Netherlands. The research approach is both quantitative and qualitative. A comprehensive review of the literature is undertaken to place the study in context. On the quantitative domain, the non-equivalent pretest-post control group quasi-experimental design is employed for a statistical determination of the significance level of any observed differences in achievement levels. The quantitative approach is triangulated with model eliciting *Sketchpad* instructional activities designed as a learning trajectory to model the derivative in a dynamic mathematics environment. The activities form the basis of the

experiment and are complemented with task-based interviews to gain a deeper understanding of students' cognitive experiences. An exit group interview for the experimental students helps to round up the data gathering process.

The analysis I report is based primarily on a teaching experiment design conducted in 2006/2007 at the National University of Science and Technology, a state university in the city of Bulawayo, Zimbabwe. Non-mathematics major students in the Faculty of Applied Sciences participated in the teaching experiment that consisted of six instructional sessions lasting one-and-a-half hours each. From a group of 18 students who volunteered to write the pre-test in the pilot phase of the study, a control group of 9 students was selected on a first-come-first-selected basis. In the final phase 20 pretest participants and an experimental group of seven students were involved. In both the pilot and final phases the researcher used already established groups of subjects, gave a pretest, administered the teaching experiment to one group, and gave the posttest. A control group was selected from the same faculty. Both groups were enrolled for the same preparatory mathematics course in which the derivative is taught. The performance of the non-major students on the course has generally been characterized by dismal test scores and many repeats suggesting the course to be relatively difficult for them.

1.4   Definition of key terms

1.4.1   The Derivative

In this study the derivative shall be construed to be that part of the calculus that describes the rate of change of one variable with respect to another. It embodies two aspects: the instantaneous rate of change and the change in the rate of change (rate of change function). According to Wikipedia (2006/06/30:1) the simplest type of derivative is the derivative of a real-valued function of a single real variable, which has several interpretations:

- The derivative gives the slope of the tangent to the graph of the function at a point. In this way, derivatives can be used to determine many geometrical properties of the graph, such as concavity or convexity.

- The derivative provides a mathematical formulation of the rate of change: it measures the rate at which the function's value changes as the function's input values (domain) change.

This latter meaning of the derivative is the kind usually encountered in an introductory course on calculus, and historically was the first to be discovered. However, there are also many generalizations of the derivative. The opposite of the derivative is the integral, which also has two main meanings, namely as in definite integration when finding area under curves, or in indefinite integration where it is seen as the inverse of differentiation. Historically the latter meaning and understanding developed much later and is encapsulated in the fundamental theorem of calculus.

1.4.2   The potential infinity and the actual infinity

The potential infinity (potential infinite or potentially infinite, uncompleted infinity) is used to indicate the Aristotelian conception that the infinite is a quantity that is literally in-finite. That is, it is endless (without end). It is alternatively referred to as the uncompleted infinitude to stress that it cannot be completed in any finite time. Cantor as quoted by Strauss (2001) describes the potentially infinite or the uncompleted infinity as follows:

> The potential infinite is preferably indicated where an indefinite variable of finite magnitude occurs, which either increases beyond all finite limits …., or decreases beyond all fine borders (Strauss, 2001:31).

From an APOS[1] perspective (see 2.2.6 and 4.3.1.3) Dubinsky, Weller, McDonald and Brown (2005a:346) describe the potential infinity more practically as the conception of the infinite as a process constructed by beginning with the first few steps (an action conception) and repeating these steps *ad infinitum* (i.e. interiorization of the action into a process).

Note 1: APOS Theory the acronym stands for: actions, processes, objects and schema

The actual infinity (or actually infinite) is used to refer to a quantity which is determined in all its parts (a definite entity) while it simultaneously exceeds every finite quantity (Dubinsky et al, 2005a). It is alternatively referred to as the completed infinitude. Cantor

as quoted by Strauss (2001) describes the actually infinite or completed infinity as follows:

> Actually infinite, though, is understood as a quantum, which on the one hand does not change, but which rather is set and determined in all its parts, a true constant, but simultaneously on the other hand exceeds in magnitude every similar finite magnitude (Strauss, 2001:31)

In other words, the uncompleted infinite is linked to the nature of a constant (as an indefinite variable of a finite magnitude) while the completed infinite is linked to the nature of a variable (as an indefinite variable of an infinite magnitude).

1.4.3   The Concept Image

In the theory developed by Tall and Vinner (1981), the concept image is the total cognitive structure associated with a mathematical concept in an individual's mind. It includes all the mental pictures or images, properties, mental associations and processes connected to a given concept, and is continually constructed and reconstructed as the individual grows and matures with new stimuli, models and representations of the concept. Giraldo, Carvalho and Tall (2003:1) view the concept image as a mental picture and contrast it with concept definition, which is a formal statement, used to specify that concept from its properties. However, Tall (2006:206) views the concept image from a cognitive biological perspective and regards it to be an embodiment of both the mental picture and the concept definition. In this study Tall's definition is adopted for its broadness.

1.4.4   Modeling

In this study, modeling is used in the sense of idealizing to simplify complex situations as conceived by Freudenthal (1991:84). That is, it is construed to be a process of grasping the essentials of a static or dynamic situation by discovering common features, similarities, analogies, and isomorphisms towards the goal of generalizing. This resonates with Mudaly's (2004:85) conceptualization of modeling as the way the learner thinks, processes, checks solution, makes a plan and executes it. Modeling with *Sketchpad* subsumes modeling as a strategy or tool use to pedagogically represent a concept or

knowledge structure so that it can be better understood and internally represented by the learner. This form of modeling is therefore distinct from mathematical modeling which focuses more on mathematization of real world situations. A basic assumption of this study is that a *Sketchpad* visualization (multiple-representational) environment can help students develop powerful conceptual models of the derivative.

1.4.5   Dynamic mathematics software environments

In this study, these are considered to be computer assisted learning environments in which the graphing interface can be manipulated by the user through enactive dragging using the mouse (or joystick) or animations that flexibly retain the defined relationships in function plots and constructed mathematical objects. Such software packages include *Cabri II*, the *Geometer's Sketchpad*, *Mathematica, Graphic Calculus, MathCad, Maple*, with varying degrees of dynamic capability. Version 4 of the *Geometer's Sketchpad* which has both CAS and dynamic properties is the dynamic mathematics software whose modeling capabilities are under investigation in this study.

1.5   Significance of the Study

There is significant research on the role of technology in the computer aided teaching and learning of mathematics in general and calculus in particular, but not as much of calculus in a dynamic software environment such as *Sketchpad*. This research is therefore neither a counter nor a ritual repetition but an effort to extend the potential into which *Sketchpad* can be instrumented to expand and deepen students' understanding of the numeric, symbolic, graphic and dynamic representations of the derivative's dual meaning of instantaneous rate of change (i.e. as the limit of the average rate of change of a function) and as the slope or gradient function (i.e. as the rate of change function). In support of efforts such as this Weignand and Weller (2001: 89) report that in a review of studies on whether computers will be a help or an obstacle to concept development, they found that only a few studies at the time had a scientific research design that investigated with experimental and control groups and pre- and posttest or on students' working styles. Carnine and Gersten (2000:139) acknowledge that well-controlled experimental and

11

quasi experimental research studies are the building blocks of scientific knowledge about teaching. While acknowledging that no one study will be flawless, it is nevertheless a virtue that knowledge accrues through systematic reviews of sets of studies (Gage, 1999).

Given that the number of studies might even be much less in developing countries, the study could be a significant contribution. Furthermore, the university environment offers more unique opportunities for pioneering curricula changes on a micro-scale than does a public school system dominated by centrally controlled and regimented curriculum and examination practices.

The ability to translate the research on learning mathematics in dynamic software environments in a participatory non-answer giving but creative and enactive problem-solving (modeling) and experimental approach in a manner as efficient as possible is a significant potential contribution of this study. Such a translation is all the more urgent given the prospect that dynamic CAS may transform many methods of problem solving and concept development from the dubious status of 'methods in principle' to the credible status of 'methods in practice' referred to by Kendal and Stacey (2001:144).

The results of this study should contribute to the improvement of the achievement of non-Mathematics majors in calculus courses they are required to take and to make such learning more enjoyable than it currently is. The anticipated results should hopefully encourage both undergraduate (even high school) calculus educators to embrace technology more enthusiastically in the teaching of mathematics, not just to non-majors but also to majors given the expanded availability of computer technology at tertiary level. To those involved in curriculum design in dynamic environments, the results of this study should be appreciated as a valuable contribution to the pool of exemplars open to utilization to make the learning of the derivative in particular and introductory calculus in general easier to understand and to orchestrate.

To the software designers, the study ought to leave a challenge on how the software programming can be refined to overcome some of the computational limitations that may

cause conflict or simply to improve capabilities. More specifically there is the challenge of using finite algorithms to represent infinitesimals and the resulting anomalies at high magnification as highlighted in the literature review and in representing division by h as $h{\rightarrow}0$ without actually reaching zero. In other words, the opportunity exists for the gap between the needs and uses of mathematicians/researchers on the one hand and educational and engineering uses of software on the other.

1.6   Organization of the thesis

The rest of this thesis is organized into five chapters. In Chapter II, a literature review is presented on the historical development of the calculus with special attention being paid to the notion of the derivative and the antecedent concepts of infinity and limit. A further review of literature is carried out on the teaching and learning of the derivative concept within the larger context of the Calculus Education Reform effort in computer-aided or Computerized Learning Environments (CLE). The chapter reaches a climax by reviewing the opportunities and challenges provided by the dynamic software environments in the didactics of the derivative concept supported by research findings in this direction. It concludes with some initial aspects of a proposed learning trajectory for the derivative concept in *Sketchpad*.

Chapter III clarifies the modeling context of the study. In particular, a philosophical background to modeling is briefly outlined to provide anchor before various perspectives on modeling are discussed and extended to how a dynamic mathematics environment can, in partnership with other strategies, model instructional opportunities to facilitate the attainment of deeper conceptual understanding of mathematical concepts in general and the concept image of the derivative in particular. Chapter IV describes and justifies the methodology adopted for the study and details and justifies the instruments/techniques used for gathering the data. Contemporary theoretical frameworks on mathematical thought development are also discussed to provide some psychological basis for the assessing or categorizing students' understanding. In Chapter V the data are analyzed in relation to the respective methods by which they were gathered and the research

questions that they are required to address. Chapter VI presents the findings, recommendations and conclusions.

1.7   Conclusion

In this chapter the role and importance of the derivative as a calculus concept has been briefly outlined. Its cognitive complexity has been highlighted by the long-standing struggle by mankind to understand the infinity. Students' difficulties in understanding the intractable concept in introductory calculus have been noted. The advent of dynamic mathematics software packages has been cast as an opportunity to re-invent the teaching of calculus in general and the derivative in particular for the attainment of a more robust understanding by students to whom the concept would otherwise have been less accessible in static textbook and chalkboard instructional genres. The purpose of the study, together with the related aims and objectives, has been enumerated (in question form). The significance of the study for a cross section of stakeholders and the knowledge base of mathematics education has been briefly argued. The research methodology adopted in this study has been briefly described and tentatively justified. Key terms have been defined in order to contextualize their meanings in the study. Finally, the general organization of the thesis has been outlined to orient the reader on the key milestones.

The next chapter reviews literature concerned with the historical development of the concept of derivative starting with the ancient Greek paradoxes of infinity, through to the emergence of the concept in the context of the drawing of tangents, analysis of non-uniform motion, and the theory of infinitesimals and transfinite numbers. The evolution of strategies for teaching this concept is examined stretching from the traditional to computer-assisted environments up to and including research findings in the nascent dynamic software environments. Insights gained from the review of the literature are summarized and an initial learning trajectory proposed.

CHAPTER II

HISTORICAL DEVELOPMENT OF THE DERIVATIVE AND ITS TEACHING

2.1   Introduction

The purpose of this chapter is to trace the historical origins of the derivative in brief and the way it has been characterized and taught both traditionally and in computer assisted instruction up to and including the limited research in dynamic mathematics software environments. The chapter begins with an account of the dichotomy between the potential and the actual infinity in Greek thought, the vexing questions of Zeno's paradoxes of the infinity and how they were tackled in Greek thought, notably by Aristotle. The discovery of the derivative concept is outlined starting with mankind's struggle with the drawing of tangents to curves after many centuries of stagnation occasioned by the stranglehold of Aristotelian views. An account is made of mankind's attempts to understand the rate of change in the context of non-uniform motion and how that effort eventually led to the discovery of the calculus by Newton and Leibniz in the 17th century. The development of the theory of limits, infinitesimals and transfinite numbers is alluded to as a final extension of the understanding of notions foundational to the concept of derivative.

The review of related literature is extended to the teaching and learning of the derivative in the traditional context as the limit of a sequence of secant lines while highlighting the conflicting characterizations of the tangent between the Euclidean and function graphing situations. The chapter then concentrates on the use of computational technology (computer algebra systems) in modeling and simulating multiple representations of the derivative. Some limitations of the computational environments are highlighted to gain insight into possible obstacles that may be encountered when integrating technology into mathematics classrooms. Special attention is paid to the limited research in dynamic mathematics software environments.

Finally, insights derived from this review are summarized. As a consequence of the synthesis, an initial *Sketchpad* mediated learning trajectory is hypothesized tapping both

from the insights gained and the assumed enablements of the dynamic mathematics software. This chapter helps us to conjecture how mathematical instruction can be modeled or sequenced in a *Sketchpad* environment in order for the dual meaning of the derivative as the instantaneous rate of change and as the rate of change function to emerge. In other words the chapter helps us to answer the following research questions: How has the concept of derivative developed historically and what pedagogical lessons can we derive and apply to a dynamic mathematics software environment? What contribution can technology potentially make towards developing deeper conceptual understanding of calculus in general and the concept of the derivative in particular? More specifically, how can dynamic mathematics software be used as a tool or instrument to support the sequential modeling of the concept of derivative and what conditionalities have to be factored in?

2.2   A brief history of the derivative

2.2.1   The dichotomy between the potential infinity and the actual infinity

The understanding of infinity has a direct bearing on the understanding of the historical evolution of the concept of derivative. Hilbert remarks as follows:

> The infinite has moved the human mind like no other question since the earliest times; the infinite has brought about mental stimulus and fruitfulness like virtually no other idea; the infinite however needs clarification like no other concept (Strauss, 2001:20).

Dubinsky et al (2005a:34) stress that from Aristotle (384–322 B.C.) on, a key aspect of the concept of infinity has been the distinction between *potential infinity*, an ongoing activity that never ends, and *actual infinity*, a definite entity encompassing what was potential. Aristotle introduced this dichotomy in an attempt to deal with the paradoxes of the infinity that he believed could be resolved by refuting the existence of the actual infinity. Aristotle mentions Zeno's four arguments against motion. Unlike his predecessors, many of whom were interested in thinking about the infinite in metaphysical terms, Aristotle wanted to determine whether anything in space and time is

16

infinite and consequently elected to define the infinite, in pragmatic terms, as that which is untraversable. That is, that which requires an endless process, an infinite sequence of steps in which each succeeding step differs from all of its predecessors.

Strauss (2001:47) prefers to refer to the potential infinite as the *successive infinite* in order to reflect the determining role of the numerical time-order of succession. In line with this definition on the one hand, a circle, despite having no beginning and no ending points, in a sense analogous to time, was not considered to be infinite, because each successive traversal of its circumference is like the first. On the other hand, the natural numbers were considered to be potentially infinite, because in constructing them, the process of adding one always produces a successor which differs from all of its predecessors. Thus, although Aristotle accepted the existence of each natural number, he argued that the totality of all natural numbers was not traversable (it would require the whole of time) and so incomprehensible (unknowable) because our existence is constrained by time. Given the untraversability of the process of counting, for Aristotle, there could be no such thing as infinite quantity (Moore, 1999).

However, Aristotle could not reject the infinite completely, as its existence was self evident in a number of manifestations: time, which appears to be infinite both by division and by addition (and subtraction); matter, which, by virtue of its continuous nature, seems to be infinitely divisible; and space (or universe), whose expanse appears to be seamless. To reconcile the incongruities Aristotle defined two different notions of infinity, *potential* and *actual*. This allowed him to acknowledge the existence of the infinite, provided that it was not present "all at once" (ibid. p. 39). Aristotle defined *actual infinity* to be the infinite present at a moment in time. Strauss (2001:47) prefers to designate this as the *at once infinite* in accordance with the *spatial time-order of simultaneity*. The infinite sequences of natural numbers, integers and rational numbers would be viewed as actual infinities, - or infinite totalities given *at once*. Aristotle argued that if the infinite were to be grasped at all, it could only be understood as being presented over time, that is, as being a *potential infinity*. As far as he was concerned, all objections to the infinite were objections to actual infinity, and were therefore valid in his

17

opinion. On the other hand, potential infinity was seen as being a "fundamental feature of reality" and thus acceptable (Moore, 1995: 114). Aristotle believed that this distinction could be used to resolve Zeno's paradoxes. In particular, he argued that when something moves it does not move in a counting manner. (c.f. continuous vs discontinuous phenomena as objects of study)

2.2.2   Zeno's paradoxes

As seen above the history of the derivative concept dates back to Zeno's (490 –430 BC) paradoxes of the infinity, mainly that of the Arrow and that of Achilles and the tortoise. Dubinsky et al (2005a:336) contend that the question of thinking about the infinity has been of interest to mathematicians, philosophers of mathematics, and mathematical historians for at least 3000 years.   In the paradox of the *Arrow*, Zeno assumed that time is made up of infinitesimal instants, and argued that at any instant of time a moving arrow is either at rest or it is not at rest (Bell 1978:50). If an instant is indivisible, then the arrow is at rest, for if it were not, the instant would still be sub-divisible further. Therefore since the arrow cannot move at any single instant, it must always remain at rest, hence the paradox. That is, if the arrow cannot move in an instant of time then, what ever time it is given should be an aggregation of zero-motions which would be equal to non-motion. Ultimately, it would be impossible to determine the velocity at an instant (because and atomic instant of distance would also be zero).

The paradox of the Achilles and the tortoise tackles the notion of continuity and the infinite divisibility of a magnitude. According to Moore, in Dubinsky et al (2005a:335) in this conundrum the swift demigod challenges the slow tortoise to a race and grants her a head start. Before he can overtake her, he must reach the point at which she began by which time she will have advanced a little. Achilles must now make up the new distance separating them, but by the time he does so, she will have advanced again. And so on, *ad infinitum*. It seems that Achilles can never overtake the tortoise. In a sense Zeno argued that if a magnitude (line segment) is infinitely divisible, then motion is impossible since in order to traverse the line segment, it is necessary first to reach the mid-point and to do this one must first reach the one-quarter point and so on ad infinitum (Smith 1996:25). It

follows that since we cannot get to the very smallest (atomic) distance to be traversed, then motion is impossible. The derivative at an instant thus implies zero distance traversed over a zero time interval, which defies commonsense intuition.

Aristotle quoted by Strauss (2001:26) confronts Zeno's problem with the following argument:

> In the act of dividing a continuous distance into two halves, one point is used twice, since we can make it the starting-point and the end-point:… But if such divisions are made, *neither* the distance, *nor* the motion would be continuous; … and although that which is continuous contains an infinite number of halves, these are not actual but only potential halves.

Strauss further explains that Aristotle rejects the existence of the actually infinite on two grounds. First, if the actually infinite consists of parts then these parts must themselves be actually infinite, which would imply the absurdity that the whole is no longer larger that a part. Secondly, if the actually infinite consists of finite parts, this would imply the impossibility that the infinite can be counted, or there would have to be transfinite (cardinal) numbers which are neither even nor uneven. In fact, Aristotle's geometrical axiom that the whole is always greater than its part held sway for centuries, and became a stranglehold that contributed to the protracted delay in alternative interpretations. The stranglehold partly explains the belated discovery of the infinitesimal calculus.

2.2.3   The Drawing of Tangents

Eves (1976:315) contends that the concept of a derivative, the end product of differentiation, may have originated chiefly as a result of many centuries of effort in drawing tangents to curves and in finding minimum and maximum values of functions. Kepler (1571 – 1630) had observed that the increment of a function becomes vanishingly small in the neighbourhood of an ordinary maximum or minimum value. Fermat (1601 – 1665) in 1629 developed Kepler's idea into a process of determining such a maximum or minimum and thereby set forth the first serious attempt at differentiation. If $f(x)$ has an ordinary maximum or minimum at x, and if h is very small, then the value of $f(x – h)$ almost equals that of $f(x)$. We can therefore temporarily set $f(x - h) = f(x)$ and then make the equality correct by letting h assume the value zero. The roots of the resulting equation

then yield those values of x for which $f(x)$ is a minimum or a maximum. Although the logic of the method is not rigorous enough, it is seen that Fermat's expression is equivalent setting the limit $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = 0$, which is equivalent to setting the derivative of $f(x)$ equal to zero. However, the Fermat Method, as it was later called, was oblivious to the fact that the vanishing of the derivative is only a necessary but not sufficient condition for an ordinary maximum or minimum. Nor could Fermat distinguish between a maximum or minimum value in this method.

Eves (1976:316) further asserts that Fermat is generally credited with the invention of a general procedure for finding the tangent at a point of a curve whose Cartesian equation is given. His idea was to find the sub-tangent for the point, that is, the segment on the x-axis between the foot of the ordinate drawn to the point of contact and the intersection of the tangent line with the x-axis. The method uses the idea of a tangent as the limiting position of the secant line when both of its points of intersection with the curve coincide (see Figure 2.1) (compare 2.8.3).
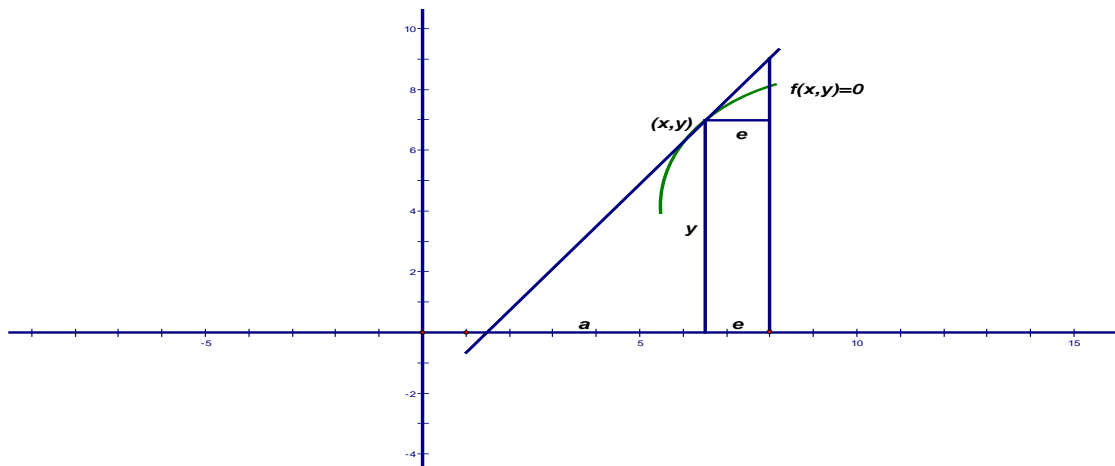
Figure 2.1: An Illustration of the Fermat Method

Using today's notation the method is as follows: Let the equation of the curve be $f(x,y)=0$, and let it be required to find the subtangent a of the curve for the general point (x,y). By similar triangles we find the coordinates of a point on the tangent in the

20

neighbourhood of the point of tangency to be $[x + e, y(1 + e/a)]$. This point is tentatively

treated as if it were also on the curve, yielding $f[x + e, y(1 + \frac{e}{a})]=0$. The equality is

then made correct by letting e assume the value zero and we then solve the resulting

equation for the subtangent in terms of the coordinates s and y of the point of contact.

Using this method Fermat was able to find the tangents to the ellipse, cycloid, cissoid,

conchoids, quadratix, and the folium of Descartes.

Eves, (ibid. p. 319) points out that in Isaac Barrow's (1630-1677) book we find a very

near approach to the modern process of differentiation, using the so-called differential

triangle, which we find in our present-day text books. In this method we let it be required

to find the tangent at a point P on the given curve in Figure 2.2. Let Q be a neighbouring
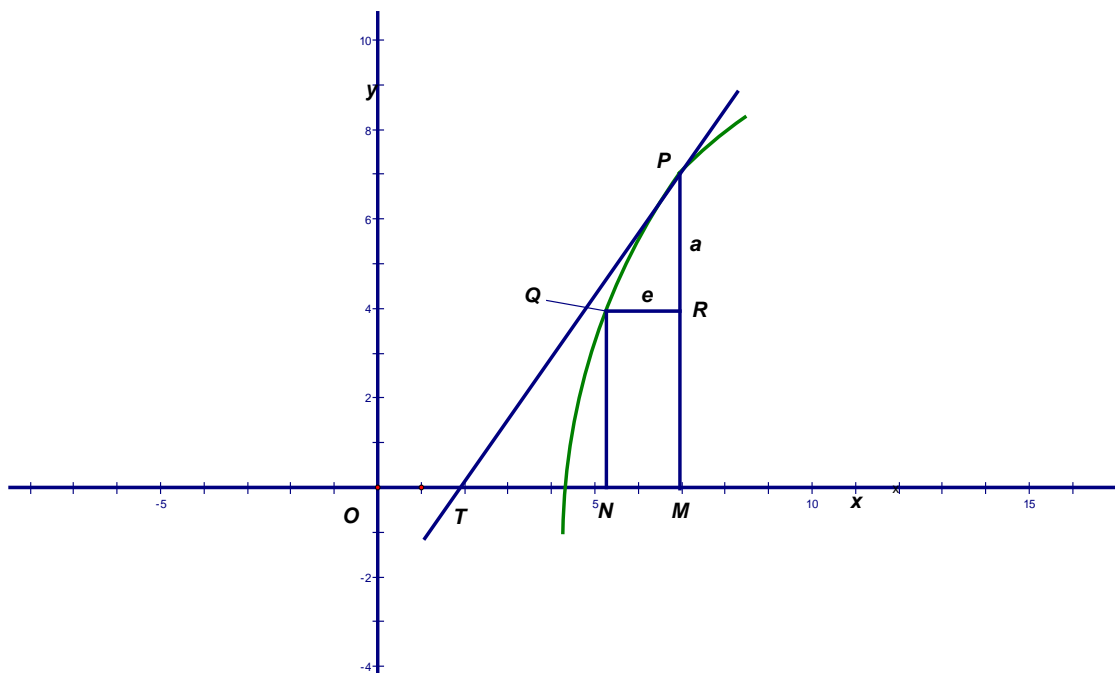
Figure 2.2: An Illustration of the Differential Triangle Method

point on the curve. Then triangles PTM and PQR are very nearly similar, and Barrow

argued that as the little triangle becomes indefinitely small, we obtain

$\frac{RP}{QR} = \frac{MP}{TM}$. Let us set $QR = e$ and $RP = a$. Then if the coordinates of P are x and y, those

of Q are $x - e$ and $y - a$. Substituting these values into the equation of the curve and neglecting squares and higher powers of both e and a, we obtain the ratio $\dfrac{a}{e}$, our

modern $\dfrac{dy}{dx}$. We then obtain $OT = OM - TM = OM - MP[\dfrac{QR}{RP}] = x - y[\dfrac{e}{a}]$, and the

tangent line is determined.   Using this method Barrow managed to construct tangents to the kappa curve, a special Lame curve, the folium of Descartes, the quadratrix and the tangent curve. Barrow's rudimentary procedure laid a solid foundation for Leibniz and Newton, and can be made more rigorous today by the use of the theory of limits which depends on infinitesimals.

2.2.4 The theory of infinitesimals

As we saw earlier, the infinity has fascinated mankind since time immemorial and Zeno revealed that, whether we consider space and time to be infinitely divisible or consisting of tiny indivisible atoms, in both cases paradoxes appear. Despite this uncomfortable problem, Tall and Tirosh (2001:199) note that practical mathematicians continued to use a range of infinitesimal and indivisible methods of calculation through to the 17th century development of the calculus and beyond. They contend that even at the beginning of the 19th century, infinitesimal methods were still widely used. To clarify the complexity of the infinitesimals (or infinitely small numbers) they note that Dedekind's construction of the real numbers suggested that the real line consists only of rationals and irrationals. That is, Dedekind's cuts of **R** 'completed' the real line by adding irrational numbers to fill the gaps between the rational numbers, leaving no room for infinitesimal quantities. As if this were not enough, the arithmetization of analysis by Riemann confirmed that there was no number α that could be arbitrarily small, for if $0 < \alpha < r$ for all positive reals r, then ½α is positive and even smaller than α. (c.f. second foundational crisis in the history of mathematics (Strauss, 2001)).

George Cantor's arithmetic of cardinals also had no use of infinitesimals since infinite cardinals do not have multiplicative inverses (ibid). By the beginning of the 20th century infinitesimal ideas were theoretically under attack, but, as already alluded to (see 2.2.1) they still continued to flourish in the practical world of engineering and science

representing a variable that could become arbitrarily small. Abraham Robinson's theory of 'non-standard analysis' attempted to reformulate infinitesimals on a logical basis but invoked the axiom of choice to assert that such entities existed without being able to give a specific finite construction. Although the axiom gives inverses to infinitesimals, it fails to do so for cardinal infinities. As a consequence of these shortcomings there is no universal acceptance of the axiom.

Dubinsky, Weller, McDonald and Brown (2005b:263) interpret the development of the derivative concept in terms of APOS theory and point out that historically the controversy regarding infinitesimals centered around the nature of the 'small' increment o in the difference quotient $f(x+o)- f(x) : o$. How could o be both regarded as nonzero, so that it was permissible to divide by it, and then discarded because it has no contributing value later in the calculation? In APOS parlance, the limit L of a function f at a domain point a can be viewed as the coordination of two processes, a domain process and a range process. These processes are coordinated by the function in that the domain process x is transformed by f to the range process $f(x)$. The domain point a is taken to be a transcendent object of the domain process, while the limit L is the transcendent object of the range process.

Newton quoted by Dubinsky et al (ibid) comments that:

> *. . .* those ultimate ratios with which quantities vanish, are not truly the ratios of ultimate quantities, but limits towards which the ratios of quantities, decreasing without limit, do always converge; and to which they approach nearer than by any given difference, but never go beyond, nor in effect attain to, till the quantities are diminished *in infinitum*.

This comment suggests that the quantities which form the denominator represent the domain process and the ratios themselves represent the corresponding range process. Thus, in terms of an APOS analysis, it is postulated that Newton intended o and $f(x + o) - f(x)$ to represent domain processes of approaching 0 and, in turn, the difference quotient as representing a range process of approaching an "ultimate ratio." But his critics, such as Berkeley, with his metaphor of "ghosts of departed quantities",

23

insisted that *o* and the difference quotient itself must always be viewed as static objects. This has a familiar ring with the *at once* infinity alluded to in 2.2.1 above. However, from an APOS theory perspective such a stance evinces failure to distinguish between an object directly produced by the process and an object that is brought into being by encapsulating the process as a result of applying the action "What is the ultimate value of the range process?" (Compare with Tall's 'procept' in 4.3.1).

From APOS theory and using more modern terminology, Dubinsky et al (ibid) argue that in the expression $f(x + \Delta x) - f(x),$ the symbol Δx represents a process. The action of evaluating the ultimate value of this process leads to its encapsulation and the selection of 0 as the transcendent object. The process of Δx approaching zero is coordinated (or better still, transformed) by f and the difference quotient to obtain a new process (approaching L). As one imagines the completion of this latter process, it can be thought of in its totality (actual infinity), and the action of determining the ultimate value of the quotient leads to an encapsulation. The 'ultimate ratio' (Newton's phrase), which results from this encapsulation, is then the derivative.

Dubinsky et al (ibid) further argue that because the derivative stands outside of, and reflects the totality of the difference quotient process (differentiation), the issue of whether Berkeley's ultimate "evanescent increment" is finite, or infinitely small, or "yet nothing" is moot. More specifically, the derivative is not determined by computing the quotient. Rather, it represents a value toward which all of the intermediate states of the process point. This is consistent with Newton's view of the derivative as the ratio of the quantities, not before they vanish, nor afterwards, but with which they vanish (ibid p. 263). It seems, though, as if the term 'process' is rather loosely used where in fact 'transformation' would probably be a more mathematical term in the sense of Kieran's transformational (rule-based) activities referred to by Lagrange (2005:149).

2.2.5   The Emergence of the Derivative in the Context of Motion.

As with the case of Zeno's paradoxes referred to earlier (see 2.2.2), the notion of derivative also arose in the context of motion. Dolan (1997:121) specifically asserts that the concept of a derivative (also) arose chiefly as a result of finding velocities of bodies

in non-uniform motion. How, for example, could the velocity of a uniformly accelerating body be ascertained at a given moment? Gravemeijer and Doorman (1999:120) travel back in time to the days of the 14th century emergence of kinematics at Merton College, Oxford. They contend that logicians and mathematicians of the time strove to find a description of the distance traversed by a body moving with uniformly accelerated motion. They acknowledge that this problem was not easy given that the velocity of the body constantly changes and that there was no universal understanding of the concept of motion and change of motion (derivative).

In spite of the conceptual obstacles, the logicians came up with the Merton Rule (Graqvemeijer & Doorman, 1999) which stated that when the velocity of an object increases uniformly, from zero to a velocity v in a time interval t , then the distance traveled is equal to half the distance traversed at constant velocity v in that time interval. Conversely, they could implicitly ascertain the velocity at the end of the interval (velocity at that instant). In the 17th century Galileo (1564-1643) applied the Merton Rule and experimented with objects in free fall to explain the relationship between time and distance as well as time and velocity at time t after the start of free fall. Galileo is credited with the discovery that aside from air resistance, light objects drop just as fast as heavy ones. He made careful measurements of time intervals that it took balls of different sizes to roll down ramps and estimated their velocities at different points. (see 3.2 for experimentation in mathematics)

According to Strauss (2001:29) Newton's first calculus of 1665 – 1666 seems to have been abstracted from the intuitive ideas of motion wherein a curve in the Cartesian plane was imagined as traced by the motion of a 'flowing point'. The 'infinitely short' path traced by the point in an infinitely short' time was called the 'momentum' (c.f. infinitesimal or atomic instant in 2.2.2) and this momentum divided by the infinitely short time was the 'fluxion'. This fluxion is nothing other than the elusive notion of instantaneous speed referred to earlier (see 1.1), the derivative describing or defining the changes in y- (the ordinate fluent or flowing point) with respect to changes in x- (the abscissa fluent or flowing point).

A Zeno like explanation of the difficulty in defining and calculating instantaneous velocity was given by Morris Kline who noted that if the distance an automobile travels in one instant is zero and the time that elapses during one instant is also zero then the velocity at an instant is 0/0, which is meaningless (Anton, 1999:171). Thus although instantaneous velocity is a physical reality, there seems to be an enduring difficulty in calculating it. Once the notion of instantaneous rate of change has been grasped it should be relatively easier to grasp the rate of change of function notion.

2.3   The traditional approach to the teaching of the derivative

The traditional teaching of the derivative has focused on the graphical ideas of rate of change and the manipulation of rules of differentiation. The initial stages usually begin with informal ideas of the limit concept in geometric, numeric and symbol form. The overemphasis on symbolism is part of broader traditional approach to algebra which Huntley, Rasmusen, Villarubi, Sangtong and Fey (2000:349) consider to have characteristically focused almost exclusively on the use of symbolic expressions to represent operations and relationships involving quantitative variables. As already highlighted earlier (compare 1.1), a major drawback of the traditional calculus curriculum has been its emphasis on rote learning of formulae and mastery of procedural manipulation of symbols to obtain correct answers to be verified with the teacher and textbook as the sole authorities. In concurrence, Tall (1997:289) notes that most traditional calculus texts include a large number of worked examples and exercises to satisfy even the most anxious student, yet the heavy diet of procedural exercises produces high failure rates.

The casting of mathematics within set theory and axiomatics during the New Mathematics Movement meant an exposure of students to the finished products of mathematics. Yet according to Hart (1983:52) the brain was designed by evolution to deal with *natural complexity*, not neat logical simplicity. In other words, the introduction of the derivative via the formal epsilon-delta concept definition of the limit is an

antididactical inversion in Freudenthal's (1983) parlance. In similar vein, Blum (1998:1) also clearly points out that formal proofs are mostly the final stage in a genetic development - historically as well as epistemologically as well as psychologically. Tall and Tirosh (2001:201) are even more rhetorical in declaring that to begin a calculus course with a definition of limit may be logically constructive but pedagogically destructive. That is, rigor for rigor's sake is more likely to defeat rather than redeem the student. But of course, these observations leave unanswered why some students have succeeded nonetheless. Tall (2006:202) explains that on the whole, as the mathematics becomes more sophisticated, more successful students tend to focus increasingly on the power of symbolism than on the sensory meaning of the embodiments. Unraveling why the 'successful students' succeed without understanding should equip us better to improve our instruction in order to bridge the conceptual gaps.

Akkoc and Tall (2005:1) similarly observe that one of the difficulties encountered by students learning mathematics in the traditional curriculum is that the logical development of the subject is not the same as the cognitive development of the students. In this connection the authors lean back on Skemp (1971) who recommended that we should teach the process of mathematical thinking rather than the product of mathematical thought. They further attest that curriculum design based on overall structural principles such as supplying students with proper set-theoretic definitions of functions hasn't been shown to work because the students don't seem to use them. Instead, students remember the experiences of what they do that proves successful in accompanying definitions rather than what they are told to remember as a formal definition. This is a reminder of the Chinese cliché: I hear I forget, I see I remember, I do and I know.

For those traditional texts that adopted the intuitive approach to the calculus (e.g. British schools), Mamona-Downs (1990) confirms previous research suggesting that the intuitive approach suffers from side-effects which clash with the formal definitions. For instance, the use of the tangent slope as the gateway to the derivative concept requires adjustment or expansion of meaning beyond the circle geometry definition as a line that touches a circle at one point, given that in calculus a tangent at a local maximum or local minimum

might cross the curve at another point. At a point of inflexion the traditional intuitive meaning of tangent is completely violated as the 'so-called' tangent line actually crosses the graph. Figure 2.3 shows some instances in which the tangent line can behave differently in the graphing of functions.

a) Tangent crossing curve on the right hand side

b) Tangent crossing curve at a point of inflexion

Figure 2.3: Tangents that cross a curve
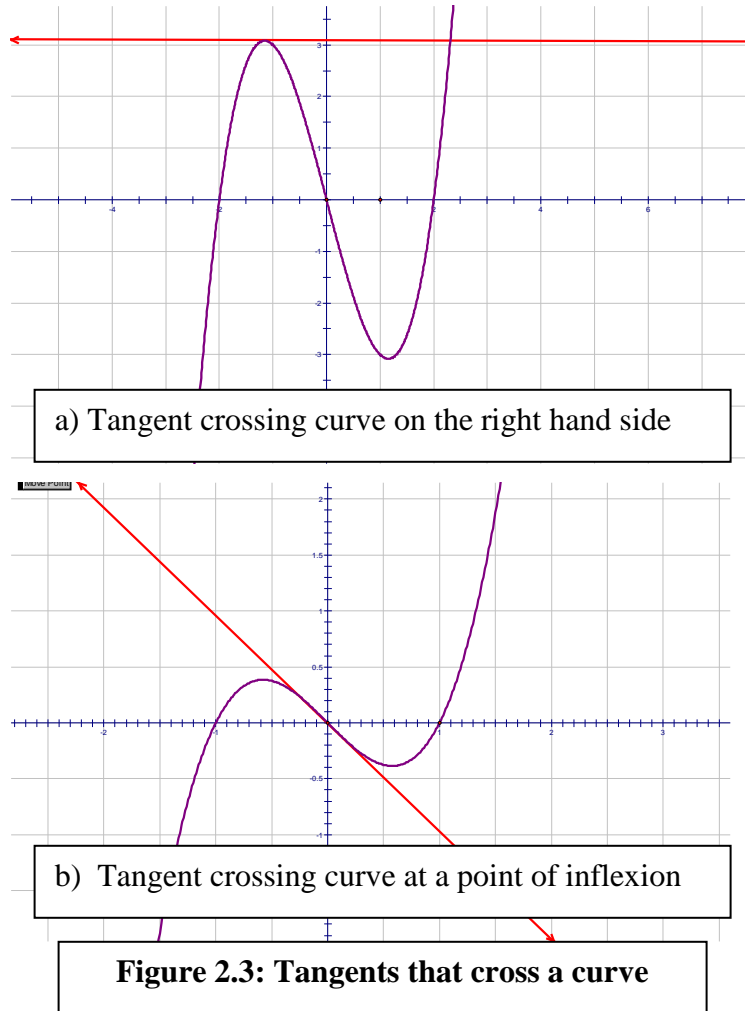
Vinner (1983) similarly observes that the circle geometry notion of the tangent produces a concept image that causes cognitive conflict not only in the cases cited above but also at the origin of a cusp graph where the existence or non-existence of a tangent is a contentious issue even among mathematicians. (See Figure 2.4).

$f(x) = \sqrt{|x|}$

Figure 2.4: (Non) Differentiability of the cusp graph of $y = \sqrt{(|x|)}$
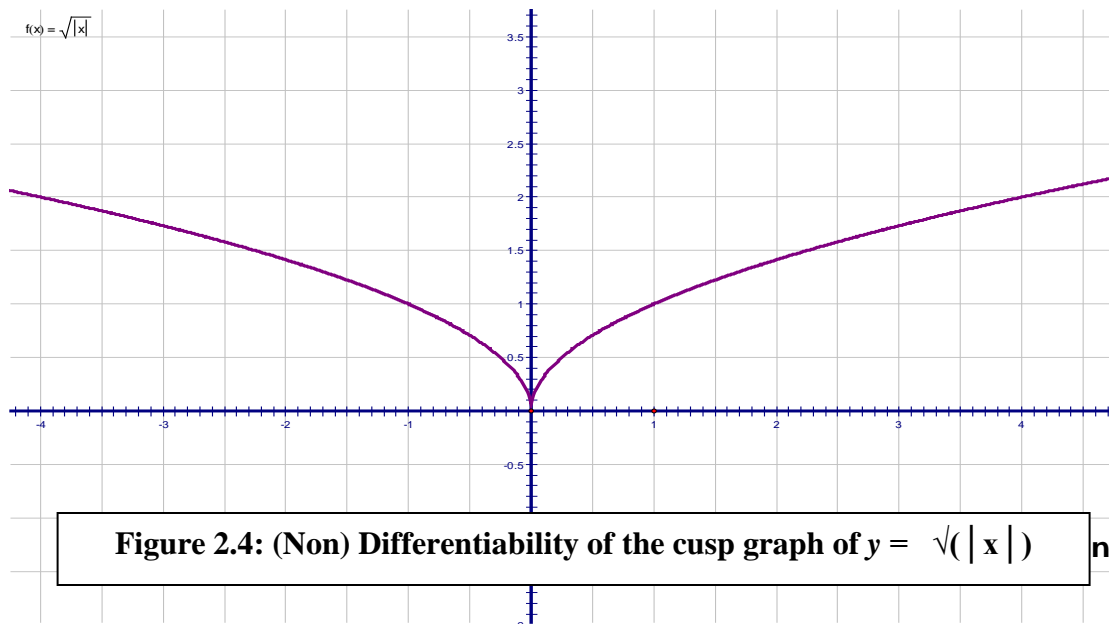
A major limitation of the traditional approach in representing the dynamic notion of derivative is the static nature of the textbook environment. The tangent as the limiting position of a convergent sequence of secant lines can only practically be represented with a finite number of secant lines in a single diagram or limited number of diagrams on the chalkboard. The conflict is that a continuous sequence of secant lines is discretely represented thus undermining the limit concept by which the derivative is defined. The potential infinity of infinitesimals is indiscreetly substituted by the cardinal infinity of Cantor which Tall and Tirosh (2001:202) acknowledge to be a notion of infinity fundamentally divorced from the idea of continuity. Put differently, an attempt to represent a continuous or dynamic concept by discrete or static means is, to a fundamental extent inherently flawed and likely to foster limited concept images of a mathematical object.

Under the traditional approach the different representations of the derivative (symbolic/ algebraic, numeric and graphic) do not always have links between them in the minds of learners. In a tribute to Skemp, Tall (2002:157) documents a study where he found that in a traditional class that he investigated students could perform the followings: they could show that secant slopes of $y = x^2$ tended to 2 numerically at $x = 1$, they could algebraically deduce the derivative as $y = 2x$ from the limit of $(x + \delta x)^2 - x^2$ divided by δx could obtain the general formula for calculating the derivative for polynomials but none

could give any relational explanation of the process. The Calculus Reform Movement sought to remedy such shortcomings of the traditional approach to make mathematics more accessible to a broader spectrum of students.

2.4   The Calculus Education Reform Movement and the Impact of Technology

Pierce and Stacey (2004:59) indicate that since their development in the 1970s and their introduction into tertiary teaching in the 1980s, the powerful technologies of computer algebra systems (CAS) have been recognized as highly valuable for doing mathematics and as potentially valuable for teaching and learning mathematics. For instance, CAS such as MACSYMA, which could simplify, factor or expand expressions, solve equations analytically or numerically, differentiate, compute definite and indefinite integrals, expand functions in Taylor or Laurent series, etc marked a turning point in calculus education. The prevailing view that, for the overwhelming majority of students, the calculus is not a body of knowledge but a repertoire of irrational imitative behaviour patterns led some to question the whole wisdom of burdening students if computer software can do all the things required of a student in an examination.

With the rallying call and subsequent proliferation of newer computer approaches (such as *MuMath, Derive, Random Grapher, etc*) curriculum designers were challenged to investigate the use of computers in mathematics in general and calculus in particular to relieve students of the perceived drudgery of traditional calculus, the Calculus Reform Movement began in the USA. While Douglas (1986) called for reform to *A Lean and Lively Calculus,* Steen (1988) called for a *Calculus for the New Century: A Pump not a Filter* to make calculus more understandable and a genuine driving force for learning instead of a bottleneck which weeded out poor students. The revolution in technology thus stimulated and catalyzed calculus reform.

Huntley et al (2000:349) share this view when they point out that one of the principal arguments for reform of the traditional approach is the conjecture that the integrated numeric, graphic and symbolic tools of modern calculators and computers provide

powerful new ways of learning and doing mathematics. The National Council of Teachers of Mathematics (NCTM) had already lent full support when it argued that students who become fluent in applying and translating among those multiple representations 'will have at once a powerful, flexible set of tools for solving problems and a deeper appreciation of consistency and beauty of mathematics' (NCTM, 1989:146). Huntley et al (2000:350) further acknowledge that one of the most important research problems raised by current reform proposals is about understanding the connections between development of conceptual and procedural knowledge of calculus. With the increased availability of numeric, graphic and symbolic tools, for instance, many mathematics educators have suggested that students might need to know only how to plan and interpret algebraic calculations, not to be proficient in the procedures themselves (Huntley et al, 2000; Stroup, 2002). Studies involving both tertiary and secondary mathematics classes (for example Heid, 1988; Atkins, Creegan & Soan, 1995; Lagrange, 2005) have also supported the contention that the symbolic manipulation features of CAS can free students from manipulation errors and thus allow them to quickly generate both exact and approximate results.

The proliferation of symbolic and graphic interfaces thus spurred practicing mathematicians and mathematics educators to engage in experimental research in the wide range of possibilities in the teaching and learning of calculus that this new technology offered. Apart from the altruistic educational goal of making calculus more accessible to a greater number of students, there were other motivating factors. Tall (1997:284) points out that there were also, *inter alia*, commercial desires to produce marketable products, practical considerations of what mathematics or calculus actually needed to be taught, and a growing aspiration to research the learning process itself in order to understand how individuals conceptualize calculus. In a similar sense Artique (2002:245) remarks that the sophisticated new tools such as Computer Algebra System (CAS) do not become immediately efficient mathematical instruments for the user. In the light of this observation, this study attempts to instrument *Sketchpad*, which has many of the CAS properties, to enrich the qualitative or conceptual understanding of the derivative.

Lagrange (2005:147) points out that software for doing mathematics can roughly be separated into numerical/graphical applications – based on approximate arithmetic, and computer algebra – performing exact symbolic calculations. He then analyses specific support to classroom practices that can be expected from each and how they fit into curriculum and practices. The arrival of the computer graphical interface which allows the user to interact in a physical way by pointing selecting and dragging or animating objects onscreen gives the possibility of extending the embodied context of real world calculus. Symbol manipulators such as *Mathematica, Maple* and *Derive* have the capacity to carry out the algorithms of calculus on behalf of the user. These applications have a largely symbolic interface, producing static graphic output on the screen, but with no dynamic character in earlier versions. Modern calculus reforms (both within and outside the USA) seek to make the computer representations of calculus concepts more practical and enactive. In this direction we now have dynamic calculators such as the TI-92 calculator, and dynamic software *Cabri II* and the *Geometer's Sketchpad* that have animation facilities which their forerunners did not have in earlier versions. This study focuses on research in dynamic mathematics software environments such as *Sketchpad* and explores and exploits the effectiveness of the same in representing the concept of derivative.

2.5   Research on the teaching of the derivative in dynamic software environments

2.5.1   Magnification of a graph to investigate its local straightness (slope).

Using his own program *SuperZoom*, in the Supergraph package, Tall (1985) discloses that to draw tangents this program draws a straight line through two close points on a curve (*s, f(x)*) and (*a + s, f(a + s)*) for *s* + 0,001. He refers to this as the 'practical tangent' since it is calculated with sufficient precision to satisfy the limited requirements of the visual display and this works satisfactorily for most standard functions.

Under high magnification the tangent and the graph seem to merge into one line attaining 'local straightness'. He further develops this magnification facility as an alternative to the traditional idea of featuring the tangent as touching one single point and argues, as

before, that the graph of a differentiable function 'looks straight' when highly magnified (Tall, 2000). He asserts that the local straightness is a primitive human perception of the visual aspects of a graph which is closely connected to the way an individual looks along a graph and apprehends the changes in gradient and 'sees' them as he follows the curve with his eyes, in a sense similar to how, through sophisticated interpretation one can 'see' infinitesimals on the real number line (Tall & Tirosh, 2001:202).

This presumably helps the student 'to see' the gradient function – the derivative as the rate of change. Further investigations might be necessary to check whether students really manage to 'see' as envisaged. It is then suggested that this embodied approach is suitable as a cognitive root for the concept of derivative. Tall (2000) further recommends that students be given guidance over this point because they easily see that a curve has a gradient at those points where it magnifies to look straight but drawing the derivative function on the same axis as its original function might help students to compare and intuitively deduce, for example that the derivative of 'cos x' is the graph of 'sin x' upside down, suggesting the gradient is '–sin x' (See Figure 2.5).
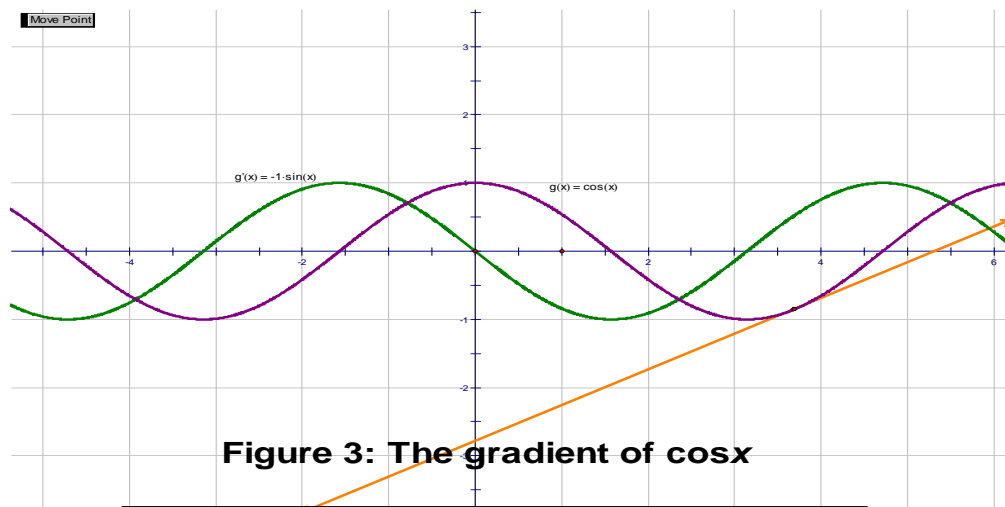
Figure 3: The gradient of cos*x*

Figure 2.5: The gradient graph of cos *x*

In this way, the gradient function (derivative) concept is apparently grasped even without

the necessity of the squeeze theorem and its proof that $\lim\limits_{h \to 0} \dfrac{\sin h}{h} = 1$.

33

The idea of local straightness under magnification can also be extended and used to investigate differentiability of a function at a point. With this tool the differentiability of the cusp at the origin, and differentiability at corners and points of discontinuity can be investigated. In summary, Tall (2003:23) concludes that local straightness is an embodied foundation (cognitive root) for the calculus while local slope of the graph as rate of change is an embodied foundation for the slope function (derivative). That is, the embodied approach as espoused presumably has the necessary conceptual power to lead to a potentially meaningful understanding of the symbolism of calculus and the axiomatic foundation of analysis.

In similar spirit it is pertinent to note that there is a broad consensus in the Calculus Reform Movement that heuristics rather than formal proof should be used as a prelude to formal axiomatics. In this regard Wu (2006:4) seems to affirm that while it is desirable to aspire to make every student learn what a proof is, it would be a grave mistake to insist that every statement in mathematics, up to and including calculus, be given a proof. In other words, there is no reason to impose the kind of training designed for future professional mathematicians on the average non-mathematics major student. In particular, he argues that epsilon-delta proofs may be best reserved for honours calculus. What appears important, however, is to give students adequate training in making logical deductions. This stance is compatible with de Villiers'(1999) distinction of explanation as a function of proof where proof may be presented as a means of explaining why results are true, even though one may already be convinced by graphical or other evidence. These need not be the formal, systematized proofs of real analysis, but can be localized in terms of intuitively acceptable assumptions.
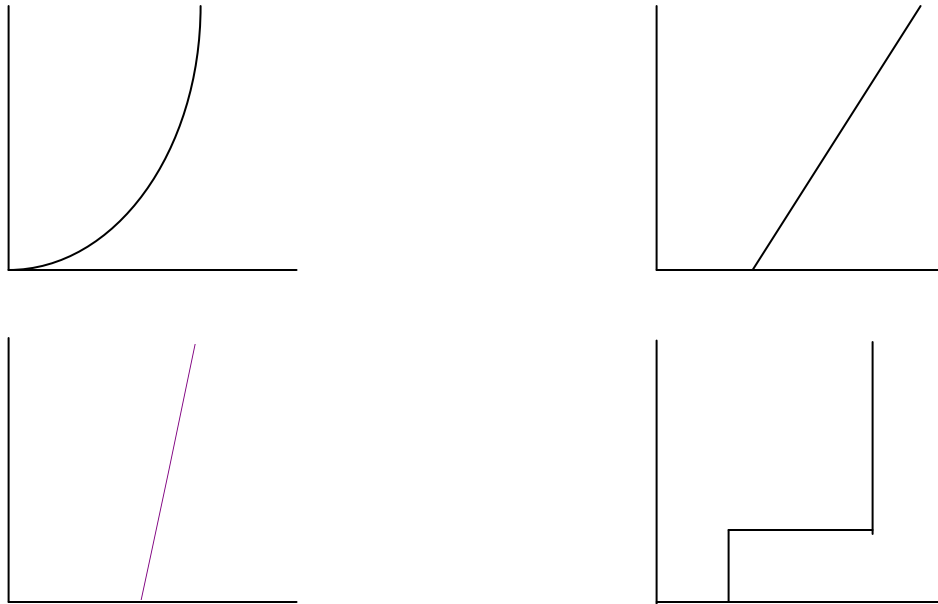
Figure 2.6: A theoretical-computational conflict on local magnification

In expressing a similar concern, Giraldo and Carvalho (2002:1) define *theoretical-computational conflict* to be any situation in which a computational representation for a mathematical concept is (at least potentially) contradictory with the associated theoretical formulation. Figure 2.6 shows an example of theoretical-computational conflict experienced in the local magnification of the graph of $y=x^2$ performed by *Maple* around the point $x_0=1$. Since the curve is differentiable, it should be locally straight when highly magnified. However, due to floating point errors for very small values of graphic window ranges it looks so disfigured as to resemble a polygon. Much the same is expected of a graphing calculator, which has low resolution.

Although *Sketchpad* does not disfigure the graph of this function to the same extent as does *Maple*, it has its own magnification problems when it comes to investigating the differentiability of oscillating functions like $x \sin\left(\dfrac{1}{x}\right)$ where the magnified graph appears differentiable at the origin but the animation of a constructed point around this vicinity more accurately reveals the oscillation than the function plot itself (see Figure 2.7).
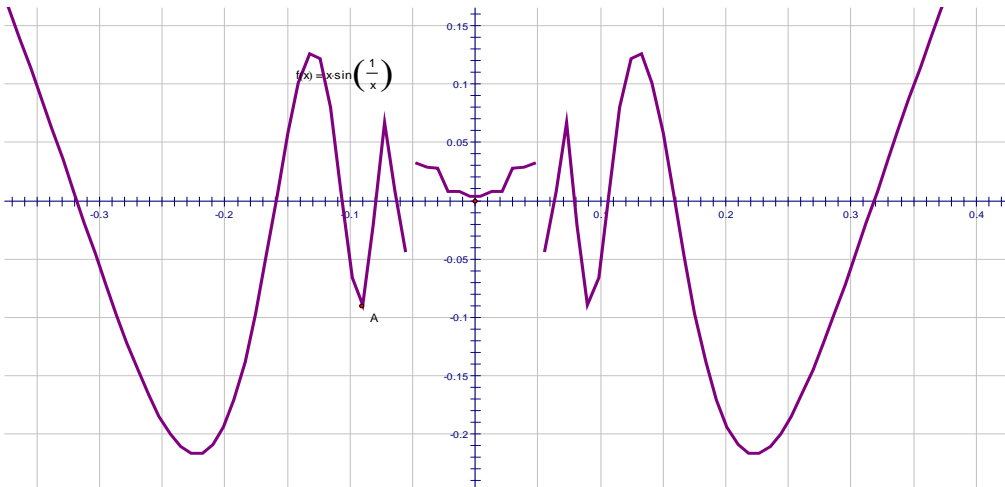
Figure 2.7: Magnification of the graph of $x \sin\left(\dfrac{1}{x}\right)$ **at the origin**

In a more or less similar situation, Giraldo et al (2003) report on a study of six first year undergraduate students in Brazil who were given two representations for the function $h(\mathrm{x}) = \sqrt{(x^2 + 1)}$, namely the algebraic formula and the graph sketched by *Maple* for $(x,y) \in [\text{-}100,100]^2$. Figure 2.6 shows the same graph generated from *Sketchpad*.
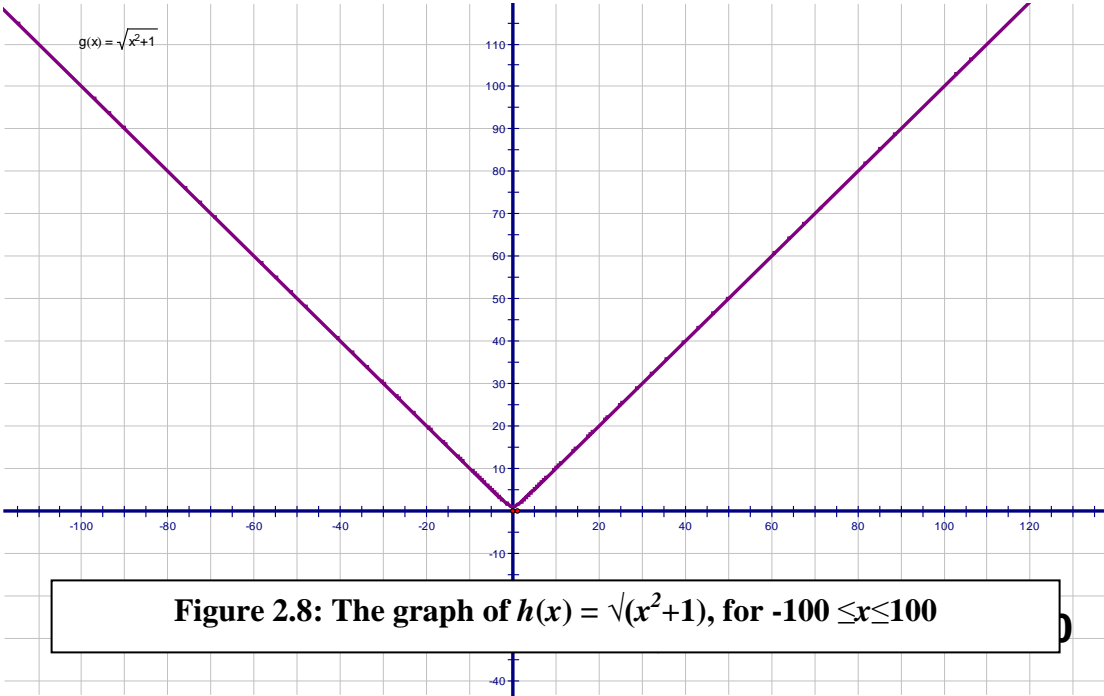
Figure 2.8: The graph of $h(x) = \sqrt{(x^2+1)}$**, for -100** $\leq x \leq$**100**

36

Due to the choice of a particular scale, the curve acquired the shape of two line segments with ends at the origin (in fact its asymptotes). The conflict in this case occurs between the curve displayed on the screen, which seemed to have a "corner", and the algebraic expression, which suggested the function was differentiable. In this case magnification revealed that the graph was actually curved and locally straight (see Figure 2.9).

$g(x) = \sqrt{x^2+1}$
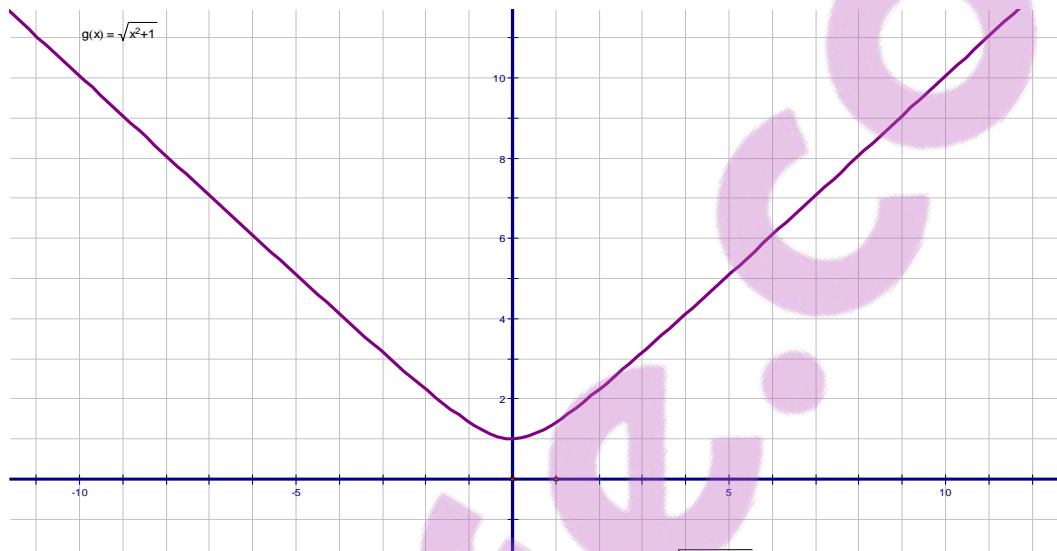
Figure 2.9: The magnified graph of $h(x) = \sqrt{(x^2+1)}$, -10 ≤x≤10

It is therefore sensible to link the magnification process with the symbolic formulae at the same time so that the visual insight supports the symbolization used for more sophisticated manipulations and computations. The reverse is also of critical importance, as noted earlier - symbolization being used to check the authenticity or mathematical faithfulness of a visual representation. Approaches such as this have been adopted in syllabuses in the UK and elsewhere (Tall, 2006).

Further insights can be gained from magnification to support complex ideas, which in the past could not be easy for beginning calculus students to imagine or convey. For example, the graph of a nowhere differentiable function such as the blancmange function looks rough at every level of magnification (Figure 2.10). The Blancmange function, also called the Takagi fractal curve, is a pathologically continuous function, which is nowhere differentiable.
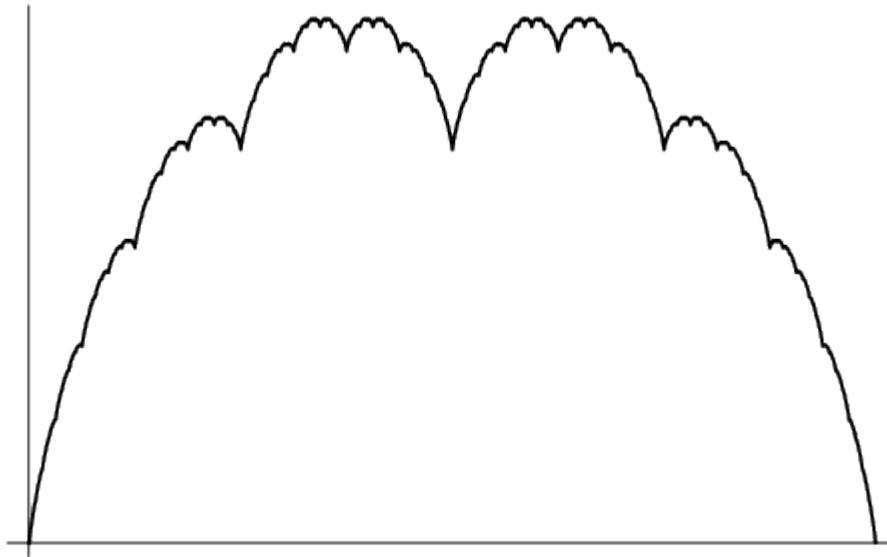
Figure 2.10: The Blancmange Function

Software representation anomalies are also reported by Belfort and Guimaraes (1998) in *Sketchpad* optimization activities to demonstrate the practical usefulness of the derivative concept. Teachers' behaviour was observed in an activity to empirically find the rectangle with perimeter 40m and largest area possible. The software permitted users to construct a rectangle, gradually varying the measurements of the sides while keeping the perimeter fixed. The teachers had to observe consequent variation of the area. Due to the built in floating point arithmetic the software could only generate approximate results. For example, when one of the teachers found a rectangle with area 100 m^2, sides 10.03 m and 9.97 m he is reported to have accepted the result given by the software as conclusive.

2.5.2   Building the gradient function (derivative) of a function

As one of the pioneering studies in the use of technological tools to enhance the teaching of mathematics, Heid (1988) showed how a Computer Algebra System (CAS) can be used to facilitate the development of calculus concepts among first year university students. The experimental condition entailed the use of a CAS to build up a rich concept image of the derivative by using graphs and combining representations. Techniques of differentiation were not introduced until the end of the course. The experimental group students demonstrated a good conceptual understanding of the derivative and performed

not worse than the control group on the technical part of the post-test. The results of this 'concept first' course seem to indicate that the development of concepts can precede the learning of techniques. As a consequence Heid conjectures that the use of a CAS might provoke a resequencing of concepts and skills in a calculus course.

Tall (1997:300) shows that it is possible to build up the gradient function of a locally straight graph by computing the numerical gradient between x and $x + c$, for small c at selected points along the graph, and plot them to be able to 'see' the graph and experimentally conjecture its formula (Figure 2.11). It seems as if by 'to see', Tall at this stage is referring to seeing plotted points and the shape they form, in which case this visualization is still partly at thought experimental level and not a dynamic facility of the software.

Figure 2.11 Building the gradient function of sin$x$
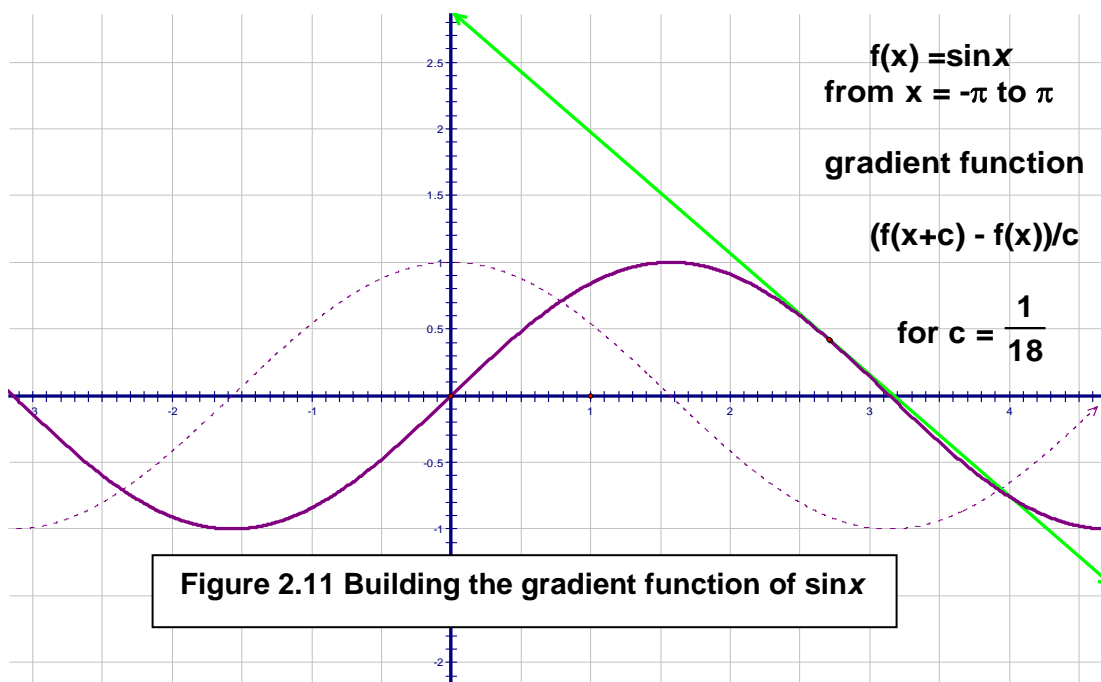
With this in mind, Tall (2003) proceeds to compare and contrast embodied local straightness with mathematical local linearity. He suggests on the one hand that 'local straightness' is a primitive human perception of the visual aspects of a graph and has global implications as the individual 'looks' along the graph and 'sees' the changes in gradient so that the gradient of the whole graph is seen as a *global entity*. On the other hand local linearity is a *symbolic linear approximation* of the slope at a *single point* on

the graph, having a linear function approximating the graph at that point. That is, local linearity is a mathematical formulation of slope, taken first as a limit at a point *x* to give the formal derivative as a function. Tall suggests that local straightness remains at an embodied level but links readily to visualizing the slope of a given graph while local linearity focuses explicitly on the 'best' local linear approximation expressed symbolically.

As this local linearity varies, its rate of change yields the gradient function, which can be symbolically represented. For instance, the derivative (function) of cos *x* as in Figure 2.3, is 'seen' as the graph of sin *x* turned upside down. Figure 2.12 visually compares the gradient function of cos *x* with that of sin *x*.
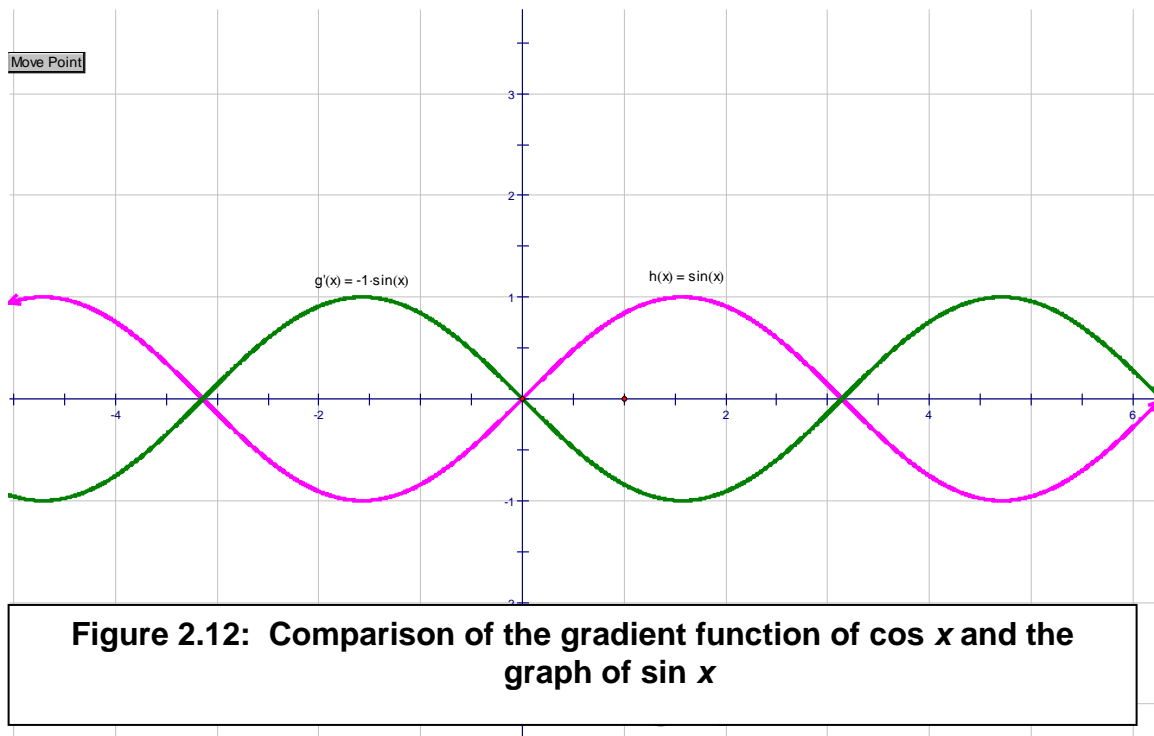
Figure 2.12: Comparison of the gradient function of cos *x* and the graph of sin *x*

Tall (ibid) emphasizes that this comparison does not amount to a proof in a formal sense. From his own research, however, he concludes that the symbolic proof of the derivative by finding the limit

$\lim_{h \to 0} \dfrac{cos(x + h) - cos\ x}{h}$ is rarely convincing to students. He notes that in practice the proof is, instead, based on the use of trigonometric formulae which are not 'proved' symbolically at this stage and on an *ad hoc* argument (usually presented visually and numerically) that $\lim_{h \to 0} \dfrac{sin\ x}{x} = 1.$ (Squeeze Theorem).

He reiterates that an embodied experience with meaning would be more appropriate at this introductory point while leaving the more manipulative and formal aspects to a later stage when the students have more chance of making sense.

It is clear that Tall's theorization is limited by the affordances of the software he used as reference points at the time, (*Supergraph, Maple, Random Grapher, and Function Probe).* The meaning of 'enactive' or 'perceptual control' in question in his discourse appears to be circumscribed to static graphical representations and plotting of the numerical chord from *x* to *x + h* in order to plot the gradient of the curve as *x* 'moves along' the curve in discrete steps rather than as continuous motion. In this regard he refers, for example, to Confrey's (1992) *Function Probe* which links with alternative forms of representation by allowing graphs to be manipulated enactively using the mouse to transform graphs by translating, stretching and reflecting. That is, the dragging and animation capabilities of *Sketchpad* 4, *Cabri II* or *Maple 10* might not have been accounted for let alone the colour effects. It is gratifying though that Tall (2003) is fully aware of the time lag between the pace at which technology is developed and the full use for educational purposes. For example, he notes that the fledging use of numeric programming and graphic visualization was overtaken by the power of computer algebra systems at a time when the power of enactive interface was still to be fully understood.

By enactive experience, however, Tall (1997:291) goes beyond graphs in acknowledging that by simulating relationships between time distance and velocity Kaput's *MathCars* software fosters an intuitive sense of concepts such as distance, velocity, acceleration that can enable the study of aspects of calculus at a far earlier age. Moreover, the simulations

involved, such as driving a car along a highway - linked to numeric and graphic displays of distance and velocity against time – allow a study of change which is not limited to functions given by standard formulae.

2.5.3   Further Research in Computer Aided Learning of Calculus Concepts

Kendal and Stacey (2001) examined how two teachers taught differentiation using a handheld computer algebra system (CAS), which made numerical, graphical and symbolic representations of the derivative readily available. (*Sketchpad* is also capable of these representations). The teachers planned the lessons together but taught their Year 1 classes in very different ways. They had fundamentally different conceptions of mathematics which influenced their teaching practices, innate 'privileging' of representations and of technology use (compare with teacher's philosophy of mathematics in 3.2). The study then linked these instructional differences to different differentiation competencies of the students who acquired them. Students of the teacher who privileged conceptual understanding and student construction of meaning were more able to interpret derivatives. Students of the teacher who privileged performance of routines made better use of the CAS for solving routine problems. Comparison of results with an earlier study showed that although each teacher's teaching approach was stable over two years, each used technology differently with further experience of CAS. The teacher who stressed understanding moved away from using CAS, whilst the teacher who stressed rules adopted it more. The study highlights that within similar overall attainments on student tests, there can be substantial variations of what students actually know. In a sense the study raises more questions than it answers. If new technologies provide more approaches to teaching and so greater variations between teaching and the consequent learning, how can these understandings be validly assessed and evaluated in a uniform way? In other words, to what extent did achievement measurement involve the technology used during instruction? Are paper and pencil techniques suitable for assessing achievement of students taught in technology intensive environments? (….)

In an earlier study, Hart (1991) reports that students using graphing calculators showed definite preferences for certain representations: those confident in symbol manipulation

skills tend to use alternate representations only when unsuccessful at finding an answer symbolically; those who do not have access to a graphing calculator do not typically choose to use the graphical representation even when it is provided, and traditional students were more likely to solve problems without considering other possibilities. Furthermore students who were not confident in symbol manipulation were more likely to use their calculator. When a solution was found, it was rarely cross-checked by using other representations even when it was wrong. Nevertheless, experimental students showed greater conceptual understanding than traditional students and there was ample evidence that success in the course was not correlated with previous grades so that

> students who might be termed as symbolically illiterate can be successful in learning and understanding calculus through the use of graphic and numeric/symbolic tools (Beckmann, 1993:112).

Ellison (1994) found that while most students improved their conceptual understanding in a calculus course that focused on multiple representations and links between them, some students developed only partial conceptual understanding. It is therefore fundamentally important to distinguish the use of the tool from the effect of teaching. This observation is consistent with Hunter et al's (1993) finding that if the use of technologies tends to emphasize certain skills at the expense of others previously in use, then the latter tend to atrophy. For example, since having graphs drawn by technology does not involve explicitly calculating and plotting values, Hunter et al (ibid) found that a third of the students in one class could answer the following question before the calculus course but not after:

> *"What can you say about u = v + 3 and v = 1"*

During the course they had no practice in substituting values into expressions and the skill seems to have receded until it was not used in the post-test. By the same token Monaghan, Sun and Tall (1994) found that some students using a CAS to carry out the process of differentiation responded to a request for an explanation of differentiation by describing the sequence of key-strokes that were necessary to get the result. It appears

that some students may simply replace one procedure, which has little conceptual meaning with another.

Roddick (2001) compared students from the calculus reform course sequence *Calculus and Mathematica* with traditional students in two areas: conceptual and procedural understanding of calculus and achievement in calculus dependent courses. Task-based interviews were conducted with students from both groups after they had completed one of two calculus sequences. The interviews were used to investigate understanding of calculus. An analysis of grades collected from the university data base was conducted to investigate understanding of calculus dependent course. Results from the interviews showed that the *Calculus and Mathematica* students were more likely to approach problems from a conceptual view point of calculus knowledge, whereas the traditional students were more likely to approach problems procedurally. The *Calculus and Mathematica* group also demonstrated a more general understanding of the derivative and integral than the traditional group.

The significance of studies in *Mathematica* environments lies in that it is also a graphic, numeric, and symbolic interface. However its differences are that it has 3D representation, which *Sketchpad* does not have. *Sketchpad* in turn, has dragging, animation and colour effects which *Mathematica* does not have. Regarding this differential, Kahng (2005:1) confirms that after 15 years of service *Mathematica* is badly in need of overhaul, for example, in its graphics that cannot support colours, animations, interactive real-time 3D representation, to name a few'.

Some research shows considerable changes when using graphic calculators over a succession of courses. Quesada (1994) introduced graphics calculators into a pre-calculus course where previously an average of 60 % of the students finished with a grade D or F, or withdrew from the course. They did not have a clear understanding of the families of functions, could not read basic graphs; and had not developed basic study habits. They were discouraged from using graphic or decimal solutions if an algebraic solution was possible. Over three courses the number of experimental students obtaining D, F or withdrawing was 43% compared with 69% in the control group. Of the totals taking the

final examination, 53% of the experimental students obtained A or B compared with 19% of the control students. When the students moved on to calculus courses, the experimental students again obtained significantly higher percentages of grades A and B in Calculus I and II though the position was marginally reversed in Calculus II (Quesada, 1995). The significance of these results is that students who are exposed to technologies might be positively influenced in their future performance in mathematics in general and calculus in particular.

However, other experiments do not always show significant improvements in performance, particularly in paper and pencil manipulative skills. Comparing students in a computer laboratory using *Derive* and a traditional course, Coulombe and Mathews (1995) found no significant differences in knowledge, paper and pencil manipulation, conceptual understanding, or higher order thinking skills, although it produced similar levels of performance whilst giving students additional familiarity with computer technology. The significance of *Derive* to this study is that it plots (draws) the graph of the derivative in a way analogous to the way *Sketchpad* does.

It is clear that although many studies have made claims that using a CAS improves understanding of calculus concepts, this is not always the case. An important consideration might be a close examination of how the new technologies are used in qualitatively different ways.

2.6   Studies describing how technological tools can be employed in qualitatively different ways

The study by Kendal and Stacey (2000) already referred to above is one illustration of how technology can be used in dichotomous ways of sequencing (privileging): the black-box/white-box sequencing, and the white-box/black-box sequencing leading to achievement levels that are similar but understandings that are qualitatively different.

In their qualitative, classroom-based design study of students' understanding of functions Doerr and Zangor (2000) found five patterns and modes of graphing calculator use: as a computational tool, transformational tool, data collection and analysis tool, visualizing

tool and checking tool. The results of the study show that the nature of the mathematical tasks and the role, knowledge and beliefs of the teacher influenced the emergence of a rich usage of the graphing calculator. The researchers also found that the use of the calculator as a personal device can inhibit communication in a small group setting, while its use as a shared device supported mathematical learning in the whole class setting.

Goos, Galbraith, Renshaw and Geiger (2000) describe a four-tier hierarchy of human computer interactions (or HCI's): where the student may be subservient to the technology, the technology can be a replacement for pen and paper, the technology can be a partner in explorations, or technology can be an extension of self, integrated into mathematical working. The last two tiers can be linked to Lagrange's (1999) theory of instrumentation which distinguishes between the use of technology as a *tool* and as an *instrument*. Transforming technology from a tool (machine or artifact) into an (mathematical) instrument involves actions and decisions based on how it is the adapted to accomplish a particular mathematical task. Trouche (2004:289) explains that this instrumentation process or 'instrumental genesis' has two complementary processes – instrumentalization (directed towards the artifact) and instrumentation (directed towards the subject). In other words, as the student shapes the instrument, s(he) too is shaped by the instrument.

With the theoretical framework of Realistic Mathematics Education and Developmental Research as a background, Drijvers' (2000) study focused on the identification of obstacles that students encountered while using computer algebra. Five obstacles were identified that have both a technical and a mathematical character. The author believes that taking these barriers seriously is important in developing insightful pedagogical strategies.

2.7 Insights from the review of the literature

It is evident that the infinity and the limit concepts, with their dependency on the infinitesimals are undoubtedly a source of conceptual difficulties for many students beginning to learn calculus. These difficulties or obstacles are borne out by the many

centuries of (unsuccessful) grappling with the potential infinity stretching from Zeno's paradoxes of 5th century BC to the time of their resolution by Newton and Leibniz in the 17th century to discover calculus. The quest for a sound conceptual understanding by beginning learners of calculus is compounded by Cantor's theory of cardinal and ordinal infinities for discrete rather than continuous processes because of its exclusion of infinitesimals.

The discovery of the derivative was underpinned by sustained efforts at drawing tangents (both on the Euclidean and Cartesian spaces) to represent the rate of change geometrically and the study of motion and change of motion on the physical reality side. By attempting to divide an indivisible magnitude of zero by an atomic instant of duration zero time the notion of velocity at an instant was a venerable conceptual hurdle to be surmounted by mathematicians of old. It still remains a slippery concept to explain and illustrate to the satisfactory understanding of the average student.

The traditional textbook approach to the introduction of the derivative uses the formal definition of the limit concept as the gateway. It has been suggested that this approach creates some conflicts between tangent definitions and some conceptual obstacles in that two points are allowed to get as close as infinitely possible but forbidden from coinciding exactly. This further complicates the work of software designers as it is not possible to use a finite algorithm to represent a potentially infinite process without encountering antimonies at some exhaustion point.

The pioneering developments in the use of the computer reveal mixed results. While some results claim that combined numeric, graphic and symbolic interfaces do improve students' conceptual understanding others show no significant changes. While others suggest that there might be some theoretical-computational conflicts leading to a narrowing effect, some report that these theoretical-computational conflicts can in fact be harnessed to enrich students' concept image of the derivative. Yet other studies show that indiscriminate use of the new digital technologies may cause neglected skills to atrophy (e.g. symbolic manipulation) (Tall, 2000). In yet other cases, studies reveal that if teachers (and students) are not fluent in the use of the technology the whole reform effort

might fall on its face and become a Frankenstein monster. This calls for an intelligent and creative use of technology as an instrument of learning which in turn, calls for a redefinition of the teacher and students' roles.

It seems that an integrated approach has to be maintained which taps on the strengths of the computer environment while retaining some of the strengths and benefits of the traditional genre. In particular, it seems that for the foreseeable future it will be relevant and necessary that the computer should not be used as a substitute for the students' thinking but rather should be a tool for thinking and experimenting with in the problem solving process.

With the change in tool options for learning of calculus there has also emerged a rallying call to review the relevance, not just of the didactic practices and relations between student, teacher and the computer, but also corresponding changes in the assessment practices and methods to reflect the learning circumstances more validly. Kendal and Stacey's (2001) study, for instance, shows that within similar overall attainment on student tests there can be substantial variations of what students know. That is, students might obtain more or less the same scores in pencil and paper achievement tests yet their knowledge base and experiences might be different in fundamental ways with a long-term effect.

While the traditional curriculum has virtues of built in precision, bringing procedural skills to the forefront and letting teachers know unambiguously what to teach and assess, the computer environment runs the risk of ambiguity of answers and expectations on both teachers and students. While the traditional curriculum is driven by algorithms without explanations the computer dominated genre is in danger of being driven by answers without explanations. When all has been said it is also evident that most of the research in dynamic software environments is at its infancy. As technologies evolve, especially in the degree of real-time enactive and perceptual control of representations it should be expected that research and curricula will inevitably lag behind necessitating rapid revisions of both software programming and learning theories.

2.8 Initial aspects of a learning trajectory in *Sketchpad*.

2.8.1 Modeling Functions in *Sketchpad*

Sketchpad can plot functions typed onto it or entered by means of a calculator pad or keyboard and mouse. Such a plotted function is displayed both symbolically and graphically. *Sketchpad* further permits the user to construct a point on the function plot (graph). When dragged or animated the point follows the function plot as its locus. The point's coordinates can be measured, displayed on screen and real-time changes observed as the point animates. Furthermore, a table of numerical values can be created as the animation progresses to give a sample of positions along the curve. In this way a function can simultaneously be represented in numeric, graphical, symbolic and dynamic forms (see Figure 2.13). This implies that the representation in *Sketchpad* moves a step further in its enactive ability once the point that traces the locus of the plotted graph is animated. The facility to vary the speed and direction of animation enables the user to keep track of the behaviour numerically and the sample table of values so created can even be re-plotted later. The re-plotting can be used to confirm that indeed they lie on the graph thus making firm linkages between the dynamic numeric and static numeric representations on the one hand and the dynamic graphic and static graphic on the other.
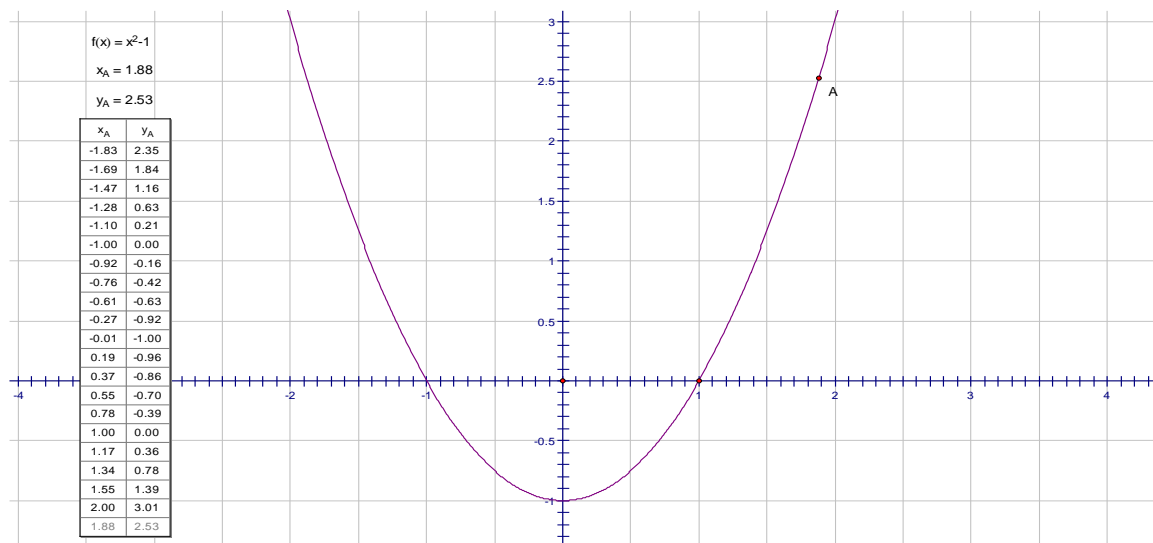
Figure 2.13:Graph of y = x^2 – 1 and a sample table of values

Students could be encouraged to animate for certain domains of x, for example $-2 \leq x \leq 2$, and to take note of changes in signs of the coordinates and explain (discuss) (e.g. at the roots). A variety of functions can be plotted and their shapes and behaviour observed and discussed in small groups or in whole class presentations. In a sense the students have the freedom to choose the examples they input, with less reliance on the teacher as explainer and demonstrator and more as facilitator and catalyst to monitor students' progress as encouraged by Lee (2005:228). This approach has some reminiscence with the black-box/white-box approach sequence of CAS utilization as a generator of examples and as an explanatory tool that may, as observed by Drijvers (2000:190), elicit curiosity and can lead to interesting discoveries.

2.8.2   Modeling the Average Rate of Change of a Function in *Sketchpad*

Sketchpad can construct (plot) more than one point on the same function plot (graph) in which case a second point, say B can be plotted. *Sketchpad* can also construct a line segment, ray or line joining or passing through any two selected points. With this facility points A and B on a function plot can be joined by a line to form a secant line.

x_A	y_A	x_B	y_B	$x_B\text{-}x_A$	$y_B\text{-}y_A$	$\dfrac{(y_B\text{-}y_A)}{(x_B\text{-}x_A)}$	Slope $\overleftrightarrow{BA}$
1.00	1.00	1.77	3.15	0.78	2.15	2.77	2.77
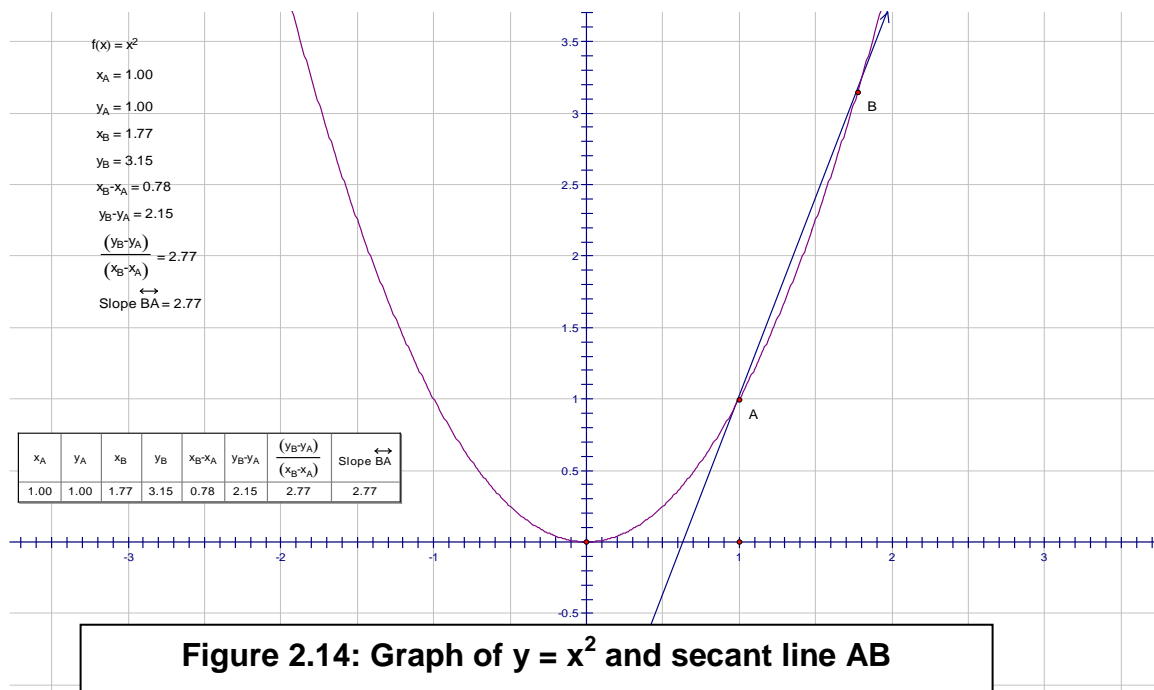
Figure 2.14: Graph of y = x^2 and secant line AB

The coordinates of point B can be measured and displayed along side those of A. Differences in the ordinate and abscissa measurements can be calculated and displayed to show changes in y relative to changes is x. The rate of change (slope) can be calculated and displayed or selected for display from the Measure menu. (See Figure 2.14).

x_A	y_A	x_B	y_B	x_B-x_A	y_B-y_A	$\dfrac{(y_B-y_A)}{(x_B-x_A)}$	Slope $\overleftrightarrow{BA}$
1.07	1.14	1.85	3.41	0.78	2.26	2.92	2.92
0.98	0.96	1.76	3.09	0.78	2.13	2.74	2.74
0.87	0.75	1.64	2.70	0.78	1.95	2.51	2.51
0.75	0.57	1.53	2.34	0.78	1.77	2.28	2.28
0.57	0.33	1.35	1.82	0.78	1.49	1.92	1.92
0.17	0.03	0.95	0.90	0.78	0.87	1.13	1.13
0.02	0.00	0.80	0.64	0.78	0.64	0.82	0.82
-0.21	0.05	0.56	0.32	0.78	0.27	0.35	0.35
-0.39	0.15	0.39	0.15	0.78	0.00	0.00	0.00
-0.44	0.19	0.34	0.11	0.78	-0.08	-0.10	-0.10
-0.52	0.27	0.26	0.07	0.78	-0.21	-0.27	-0.27
-0.62	0.39	0.15	0.02	0.78	-0.36	-0.47	-0.47
-0.78	0.61	-0.01	0.00	0.78	-0.61	-0.79	-0.79
-0.83	0.68	-0.05	0.00	0.78	-0.68	-0.88	-0.88

Figure 2.15: Animation of secant line AB

Figure 2.15 shows that a dynamic table of values can be constructed and changes observed taking place in real time as the secant line is animated or dragged forwards and backwards to obtain both positive and negative values of the slope. The changes in the slope signs can be related or connected to the orientation of the secant line as it animates.

x_A	y_A	x_B	y_B	x_B-x_A	y_B-y_A	$\dfrac{(y_B-y_A)}{(x_B-x_A)}$	Slope $\overleftrightarrow{BA}$
0.55	0.31	1.77	3.15	1.22	2.84	2.33	2.33
0.22	0.05	1.77	3.15	1.55	3.10	2.00	2.00
-0.11	0.01	1.77	3.15	1.88	3.14	1.66	1.66
-0.84	0.71	1.77	3.15	2.62	2.44	0.93	0.93
-1.30	1.69	1.77	3.15	3.07	1.46	0.47	0.47
-1.42	2.02	1.77	3.15	3.20	1.12	0.35	0.35
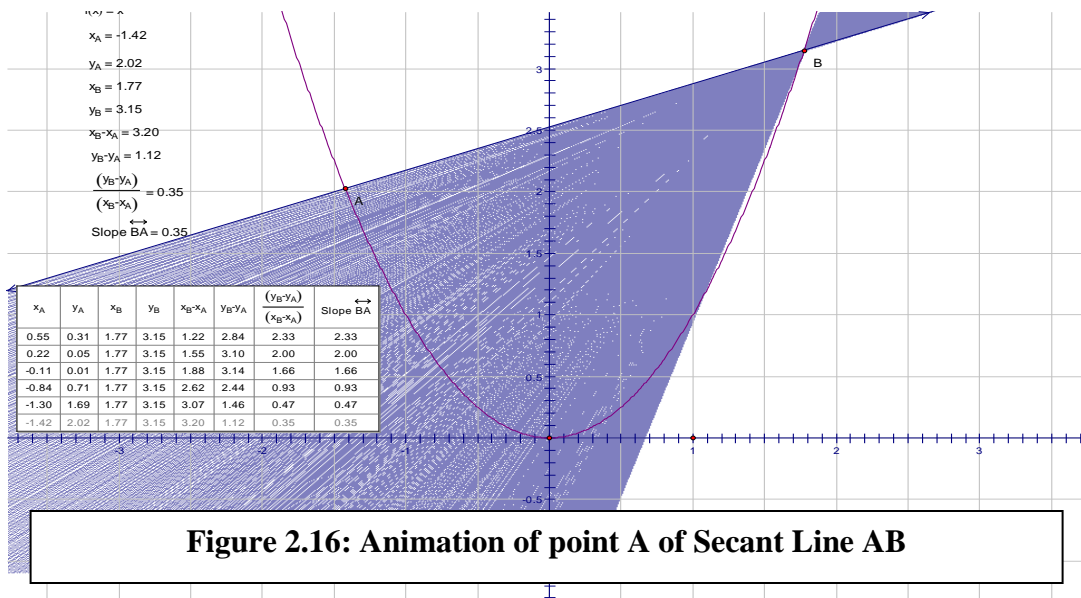
Figure 2.16: Animation of point A of Secant Line AB

Again a sample table of values can be constructed as the animation progresses to keep track of the changes. The concept of average rate of change of a function between two points on its graph can be introduced as represented by the secant line slope.

$f(x) = x^2$

$x_A = 1.00$

$y_A = 1.00$

$x_B = -1.61$

$y_B = 2.60$

$x_B - x_A = -2.61$

$y_B - y_A = 1.61$

$\dfrac{(y_B - y_A)}{(x_B - x_A)} = -0.61$

Slope $\overleftrightarrow{BA} = -0.61$

x_A	y_A	x_B	y_B	x_B-x_A	y_B-y_A	$\dfrac{(y_B - y_A)}{(x_B - x_A)}$	
1.00	1.00	-0.22	0.05	-1.22	-0.95	0.78	
1.00	1.00	-0.43	0.18	-1.43	-0.81	0.57	
1.00	1.00	-0.73	0.53	-1.73	-0.46	0.27	
1.00	1.00	-0.96	0.91	-1.95	-0.08	0.04	0.04
1.00	1.00	-1.01	1.02	-2.01	0.03	-0.01	-0.01
1.00	1.00	-1.15	1.32	-2.15	0.33	-0.15	-0.15
1.00	1.00	-1.50	2.25	-2.50	1.26	-0.50	-0.50
1.00	1.00	-1.61	2.60	-2.61	1.61	-0.61	-0.61
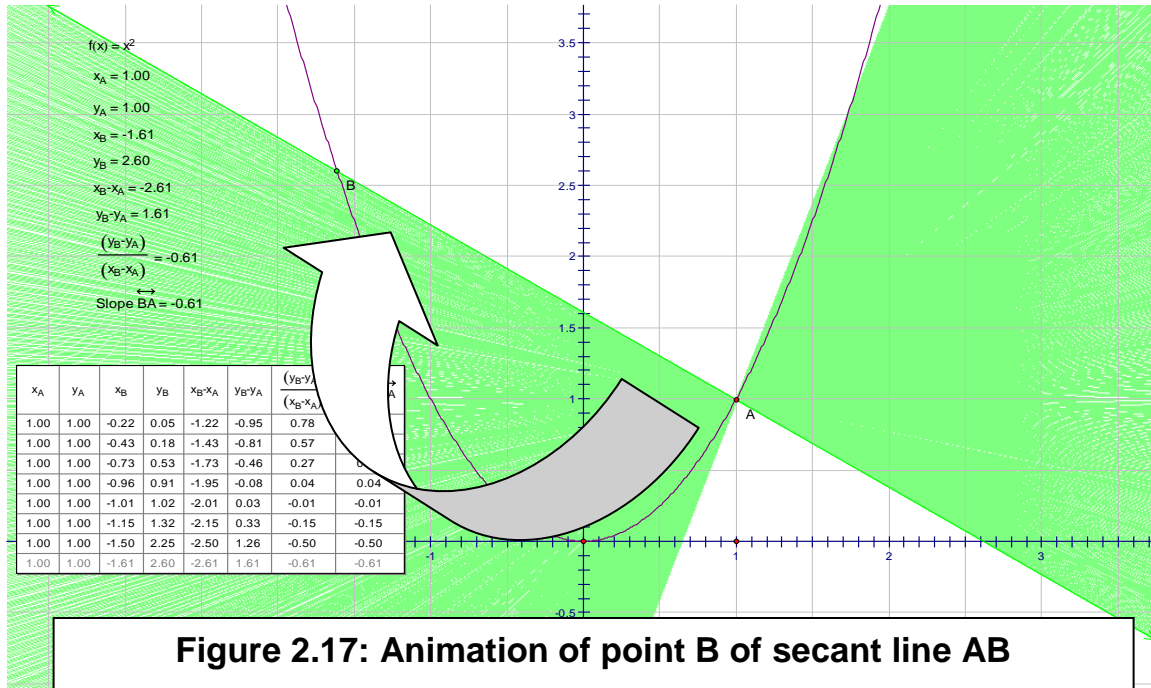
Figure 2.17: Animation of point B of secant line AB

Students can also experiment with animating one point at a time for a richer variety of possible secant line positions (See Figures 2.16 and 2.17). For use later, the differential

quotient $\dfrac{f(x + h) - f(x)}{h}$ can be introduced as an extended abstraction of the visible

particular diagram (Bakker & Hoffmann, 2005:340) to an algebraic system of representing the secant line.

2.8.3   Modeling the Derivative as the Instantaneous Rate of Change in *Sketchpad*

Sketchpad can move one constructed point to another along a function plot either by dragging, direct animation or by creating an appropriate (animation) action button. This movement allows the user to construct a Tangent Line at the second point, as the line segment joining the two points gets shorter and shorter. Initially, the animation enables the students to treat the tangent as a line that touches a curve at one point (see Figure 2.18).

f(x) = x²

Move B -> A

$x_A = 1.00$

$y_A = 1.00$

$x_B = 1.00$

$y_B = 1.00$

$x_B-x_A = 0.00$

$y_B-y_A = 0.00$

$\frac{(y_B-y_A)}{(x_B-x_A)} = 2.00$

Slope $\overleftrightarrow{BA} = 2.00$

x_A	y_A	x_B	y_B	x_B-x_A	y_B-y_A	$\frac{(y_B-y_A)}{(x_B-x_A)}$	Slope $\overleftrightarrow{BA}$
1.00	1.00	1.68	2.83	0.68	1.83	2.68	2.68
1.00	1.00	1.49	2.21	0.49	1.21	2.48	2.48
1.00	1.00	1.31	1.72	0.31	0.72	2.31	2.31
1.00	1.00	1.12	1.27	0.13	0.27	2.12	2.12
1.00	1.00	1.02	1.04	0.02	0.04	2.02	2.02
1.00	1.00	1.00	1.00	0.00	0.00	2.00	2.00
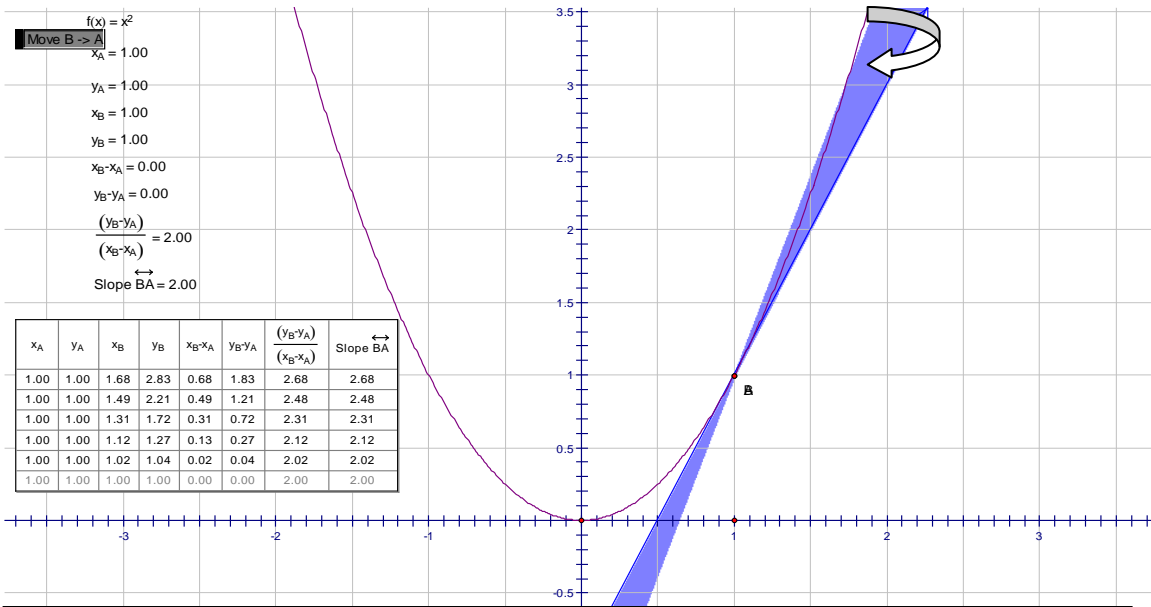
Figure 2.18: Constructing a Tangent Line from the right hand side of Point A

f(x) = x²

Move B -> A

$x_A = 1.00$

$y_A = 1.00$

$x_B = 1.00$

$y_B = 1.00$

$x_B-x_A = 0.00$

$y_B-y_A = 0.00$

$\frac{(y_B-y_A)}{(x_B-x_A)} = 2.00$

Slope $\overleftrightarrow{BA} = 2.00$

x_A	y_A	x_B	y_B	x_B-x_A	y_B-y_A	$\frac{(y_B-y_A)}{(x_B-x_A)}$	Slope $\overleftrightarrow{BA}$
1.00	1.00	1.68	2.83	0.68	1.83	2.68	2.68
1.00	1.00	1.49	2.21	0.49	1.21	2.48	2.48
1.00	1.00	1.31	1.72	0.31	0.72	2.31	2.31
1.00	1.00	1.12	1.27	0.13	0.27	2.12	2.12
1.00	1.00	1.02	1.04	0.02	0.04	2.02	2.02
1.00	1.00	1.00	1.00	0.00	0.00	2.00	2.00
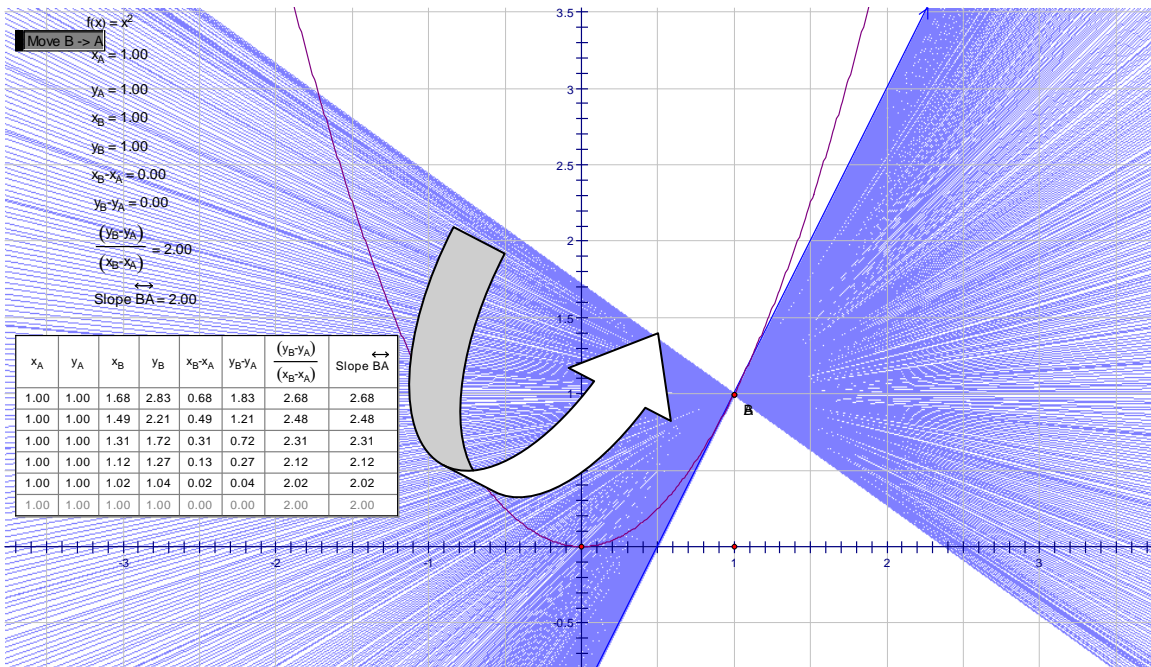
Figure 2.19: Constructing a Tangent Line from the left hand side of Point A

Dragging point B to the left hand side and animating again while point A remains stationery would help emphasize that the limiting position of the secant line should be the same whether from above or from below (See Figure 2.19).

It should make better sense in a dynamic software environment to redefine the tangent perceptually as the limiting position of a mobile secant which should be equivalent to defining it at the limiting position of a convergent sequence of infinitely many secants as B approaches A. The smoothness of the movement and the continuousness of the path traced should aid a better appreciation of the continuity of motion and hence an appreciation of infinitesimally small distances as B approaches A (or as x_B-$x_A = h$) tends to zero). The fact that the speed at which B approaches A can be varied using a motion controller makes it possible to demonstrate numerically and visually as B approaches A very slowly that the distances can be so infinitesimally small that motion may not be visible by the naked eye (c.f. the movement of the hour and the minute hands of a clock).

Students could be led to discuss the values of x_B-x_A, y_B-y_A, $\dfrac{y_B - y_A}{x_B - x_A}$ and the

measurement of the slope of AB on the table of values when point B reaches point A both from the left and from the right. At the default precision of two decimal places all these values ought to read 0.00 each except for that of the slope of AB suggesting that at

A (1,1) on the graph of $y=x^2$, the slope of the tangent is $\dfrac{y_B - y_A}{x_B - x_A} = \dfrac{0.00}{0.00} = 2 = \text{Slope } \overleftrightarrow{AB}$.

The precision could be increased to five decimal places to reveal that what appeared to be

$\dfrac{0.00}{0.00} = 2$ was only a matter of precision differentials that created a theoretical-

computational conflict.

Using the motion controller points A and B can be brought slowly together' until l they are 1/100000 or 0.00001 apart. Expressed as a fraction of a centimetre students can come to better appreciate the concept image of infinitesimals in calculus.

$f(x) = x^2$

$x_A = -1.82094$

$x_B = -1.82094$

$y_A = 3.31583$

$y_B = 3.31583$

$y_B - y_A = 0.00$

$x_B - x_A = 0.00000$

$\dfrac{(y_B - y_A)}{(x_B - x_A)} = \text{undefined}$

x_A	x_B	x_B-x_A	y_A	y_B	y_B-y_A	$\dfrac{(y_B-y_A)}{(x_B-x_A)}$	Slope $\overleftrightarrow{AB}$
1.00	1.90	0.89	1.00	3.59	2.59	2.90	2.90
1.00	1.77	0.77	1.00	3.13	2.13	2.77	2.77
1.00	1.50	0.50	1.00	2.25	1.25	2.50	2.50
1.00	1.28	0.28	1.00	1.63	0.63	2.28	2.28
1.00	1.08	0.08	1.00	1.16	0.16	2.08	2.08
1.00	1.03	0.03	1.00	1.07	0.06	2.03	2.03
1.00	1.00	0.00	1.00	1.01	0.00	2.00	2.00
1.00	1.00	0.00	1.00	1.00	0.00	2.00	2.00
-1.82094	-1.82094	0.00000	3.31583	3.31583	0.00	undefined	

Figure 2.20: Mystery of the disappearing tangent

Move B -> A$) = x^2$

$x_A = -1.53$

$y_A = 2.35$

$x_B = -1.53$

$y_B = 2.35$

$x_B - x_A = 0.00000$

$y_B - y_A = 0.00000$

$\dfrac{(y_B - y_A)}{(x_B - x_A)} = -3.07$

Slope $\overleftrightarrow{BA} = -3.07$

x_A	y_A	x_B	y_B	x_B-x_A	y_B-y_A	$\dfrac{(y_B-y_A)}{(x_B-x_A)}$	Slope $\overleftrightarrow{BA}$
-1.53	2.35	-0.54	0.29	0.99	-2.06	-2.08	-2.08
-1.53	2.35	-0.84	0.71	0.69	-1.64	-2.38	-2.38
-1.53	2.35	-1.26	1.60	0.27	-0.76	-2.80	-2.80
-1.53	2.35	-1.53	2.35	0.00	0.00	-3.07	-3.07
-1.53	2.35	-1.53	2.35	0.00016	0.00	-3.07	-3.07
-1.53	2.35	-1.53	2.35	0.00009	0.00	-3.07	-3.07
-1.53	2.35	-1.53	2.35	0.00010	0.00	-3.07	-3.07
-1.53	2.35	-1.53	2.35	0.00000	0.00000	-3.07	-3.07

Figure 2.21: Stubborn tangent

When the two points are brought together this way two things are likely to happen in *Sketchpad*. Either the tangent disappears as in Figure 2.20 or it remains visible as in

55

Figure 2.21. Each scenario can be a rich source of discussion. If the tangent disappears, the implication would be that A and B have coincided exactly and the defining property of AB would reduce to a single point hence the gradient at this one point becomes

undefined confirming 0/0 as undefined. The *Sketchpad* answer for $\frac{y_B - y_A}{x_B - x_A}$ in this case

is 'undefined'. The slope measurement for line AB fires blank on the table of values while its caption disappears confirming that there is no more slope to consider for a single point.

In the second instance where the tangent remains visible the slope is given as 2 and the students can suggest reasons before they are told that *Sketchpad* 's precision extends to 15 significant figures and there might still be a residual difference between points A and

B beyond the displayed 5 decimal answer to $\frac{y_B - y_A}{x_B - x_A}$ Such discussions should lead to

a deeper understanding of infinitesimals if students are asked to imagine how close to each other these points must be. It should, as a consequence be adequate grounds for them to justify why it is necessary to adopt Tall's proposition of a 'practical tangent' being defined as a line joining two very close points. (See 2.3). The gradient at this point can then be linked to Newton's fluxion and defined as the Instantaneous Rate of Change of *f(x)* at (*x, y*) (compare 2.2.5).

At this point the opportunity could be seized to define the tangent algebraically as the

limit of the differential quotient $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. The derivative as the

limit $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ for $f(x) = x^2$ could then be introduced explicitly more

meaningfully algebraically as $\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$ which reduces to $2x$ for $(h \neq 0)$.

2.8.4   Modeling the Derivative as the Rate of Change Function in *Sketchpad*

Once constructed, the tangent line in *Sketchpad* can be animated backwards and forwards as done with the secant line in 2.7.2. Taking advantage of this capability, the slope of the tangent line at 'point' A (or B) can be plotted against the abscissa to obtain plotted point C. This plotted point C can be traced by a light colour (say green). As the tangent line animates point C traces the locus of the Gradient Function. While this happens, a table of values can be constructed to monitor how the gradient of the tangent line changes numerically with respect to changes in *x*. (See Figure 2.22).

The path traced by point C is no longer seen in the imagination (i.e. not as a thought experiment referred to in 2.2.5) but can now be observed and seen with the naked eye. Newton's envisioning of the fluxion as a flowing point is given a physical existence (life) in *Sketchpad.* Students can be asked to come up with an equation for the path traced by point C by means of the gradient and intercept method learnt in pre-calculus. This should

Figure 2.22: Plotting and tracing the tangent slope against changes in *x*
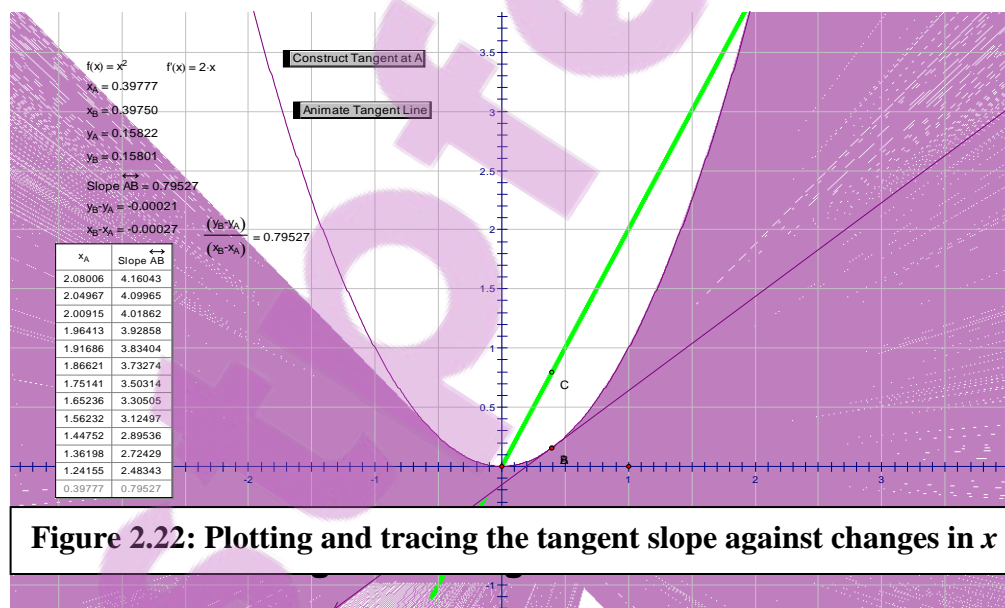
be easily seen to be the same $y = 2x$ calculated in 2.7.3. The traced path can then be erased and the derivative of $y=x^2$ found in *Sketchpad* algebraically and comparison of this answer with previous expressions made, Plotting the derivative function should also

confirm that it lies exactly on the trace of Point C. A resumption of the animation further confirms dynamically that the rate of change a (gradient) function is the derivative.

Further explorations can be made with the graph of $f(x) = x^3$, and higher powers to confirm the power rule nx^{n-1}. The graph of $f(x) = x^3$ can be drawn together with its first , second and third derivatives to demonstrate that being a (gradient) function a derivative can in turn have its own derivative (see Figure 2.23). More comparisons can even be made for trigonometrical, exponential and logarithmic functions.

Figure 2.23: The graph of f(x) = x² and its 1ˢᵗ, 2ⁿᵈ 3ʳᵈ and 4ᵗʰ derivatives
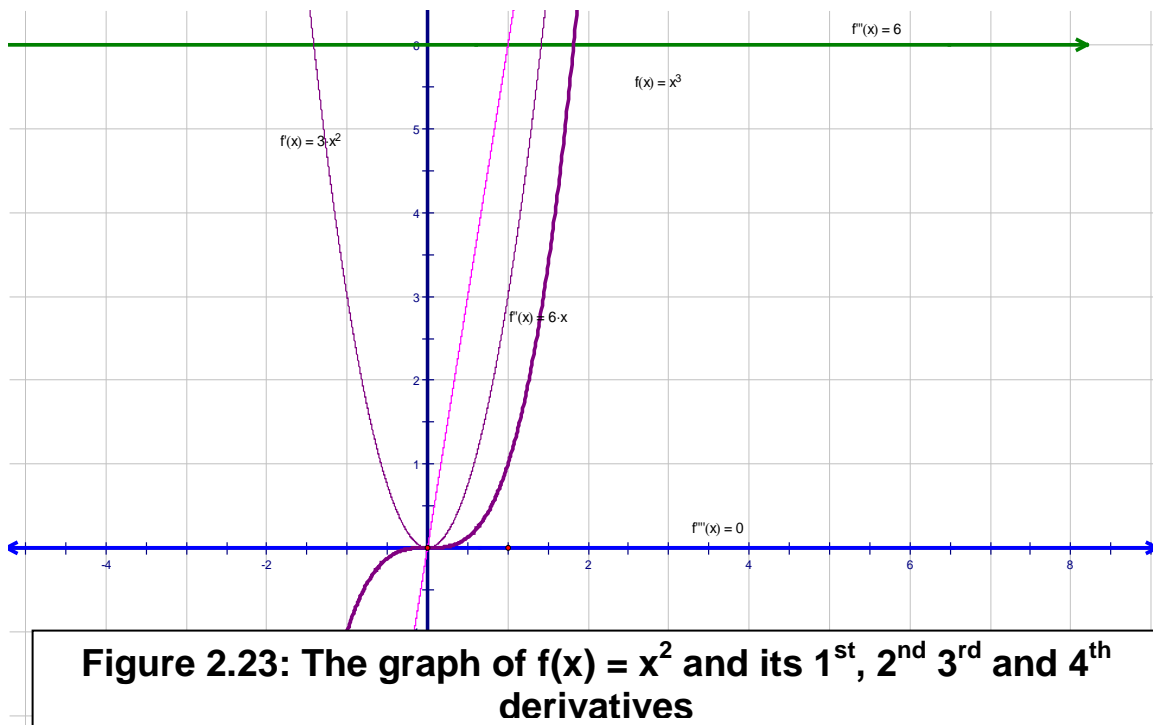
2.8.5   Investigating Differentiability in *Sketchpad*

A function is differentiable at a point if its derivative exists at that point. Since a derivative is a limit, for it to exist both the left and the right derivatives must be the same. A function is differentiable on an interval if it is differentiable at every point of the interval. As a corollary, we can say to establish the differentiability of a function at a point we recall that the points of differentiability of a function *f* are the points where the curve $y = f(x)$ has a tangent line, and the points of non differentiability are points where the curve does not have a tangent line.

Informally stated, the most commonly encountered points of non-differentiability can be classified as corner, points of vertical tangency, and points of (jump) discontinuity. Figures 2.24 – 2.26 illustrate these special cases. Students can then be given the opportunity to construct left and right tangents in each case so as to establish non-differentiability.
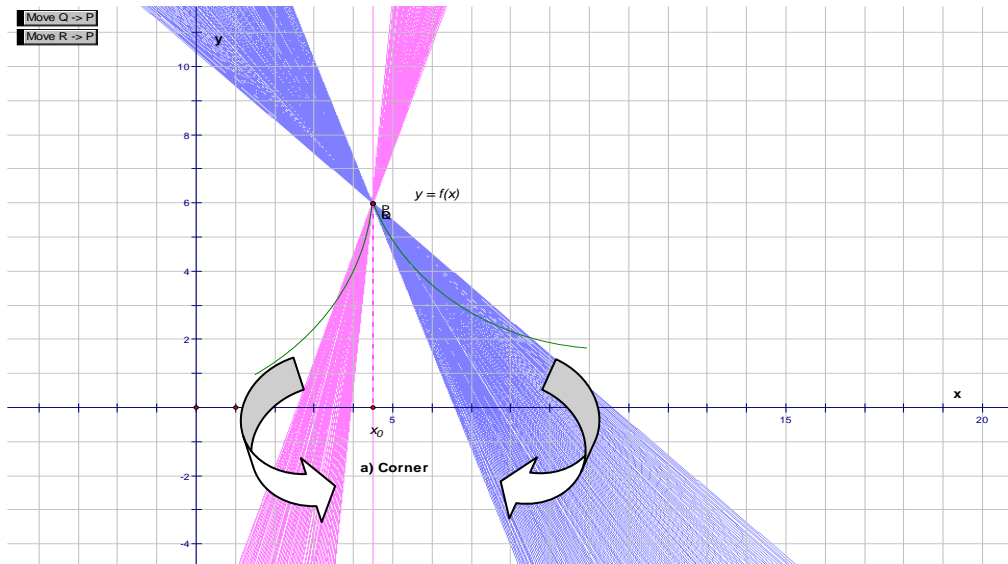
Figure 2.24: Differentiability at a corner point
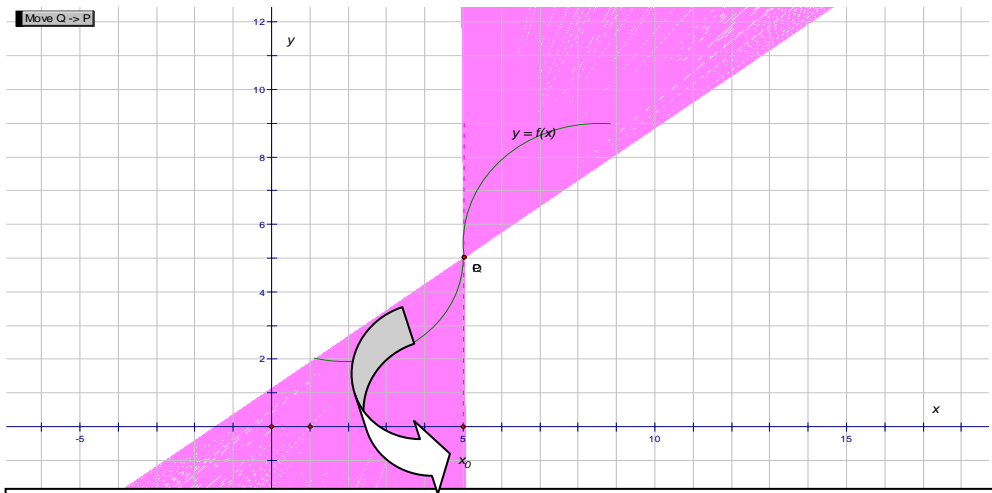
Figure 2.25: Differentiability at a point of vertical tangency
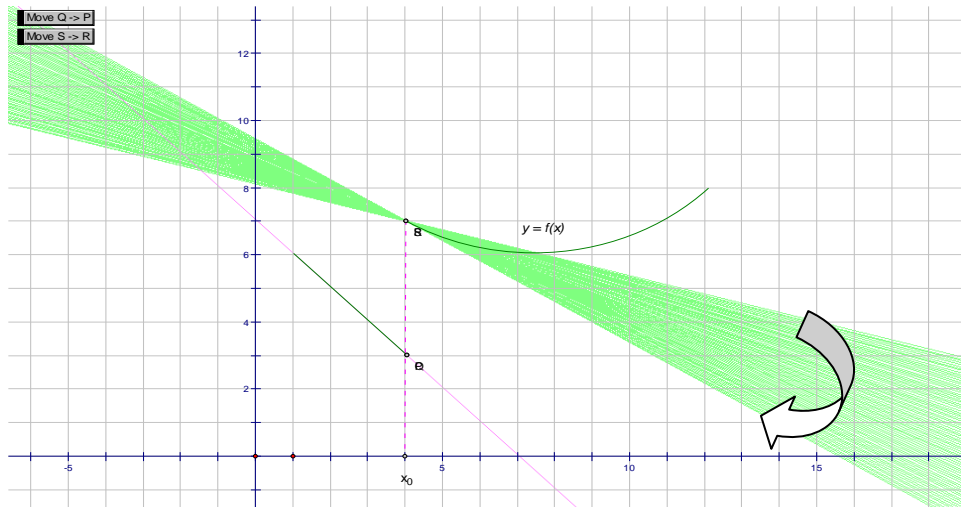
Figure 2.26: Differentiability at a point of discontinuity

More drill-and-practice exercises could be given offering a fairly wide repertoire of experiences in using the dynamic software as a cognitive tool with which the students can enrich their concept image of differentiability or lack of it and consequently extend their qualitative/conceptual understanding of the derivative in a graphical representational context. If interaction with peers were encouraged, the students would learn to explain and justify their own thinking as envisaged by Hurme and Jarvela (2005:50).

2.9   Conclusion

In this chapter the historical origins of the derivative were traced and the way it has been characterized and taught both traditionally and in technology aided instruction up to and including the limited research in dynamic mathematics software environments. The chapter began with an account of the intractable concept of the infinity by which the derivative is defined as a limit. The nature of two of Zeno's paradoxes of the infinity and how they were tackled by Aristotle in Greek thought was outlined. The historical genesis of the derivative concept was further detailed starting with mankind's struggle with the drawing the tangent to curves to determine maxima and minima after many centuries of stagnation, up to Barrow's differential triangle - a case of the derivative as the instantaneous rate of change.

An account was given of how mankind tried to figure out the instantaneous rate of change of non-uniform motion and how that sustained effort, starting with the Merton rule eventually culminated in the discovery of the calculus by Newton and Leibniz in the 17th century. In particular, the characterization by Newton extended the derivative concept to the rate of change function notion. However, it was Leibniz's notation that endured. The development of the theory of limits, infinitesimals and transfinite numbers was alluded to as a final solidification of the understanding of the concept of the derivative as a limit.

The review of related literature was extended to the teaching and learning of the derivative in the traditional context as the limit of a sequence of secant lines in the static textbook environment. The chapter then concentrated on the use of computational technology (computer algebra systems) in encapsulating multiple representations of the derivative in graphical, numeric and algebraic forms. Some limitations of the computational environments were highlighted to caution against possible obstacles that may be encountered when integrating technology into mathematics classrooms. Special attention was paid to the limited research on the use of dynamic mathematics environments which by and large have all of the CAS properties in the first instance. Finally, insights derived from the literature review were synthesized into an initial *Sketchpad* mediated learning trajectory for the derivative capitalizing on the assumed affordances of the dynamic mathematics software. The trajectory took the following sequence: modeling the function concept, modeling the average rate of change in, modeling the derivative as the instantaneous rate of change, modeling the derivative as the rate of change function and modeling differentiability at a point. The chapter has therefore met its objectives.

The next chapter contextualizes the study in a modeling context. The philosophical background underpinning the instructional design is sketched out and the various perspectives on modeling discussed is considerable detail and interweaved into to the envisaged learning trajectory. Special attention is paid to how dynamic mathematics software can help students to develop deeper concept images of the derivative, under what relationships with each other, the artifact and mathematical problem solving. The

various perspectives on modeling are synthesized and a modeling proffered as a viable

instructional model.

CHAPTER III
MODELING AND CONTEXTUALIZATION IN *SKETCHPAD*

3.1   Introduction

In this chapter a philosophical background to modeling is presented to underscore the view that our views about the nature of mathematics influence our methodological and tool choices in the classroom. Various philosophical positions are discussed in a nut shell and their consequences for mathematics instruction evaluated. The meaning of a 'model' is revisited and further elaborations made influenced largely by the Realistic Mathematics Education philosophy of the Freudenthal Institute in the Netherlands. Various perspectives on modeling are examined as explicated in the mathematics education literature in an attempt to situate modeling within the mathematical, cognitive and didactic contexts. Modeling is then situated in a dynamic mathematics learning environment where *Sketchpad* is used as a modeling and simulation tool to enhance the understanding of the derivative concept. The modeling perspectives are threaded into the conjectured teaching/learning trajectory presented in the previous chapter in line with the chosen philosophical direction (compare 2.5). The various perspectives on modeling are then synthesized and an integrated meaning of modeling mooted.

Ultimately modeling is cast as a teaching strategy to permeate and characterize the envisaged didactic practice entirely. In a sense this chapter aims at answering the following research question: What instructional model and didactical relationships are conducive to a successful orchestration of dynamic mathematics software to enrich students' concept image of the derivative?

3.2   Philosophical background to modeling
3.2.1   The Importance of a Philosophical View in Mathematics Education
Philosophical views influence perspectives about methodological and tool choices.
Dossey, McCrone, Giordano and Weir (2002:8) remark that the conception of

mathematics held by teachers significantly influences how they teach it. This is clearly an echo of Rene Thom's assertion:

> Whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics (Thom, 1973:204).

This suggests, among others, that the teacher's understanding of the manner in which mathematical knowledge is acquired inevitably influences the way (s)he teaches it. Brousseau's (1997) analysis of mathematics teaching supports Thom's assertion. Brousseau's notion of the didactical contract explains how the role of mathematics teacher is shaped by its institutional context. In his view, the role of the mathematics teacher is defined and shaped by the responsibility of teaching mathematics and the justification that any activity (modeling) in mathematics classrooms must include an explanation of how the activity is mathematical. Freudenthal's (1991:14-15) view of mathematics as an activity buttresses the individual learner's contribution, 'his/her activity' in the mathematical learning process, which includes not just reading, listening, reproducing mathematics as given, but also the aspects of producing mathematics and coming up with his/her own products. These products are largely representations, which are in themselves models achieved through the activity of modeling. In both senses, the pedagogy of the mathematics classroom rests to a lesser or greater degree on a philosophy of mathematics.

3.2.2 Logicism and Mathematics Education

Strauss (2001:19) notes that although it may seem natural to relate mathematics as a special science to the aspects of number and space in the first instance, the logicist, Russell wants to stress that mathematics is not concerned with *quantity*, but with *order*. A teacher with a logicist view might, as Russell aspired, want to reduce all of mathematics to logic, by declaring that mathematics and logic are identical. The view of mathematics as critically concerned with order is shared by Hamilton who defined algebra as the science of pure time and order (Cassier in Strauss, 2001:19). Cassier held the view that the main purpose of the study of the history of mathematics is 'to illustrate and confirm the special thesis that ordinal number is logically prior to cardinal number, and more

generally that mathematics itself can be defined, in Leibnizian fashion, as the science of order (ibid).

The whole goal of mathematics learning and modeling could then be to establish order. But then we are quickly reminded of retorts to these claims, as evidenced by the formalists and, more recently, the failure of the New Math programme, which attempted to rest all school mathematics on set theoretic logic with disastrous consequences. Logicism had to concede that it failed in providing a successful reduction to logic of the notion of infinity because the logic of infinity was not considered to be an axiom of logic. Yet a clear understanding of the completed/actual infinity underlies the notion of derivative as a limit.

3.2.3   Formalism and its influence on Mathematics Education

A teacher with an axiomatic formalist view of mathematics might, as Hilbert did, argue against the logicists that no science can exclusively be based or couched in logic hence:

> Mathematics has a guaranteed content independently of all logic … there is a further prerequisite for the application of logical conclusions ….namely that something must be given in the conception: specific extra-logical objects intuitively present as immediate experience prior to all thinking (Strauss 2001:39).

It is clear that although formalism arose as a response to logicism, it is strong in its desire to axiomatize and formalize. Formalism works on the foundations of set theory, and those dealing with the philosophy of mathematics, often refer to mathematics as "the science of formal systems". This suggests that to the axiomatic formalist teacher mathematics is set theory. These inclinations are precisely a didactic problem in the sense that they lead to an emphasis of starting with the end-products of mathematics: definitions, theorems, etc which becomes an antididactical inversion as observed earlier. Robinson (1967:39) points out that Cantor, as founder of set theory, was convinced that set theory deals with the actual infinite. This suggests that a teacher defining mathematics as set theory places the problematic dichotomy between the uncompleted/potential infinity and the completed/actual infinity at the heart of the definition. His/her practice will be coloured

and refracted through the same prism. This formal view does not seem to be in tandem with a modeling perspective to mathematical instruction because pure mathematics with no consideration of any applications, invariably excludes applied mathematics and mathematical modeling.

3.2.4 Platonism and its Influence on Mathematics Education

Regarding the relationship between the *universal* and the *individual*, platonic realism designates 'objective reality' to an independent/universal existence outside the knowing human soul/individual. A teacher with a platonic view might, therefore, present mathematics as a structure 'existing outside the mind and experiences of the student' (Wessels 2006:5). If mathematics is out there, ready made and waiting to be discovered, then it still leaves room for the practitioner to guide students to rediscover mathematics for themselves during the learning process. The platonic view would therefore not necessarily favour the transmission model of teaching which takes the learner to be an empty vessel to be passively and unresponsively filled up. That would not necessarily fly in the face of instrumenting computational tools into artifacts with which to model mathematical phenomena.

3.2.5 Positivism and its influence on Mathematics Education

Positivism's epistemological and philosophical idolization of the *experimental method* on the basis of *sensory perception* blocks out insight (thought) or practical reason as a possible source of mathematical knowledge by means that transcend the domain of sense perception and logical understanding. Strauss (2001:53) notes that by the end of the nineteenth century and the beginning of the twentieth century positivism emerged as a philosophical trend with the explicit purpose to abolish whatever supersedes sense perception. While a teacher who subscribes to positivism might therefore adopt a rigid formal naturalistic approach to methodology, such a possibility cannot just be uncritically attached to 'positivism' as it may equally apply to an experimental or empirical approach. In a sense it is more about a view of learning math that influences teachers' decisions. Again such a teacher would may not necessarily be inclined towards the transmission model as there are many things that impact on teachers' decisions and choices of a certain

approach, only one of which is their philosophical view of math. A teacher who holds an empirical/empiricist view may teach directly in the transmission mode, not because of their philosophical view but because they believe from their understanding of how students learn but that they learn quickest. Another teacher with a formalist perspective, may choose to slowly guide and develop students through different stages, to finally cut the ontological bonds, and to study math as an axiomatic system, divorced from the contexts from which they were originally developed.

3.2.6   Aristotelian views and their influence on Mathematics Education

Aristotle believed that our knowledge of the individuality of entities closely coheres with the way in which we experience the identity of those things (Strauss, 2001:74). That is, this identity is something given to us in our experience and can therefore not be construed or accessed via other modes of knowing reality. A teacher with an Aristotelian view of mathematics might, therefore, rely heavily on experiences, and perhaps to the detriment of developing relations in a more abstract and general setting (Wessels 2006:7). Yet some of the greatest scientific breakthroughs have been achieved by defying commonsense intuition of the experienced world. For example, the Copernican discovery that it is the earth that goes round the sun and not the sun around the earth as daily commonsense experience suggests is a classic case in point. Galileo's ex-communication by the church symbolizes how intolerant an Aristotelian teacher might tend to be towards students experimenting with new mathematics or unconventional ways of representing mathematical concepts in an environment laden with new digital technologies.

3.2.7   Empiricism and its Influence on Mathematics Education

Empiricism, is a (neo-) positivist philosophy which, with its stress on experience as a source of knowledge, designed a scientific methodology which begins with particular sensory data/sensory impressions on the one hand and the logical construction of entities from these impressions, on the other. This resulted in the well-known progression of empirical perception-hypothesizing-testing (verification) as the accredited methodological approach to theory formation (verified hypothesis) (Strauss, 2001:3) An empiricist teacher, thus, who prefers total reliance on experience-based learning might,

like the Aristotelian, be more inclined to deprive students the opportunity to abstract and generalize to the formal level of mathematical cognition, yet it should be a question of when and how to get them there. Murray, Olivier and Human (1993:73) criticize the empiricist view of teaching as the transmission of knowledge and learning as absorption of knowledge and advocate that students should be given opportunities to construct their own mathematical knowledge. However, they do so from a learning theory point of view, rather than a philosophical point of view.

3.2.8   Constructivism and its Influence on Mathematics Education

Within the foundations of mathematics 'constructivism' means something completely different from 'constructivism' as a theory of knowledge and knowledge acquisition. In relation to the first, constructivism rejects the law of the excluded middle (and therefore all proofs based on contradiction (*). Its goal is to systematize mathematics without having to prove the existence of objects without showing how they can be constructed, hence the name "constructivism". The second constructivism, which is of interest in this study, comes from general philosophy and can be described as an epistemology of how knowledge is gained (**). In the latter sense, Ernst von Glasersfeld's basic principles of radical constructivism are the following:

1. Knowledge is not passively received either through the senses or by way
    of communication, but it is actively built up by the cognizing subject.
2. The function of cognition is adaptive and serves the subject's organization of
    the experiential world, not the discovery of an objective ontological reality.
    (von Glasersfeld, 1988:83)

Von Glasersfeld (1993) acknowledges that his principles are built on the ideas of Jean Piaget, who applied the biological concept of adaptation to epistemology. He refers to his ideas as "postepistemological" because his radical constructivism posits a different relationship between knowledge and the external world than does traditional epistemology (Johnson, 2008:1). By emphasizing the knower as an active cognizing agent the radical constructivist teacher might have links to teaching and learning in ways

that are an alternative to the transmission model and therefore supportive of modeling. Conversely, by denying the objective, mind-independent existence of the world, the radical constructivism might entail a return to a magical, capricious world-view that might stifle knowledge growth Hom (2000:156) and therefore might appear to be against mathematical modeling.

As a theory of learning social constructivism recognizes that mathematical knowledge is a product firstly of an individual human activity and secondly a social activity in that the individual's subjective knowledge must be shared with others to become accepted objective knowledge or joint activity (Hurme & Jarvela, 2005). Furthermore students are expected to construct or model their own knowledge according to prior understandings, which envelop and colour their interpretation of new knowledge. This philosophy is therefore promotive of experimentation in the classroom. Lagrange (2005:147) points out that although experimentation is a basic choice in physics teaching it is more and more mentioned in Mathematics teaching as well. This suggests that in class the teacher has to present the observational basis ('the experiment') to the students without theorization and make theorization appear as built from experiment. This enables the classroom to be both a scientific research institution devoted to knowledge production and a didactic institution devoted to apprenticeship. In other words the constructivist views the learner as an active participant in the construction of his own models and processes of understanding.

3.3   An analysis of the meaning of a model

Lesh and Doerr (2000:362) observe that in physics, mathematics, chemistry or other physical sciences, a model is a system consisting of elements, relationships among elements, operations that describe how the elements interact and patterns or rules (e.g. symmetry, commutativity, transitivity, etc). A model in this sense has a structure made up of components and relationships and dynamism of operations that interrelate or connect the workings of the component elements one to another to represent a physical reality. We can view this as an explanatory role of a model. Within a problem solving context, Lesh, Hoover, Hole, Kelly and Post (2000:598) consider a model to be a simplified

description that focuses on significant relationships, patterns, and trends. It simplifies the information in a useful form, while avoiding or taking into account difficulties related to surface-level details or gaps in the data. This seems to be a descriptive view of students' ways of thinking.

Freudenthal (1991:34) refers to a model as an intermediary by which a complex reality or theory is idealized or simplified in order to become accessible to more formal treatment. It enables one to grasp the essentials of a static or dynamic situation by discovering common features, similarities, analogies, and isomorphisms towards the goal of generalizing. In this sense a model is seen as a simplified (generalized) representation of the structure and dynamics of a complex situation. He rejects the term 'mathematical model' in a context where it wrongly suggests that mathematics directly or almost directly applies to the environment. Rather a model can be considered to be an instrument or tool by which to gain clearer understanding of an otherwise complicated situation. English and Halford (1995:13) note that in cognitive science a model is a hypothesized knowledge structure and processes underlying the learning and application of mathematics. This view of a model shifts attention towards a different kind of object to be modeled - from physical phenomena or situations to modeling of thought processes learning scenarios and applications of mathematical knowledge to solve real world problems. This conjures up a problem-centredness dimension to modeling, a characteristic that is in harmony with the Realistic Mathematics Education vision.

Mudaly (2004:85) shares a similar view when he regards a model as a theory of the way the learner thinks, processes, checks solutions, makes a plan and executes it. This view mixes up the psychological processes of mental models or schemes with Polyan problem solving as aspects of modeling. By contrast, Wessels (2006) more categorically considers problem solving to be a simplified version of modeling which suggests that it is a sub-set of the larger modeling process and cannot, therefore, be equivalent to it (compare with Aristotle's whole-part relationships in 2.2.2).

Taken together these definitions and views on modeling broaden the scope of modeling in this study to include how mathematical knowledge (the concept of derivative) can be effectively represented (the teaching dimension) and its meaning negotiated (student and teacher roles) in *Sketchpad* (the modeling tool), to deepen students' understanding (concept images) of the derivative so that they can competently and confidently solve problems which require this knowledge (applications). In this study therefore modeling extends beyond the cognitive activity of formulating mathematical models, to a more general teaching methodology, and to how learners can make mathematical meaning in a dynamic mathematics environment and how these meanings and understandings can be represented, adjusted and refined.

3.4   Modeling as a process in Mathematics Education

3.4.1   Mathematical modeling and the Scientific Method

According to Dossey et al (2002:114) a mathematical model is a mathematical construct designed to study a particular real-world system or phenomenon and includes graphical and symbolic representations, simulations and experimental constructs of a model. This characterization is compatible with *Sketchpad*'s capabilities but limited in that verbal and numerical representations are not explicitly included but left implied. The real-world dimension is echoed by Chaachova and Saglam (2006:16) who view modeling as indicating the translation of a real phenomenon to a model, analysis of the model and a translation back to reality. This view is in natural synchrony with the context of discovery and invention of the derivative (rate of change of motion system) referred to in Chapter II (see 2.2.3).

The mathematical modeling process of Dossey et al (2002) is itself further characterized as a cognitive system (see Figure 3.1). Data gathered from the real world are converted to a model, which in turn is analyzed and mathematical conclusions drawn. *Sketchpad,* for instance enables the user to convert data into data plots of the relationship between two variables, x and y, which can be linked to produce a graphical representation to facilitate

analysis and drawing of conclusions (compare 2.8.4). In the context of this study such conclusions are the determination of the rate of change of a functional relationship.
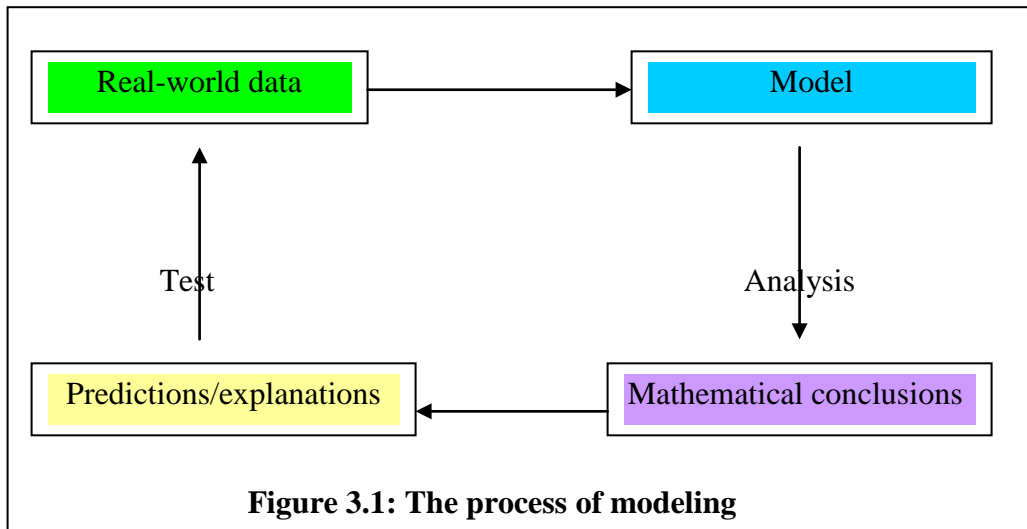
Figure 3.1: The process of modeling

According to Dossey et al (2002) these conclusions of the modeling process can then be used to make predictions and explanations, which are tested against real-world data (phenomena) again to complete the modeling cycle. Given a value of x, the plotted function can be used, to predict a corresponding y-value to inform us how any y-(instantaneous rate of change) will vary with respect to x. Additional samples of real world (or simulation) data can be drawn and plotted again (compare 2.8.4) – to check the graph model against real world data to complete the modeling cycle.

Dossey et al (2002) further summarize the model construction process into six steps (see Figure 3.2). The first step is problem identification (in real world data or phenomena), which should be sufficiently precise so that it is translatable into mathematical statements. In the second step assumptions about hypothesized functional relationships between variables are made. For instance, hypothesizing whether the relationship is linear, cubic, quadratic, or curvilinear. The third step interprets the mathematical model by putting together all sub-models to see what the model is telling us as the 'best' or 'optimal' solution. The fourth step verifies or tests the model before use to check if it is answering the problem faithfully and sensibly. That is, a check is made as to whether the model makes sense, is reasonable or can be corroborated. The fifth step implements the model and applies it in a user-friendly manner. That is, the model is trial run. In the sixth

and final step the model is maintained as changes occur. It is clear that the six steps can be supported by *Sketchpad*, in so far as the functional relationship being dealt with in this study is concerned i.e. continuous functions, differentiable functions, etc.

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Step 1:   **Identify the problem**

Step 2:   **Make assumptions**
- Identify and classify the variables
- Determine interrelationships among variables and sub-models

Step 3:   **Solve or interpret the model**

Step 4:   **Verify the model**
- Is it really addressing the problem?
- Does it make sense?
- Test it with real-world data

Step 5:   **Implement the model**

Step 6:   **Maintain the model**

Figure 3.2: Steps in mathematical model construction

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The six-step model construction process has clear reminiscences with the Scientific Method in that both make assumptions or hypotheses, gather real-world data, and test or verify hypotheses using that data (Dossey et al, 2002:118). The similarities or parallels confirm modeling as a scientific and objective undertaking. However, Dossey et al (ibid), are of the opinion that modeling and the scientific method differ in their primary goals. Whereas the goal in modeling is to hypothesize a model through evidence (real world data) the objective is neither to confirm nor deny but to test the model's reasonableness or plausibility. Many functions to be differentiated in introductory calculus are not necessarily an exact mathematical representation but an idealization or model. Hence there is compatibility between modeling and the mathematical objectives of using *Sketchpad.*

A relevant feature of *Sketchpad* is that even in the absence of real-world data students can graph/plot functions, which they can use as examples to acquaint themselves with the structure and shapes of the wide range of functional relationships that are possible between physical phenomena. In other words, *Sketchpad* not only facilitates modeling by way of fitting real world data (in fact it is less suitable for this) but more importantly it offers model fitting opportunities to real world data or situations by offering a repertoire of graphical, numeric and dynamic representations for hypothesized symbolic relationships between variables.

3.4.2   A systems perspective of the modeling process

In their elaboration of a model as a system, Lesh and Doerr (2003:362) contend that for a system to be a model it must be used 'to describe some other system or to think about or to make sense of it or to explain it or to make predictions about it.' (Lesh & Doerr 2003: 362). There seems to be some equivalence with the scientific model perspective of Dossey et al discussed above, in terms of sense making reasonableness of representation, and predictive validity or usefulness.

There is also convergence regarding the instrumental value of a model as a tool to simplify a complex situation. However what shines through as a distinguishing feature of the systems model is that the system-cum-model can be used to describe some other system or to think about or make sense of it. This functionality of the model as a reasoning tool is a subtlety that is critical to this study. It is particularly suited to the interpretation of 'modeling with *Sketchpad*' envisaged in this study where the dynamic software is used to model the physical derivative concept, not only graphically and symbolically, but also in dynamic numeric and dynamic graphic senses, simultaneously.

Lesh and Doerr (2003:363) take a further step to examine the question of the model: is it inside or outside the mind? On the one hand they consider models to be conceptual systems (which places them in the cognitive domain) that function with the support of powerful tools (which are external elements), or representational systems (external systems) which places them in the exterior (real world) domain, each of which

emphasizes and de-emphasizes (or ignores or distorts) somewhat different aspects of the underlying conceptual system (which makes them subjective).

On the other hand, man-made conceptual systems (mental networks, or theoretical frameworks, including mathematical structures and systems) are viewed to be partly embedded in conceptual tools that involve electronic gadgets (computers), specialized symbols (e.g. *Sketchpad* and calculus symbolization), language, diagrams, organizational systems or experience-based metaphors. What stands out here is the function and role of a 'tool' in the modeling process. The use of a tool is an integral part of the act or process of reasoning. Because reasoning is a cognitive process, the tool is therefore not entirely outside the mind, and because the tools can be independently external, it means the reasoning process is not entirely confined to the head. Again these interpretations are in accord with the cognitive science view of a model and the intended use of *Sketchpad*. In other words, the modeling we envisage with *Sketchpad* is as a conceptual system with both internal (cognitive) and external dimensions in interaction with one another to solve real-world or quasi real-world problems. In their turn, conceptual tools have both an external and internal existence.

3.4.3   Modeling and representations

In comparing models with representations, Lesh and Doerr (2003:363) note that the meaning of a model (or conceptual system) tends to be distributed across a variety of interacting systems which may involve written symbols, spoken language, pictures or diagrams, concrete manipulatives or experience-based metaphors. These components or forms of representation systems seem identical in description to the characterization of conceptual tools made in 3.4.2 above. The differences are that while on the one hand models emphasize the dynamic and interacting characteristics of systems being modeled, representations draw attention to the objects within these systems. On the other hand, while models refer to functioning whole systems, representations tend to be treated as inert collections of (static) objects to which manipulations (dynamism) and relationships must be added in order to function.

The above characterization of representations seems to be partly compatible with the representational capabilities of *Sketchpad.* On the one hand the graphic, symbolic and numeric representations in static form seem to fit the characterization of representations quite perfectly. On the other hand the dynamic numeric, and the dynamic graphic representation capabilities could pass the test of being systems in these criteria because they are imbued with motion (dynamism) to illustrate the relationship between variables. Alternatively they can be interpreted as dynamic objects, in which case the restriction to static would need to be revised. When the tangent is animated, the derivative of a function is literally set in motion in *Sketchpad*, which is a greatly extended visualization or degree of modeling. In other words, under animation, the representational objects transmute to a representational system.

Lesh and Doerr (2000) summarize their comparison of models and representations by stressing that the modeling process involves the interaction among three types of systems: a) the internal conceptual systems, b) the representational systems which function both as externalizations of internal systems and as internalizations of external systems and c) external systems that are experienced in nature or are man-made artifacts. This summary accurately represents the spirit of modeling envisaged with *Sketchpad* in this study. The summary diagram however mismatches the verbal summary (See Figure 3.3).

Figure 3.3: Modeling interactions among three types of systems
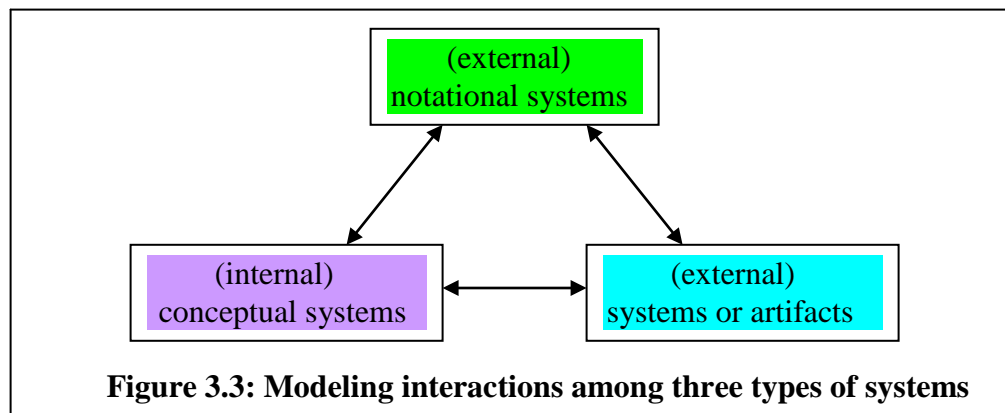
The authors argue that conceptual systems seem to exist in the head, while representational systems are embedded in spoken language, written symbols, pictures, diagrams and concrete models that people use to both express their mental (conceptual) systems and to describe external systems. Van Oers (1996:93) shares a somewhat similar

view in suggesting that representational systems are an activity of figuring out, refining their representativity and communication value with others. He further contends that although they are carriers of meaning, they are not themselves the creator of meanings. In other words the cognising agent in interaction with the symbol system creates the meanings.

External systems are considered to be man-made artefacts (e.g. economic systems, communications systems, mechanistic systems, mathematical structures and systems) projected into the world to become part of the experienced world of others. It is further suggested that the boundaries between the systems are fluid, shifting and at times

Figure 3.4: Interaction between internal and external systems
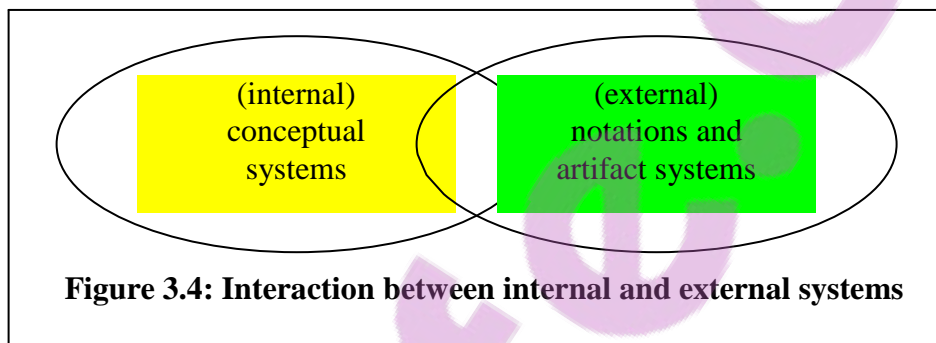
ambiguous (ibid. p. 363). In other words the systems are viewed as partly overlapping, interdependent and interacting and Figure 3.4 can be a plausible alternative diagrammatic representation.

3.4.4   Modeling and mathematizing

Lesh and Doerr (2000:366) consider mathematizing (e.g. quantifying, visualizing or coordinating) to be a form of modeling involving the use of special language, symbols, graphs, pictures, concrete materials, and other notation systems to develop mathematical descriptions and explanations that make obvious heavy demands on learners' representational capabilities. However, if we make a distinction between horizontal and vertical mathematization as proposed by Treffers (1987) then we obtain two interpretations of modeling in the sense of mathematizing.

Horizontal mathematization has to do with real-world, or realistically imaginable, applications of mathematics and to this extent resembles mathematical modeling in a problem-centred or situated cognition sense. Van den Heuvel-Panhuizen (2003:12) similarly point out that to mathematize horizontally means to go from the world of symbols to the world of life. Vertical mathematization, however, refers to the progressive sophistication of mathematical subject matter from one level to the next in the sense of mathematizing mathematics (ibid p. 11). The modeling process in *Sketchpad* can support both horizontal and vertical mathematization. In the activities on modeling the derivative in this study (Appendix B), the average rate of change of a function (Activity B) is used as a stepping-stone to the instantaneous rate of change (Activity C), which in turn is used as a stepping-stone to the rate of change function (Activity D) constituting a cycle of vertical mathematization.

The real-world applications of these sub-concepts of the derivative are also dealt with in practice exercises in activities thus enabling horizontal mathematization to generalise concepts (see Figure 3.5). The proposed sequence of activities in this study is such that functions are dealt with in the first activity to familiarize students with various change models, and the use of *Sketchpad.* In the second activity the secant line is introduced and its slope used to model the average rate of change of a function with applications in the

Figure 3.5 : Vertical and horizontal mathematization
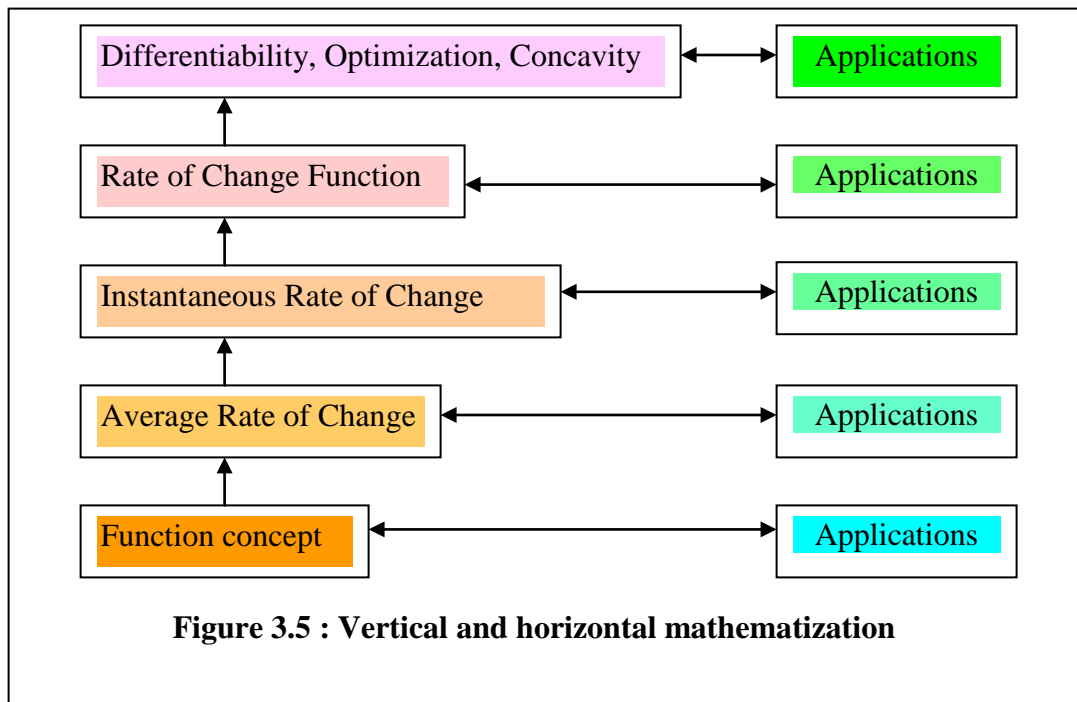
exercises the difference quotient is introduced as the symbolic model for the secant line. In the third activity, the tangent line is introduced as the limiting position of the secant line and its slope used to model the instantaneous rate of change. The derivative at this stage is represented numerically as a value designating the slope at a point and symbolically as the limit of the differential quotient as h tends to zero. Real-world applications of instantaneous rate of change between physical quantities are dealt with in practice exercises. In the fourth activity, the instantaneous rate's variation with respect to changes in the independent variable is introduced as the change of rate function and the derivative function models this scenario. Eventually additional experiences and uses of differentiability and the derivative are used to expand students' concept image of the derivative.

From the proposed instructional sequence it is clear that what is operational matter at one level becomes the subject matter at the next level thus defining and characterizing vertical mathematization or growth in sophistication and representational models, while applications permeate and illustrate usefulness and relevance at all levels. This is in line with Freudenthal's (1991) recognition that learning mathematics amounts to creativity and active construction of models to generalize or idealize conjectures. In support, Cobb et al (2003:240) reaffirm that the term model in the Realistic Mathematics Education (RME) is understood in a dynamic, holistic sense. That is, symbolizations (algebraic, graphical, numerical, etc) are embedded in the process of modeling. Mathematizing thus encourages model-eliciting activities wherein solutions to some problems involve a series of modeling cycles (formulations and reformulations) in which progressively more sophisticated ways of thinking are introduced, tested and refined. The activities under discussion should afford students the opportunity to be deeply engaged with the modeling process in multiple ways and, more importantly, for use as prototypes for thinking and reasoning about other similar manifestations of the derivative in new situations.

3.4.5   The Emergent Model Perspective of modeling
In their proposal for an alternative to the transmission model, Cobb, Yackel and McClain (2000:277) highlight that in mathematical modeling students begin with informal

understandings and progressively build on them and thus their development of mature scientific understanding occurs through incremental reformulations of commonsense knowledge. This is reminiscent of progressive schematization or mathematization as discussed above. Kaput's (1993) *MathCars* programme alluded to earlier (see 2.5.2), is cited as a paradigmatic exploratory case to help students bridge the gap between formal mathematics activity and authentic experience. Instead of the unnatural model of a straight line distance-time graph often presented in text books, the *MathsCars*, as already noted, have a 'dashboard' that records, symbolizes and even graphs car movements simulated by the programme bringing students as close to the authentic driving experience of variable speed as possible (compare with dynamic numeric representations in *Sketchpad*, in 2.8). In a sense the island of mathematics is brought to the mainland of the experiential world.

It is contended that as the model develops, the mental model based on the mathematical representations or symbolizations comes to relate more directly to conceptualizations of the experiential real settings (c.f. Talls's 'enactive' and 'perceptual control' in 2.5.2). The close resemblance between *MathCars* and *Sketchpad*'s dynamic numeric and dynamic graphics generates some optimism about the latter's potential in developing richer concept images of the derivative. Further analysis of the emergent model yields a distinction between symbol system and symbol use. Nemirovsky (1993) for instance, clarifies that a symbol system is an object typically analyzed apart from activity while symbol use is a meaningful, situated activity. The Cartesian plane, as a case in point, is a coordinate system whereas reasoning about functional relationships with the Cartesian graphs is symbol use. In a sense the Cartesian plane can be viewed as some form of platform, workbench or operating system upon which functions can be modeled graphically. With its animation and dragging properties, *Sketchpad* can be characterized as a more sophisticated Cartesian system in which functions can be modeled not only symbolically and graphically but dynamically as well and hence creating wider scope for meaningful symbol use.

Instead of attempting to bridge the gap between the informal and conceptual ways of symbolizing or modeling by assimilating the latter into the former, emergent models fulfill the bridging function between the informal and the formal level by shifting from a '*model of*' to a '*model for*' (Van den Heuvel-Panhuizen, 2003:14). They seek to support an emergence of meaningful symbolizations arising during collective negotiation of meaning. Students first model situations in an informal way (for example, determining speed at point of collision, or crossing race lines at different speeds but same average velocity) and this constitutes a *model of* the situation (instantaneous rate of change). They mathematize their informal modeling activity (through tables of values representing real world data as the limiting position of the secant line (s) is approached, and/or drawing of secants and tangents) and this constitutes a *model for* mathematical reasoning. The model that emerges as students formalize their reasoning gradually assumes an independent form in a sense similar to Tall's (2003) transition from an embodied foundation to proceptual symbolism by means of mental abstraction and compression (compare 4.3.1).

The transition from a *model of* a scenario to a *model for* reasoning with it is also similar to Sfard's (1991) historical analysis of the process of reification, which starts with action, moves to objectification and arrives at pure mathematics. For example, from functions to linear and quadratic functions, to continuous and discontinuous functions and, in the context of this study, arriving at differentiable and non-differentiable functions. Four levels of activity are identified in the transition from *models of* to *models for* (Figure 3.6).
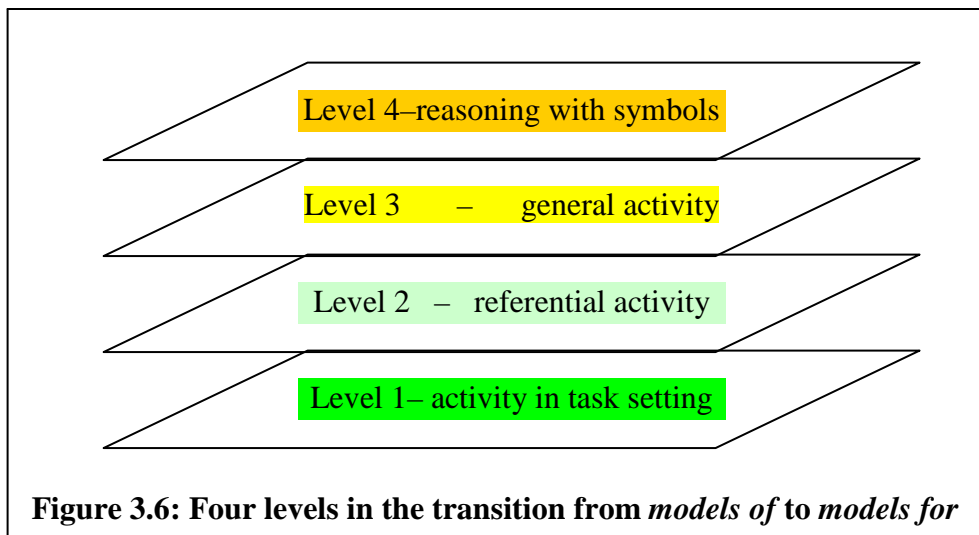
Figure 3.6: Four levels in the transition from *models of* to *models for*

Level 1 refers to activity in the task setting, in which interpretations and solutions depend on the understanding of how to act in the setting and often out of school settings. For example, the rate of change of speed or position of an object in free fall. Level 2 is referential activity where *models of* refer to activity in the setting described in instructional activities (posed more formally mostly at school such as the drawing of tangents). For example, if a train starts from rest, and accelerates uniformly at 10 m/s^2, what is its velocity at the end of the 10th second?

Level 3 consists of general activity in which *models for* make possible a focus on interpretations and solutions independently of situation specific imagery (e.g. graphical representations of prototypical functional relationships such as $y=x^2$, $y=e^x$, etc). Level 4 consists of reasoning with conventional symbolizations, which no longer depend on the support of *model for* activity. For example, symbolic differentiation to obtain the derivative function. In other words, the fourth level completes the reification or abstraction process.

3.4.6 Modeling and cognitive construction

To Lesh and Doerr (2000:365) models are interacting systems based on complex conceptual systems and as such cannot be meaningfully handed over or transmitted to students. Rather, models (**as constructs**) must be constructed by students themselves and not handed over. This implies that for meaningful learning to take place learners must be given the opportunity to construct their own understandings (cognitive models). In other words, a cognitive model is a mental construction or concept image formed by the student. Through a process of construction and reconstruction these mental models or images can be reorganized to achieve stable conceptual systems out of what were initially unstable incomplete models. *Sketchpad* affords students an opportunity to feel in control of the construction and reconstruction processes through a multiplicity of illustrative examples chosen by the student as well. This is partly in harmony with Bakker and Hoffmann's (2005:333) remark that by reflecting on the role of signs from a semiotic

point of view we realise that the possibility of mathematical knowledge depends on the possibility of representing it.

Activity 2.8.5 to explore differentiability at a point is an illustrative case in this study that enables students to investigate differentiability at a corner point, at a point of vertical tangency and at a point of (jump) discontinuity by drawing tangents from the left and from the right of the point being investigated. Ultimately students invent or construct their own generalizations of non-differentiability.

Lesh and Doerr (2000), however, have some few warnings about the model building process. First, construction *per se* can lead to a great many things (such as complex systems of low level facts and skills) that are not models for *making sense* of experiences. Secondly, 'construction' isn't the only process that contributes to the development of models because models can also evolve by being sorted out, refined, or organised, or at least, as often as they evolve, by being (re-) assembled or (re-) constructed. Thirdly, sorting out and refining *unstable* conceptual systems is not the same as assembling stable conceptual systems.

In a sense then, model development ought not to be an end in itself. The authors argue that instruction and learning inspired by the models and modeling perspective should critically focus on 'models that correspond to the "big ideas" or main constructs and conceptual systems that underpin the mathematics curriculum. This emphasis seems to resonate with Tall's (2003) emphasis on a 'global gestalt'. This entails, for instance, exposing students not just to disparate cases of points of non-differentiability but also to examples of functions that are everywhere differentiable (such as polynomials, $\sin x$, $\cos x$, etc), those that are everywhere non-differentiable (such as the blanchmange function, the Koch curve, etc) those that are 'in-between' or differentiable everywhere on the domains they are defined.

3.4.7   Modeling and social constructivism

In real life, groups and not individuals, often develop models in collaboration. The development of models often involves not just (individual) cognitive activity but also social activity. Since individuals can have different understandings of the same shared model there is need for interpersonal negotiation of meaning as espoused by the social constructivism of Ernest (1996). In a *Sketchpad* mediated environment students can be encouraged to work in pairs or in small groups sharing their experiences, discoveries and insights with their colleagues to refine their own as well as their collective cognitive models. These collaborative or cooperative learning strategies are even more pertinent in a computer environment where students work on different PCs at different paces and might be experiencing different versions (variations) of the same concept or representations. For example, students who animate the tangent line in different directions in Activity 2.8.4 or on different scale sizes will experience the derivative of $y=x^2$ in idiosyncratic ways. Those who use big scales will have to deal with the premature disappearance of the gradient function trace for point C for negative values of x while those able to reduce the scale are likely to see more of the gradient function on both sides of the y-axis.

The importance of cooperative learning is supported by Costain's (1995) study of the effects on grades of cooperative learning with or without the use of technology. The results showed no significant differences using technology alone but technology plus cooperative learning produced a significant improvement in attitude whilst cooperative learning by itself produced a highly significant improvement in problem solving. Whole class discussions can then help bring together the class's experiences and thereby opening avenues for broader and deeper understandings of the behaviour of the derived function.

Lesh and Doerr (2003:366) further point out that individuals do not necessarily even have direct access to their own models or conceptual systems because the internal conceptual system often changes as one experiences the external representation created. That is, the more external representations or examples of a concept are experienced the richer the internal representations or concept image. In other words the unstable internal conceptual

systems stabilize as the individual experiences a wider variety of external symbolic/graphic or numeric representations.

3.4.8   The modeling perspective of learning and instruction

In consonance with the cognitive science view of a model, Lesh and Doerr (ibid. p. 364) argue that ways of thinking that are involved in the kinds of models of producing shareable and usable descriptions and explanations represent important forms of learning. In a sense, to model is to learn. Helping students to develop powerful conceptual models (concept images) is viewed to be the most important goal of instruction and learning since it is a means for them to make meaningful sense of their world. A hypothesis of this study is that a *Sketchpad* visualization (multiple-representational) environment can help students develop powerful conceptual models of the derivative.

In mathematics, where the emphasis is on representations (more so in calculus), Lesh and Doerr (ibid) observe that it is often assumed that students do not or cannot develop their own models and articulate them to make sense of systems that involve mathematical entities. Yet when confronted with the need to create meaningful models of experientially real situations all learners can invent appreciable quantities of mathematics by themselves. Given a *Sketchpad* mediated environment that permits multiple representations, experimentation and immediate verification of results it seems possible that students might develop substantial capacities to deal with novel problem situations and therefore can be more successful in solving routine and non-routine problems. This is a plausible view that is in harmony with Freudenthal's (1991) advocacy for guided reinvention by letting students explore and experiment as deeply and as widely as their abilities can afford them.

Given the constructivist belief that students are trying to make sense of their experiential worlds Lobato, Clarke and Ellis (2005) emphasize that a teacher's pedagogical content knowledge might have to shift focus towards understanding what models students are constructing. In this direction, teachers have to use new teaching actions to help students develop conceptual understanding of mathematics. Such actions might include but may

not be limited to the following: constructing models of students' ways of thinking about and understanding of mathematics as suggested by Confrey (1990) and Shifter (2001); choosing and sequencing rich mathematical tasks (Henningsen & Stein, 1997; Lampert, 1990); predicting student reasoning (Smith, 1996); generating and revising hypothetical learning trajectories (Shifter, 2001; Simon 1995) and directing classroom discourse.

Although this list may sound generic, it applies equally well to a technology intensive classroom discourse. Even in a *Sketchpad* environment, for instance, teachers are encouraged to listen more carefully to their students while allowing them the opportunity to engage independently in reflectively abstracting from and reorganizing problem-solving activities. Tall (2006:2) further highlights that a laboratory with computers in it has a very different dynamic from a lecture theatre with a teacher in front and students in serried ranks (compare 1.1). Rather more emphasis should rest on 'hands on' sessions and student-to-student sharing of insights which can add substantial understanding to the introduction of mathematical concepts and help students build more comprehensive conceptions.

3.5   A synthesis of the various perspectives on modeling

In essence modeling is a theoretical framework for the learning and teaching of mathematics. It taps mainly from the Realistic Mathematics Education philosophy and is preferred in the place of the transmission or formalistic models of teaching and learning mathematics. Modeling casts the process of learning mathematics as starting from informal everyday experiences (reality-based) and progressing through a sequence of hypothesis tests, formulations and reformulations (of models) and ways of acting, thinking and reasoning until pure mathematical models or formal understandings are achieved, represented and communicated (shared). This captures the substance of mathematical modeling as a scientific and problem-solving endeavour.

Representation systems support the progressive differentiation and refinement of conceptual systems (organization, simplification, and generalization). Internal conceptual systems and their corresponding external representational systems evolve continually

(through interaction) from unstable, less defined systems to become stable, more clearly defined systems. The general cycle of development is seen as repetitive or iterating until the match between the evolving model and the situation being modeled is as close as possible. This captures the systems perspective of modeling. Lack of stability manifests as cognitive conflict together with adaptation and resolution techniques aimed at increasing stability and attaining cognitive harmony. This is the main motivation behind model development. That is, to attain clearer understanding, and captures the emergent and/or reification perspective of modeling. Generalizing and refining are key activities in the model building process.

The objects of modeling appear to be cognitive processes involved in learning, real world applications problem solving (horizontal and vertical mathematization), mathematical concept representations and the teaching styles suitable for effective learning. Doerr and English (2003) make the following summative view:

> Thus, a modeling perspective leads to the design of an instructional sequence of activities that begins by engaging students with (non-routine) problem situations that elicit the development of significant mathematical constructs and then, extending, exploring, and applying those constructs in other problem situations leading to a system that is reusable in a range of contexts.

3.6   Towards an integrated meaning of modeling in this study

In this study, however, modeling is extended to include the teaching strategies, the mathematical and conceptual development of learners' cognitive processes and models, the modeling and representational/simulation capabilities afforded by a dynamic software environment. In this study it is envisaged that the modeling process starts with the design of an instructional sequence, which engages students in activities that start with real world problems and *Sketchpad* is then used to solve the problem in static and dynamic forms of representation.

The proposed sequence starts with informal real world situations of functional relationships which are then exemplified and solved in *Sketchpad* (see appendix B). Students then familiarize themselves with a variety of relationships in the real world that

87

functions can represent – from polynomials, to exponentials and trigonometric functions. If functions themselves are taken as models of relationships between phenomena or variables then an exposure to a wide variety of information rich examples, implies an enrichment of students' repertoire of differentiable functions.

The average rate of change of a function is introduced informally through the car problem and exemplified in *Sketchpad* through the secant line which is introduced, compared and contrasted with the chord in circle geometry, and its slope explained and illustrated as representing or modeling the average rate of change in an interval. Real world examples such as average speed average temperature, average growth rates of ecosystem population components, average pollution rates, etc are considered in the wider familiarization with the concept of average rate of change. In other words, students are exposed to situations that are modeled or can be modeled by the average rate of change as a precursor to the derivative. Animation or dragging enables students to see multiple possibilities of negative, positive and non-negative gradients, graphically, numerically and dynamically. The differential quotient, $\dfrac{f(x+h)-f(x)}{h}$ is introduced as the formal symbolic model (or representation) of the average rate of change.

The instantaneous rate of change is introduced informally through a ladder problem. The instantaneous rate is then exemplified in *Sketchpad* through the tangent line as the limiting position of a moving/dynamic secant line. Local magnification is exploited to demonstrate that the tangent represents the local straightness of a differentiable function. The slope of that tangent is introduced as representing the derivative at the point of tangency. That is, the derivative at a point or instantaneous rate of change is introduced or modeled as a limit. This limit, if it exists, is illustrated and demonstrated to be numerically the same both from below and from above (or average of left and right approximations) and the opportunity is seized to deepen students' understanding of infinitesimal change. The tangent line is also re-defined practically to be a line joining two microscopically or infinitesimally close points in a dynamic software environment. The algebraic or symbolic derivative is demonstrated/modeled from first principles to

88

be $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$. Further specific real world examples of applications are given to

illustrate that real world phenomena can be modeled mathematically.

A falling cup problem is used to informally introduce the rate of change function. The rate of change function (slope function) is then exemplified in *Sketchpad* by plotting the instantaneous rate of change against changes in *x*. The function is modeled numerically and dynamically as the tangent line is animated or dragged. From the sample table of values (numerical data, which could be derived from the real world) the rate of change can be plotted and represented graphically. From the numeric and graphic dynamic simulations, the rate of change can be conjured up and modeled symbolically as a function. This model can then be used to predict future rates of instantaneous changes and those predictions can, in turn, be checked against the hypothesized relationships (rates of change) to complete the modeling cycle.

A bouncing ball problem is used as an informal situation in which non-differentiability can occur. The non-differentiability of a function at a given point is exemplified in *Sketchpad* to broaden students' appreciation that not all relationships (models) of change are differentiable. Finally an exposure of students to more examples of graphs of functions and their derivatives, and the uses of the derivative in curve sketching and applications to solve optimization problems ought to further enrich students' understanding of the derivative. Of critical significance is that each activity can be treated as a problem-solving cycle and these cycles together form a system of instantaneous rate of change model that is reusable in a range of contexts. This confirms Wessels' (2006:14) conclusion that modeling consists of more than one cycle of problem solving. In this study's conjectured trajectory for learning the derivative concept in *Sketchpad* problem solving is not just a cycle but also a repeated or iterated cycle throughout all activities.

Both horizontal and vertical mathematizations are built in and the end-point of each problem solving cycle or modeling sequence is always a mathematical representation, internally in the learner and externally in *Sketchpad*. In our case each of the four or five

cycles respectively ends up with numeric, symbolic, graphic and dynamic representations of the average rate of change, the instantaneous rate of change, the rate of change function. The external representations make explicit what the student implicitly holds as an internal representation. The external representations in *Sketchpad* help the students to sharpen the contours of their internal representations of the real world systems being modeled. In a sense there is a dialectic relationship between the internal and the external representational systems.

3.7   Modeling as a teaching strategy in each cycle

Each session should begin with a whole class review of previous work and brief explanation of what new material to expect and the functionalities of *Sketchpad* required in the execution of the task(s). Students should then work individually or in pairs to follow instructions on worksheets and to answer questions at each stage where required giving reasons for their answer. The teacher goes round checking progress and offering any *Sketchpad* help where required. After going through in small groups a whole class discussion of findings/results is engaged in and wound up with some extension work being given for further practice.

Group reports are submitted for further analysis and interviews held with selected students. Clearly, the teacher's role is re-defined from that of transmitter of knowledge to facilitator of learning while students use their prior understandings to build new knowledge in each cycle for a deeper and richer understanding of the derivative concept – starting with its preconcepts (average rate of change), then to the notion of gradient at a point of tangency, gradient function and ultimately differentiability and uses of derivatives for curve sketching and optimization purposes. Students assume greater responsibility for their own learning. Specifying the process of learning amounts to gaining insight into the forms of representation of a particular concept. Personal meaning cannot be taught directly but can only be built up by involvement in an educative relationship in which *Sketchpad* is the personal sense-making tool.

3.8   Conclusion

In this chapter an attempt was made to elaborate on the philosophical foundations of modeling in mathematics education with an illumination from various paradigms. Logicist, positivist, empiricist, formalist, constructivist and Aristotelian views of the nature of mathematics have been briefly stated and their influence on modeling assessed. The idea of a model was revisited and elaborated upon. Various perspectives on modeling as a process in Mathematics Education were considered and included 'modeling as a scientific method', modeling as mathematical modeling, modeling as idealization or simplification of complex representational systems, modeling as mathematization, modeling as problem solving and modeling as a teaching strategy. Modeling was also viewed from an emergent perspective, and an encapsulation or transition from *models of* to *models for*. The roles of cognitive construction, social constructivism and learning and instruction in a modeling context were considered. The various perspectives on modeling were synthesized and an integrated meaning proposed for the purposes of this study. Modeling was finally proposed as a teaching strategy in its own right with *Sketchpad* as the modeling instrument.

The next chapter describes, explains and justifies the research approach (es) used in this study together with the data gathering techniques and instruments enlisted. The classroom design experiment is highlighted as the major approach amenable to both qualitative and quantitative analysis. Model-eliciting instructional activities are stated as forming the core of the investigation technique and as an operationalization of the learning trajectory proposed in Chapter II. Post task-based interviews are described as an important qualitative data gathering technique for gaining deeper insight into students' thought processes as they model the derivative. The pre-test posttest design is built into the classroom design experiment methodology in order to compare the performances of the experimental and control groups prior to and after the treatment. Some theoretical frameworks for evaluating students' mathematical knowledge and skills are briefly outlined to preface framework for coding students' responses to the model-eliciting activities, task-based interviews scripts and the pre-test posttest responses.

CHAPTER IV
RESEARCH METHODOLOGY AND DESIGN OF INSTRUMENTS

4.1   Introduction

The purpose of this chapter is to explain and justify the research methodology, and techniques employed in this study. The methodology has the general form of a classroom design experiment or curriculum/instructional design research (Cobb et al 2003, Gravemeijer, 1994). This design is triangulated with pretest-posttest and task-based interview designs. The chapter starts by describing the theoretical underpinnings of design studies, model-eliciting activities and clinical/task-based interviews as legitimate qualitative and quantitative methods in mathematics education research. An assessment framework for the microgenesis of students' concept image of the derivative is described against the backdrop of the Van Hiele thought-level model, the Action-Process-Object-Schema (APOS) Theory, Tall's (2003) Embodied Approach to the Calculus, the Structure of Observed Learning Outcome (SOLO) response level Taxonomy, and Carlson, Larsen and Lesh's (2003) framework for evaluating the development of students' covariational reasoning abilities.

The pre-test is included in order to help determine the experimental and control group students' entry knowledge to the study. That is, it helps us to answer the following question: What prerequisite knowledge and skills do students bring to the study and how can it can be capitalized upon to benefit meaningfully from dynamic software environments? The model-eliciting activities that operationalize the proposed learning trajectory are intended primarily to reiterate answers to the following fundamental research questions: How can dynamic mathematics software be used as a tool or instrument to support the sequential modeling of the derivative? What constraints have to be factored in? What instructional model and didactical relationships are conducive to a successful orchestration of dynamic mathematics software to develop students' concept image of the derivative?

The task-based interviews are intended to answer the following research questions: How do students perceive a technology intensive environment as a medium for organizing and supporting the learning of mathematics? How do students construe the processes induced by dynamic mathematics software to help them construct quality concept images of the derivative? What advantages or obstacles do students experience as they attempt to make sense of the real world with the aid of dynamic mathematics software? The posttest was designed to help determine the exit knowledge of the experimental and control group students. It therefore helps us to answer the following research questions: How do the experimental group students' post learning concept images of the derivative and the associated concepts of function, variability, slope, limit, etc, compare with their pretest understandings? How do experimental students' concept images of the derivative and the associated concepts compare with those of control group students at the end of the study?

4.2   Research Methodology

4.2.1   The Relevance of Design Studies to Mathematics Education

Lehrer and Schauble (2005:636) point out that design studies are a means for building understanding of learning by designing the elements of a learning system and then studying the functions of these interrelated elements as the design is put into play or operationalised. The purpose is to contribute to a greater and deeper understanding of domain-specific instructional processes of learning, not to test large scale Theories (capitalized to distinguish, say Theories of motivation or conceptual development from theories about the development of covariational reasoning or historical reconstruction of the derivative concept in the context of new dynamic software modeling tools). Nor is the purpose to empirically tune 'what works' or to evaluate one design against another except the traditional textbook oriented teacher-centred approach. Rather the intent is to construct and refine what Cobb et al (2003) call a "humble theory", that is an amount of knowledge or model development tightly tied to the particulars. In similar vein the classroom-design-experiment methodology is used in mathematics education to study the process of students' mathematical learning as it occurs in the classroom and the means by

93

which that learning is organized and supported (Cobb, McLain & Gravemeijer, 2003). From a methodological perspective, classroom design experiments are part of the broader research orientation known as design research (Cobb et al, 2003). The distinguishing characteristic of design research resides in how it links the development of humble theory about learning concepts to the design of educational applications and products like curricula.

In the case of classroom design experiments it is argued that the main theoretical products developed are domain–specific instructional theories that consist of substantiated trajectories for students' mathematical learning, and the documented means of supporting and sequencing/organizing learning along those trajectories (Cortina, 2006; Cobb, McLain & Gravemeijer, 2003). A domain specific instructional model is developed by generating theoretical conjectures that inform the design of instructional sequences and other resources supporting students' learning. Such conjectures include but are not limited to historical reconstruction of the way a concept developed, or the "genetic decomposition" of each basic mathematical concept into developmental steps following a Piagetian theory of knowledge (Swingendorf, 2006).

In advocating for transformative teaching experiments through conjecture-driven research design, Confrey and Lachance (2000:231) outline that one of the purposes that educational research must serve is to invent, develop and test novel ways of teaching mathematics. The transformative and conjecture-driven teaching experiment is motivated by a commitment to equity - equal opportunity for all students to participate in and succeed at mathematics. This view is in synchrony with the student population choice made in this study. In supporting the invention, development and testing of effective ways of teaching mathematics, Kelly and Lesh (2000:36) remark that perhaps the most important consideration not yet factored into the designs and expectations of most researchers is the hierarchical nature of the subject matter, i.e. every idea appearing in later mathematics has conceptual and developmental roots in earlier years. This view tallies with the genetic decomposition perspective of mathematical curricula development. As we have seen before, traditional mathematics is taught with undue and

premature emphasis on the abstract and formal, which prevents many students from accessing mathematical concepts, including introductory calculus.

Kelly and Lesh (2002:1) observe that working against current tendencies to establish randomized trials as the 'sine qua non' of scientific methods is a quiet revolution in design-based research methods which are typically applied when describing the iterative refinement of some innovation (often involving technology) in teaching and learning environments. It is a primary goal of this study to attempt to describe the iterative refinement of modeling with *Sketchpad* to expand students' understanding of the concept image of the derivative.

4.2.2   The Relevance of Model–eliciting Activities

Lesh et al (2000:592) define model-eliciting activities to be thought revealing activities that focus on the development of constructs (models, or conceptual systems that are embedded in a variety of representational systems) that provide the conceptual foundations for deeper and higher order understandings of mathematical concepts. That is, the descriptions, explanations and constructions that students generate while working on the tasks reveal how they are interpreting the mathematical situations that they encounter by disclosing how these situations are being mathematized (e.g. quantified, organized, coordinated, dimensionalized) or interpreted (Carlson et al, 2003).

In this study, the researcher designed/developed activities to promote and reveal acts of 'building' or modeling the derivative concept and representing it graphically, numerically, symbolically and dynamically. Since model-eliciting activities are assumed to lead to significant forms of learning, they involve 'local conceptual development' a property that is especially pertinent for modeling the derivative in a *Sketchpad* dynamic software mediated environment. Carlson et al (2003:470) further point out that since significant forms of conceptual development can occur within relatively brief periods of time in model-eliciting activities, it is often possible to observe the processes that students use to extend, differentiate, integrate, refine, or revise the relevant constructs or mental models. Consequently, if activities can be fashioned to promote significant

development of the derivative concept in a relatively short period of time, they have the potential to provide meaningful insight into the development of students' concept images of the concept.

When working through a model-eliciting activity students are expected to produce descriptions, explanations, procedures and constructions of the processes they engage in. Bergen (1999:53) decries the mistaken understanding of the dominant paradigm that the scientific method implies that what exists, in some degree, can therefore be measured in numerical categories. He is also skeptical about the view that if something cannot be measured, it does not exist, except in the imagination of the individual. In contradistinction to these rigid approaches stands the view that the observation of human behaviour in 'natural' settings is an appropriate means by which to understand that behaviour. The observation of students' (cognitive) learning behaviour during their performance or execution of model-eliciting activities can be a rich source of insights about how their understandings evolve as they make sense of the situations at hand. In other words, quantitative research is not the only path to scientific wisdom but a part of multiple means of knowing about an event.

If intermediate solution steps are externalized in forms that can be intelligibly examined by researchers, teachers or students themselves, then the by-products of these learning or problem–solving episodes (or solution models) can generate trails of documentation that transcend beyond mere provision of information about final results. That is, they can reveal important information or clues about the processes that contributed to the results (Lesh et al, 2000:593). When a series of trial ways of thinking is externalized, tested and refined or extended repeatedly model –eliciting activities can support the productivity of ongoing learning or model development experiences.

Appendix B contains the instructional activities used in this study with special focus on modeling function representation (Activity A) , the average rate of change of a function (Activity B), the derivative as the instantaneous rate of change (Activity C), the derivative as the rate of change function (Activity D), differentiability and optimization

(Activity E) and more practice in modeling functions and their derivatives (Activity F), in a *Sketchpad* dynamic software environment. In other words, the activities help to answer the research question about how dynamic mathematics software can be used as a tool or instrument to support the sequential modeling of the concept of derivative and what constraints have to be factored in.

4.2.3   The Relevance of Task-based Clinical Interviews

Task-based interviews are included as part of the design in order to complement model-eliciting activities (compare 4.3.2.3). Genetic decompositions in the model-eliciting activities initially hypothesized by the researcher based on the underlying learning theory are modified and refined. The modification is based on in-depth student interviews together with observations of students' activities and hypothesized underlying thought processes as students attempt to make sense of the concept of derivative. Kelly and Lesh (2000:35) point out that while in the traditional approach educational phenomena derived their status by surviving a variety of statistical tests, today nascent educational phenomena are accorded primacy and the onus is on researchers to describe them in rich and systematic ways. Task-based interviews offer this opportunity.

In encouraging a shift from traditional approaches, Kelly and Lesh (ibid p. 36) point to the changing nature of research questions, not just from investigating instructional approaches that lead to the greatest growth in learning (under controlled conditions) but also to involve model development and model validation. Task-based interviews, offer researchers the opportunity to develop, refine and validate conjectured learning models or trajectories (compare 2.8) about productive ways of thinking about problem solutions. The simplistic use of number and its apparent precision is increasingly disfavoured in mathematics education research because of a condemnation of the thoughtless use of statistics and the immense loss of information when numbers are used (Bergen 1999:53)

Goldin (2000:522) points out that task-based interviews help researchers to make qualitative descriptions of inferred competencies and competency structures, rather than quantitative reduction to scores. Task-based interviews are more useful in exploring the

growth of internal representational capabilities of the students and their interplay with external representations of the derivative in *Sketchpad.*

Task-based clinical interviews enable researchers to pay greater attention to models that learners construct using the process of model development and outcomes of creating representations of conceptual development compared with the descriptions of behaviour. Clinical interviews further enable researchers to transcend the limitations of single measures of achievement by replacing these with more attention being paid to iterative cycles of observation of complex behaviours. In other words, task-based interviews help the researcher to answer the question of how the students' concept image of the derivative evolves in a *Sketchpad* mediated environment.

Kelly and Lesh (2000:515) contend that one reason why teaching experiments have become popular in mathematics and science education is that one of the best ways to learn about the nature of students' state of knowledge is to try to teach something new by inducing changes in existing states of knowledge. In many ways teaching experiments can be viewed as extended clinical interviews or conversely, clinical interviews often can be viewed to be brief teaching experiments. Hence task-based clinical interviews and teaching experiments are mutually supportive qualitative research approaches. In corroboration, Chazan and Ball (1996:16) attest that clinical interviewing techniques have been extremely successful in bringing student thinking to the surface. In other works, task-based interviews help to unravel the students' models of concept images they construct and reconstruct during the learning process. Further, Goldin (2000:520) notes that in comparison with conventional, paper-and-pencil test-based methods, task-based interviews make it possible to focus research attention more directly on subjects' processes of addressing patterns of correct and incorrect answers.

4.2.4   The Relevance of a Pretest-Posttest Design and Sampling Design
The pretest-posttest quasi experimental dimension has been built into the study design to compare achievement between the treatment and the control groups. McMillan and Schumacher (1993:316) observe that while true experimental designs provide the

strongest, most convincing arguments of the causal effect of the independent variable, many circumstances in educational research are not amenable to true experimental design because random assignment of subjects to control and experimental groups is impossible. A control or comparison group is also usually unavailable, inconvenient, or too expensive. Quasi-experimental designs provide reasonable control over most sources of invalidity and are stronger than pre-experimental designs.

In this study the non-equivalent pretest-posttest control design was employed. From a group of 18 students who volunteered to write the pre-test in the pilot phase of the study, a control group of 9 students was selected on a first-come-first-selected basis. (The final phase consisted of 20 pretest participants and an experimental group of seven students). In both the pilot and final phases the researcher used already established groups of subjects, gave a pretest, administered the teaching experiment to one group, and gave the posttest (compare 1.3). The only difference between this design and the pretest-post test control group design is in the lack of random assignment of subjects. The design is represented in Figure 4.1.

Group		Pretest		Teaching Experiment		Posttest
A	⟶	**O**	⟶	**X**	⟶	**O**
B	⟶	**O**			⟶	**O**
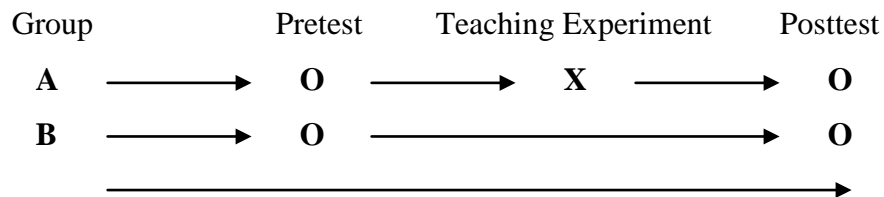
Figure 4.1:   Non-equivalent Pretest–Posttest Control Group Design

The most serious threat to the internal validity of research conducted with this design is that the groups may differ in characteristics that affect the dependent variable. In particular, even though the Applied Sciences students belonged to the same class for the preparatory mathematics lessons, they held different mathematical qualifications. Some had GCE Ordinary Level while others held GCE Advanced level. The latter had considerable calculus background while the former had only pre-calculus knowledge. To minimize the threat to validity, matching of subjects' results became necessary.

The experimental and control group students both volunteered to take the pre-test. However the nine pilot and six final phase experimental group students selected on a first-past-the-post basis could have been more motivated to participate. This is a limitation as more motivated students tend to perform better. Most of the students either had no computer competency or had beginner level only when they enrolled at the University. None of the students had used computers in the learning of mathematics at high school. However, all students had had access to the computer laboratory at university level by the time they took part in the study.

4.3   Design of Instruments

4.3.1 A Theoretical Framework for Assessing Students' Understanding

A primary goal of educational research is to improve instruction and learning which are at the core of educational practice. Educational research is thus directed at an issue of central interest to society: how to understand and improve teaching and learning systematically (Kelly & Lesh, 2002:1). Improvement in teaching can only be meaningfully measured in terms of improvement in learning – which is improvement in understanding of relevant concepts and their applications and implications. Understanding is the quality or integrity of internal mental representations or networks of connections as well as a product that flows from core meanings, not just connections (Williams, 2001:345). Bakker and Hoffmann (2005:333) stress that understanding mathematics presupposes the ability both to transform representations within given representational systems and to change between different systems. Several models of assessment/learning exist in the mathematics education literature as part of a broader effort to measure and/or describe students' level of understanding of mathematical concepts.

4.3.1.1 *The van Hiele Geometric Levels of Thought*

The van Hiele theory is a three-part model that a) describes five sequential and discrete levels that students pass through as geometrical thought develops b) discusses the nature of insight into geometric concepts and c) presents a guide to the phased development of

geometric lessons. Although the descriptions are specific to geometry they are actually stages of cognitive development as van Hiele concedes that the levels are 'situated, not in the subject matter but in the thinking of man' (van Hiele, 1986:41). However progression from one level to the next is not a fortuitous result of maturation or natural development.

LEVEL 5: RIGOR (MATHEMATICAL)
FORMAL REASONING ABOUT *POSTULATIONAL SYSTEMS* POSSIBLE
Phases: 5.1→5.2→5.3→5.4→5.5

LEVEL 4: FORMAL DEDUCTION
DIFFERENT *ELEMENTS OF AXIOMATIC SYSTEMS* UNDERSTOOD *FORMAL PROOFS* POSSIBLE
Phases: 4.1→4.2→4.3→4.4→4.5

LEVEL 3: INFORMAL DEDUCTION (RELATIONAL)
LOGICAL CLASSIFICATION OF FAMILIES OF SHAPES
Phases: 3.1→3.2→3.3→3.4→3.5

LEVEL 2: DESCRIPTIVE (ANALYTICAL)
SHAPES CHARACTERISED BY THEIR *PROPERTIES*
Phases: 2.1→2.2→2.3→2.4→2.5

LEVEL 1: VISUAL (RECOGNITION)
SHAPES OPERATED ON ACCORDING TO THEIR *APPEARANCE*
Phases: 1.1→1.2→1.3→1.4→1.5

Figure 4.2: Van Hieles' Geometric Thought Levels

Rather it is the nature and quality of the experience in the teaching/learning programme that influences a genuine advancement from a lower to a higher level as opposed to the learning of routines and procedures as a substitute for conceptual understanding. This

101

focus of the theory on teaching as facilitation of greater understanding is a relevant accent.

The Van Hiele Geometry Test designed to test the van Hiele theory has five subtests, each of which contains five questions written to correspond with characteristic behaviours or statements associated directly, with a specific and unique van Hiele level (Crowley, 1990; Usiskin & Senk, 1990). Mastery of a subtest is determined by scoring at or above an established cutoff score and an examinee is assigned a van Hiele level based on the sequence of subtests mastered. Through these tests, administered on 2699 students, and the resultant cross tabulation analyses of fall and spring results, Usiskin and Senk (1990) found that the theory was descriptive enough as to be universally used, and predictive enough about students' performance in proof-writing. Land (1990) investigated the appropriateness of the theory in algebra and came to the conclusion that the theory was applicable. Figure 4.2 is a schematic summary of the van Hiele model of geometrical thought development.

4.3.1.2 The Structure of Observed Learning Outcome (SOLO) Model

Pegg and Davey (1998:116) criticize the van Hiele theory for being adequate only as an important (even necessary) first step in understanding cognitive growth. They argue that new tools or different frameworks are needed to move the model to its next phase. The Structure of Observed Learning Outcome (SOLO) Taxonomy, originally mooted by Biggs and Collis (1982), appears to offer a neo-Piagetian general model of intellectual development by postulating that all learning occurs in one of five modes of functioning. These modes correspond reasonably closely to Piaget's stages of development and are not applied to geometry or spatial thinking.

A fundamental departure from the Piagetian formulation is that student growth in understanding is not seen in terms of stages related to some overall age- or mind-related logical structures. Rather, understanding is viewed as a more individual characteristic that is both content and concept specific: the amount of information that can be retained, and features specific to the task, are important determining variables. Tall (2003:3)

further notes that the SOLO Taxonomy differs from Piaget's theory in that it is intended to provide a template for assessment. Coding a specific performance (response) is dependant on its nature or abstractness (mode of thinking) and on the individual's ability to handle, with increased sophistication, relevant cues (levels of response). Five modes of thinking are identified: sensori-motor (soon after birth); ikonic (from 2years), concrete symbolic (6 or 7 years): formal (15 or 16 years) and the post formal (possibly around 22 years (Pegg, 2002:242). Within each mode responses become increasingly complex as the cycle of learning develops and three levels of mastery are identified as unistructural, multistructural and relational understanding.

Formal Mode					U_1
Concrete			R_1		R_2
Symbolic		M_1			M_2
Mode	U_1			U_2	
Ikonic Mode	R_2				
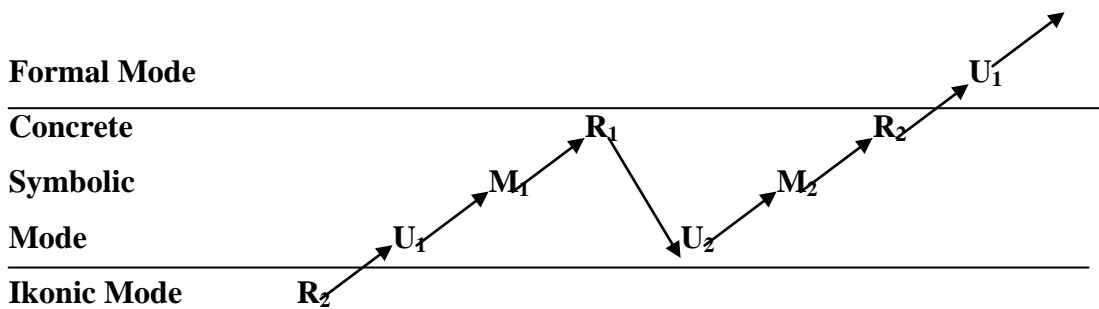
Figure 4.3: Diagrammatic representation of levels associated with the concrete symbolic mode (Adapted from Pegg, 2002:245)

A strength of the SOLO model is the linking of the cyclical nature of learning and the hierarchical nature of cognitive development. Each level of functioning within a cycle has its own integrity, its own idiosyncratic selection and use of data. Figure 4.3 summarizes this duality of the model which, appears to be a general non-domain specific theory.

4.3.1.3 *APOS Theory*

The acronym APOS stands for 'Actions, Processes, Objects and Schemas'. It is a particular theory of how mathematics is learned and, according to Dubinsky et al (2005a) can be used to analyze, from a cognitive perspective, classical issues related to the concept of mathematical infinity. The understanding of the infinity, especially the actual or completed infinity, is crucial to the understanding of the limit concept in calculus and, by proxy, the derivative as a limit. APOS Theory involves the mental mechanisms of

encapsulation and interiorization and uses the principle that *actions* become repeatable *processes* which are interiorized and encapsulated as *objects* and then related in a wider *schema* (Tall, 1997). That is, there is a close relationship between the nature of a mathematical concept and its development in the mind of the individual.

Through interiorization and encapsulation cognitive structures (actions, processes, objects and schemas) are built by which the development of a mathematical concept can be described in terms of transformations on objects to obtain other objects. A transformation is first conceived as an action in that it requires specific instruction as well as ability to perform each step of the transformation explicitly. As an individual repeats and reflects on an action, it may be interiorized into a mental process. A process is the mental structure that performs the same operation as the action being interiorized, but wholly in the mind of the individual (cf. thought experiment). For example an individual with a process-level understanding of functions will construct a mental process for a given function and think in terms of inputs and transformation of those inputs to produce outputs (c.f. black box approach to functions as machines).

If however, one becomes aware of a process as a totality, one realizes that transformations can act on that totality and can actually construct such transformation, then the individual is adjudged to have encapsulated the process into a cognitive object. For the function concept, for example, concept encapsulation allows the individual to apply transformations of functions such as a formal set of functions, defining arithmetic operations on such a set, equipping it with a topology etc (Dubinsky et al, ibid). If the individual is further able to organize and relate/link the many actions, processes and objects involved in a particular mathematics topic into a coherent framework, the individual is adjudged to have formed a schema (c.f. Freudenthal's schematization) which can be used to decide which mental structures to employ/utilize in mathematical problem solving. Figure 4.4 is a schematic summary of the Theory which Dubinsky et al (2005) are optimistic can be used to formulate a new explanation of how people might think about the mathematical concept of infinity. Hence it is assumed possible to interpret the concept of derivative as a limit similarly.

| Cognitive Structures | Mental Mechanisms |
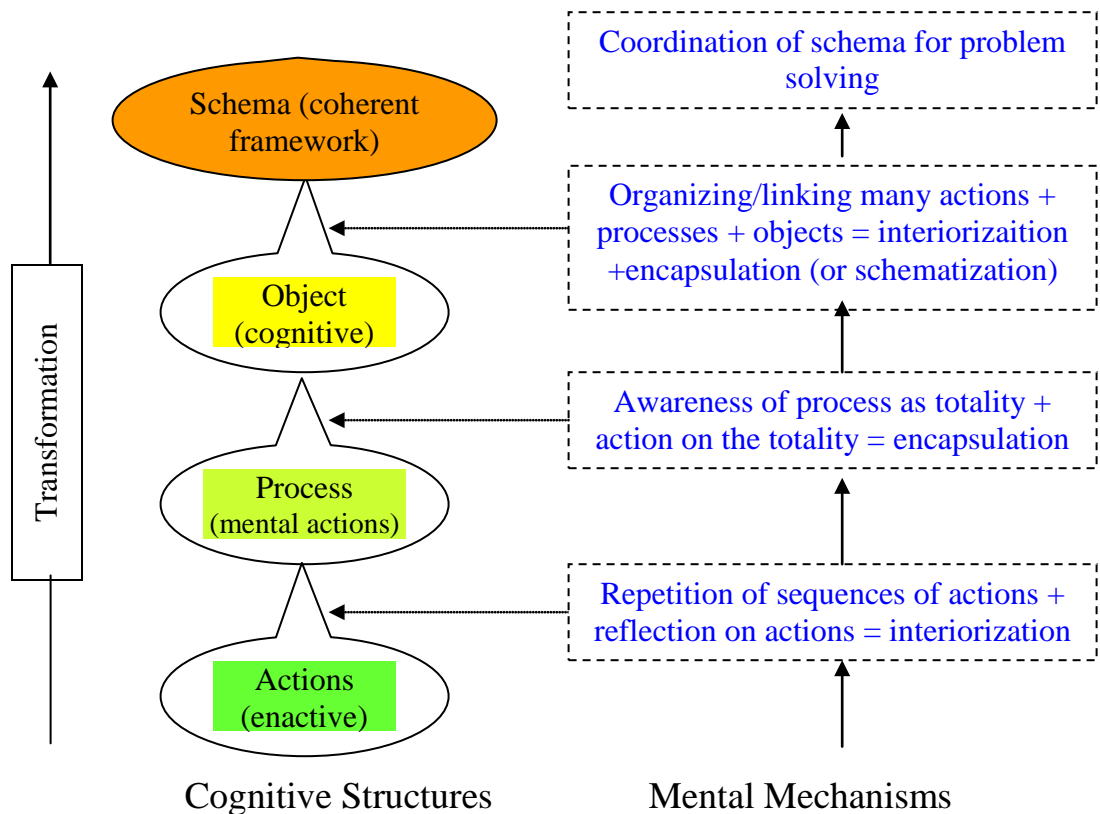
Figure 4.4: Schematic Representation of the APOS Theory

4.3.1.4 *An Embodied Approach to the Calculus*

Tall proposes an embodied approach to the calculus by drawing inspiration from Bruner, Piaget, the van Hiele and SOLO models, and APOS Theory (Tall, 1997; Tall, 2003; 2006; Tall & Tirosh, 2001; Tall, 2006 and Gray & Tall, 2006). In this framework mathematical thinking is categorized into three significantly different worlds (Figure 4.5). The first is an object-based conceptual-embodied world reflecting on the senses to observe, describe, define and deduce properties developing from experiment and thought experiment to Platonic Euclidean thought and proof (Tall, 2006). It is based on human perceptions and actions in a real world context including but not limited to enactive and visual aspects (Tall, 2003:3).
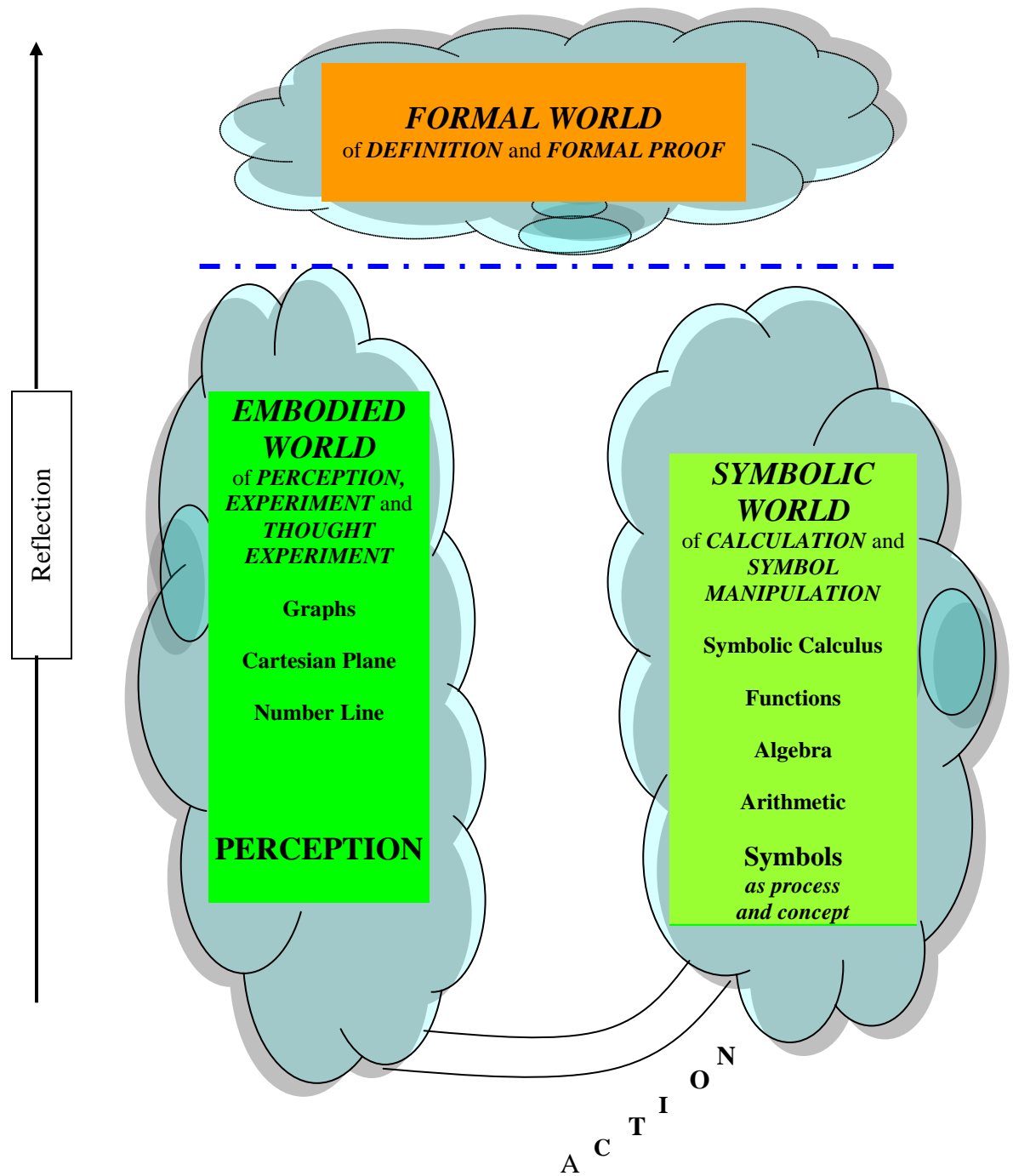
Figure 4.5: Tall's Three Operational Worlds of Mathematics

Empirical and reflective abstraction in shape and space lead to a van Hiele type development through a conceptual embodied world of thought in which visual perception is verbalized leading to growing dominance of verbal description (Gray & Tall, 2006).

The second is an action-based symbolic-proceptual world that begins with real world actions that are symbolized and considered as concepts to lead successfully to symbolic calculus. The symbols, called 'procepts' operate dually as processes and concepts to give information beyond the embodied world (Tall & Ramos, 2004:3). That is, the symbol acts as a pivot representing the process and the output of that process as a concept to form a procept (Tall, 2000, 2003). For example, the process of differentiation yields the derivative, while integration yields the integral as the output.

The third is the property-based axiomatic world of axiomatic systems, definitions and formal, set-theoretic, proofs (Tall & Ramos 2004:3). The formal approach starts from selected axioms and makes logical deductions to prove theorems, which may then be used as steps in building up a systematic formal theory (Tall, 2003; Gray & Tall, 2006). There are obvious similarities with the fourth and fifth levels of the van Hiele theory here but no specific scheme of student assessment or descriptors for level identification.

It is critical thus to observe that both the van Hiele and the SOLO models as well as the APOS frameworks are Theories in capital letters rather than domain specific developmental trajectories within a specific topic level. Tall's Embodied Approach is also a broad theoretical framework for the development of mathematical reasoning. In this regard, Pegg (2002:245) concedes that over the recent past a range of learning theories has appeared in which cycles of learning with a 'local' flavour have been used to explain and predict cognitive development in specific topic domains in mathematics education.

Kelly and Lesh (2002:2) similarly point out that standardized paper and pencil measures are often, politically, the "gold standard" of learning yet they are too generic for innovations in specific technological environments such as envisaged in this study. How can we measure learning and cognitive change objectively as well as qualitatively while students are actively involved in the design study? In this study students' responses in the pretest and posttest were assessed through a system of coding the concept images inferred

from their mental actions. This builds up on the coding of mental actions in Carlson et al's (2003) Covariation Framework.

4.3.1.5 *The Covariation Framework*

In the *Covariation Framework* six categories of mental actions have been observed in students when applying covariational reasoning in the context of representing and interpreting a graphical model of a dynamic function event. Figure 4.6 summarizes the categories. A form of *Differentiation Competency Framework* (Kendal & Stacey, 2002:148) has been designed for this study to provide direction and organize data collection, analysis and interpretation more in a manner resembling the micro-genesis of the derivative from the average rate of a function, to the instantaneous rate of change and ultimately the rate of change function concept in a graphical context.

<div style="border:1px solid black; padding:10px;">

Covariation Framework

Categories of Mental Actions (MA)

MA1 An image of two variables changing simultaneously;
MA2 A loosely coordinated image of how variables are changing with respect to each other (e.g. increasing, decreasing);
MA3 An image of an amount of change of the output variable while considering changes in fixed amounts of the function's domain;
MA4 An image of rate/slope for contiguous intervals of the function's domain;
MA5 An image of continuously changing rate over the entire domain;
MA6 An image of increasing and decreasing rate over the entire domain

Figure 4.6 Categories of Mental Actions in the Covariation Framework

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The framework is exemplified in the context of The Bottle Problem (see Figure 4.7) where *MA1* is the emergence of an image of the water level changing while imagining increasing amounts of water in the bottle; *MA2* is the emergence of an image of the height increasing, as the amount of water in the bottle increases; *MA4* is the emergence of an image of the slope/rate of height change with respect to an imagined fixed amount of water until the bottle fills up; *M5* is the emergence of an initial continuous image

The Bottle Problem (adapted from Carlson et al, 2003)

Imagine this bottle filling with water. Sketch a graph of the height as a function of the amount of water that is in the bottle
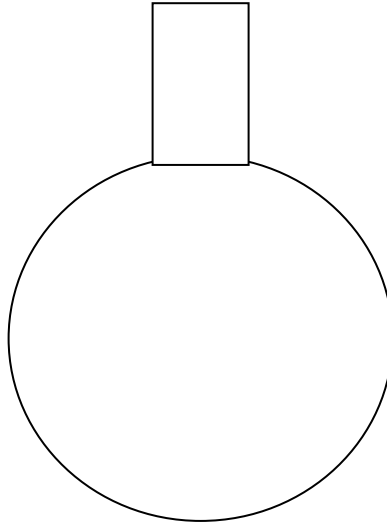
Figure 4.7: The Bottle Problem

slope/rate of height change with respect to volume i.e. As the volume of water is imagined to change continuously, the rate of change of height is continuously adjusted suggesting a smooth curve on the graphical representation and *MA6*.

4.3.2 Coding of students' responses

4.3.2.1 Guidelines for Coding of Pretest and Posttest Items

With the benefit of the discussed models, the pre-test /posttest items in Appendices A and C were scored/coded as follows:

Question 1: The Function Concept

Rationale: To ascertain students' graphical representational competencies for single variable functions. Functions are a central building block of calculus.

Coding: Wrong graph or no graph at all   0

Correct graph shape (concavity)   1

Correct shape and sense of scale (proportionality)	2
Correct graph shape plus intercepts	3
Correct graph shape, intercepts and partly correct explanation	4
Correct graph shape, intercepts and complete explanation	5

Question 2: Calculation of the Average Rate of Change of a Function

Rationale: To establish students' procedural ability to calculate the gradient between any two points on a graph, hence the ability to calculate the average rate of change of a function appreciation of the average rate of change.

Coding in each case: Incorrect response	0
Correct response	1

Question 3: Gradient of Tangent Line as limit of Secant Line gradient.

Rationale: To test qualitative appreciation of continuous change in rate of change and the existence of a limit as a function approaches a given point/ limiting value. This is a fundamental concept to understanding the derivative from first principles.

i) Calculation of the gradient of a straight line through two given points.
Coding:

No answer or wrong answer given	0
Correct numerical answer	1
Correct algebraic answer	2

ii) Representation of the gradient of line AT
Coding:

No answer or wrong answer given	0
Idea of limit evident but inadequately explained	1
Idea of limit adequately explained	2

Question 4: Differentiability at a point

Rationale:	This question was phrased so that students could reveal knowledge of	
	differentiability, non-differentiability via the existence, computability or	
	otherwise of the gradient at a designated special point.	
	Coding in each case:	
i)	No answer or wrong answer given	0
	Correct answer given	1
ii)	No explanation or wrong explanation given	0
	Correct explanation given	1

Question 5 (Pretest Bottle Problem):

Rationale:	Demonstration of qualitative understanding of variable rate of change in a	
	real-world application context.	
Coding:	No response/wrong response	0
	Concept image of two variables varying simultaneously	1
	Concept image of the general direction and strength of covariation	2
	Concept image of changing rate of covariation	3
	Concept image of turning points	4
	Concept image of proportionality of rates of change	5

Question 5 (Posttest): A graph and the qualitative graphical representation of its derivative.

Rationale:	This item was selected to test students' ability to make connections	
	between the graph of a function and that of its derivative and to justify	
	their choices.	
Coding:		
a)	Identifying a graph with a given graphical representation of it derivative	
	Wrong choice of graph with given derivative	0
	Correct choice of graph with given derivative	1
b)	Explanation of choice	
	Wrong explanation or no explanation	0
	Correct verbal explanation	1
	Correct differentiation explanation	2

	Correct integration explanation	3

Question 6 (Posttest): Sketching derivatives of given graphs

Rationale:	This item was included to assess students' abilities to draw sketch graphs for the slope function (derivatives) of given graphs. In a geometrical context, this would be indicative of the slope/rate of change perception.	
Coding:	Incorrect graph	0
	Correct left **or** right graph	1
	Correct left **and** right graph	2
	Additionally aware of strength of rate of change	3

Question 7 (Posttest): Modified Bottle Problem

Rationale: To assess students' qualitative understanding of differing rates of change

a) i.	Concept image of positive covariation	1
	Concept image of constancy of covariation	2
	Concept image of strength of covariation	3
ii.	Concept image of continuous positive covariation	1
	Concept image of changing strengths of covariation	2
	Additional concept image of inflexion point	3
	Additional concept image of terminal constancy of covariation	4
	Additional concept image of proportionality	5
b) i.	Concept image of constant rate of change (derivative)	1
	Additional concept image of correct *y*-intercept	2
ii.	Concept image of rate of change as dependant variable	1
	Concept image of y-intercept of the derivative	2
	Concept image of changing derivative	3
	Concept image of turning points of derivative (concavity)	4
	Concept image of proportionality of the whole derivative	5

4.3.2.2 Coding students' written responses to *Sketchpad* tasks in Appendix B

Artigue (2002:254) stresses that activities which are posed to students and their didactic management certainly play an essential role in shaping the nature of instrumented actions. This suggests that the outcome of the instrumented or modeled actions is the end-product of the dialectical relationship between the technical (processes) and the conceptual (representations) in *Sketchpad*. Table 4.1 summarizes the laboratory activities in the proposed learning trajectory (see Appendix B). Students' responses are coded in the following chapter and categorized in terms of technical processes in *Sketchpad* and conceptual (mathematical) representations enabled/performed (see 5.3.3).

Activity	Brief Description of Activity
Activity A	Modeling the function concept in *Sketchpad*
Activity B	Modeling the average rate of change in *Sketchpad*
Activity C	Modeling the derivative as the instantaneous rate of change in *Sketchpad*
Activity D	Modeling the derivative as the rate of change function in *Sketchpad*
Activity E	Modeling differentiability and optimization in *Sketchpad*
Activity F	Further practice in modeling functions and their derivatives in *Sketchpad*
Table 4.1: Summary of *Sketchpad* Activities for modeling the derivative	

4.3.2.3 The task-based interview script/guide based on activities in Appendix B

Goldin (2000:523) notes that the rationale for the task-based interview guide is to elicit a complete, coherent verbal justification for each student's responses and a coherent external representation constructed by the student through as uniform a questioning sequence as possible. Huntley, Rasmusen, Villarubi, Sangtong and Fey (2000:358) also strongly contend that one-to-one conversations with students help us to see if their abilities to interpret algebraic calculations are as limited as their written work often suggests. In the context of this study those conversations are not circumscribed to symbolic manipulations, if any thing they lean towards multiple representations that *Sketchpad* affords the students and the link between them. This should help answer the following research questions: What is the quality and types of concept images of the derivative that students are able to form with the mediation of *Sketchpad*? What connections are students able to make between different *Sketchpad* dynamic mathematics

representations of the derivative? What advantages or obstacles do students experience as they attempt to make sense of the derivative with the aid of *Sketchpad* dynamic mathematics software? (compare 1.2 & 5.4.1)

Activity A: Modeling the Function Concept in *Sketchpad*

Stage 1:   Can you describe to me how you would plot a function in *Sketchpad*? Choose a function of your choice.

Stage 2:   How would you represent the function in dynamic graphic form?

Stage 3:   How would you demonstrate to a friend how to represent the function in static and dynamic numeric forms?

Stage 4:   Can you give examples of real-world situations that can be represented by the function you have chosen?

Activity B: Modeling the Average Rate of Change of a Function

Stage 1:   Can you describe to me how you could use *Sketchpad* to demonstrate and calculate and/or measure the average rate of change of a function between any two points of its domain?

Stage 2:   Can you demonstrate to me how you would construct a secant line in *Sketchpad*?

Stage 3:   How would you find the slope/gradient of the secant line in *Sketchpad*?

Stage 4:   How would you explain to a friend what this gradient represents? Can you give real world examples?

Stage 5:   Can you give real world examples of the average rate of change?

Activity C: Modeling the Derivative as the Instantaneous Rate of Change

Stage 1:   Can you describe to me how you would use *Sketchpad* to demonstrate /calculate the slope of a graph at a given point of its domain?

Stage 2:   Can you show me how to construct a tangent line when given a secant line in *Sketchpad*? What is the value of the gradient of the tangent line at the point of contact?

Stage 3:   Explain the meaning of the value of the gradient of the tangent at a point. Is the value always positive, negative or both? Explain your answer.

Stage 4:   Can you give real world examples of the instantaneous rate of change?

Stage 5:   How would you express the gradient of the tangent symbolically as a limit of the secant line gradient? How many ways?

Activity D: Modeling the Derivative as the Rate of Change Function

Stage 1:   In *Sketchpad*, how would you construct the graph of the function showing how the gradient of the tangent line changes with respect to *x*?

Stage 2:   How would you plot the slope of the tangent line against the *x*-values in *Sketchpad*?

Stage 3:   If you trace Point C and animate the tangent what pattern is the path traced by the plotted point?

Stage 4:   How would you represent this path (pattern) numerically in *Sketchpad*?

Stage 5:   How can you express the function (path) symbolically as a limit of the differential quotient for the general point on the graph of *f(x)*?

Stage 6:   How can you verify your findings in Stage 5 using *Sketchpad*?

Activity E: Modeling Differentiability and Function Optimization

Stage 1:   Can you demonstrate to me how you could use *Sketchpad* to explore the differentiability of a graph's corner, point of vertical tangency and point of discontinuity?

Stage 2:   What can you say about the value(s) of the left and right tangent gradients (limits) at a corner point?

Stage 3:   What can you tell me about the value(s) of the left and right gradients (limits) at a point of discontinuity?

Stage 4:   What is the value of the tangent/slope at a point of vertical tangency? Explain your answer.

Stage 5:   What conditions must be satisfied for a function to be differentiable at a point in an interval?

Activity F: More Practice in Modeling Functions and their Derivatives

Stage 1:   Can you describe to me how you would graph a function and dynamically generate its derivative in *Sketchpad*?

Stage 2:   How would you represent the derivative in static and dynamic numerical forms?

Stage 3:   How would you verify the accuracy of the graph of your derivative?

Stage 4:   What advantages do you think *Sketchpad* has over the textbook approach to teaching/learning the derivative?

Stage 5:   What difficulties did you have with laboratory sessions that we conducted?

4.4   Conclusion

In this chapter the study has been projected as attempting to utilize both quantitative and qualitative methods of investigating the research questions. To this end design studies in Mathematics Education have been highlighted as a relevant means of gaining deeper understanding of domain-specific instructional processes of learning. Model-eliciting activities in a *Sketchpad* environment have been proposed starting with the modeling of the function concept in static and dynamic graphic, static and dynamic numeric, and static and dynamic symbolic representations. The average rate of change of a function has been modeled as the differential quotient in the same modes of representation as for the function concept. The derivative has then been modeled in static and dynamic graphic, numeric and symbolic forms, first as the instantaneous rate of change of a function where the tangent slope is treated as the limiting position of a convergent sequence of secant lines and then as the rate of change (gradient/slope) function. These activities have been justified as forming the core of the instructional treatment effort to support a deeper conceptual understanding of the derivative in introductory calculus Task-based interviews have been proposed as a legitimate and complementary means of gaining in-depth understanding of students' thought processes as they construct models of the derivative in *Sketchpad.*

The incorporation of the pretest-posttest quasi-experimental design into the study has been justified as a suitable alternative to the less practicable true experimental design for

measuring both entry knowledge as well as learning progress. The van Hiele geometric thought level model, the APOS Theory of mathematical learning, the Embedded Approach to the Calculus, the SOLO model and the Covariation Framework for assessing students' learning have been outlined as theoretical guidelines for evaluating and analyzing student performance. A more domain specific assessment framework has been chosen and suggested for the coding of students' knowledge and skills in the micro-learning of the derivative in a *Sketchpad* intensive dynamic mathematics environment. Task-based interview scripts used in this study have been described in detail to complete the triangulation of data gathering techniques.

In the next chapter the data gathered are presented and initial findings discussed. Pretest results are presented first and analyzed to determine experimental and control students' pre-calculus knowledge both qualitatively and quantitatively. Experimental students' responses to the *Sketchpad* modeling activities are then analyzed to ascertain how students cope with them. The students' responses are eventually categorized according to how they are able to construct multiple representations and make connections between them. The dialectic relationship between technical and conceptual skills/knowledge is also analyzed in the process. Anecdotes of task-based interviews are reported upon in an attempt to get to grips with the evolution of the experimental students' concept images of the derivative and antecedent concepts. Finally the posttest results are presented and compared with pretest results for both the experimental and control groups.

CHAPTER V
DATA AND DISCUSSION OF DATA

5.1   Introduction

In this chapter the pretest results are analyzed to determine and report on students' entry knowledge of (pre-) calculus concepts prior to the commencement of the classroom design experiment. Such knowledge should on the one hand help to ensure that any differences in attainment levels in the posttest are not due to previous learning. On the other hand, such knowledge should help obviate exaggerated claims about changes that may appear to have occurred. The analysis of the classroom activities with the experimental group dwells crucially on the individual paper trails of the *Sketchpad* instructional activities. Although paper trails cannot give us clues about any thinking that was not put down on paper, they nevertheless serve as important referents for the researcher to make inferences about the evolution of students' thought processes. The focus of the activity trail analysis was therefore to ascertain the manner and extent to which the *Sketchpad* dynamic mathematics software fostered the construction of richer concept images of the derivative in introductory calculus. Classroom observations also helped the researcher to come to grips with the interactions and constraints undertaken or encountered during the activities. From the benefit of hindsight, the activity trail analysis sheds further illumination on the question of what contribution technology can potentially make towards deeper conceptual understanding of the derivative and what revisions might be necessary to improve the appeal of the activities to the students.

The (post) task-based (clinical) interviews are analyzed with close attention being paid to how students managed to articulate their understandings of the derivative concept and how they construed the processes induced by dynamic mathematics software to help them construct rich concept images of the derivative. The interviews also helped to determine how students perceived a technology intensive environment as a medium for learning mathematics including opinions on the affordances and obstacles experienced as they attempted to make sense of the derivative.

Finally, the posttest results are presented and compared with the pretest results to establish the extent to which the exit knowledge of the experimental group differed from that of the control group students.

5.2 Pretest investigations

5.2.1 The Function Concept: Question 1

Sketching of the curve of $y = x^2$ in this question was successfully performed for the shape by all students who took the pre-test except two suggesting that students were generally familiar with the quadratic function (parabola). Most of the students (16/18) or 89% had no difficulty with the intercepts for the graph. Final phase pretest results were even better at 94 %. Descriptions of how the first graph changes when the parameters are

	Student	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	% success
Pilot Phase	Level	A	A	O	A	A	A	A	O	A	A	A	A	A	O	A	O	A	A	
	Sex	M	M	F	F	F	F	F	M	M	F	M	M	M	M	M	M	M	M	
	Literacy	C_1	C_1	C_2	C_1	C_1	C_1	C_2	C_1	C_1	C_2	C_1	C_3	C_1	C_1	C_1	C_1	C_1	C_1	
	$y = x^2$	3	3	3	3	3	3	3	0	3	3	3	3	3	nr	3	3	3	3	89%
	$Y=x^2+1$	4	5	3	3	3	3	2	0	5	5	5	1	1	0	4	5	1	5	61%
	$Y=x^2-1$	5	4	3	3	3	3	3	0	5	5	5	1	1	0	4	5	1	5	62%
	$Y=2x^2$	5	3	2	2	1	3	3	0	4	3	0	3	0	0	4	4	3	5	50%
	$Y=x^2+x$	3	3	2	2	3	1	1	0	5	0	2	3	0	0	0	1	1	5	36%
	Total	20	18	13	13	13	13	12	0	22	16	15	11	5	0	15	18	9	23	57%
	Student	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16			% success
Final Phase	Level	A	O	A	A	O	A	A	A	A	A	A	A	A	A	A	O			
	Sex	F	F	M	M	M	F	F	M	M	M	M	M	M	F	F	F			
	Literacy	C_1	C_1	C_1	C_1	C_1	C_1	C_1	C_1	C_1	C_1	C_1	C_1	C_1	C_1	C_1	C_1			
	$y = x^2$	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	nr			94%
	$y=x^2+1$	5	3	4	3	3	3	3	3	5	5	5	3	4	3	5	nr			71%
	$y=x^2-1$	5	2	4	3	3	3	3	3	5	5	5	3	4	3	5	nr			70%
	$y=2x^2$	4	1	4	3	2	2	2	3	1	3	4	3	2	2	3	nr			49%
	$Y=x^2+x$	3	3	3	3	1	0	0	3	1	3	0	1	0	0	3	nr			30%
	Total	20	12	18	15	12	11	11	15	15	19	17	13	13	11	19	0			60%

Table 5.1: Pretest responses to the graphical representation of the function concept

changed, however, presented challenges to some of the students, especially for $y= 2x^2$ and $y= x^2+x$ where the effect of adding x to x^2 causes a shift of the parabola's turning point to the left. The success rates were 50% (pilot phase) and 49% (final phase) for $y= 2x^2$ and 36%/30% for the graph of $y= x^2+x$. On the whole the success rate for the questions on the function concept was 57%/60% suggesting that the participants were reasonably ready to tackle the calculus activities (see Table 5.1).

The two pilot students who could not represent the graphs of the functions correctly had passed Ordinary Level Mathematics. The first of these (Student 8) could not produce the correct shape of the graphs. The second, Student 14, only drew the x- and y- axes and offered no further responses. However, two other students with only 'O' Level background (respondents 3 and 17), performed satisfactorily in curve sketching but could not explain the transformations of the graph of $y=x^2$. This was not peculiar to the O level holders. Final phase results were hardly different. (c.f. zone of proximal development)

Graphs of $y=x^2 + 1$ and $y = x^2 - 1$

Twelve pilot students sketched the graph of this function correctly with the appropriate y-intercepts and curvature (concavity). Two candidates with 'O' level Mathematics could

not sketch this graph while a further four with A level Mathematics also did not draw sufficiently accurate graphs. Students12 and 18 with A level background had the correct shapes but incorrect intercepts for $y = x^2 + 1$ and $y = x^2 - 1$ and seemed remediable. They confused the axes along which the subsequent modifications should be translating. Overall attainment on these graphs was 61% and 62%. The final phase attainments were similar to each other at 71% and 70% respectively.

The graph of $y = 2x^2$

Only six out of 18, or 33% of the pilot students drew a correct graph for this function. Increasing the graph by a factor of two meant the graph should be 'taller' to show faster (steeper) growth of y per unit increase in x. Only two students gave a correct explanation of how the original graph is 'transformed'. The rest of those who attempted this item provided no explanation for the nature of the transformation. As already noted, the final phase attainment on this question was hardly different from the pilot findings.

There was no significant difference in performance between males and females both in the pilot and final phases. The six pilot female students scored an average of 13,3 while the boys scored an average of 13. However, there was more spread of marks for males with a Standard Deviation of 8.034 compared with a Standard Deviation of 1.366 for females and an overall Standard Deviation of 6.507. An analysis by Mathematical level background showed that the average for the pilot 14 A Level Mathematics holders was higher at 14.64 compared with 7.75 for the four O Level Mathematics holders. The t-test value of 6,585 for independent samples shows that the A Level holders performed better at a significance level of $p < 0,001$ for 16 degrees of freedom.

On balance, however, students seemed to be conversant with the notion of function, an important pre-calculus concept for the learning of the derivative. Taking note of students' readiness to tackle higher order concepts in mathematics is a crucial Vygotskian injunction that students should be presented with problems that lie within their zone of proximal development (Murray, Oliver & Human, 1993;73). Szydlik (2000) reaffirms that functions are the primary objects of interest in calculus (compare1.1 and 4.3.2.1).

5.2.2 The Gradient Concept (as Average Rate of Change)
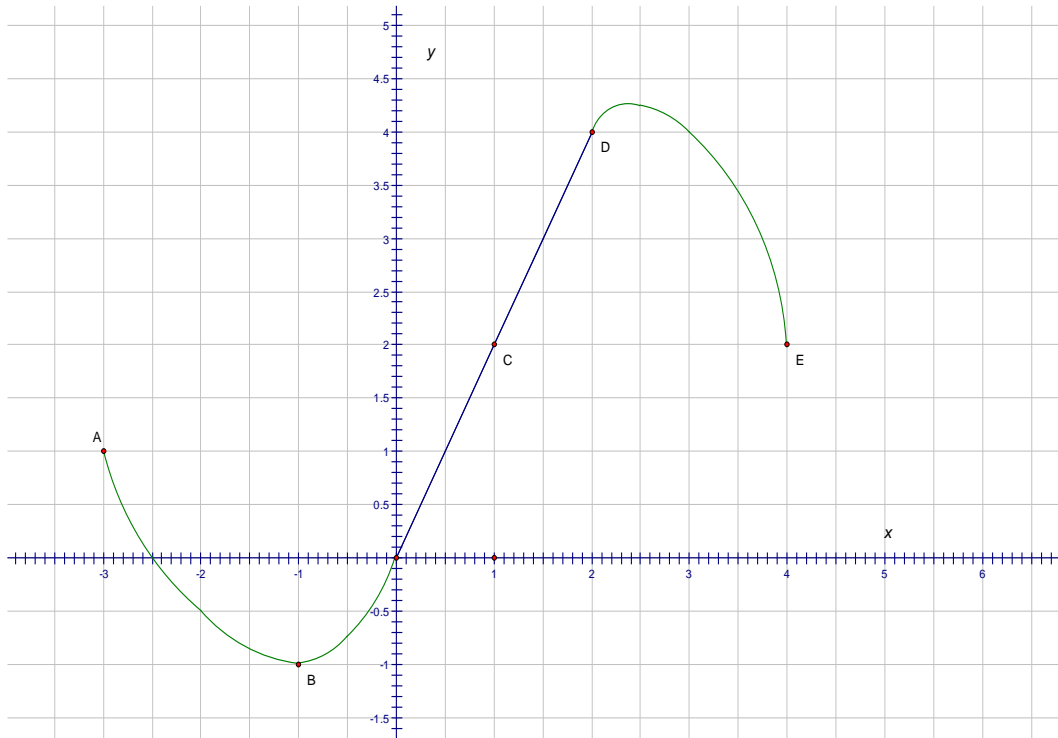
Find the average rate of change between the following points on the graph:

(Note: the "average rate of change" from P to Q means the gradient of PQ)

i)   from C to D …………………..
ii) from D to E …………………..
iii) from A to B …………………..
iv) from B to C …………………..
v)   from C to E …………………..
vi) from D to C …………………..

Figure 5.1: Identifying the gradient of a curve through various points

Figure 5.1 shows the diagram and the questions used to test students' gradient calculation skills. Table 5.2 summarizes the participants' responses.

	Gradient	C to D			D to E			A to B			B to C			C to E			D to C			% success
Pilot Phase		Y	N	T	Y	N	T	Y	N	T	Y	N	T	Y	N	T	Y	N	T	
	'O' Level	4	0	4	1	3	4	1	3	4	2	2	4	3	1	4	3	1	4	58%
	'A' Level	14	0	14	9	5	14	10	4	14	9	5	14	13	1	14	8	6	14	75%
	Totals	18	0	18	10	8	18	11	7	18	11	7	18	16	2	18	11	7	18	71%
Final Phase	Gradient	C to D			D to E			A to B			B to C			C to E			D to C			% success
		Y	N	T	Y	N	T	Y	N	T	Y	N	T	Y	N	T	Y	N	T	
	'O' Level	1	2	3	2	1	3	1	2	3	1	2	3	1	2	3	1	2	3	39%
	'A' Level	11	2	13	9	4	13	6	7	13	10	3	13	12	1	13	7	6	13	71%
	Totals	12	4	16	11	5	16	7	9	16	11	5	16	13	3	16	8	8	16	65%

<u>Key to Table</u>

Y = Correct Answer, N = Incorrect Answer, T = Total, 'O' Level = GCE Ordinary Level Mathematics Holders, 'A' Level = GCE Advanced Level Mathematics Holders

Table 5.2: Participants' Responses to the Gradient Calculation Question

All pilot participants (100%) calculated the gradient from C to D correctly, suggesting that it was a familiar pre-calculus concept even among the Ordinary Level holders. In a sense, where x and y are positively correlated the calculation of the gradient appeared easier for most students. There were, however, four casualties in the final phase with the O level holders suffering a higher percentage. By contrast the gradient from D to E, was calculated correctly only by 56% of the pilot and 69 % of the final phase students. The rest of the candidates seemed to have difficulties with the sign (or direction) and magnitude (or intensity) of the gradient. This suggests that when two variables vary inversely, (y decreasing when x is increasing), the determination of the gradient causes difficulties for weak students. Examples of wrong answers included '2/2=1' instead of '-2/2=-1' suggesting difficulty in noticing negative slope or determination of decreases in y from the orientation of the slope line. Other students could not reduce the gradient to its lowest terms. For example, participants 9 and 13 who gave their gradients as 2, both missed the sign and the sense of scale. Student 10 obtained the correct sign but miscalculated to get -1,35 as the gradient possibly as a result of miscounting and/or misreading of the scale.

The slope from A to B was calculated correctly by 11 students out of 18, constituting a success rate of 61% suggesting that weak students have similar difficulties for this question as for the previous one. Those students who could handle negative change along the *y*-axis continued to perform better. However, students 10 and 15 who had wrongly calculated the previous gradient successfully calculated that of A to B. This suggested that they could have changed their strategy and/or had two conflicting methods of interpreting the same gradient. The quadrant in which lines joining A to B and D to E appears to influence awareness about the sign of the slope. The pre-test items in this item may need to be adjusted to request students/participants to show their working in order that the strategies they employ may be deduced with more certainty.

The slope from B to C was calculated correctly by 11/18 pilot students (61%) and 7/16 final phase students (44%). The positive orientation of the slope line could explain the fairly satisfactory calculation by pilot students. However, Student 1 obtained 2/3 as the gradient suggesting an interchange of axes or variables. Student 2 gave 12/7 as the answer implying some miscounting of both the vertical rise (change in *y*) and the horizontal run (change in *x*) and/or failure to interpret the scale correctly. It again might be prudent to require students to show their working in order that their strategies can be deduced more accurately. The slope from C to E was calculated correctly by 89% pilot and 81% final phase participants. This evidently suggested considerable awareness of a 'no slope' situation among the participants. However weak candidates like Student 8, with an 'O' level background, and Student 13, with 'A' level background, obtained answers like '19' and '2' respectively. The value '2' corresponded to the *y*-coordinate suggesting some confusion with the height of the graph. This confusion is not a new phenomenon. Zaslavsky, Sela and Leron (2002:120) indicate that problems of understanding the notion of slope are addressed in a number of studies, mostly in connection with the common misconception of confusing height for slope.

The slope from D to C (a reversal in direction with C to D) was calculated correctly by 55% of pilot and 50 of final phase students. This contrasted sharply with the slope from C to D (same line), which was calculated correctly by the same candidates earlier, and

suggests that a reversal in the direction of the line might be perceived by some students to have an effect on the slope. Tall (1986) also observed this difficulty in students learning introductory calculus but does not discuss possible causes. In the pilot phase of this study, the answer '-2' given by five students (Students 2, 8, 10, 15, and 18) strongly suggested that the reversal in direction implied a change of sign as happens in vector geometry. It might therefore be a result or the overgeneralization of the rules for vectors (directed line segments) to line segments. However, this is clearly one of the fundamental tenets of constructivism when it claims that new concepts emerge from more general concepts, class extension in which existing concepts become cases of another subsuming concept, and re-conceptualization in which the nature and relationship between concepts changes significantly (Dykstra, Boyle & Monach, 1992).

5.2.3 Concept Image of the Tangent Line Gradient as the Limit of the Secant Line Gradient. (Question 3)
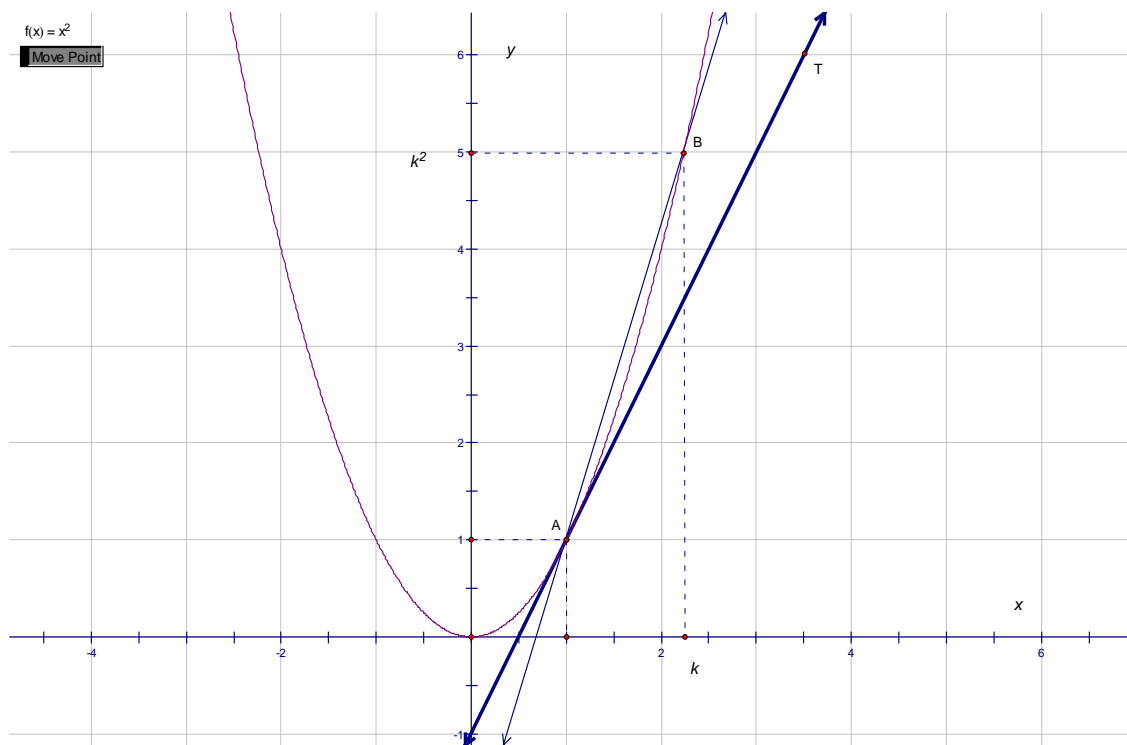
Figure 5.2: The relationship between the gradient of a secant and that of a tangent

Figure 5.2 shows the diagram that accompanied the question.

Gradient of the straight line through A, B.

This was a difficult question for most students. Only one pilot student (or 6%) managed to obtain a correct numerical answer. After some re-wording in the final phase, only three students (or 19%) managed to express the gradient algebraically in terms of *k*. Given that the previous question (Question 2) had predominantly sought numerical answers to the gradient, most pilot students (94%) seemed to have rushed for a numerical rather than algebraic solution. The revised final phase wording suggested that stressing the method (expression in terms of *k*) rather than the answer could guide students better to shift from an arithmetic to an algebraic mode of reasoning (see Appendix A Question 3). Eight of the wrong answers were algebraic attempts (50%).

Calculation of the gradient of AT

The gradient of AT was correctly computed by 78% of the pilot students (14 out of 18). This was evidence of considerable fluency in the arithmetic computation of the gradient. The numerical answer appeared to be easier to understand and to calculate. By contrast the final phase students, performed dismally in this question achieving a success rate of

	Student	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	% P
Pilot Phase	Q 3 i)	1_n	0_n	0_n	0_n	0_n	0_n	0_n	0_n	0_n	0_n	0_n	0_n	0_n	0_n	0_n	0_n	0_n	0_n	6%
	Q 3 ii)	1_n	1_n	0	1_n	1_n	0	1_n	0	1_n	1_n	1_n	1_n	1_n	1_n	0	1_n	1_n	1_n	78%
	Q 3 iii)	0	0_t	0	0_d	0	0	0	0	0_d	0_d	0_n	0_d	0_d	0	0	0	0	0_d	0%
	Totals	2	1	0	1	1	0	1	0	1	1	1	1	1	1	0	1	1	1	21%
	Student	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16			%P
Final Phase	Q 3 i)	0_a	0_n	0_a	1_a	0_a	0_a	0_a	0_n	0_n	1_a	1_a	0_a	0_a	nr	0_a	nr			19%
	Q 3 ii)	0_a	0_n	1_n	nr	0_a	0_n	nr	0_a	0_n	1_a	1	1	0_n	nr	nr	nr			25%
	Q 3 iii)	0	0	0	0	0	0	0_d	0	0	0	1_a	0	0	0	0	0			6%
	Totals	0	0	1	1	0	0	0	0	0	2	3	1	0	0	0	0			17%
Key to Table: 0_a=wrong algebraic answer, 0_n=wrong numerical answer, 1_a=correct algebraic answer; 1_n = correct numerical answer, 0_d=wrong differentiation answer, 0_t=wrong tangent answer, % P= % success																			

Table 5.3: Performance in the tangent slope as limit of secant line slope question (Q3)

only 19%. Thirty-one percent of the answers were algebraic (only one correct and four incorrect). This group of participants appeared to have been working in the algebraic/symbolic mode required in the previous sub-question, instead of the numerical or arithmetic mode. A further 31% of the respondents to the questionnaire did not respond to this sub-question. This was a relatively high abstention rate which leaves open the question whether the students were reasoning in an algebraic or numerical mode.

Gradient of AT from First Principles

In the pilot phase of the study, no student managed to answer this question correctly. One third of the responses had alluded to differentiation but failed to bring forth the limit concept. The wording of this question was then assumed to have affected the students' performance. That is, the expression 'from first principles' appeared not to have been universally understood. In the final implementation phase the question was extended to raise the expectation that students must deduce an expression of the tangent gradient at A from the expression of secant AB's gradient as k -1 tends to 0. The performance did not improve in any meaningful way as only one student managed to express the tangent gradient as a limit of the secant gradient. This was despite the fact that some of the students had strong calculus background (A level holders). The poor performance in both phases suggested that the question required deeper conceptual understanding. Tall (2000) similarly found that few students would naturally invent the limit concept for themselves. Only one out of a hundred undergraduate students (who had been taught the limiting notion) in his study had successfully produced a limiting argument as $k \to 1$.

Familiarity with tangents and the limit of the Secant Line slope is critical in understanding the derivative as the instantaneous rate of change. Pilot Student 2 gave an answer stating that to find the gradient of AT from first principles we would have to

find $\dfrac{\Delta_y}{\Delta_x}$. This was illustrated with the labeled sketch in Figure 5.3.

127
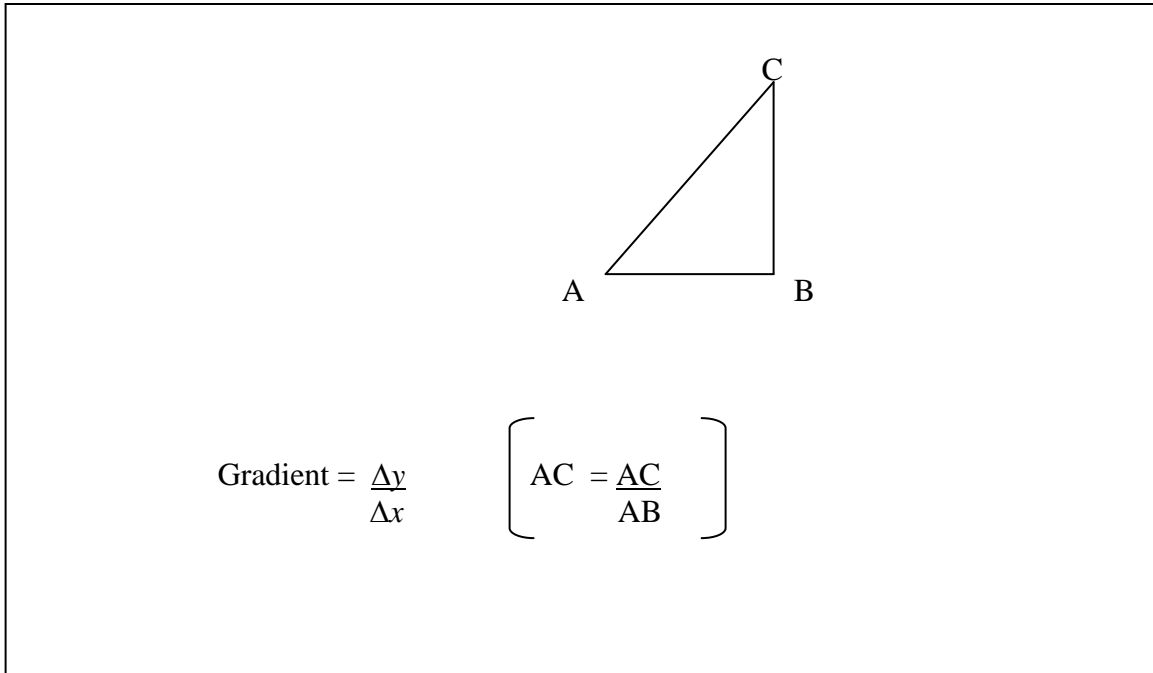
Figure 5.3: Pilot Student 2's gradient from first principles

The slope of AT was illustrated statically by dropping a perpendicular from T and drawing a horizontal line stretching to the right of A to meet the perpendicular dropped. The solution or proof 'from first principles' was not in evidence in the student's exposition/illustration. Pilot Student 12 gave the 'first principles explanation' more precisely as $\dfrac{\text{small change in } y}{\text{small change in } x}$, and made reference to the limit of this slope as δx

tends to zero $= \dfrac{\delta y_2 - \delta y_1}{\delta x_2 - \delta x_1}$ lim $\delta x \to 0$. This was the closest student and his work is shown

1st principle

$$\frac{\text{small change in } y_2 - \text{small change in } y_1}{\text{small change in } f(x_2) - \text{small change in } x_2}$$

$$= \frac{\delta y_2 - \delta y_1}{\delta x_2 - \delta x_1}$$

$$\lim \delta x \to 0$$

Figure 5.4: Pilot Student 12's gradient from first principles

in Figure 5.4. Pilot Student 21 gave the gradient of AT from first principles as follows:

$$\Rightarrow \frac{\text{increase in } y}{\text{increase in } x} = \frac{y_1 - y_2}{x_1 - x_2}$$ which fell short of the limit as $x_2 - x_1 \longrightarrow 0$ and reversed

the order of y_1 and y_2 in the numerator and that of x_1 and x_2 in the denominator. Students 4, 9, 10, 13 and 14 made specific reference to differentiation as the method by which to determine the gradient of AT from first principles. The responses in Figure 5.5 attest to this assertion

"First differentiate the equation $f(x) = x^2$, You get $f'(x) = 2x$ Substitute the value of x on which the line is in contact with curve to find gradient. Substitute $x = 2$ to get the gradient as 2"	**Student 4**
'Differentiate x^2 and substitute x values by that of A'	**Student 9**
'Find $f''(x)$ of AT'	**Student 10**
By differentiating $f(x) = x^2$, let $y = f(x)$, $y = x^2$, $$\frac{d(x^2)}{dx} = 2x$$	**Student 13**
If $f(x) = x^2$, $$\frac{d(x^2)}{dx} = 2x$$ 	**Student 14**

Figure 5.5 Sample responses to the 'first principles' question

Student 4 substituted a wrong value for x but got the correct gradient of 2. In this case the correct answer was obtained wrongly if not accidentally and conceptual understanding could not be guaranteed. Student 9's response was similar but more deficient in that the specific value ($x = 1$) and the differentiation results were not specified. Student 10 was less certain about what had to be done. The second derivative was further from the question asked. Students 13 and 14 got the correct algebraic answers and stopped short of substituting to obtain the numerical value. Student 15 proceeded to substitute a wrong value of x in the 'derivative'. In sum these five cases revealed that the idea of 'first principles' or limit was a difficult concept to demonstrate.

Both the pilot and final phase pre-test results for this question suggest that although most 'A' level Mathematics holders do not exhibit significant conceptual understanding of the derivative at a point (instantaneous rate of change) some possess considerable procedural knowledge of how to calculate the derivative. The limit concept underlying the derivative was conspicuous by its absence. The dim memories seemed to betray a dearth of conceptual understanding in the traditional drillmaster's examination oriented pedagogics (compare 1.1 & 2.3). Schoenfeld (1988), concurs that what sometimes appears to be successful mathematics instruction on the surface (as evidenced by high student scores on standardized tests) may actually inculcate in students the misconception that understanding is not necessary when solving mathematics problems: one simply follows an answer yielding procedure whether it makes sense or not. Yet the limit concept is perhaps the most fundamental idea in the standard calculus (compare 2.3). Szydlik (2000:259), acknowledges that, coupled with this centrality, the limit concept is a difficult concept for students to acquire. This suggests that a solid conceptual understanding of the limit concept cannot be left to 'procedural' chance. We therefore conclude that while both the pilot and final phase students exhibited a shaky grasp of the limit concept when entering the study, the predicament was not unique to their circumstance, but posed a serious impediment to be surmounted.

5.2.4   Differentiability of a function at a given point (Question 4).

If a function is differentiable at a point it means that a) it magnifies to look straight at that point and b) that a tangent can be drawn at the point of contact. In other words, the gradient at the point of tangency can be ascertained assuring us that the derivative of the function at that point exists. If, however, the function is differentiable at a point, its left and right tangents must have the same limiting position. In other words, the derivative of

a function f is defined at those points where the limit of $\dfrac{\Delta_y}{\Delta_x}$ exists as Δx approaches

zero (Anton, 1999:181). Conditions of existence or non-existence of the derivative are an integral part of a rich concept image of the derivative. Geometrically, the points of non-differentiability are the points where the curve $y=f(x)$ does not have a tangent. Informally

130

stated the most commonly encountered points of non-differentiability can be classified as a) corners, b) points of vertical tangency and c) points of discontinuity (Anton, ibid) (compare Activity D in Appendix B).

Differentiability of the graph of $y=x^2$-x at x = 0.

The results of this question showed that, for the parabola or quadratic function $y=x^2$-x , 89% of the pilot students and 81% of final phase students, deduced correctly that the graph was differentiable (i.e. a tangent can be drawn) at $x = 0$ (see Figure 5.5 for graph) However, only 50 % of pilot and 38% of final phase students managed to calculate the tangent gradient (derivative) correctly (see Table 5.4).

Graph of x²-x

Can you calculate the gradient at x=0? YES/NO
If YES, what is the gradient, if NO, why not?
....

a) Graph of $f(x) = x^2 - x$

Graph of y=|x|

Can you calculate the gradient at x=0? YES/NO
If YES, what is the gradient, if NO, why not?
....

b) Graph of $f(x) = \left| x \right|$

Graph of y=√|x|

Can you calculate the gradient at x=0? YES/NO
If YES, what is the gradient, if NO, why not?
....

c) Graph of $f(x) = \sqrt{(\left| x \right|)}$

Graph of y=|x³|

Can you calculate the gradient at x=0? YES/NO
If YES, what is the gradient, if NO, why not?
....

d) Graph of $f(x) = \left| x^3 \right|$

Figure 5.6: Differentiability test items

Some of the students were quite confident and went on to differentiate and find the gradient at $x = 0$ by substituting for x in the derived function $f'(y) = 2x - 1$ (for example

pilot Student 1). Only two (11%) of the pilot and three (19%) of the final phase students considered the function to be non-differentiable. Pilot Student 8 argued that the gradient could not be calculated because 'there was no change in the *x*-axis'. This suggested some confusion between gradient of a line being rise (change in the *y*-axis direction) over run (change in the *x*-axis direction) with gradient at a point on a curve being the slope of the tangent at that point and not the dimensionless point itself. That is, the reasoning is in conflict with the notion of the gradient at a point on a curve as the slope of local straightness (compare 2.5.1, 3.6, 5.3.5 and 5.4.4). However, as noted both earlier and later in the study, this reasoning acts as a pivot to explain why the tangent disappears in *Sketchpad* and why we have to re-define the tangent practically 'as a line joining two very close points' as suggested by Tall (1997) (compare 5.3.5, 5.4.4. and 2.8.3).

	Student	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	% P
Pilot Phase	Q.4a) i	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	0	1	89%
	Ii	1	1	1	1	0	1	0	0	1	1	1	0	0	0	0	0	0	1	50%
	b) i	1	0	1	1	1	1	1	1	1	0	0	0	1	0	1	0	0	0	56%
	ii.	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	6%
	c) i.	1	1	0	0	1	1	0	0	1	0	0	0	1	0	1	1	1	1	56%
	ii.	0	0	1	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	17%
	d) i.	1	1	1	1	1	1	0	0	1	0	0	1	1	0	1	1	1	1	100%
	ii.	1	0	0	1	1	1	0	1	1	0	1	1	1	0	1	1	0	1	61%
	Total	6	4	5	6	5	6	3	3	5	4	4	3	5	3	5	4	2	5	54%
	Student	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16			% P
Final Phase	Q.4.a)i.	1	0	1	1	1	1	0	1	1	1	1	1	1	1	1	nr			**81%**
	ii.	nr	nr	1	1	0	0	0	1	0	1	1	1	0	nr	1	nr			**38%**
	b)i.	0	nr	0	0	1	1	1	nr	0	0	0	0	1	1	nr	nr			**31%**
	ii.	nr	nr	0	0	1	0	0	nr	0	0	0	0	nr	nr	nr	nr			**6%**
	c)i.	1	nr	0	1	1	1	1	1	1	1	1	0	1	0	nr	nr			**63%**
	ii.	1	nr	0	1	0	0	0	0	0	0	1	0	nr	nr	nr	nr			**19%**
	d)i.	1	nr	1	1	0	0	1	0	1	0	1	1	0	nr	0	nr			**44%**
	ii.	1	nr	0	0	0	0	0	1	0	0	1	1	nr	nr	0	nr			**19%**
	Total	5	0	3	5	4	3	3	4	3	3	6	4	3	2	2	0			**39%**
	Table 5.4: Individual Performance on Differentiability Questions																			

The only other pilot candidate (Student 17) to claim that the gradient could not be calculated gave a similar reason when she agued that 'there is no gradient at a point but between points'. This reemphasizes the epistemological importance of framing differentiability in terms of existence or non-existence of a tangent at a point in a function's domain.

N = 18	Gradient Exists			Gradient Value or Reason for						
	Y	N	Nr	-1	1	½	other			
a) At $x = 0$, for $y = x^2-x$	16	2		9	2	1	3			
	Y	N	Nr	SL	0	N/A	1	TP	-1	±1
b) At $x = 0$, for $y = \lvert x \rvert$	7	11		1	1	1	2	1	1	1
				0	U	Inf	TP	N/A	Nil	Other
c) At $x = 0$, for $y=\sqrt{(\lvert x \rvert)}$	9	9		5	1	5				1
	Y	N	nr	1	0,1	TP	SP	0	Nil	Other
d) At $x = 0$, for $y= \lvert x^3 \rvert$	11	7		1	1	2	1		1	1

<u>Key:</u> Y = Yes, N = No, nr = no response, SL = Straight Line, TP = Turning Points, SP = Stationary Point, Inf = Infinity, N/A = Not Applicable

Table 5.5: Collective performances on differentiability questions

The Differentiability of the graph of $y = \lvert x \rvert$ at x = 0.

The results for this graph show that only seven out of 18 students (or 39 %), correctly disagreed with the fact that the gradient could be calculated the (sharp) turning point of $x = 0$. The reasons given were varied though, as the Figure 5.7 shows.

"There is no tangent of a straight line"	**Student 2**
"It is a turning point"	**Student 10**
"It's a stationary/turning point, there is no change hence no rate of change"	**Student 11**
"There is no increase or decrease, no line passes through the point"	**Student 12**
"Never seen this before"	**Student 14**
"No gradient of a point"	**Student 17**

"When $x = 0$, it is the minimum value, or the turning point of graph so it is a straight line. Gradient of a straight line is always zero. There is no tangent for a straight line."
                                                                    Student 18

Figure 5.7: Sample responses to the differentiability of $y = \lvert x \rvert$ at x =0

It is unclear why students think that a stationary point or turning point should be reason enough for a gradient to be non-existent because that argument would also apply to the first two cases. However there is some parallel with the previously observed conceptualization of the slope at a point (where the slope is zero). The argument that there is no increase or decrease at a point (Student 12) seems to complement Student 17's insistence that there is no gradient of a point. However, Student 12 added that for this graph no line passes through point $x=0$. Once again this seems to be at odds with the notion of local straightness and draws attention to the practical usefulness of defining differentiability in terms of constructability of a tangent at a point. Student 18 struggled and surprisingly concluded that since '(0,0)' was the minimum value or turning point of the graph it must be a straight line whose gradient is always zero. This suggests a previous exposure to maxima and minima of 'smooth' graphs where by the tangent will always be horizontal and therefore be of gradient zero. However, the generalization to a corner point seems to be an over generalization that creates conflict with previous learning. We can conclude that although the students have previously encountered turning points (maxima and minima), the special case of a corner point has not been well articulated, thus creating cognitive conflict when first encountered.

The differentiability of the graph of $y = \sqrt{(|x|)}$

Ten out of 18 pilot students (56 %) and 63% final phase students correctly stated that the gradient could not be calculated. There were two non-responses from the pilot group and four from the final phase group suggesting that the abstention rate was much higher for the final phase participants. Again a variety of reasons for failing to calculate the gradient

"I'm failing to find the tangent that touches the curves of the graph"	**Student 3**
"There are many gradients because many lines/tangents can be drawn at $x=0$"	**Student 4**
"The graph is not defined for $x = 0$"	**Student 7**
"No because the change in x is infinite"	**Student 8**
"The graph does not pass through point $x = 0$."	**Student 10**
"It's a turning point on the mirror line"	**Student 11**
"There is no increase or decrease."	**Student 12**
"Never attempted such."	**Student 14**
Gradient of straight line is 0 since x is a point on a straight line"	**Student 16**

Figure 5.8: Sample responses on the differentiability of $\sqrt{(|x|)}$ at x = 0.

was offered as Figure 5.8 shows, but again only a few hint at mathematically valid reasons.

Pilot Students 3 and 14 appeared to be in a state of despair as they could not find a suitable tangent and/or had no previous experience with the case of vertical tangency as both the left and right limit. Student 4 saw exactly the opposite and argued that there were, in fact, 'many' gradients because many tangents could be drawn. The argument would have been more plausible for $y = \mid x \mid$ where a tangent defined as a line touching the graph at a point would definitely have allowed an infinite number to be drawn. Students 7 and 8 referred to the notion of 'undefined' which is intuitively relevant to non-differentiability at a point of vertical tangency. Student 7 referred to the graph as 'undefined' while Student 8 referred to changes in x as 'infinite' instead of zero thus forcing any change in 'y' to be divided by zero. From the range of answers given it is clear that the differentiability of the 'cusp' at $x = 0$ causes difficulties to many able students as confirmed by Tall's (1986) observations (compare 2.3 and 2.5.1). However, contrary to Tall's argument that there is no tangent, this is a case of vertical tangency which means that the gradient is undefined thus rendering the function non-differentiable at this point.

The differentiability of the graph of $y = \mid x^3 \mid$

Thirteen out of 18 pilot (72%) and 44 % of final phase students correctly stated that the gradient could be calculated. Seventy-two percent pilot and 25% final phase students calculated the gradient correctly but in both cases it was not necessarily those students who had indicated the gradient to be computable. Although convinced that the gradient was computable, the procedure remained imprecisely understood by some students. We can conclude that while pilot phase students had a firmer grasp of the horizontal tangent situation, the final phase students had a shakier grasp of this concept. Tall (1986) notes a similar ambiguity in the calculation of the gradient in this case. In his study, he reports that although most students obtain the result 0, a significant number, carry out the calculation through an erroneous differentiation, explicitly noting the derivative of $|x^3|$ to be either $|3x^2|$ or $3x^2$ (a correct formula being $3x|x|$). In this pretest, a number of students

(three in the pilot phase and two in the final phase) start by denying that the gradient can be calculated but proceed to calculate or state the correct gradient of 0 for the horizontal tangent.

5.2.5   Sketching derivatives of given graphs (New Questions 5 and 6)

Questions 5 and 6 which had not been included in the pilot phase were included in the final phase pretest to enhance the validity of comparison of pretest and posttest results. That is, in the final phase both the pretest and posttest items were identical. This helped to account for all previous calculus learning. Figures 5.9 and 5.11 show the diagrams used for the pretest (see Appendix A, Question 5, for more details). In Question 5 students were required to select a graph whose derivative was represented by the first graph in Figure 5.9.

Figure 5.9: Identifying the graph with a given derivative
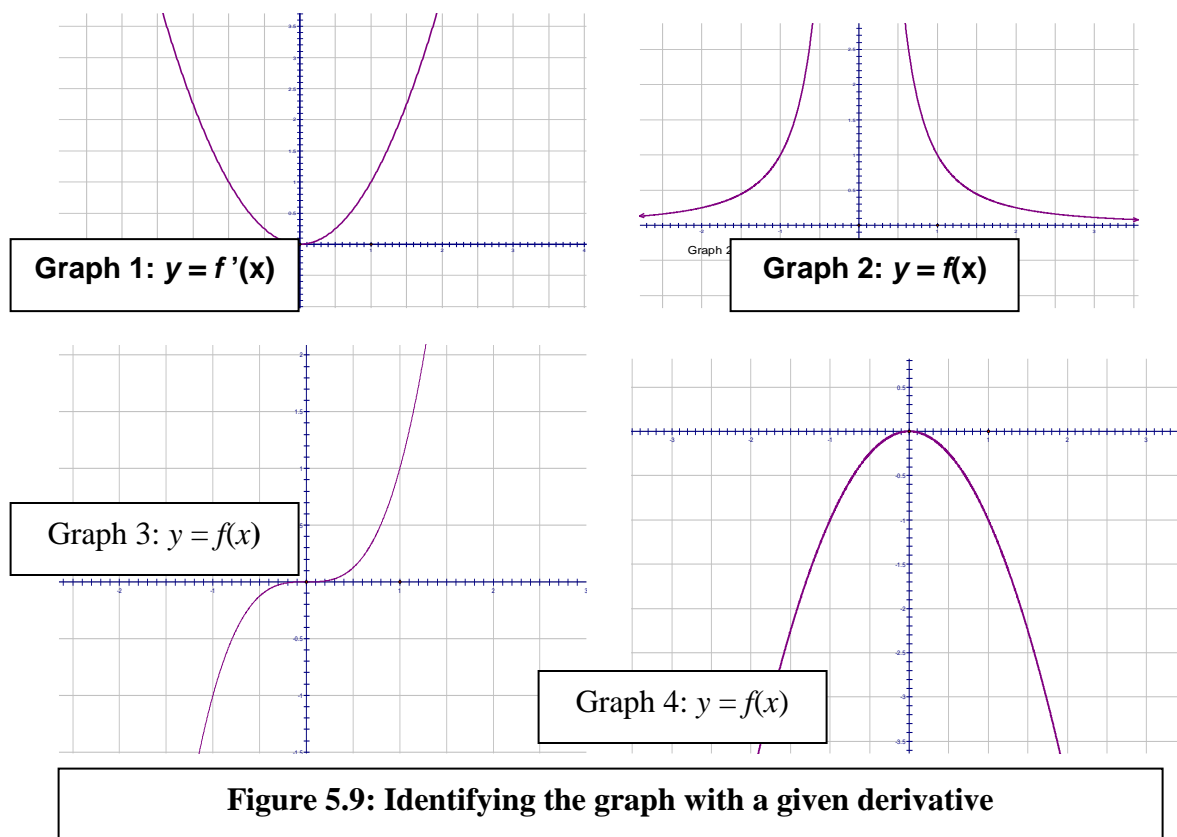
Seventy-five percent of the participants gave correct responses to this question suggesting that they had considerable calculus knowledge that they brought to the study as borne out

136

by their A level qualifications. However, only 31% gave plausible reasons. Figure 5.10 shows selected samples of incorrect responses. Student 3 correctly reasons that Graph 3 is cubic but then deduces that the derivative is equal to Graph 3 (instead of Graph 1).

"The graph of 3 is $y = 2x^2$. The derivative of $f'(x)$ = Graph 3"	**Student 3**
"Its line of gradient is $y = x$"	**Student 4**
"It gives a linear graph as its derivative"	**Student 5**
"It will have its derivative as the positive have been reflected to the left and the left to the right"	**Student 6**
"Derivative is reflected the negative sides through x-axis	**Student 7**
"It is because the other part of the graph is the reflection of the other and another is maintained as it is"	**Student 8**
"The graph of $y = f'(x)$ is a reflection of the graph of $f(x)$" referring to Graph 4.	**Student 13**
"Because the part of the graph on the negative side will be reflected to the positive side"	**Student 14**
Figure 5.10: Sample responses to matching a graph to its derivative (Question 5)	

The student had the correct idea but seemed to fail to express himself and manifested some circularity in reasoning or it was simply a careless error. Student 5 said 'it produces a linear graph as its derivative' when in fact it should be a quadratic and not a linear graph. This seems to be a problem of semantics. Student 4 argued that 'its line of gradient is $y = x$'. This could be either a semantic or a conceptual error. Students 6, 7, 8, 13 and 14's responses had a surprisingly common characteristic of likening the derivative to the geometric transformation of reflection. Although a geometric transformation is a function in its own right, a reference to it in a different mathematical field suggests a 'modal error'. That is, the students might not be responding or operating in the calculus mode. Alternatively, they might have noted, but expressed poorly, that because graph 3 is point symmetric, the slope of the tangent will be reflective symmetric, which is a general

137

calculus property, not generally taught, but easily proved. In other words, they might have noticed that the slope of the tangent as one moves from left to right on the cubic decreases from large positive to zero at $x = 0$, and then in exactly the same way, increases from zero to infinity, which is exactly what's depicted by the quadratic graph. However, without interviewing the students, because of the confidentiality of the pretest, its all speculation about what they were actually thinking of.

In Question 6 students were required to sketch the derivatives of the given graphs in Figure 5.11. This was basically the reverse of Question 5 (see Appendix A, Question 6 for more details). Table 5.6 is a summary of scores to students' responses to both Questions 5 and 6. The key to the table shows the scoring rubric for the students'

Figure 5.11: Sketching the derivatives of given graphs
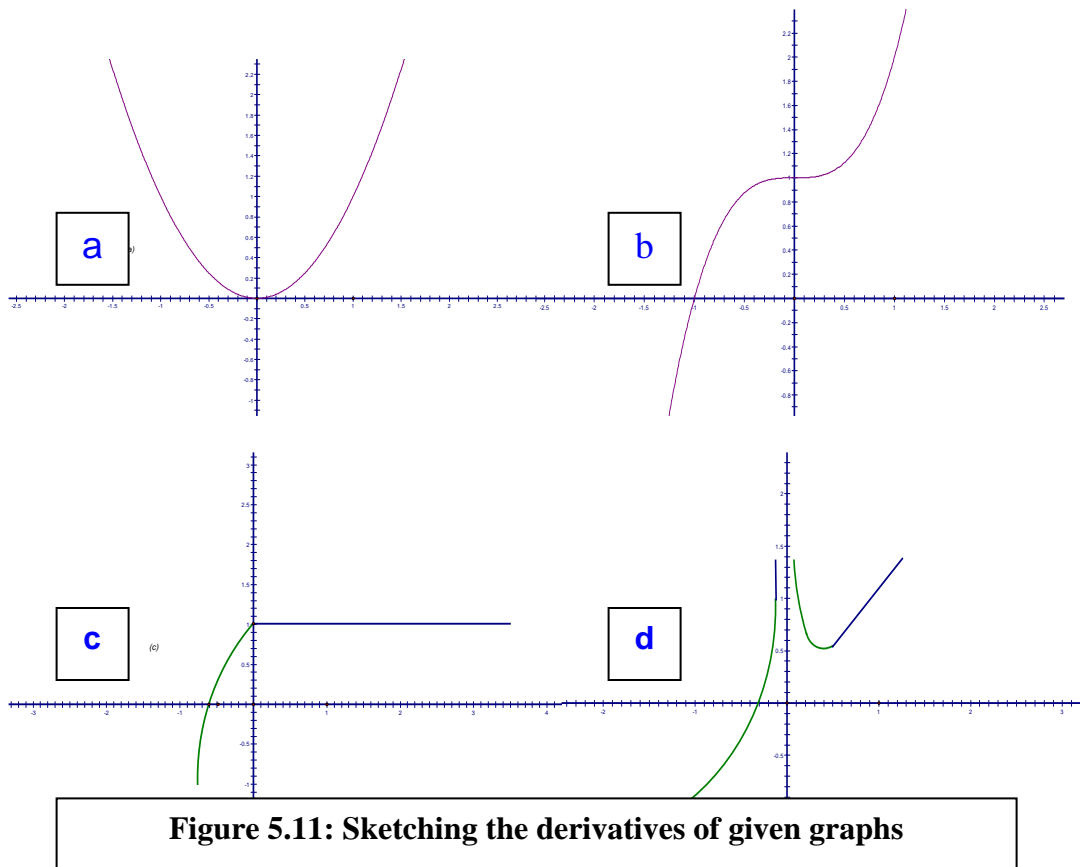
sketches for Question 6. The rubric has emphasized concavity, domain appropriateness and intercepts on both the left hand side (LHS) and the right hand side (RHS). Domain and image (range) appropriateness encompasses appropriate left and right y-intercepts.

138

	Student	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	% P
Final Phase	Q. 5 a	1	nr	1	1	1	1	1	1	nr	1	1	1	0	1	1	nr	**75%**
	b	nr	nr	1	0	0	0	0	1	nr	1	0	1	0	0	1	nr	**31%**
	Q. 6 a	3	nr	2	0	2	3	2	5	0	5	2	5	3	3	1	Nr	**45%**
	b	2	nr	1	1	5	1	1	nr	1	nr	4	4	1	5	0	Nr	**33%**
	c	3	nr	1	0	0	0	1	nr	0	nr	3	nr	1	0	nr	Nr	**9%**
	d	1	nr	1	0	0	1	1	nr	0	nr	0	nr	0	0	nr	Nr	**4%**
	Total	10	0	7	2	8	6	6	7	1	7	10	11	5	9	3	Nr	**24%**

Key to Table

nr =no response, 0 = wrong response, LHS= left hand side, RHS=right hand side

correct LHS or RHS concavity	1	+ correct LHS + RHS intercept	4
correct LHS + RHS concavity	2	+ correct LHS or RHS range	5
add correct LHS or RHS intercept	3	+ correct LHS + RHS range	6

Table 5.6: Pretest responses to derivative questions (Questions 5 and 6)

Concavity includes appropriate turning points where applicable. On the whole therefore, the rubric is on the one hand, an adaptation of the Covariation Framework (compare 4.1, 4.2.1, 4.3.1,5 & 5.2.6). On the other hand, it takes into cognizance Huntley et al's (2000:340) advice that partial credit should be given for responses that show evidence of progress towards a correct solution.

5.2.6   Solutions to the Modified Bottle Problem

The Modified Bottle Problem (Question 7 in Appendix A) was intended to assess students' ability to apply covariational reasoning in non-routine situations as part of the broader goal of ascertaining students' entry knowledge into the experiment (compare 4.1, 4.2.1). As already noted in the preceding section, students' responses were categorized according to the Covariation Framework (compare 4.3.1.5). The modifications were the introduction of a cylindrical bottle and the additional requirements of students to sketch derivatives for the graphs. The graphs of the resultant derivatives were coded/scored in the same way as for Question 6 above. Table 5.7 summarizes the students' performances in the modified bottle problem.

	Student	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	% P
Final Phase	Q. 7a)i.	3	nr	nr	2	3	2	3	3	nr	nr	3	nr	3	2	nr	nr	**50%**
	ii	2	nr	nr	0	0	0	2	0	nr	nr	1	nr	0	0	nr	nr	**10%**
	b)i.	0	nr	nr	4	2	2	3	2	nr	nr	2	nr	2	2	nr	nr	**24%**
	ii.	0	nr	nr	0	0	1	1	0	nr	nr	0	nr	0	0	nr	nr	**3%**
	Total	5	0	0	6	5	5	9	5	0	0	6	0	3	4	0	0	**22%**

<div align="center">Key to Table</div>

	Level		Level
No response	0	Concept image of the intensity (concavity) of covariation	3
Concept image of covariation of two variables	1	Concept image of varying intensity of covariation	4
Concept image of direction of covariation	2	Concept image of proportionality of the covariation throughout the domain.	5

Table 5.7: Final phase students' pretest responses to the Modified Bottle Problem

Level 0   Four pilot Students (3, 13, 14, & 16), and seven final phase Students (2, 3, 9, 10, 12, 15 & 16) fell into this category since they did not respond to the question. The abstention ratio for this question was much higher during the final phase (44%) than during the pilot phase (22%). Although a non-response in no basis for presuming the students knew absolutely nothing about the co-variation of variables it remains a non-event. This stance is supported by the SOLO model which lays stress on the structure of **observed** learning outcome - not an unobserved one (compare 4.3.1.2). A closer look revealed that three of the pilot students that did not respond to this question had 'O' level mathematics background only suggesting that their non-response could be indicative of skills and knowledge not yet acquired/mastered. This observation is only of speculative value since only two of the seven final phase students who abstained were 'O' level holders.

Level 1: Pilot Students 5 and 10 could be classified as level 1 only because they were aware that two quantities were co-varying but were neither aware of the direction of covariation nor the whereabouts of the *y*-intercepts. There were no level 1 responses in the final phase. Figure 5.12 shows the two responses categorized to be at this level.
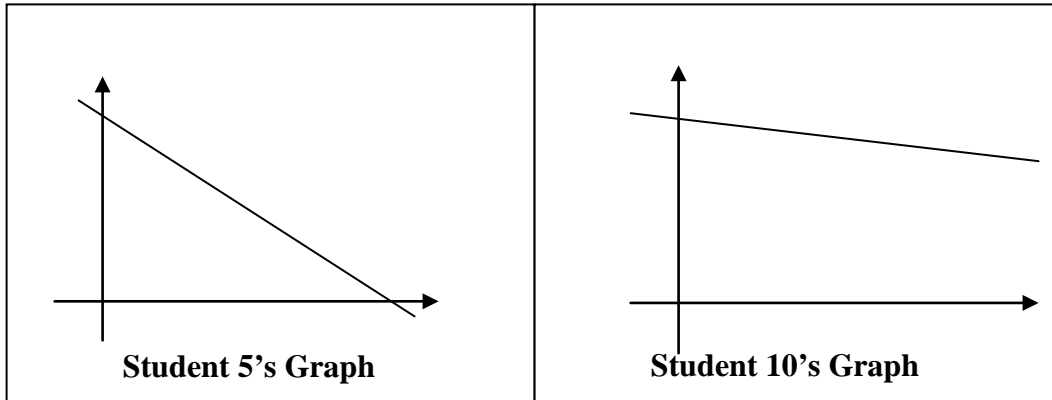
Figure 5.12: Sample Level 1 Responses to the Bottle Problem

Level 2: The responses by pilot Students 8 and 1 could be classified as Level 2. Both students got the correct *y*-intercepts. While Student 8's graph was concave down (convex) throughout, that of Student 1 was concave up throughout as Figure 5.13 shows.
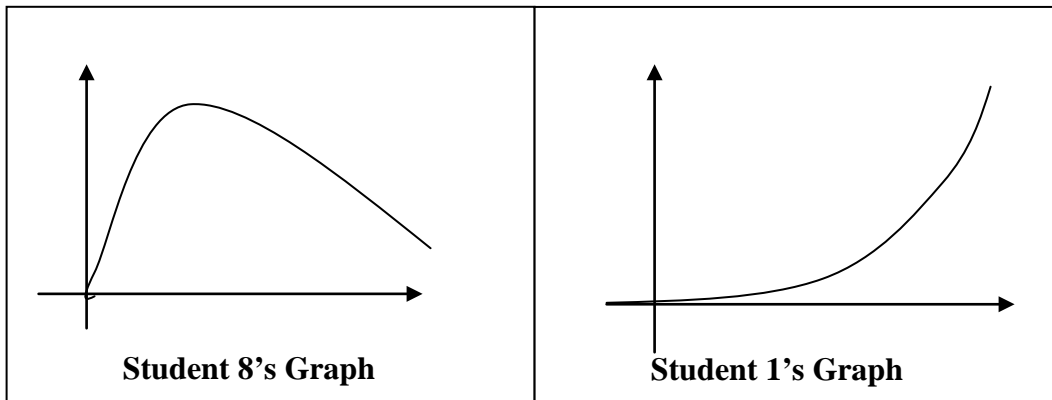
Figure 5.13: Sample Level 2 Responses to the Bottle Problem

Student 8's response was less insightful in the sense of having a maximum turning point followed by an unrealistic reduction in height. Final phase Student 5 had a graph similar to that of Student 1. In both cases the graphs resembled the shape of the bottle.

Level 3: Pilot Student 4 and final phase Student 6's responses (see Figure 5.14) could be classified as Level 3. Student 4's response was positive linear but with an incorrect domain that spilt over to negative numbers. Student 6's response was also linear but with

141

the correct domain. Student 4's response could be placed in a level 3a category while that of Student 6 could be placed in a 3b category to differentiate domain awareness.

Figure 5.14: Sample Level 3 Responses to the Bottle Problem
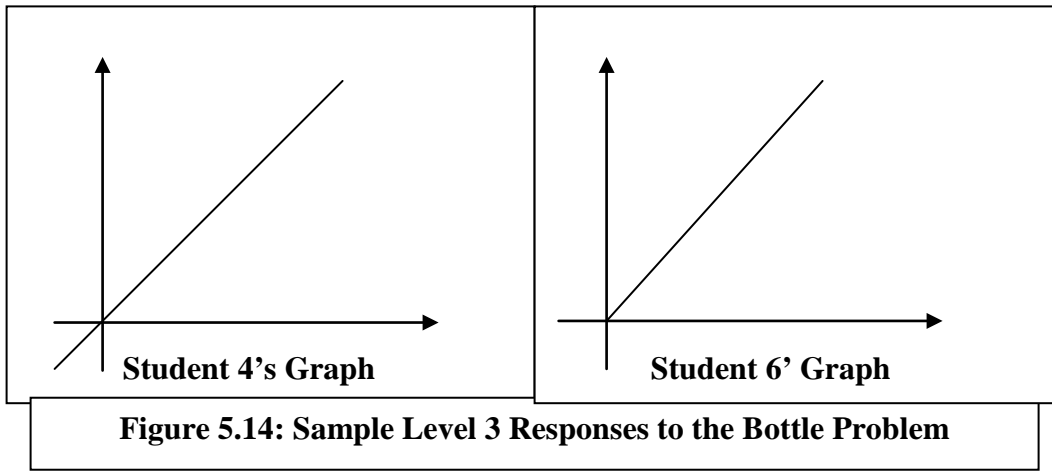
Level 4: Pilot Students 6 and 9's responses could be categorized as Level 4. For Student 9 there was an awareness of a two way variation in the rate of change to reveal a point of inflexion (i.e. concave up followed by concave down). However, the concavity was still reversed. On the other hand, for Student 6 the concavity was correct for the bottom part of the bottle so the point of inflexion was still missing. Student 9's response could be classified as 4a while that of Student 6 could be classified as 4b (see Figure 5.15).

Figure 5.15: Sample Level 4 responses to the Bottle Problem
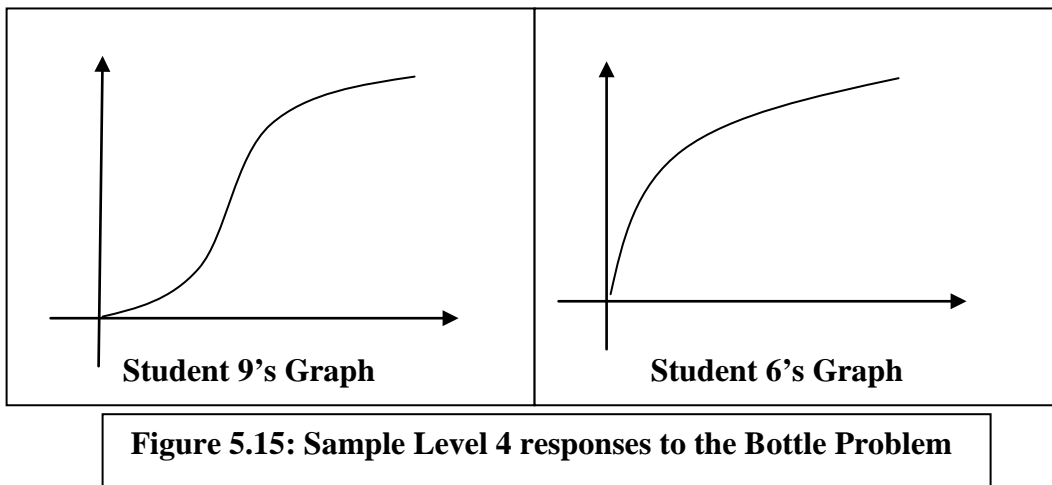
Level 5: Pilot Students 11 and 18's responses (see Figure 5.16) could be categorized as level 5. Student 11's graph had correct concavity, namely concave down, concave up with a clear point of inflexion but without an adequate sense of proportion and could be classified as Level 5a. Student 18's response was closest to the expected concavity and proportionality and could therefore be classified as Level 5b.

142
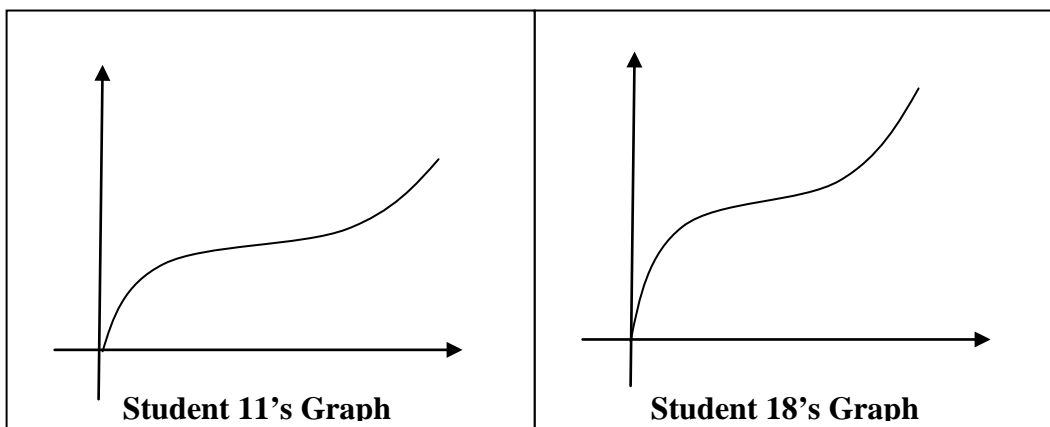
Figure 5.16: Sample Level 5 responses to the Bottle Problem

From the pre-test findings it can be concluded that although those students with A level calculus background performed significantly better than the O level Mathematics holders, both groups had fairly adequate pre-calculus background to be competent enough to understand the mathematical content in the *Sketchpad* Activities. However, knowledge of the limit concept and skills in sketching the derivatives of given graphs were still underdeveloped. Regarding the high abstention ratio in some questions, Chaachoua and Saglam (2006:21) remark the students' abstention ratio shows at which point students have difficulties in establishing links between knowledge acquired on a common concept in two different disciplines. We might as well add that this may occur even to knowledge acquired in different contexts and/or at different times. Thus although the experimental students were expected to benefit from the *Sketchpad* Activities they were not expected to benefit evenly given their varied baseline knowledge.

5.3   *Sketchpad* **Activities to enhance students' concept image of the derivative.**

5.3.1   The Didactic Method employed in the *Sketchpad* Activities

The didactic aim of this experiment was the modeling of the derivative using *Sketchpad* as a dynamic graphic, numeric and symbolic tool for modeling functions. These activities were done individually in the computer laboratory and in the initial phase nine students who had taken part in the pretest participated consistently. In the final phase six students participated consistently in the laboratory activities. The students were provided with

work sheets to follow as they used *Sketchpad* to model the derivative and to respond in writing to questions posed. Students also had the opportunity to interact with their colleagues and were encouraged to discuss findings and/or difficulties with their classmates first before reverting to the teacher-researcher.

In their review of undergraduate mathematics online course designs, Engelbrecht and Harding (2005b: 254) caution that in a web-based course one should not take constructivism too far, since learning cannot be expected to generate students' own ways of collaborating. Half as much of this advice might be applicable to a lab-based instructional setting where each student has his/her own PC and therefore susceptible to idiosyncratic experiences. By contrast, though, Tall and Ramos (2004:2) observe that in the different dynamic of a computer the tradition of each student being responsible for their own work and not cheating by copying from others is transformed to a corporate enterprise where we learn better if we share the insights of others in collaborative work.

In regards to the role of the teacher in a technology intensive environment, Tall and Ramos (ibid.) lament that the relationship between the teacher's knowledge, pedagogical strategies, and the use of graphing calculator technology is largely unexamined. They insist that many studies do not report or describe the teacher's role in the classroom or the teacher's graphing calculator knowledge and skills or the teacher's beliefs about the efficacy of using graphing calculators in mathematics learning. By contrast Ndlovu (2004) identifies five competencies that are necessary for teachers to orchestrate geometry instruction in a *Sketchpad* environment. The disclosure of the teacher-researcher's role in this study is an attempt to avoid the blind spots of previous studies.

The teacher-researcher believed in the efficacy of *Sketchpad* in helping students to mentally construct richer concept images of the derivative. The teacher-researcher privileged technology from the onset and consequently this could be regarded as a technology intensive didactic environment in the sense described by Engelbrecht and Harding (2005a:243) as high interaction with software, and high mathematics content. Accordingly *Sketchpad* activities were intended to investigate the following questions

research questions: How effective is the use of dynamic mathematics software as a tool to model the concept of derivative and what constraints or limitations are found? How effective or successful is the chosen instructional model and didactical relationships designed with dynamic mathematics software to enhance students' conceptual understanding of the derivative?

5.3.2   Student Familiarity with the Computer Environment

Although virtually all students had no prior high school learning of mathematics in a computer-aided environment, or Computerized Learning Environments (CLE) according to Trouche's (2004:285) terminology, all of them had had preliminary exposure to computers as part of initiation into e-learning resources at the university level environment. Unlike in a previous study by this author (Ndlovu, 2004), where technophobia was a major obstacle to overcome, the students in the experimental groups of this study were very keen learner users of computers. They were all younger learners by comparison and still malleable enough to adjust quickly to a new technological environment. The only disadvantage was that when they got stuck with *Sketchpad* technicalities, the students digressed to Internet surfing and e-mailing which they were still extremely excited about. There was therefore need for alertness on the part of the teacher researcher to ensure that students remained on task. Otherwise the general interest both in the pilot and final phase sessions was quite high.

5.3.3   **Activity A:** Modeling the Function Concept in *Sketchpad*

These activities were intended to introduce the experimental group students to *Sketchpad's* functionalities, with specific regard to multiple representations of functions. Arcavi and Hadas (2000:25) point out that dynamic computer environments can play a significant, and possibly unique, role in supporting innovative learning trajectories in mathematics in general and geometry in particular. This is also true for functions and the dynamic graphic/numeric interface in *Sketchpad* which is essentially a visualization characteristic of the computer environment in which students have the ability to 'represent, transform, generate, communicate and reflect on visual information' (ibid).

The computer environment also affords experimentation, surprise, discovery and immediate feedback. The experimental students were exposed to various linear, cubic, quadratic and polynomial, trigonometric, exponential, logarithmic and inverse functions (compare with empiricism in 3.2). Pierce and Stacey (2004:68) contend that the ability to move quickly between algebraic, graphical and tabular representations of functions (swapping representations) can be a source of difficulty. Yet it is through the ability to coordinate representations that one engages in mathematics (Stroup, 2002:172). Figure 5.17 shows the multiple representations of a function that *Sketchpad* is capable of. The static graphic, numeric and symbolic interfaces are quite visible. Missing is the dynamic symbolic representation of a straight line equation (compare Figure5.18). *Sketchpad* therefore appears to offer a rich environment or 'thinking space' that students can 'dwell in', and 'instrumentalize' to statically and dynamically model, not just the function concept, but the average rate of change, the derivative as the instantaneous rate of change, the derivative as the rate of change function among other possibilities.
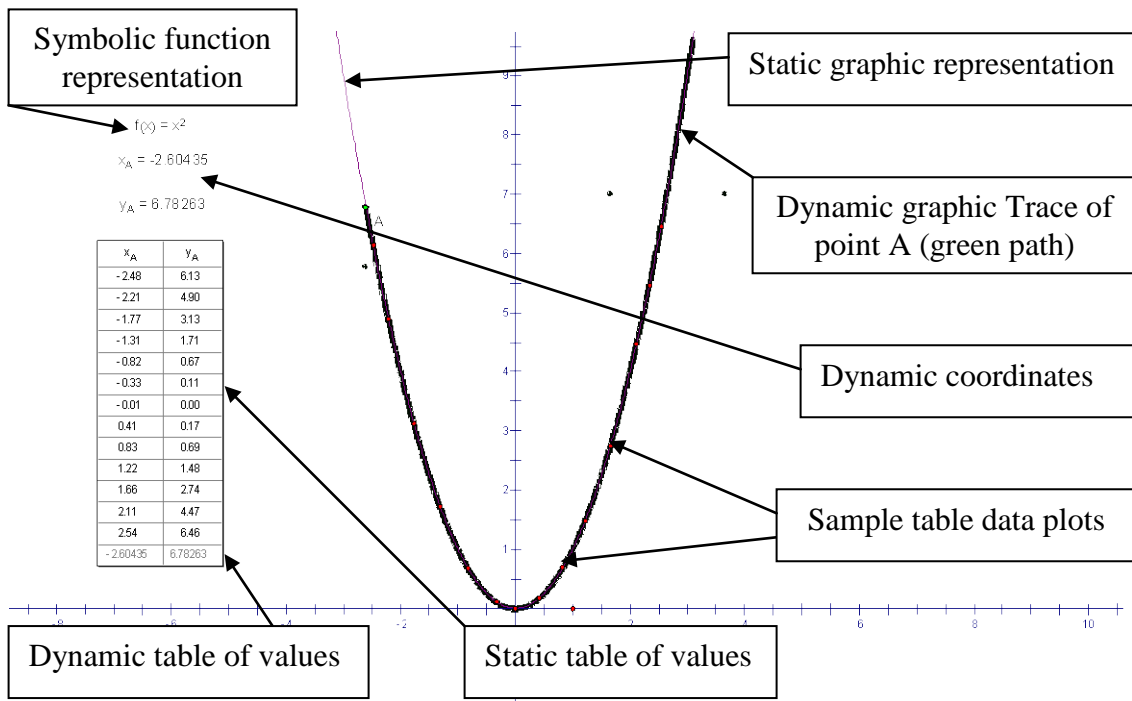
Figure 5.17: Function modeling and simulation in *Sketchpad*

Table 5.8 is a summary of *Sketchpad* techniques and the foreground and background mathematical representations that this introductory activity enabled students to handle

146

Sketchpad Syntax/Technical Skill (Enactive)	Representation in Foreground	Background Representation to link with	Representation or concept to contrast with
Defining a function	Symbolic representation		
Plotting a function	Static graphic representation	Symbolic representation	
Constructing a point on function plot	Static geometric/graphic	Any point on graph satisfies equation/function	Domain & range
Animation of plotted point	Dynamic graphic	Static graphic	Variables
Measuring coordinates of plotted	Static numeric	Cartesian plane	
Tabulating coordinates	Static numeric	Function domain and range	Run
Animating plotted point	Dynamic graphic	Dynamic numeric	Domain & Range
Tracking tabulated values	Static numeric	Snapshots of dynamic	
Plotting table data	Static graphic	Static numeric/ compare domain and range of values	One of infinitely many possible samples
Practice with more functions	Multiple representations	Multiple representations	
Combinations of the above	Real world applications	Algebraic/symbolic/graphic /numeric as applicable	

Table 5.8: *Sketchpad* **skills and mathematical representations for the function concept**

147

with varying degrees of success. The menu command details are treated as routine background technical material and excluded from the table for the sake of economy but included in subsequent *Sketchpad* activities in Appendix B. In other words the software syntax remains an important constraint if not adequately mastered or automated by students. In this connection Artigue (2002:250) notes that in order to understand and promote instrumental genesis for learners it is necessary to identify both the command (or internal) and the organizational (interface) constraints induced by a CAS instrument. By instrumental genesis in respect of the software, Artigue seems to be echoing the use of software as a tool for modeling mathematical concepts (compare 2.8, 3.1, 3.4.1, 3.4.2, 3.4.4, 3.4.5, and 4.2.2). The combination of dynamic graphic, dynamic numeric and dynamic symbolic features give *Sketchpad* its distinguishing characteristic. Falcade, Marrioti and Laborde (2004:368) remind us that the primitive metaphor of co-variation is motion and suggest that dynamic software might provide a semantic domain of space and time within which variation can be experienced as motion.

5.3.4   **Activity B**: Modeling the Average Rate of Change of a Function

This activity was intended to introduce or refresh and consolidate participants' understanding of the slope between two points on a graph as representing the average rate of change between two points on a function's domain (compare 2.8.2, 3.4.4 and 4.2.3). Figure 5.18 shows the graph of $y=x^2$ which was used in this activity. Table 5.9 summarizes participants' responses to Activity B.

Question B1 (Step 8a): *Knowledge of the Secant Line*. Eight out of nine pilot students responded correctly to this question referring to the line joining the two points, A and B, on the curve as a secant line and not the chord. One student did not respond to this question. However, as to what the line represented, only Student 9 responded correctly that it represented the 'gradient'. More strictly though, the line does not represent the gradient. Rather it represents the secant, and the gradient of the line represents the average change in the curve in the interval. Put differently, the slope of the line represents an approximation of the slope of the curve.   The low response to this question in the pilot

f(x) = x² $x_A = -0.50$ $x_B = -1.83$

$y_A = 0.25$ $y_B = 3.34$ $x_B-x_A = -1.33$ $y_B-y_A = 3.09$

$\dfrac{y_B-y_A}{x_B-x_A} = -2.33$ Slope $\overleftrightarrow{AB} = -2.33$ $\overleftrightarrow{AB}$: y = -2.33x-0.91

x_A	x_B	x_B-x_A	y_A	y_B	y_B-y_A	$\dfrac{y_B-y_A}{x_B-x_A}$	Slope $\overleftrightarrow{AB}$	$\overleftrightarrow{AB}$
2.16	0.84	-1.33	4.68	0.70	-3.98	3.00	3.00	y = 3.00 x-1.81
2.03	0.70	-1.33	4.10	0.49	-3.61	2.72	2.72	y = 2.72 x-1.42
1.79	0.46	-1.33	3.19	0.21	-2.98	2.24	2.24	y = 2.24 x-0.82
0.96	-0.37	-1.33	0.92	0.14	-0.78	0.59	0.59	y = 0.59 x+0.35
0.59	-0.74	-1.33	0.34	0.55	0.20	-0.15	-0.15	y = -0.15 x+0.43
0.45	-0.88	-1.33	0.20	0.78	0.58	-0.44	-0.44	y = -0.44 x+0.39
-0.03	-1.36	-1.33	0.00	1.84	1.84	-1.39	-1.39	y = -1.39 x-0.04
-0.36	-1.69	-1.33	0.13	2.85	2.72	-2.05	-2.05	y = -2.05 x-0.61
-0.50	-1.83	-1.33	0.25	3.34	3.09	-2.33	-2.33	y = -2.33 x-0.91

Figure 5.18: Graph of *y=x²* and the Secant Line AB

phase suggested the need for answer spaces to be provided to prompt responses. When this was done in the final implementation phase the response rate improved to 100% with only one incorrect response. The responses included: 'gradient' (x2) 'gradient between two points', 'average gradient from A to B', 'the slope of the curve'.

Question B2 (Step 9: *Slope calculation*)

The calculation of the slope between A and B was not a conceptual problem as all students used the correct method and 8/9 students (89%) obtained the correct value of 3 in the pilot phase. In the final phase all five participants obtained correct answers suggesting strong pre-calculus knowledge (compare 5.2.2).

	Pilot Phase										Final Phase					
Question	MT	NM	HS	JM	BT_1	BT_2	ES	NS	CJ	% P	EM	TM	DJ	RI	MN	% P
Step 8a)	1	1	1	1	nr	1	1	1	1	**89**	1	1	1	1	1	**100**
B8b)	0	n	nr	0	nr	0	nr	nr	1	**11**	1	1	0	1	1	**80**
B9a)	1	1	1	1	1	1	1	1	1	**100**	1	1	1	1	1	**100**
B9b)	1	0	1	1	1	1	1	1	1	**89**	1	1	1	1	1	**100**
B10a)	1	0	1	1	1	1	nr	1	1	**78**	1	1	1	1	1	**100**
B10b)	1	1	1	1	1	1	nr	1	1	**89**	1	1	1	1	1	**100**
B11a)	1	nr	nr	nr	nr	nr	nr	1	nr	**22**	1	1	1	1	1	**100**
B11b)	1	nr	nr	nr	nr	nr	nr	1	nr	**22**	1	1	1	1	1	**100**
B12a)	1	nr	nr	nr	nr	nr	nr	nr	nr	**11**	0	1	1	1	1	**80**
B12b)	1	nr	nr	nr	nr	nr	nr	nr	nr	**11**	0	1	1	1	1	**80**
B13a)	1	nr	nr	1	nr	1	1	1	1	**67**	1	1	1	1	1	**100**
B13b)	1	0	nr	1	nr	1	1	1	1	**67**	1	1	1	1	1	**100**
B15a)	1	1	1	nr	nr	1	1	nr	nr	**56**	0	0	1	1	1	**60**
B15b)	1	1	1	nr	nr	1	1	nr	nr	**56**	0	0	1	1	1	**60**
B15c)	nr	1	nr	nr	nr	1	1	nr	nr	**33**	0	1	1	1	1	**80**
B15d)											1	1	1	1	1	**100**
B16											1	1	1	1	1	**100**
Total	**13**	**6**	**7**	**7**	**4**	**10**	**8**	**9**	**8**	**53**	**12**	**15**	**16**	**17**	**17**	**91**
Key to Table 5.7																
1 = correct response; 0 = wrong response; nr = no response																
Table 5.9: Student responses to the average rate of change activity																

Question B3 (Step 10: *Sketchpad confirmation of the slope value*)

During the pilot phase 7/9 students successfully verified the slope of Secant Line AB using *Sketchpad*. Only student NM calculated a wrong value of 2. Student ES did not respond to this question. When answer spaces were provided in the final phase the response rate improved to 100%. The main thrust of the question was to compare the hand calculated slope with the *Sketchpad* measured slope (see Figure 5.19).

Sample Responses to Step 10	
'calculated slope = slope from measure menu'	Student JM
'The gradient is the same'	Student BT_1
'The gradients are the same meaning computer is equally accurate'	Student BT_2
'The slope measurements are the same'	Student CJ

Figure 5.19: Comparison of hand calculated and computer generated results

Of particular interest was Student BT_2's response where he checked the computer's answer against the hand-calculated one suggesting an appropriate orientation to habitually test computer answers for reasonableness and plausibility rather than believing them wholesale to be accurate. This is in tandem with Artigue's (2002:208) warning that one should be careful not to leave the student with a feeling of dependence on the technological tool – working with an 'oracle' instead of an instrument. In other words the student rejects the slavish use of the computer as the arbiter of pure reason (compare 1.1).

Question B4 to B5 (Steps 11 – 15)

These questions were to enable students to create an expression of the numerator and denominator of the difference quotient in *Sketchpad.* This question epitomized a transformation from mechanical actions in the enactive world to algebraic representation – a shift from the graphic world to the symbolic embodiment (compare 4.3.1.4).

Question B6 (Step 14)

The purpose of the first step was to confirm the equivalence of the difference quotient, the average rate of change between two points on a graph, and the *Sketchpad* calculated slope. In a sense this fulfils the transformational (rule-based) role of algebraic activity to change the form of an expression (through say substitution) while maintaining equivalence (Lagrange, 2005:149). The purpose of the second step was to help students

Sketchpad syntax/Technical Skill (Enactive)	Mathematical Representation in Foreground	Background Representation to link with	Representation or concept to contrast with
Constructing line segment	Static geometric line segment	line as shortest distance between two points	chord, tangent, arc
Constructing secant line	Static geometric/graphic	Chord, tangent, arc	
Measuring secant slope	Static numeric	Static geometric steepness	rise, run
Measuring change in x (run)	Static numeric	Change in x-axis direction	rise
Measuring change in y (rise)	Static numeric & symbolic	Change in y-axis direction	Run
Calculating the slope as $\delta y/\delta x$	Static symbolic & numeric	$-\delta y/-\delta x$	
Equation of secant line	Static algebraic/symbolic	Slope & y-intercept	
Animation of secant line	Dynamic graphic, dynamic numeric, dynamic symbolic	Orientation and steepness of orientation of line	Range of slope values
Tabulation of values	Static numeric/static graphic/ static symbolic		
Tracking of values as animation progresses	Dynamic numeric/graphic/ symbolic	Static numeric/graphic/symbolic	
	Real world applications		

Table 5.10: *Sketchpad* skills and mathematical representations for modeling the average rate of change

152

utilize the dynamic *Sketchpad* properties to demonstrate that a change of point B's position simultaneously affects the other calculated components of the differential quotient. The third part of the question was to reinforce the idea that an increase or decrease (depending on whether one drags point B upwards or downwards) by dragging visibly affects the orientation of the secant line AB (compare Tall's perceptual control in 1.1 & 2.5.2).

The opportunity was ceased at this stage to introduce the differential quotient in its conventional algebraic form: $\dfrac{f(x + h) - f(x)}{h}$, where $f(x + h) = y_B$, $f(x) = y_A$ and

$h = (x_B - x_A)$. This question was intended to consolidate the symbolic embodiment of the average rate of change of change as the differential quotient. Once again this brings to the fore the transformational activity of algebra as a transposition from *Sketchpad* to an algebraic/symbolic representation. This does not lead to a deteriorated version but to inductive reasoning from data facts to ideas or laws (Lagrange, 2005:181).

Table 5.10 summarizes the interplay between *Sketchpad* procedures, the static and dynamic graphic, numeric and symbolic representations afforded and the connections between them.

5.3.5   Activity C: Modeling the Derivative as the Instantaneous Rate of Change

Figure 5.20 shows the *Sketchpad* diagram for this activity which focused on moving point B to point A which changes the 'secant line' to the tangent line at point A. Szydlik (2000:260) fears that the definition of the limit in terms of 'the function *f* goes to the *L* as *x* goes to *a*' is likely to create in students the belief that limits are approached forever (compare potential infinity in 1.4.2, 2.2.1). *Sketchpad*'s dynamic representational capability allays such fears by giving concrete expression and breathing life to the definition when students observe the limit of the secant line being reached in real time as point B moves to point A. It is the motion metaphor that gives the limit notion its primary meaning (Williams, 2001:294).

f(x) = x² $x_A = 1.03739$
Move A -> B $x_B = 1.03768$
$y_A = 1.07617$ $y_B = 1.07678$
$x_B-x_A = 0.00$ $y_B-y_A = 0.00$
$\dfrac{y_B-y_A}{x_B-x_A} = 2.08$ ↔ Slope $\overleftrightarrow{AB} = 2.07507$
AB: y = 2.08x-1.08

x_A	x_B	y_A	y_B	$\dfrac{y_B-y_A}{x_B-x_A}$	Slope $\overleftrightarrow{AB}$
0.02142	-0.75089	0.00046	0.56383	-0.73	-0.72947
-0.08251	-0.75089	0.00681	0.56383	-0.83	-0.83340
-0.18772	-0.75089	0.03524	0.56383	-0.94	-0.93861
-0.30516	-0.75089	0.09312	0.56383	-1.06	-1.05605
1.03739	1.03768	1.07617	1.07678	2.08	2.07507

x_A	x_B	y_A	y_B	x_B-x_A	y_B-y_A	Slope $\overleftrightarrow{AB}$
1.21624	0.97775	1.47925	0.95599	-0.24	-0.52	2.19399
1.14707	0.97775	1.31577	0.95599	-0.17	-0.36	2.12482
1.07794	0.97775	1.16196	0.95599	-0.10	-0.21	2.05569
1.01516	0.97775	1.03056	0.95599	-0.04	-0.07	1.99291
1.03739	1.03768	1.07617	1.07678	0.00	0.00	2.07507

x_A	x_B	y_A	y_B	x_B-x_A	y_B-y_A	$\dfrac{y_B-y_A}{x_B-x_A}$	Slope $\overleftrightarrow{AB}$
-1.12429	-0.75089	1.26403	0.56383	0.37	-0.70	-1.88	-1.87518
-1.07142	-0.75089	1.14794	0.56383	0.32	-0.58	-1.82	-1.82231
-1.00744	-0.75089	1.01494	0.56383	0.26	-0.45	-1.76	-1.75833
-0.87834	-0.75089	0.77148	0.56383	0.13	-0.21	-1.63	-1.62923
1.03739	1.03768	1.07617	1.07678	0.00	0.00	2.08	2.07507

x_A	x_B	y_A	y_B	x_B-x_A	y_B-y_A	$\dfrac{y_B-y_A}{x_B-x_A}$	Slope $\overleftrightarrow{AB}$	$\overleftrightarrow{AB}$
1.59278	1.03768	2.53696	1.07678	-0.56	-1.46	2.63	2.63046	y = 2.63x-1.65
1.48832	1.03768	2.21511	1.07678	-0.45	-1.14	2.53	2.52600	y = 2.53x-1.54
1.37832	1.03768	1.89977	1.07678	-0.34	-0.82	2.42	2.41600	y = 2.42x-1.43
1.27387	1.03768	1.62273	1.07678	-0.24	-0.55	2.31	2.31155	y = 2.31x-1.32
1.16386	1.03768	1.35456	1.07678	-0.13	-0.28	2.20	2.20154	y = 2.20x-1.21
1.03739	1.03768	1.07617	1.07678	0.00	0.00	2.08	2.07507	y = 2.08x-1.08

Figure 5.20: Dynamic generation of tangent as limit to secant line position

Table 5.11 shows the response rate to this activity.

Question C1 (Step 5) *The Effect of Moving Point B to Point A*

All the six respondents in the pilot phase and all five in the final phase correctly pointed out that when point B reaches point A's position, the secant line coincides with the tangent drawn at A. The value of the gradient remains 2, and again all five of the six pilot students who responded to this question obtained the correct gradient value. Five out of six students correctly identified the new secant line position to be the tangent at A. The fact that the tangent line position acted as the limiting position for the secant line helped to understand the concept of limit as a boundary value. The opportunity was seized at this stage to express the derivative at a point as the limit of the differential quotient obtained in Activity B as point B tends to point A. That is, $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

5.11	Pilot Phase							Final Phase					
Question	MT	HS	JM	BT$_2$	NS	CJ	%	EM	TM	DJ	RI	MN	% P
Step 5a)	1	0	1	1	1	1	83%	1	1	1	1	1	100%
5b)	0	1	1	1	1	nr	67%	1	1	1	1	1	100%
5c)	1	1	0	1	1	1	83%	1	1	1	1	1	100%
6a)	1	1	nr	1	1	1	83%	1	1	1	nr	1	80%
6b)	1	1	1	1	1	1	100%	1	1	1	nr	0	60%
7a)	1	1	1	0	0	0	50%	1	1	1	1	1	100%
7b)	1	0	nr	nr	1	0	33%	1	1	1	nr	1	80%
8a)	1	1	1	1	1	1	100%	1	1	1	1	1	100%
8b)	1	1	1	1	1	1	100%	1	1	1	1	1	100%
8c)	1	nr	nr	nr	1	nr	33%	1	0	1	0	0	40%
8d)	1	1	1	1	1	1	100%	1	0	nr	1	1	60%
8e)	1	1	1	1	nr	nr	67%	1	0	1	1	0	60%
8f)								1	1	1	1	0	80%
11a)	nr	nr	nr	0	nr	1	17%	1	1	1	1	1	100%
11b)	nr	nr	nr	1	nr	1	33%	1	1	1	0	0	60%
13a)	1	1	nr	1	nr	nr	50%	1	0	1	1	1	80%
18								nr	1	1	1	1	80%
Total	12	10	8	11	10	9	67%	16	13	16	12	12	81%
Key to Table: 1 = correct answer; nr = no response; 0 = wrong response													
Table 5.11: Student responses to the instantaneous rate of change activity													

Question C2 (Step 6) *Confirmation of conjecture that the secant line becomes the tangent*

Again all pilot and final phase students had no difficulty in confirming that the limiting position of the secant line was the tangent at A. This was in contrast to the pretest responses to Question 3 (compare 5.2.3). However, although all students correctly said the secant line could not go beyond the tangent at A, no explanation was sought in the pilot materials. The traces of the secant line path, appeared to confirm the mobile or dynamic nature of the secant line. In other words, the trace tool serves as a semiotic mediator for the limit concept (compare Activity A) of the trace tool (To be erased, though, after illustration in order to reduce cluttering as Figure 5.21 shows, otherwise without erasure students might mistake the shaded area for the tangent).

$f(x) = x^2$

$x_A = 1.00$

$x_B = 1.00$

$y_A = 1.00$

$y_B = 1.00$

Slope AB = 2.00

$y_B\text{-}y_A = 0.00$

$x_B\text{-}x_A = 0.00$

$\dfrac{(y_B\text{-}y_A)}{(x_B\text{-}x_A)} = 2.00$

x_A	x_B	$x_B\text{-}x_A$	y_A	y_B	$y_B\text{-}y_A$	$\dfrac{(y_B\text{-}y_A)}{(x_B\text{-}x_A)}$	Slope AB
1.00	1.90	0.89	1.00	3.59	2.59	2.90	2.90
1.00	1.77	0.77	1.00	3.13	2.13	2.77	2.77
1.00	1.50	0.50	1.00	2.25	1.25	2.50	2.50
1.00	1.28	0.28	1.00	1.63	0.63	2.28	2.28
1.00	1.08	0.08	1.00	1.16	0.16	2.08	2.08
1.00	1.03	0.03	1.00	1.07	0.06	2.03	2.03
1.00	1.00	0.00	1.00	1.01	0.00	2.00	2.00
1.00	1.00	0.00	1.00	1.00	0.00	2.00	2.00
1.00	1.00	0.00	1.00	1.00	0.00	2.00	2.00

Move B -> A

Figure 5.21: Erasing the secant line path to reveal tangent as limiting position

Question C3 (Step 7)

Three out of six pilot students (50%) computed the correct gradient value of the tangent

to be 2. Student BT_2 computed the gradient as follows: $\text{gradient} = \dfrac{1\text{-}1}{0\text{-}0} = 0$ which

erroneously considered the result of dividing by zero to be zero. Student CJ similarly argued that the gradient at (1.00; 1.00) was zero 'as the points are combined there is no change'. At this stage the idea of local straightness can be articulated and illustrated by local magnification as suggested by Tall (1986). The students have a valid notion that since point B would have been merged with point A then there cannot be any change in y nor in x and therefore no gradient. (NB: need felt to re-define the tangent practically to be a line joining two very close points then arises. Otherwise the same argument that there is no change in y or x also explains why the tangent disappears when point B coincides exactly with point A).

The meaning of the gradient at the point of tangency was explained by MT to be that

'the gradient of the curve $f(x) = x^2$ at (1;1) is 2,02.' (same as that of the tangent at that point). Student NS said 'It means that there is no change'. We can see here conceptual hurdles as students get stuck with the meaning of the gradient of the tangent disconnected from the rate of change at the point. The teacher-researcher explained that the gradient at the point of tangency represented the rate of change (derivative) at that instant (instantaneous rate of change). That is, the derivative as the limit of the differential

quotient $\dfrac{f(x + h) - f(x)}{h}$ as h tends to zero either from below or from above.

Question C4 (Step 8) *Interpretation of the table of values*

Most students had no problems reading the table of values. All six pilot and five final phase students who responded to this question gave the values of $y_B - y_A = 0{,}00$, $x_B - x_A = 0.00$. Four of the six students, however, could not explain the meaning of $\dfrac{0.00}{0.00}$.

Student MT reasoned as follows: ' $\dfrac{0.00}{0.00}$ means that the 2 points (A and B) are at the same point, they are appearing as a single point.'

Asked if the result was correct student MT replied plausibly:
'No, the answer is wrong because it is not for the line joining points A and B but for the gradient of the tangent to the curve at that point'

This student ingeniously tried to separate the meaning of the gradient at a point (which is meaningless) from the gradient of a curve at a point of tangency. In fact the student stumbled on the double meaning of point of tangency which in essence reduces to a

gradient at a point and yields 'undefined' as the answer to $\dfrac{0.00}{0.00}$ when A and B

coincide exactly.

157

Student HS simply adjudged that $\dfrac{y_B\text{-}y_A}{x_B\text{-}x_A}=0.00$ was a wrong answer without elaborating.

(NB the answer on the table of values was actually reflected as 2.00 suggesting that the student did not believe the computer's answer to be correct).

Question C5 (Step 11) *Increased precision for* $\dfrac{0.00}{0.00}$ *in Sketchpad.*

Only three pilot students (HS, BT_2 and CJ) managed to respond to this question. Student HS correctly predicted that with increased precision $\dfrac{y_B\text{-}y_A}{x_B\text{-}x_A}=0.00$ 'may end up having some value'. In fact it already had a value of 2.00 on the table which had been incorrectly given as 0.00 in the worksheet as referred to above. (In the final implementation phase the error was corrected to reduce confusion). Student BT_2, however, noticed the error in the printed value and argued as follows: 'If we increase the precisions, the values of x_B-x_A and y_B –y_A increase but $\dfrac{y_B\text{-}y_A}{x_B\text{-}x_A}$ remains equal to 2.00'

Student CJ similarly noticed that 'there are more numbers before the decimal hence the value was rounded off'. She realized that the answers given by the computer were approximations to fewer decimal places and were otherwise extensible. Encouraging students to evaluate the accuracy of computer generated answers obviates the slavish use of technology. With this in mind, Drijvers (2000:208) cautions that one should be careful not to leave the student with a feeling of dependence on the technological tool to the extent that it becomes an oracle instead of an instrument. In support of this view, Noble, Nemirovsky, Wright and Tierney (2001:87) assert that concepts reside not in physical materials, computer software, or prescribed classroom activities but in what students do and experience (technology by itself not a panacea 1.1 & 6.1).

<u>Question C6</u> (Step 13): *Increased precision of the value of y_B-y_A and the slope of AB.*

Only three pilot students managed to respond to this question. Student BT$_2$ proceeded as

follows: ' y_B-y_A = 0.0007 in ten thousandths

= 0.00070 in hundred thousandths

Slope AB = 2,0091' and concluded that

'as we increase the precision the values also increase.

It was clear to the student that what was given as the value of y_B-y_A was an

approximation to two decimal places. Similarly, the value of the slope of AB = 2.00 was

shown to be an approximation. Hidden behind the two decimal place answers for the

numerator and the denominator, were residual fractions in the ratio of 2:1 respectively.

Students HS and MT commented respectively as follows: 'There are some values which

will come after zero meaning the number was rounded off'; The slope is not 2 but is

2.00365. I can conclude that the accuracy was truly limited'. Both the slope value

sequence in the table of values and the sequence of secant line equations converge to the

tangent line position slope and equation. (NB: A Cauchy sequence is a sequence whose

terms become arbitrarily close together as the sequence progresses while a convergent

sequence is one whose terms become arbitrarily close to some arbitrary number, the limit,

as the sequence progresses (Bell, 1978:39)). (NB: It was an oversight for the question to

provide for point A to be approached only from the right hand side. Allowing it to be

approached from both the left hand side and the right hand side would reinforce the idea

that the left and right limits have to be the same for the limit of the differential quotient to

exist. Compare 2.8.5 & 3.6)

Artigue (2002:266) encourages a distinction to be made between exact and approximate

computations in a CAS environment. In the same regard, Cuoco (2002:294) sounds a

word of caution that although CAS environments have the potential to help students

better understand the connections between the analytic behaviour of functions and their

algebraic representations, algebraic calculations in a CAS environment know nothing of

approximating. Their results are purely formal and exact, and therefore only numerical

computations are amenable to approximations.

Sketchpad **Syntax/Technical Skill** (Enactive)	**Mathematical Representation in Foreground**	**Background Representation to link**	**Representation or concept to contrast with**
Moving Point A to Point B along the function plot	Dynamic graphic	Dynamic numeric	Coordinates (numeric)
Tabulation of coordinates	Static numeric	Cartesian plane	
Tracking of coordinates	Dynamic + static numeric	Point B as destination of Point A(dynamic graphic)	Point B as limit of Point A
Tabulation of Secant Line Slope	Static numeric	Static graphic	
Indirect animation of Secant Line	Dynamic graphic	Dynamic numeric	
Tracking Secant Slope & Equation from above (for +ve and –ve slope)	Static numeric	Dynamic graphic + dynamic numeric +dynamic symbolic	Limit from above
Magnification of point of tangency 'from above'	Static graphic	Symbolic Tangent Equation	Equation from above
Tracking Secant Slope & Equation from below (for +ve and –ve slope)	Static numeric	Dynamic graphic + dynamic numeric + dynamic symbolic	Limit from below
Magnification of point of tangency from below	Static graphic	Symbolic Tangent Equation	Equation from below
Increasing slope precision	Static numeric/static symbolic	Numeric slope	
Resurfacing a disappeared Tangent	Tangent as line joining two very close points	Infinitesimally close points	Infinitesimals
Multiple representations	Real world applications (verbal)	Algebraic/symbolic/graphic /numeric as applicable	Motion context of derivative

Table 5.12 *Sketchpad* **Skills and Mathematical Representations for Modeling the Derivative as the Instantaneous Rate of Change (Tangent Slope)**

Table 5.12 summarizes the interplay between *Sketchpad* procedures, the static and dynamic graphic, numeric and symbolic representations afforded and the connections between them. In this dynamic the *Sketchpad* syntax performed by the student enactively forms the input background while the on-screen display as input occurs forms the input foreground. The output or mathematical representations that the student works on directly becomes the foreground mathematical representation while the representation that is achieved indirectly forms the background. The challenge for the student is to coordinate and make mutual sense of these multiple representations.

5.3.6   Activity D: Modeling the Derivative as the Rate of Change Function

Figure 5.18 shows the *Sketchpad* graph of $y=x^2$ and its derivative being the green line generated as the trace of the plot of the tangent slope against x-values as the tangent is animated. A central aspect of this activity was therefore the ability to plot the slope at any point of the curve against *x*-values on the same axes. To achieve this, the slope/gradient
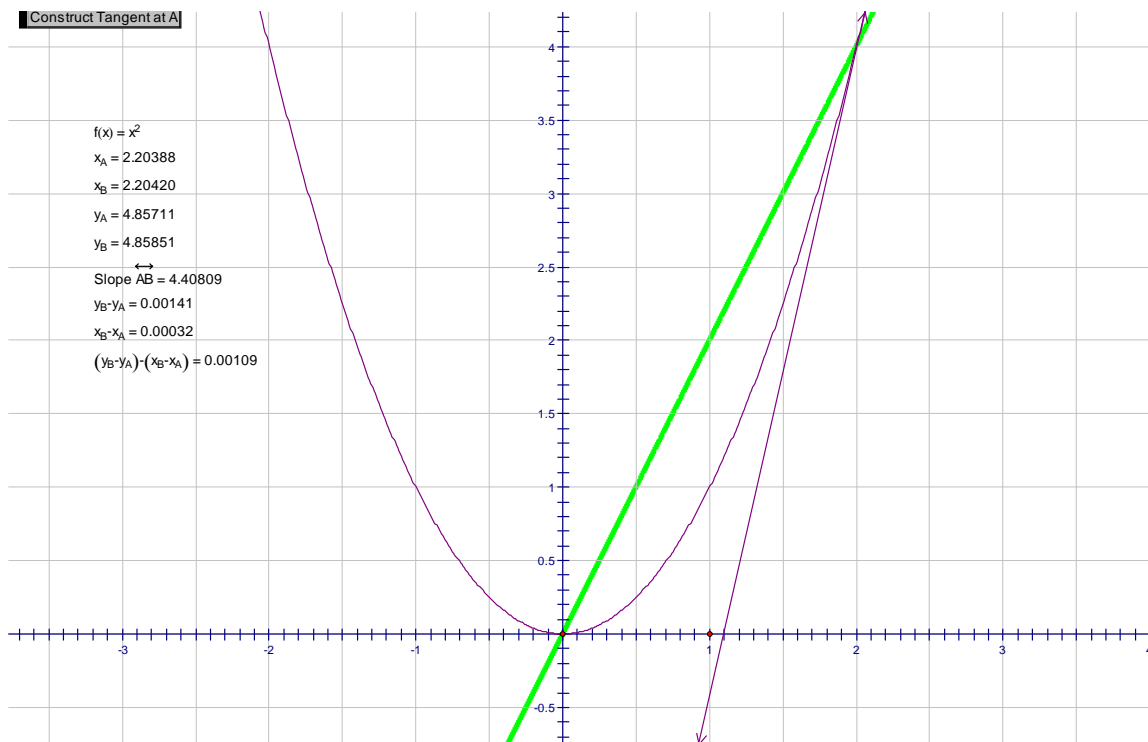
Figure 5.22: Generating the Gradient Function Trace

of the tangent at every point (instantaneous rate of change) was plotted against the *x*-coordinate of the dynamic (moving) point of tangency. When the tangent is animated in *Sketchpad*, and the plotted point traced, the trace is the green straight line graph of *y = 2x* in Figure 5.22. Table 5.13 shows the responses to this Activity by eight pilot students in the experimental group and five in the final phase experimental group.

Question D1(Step 8):

Three out of eight pilot students responded correctly to this question. One could not produce the desired graph while the remaining four provided no responses. The teacher-researcher helped all those students that got stuck with their work. Only one final phase student responded to this question and did so correctly.

Question D2 (Step 9):

By clicking and tabulating values of x_A and the tangent slope then double clicking periodically as the animation progresses a sample table of values is generated in *Sketchpad* as shown in Figure 5.23.
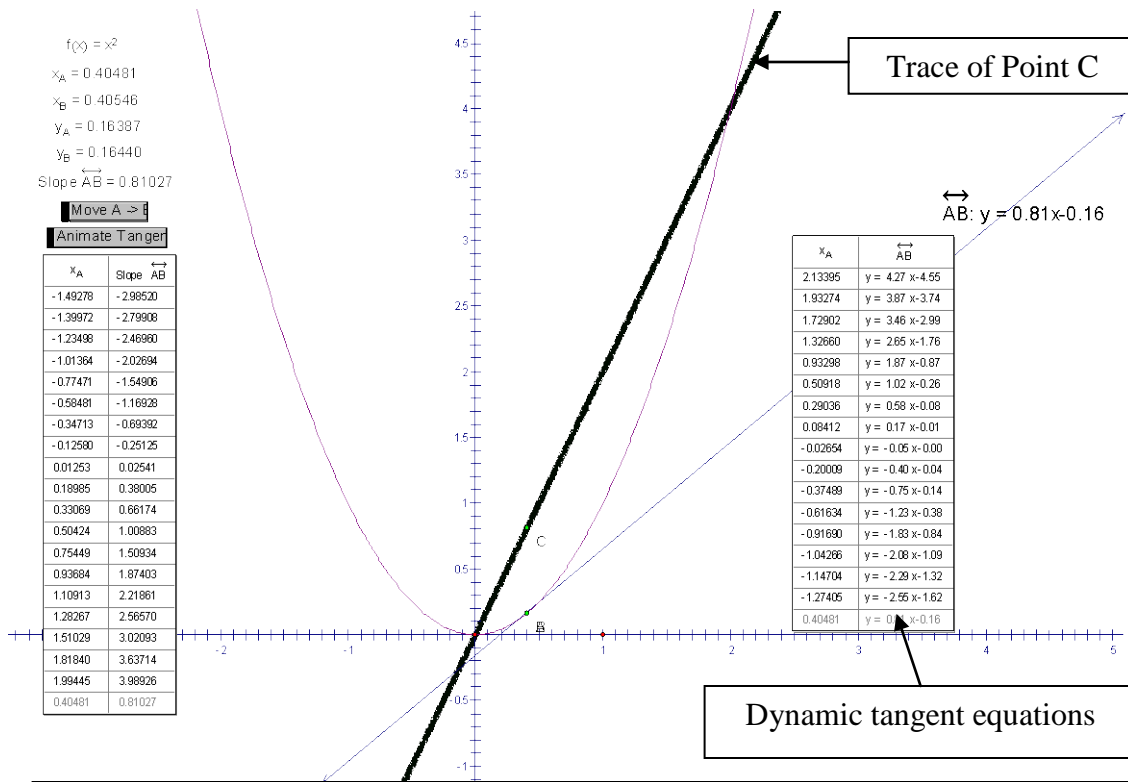
Figure 5.23: The graph of the gradient function of *y=x²* with table of values

5.13	Pilot Phase									Final Phase					
Question	MT	NM	HS	JM	KR	BT$_1$	BT$_2$	NS	%	EM	TM	DJ	RI	MN	%
Step 8a)	1	1	nr	nr	nr	nr	1	0	38%	nr	nr	nr	1	nr	20%
9a)	1	nr	nr	nr	1	nr	1	1	50%	1	1	1	1	1	100%
9b)	0	1	1	nr	1	nr	0	nr	38%	1	1	1	1	1	100%
10a)	1	1	nr	1	1	nr	1	1	75%	1	1	1	nr	1	80%
10b)	1	1	1	1	nr	nr	1	1	75%	1	1	1	nr	1	80%
10c)	nr	nr	1	nr	nr	nr	1	1	38%	1	1	1	nr	1	80%
10d)	1	nr	0	nr	nr	nr	1	1	38%	1	1	0	nr	1	60%
Step 12	1	1	1	1	1	nr	1	nr	75%	1	1	1	nr	0	60%
13	1	1	1	nr	1	nr	1	1	75%	nr	1	nr	nr	1	40%
16	nr	1	1	1	nr	1	1	nr	63%	1	1	1	nr	1	80%
17a)	1	nr	nr	1	nr	nr	1	nr	38%	1	1	1	nr	1	80%
17b)	nr	nr	nr	1	nr	1	1	nr	38%	1	1	1	nr	1	80%
17c)										1	1	1	nr	1	80%
18	nr	nr	nr	nr	nr	0	nr	nr	0%	1	1	1	nr	1	80%
19a)										1	1	1	nr	1	80%
19b)										1	1	1	nr	1	80%
20										1	1	nr	nr	1	60%
Total	8	7	6	6	5	2	11	6	49%	15	16	13	3	15	73%
Key to Table: 1 = correct response; 0 = wrong response; nr = no response															

Table 5.13: Summary of Responses to the Rate of Change Function

This question was intended to encourage students to read/decipher a pattern or functional relationship between the *x*-values and the *y*-values on the table. Five of the eight pilot

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Sample responses to Question 2 (Step 9)

'As x_A decreases the slope also decreases'   **Student HS**

'As x_A decreases, the slope off AB also decreases' **Student NM**

'They both increase or decrease at the same time' **Student BT$_2$**

'As x_A values increase as the values of the slope also increase' **Student NS**

'Slope is almost twice the value of x_A' **Student KR**

Figure 5.24: Sample Responses to the Derivative as Rate of Change Function

final phase the question was re-worded to require the proportion and all five final phase students that responded gave correct answers. Figure 5.24 provides some samples of students' responses.

The first four pilot students would only indicate that the slope increases (or decreases) as x_A increases (or decreases) without quantifying the strength (proportion) of the rate of increase in the slope caused by a unit increase in x_A. Student KR was therefore the only one who came closest to detecting that $y = 2x$ (by using 'almost suggesting some hesitation). The rest of the respondents would not hazard a guess or conjecture The question was reworded in the final implementation phase to elicit first, the direction of relationship and then to quantify the strength or intensity of the relationship (c.f. Stroup's (2002) metaphor of rate as intensive quantity and slope as steepness). After the re-wording, the students' achieved 100% success on both the direction and the strength of the relationship (see Table 5.13).

Question D3 (Step 10): *Determining the gradient and equation of the path of point C.*
As Table 5.13 shows, six out of eight pilot students and all of the final phase students who attempted the question obtained the correct gradient equation for the linear path

Student MT's Response	**Student BT$_2$'s Response**
'Gradient of path C = $$\frac{\text{increase in } y}{\text{increase in } x} = \frac{2}{1} = 2.$$ Equation $y = mx + c$ $\quad\quad y = 2x + c$, where $c = o$ $\quad\quad => y = 2x$ The function represents the derivative of the curve	
Student NS's Response	Grad of C = $\frac{2 - 0}{1 - 0} = 2$ Equation = $\frac{y - 2}{x - 1} = 2$ $y = 2x = 2x - 2$ $y = 2x$
$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{3 - 2}{1,5 - 1} = \frac{1}{0,5} = 2.$$ Intercept $= 0$, $=> y = 2x$ is the equation of line. The Derivative is $y = 2x$	It represents the gradient of $f(x) = x^2$

Figure 5.25: Sample responses to the equation of the gradient function trace

traced by point C (c.f. link concept of second derivative here) and to obtain the equation represented by the line. This question provided students with the opportunity to shift from the embodied world of perception to the symbolic world of symbol calculation and manipulation (compare 4.3.1.4).

From the sample responses students were by then quite familiar with the determination of the gradient of a line (or between two points). The students, however used different strategies for coming up with the equation represented by the line, an indication of different prior learning backgrounds. The meaning of the equation was interpreted differently as: 'representing the derivative', 'derivative of the curve', or 'the gradient'. The use of the term 'derivative' was further testimony to some previous calculus background by A level holders. The 'gradient' student probably meant 'gradient function' which is the equivalent of 'derivative', presumably due to a weaker pre-calculus background (NB: in the pilot phase the term 'derivative' was avoided in the activities to accommodate non A level holders). All pilot students, however, were unable to link their conclusion in Step 10 to the conjecture made in Step 9. It was the opposite in the final phase after answer spaces had been provided. All students established the ratio. For a example, 'x_A:slope = 1:2' given by Student MN.

Question D4 (Step 12):
The rationale for coming up so late with the method of obtaining the derivative straight from *Sketchpad* was to allow for a solid foundation to be laid on what the derivative represents as chronicled in the previous steps in a dynamic graphic context. As Table 5.9 shows, six out of eight pilot students (75%) and 100% of final phase students confirmed that the derivative given by *Sketchpad* 'at an instant' for $f(x) = x^2$ was the same as the calculated one. This strategy was in consonance not just with the embodied approach to the calculus, but also APOS theory (compare 4.3.1.3 & 4.3.1.4). The actions of constructing the derivative are interiorized into a process (graphic differentiation) leading to a trace (object) that must be encapsulated into a derivative (differentiation) schema.

Question D5 (Step 13): *Plotting the Sketchpad given derivative.*

The idea of plotting the derivative was to check the equivalence between the plot and the trace of point C. All six pilot students who reached this stage of the activity confirmed that the plot of the derived function fitted the trace of point C in the perspectives reflected in Figure 5.26. This use of *Sketchpad* fits within the verification (checking) function of a computational tool referred to by Doerr and Zangor (2000) (compare 2.6).

Students' Sample Responses to Question 6 of Activity D

'They represent the same line'   **Student NT**

'Plotted derivative is the same as the trace of point C'   **Student NM**

'Its similar to path of C'   **Student HS**

'Fits perfectly the same'   **Student KR**

'The plotted derivative and traces of point C are represented by the same line'
                                                                                                  Student BT$_2$

The line plotted by C is the same line plotted by the derivative of $y = x^2$ i.e. *2x*'
                                                                                                  Student NS

Figure 5.26: Sample Responses to a Comparison of the Trace of Point C with the Graph of the Derivative of $y= x^2$.

After 'returning' the trace of point C and re-animating the tangent line, point C was observed to follow the graph of the derivative and all pretest and posttest respondents to this question correctly affirmed that their conjectures in steps 10 and 12 had been confirmed.

Question D6 (Step 17): *Expansion of the differential quotient*

Only three out of the eight pilot students successfully expanded the differential quotient as shown in Figure 5.27. Once again, the students who succeeded used different expansion strategies and cues, which seemed reflective of the extent high schools go to ensure students have elaborate procedural understanding to pass examinations. However, all of these students did not respond to the final part of the question regarding comparison. Student BT$_2$ provided the most technically comprehensive answer to the

other stages while MT only simplified and did not follow through. Student JM was on the right path even though she did not bother to use the equal sign or even to indicate what stage of the question she was answering at a given point. She left everything assumed instead. Once again provision of adequate answer spaces appeared necessary in the final implementation stage to increase the response rate. (All four final phase respondents provided correct solutions to this question)

Sample Responses to Question 7 (Step 17)

$$\frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2}{h}$$
$$= \frac{2xh + h^2}{h}$$
$$= \frac{h(2x+h)}{h}$$
$$= 2x + h$$

Student MT

$$\frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h}$$
$$= \frac{2xh + h^2}{h} = \frac{2xh}{h} + \frac{h^2}{h}$$
$$= 2x + h$$

as $h \to 0, y \to 2x$

Student BT$_2$

$$a^2 + b^2 =$$
$$(x+h)^2 - x$$
$$\frac{x^2 + 2xh + h^2 - x^2}{h}$$
$$\frac{2xh + h^2}{h}$$
$$\frac{h(2x+h)}{h}$$
$$2x + h$$
$$2x$$

Student JM

Figure 5.27: Sample Responses to the Expansion of the Differential Quotient to Determine the Limit Symbolically from First Principles

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Student BT$_1$, however, illustrated the limit as $\underset{h \to 0}{2x + h} \to 2x$ instead of

$lim_{h \to 0}\ 2x + h = 2x$, which was more of a notational rather than conceptual or

procedural knowledge problem. The student went on to attempt the 'Further Exploration' additional/extension work. As to why the point C seemed to wobble when animated he gave the answer 'It is the derivative function of $f(x)$', which of course, was inadequate.

167

In response to the question about whether the power rule of differentiating applied to negative and fractional indices, the student ticked the work sheet to affirm and for the real world problems he had the following responses:

Problem 1: Given that $y = x^2 + 1$, the student evaluated $\frac{dy}{dx} = 2x$, a sign of familiarity with differentiation rules.

a) For the average rate of change of y with respect to x over the interval (3,5) he erroneously computed $\frac{dy}{dx} = 2(3) = 6$ instead of evaluating the differential quotient. More practice appeared necessary for a clearer algebraic distinction between the givens of the differential quotient (average rate of change) and the derivative as a limit of the differential quotient (limit of the average rate of change).

b) For the instantaneous rate of change of y with respect to x at the point $x = -4$, he correctly evaluated $\frac{dy}{dx} = 2(-4) = -8$ and For the instantaneous rate of change of y at a general point he gave $\frac{dy}{dx}(0,0) \rightarrow 0$, instead of the general (indefinite) derivative, $\frac{dy}{dx} = 2x$, which he had actually obtained earlier on without being requested.

Reconstructed Interview with Student BT$_1$

The following episode with Student BT$_1$ illustrates some of the dialectical relationship that can arise as cognitive conflict when we shift from a textbook oriented curriculum to a *Sketchpad* dynamic mathematics environment. (NB: TR = Teacher Researcher)

Student BT_1:   Excuse me sir, the table I have generated is not the same as that in the worksheet.

TR: How is it different? (seemingly equally surprised)

Student BT_1: Look, the values for x_A and for the slope of AB that my table shows are different.

TR: Can you figure out why?

Student BT_1: I can't explain, I expected them to be the same as those on the work sheet.

TR: Those in the worksheet are those that I generated myself and not you.

Student BT_1: But shouldn't they have been the same all the same?

TR: They could only have been the same if you had ticked them at the same rate (intervals) as mine. Yours is therefore a different set of points along the curve. Check whether your friend's are the same as yours.

Student BT_1: No, they are not.

TR: Why do you think they are not the same?

Student BT_1: Does it mean we sampled different sets of points, so each one of us in the class has a different table of their own?

TR: Precisely, unless there people who sampled/clicked exactly the same sequence of points.

In this way the researcher was able to focus student's attention to the uniqueness of their sample table of values and by asking them to plot them it was clear from the plots along the curve that all sample points lay on the graph (see Figure 5.17). This interview revealed two principles: the uniqueness of students' creations (compare 3.4.7, 4.3.1.2 & 5.3.1) and as well as the coordination of multiple representations (compare 5.3.3) and the idea of dwelling in the tool or transition from *model of* to *model for* reasoning with (compare 3.4.5 & 5.3.3).

Summary

On the whole, this activity marked the zenith of the instrumentalization of *Sketchpad* to model students' concept image of the derivative (compare 2.8). Table 5.14 summarizes the foreground *Sketchpad* instrumentalization processes and both foreground and background mathematical representations afforded. The arrows emphasize the dialectical

Sketchpad **Syntax/Technical Skill** (Enactive)	**Mathematical Representation in Foreground**	**Background Representation to link with**	**Representation or concept to contrast with**
Plotting tangent slope against x_A	Static numeric	Static graphic	Tangent slope
Labeling and tracing plotted point	Static graphic		
Tabulating coordinates of plotted point	Static numeric	Static graphic	
Animating tangent	Dynamic graphic (x2)	Dynamic numeric	
Tracking table data	Static numeric snapshots	Dynamic numeric + dynamic graphic	
Plotting table data	Static graphic	Static numeric	
Comparing input & output values	Static numeric	Static graphic	Functional relationship
Constructing lines joining plotted table data	Static graphic	Static graphic (Trace of C)	
Measuring line equations	Static symbolic (equations)		
Calculating equation of trace of C (pen and paper)	Algebraic/symbolic		
Finding symbolic derivative	Static Symbolic	Symbolic equation for Trace of point C	
Plotting derivative	Static graphic		
Animating tangent	Dynamic graphic	Dynamic numeric, dynamic symbolic	
Problem solving	Real world applications (verbal)	Algebraic/symbolic/graphic /numeric as applicable	Motion context of derivative, etc

Table 5.14 *Sketchpad* Skills and Mathematical Representations for Modeling the Derivative as the Rate of Change Function

relationship between the technical or instrumented actions and the mathematical representations enabled.

5.3.7 Miscellaneous Activities (Activities E & F)

Activity E: Modeling the Differentiability and Optimization of a Function

This activity served more as a demonstration/explanation of various specific situations when a function is not differentiable. That is, when the derivative does not exist: at a corner point, point of vertical tangency and point of jump discontinuity.

Activity F: Further Practice in Modeling Functions and their Derivatives

In the final implementation phase this activity was done to enable students to gain further practice in graphing functions and their derivatives in *Sketchpad*.

5.3.8 A synthesis of activity trail analyses

Figure 5.28 is a two dimensional schematic representation of the degree of technical/conceptual mix into which students' work can be categorized. Students in the and/or peer support during activities. Students in the high-high category are mathematically and technically strong and can explore and experiment with the software's capabilities beyond the set tasks and thus reach a high degree of control over the artifact. Students low in technical fluency but high in mathematical competence can work with minimum technical/machine support.

This instrumentalization process lends credence to Verillan and Rabardel's(1995) assertion that a technological artifact (given object) ceases to exist in itself and becomes an instrument (a psychological construct) when the subject has been able to appropriate it for himself and has integrated it with his activity. In other words *Sketchpad*, becomes an extension of the body, an organ so to speak, made up of an artifact component and a psychological component in the sense envisioned by Trouche (2004:285). In support, Artigue (2002:250) reports on the dialectical relationship between the artifact and the subject when instrumental genesis takes place: the instrument is shaped by the user when progressively loading it with potentialities (instrumentalization) while the user is in turn

| HIGH TECHNICAL LOWCONCEPTUAL | HIGH TECHNICAL HIGH CONCEPTUAL |
| LOW TECHNICAL LOW CONCEPTUAL | LOW TECHNICAL HIGH CONCEPTUAL |

Technical Dimension

Conceptual Dimension
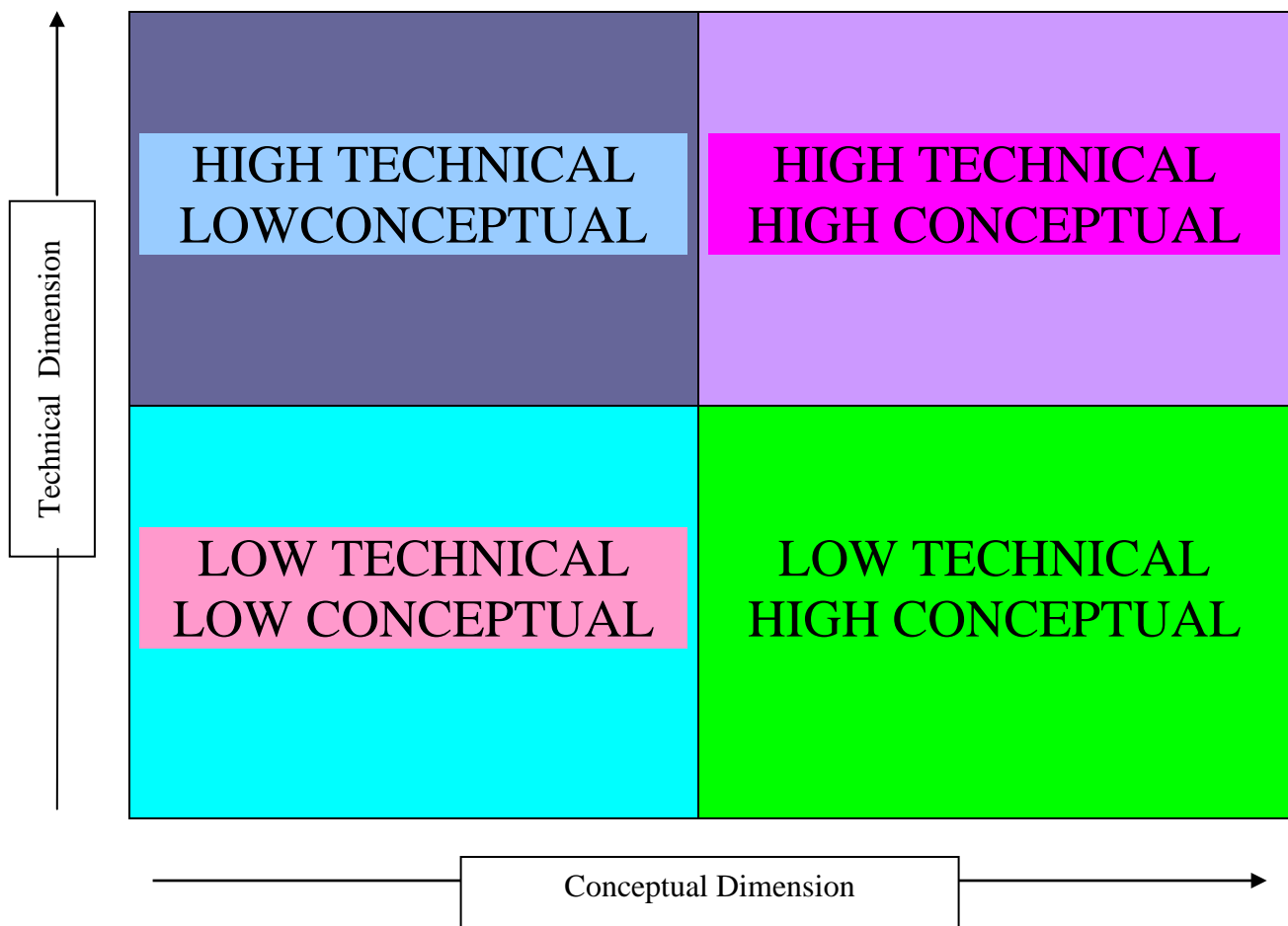
Figure 5.28: Two dimensional categorization of students' instrumentation of *Sketchpad*

shaped by the instrument when developing and appropriating schemes of techniques to respond effectively to given tasks (instrumentation)(compare 2.6).

5.4   Analysis of Post Task-based Interviews

5.4.1   Research questions addressed by post-task-based interviews.

The purpose of the task-based interviews was to enable students to reveal the scope and nature of understanding about the derivative gained in the *Sketchpad* activities. Accordingly the interviews helped to answer the following research questions: What is the quality and types of concept images of the derivative that students are able to form with the mediation of *Sketchpad*? What connections are students able to make between

different *Sketchpad* dynamic mathematics representations of the derivative? What advantages or obstacles do students experience as they attempt to make sense of the derivative with the aid of *Sketchpad* dynamic mathematics software? (compare 1.2 & 4.3.2.3).

5.4.2   Sample Interview on Modeling the Function Concept in *Sketchpad*

Interview protocol for pilot Student MT

A1. TR: Can you describe to me how you would plot a function in *Sketchpad*? Choose a function of your choice.

A2. Student MT: From the Graph menu I would select '**Plot New Function'** and define $f(x) = x^2$ using the calculator keypad that appears.

A3. TR: How would you represent the function in dynamic graphic form?

A4. Student MT: I would first plot a point on the graph of $f(x) = x^2$ by highlighting the plotted graph of $f(x) = x^2$ and choosing '**Plot Point of Function Plot**' from the Construct menu and animate or drag the point.

A5. TR: Do you mean Construct Point on Function Plot?

A6. Student MT: Yes construct, not Plot.

A7. TR: What happens when you animate the constructed point?

A8. Student MT: The point follows the graph of the function.

A9. TR: How would you create a table of values for $f(x) = x^2$ in *Sketchpad*?

A10. Student MT: I would select the plotted point and label it A, then select **Abscissa(x)** and **Ordinate (y)** from the Measure menu to display the coordinates.

A11. TR: How do you create the table after that?

A12. Student MT: I would select the values for x_A and y_A and select **Tabulate** from the Graph menu to create a table of the two values. Then animate and double click the table as the point moves along the graph.

A13. TR: Can you give examples of real-world situations that can be represented by the function you have chosen?

A14. Student MT: Maybe speed, growth rate, etc.

Figure 5.29 provides a framework for analyzing and modeling student's software competencies and cognitive processes of transforming mechanical/technical actions (c.f *models of* to *models for* and instrumental genesis in 1.2, 1.7, 2.7, 3.4.5, 3.8, 4.3.2.2, 5.3.3 and 5.3.6 and) into mathematical concepts. That is, the use of the software as a generator of mathematical concepts. In this connection, Noble (2001:87) reminds us that mathematical concepts reside not in physical materials but in what students do and experience. Similarly the ability to make connections between representations clearly lies outside the materials themselves and we can only gain access to them from the individual learner's interpretations.

Real-world phenomena	→	Static symbolic model $f(x) = x^2$ (on screen symbolic interface)	→	Static graphic model (visual display on screen graphic interface)

Static numeric model (static table of values)	←	Dynamic numeric model (creating a dynamic table of values)	←	Dynamic graphic model (animating point on function plot)
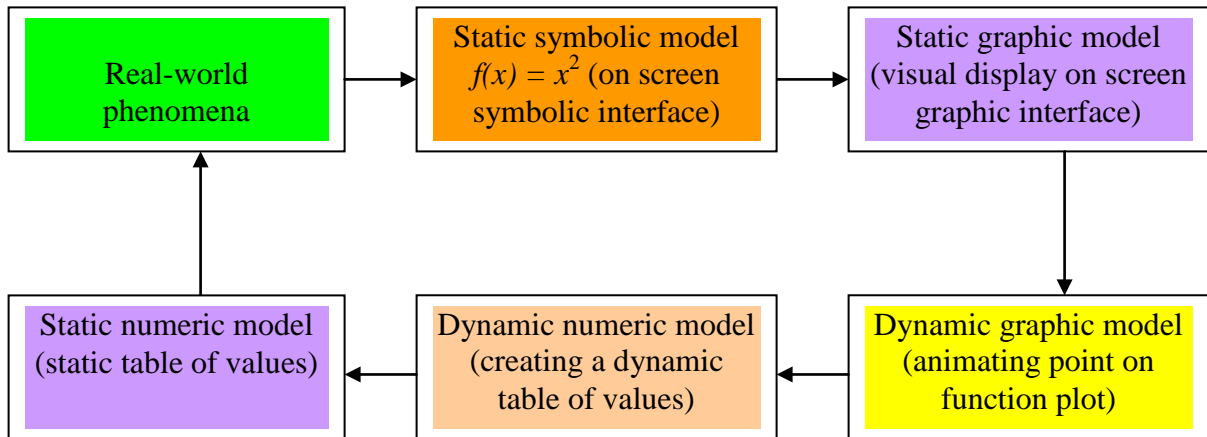
Figure 5.29: A *Sketchpad* Modeling Cycle for the Function Concept

In the interview there is a satisfactory description of the *Sketchpad* command processes for defining and transforming a function from its static symbolic form through a correct background sequence of keystrokes and menu commands (A2). The static symbolic form, $f(x) = x^2$, in the foreground, is transformed to the static graphic visualization on the graphic interface by the appropriate Graph menu command. As this happens, the graphic model surges to the foreground while the symbolic representation recedes to the background receiving less active attention. The static graphic representation is transformed to a dynamic representation by constructing a point (A4), struggling through relevant command sequences (A6), on the function plot. The plotted/constructed point is successfully animated (A4). The connection between the graphic and dynamic

representation remains in the foreground since the locus or path of the animation is the static graphical representation (A8).

The transformation of the dynamic graphic model to the dynamic numeric is done appropriately (A10) by first plotting the coordinates of the constructed using the Graph menu commands. There is however a technical overload of syntax as evidenced by the attempt to abridge the explanation. The link between the dynamic graphic and dynamic representations is coordinated visually in the foreground as the coordinates change in real time during the animation. The transformation of the dynamic numeric model is created through a conversion of the dynamic table of values to a static table (sample table data). (A10 & A 12). The static table data become the focus of attention and dominate the foreground while the other representations recede to the background. The table data (which can be plotted back to function plot), represent real world data that can be obtained through appropriate measurements of physical phenomena. By observing the pattern of the static table data a real world pattern of relationships can be hypothesized thus providing a link with the real world (compare 3.4.4). It is clear that whereas the student is able to make interconnections between the multiple representations there are still some software syntax overload problems to contend with.

5.4.3 Sample Interviews on Modeling the Average Rate of Change in *Sketchpad*
Interview protocol for final phase Student TM
B1. TR: Can you describe to me how you could use *Sketchpad* to demonstrate, calculate or measure the average rate of change of a function between any two points of its domain?
B2. Student TM: Firstly select **Plot Point on Function** from the construct menu, twice and click OK each time to plot 2 points on the graph of the function.
B3. TR: Is it **Plot Point** or **Construct Point**? (*syntax difficulties*)
B4. Student TM: Thank you for that correction, it should be '**Construct**', not '**Plot**'
B5. TR: That's OK you can proceed.
B6. Student TM: Then select the points and choose label points from the Display menu to label the left point and the point on the right.

B7. TR: Can you demonstrate to me how you would construct a secant line in
             Sketchpad?

B8. Student TM: Choose **Line** from the Construct menu to construct a line passing
             through both points and this is the secant line.

B9. TR: How would you find the slope/gradient of the secant line in *Sketchpad*?

B10. Student TM: To find the slope in *Sketchpad* I will select the secant line and choose
             Slope from the Measure menu, and this gives me the measurement. (*syntax*)

B11. TR: How would you find the equation of the secant line in *Sketchpad*?

B12. Student TM: I would select **Equation** from the Measure menu, instead of **Slope** this
             time.

B13. TR: How would you explain to a friend what this gradient represents?

B14. Student TM: To my friend I can say change in *y*-values divide by change in *x*-
             values.

B15. TR: How can you express the slope symbolically?

B16. Student TM: To symbolically express the gradient, let there be two points A and
             B such that their coordinates are $(x_1;y_1)$ and $(x_2;y_2)$. Then

$$secant\ line\ gradient = \frac{y_2 - y_1}{x_2 - x_1}$$

B17. TR: Could this be the same as $\dfrac{f(x+h) - f(x)}{(x + h) - x}$?

B18. Student TM: Yes, if we substitute y_2, y_1, x_2, and x_1 for $f(x+h)$, $f(x)$, $(x+h)$ and x
             respectively.

B19. TR: What happens when the secant line is animated?

B20. Student TM: The coordinates for points A and B, the slope and equation of the
             secant line all change as the animation goes on.

B21. TR: Can you give real world examples of the average rate of change?

B22. Student TM: Average speed of a bus from Bulawayo to Gwanda on its way to
             Beitbridge or any two points of its journey.

B19. TR: Can you describe to me how you could use *Sketchpad* to demonstrate and
             calculate and/or measure the average rate of change of a function between
             any two points of its domain?

B20. Student DJ: Firstly I select **Plot Function Plot** from the Construct menu twice
             and click OK to plot two points on the graph of the function.

B21. TR: Can you demonstrate to me how you would construct a secant line in
             Sketchpad?

B22. Student DJ: I then select the points and choose Label Points from the Display
             menu to label the left point as A and the point on the right B.

B23. TR: How would you find the slope/gradient of the secant line in *Sketchpad*?

B24. Student DJ: I then select both points and choose **Abscissa (x)** from the Measure
             menu to display the y-coordinates, y_B and y_A. I drag the points to rounded
             figure. For example, (1;1) and (2;4).

B25. TR: Would that be the quickest way to find the slope in *Sketchpad*?

B26. Student DJ: No…… I then deselect everything and select A and B in that order.
             I choose **Line** from the Construct menu to construct a line passing through
             both points (this is the secant line). To find the slope in *Sketchpad* I will
             select line and choose **Slope** from the Measure menu.

B27. TR: How would you explain to a friend what this gradient represents?

B28. Student DJ: To my friend I can explain that the gradient represents the increase
             in y, or vertical distance, divided by horizontal distance (x).

B29. TR: Can you give real world examples?

B30. Student DJ: For example, to find how steep land is between two distinct points.

B31. TR: How can you represent the slope symbolically?

B32. Student DJ: To symbolically express the gradient let there be two points A and
             B that with coordinates $(x_1;y_1)$ and $(x_2;y_2)$ respectively. The gradient is given

             by $\dfrac{y_2 - y_1}{x_2 - x_1}$.

B33. TR: What happens when the secant line is animated?

B34. Student DJ: The measured slope of the secant line changes as the animation

 progresses

Protocol Analyses

Both Students TM and DJ start with the construction of two points on the function plot
(B2 & B20). In both cases there are semantic conflicts and difficulties. The 'Construct
Point' command is referred to as 'Plot' (B4 & B20). There is a clear semantic conflict
between the colloquial mathematical language (that a point is 'plotted' not 'constructed')
and *Sketchpad* mathematical language. In their narratives, students seem to be
rationalizing this semantic differential. Both are, however, clear that they have to
construct two points (B8 & B20) on the graph (Function Plot). In other words, the
students start with a graphic representation and proceed to the numeric (slope) which is
then expressed symbolically (B10 & B10) as a differential quotient. (NB: also expressible
as an equation of the secant line). The activity thus gave students options of representing
the average rate of change as a measure of slope or as a method of calculating. In Tall's
Embodied Approach to the Calculus (compare 4.3.1.3) the method would be the process
while the measure is the product which presents the slope as a precept. In APOS Theory
parlance (compare 4.3.1.2) we could note that actions (of constructing the secant line and
measuring the slope are interiorized into a process of determining the slope which is in
turn encapsulated as a mathematical object (the slope).
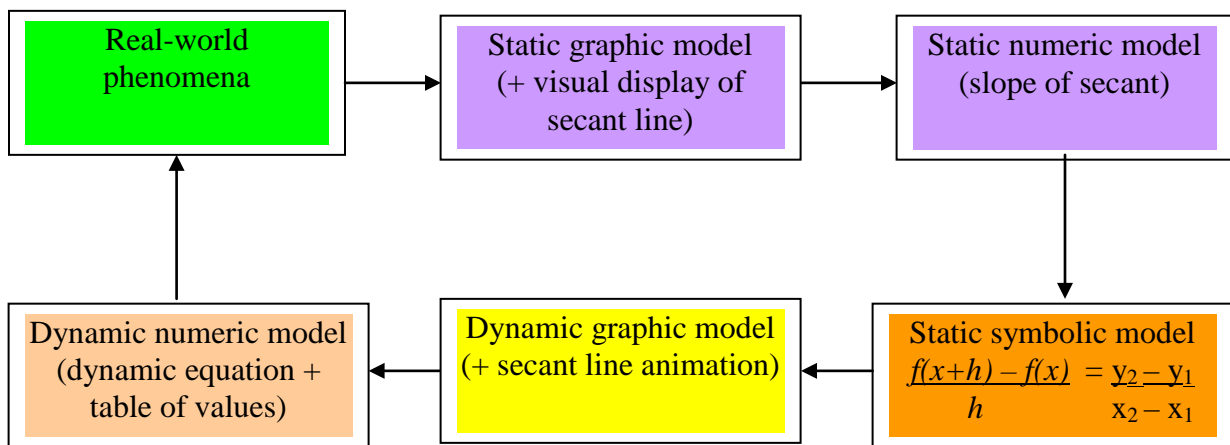
Figure 5.30: A *Sketchpad* Modeling Cycle for the Average Rate of Change Concept

When the secant line is animated (B16 & B33) both the static graphic, symbolic and static numeric representations of the average rate of change are transformed into dynamic graphic, symbolic and numeric representations. When asked about real life applications or meanings, Student TM refers to average speed (B18) which is the primitive context while Student DJ refers to the geographical slope (B30) which is the literal meaning of slope. Both students are able to represent the differential quotient in algebraic terms demonstrating some understanding of the equivalence of representations of the average rate of change concept. Figure 5.30 is a schematic representation of the sequence of representations described by the students.

5.4.4. Sample Interviews on Modeling the Derivative as the Instantaneous Rate of Change in *Sketchpad*

Interview protocol for final phase Student MT

C1. TR: Can you describe to me how you would use *Sketchpad* to demonstrate/ calculate the slope of a graph at a given point of its domain?

C2. Student TM: By drawing a tangent line through that point by choosing **Line** from the Construct menu. To calculate slope using *Sketchpad*, I select the tangent line through given point then I choose '**Derivative**' from the Graph menu. *(NB: procedural error).*

C3. TR: Can you show me how to construct a tangent line when given a secant line in *Sketchpad*?

C4. Student TM: By selecting point B and dragging it towards A it's the rate of change of function at that point.

C5. TR: What is the value of the gradient of the tangent line at the point of contact?

C6. Student TM: This refers to the increase in y divided by increase in x of a tangent.

C7. TR: Can you explain the meaning of the value of the gradient of the tangent at a point?

C8. Student TM: Instantaneous rate of change at that point.

C9. TR: Is the value always positive, negative or both?

C10. Student TM: The value of the gradient on a graph can be positive, negative or non-negative.

C11. TR: Can you explain your answer?

C12. Student TM: It cannot be both positive and negative at the same time.

C13. TR: Can you give real world examples of the instantaneous rate of change?

C14. Student TM: Speed at an instant.

C15. TR: How would you express the gradient of the tangent symbolically as a limit of the secant line gradient?

C16. Student TM: $\lim_{\triangle x \to 0} \dfrac{f(x + \triangle x) - f(x)}{\triangle x}$.

C17. TR: Is there any other way of expressing the limit?

C18. Student TM: $\lim_{\triangle x \to 0} \dfrac{f(x^2 + \triangle x) - f(x)}{\triangle x}$.(*note error*)

Interview protocol for final phase Student DJ

C19. TR: Can you describe to me how you would use *Sketchpad* to demonstrate/ calculate the slope of a graph at a given point of its domain?

C20. Student DJ: By drawing a tangent line through that point by choosing **Line** from the Construct menu. To calculate slope using *Sketchpad*, I choose the tangent line through given point then I choose **Slope** from the Measure menu. It will then be shown on the screen.

C21. TR: Can you show me how to construct a tangent line when given a secant line in *Sketchpad*?

C22. Student DJ: By selecting point B and dragging it towards the A. It's the rate of change of the function at that point A.

C23. TR: What is the value of the gradient of the tangent line at the point of contact?

C24. Student DJ: This refers to the increase in *y* divided by increase in *x* of a tangent.

C25. TR: Is it always an increase?

C26. Student DJ: No….

C27. TR: Can you explain the meaning of the value of the tangent gradient?

C28. Student DJ: Instantaneous rate of change at that point.

C29. TR: Is the value always positive, negative or both?

C30. Student DJ: No.

C31. TR: Can you explain your answer?

C32. Student DJ: It can be negative if x- and y- are changing in opposite directions cannot be both positive and negative at the same time.

C33. TR: Can you give real world examples of the instantaneous rate of change?

C34. Student DJ: Speed at point of impact in a car accident.

C35. TR: How would you express the gradient of the tangent symbolically as a limit of the secant line gradient?

C36. Student DJ: $\lim_{B \to A} Gradient\ of\ Secant = Gradient\ of\ Tangent$

C37. TR: Is there any other way of expressing the limit?

C38. Student DJ: $\lim_{\triangle x \to 0} \dfrac{f(x + \triangle x) - f(x)}{\triangle x}$.

Protocol analyses

Both Students TM and DJ said they would 'draw a tangent line' in *Sketchpad* by choosing **Line** from the Construct menu. These descriptions are more like paper-and-pencil methods. Student TM proceeds to find the '**Derivative** of the tangent line' in *Sketchpad* (C2) while Student DJ proceeds to find the '**Slope** of the tangent' (C20). In both cases there were procedural errors. The second question of the interviews (C3 and C21) redirected the students towards using the secant line as the starting point. (NB: This appears to be interference by a previously familiar genre) Both students (C4 & C22) then reverted to dragging point B towards point A (in Figure 5.16). In other words, they restarted with static graphic and numeric representations and transformed these to dynamic graphic and numeric representations. That is, the tangent position and slope are treated and perceived as the limiting position and slope for the moving secant line respectively (compare 2.8.3 & 5.3.5).

The symbolic representation of the instantaneous rate of change was expressed correctly by Student TM in general terms indicating ability to link with previous learning in a textbook environment (C16). But the student wrongly substituted for the specific case of $f(x) = x^2$ (C18). Student DJ showed deeper conceptual understanding by expressing the limit verbally (C36) in *Sketchpad* labeling form. He was also able to confirm the verbal representation to be equivalent to the symbolic limit in C38. Both students, however, were able to relate the derivative as instantaneous velocity (C14 & C34). Figure 5.31 is a schematic representation of the transformations in representations.

Real-world	Dynamic graphic model (dynamic visual display during animation)	Dynamic numeric model (dynamic numeric secant limit +equation)
Static symbolic model $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$	Static numeric model (table of secant slope values + equations)	Static graphic model (Trace of secant line path + tangent as limit)

Figure 5.31: A *Sketchpad* modeling cycle for the instantaneous rate of change concept

5.4.5   Sample Interviews on Modeling the Derivative as the Rate of Change Function in *Sketchpad*

<u>Interview protocol for Student TM</u>

D1. TR   In *Sketchpad*, how would you construct the graph of the function showing how the gradient of the tangent line changes with respect to *x*?

D2. Student TM: I would start with the tangent I drew in Activity C and select the tangent and choose **Slope** from the Measure. Then select Point A menu and choose **Abscissa (x)** from the Graph Menu.

D3. TR: How would you plot the slope of the tangent line against the *x*-values in *Sketchpad*?

D4. Student TM: Select the slope and the abscissa and choose **Plot as (x,y)** from the Graph menu. Point C is plotted

D5. TR: If you trace Point C and animate the tangent what pattern is the path traced by the plotted point?

D6. Student TM: It's a straight line.

D7. TR: How would you represent this path (pattern) numerically in *Sketchpad*?

D8. Student TM: Use the tables formed during the clicking

D9. TR: How can you express the function (path) symbolically as a limit of the differential quotient for the general point on the graph of *f(x)*?

D10. Student TM: Limit x everywhere $\lim_{h \to 0} \dfrac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \dfrac{x^2 + 2xh - x^2}{h} = 2x$

D11. TR: How can you verify your findings in Stage 5 using *Sketchpad*?

D12. Student TM: Choosing **Derivative** from the Graph menu as we did in the first step.

Interview protocol for Student MN

D13. TR: In *Sketchpad*, how would you construct the graph of the function showing how the gradient of the tangent line changes with respect to *x*?

D14. Student MN: Follow Activity C. select $f(x) = x^2$ and select **Derivative** from the Graph menu then select **Plot Function** from the Graph menu. (*less efficient but good implicit understanding of the derivative*).

D15. TR: What does the plotted function represent?

D16. Student MN: The function plotted represents the gradient at any point.

D17. TR: How would you plot the slope of the tangent line against the *x*-values in *Sketchpad*?

D18. Student MN: Move point B closer to point A. Plot *x* against slope as (*x,y*). That is, point C. (*see Figure 5.18*). Select point C and choose **Trace Point** from the Display menu. Deselect every thing and select line AB (*tangent*) and select **Animate Line** from the Display menu (*heavy demand on syntax*). The path taken by point C represents the gradient of the function at any point.

D19. TR: If you trace point C and animate the tangent what pattern is the path traced by the plotted point?

D20. Student MN: The path traced by C is a straight line.

D21. TR: How would you represent this path (pattern) numerically in *Sketchpad*?

183

D22. Student MN: Select **Abscissa (*x*)** and **Ordinate** (as slope of AB) from the Graph menu. Select **Plot as (x,y)** and **Tabulate** *x* and slope values to form a table of values.

D23. TR: How can you express the function (path) symbolically as a limit of the differential quotient for the general point on the graph of *f(x)*?

D24. Student MN: $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \dfrac{(x+h)^2 - x^2}{h} = \dfrac{x^2 + 2xh - x^2}{h} = 2x + h = 2x, \, h \neq c$

D25. TR: How can you verify your findings in the previous step using *Sketchpad*?

D26. Student TM: Select *f(x) = x²* and then select **Derivative** from the Graph menu. Select **Plot Function** from the Graph menu.

Protocol analyses

Student TM started by selecting the tangent drawn in the previous Activity (C) and made the appropriate selection of tangent slope and an abscissa (x) for the point of tangency, point A (D2) and the plotting point C. Student MN also starts with work in Activity C for continuity purposes, and specifies the function whose tangent is under investigation to be $f(x) = x^2$. He, however, selects **Derivative** from the Graph menu straight away (D14) and is clear that the function represents the derivative (gradient at any point) (D14). It is evident that student MN takes advantage of the recently introduced *Sketchpad* shortcut for finding and drawing the derivative of a function. This is a less illuminating top-down, black box use of software referred to by Drijvers (2000:194).
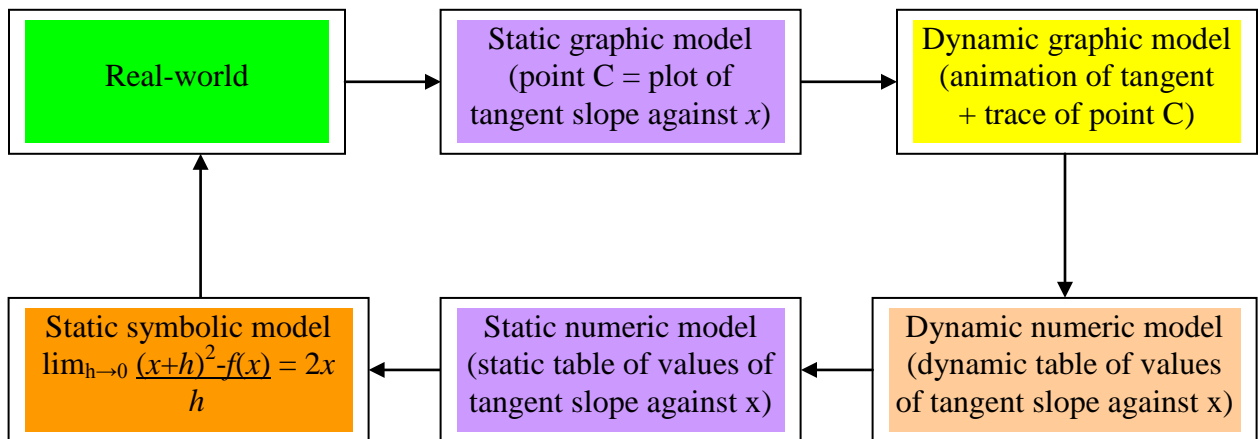
Figure 5.32: A *Sketchpad* Modeling Cycle for the Gradient Function Concept

Student TM was able to plot the tangent slope against the *x*-coordinate of the point of tangency (D4). He managed to trace Point C and to animate the tangent (D6). Student MN on the other hand was more precise about the **Trace** and **Animation** menu commands and appeared more confident about what the trace represents (D18), a good sign of growing fluency in the use of the software syntax. Both students managed to recognize the pattern of the trace to be a straight line graph (D6 & D 20). Both students were able to form sample tables of values (D8 & D22) but once again student MN was more elaborate in explaining the *Sketchpad* syntax involved (D22).

Expressing the path of C symbolically as a limit of the differential quotient was performed fluently (D10 & D24). However, Student TM's solution was more accurate (D10). The verification of the derivative using *Sketchpad* was not a problem to the students (compare 2.6, 3.4.8 & 5.3.6). Figure 5.32 is a schematic summary of the successive transformations that epitomized students' modeling cycle for the derivative as the rate of change function (compare Chapter 3).

5.4.6   Summary of Interview Protocol Analyses

It can be conjectured from the protocol analyses that the students were able to model the function concept, the average rate of change concept in the buildup to the instantaneous rate of change concept image and to build a concept image of the derivative as a rate of change function using the instantaneous rate as a pivot. One level of modeling became a scaffold to the next level (compare van Hiele Theory in 4.3.1.1). Within each cycle of conceptual development there was considerable opportunity to interrelate and move from one form of representation to the other in a versatile manner. The multiple links between representations took heed of Thompson's (1994:39) warning that if students do not realize that something remains the same as they move among different representations then they see each representation as a 'topic' to learned in isolation. The dynamic features of the software enable students to experience calculus as the mathematics of motion, thus reconnecting reconnecting with the discovery of the derivative in the context of motion

(compare 1.1, 2.2.2 & 2.2.5). The process of generating the rate of change function coincidentally transforms Newton's theory of fluxions from the world of the imagination to virtual reality (compare 2.2.5 & 2.8.4).

In the exit interviews, the students reiterated some of the difficulties they encountered with the software in the initial phases of the study as related mainly to syntax. Examples of comments included the following: 'it can be very frustrating if you get lost or when you cannot obtain the results expected in the work book, such as a different table of values from the one in the worksheet', the disappearance of a tangent when you have followed the correct construction method', 'failing to create a new page in *Sketchpad*', 'inability of the software to show you the working yielding the derivative', 'failure to reverse an animation', etc. To cope with some of these difficulties students cited some of the following: 'reading the worksheet instructions again', 'seeking help from colleagues', 'seeking help from the teacher', 'using trial and error methods', 'using the help menu'.

In the exit interviews the students who participated also gave positive comments about their experiences: 'it was an opportunity to gain experience in working using a computer', 'the computer environment itself has been very exciting for me', 'if only we could learn more mathematics this way', ' in the end it was difficult stop attending'. In spite of the obstacles the students who participated confessed to have developed a unique understanding of the derivative in the *Sketchpad* mediated environment.

5.5   Posttest Results

5.5.1   Research Questions Addressed by the Posttest

The purpose of the posttest was to answer the following research questions: Is there a significant difference in the conceptual understanding and modeling of the derivative between the pretest and posttest scores of the experimental and the control groups, at the 5% level? More specifically: How do the experimental group students' post teaching experiment concept images of the derivative and the associated concepts of function, variability, slope, limit, etc, compare with their pretest understandings?

How do experimental students' concept images of the derivative and the associated concepts compare with those of control group students at the end of the study?

A total of eight pilot students and six final phase students who had taken part in the *Sketchpad* design experiment activities took part in the posttest. A total of twelve final phase students took part in the posttest. All of these students had also taken part in the pre-test. All of the eight pilot students had also done A level Mathematics before enrolling at the university. A total of seven students who had not taken part in the *Sketchpad* activities but had taken the pre-test, took the posttest. A total of six students who had taken part in the final phase *Sketchpad* activities and the pretest took the posttest. The following results were obtained.

5.5.2   Students' familiarity with the function concept (Question 1)

Table 5.15 compares both the pilot and final phase experimental group performance in the posttest with their performance on the pretest. The posttest item (question) was identical to the pretest item and the purpose was to ascertain the level of significance of change in the structure of the observed learning outcome. The *t*-test statistic for dependent samples was used to test the significance level of the difference in means.

	EXPERIMENTAL	1	2	4	6	12	16	17	21	MEANS	∑D	∑D²	(∑D)²
Pilot Phase	**PRETEST (x_1)**	19	17	10	11	9	12	17	18	14.125			
	POSTTEST(x_2)	22	21	15	12	15	17	18	23	17.875	$t = 5.5638$, N=8		
	$x_2 - x_1$	3	4	5	1	6	5	1	5	3.75	30		900
	$(x_2 - x_1)^2$	9	16	25	1	36	25	1	25			138	
	EXPERIMENTAL	4	5	6	7	11	14			MEANS	∑D	∑D²	(∑D)²
Final Phase	**PRETEST (x_1)**	15	12	11	11	17	11			12.833			
	POSTTEST(x_2)	19	17	19	19	23	19			19.333	$t = 9.0429$, N=6		
	$x_2 - x_1$	4	5	8	8	6	8			6.500	39		1521
	$(x_2 - x_1)^2$	16	25	64	64	36	64					269	
Table 5.15: Pretest-posttest function concept statistics for dependent samples.													

In the pilot phase 3,75 was obtained as the mean of the differences for all pairs of scores, $\sum D^2 = 138$, as the sum of squares of differences, $(\sum D)^2 = 900$, as the square of the sum

187

of differences, N = 8, as the number of matched/dependent pairs which when substituted into the formula (see Appendix C) for calculating the one-tailed t-statistic for dependent samples yielded $t(7)=5.5638$, $p<0.05$ for $t_{crit} =1.895$ (where $t(7)$ is the calculated t-statistic for 7 degrees of freedom, $p<0.05$ is the 5% level of significance, and t_{crit} is the critical value in the t Distribution table (see Appendix D)). That is, the calculated one-tailed t-test statistic of 5.5638 is located in the critical region of the t-table, with 7 degrees of freedom ($N-1 = 8-1 =7$). The result from the table is that the posttest mean is greater, by a statistical significance of $p<0.05$. So we can reject the null hypothesis (H_0 = there is no difference between the posttest mean and the pretest mean), accept the alternative hypothesis (H_1= the experimental group posttest mean is greater), and conclude with 95% confidence that the change in performance was a result of exposure to the *Sketchpad* activities (as the treatment condition). In the final implementation phase the calculated one-tailed t-statistic of $t(5)=9.0429$ also fell in the rejection region suggesting that the posttest mean was also greater at a significance level of $p<0.05$. In both cases we can therefore reject the null hypothesis.

	RESPONDENT	1	2	4	6	12	16	17	21	MEANS	$\sum x^2$
Pilot Phase	EXP GROUP (x_1)	22	21	15	12	15	17	18	23	17.875	
	x_1^2	484	441	225	144	225	289	324	529		2661
	RESPONDENT	3	5	7	10	11	13	18		$n_1=8,n_2=7$	
	CONTROL (x_2)	20	13	12	10	7	8	5		10.714	
	x_2^2	400	169	144	100	49	64	25		t=0,8301	951
Final Phase	RESPONDENT	4	5	6	7	11	14			MEANS	$\sum x^2$
	EXP GROUP (x_1)	19	17	19	19	23	19			19.333	
	x_1^2	361	289	361	361	529	361				2262
	RESPONDENT	1	8	9	12	13	15			$n_1=6,n_2=6$	
	CONTROL (x_2)	15	16	16	13	12	15			14.500	
	x_2^2	225	256	256	169	144	225			t=0,4064	1275

Table 5.16: Comparison of the function concept posttest results for the experimental and control groups (independent samples)

However, when compared with the seven pilot and six final phase students that did not take part in the *Sketchpad* activities a slightly different picture emerged. Table 5.16

compares both the pilot and final phase experimental group performances in the posttest with the performance of their control group counterparts. The *t*-test statistic for independent samples was used to test the significance level of the difference in means.

In the pilot phase 17.875 and 10.714 were obtained as means for the experimental and control groups respectively. $\sum x_1^2 = 2661$ and $\sum x_2^2 = 951$ were obtained as the sums of the squares of the experimental and control group scores. The degrees of freedom were obtained as $(n_1 + n_2) - 2 = (8 + 7) - 2 = 13$, where n_1 is the number of scores in the experimental group and n_2 is the number of scores in the control group. When substituted into the formula for calculating the *t*-statistic for independent samples (see Appendix C) the parameters yielded $t(13)=0.830$, $p<0.05$ for $t_{crit} =1.771$. That is, the calculated one-tailed *t*-test statistic of 0.830 is located outside the critical region of the *t*-table. So we can accept the null hypothesis (H_0 = there is no difference in the means) reject the alternative hypothesis (H_1 = the experimental group mean is greater). We can conclude that the experimental group students do not perform worse than the control students even though the experimental group mean appears to be 'much' bigger. In the final phase the *t*-statistic of 0,4064 for 10 degrees of freedom was also statistically insignificant which shows that although the experimental students performed better than the control group, their superior performance could not be ascribed solely to the treatment. The statistical results for independent samples point to the need for guarded optimism and keen sensibility to other conditionalities and factors that may have contributed to the success of the experimental group students (c.f. Ellison's caution about the context of teaching in 2.5.3). (NB: The small size of the samples could also have led to the inconclusiveness of the independent samples *t*-statistic – a limitation to be noted).

5.5.3   T-test results for performance in the remainder of the posttest questions
Table 5.17 was obtained through an iteration of the *t*-test calculations for the rest of the posttest questions and the pattern of rejection remained essentially the same as for the function concept (the tables for the calculations are contained in Appendix C). Notable exceptions were the rejection of the null hypothesis for the final phase graphical representation of the modified Bottle Problem for independent samples and the marginal

acceptance of the null hypothesis for the derivative question(s) (Question 6). We zoom on the results for these questions for closer analysis.

Q	Phase	Content description	Indep t-stat (df)	p<0.05 value	Verdict on H_0	Dependent t-stat (df)	p<0.05 value	Verdict On H_0
1	Pilot	Function Concept	0.8301(13)	2.160	Accept	5.5638 (7)	1.895	Reject
	Final	Function Concept	0.4451(10)	1.812	Accept	9.0429 (5)	2.015	Reject
2	Pilot	Gradient Concept	0.0795(12)	1.782	Accept	1.8248(7)	1.895	Accept
	Final	Gradient Concept	1.0892(10)	1.812	Accept	3.7268(5)	2.015	Reject
3	Pilot	Limit Concept	0.6673(12)	1.782	Accept	2.6458(7)	1.895	Reject
	Final	Limit Concept	1.0325(10)	1.812	Accept	7.9057(5)	2.015	Reject
4	Pilot	Differentiability	0.4588(13)	1.771	Accept	1.8825(7)	1.895	Accept
	Final	Differentiability	1.2871(10)	1.812	Accept	3.5571(5)	2.015	Reject
5	Final	Identifying derivative of given graph	1.3109(10)	1.812	Accept	5.4772(5)	2.015	Reject
6	Final	Sketching derivative of given graph	1.3109(10)	1.812	Accept	5.4772(5)	2.015	Reject
7	Pilot	Bottle Problem	0.6411(13)	1.771	Accept	N/A	N/A	N/A
	Final	Bottle Problem	1.8433(10)	1.812	Reject	6.0461(5)	2.015	Reject
Key to Table: Indep = independent, df = degrees of freedom, p = probability, *t*-stat = *t*-test statistic, H_0=null hypothesis Table 5.17: Summary of comparisons of pretest and posttest *t*-test results								

5.5.4 Additional analysis of results on Sketching Derivatives of given Functions (Question 6)

Table 5.18 summarizes pilot students' responses with the coding of responses explained in the key to the table. Table 5.19 shows results for the final phase. Although the *t*-test statistic leaves room for chance factors, the percentage pass rates of 50% and 18% for the pilot experimental group and the control group, and 50% and 20% for the final phase seem to be far apart.

The graphs whose derivatives were difficult to sketch appeared to be the third (c) and fourth (d) in Figure 5.11. While none of the control group students scored a single mark for these graphs, experimental group obtained 47% and 25% in the final phase.

5.18	EXPERIMENTAL GROUP									CONTROL GROUP							
Q.	1	2	4	6	12	16	17	21	%	**3**	**5**	**7**	**10**	**11**	**13**	**18**	%
6a)	cid	cid	cid	cid	cid	c_2id_2	id_2	Id		c_2id_2	wr	c_2id_2	c_2id_2	cid	c_2id_2	wr	
	5	5	5	5	5	3	2	3	83	3	0	3	3	3	3	0	38
6b)	cid	c_2id_2	c_2id_2	cid	cd	cid	cd	d		cd	wr	c_2id_2	nr	cid	c_2id_2	wr	
	5	3	3	5	4	5	4	2	78	4	0	3	0	0	4	0	28
6c)	c_2id_2	nr	nr	nr	c_1i_1	c_1i_1	$c_2i_2d_2$	c_1i_1d		nr	wr	c_2d_2	nr	wr	wr	wr	
	3	0	0	0	2	2	3	4	35	0	0	2	0	0	0	0	5
6d)	wr	nr	nr	nr	wr	c_1d_1	nr	wr		nr	wr	wr	nr	wr	wr	wr	
	0	0	0	0	0	2	0	0	5	0	0	0	0	0	0	0	0
Total	**13**	**8**	**8**	**10**	**11**	**12**	**9**	**9**	50	**7**	**0**	**8**	**3**	**3**	**7**	**0**	18

Key to Table: (LHS = left hand side, RHS = right hand side); c=correct concavity on both LHS and RHS (2 marks); c_1=correct concavity on LHS only (1 mark); c_2 = correct concavity on RHS (1mark); I = correct y-intercepts where applicable (2); i_1 = correct LHS *y*-intercept (1 mark); i_2 = correct RHS *y*-intercept (1 mark); d = correct domain/range both on LHS and RHS (for correct concavity) (2 marks); d_1= correct domain/range on LHS (1 mark); d_2=correct domain/range on RHS (1 mark); tp=correct turning point (where applicable, 1 mark each); wr = incorrect response (0); nr = no response (0).

Table 5.18: Pilot posttest results to derivatives of given graphs

	Student	4	5	6	7	11	14	% success	1	8	9	12	13	15	% success
	Level	A	O	A	A	A	A		A	A	A	A	A	A	
	Sex	M	M	F	F	M	F		F	M	M	M	M	F	
Final Posttest	Literacy	EXPERIMENTAL							CONTROL						
	Q. 5 a	1	1	1	1	1	1	100%	0	1	1	1	1	0	67%
	b	1	1	1	0	1	1	83%	0	0	1	0	1	0	33%
	Q. 6 a	5	3	5	5	5	5	93%	0	2	4	4	5	0	50%
	b	1	4	5	4	3	2	63%	0	0	4	1	3	0	27%
	c	3	4	3	2	2	3	47%	0	0	0	0	0	0	0%
	d	1	0	5	0	1	2	25%	0	0	0	0	0	0	0%
	Total	12	13	20	12	13	14	58%	0	3	10	6	10	0	20%

Key to table:
nr =no response, 0 = wrong response, LHS= left hand side, RHS=right hand side

correct LHS or RHS concavity	1	+ correct LHS + RHS intercept 4
correct LHS + RHS concavity	2	+ correct LHS or RHS range 5
add correct LHS or RHS intercept	3	+ correct LHS + RHS range 6

Table 5.19: Final phase posttest responses to questions on derivatives

5.5.5 Results on the graphical representation of the Modified Bottle Problem

Table 5.20 summarizes pilot phase responses to the modified Modified Bottle Problem.

Student 7's Graph   **Student 6's Graph**
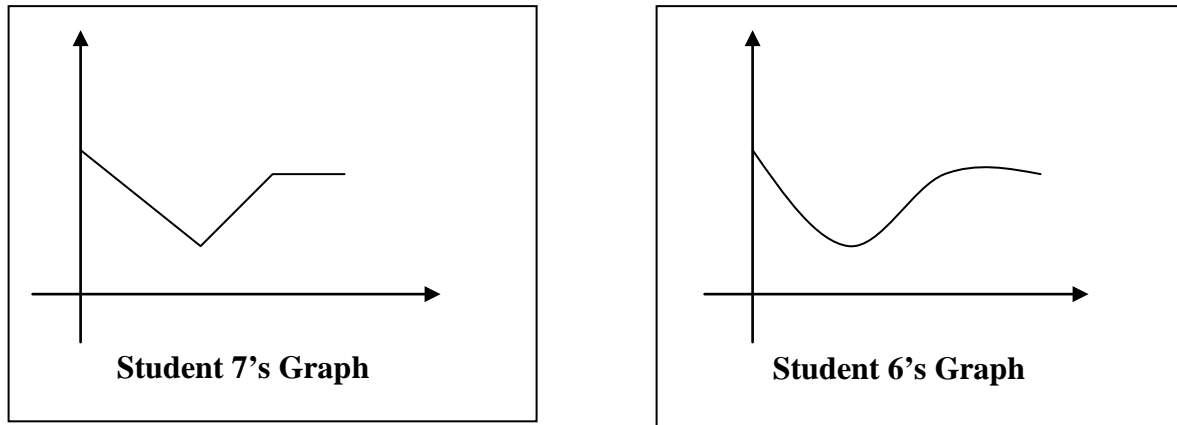
Figure 5.33: Sample responses to the derivatives of the Modified Bottle Problem

5.20	EXPERIMENTAL GROUP									CONTROL GROUP							
Stud	1	2	4	6	12	16	17	21	%	**3**	**5**	**7**	**10**	**11**	**13**	**18**	%
7a)i	cid	cid	cid	cid	id	cid	cid	id		cid	d	cid	id	id	cid	wr	
	4	4	4	3	2	4	4	2	84	4	1	4	2	2	3	0	50
ii	c_1id	c_2id	$c_{13}idt_2$	c_1id	id	cidt	$c_{12}idt$	c_2id		c_1di	d	c_2id	c_2id	$c_{12}idt$	c_2id	wr	
	3	3	5	3	2	7	6	3	57	3	1	3	3	6	3	0	34
7b)i	cid	d	cid	nr	d	cid	cd	d		cid	d	d	nr	d	cid	wr	
	3	1	3	0	1	3	2	1	58	3	1	1	0	1	3	0	38
ii	d	c_1d	$c_{23}dt_2$	nr	d	cidt	$c_{12}idt_1$	d		d	d	d	nr	c_1idt_1	d	wr	
	1	2	4	0	1	6	5	1	42	1	1	1	0	4	1	0	17
Total	**11**	**10**	**16**	**6**	**6**	**20**	**17**	**7**	58	**11**	**4**	**9**	**5**	**13**	**10**	**0**	33

<u>Key to Table:</u>
(LHS=left hand side, RHS=right hand side) c=correct concavity on both LHS and RHS (2 marks); c_1= 1st concavity (1 mark); c_2 = 2nd concavity (1mark); c_{23} = 2nd and 3rd concavities (2 marks); i = correct y-intercepts where applicable (2 marks); i_1 = correct LHS *y*-intercept (1 mark); i_2 = correct RHS *y*-intercept (1 mark); d = correct domain/range both on LHS and RHS (for correct concavity) (2 marks); d_1= correct domain/range on LHS (1 mark); d_2=correct domain/range on RHS (1 mark);t=correct turning point (where applicable, 1 mark each); (t_{12} = 1st and 2nd turning points); wr = wrong (incorrect) response (0); nr = no response (0).

Table 5.20: Pilot Posttest Results for the Modified Bottle Problem

Performance in the pretest version was below expectation suggesting that students had difficulty in interpreting and representing the problem graphically. The modified problem presented an initial example where the increase was constant in order for students to relate the co-variation (see Appendix A). The Bottle Problem attempts to model a real life situation which students may encounter, say in an industrial engineering situation.

A further analysis of the posttest results for this question showed that while the control group had a success rate of 10%, the experimental group had a success rate of 80% which included the sketching of the derivative of the initial graph of the problem. This was vindicated as significant enough by the t-test statistic as already seen above. Table 5.22 shows that the performance in sketching the derivative of the cylindrical bottle was 78% for the experimental group while that for the control group was 8%. It further shows that the experimental group scored a success rate of 53% compared with 0% for the control group in the case of the derivative of the spherical bottle (see Question 7 in Appendix A).

	Student	4	5	6	7	11	14	% success	1	8	9	12	13	15	% success
Final Posttest	Level	A	O	A	A	A	A		A	A	A	A	A	A	
	Sex	M	M	F	F	M	F		F	M	M	M	M	F	
	Literacy	EXPERIMENTAL							CONTROL						
	Q. 7a)i.	3	3	3	3	3	3	100%	0	0	3	3	3	0	50%
	ii	2	3	3	3	3	3	94%	0	0	0	0	0	0	0%
	b)i.	5	3	6	4	6	4	78%	0	0	1	1	1	0	8%
	ii.	3	1	6	5	2	4	53%	0	0	0	0	0	0	0%
	Total	13	10	18	15	14	14	80%	0	0	4	4	4	0	11%

Key to Table

Correct intercept	1	Concept image of the intensity (concavity) of covariation	4
Concept image of covariation of two variables	2	Concept image of varying intensity of covariation	5
Concept image of direction of covariation	3	Concept image of proportionality of the covariation throughout the domain.	6

Table 5.21: Final posttest responses to the Modified Bottle Problem

This was apparently one of the most demanding problems. Students who were able to transfer their learning to solve it successfully appeared to have benefited immensely from the *Sketchpad* activities (see Figure 5.33).

193

5.5.6   Summary of posttest findings

From the discussion of the posttest results, it is clear that the pretest-posttest performances of the experimental groups were significantly different. The null hypothesis was rejected at a significance level of 5% suggesting that the Sketchpad activities were an effective treatment condition which led to a deeper understanding of the derivative and related concepts. The research question about the difference in the pretest and posttest performance is therefore answered in the affirmative. However, with regards to the performance comparisons between the experimental and the control group, only the performance in the Modified Bottle problem could be reported to have been significantly different at the 5% level of significance. It can be concluded therefore that the experimental students were able to sketch the derivative of a given graph better than the control group students. This new found ability can be ascribed to the exposure to the experimental conditions of *Sketchpad*. Accordingly the second research question that the posttest set out to investigate can be answered in the affirmative only with respect to the sketching of derivatives. On the whole however, for the rest of the concepts it can be concluded unequivocally that the experimental students did not perform worse than the control group of students.

5.6   Conclusion

In this chapter, the pretest results were presented and varying levels of pre-calculus competencies identified among the participating students. It was noted that, on balance, the participating students had sufficient pre-calculus background to benefit meaningfully from the *Sketchpad* activities even though A level holders had a palpable edge over their O level counterparts. There were no gender disparities in performance in the pretest. The didactic approach adopted for the *Sketchpad* activities was briefly spelt out as laboratory and worksheet based, technology enhanced, largely individual and collaborative with the teacher playing the role of facilitator rather than authority. In other words, the students took centre-stage in performing the activities either by themselves or with the help of their colleagues and with minimum help from the teacher. Students' performances in the *Sketchpad* activities have been reported on in detail starting with preliminary activities to

familiarize them with function modeling in *Sketchpad*. The average rate of change was modeled as the differential quotient and students' responses indicated considerable familiarity with the gradient notion.

Modeling the derivative as the limit of a sequence of secant lines was successfully understood as a visual event but the disappearance of the tangent created some computational conflicts that were resolved through a reconstruction of the tangent by separating the points that had initially been joined. Increased precision of slope calculations helped to resolve the problem of approximated answers. The derivative as the rate of change function was easier to grasp using the dynamic generation and trace visualization properties of *Sketchpad*. The ability of the dynamic software to make multiple representations of concepts emerged as a powerful feature enabling students to make linkages and connections between multiplicities of representations. Students' abilities to instrumentalize *Sketchpad* were categorized into two dimensions leading to four categories.

Exemplar task-based interviews were presented in which deeper understandings of the derivative in a *Sketchpad* mediated environment have been explored. The interviews shed considerable light on how or what sort of connections between representations have to be made leading to an emergence of coherent modeling cycles for each of the contributory concepts. Six forms of representation that interact or are available for interrelation emerged as static graphic, static symbolic, static numeric, dynamic graphic, dynamic numeric and, occasionally, dynamic symbolic with the real world being the external system being modeled. While *Sketchpad* was an external representation of an otherwise external system the students' cognitive processes of understanding and making sense were internal concept images or representations (compare 3.4.3). Finally the posttest results were discussed and it was noted that whereas there was a significant improvement in performance by the experimental group when posttest and pretest results for dependent samples are considered, the improvement in performance between the experimental and control groups has been selective and not across the board.

The next chapter summarizes the findings and makes recommendations and conclusions based on this study.

CHAPTER VI
SUMMARY AND CONCLUSIONS

6.1 Introduction

It was a central purpose of this study to explore the feasibility of employing *The Geometer's Sketchpad* dynamic mathematics software as a modeling tool to enhance undergraduate students' concept images (conceptual understanding) of the derivative in introductory calculus. Although the software was designed initially for use in high school geometry classes testing has shown that its ease of use makes it attractive to instructors of college-level mathematics and teacher pre-service and in-service (Key Curriculum Press, 2002:3). College instructors are drawn particularly to *Sketchpad*'s powerful transformation capabilities allowing students to explore non-Euclidean geometries (ibid). This study acknowledges that the development of mathematics as a discipline has always been dependent upon the material and symbolic computational tools available for mathematical computations and representations. Nobody would deny the revolutionary role played by the introduction of the decimal system, logarithmic tables, tabulation of elementary functions, the invention of the abacus, the adding machine, the slide rule and symbolic algebra, among the classical computational devices and technologies.

With the proliferation of computer technology, advances in dynamic computerized environments constitute virtual laboratories in which students have the opportunity to play, investigate and learn mathematics through visualization, experimentation, surprise and immediate feedback to attain sophisticated concept images. It is hypothesized that being one of such dynamic computerized environments *Sketchpad*'s transformation capabilities may be harnessed not just to enhance the learning of geometry but a great deal of introductory calculus. Such a prospect has the potential of making calculus concepts, which have traditionally been leant by rote, more accessible to learners. However, professional mathematicians and engineers know that these sophisticated new tools do not become immediately efficient mathematical instruments for the user (Artigue, 2002:245). If anything their complexity does not make it easy to master and fully benefit from at the click of a button. It can be asserted that traditional mathematical

197

practices have, as a consequence, only slowly come to terms with the evolution of mathematical practices linked to technological evolution.

In this chapter the findings of how *Sketchpad* can be used as an instrument to model the derivative, a central concept in calculus, in a more accessible way are synthesized, limitations are spelt out and recommendations and conclusions are put forward. The strategy adopted in the present study to instrument *Sketchpad* into an environment for learning the derivative in innovative ways was to first conduct a review of the literature to gain insight on how the derivative concept has developed historically and how it has been taught traditionally as well as in the fledgling computer aided didactic environment. In the process it was hoped that insight would also be gained into the historical and didactical obstacles to a smooth understanding of the dicey concept. The insight so gained would help us to conjecture how *Sketchpad* capabilities could be utilized to overcome some of the pedagogical handicaps of the traditional approaches. In keeping with the Realistic Mathematics Education philosophy practiced in the Netherlands, a models and modeling perspective was adopted because, as English (2003:229) points out, it has been shown to be a powerful conceptual framework for research in mathematics education. As a key feature of design studies, a learning trajectory in *Sketchpad* was hypothesized and instructional activities designed for piloting with non-mathematics major undergraduate science students.

A pretest questionnaire was devised for the critical assessment of student's 'entry' pre-calculus and calculus knowledge. The hypothesized learning trajectory in *Sketchpad* was operationalized to model an instructional sequence of model eliciting activities for building concept images of the derivative and its antecedent or contributory concepts. Students were required to follow worksheets in which the model eliciting activities had been designed. The students were to give feedback during classroom observations and record their experiences and findings in the worksheets provided during the implementation phases.

Post task-based interviews were conducted for an in-depth knowledge of how students were experiencing and making sense of the assumed software efficacy in the process of modeling clearer understandings or rich concept images of the derivative. In a models and modeling perspective design experiments are implemented with "a hypothesized learning process and the means for supporting it in mind in order to expose the details of that process to scrutiny" (Cobb et al, 2003:10). At the same time the researcher looked for potential pathways for learning as the design was implemented by capitalizing on contingencies that emerged. By reflecting on initial conjectures about how student learning can be supported, more specific conjectures were developed as part of the overall iterative design process.

A posttest was administered at the end of the pilot phase to assess the improvement in the students' concept images of the derivative and its ancillary concepts. After the piloting the instructional materials were revised and a final implementation phase undertaken commencing with a revised pretest, through revised *Sketchpad* activities, and concluding with a revised posttest.

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6.2   Summary of the findings

6.2.1   A Brief History of the Derivative and its Teaching

The historical origins of the derivative were traced and the way it has been characterized and taught both traditionally and in technology aided instruction up to and including the limited research in dynamic mathematics software environments. An account of the intractable concept of the infinity by which the derivative is defined as a limit was given. The nature of two of Zeno's paradoxes of the infinity and how they affected Greek philosophical thought notably predominated by Aristotle's conceptions of the potential and actual infinitude. The potential division by zero was placed at the centre of the problem of defining the derivative as a limit to determine motion at an instant. The historical genesis of the derivative concept was extended to mankind's struggle with the drawing of tangents to curves to determine maxima and minima after many centuries of

stagnation, up to Barrow's differential triangle - a case of the derivative as the instantaneous rate of change.

An account was made of how mankind tried to figure out the instantaneous rate of change of non-uniform motion and how that effort, commenced with the Merton rule eventually culminated in the discovery and invention of the calculus by Newton and Leibniz in the 17th century. In particular, the characterization by Newton extended the derivative concept to the rate of change function notion through his theory of fluxions although it was Leibniz's notation that stood the test of time. The development of the theory of limits, infinitesimals and transfinite numbers was alluded to as a final solidification of the understanding of the concept image of the derivative as a limit. The theory of infinitesimals revealed difficulties in finding space for infinitesimals in the real number system and, consequentially in human mathematical thought. Cantor's theory of transfinite numbers could also not salvage the situation pointing to the complexity of the derivative concept (compare 2.2).

6.2.2   Initial learning trajectory in *Sketchpad*

The review of related literature was extended to the teaching and learning of the derivative in the traditional static textbook and chalkboard context as the limit of a sequence of secant lines in the static didactic context. The use of computational technology (computer algebra systems) in encapsulating multiple representations or concept images of the derivative in graphical, numeric and algebraic forms was examined. Some limitations of the computational environments were highlighted to caution against possible theoretical computational conflicts, and cognitive and organizational obstacles that may be encountered when integrating technology into mathematics classrooms. Special attention was directed to the limited research on the use of dynamic mathematics environments which by and large have most of the CAS properties in the first instance (compare 2.3-2.7).

Insights derived from the literature review were synthesized into an initial *Sketchpad* mediated learning trajectory for the derivative drawing inspiration from the assumed

affordances of the dynamic mathematics software. The initial learning trajectory started with how functions can be modeled graphically, numerically and symbolically in *Sketchpad* in both static and dynamic terms (compare 2.8.1). Finney and Thomas (1990:1) acknowledge that in calculus, functions are the major tools for describing the real world in mathematical terms. The trajectory then moved onto the static symbolic, graphic, numeric and dynamic graphic, symbolic, numeric representations of the average rate of change (differential quotient), the derivative as instantaneous rate of change, and crucially the derivative as the rate of change function(compare 2.8.2-2.85). The two basic conceptualizations (concept images) of the derivative were therefore captured in the trajectory. This initial trajectory formed the basis for designing the *Sketchpad* activities (worksheets) in Appendix B.

6.2.3   Pretest Findings

From the pre-test findings (it can be concluded that although those students with A level calculus background performed significantly better than the O level Mathematics holders, both groups had fairly adequate pre-calculus background to be competent enough to understand the mathematical content in the *Sketchpad* activities. However, concept images of the limit notion and skills in sketching the derivatives of given graphs were still underdeveloped. Regarding the high abstention ratio in some questions, it was highlighted that the abstention ratio shows at which point students have difficulties in establishing links between knowledge acquired on a common concept in different contexts and/or at different times. Or knowledge not yet acquired at all. Thus although the experimental students were expected to benefit from the *Sketchpad* activities they were not expected to benefit by equal margins given the myriad of baseline knowledge (compare 5.2). The covariation framework was utilized to classify students' levels of understanding (concept images) of the relationships between two variables. Performance in the non-routine Bottle Problem was generally dismal in the pre-test phase.

6.2.4   Results for the Implementation of *Sketchpad* Modeling Activities

On the whole, these activities marked the zenith of the instrumentalization of *Sketchpad* to model students' concept images of the derivative. The process of modeling with

Sketchpad was analyzed horizontally as the building of connections between five or six representational modes interacting dynamically as foreground (forestage) and background (backstage) eminences. The coordination of the connections emerged as the active attention given to representations within and between the foreground and the background eminences. The vertical actualization of the modeling process was manifested by the use of the previous activity as the steppingstone to a higher level of mathematical abstraction (c.f. vertical mathematization) as envisaged in the initial learning trajectory.

The technical or machine procedures were also categorized as background (or input sequence of key strokes, syntax and menu commands) and foreground (on screen animation, dragging, tracing effects etc).   A dialectical relationship between the technical or instrumented actions and the mathematical representations afforded was noted according to which students' learning styles could be categorized (Figure 5.18). Students in the low-low category indicated weak conceptual and technical skill and needed a lot of teacher and/or peer support during activities. Students in the high-high category were mathematically and technically strong and could explore and experiment with the software's capabilities beyond the set tasks and thus attained a high degree of control over the dynamic software. Students low in technical fluency but high in mathematical competence were able to work with minimum technical/machine support. This dialectical relationship between mathematical and technical competencies lent credence to Verillan and Rabardel's (1995) pronouncement that a technological artifact ceases to exist in itself and becomes an instrument when the subject has been able to appropriate it for himself or herself (compare 5.3).

6.2.5   Results for the semi-structured task-based interviews

From the analysis of interview protocols it was conjectured that the students would be able to model the function concept, the average rate of change concept in the buildup to the instantaneous rate of change concept image of the derivative and to model the derivative's rate-of-change-function concept image using the instantaneous rate notion as a pivot. One level of modeling became a scaffold to the next level (compare van Hiele Theory in 4.3.1.1). Within each level of conceptual development, students had

considerable choices and opportunity to move or translate 'horizontally' from one form of representation to another in a versatile manner as afforded by the software. The multiple links between representations took cognizance of Thompson's (1994:39) warning that if students do not realize that something remains the same as they move among different representations then they see each representation as an isolated event. The dynamic features of the software enabled students to develop sophisticated concept images of the derivative and to experience calculus as the mathematics of motion thus reconnecting it remarkably with its primitive context of discovery (compare 1.1, 2.2.2 & 2.2.5). The process of generating the rate of change function succeeded in transforming Newton's theory of fluxions from the domain of the imagination to virtual reality in real time (compare 2.2.5 & 2.8.4) thus revolutionizing and reinventing the concept image of the derivative in learners. Figure 6.1 summarizes the modeling process(es) as described in the *Sketchpad* model- eliciting activities and the semi-structured task-based clinical interviews (compare Figure 3.5).
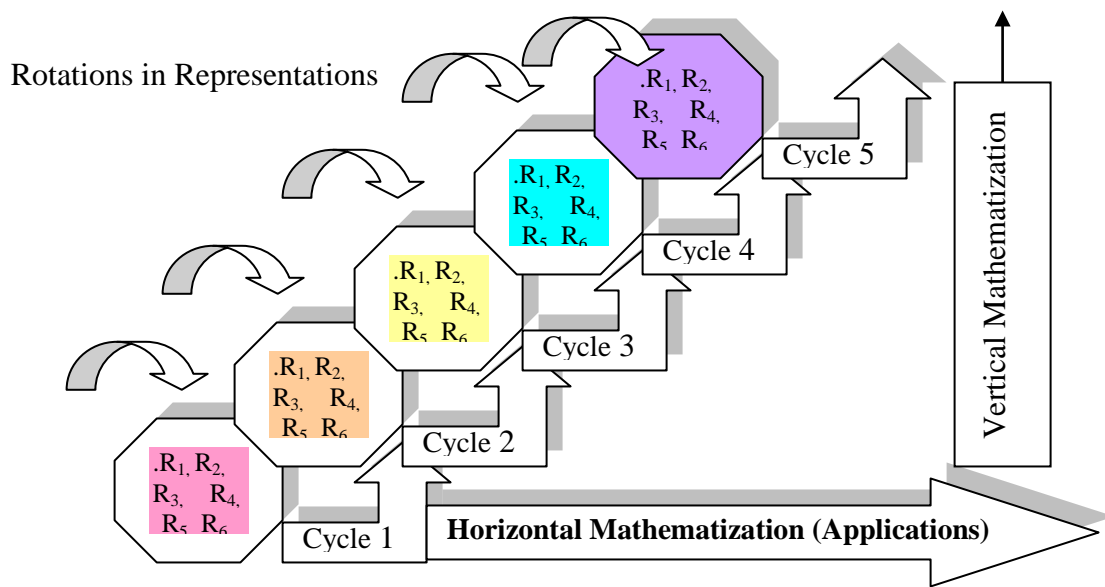
Key to Figure
R_1, R_2, R_3, R_4, R_5, R_6 = Static and dynamic representations enabled
Cycle 1=Function Concept, Cycle 2=Gradient Concept, Cycle 3= Derivative as Instantaneous Rate of Change, Cycle 4= Derivative as Rate of Change Function

Figure 6.1: A Schematic Representation of the Modeling Cycles in *Sketchpad*

In the exit interviews, the students sounded that the utilization of technology is not necessarily a comfortable ride at the beginning. Some of the difficulties they encountered with the software in the initial phases of the study related mainly to syntax sequences required to represent the desired mathematical concepts and to shift from one representation to another. Theoretical computational conflicts such as the disappearance of the tangent seemed to create some cognitive conflict that needed reconciling. Students developed coping strategies consistent with their learning styles and the newly established classroom/laboratory norms of discourse. In the exit interviews the experimental students also gave positive comments about their experiences in the computer-aided learning environment. Thus, in spite of the obstacles, the students who participated confessed to have developed a unique understanding (or concept image)of the derivative in the *Sketchpad* mediated environment and a privileged exposure to the university's computer resources. Although some students managed to relate the derivative concept to real life events or situations it was not always easy for them to come up with ready examples (compare 5.4).

6.2.6   Posttest Results

The posttest results showed that pretest-posttest performances of the experimental groups improved significantly as a consequence of exposure to the *Sketchpad* activities. The null hypotheses were rejected at a significance level of 5% suggesting that the *Sketchpad* activities were an effective tool for enhancing students' understanding of the derivative and related concepts. The research question about the difference in the pretest and posttest performance was therefore answered in the affirmative. However, with regards to the experimental and the control group performances, it was only in the Modified Bottle Problem that the experimental group performed overwhelmingly better at the 5% level of significance. It can be concluded therefore that given a real world context to model by drawing a graph to represent the relationship between two variables, and to model the derivative of that relationship, the experimental students performed better than the control group students as a result of exposure to the *Sketchpad* activities. This newfound ability can presumably be ascribed to the exposure to the experimental conditions of the

Sketchpad activities. Accordingly the second research question that the posttest set out to investigate was answered in the affirmative only with respect to the sketching of the derivative as the rate of change function. On the whole however, for the rest of the concepts it can be concluded convincingly that the experimental students did not perform worse than the control group of students (compare 5.5).

6.3   Implications of the results for mathematics learning

The epistemological value of technology does not necessarily reside in the technology itself. This observation has been a recurrent theme throughout the study. As a consequence technology should be viewed as a tool at the teacher's disposal. This tool remains a machine or artifact until it is utilized to further curricula objectives. When the tool has been utilized it transforms to an instrument. The teacher's fluency in the use of the tool is critical in this transformation. Just as a musical instrument, the type and taste of music are a product of the artiste's creativity, ingenuity, experience and sensitivity to the audience. In a sense, the instructional responsibility of the teacher is to customize the software such that it represents or models mathematical concepts in a manner that is accessible and interesting to the learners. Students themselves should be empowered to develop control over the instrument so that they feel comfortable instrumenting to generate as many examples as they can afford, illustrate, communicate and justify them. That is, they should be given the opportunity to think with the tool, reasoning with the tool as if it were a part of their body - a sense of dwelling in the tool. Once the students have internalized and automated their command of the software syntax, the software becomes an instrument of experimentation, exploration, justification and discovery stimulating and catalyzing mathematical ways of thinking in cooperation with other students. Parallel to the students' instrumentalization is the teacher's responsibility to listen to students' ways of modeling their thinking.

Curriculum designers are challenged by the new digital technologies, not just to keep abreast, but to be a step ahead and remain open to more creative uses of technologies by students. In other words the technologies challenge curriculum designers to construct

open-ended tasks. To the software designers, the challenge is to keep in constant dialogue and conversation with the mathematics educators and curriculum designers to continually enrich software customization to educational needs of the students

6.4 Limitations of the study

6.4.1 Translation of *Sketchpad* into a tool for modeling the derivative

The translation of *Sketchpad* into an instrument for the teaching and learning of the derivative in introductory calculus requires the assumption that the dynamic software has the properties that make it an efficient dynamic medium for visualizing, demonstrating, reasoning and communicating about the derivative in a multiplicity of representations. The *Sketchpad* activities were assumed to be designed clearly enough to be followed with minimum difficulty and for students to make meaningful sense of the targeted mathematical concepts and the constitutive software syntax. The behaviour of the software was expected to be uniform or at least consistent across participating students. The final activities were one example of how *Sketchpad* could be used among other possibilities

6.4.2 Time constraints

The time constraints and duration of activities were assumed to be adequate and no hitches were expected when implementing the activities. Both of these assumptions were limitations in their own right. For example, as already noted in 5.3.5, the activity used for the slope of the tangent at a point, in hindsight, should be done from both the left and the right hand side. The assumption was also made that the pretest and posttest would measure reliably within a relatively short space of time. Students were not interviewed on their answers to the bottle problem used for the pretest and the posttest. This could have provided valuable deeper insight into their reasoning and how and if *Sketchpad* had played a role in their conceptualization.

6.4.3 Size of the samples

The voluntary nature of the participation in the study imposed a limit on the size of the sample(s), more so when combined with the endemic fear of mathematics by many non-

major students, and the fact that these sessions would not contribute any credits towards the regular curriculum.

6.5   Implications for Future Research

a)   Many students have succeeded in passing through high school algebra 'untouched by mathematical understanding; they have succeeded in learning by rote' (Land, 1990:171). In constructivist language these students have remained 'novices' rather than becoming 'experts' in handling calculus concepts let alone applying them effectively to solve problems. That is, they have processed material by rote without perceiving its essence and without attaining rich enough concept images. In the context of the derivative, they have mastered the differentiation rules and procedures without coming to terms with what the product, the derivative, actually represents (c.f. derivative as that which changes x^n to nx^{n-1} in 1.1).

b)   Dynamic mathematics software such as *Sketchpad* seem to offer a way out of the current mathematics atrophy in non-mathematics major students who resign themselves to their fate and repeatedly fail introductory calculus courses that are otherwise a rehearsal of what they have previously covered at high school (c.f. they are not necessarily ready to begin courses for which they have already passed at the Advanced level).

c)   Multiple representations in *Sketchpad* should be linked and related one to another so that what remains constant or invariant is grasped appropriately as the concept image of the derivative and not a disparate collection of representations.

d)   Students relate themselves differently to the software, there are some who are not confident and will quit saving face rather than withstanding the frustration and the inconvenience. Different learning styles might require further investigation.

207

e)   The theoretical computational conflicts encountered (such as the disappearing tangent), limited window size and scale, require the attention of software designers.

For the future, a larger scale longitudinal study is suggested with revised, improved and extended materials. What has been provided as the possible connections that students can make between representations of the derivative (see 5.4.) has focused more on the technical aspects of *Sketchpad* thus providing a conceptual framework that leaves scope for further development through a more detailed qualitative study of the quality and types of students' conceptual analyses.

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Appendix A₁ - Revised Pretest Items

PARTICIPANT'S BIOGRAPHICAL DETAILS

Please note that information provided in this questionnaire shall be treated in the strictest of confidence and used only for research purposes.

Instructions:

Attempt all questions in this questionnaire.

1.   Title Mr/Mrs/Miss ………………………………….

2.   Age ………………………… years and ………………….months.

3.   Have you used a computer before   YES/NO

      If YES, how do you rate your level of proficiency?

      BEGINNER/MODERATE/HIGH

4.   Have you previously used a computer to learn mathematics? YES/No

      If YES, name programme(s) used and topic(s) covered……………………

      ………………………………………………………………………………….

5.   What is your highest qualification in Mathematics?

      'O' Level/'A' Level/Other (Specify………………………………….)

Appendix A₂ – Revised Posttest Items

POST-TEST PARTICIPANTS' BIOGRAPHICAL DETAILS

Please note that information provided in this questionnaire shall be treated in the strictest of confidence and used only for research purposes.

Instructions:

Attempt all questions in this questionnaire by giving your answers on the questionnaire

1.   Title Mr/Mrs/Miss (Delete the inapplicable)

2.   Age ………………………… years and ………………….months.

3.   Did you take part in the pre-test of this study?   YES/NO

      If YES, please specify your pre-test questionnaire number:……………….

4.   Did you take part in the teaching experiment with computers?
      YES/NO

5.   What was your highest qualification in Mathematics before enrolling at NUST? (Tick the correct level)

      'O' Level'   ☐

      A' Level   ☐

      Other (Specify)………………………………

Post-test/Posttest Items

3.   Sketch the graphs of the following group of functions and explain how the
     first is affected

   $y = x^2$, $y = x^2 + 1$, $y = x^2 -1$, $y = 2x^2$, $y = x^2 + x$

2.
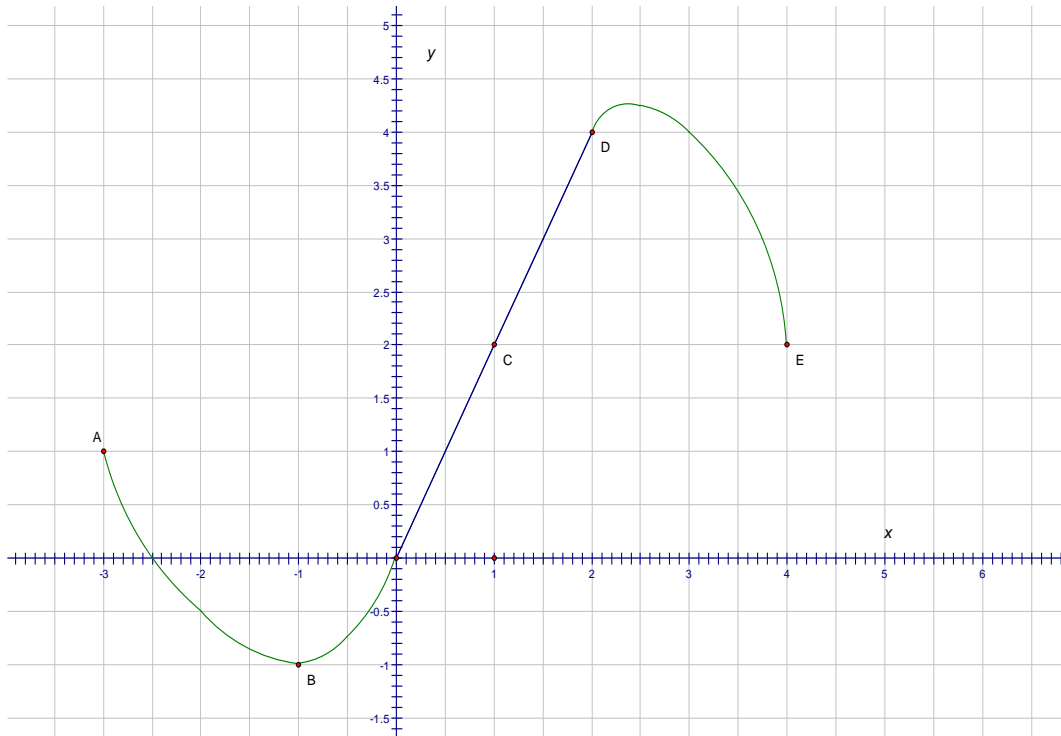
Figure C_2 1: Identifying the gradient of a curve through various points

Find the average rate of change between the following points on the graph:

(Note: the "average rate of change" from P to Q means the gradient of PQ)

i)   from C to D ……………….....
ii)   from D to E ……………….....
iii)   from A to B ……………….....
iv)   from B to C ……………….…
v)   from C to E ……………….…
vi)   from D to C ……………….…

3.

i)   Write down the gradient of the straight line through *A* and *B,* in terms of *k* …………………………………………… …………..…

ii)   Write down the gradient of *AT* …………………………………..
        Explain how you might find the gradient of *AT* from first principles.
        (i.e. as a limit of the gradient of AB as *k* -1 approaches zero).
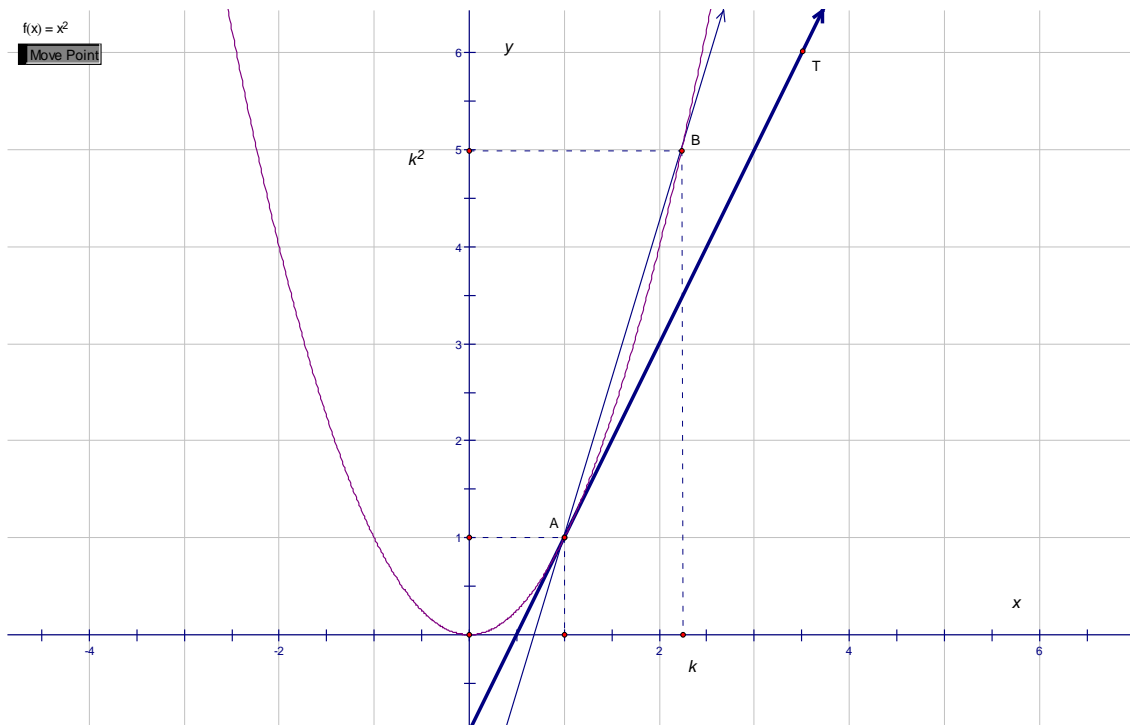
Figure C$_2$ 2: The relationship between the gradient of a secant line and of a tangent

………………………………………………………………………………………………
………………………………………………………………………………………………
………………………………………………………………………………………………
………………………………………………………………………………………………

4.   Answer the following questions for each of the graphs that follow:

$f(x) = x^2 - x$
Graph of x^2-x

Can you calculate the gradient at x=0? YES/NO
If YES, what is the gradient, if NO, why not?
....

$g(x) = |x|$
Graph of y=|x|

Can you calculate the gradient at x=0? YES/NO
If YES, what is the gradient, if NO, why not?
....

Graph of y=√(|x|)

Can you calculate the gradient at x=0? YES/NO
If YES, what is the gradient, if NO, why not?
....

Graph of y=|x³|

Can you calculate the gradient at x=0? YES/NO
If YES, what is the gradient, if NO, why not?
....

Figure C₂ 3: Gradient investigation

5.   Graph 1 is the derivative $y=f'(x)$ of a function $y = f(x)$ defined for $-2 \leq x \leq 2$.

Graph 1: $y = f'(x)$

Graph 2
Graph 2: $y = f(x)$

Graph 3: y = f(x)

Graph 4: $y = f(x)$
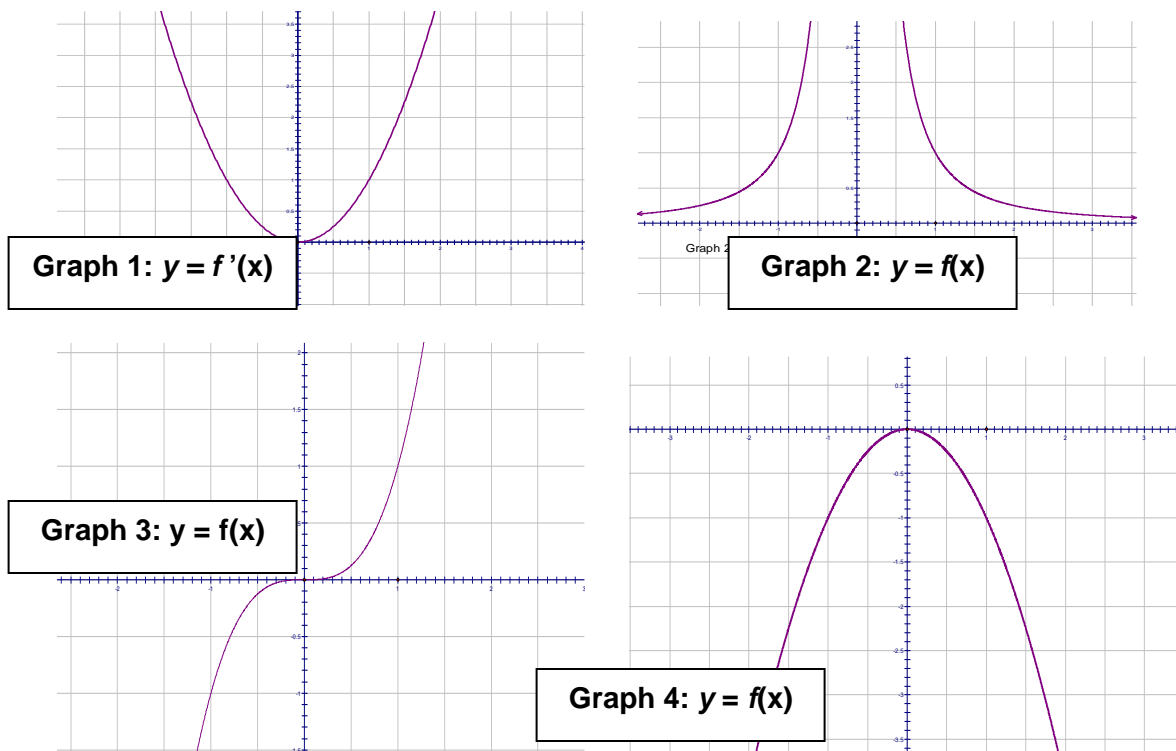
Figure C₂ 4: Identifying the graph with a given gradient

Which of the graphs, 2, 3, 4 has graph 1 as its derivative? ………..…………….

Give reasons for your choice……..………………………………………………

……………………………………………………………………………………………

6.   Sketch the derivatives of the following graphs in the spaces below.
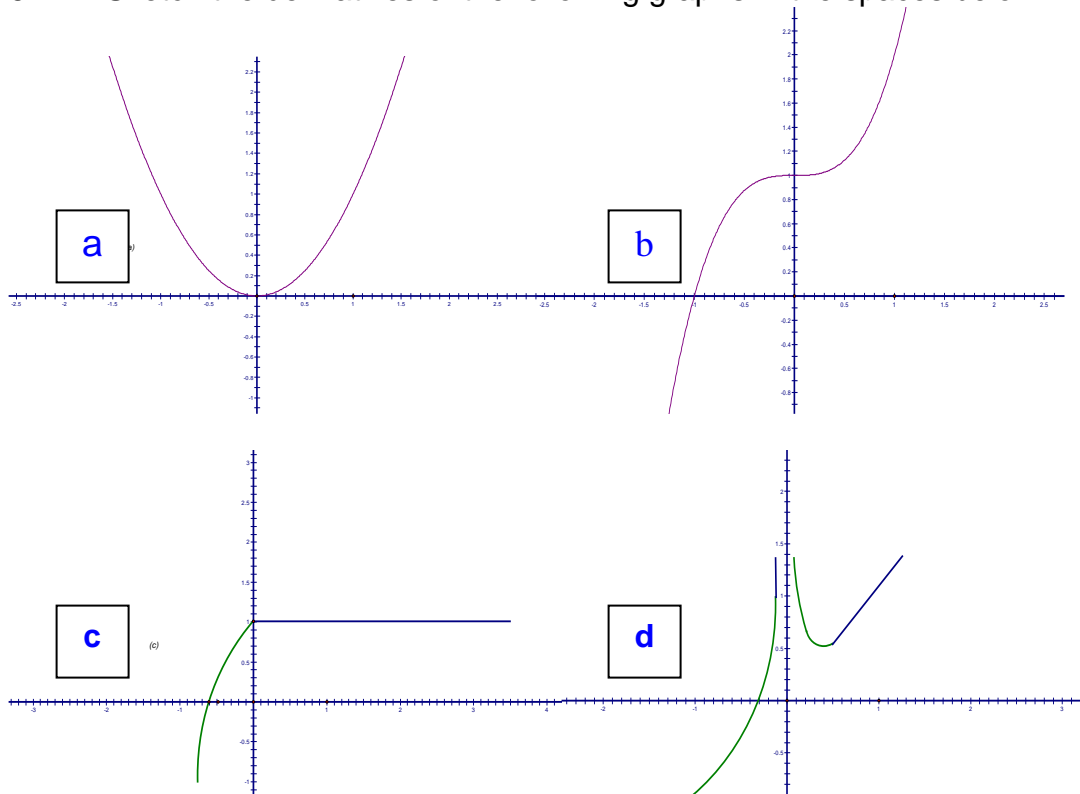
Figure C$_2$ 5: Sketching the gradient graphs for given graphs

a)

b)

c)

d)
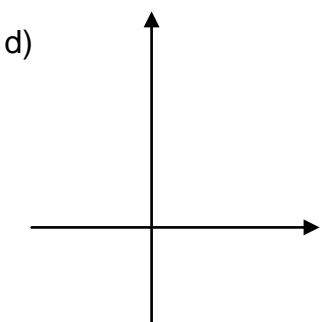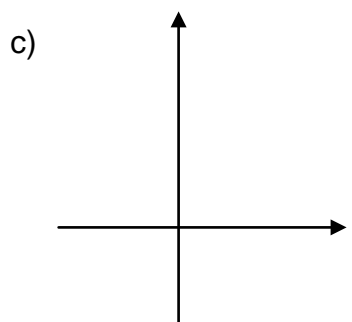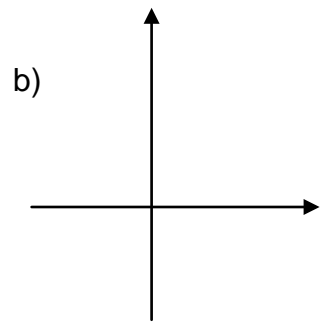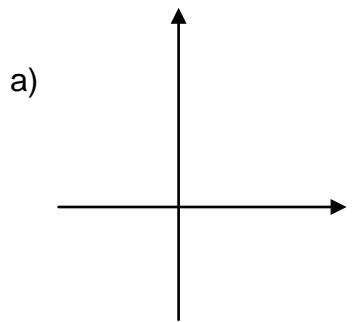
7 ()

i)   ii)

Figure C$_2$ 6: Change of height with respect to constant change in volume

    a)   Draw a graph to show how the height of water in each of the containers varies with an imagined constant increase in water volume. (Use the x-axis to represent the volume and the y-axis to represent the height)

    b)   Draw a graph to show the rate of change of the gradient/slope of the tangent (derivative) of the graphs in a) above.

a) i.

a) ii.

b) i.

b) ii.

227
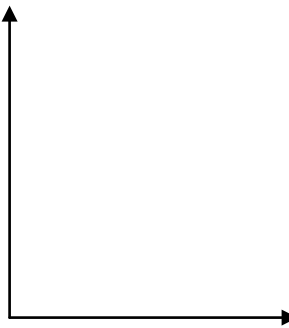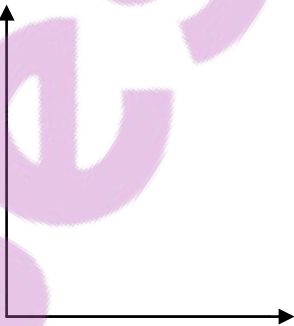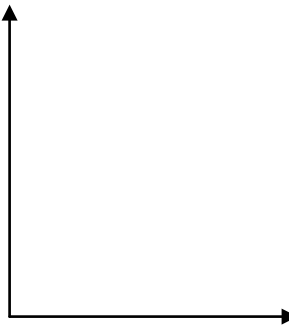
Appendix B₂ – Revised *Sketchpad* Activities

Modeling with *Sketchpad* to Enrich Students' Concept Image of the Derivative. – *Sketchpad* Activities

Activity A: Modeling the Function Concept in *Sketchpad*

> Example Problem Situations
> A loaf of bread is forgotten in a cupboard when a family goes on a week long vacation. How much mould would have accumulated on any day that the family is away? (How safe will the bread be?)

1. Use *Sketchpad* to plot the following functions

    $y = x^2,\ \ y = x^2 + 1,\ y = x^2 -1,\ y = 2x^2,\ y = x^2 + x$

    $y = x^2,\ y = x^2 - x,\ y = x^2 -2x,\ y = x^2 + x,\ y = x^2 + 2x$

    $y = x^3,\ y = x^3 - 1,\ y = x^3 + 2,\ y = x^3 + 2x,\ y = x^3 + x^2$

    $y = x^4,\ y = x^5,$

    $y = |x|,\ \ \ \ y = \sqrt{(|x|)},\ \ y = |x^3|$

    $y = \sin x,\ y = \cos x,\ y = \tan x,\ y = e^x$

2. Use *Sketchpad* to represent these functions dynamically.

3. Use *Sketchpad* to represent these functions numerically.

4. Create table data and plot for each of the functions.

5. Use parameters to plot the following families of functions.

    a) $y = ax^2 + bx + c$

    b) $y = ax^3 + bx^2 + cx + d$

Activity B: Modeling the Average Rate of Change of a Function

> ### Example Problem Situations
> A car starts at rest and increases its velocity in such a way that $d = t^2$. What is the average speed between the second and the third minute of the journey?
> (How safe is the speed?)

Step B1:   Open a New Sketch in the File menu.

Step B2:   Select **Plot New Function** in the Graph menu and define $f(x) = x^2$ using the pop-up Calculator keypad. (i.e. enter in x^2) and click **OK** to plot the graph. (See Figure 1)

$f(x) = x^2$

Figure B₂ A1a: Graph of y = x²
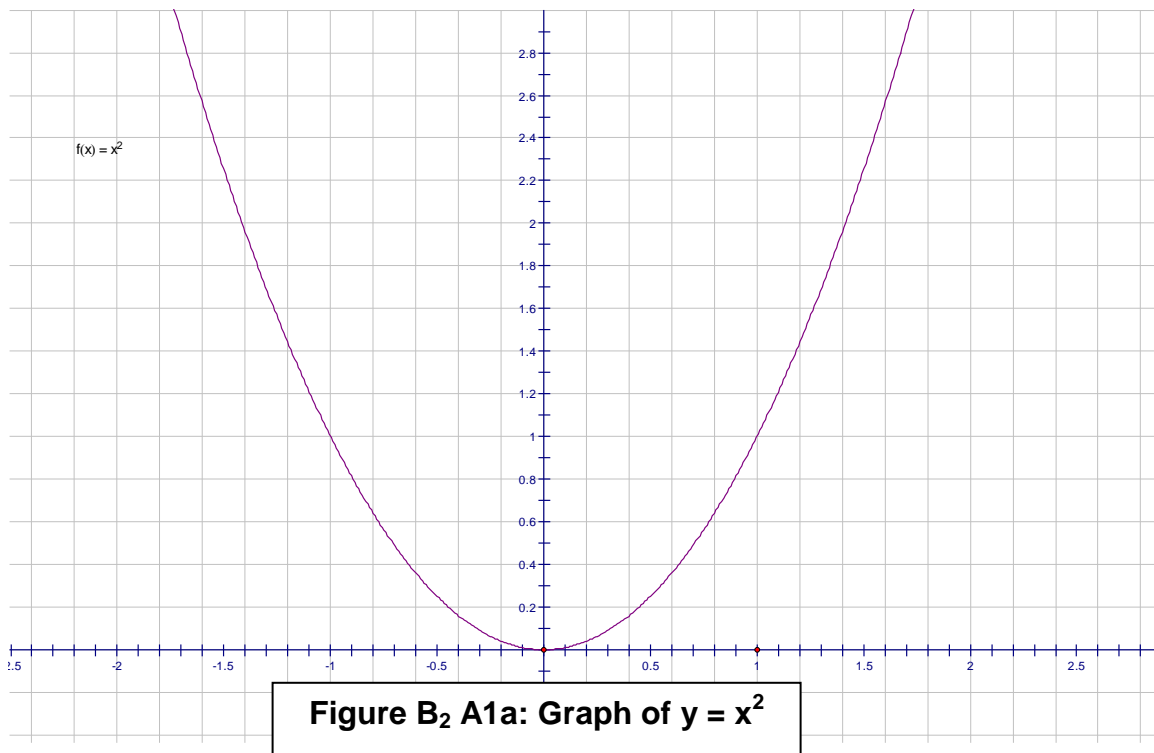
Step B3:   Select **Plot Point on Function Plot** from the Construct menu twice and click **OK** each time to plot two points on the graph of f(x). (OR use the Point Tool to plot both points.)

Step B4:   Select the two plotted points by clicking on them using the **Selection Arrow.** Choose **Label Points** from the Display menu to label the left point as A and the point on the right B.

Step B5:   Drag points A and B to the positions (1,1) and (2,4) respectively.

(See Figure 1b).
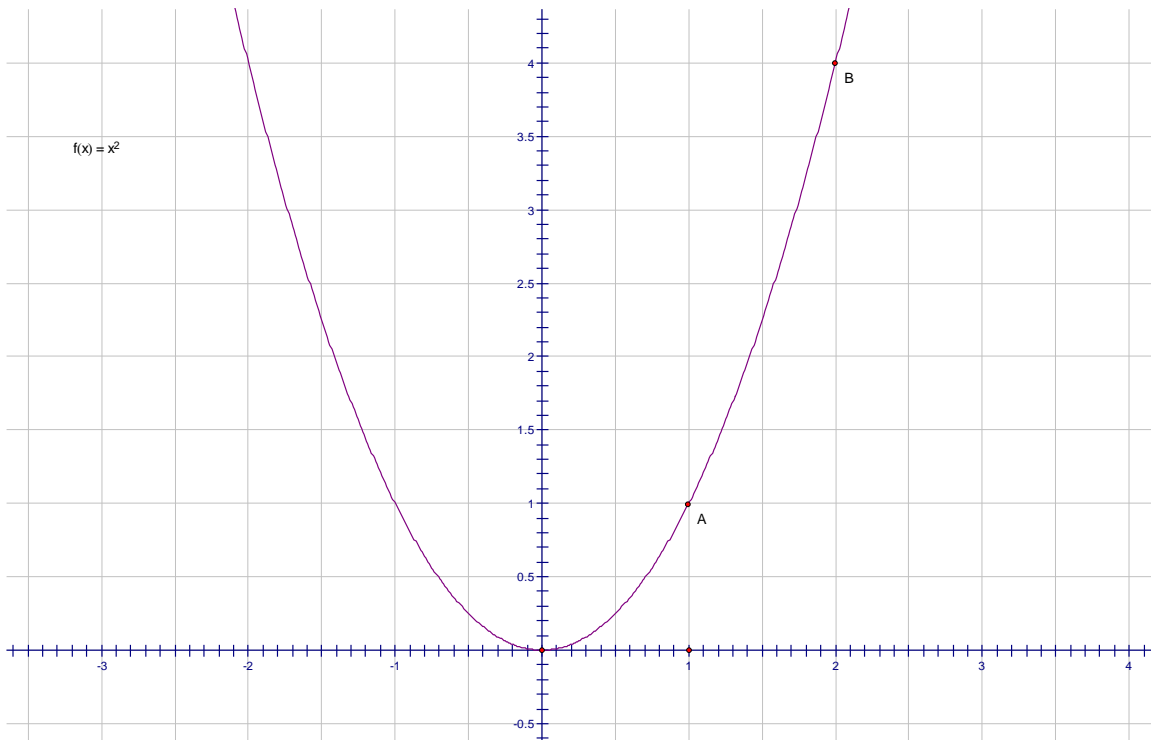
Figure B₁ A1b: Points A and B on graph of $y=x^2$

Step B6:   Select both points and choose **Abscissa (x)** from the Measure menu to display the measurements of the x-coordinates, x_B and $x_A.$ Choose **Ordinate (y)** from the measure menu to display the y-coordinates, y_B and $y_A.$ By dragging, adjust points A and B coordinates to read (1.00;1.00) and (2.00;4.00) for more precise plotting.

Step B7:   De-select everything (by clicking on blank space) and then reselect the plotted points starting with point A.

Step B8:   Choose **Line** from the Construct menu to construct a line passing through both points. (See Figure 1 c).

a) What name is given to this line? (Secant or Chord?)…………..

b) What does the line represent? ....

Step B9: Calculate the slope of this line. (Remember slope to be change in y divided by change in x).

Method:…………………………………….. Answer:…………………..

f(x) = x^2
x_A = 1.00
x_B = 2.00
y_A = 1.00
y_B = 4.00
Slope AB = 3.00
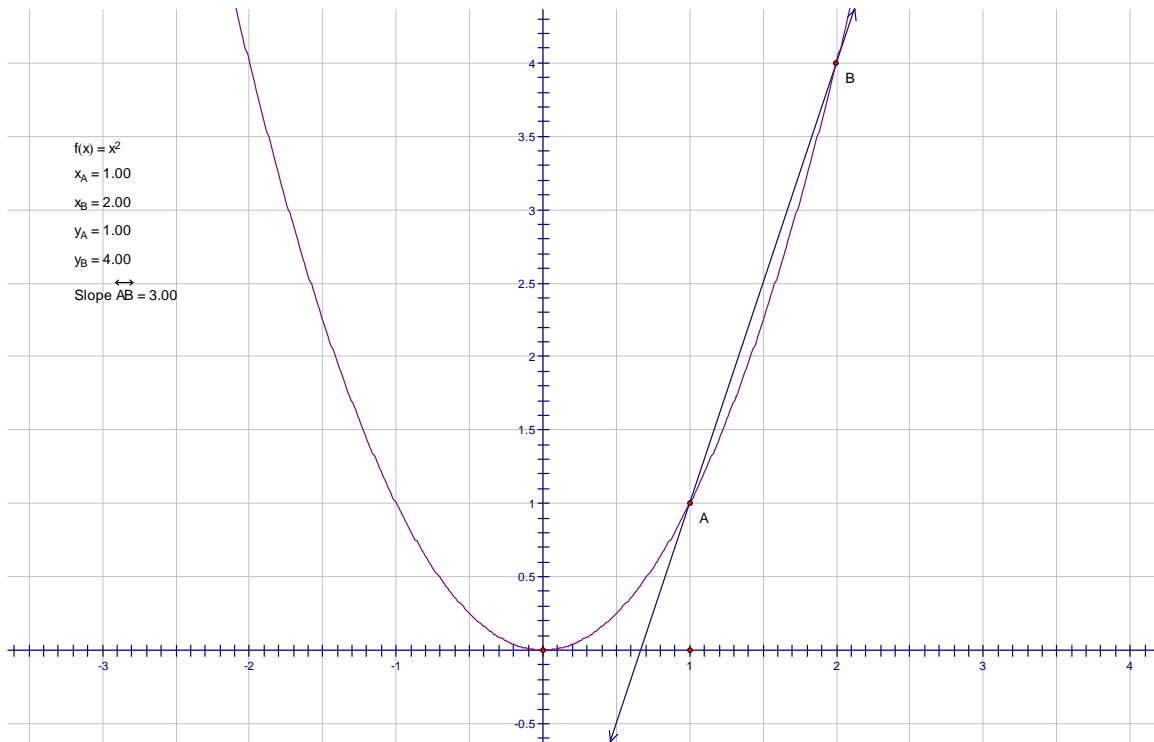
Figure B$_2$ A1c: The secant line through A and B

Step B10: Select the secant line and choose **Slope** from the Measure menu. Compare the slope measurement with your results in step 9. What do you notice?...
....

Step B11: Choose **Calculate** from the Measure menu and create a measurement for $y_B - y_A$ by selecting the measurement for y_B, followed by the minus sign on the calculator's keypad and then by the measurement for y_A. What value do you obtain?....................................Is this value fixed?...............

Step B12: Choose **Calculate** from the Measure menu and create a measurement for $x_B - x_A$ by selecting the measurement for x_B, followed by the minus sign on the calculator's keypad and then by the measurement for x_A. What value do you obtain?……………………..Is this value fixed?...............

231

Step B13: Choose **Calculate** from the Measure menu and create a measurement for $(y_B - y_A)/(x_B - x_A)$ by selecting the measurement for $y_B - y_A$ obtained in Step 11 , followed by the division sign on the calculator's keypad and then by the measurement for $x_B - x_A$ obtained in Step 12. (See Figure 1 d).
a) What is this measurement?............... ………………. ....
Compare with the slope of Secant Line AB in step 10.
b) What do you notice?..
....

Step B14: Select values for x_A, y_A, x_B, y_B, $y_B - y_A$, $x_B - x_A$, $(y_B - y_A)/(x_B - x_A)$ and Slope AB respectively. Select **Tabulate** from the Graph menu to create a table of these values.

$f(x) = x^2$

$x_A = 1.00$

$x_B = 2.00$

$y_A = 1.00$

$y_B = 4.00$

Slope $\overset{\leftrightarrow}{AB} = 3.00$

$y_B\text{-}y_A = 3.00$

$x_B\text{-}x_A = 1.00$

$$\dfrac{(y_B\text{-}y_A)}{(x_B\text{-}x_A)} = 3.00$$

x_A	x_B	$x_B\text{-}x_A$	y_A	y_B	$y_B\text{-}y_A$	$\dfrac{(y_B\text{-}y_A)}{(x_B\text{-}x_A)}$	Slope $\overset{\leftrightarrow}{AB}$
1.00	2.00	1.00	1.00	4.00	3.00	3.00	3.00
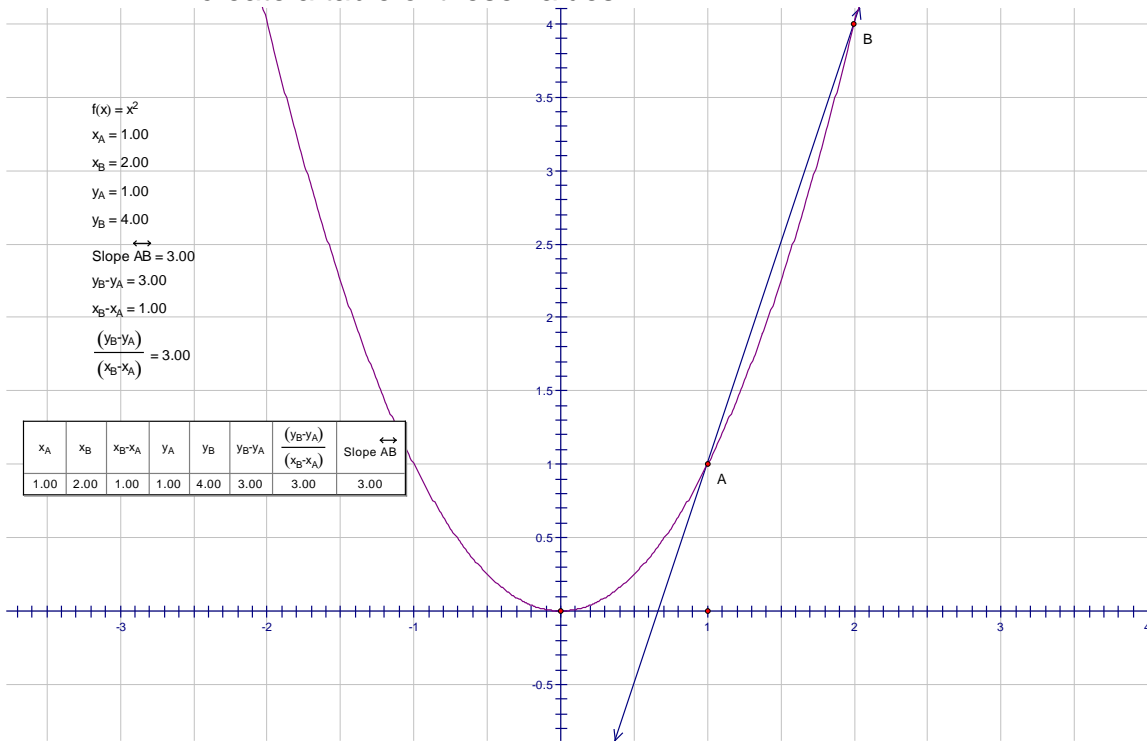
Figure B₂A1d: Graph of y=x² with table of values.

Step B15: Drag point B backwards and forwards and observe changes in the slope measurements.
a) What can you say about the average rate of change when point B is dragged upwards?
Increasing/Decreasing...
b) What can you say about the average of change when point B is dragged downwards?
Increasing/Decreasing…………………………….…………….…….
c) Is it positive or negative in this interval?..
d) Can the average rate of change be negative?..................Yes/No.

232

Step B16:   Re-write the formula for the slope by making the following substitutions to obtain the difference quotient (symbolic representation of the average rate of change):

$f(x + h)$ for y_B, $f(x)$ for y_A and h for $(x_B - x_A)$

Answer:………………………………………….

Step B17:   Save this sketch as 'Rate of Change of a Function'

Further Exploration

Step BFE1: Repeat Step 7 but start by selecting point B.

Step BFE2: Repeat steps 8 to 10. Is there a change in the slope value? If yes, why? If not, why?..

Step BFE3: Compare your conjecture in Step 2 with the direction of the bar line above AB in the slope measurement. What can you conclude?

The Derivative as the Instantaneous Rate of Change of a Function

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Example of Informal Problem Situations

A man weighing trips and drops from a ladder, at what velocity does the man hit the ground? (Risk of injury/death)

Step C1: Open the Sketch 'Rate of Change of Function' created in Activity B.

Step C2: Create a duplicate page of the sketch by choosing the **Documents Options** from the File menu. Re-label the resulting pages 1and 2 as 'Average Rate of Change' and 'Instantaneous Rate'.

Step C3: Select the Secant Line on the Instantaneous Rate sketch and choose **Trace Line** from the Display menu. Choose **Color** and select green colour from the Display menu (or any other colour that will contrast satisfactorily later).

Step C4: De-select the secant line and select points B and A in that order.

Step C5: Choose **Movement** from the 'Action Buttons' sub-menu of the Edit menu and select 'medium' from the pop-up menu as your speed. Click **OK** and observe.

Move B -> A

$f(x) = x^2$

$x_A = 1.00$

$x_B = 1.00$

$y_A = 1.00$

$y_B = 1.00$

Slope $\overleftrightarrow{AB}$ = 2.00

$y_B\text{-}y_A = 0.00$

$x_B\text{-}x_A = 0.00$

$\dfrac{(y_B\text{-}y_A)}{(x_B\text{-}x_A)} = 2.00$

x_A	x_B	$x_B\text{-}x_A$	y_A	y_B	$y_B\text{-}y_A$	$\dfrac{(y_B\text{-}y_A)}{(x_B\text{-}x_A)}$	Slope $\overleftrightarrow{AB}$
1.00	1.90	0.89	1.00	3.59	2.59	2.90	2.90
1.00	1.77	0.77	1.00	3.13	2.13	2.77	2.77
1.00	1.50	0.50	1.00	2.25	1.25	2.50	2.50
1.00	1.28	0.28	1.00	1.63	0.63	2.28	2.28
1.00	1.08	0.08	1.00	1.16	0.16	2.08	2.08
1.00	1.03	0.03	1.00	1.07	0.06	2.03	2.03
1.00	1.00	0.00	1.00	1.01	0.00	2.00	2.00
1.00	1.00	0.00	1.00	1.00	0.00	2.00	2.00
1.00	1.00	0.00	1.00	1.00	0.00	2.00	2.00

Figure B$_2$B2a: Effect of moving point B to point A

a) What can you say about the secant line now?...........................

....

b) What is its gradient?..

c) What special name do we give to this secant line after reaching point A?   ....

Step C6:   Select **Erase Traces** from the Display menu. Can you confirm your conjecture in Step 5? (See Figure 2b).

x_A	x_B	x_B-x_A	y_A	y_B	y_B-y_A	$\dfrac{(y_B$-$y_A)}{(x_B$-$x_A)}$	Slope $\overset{\leftrightarrow}{AB}$
1.00	1.90	0.89	1.00	3.59	2.59	2.90	2.90
1.00	1.77	0.77	1.00	3.13	2.13	2.77	2.77
1.00	1.50	0.50	1.00	2.25	1.25	2.50	2.50
1.00	1.28	0.28	1.00	1.63	0.63	2.28	2.28
1.00	1.08	0.08	1.00	1.16	0.16	2.08	2.08
1.00	1.03	0.03	1.00	1.07	0.06	2.03	2.03
1.00	1.00	0.00	1.00	1.01	0.00	2.00	2.00
1.00	1.00	0.00	1.00	1.00	0.00	2.00	2.00
1.00	1.00	0.00	1.00	1.00	0.00	2.00	2.00
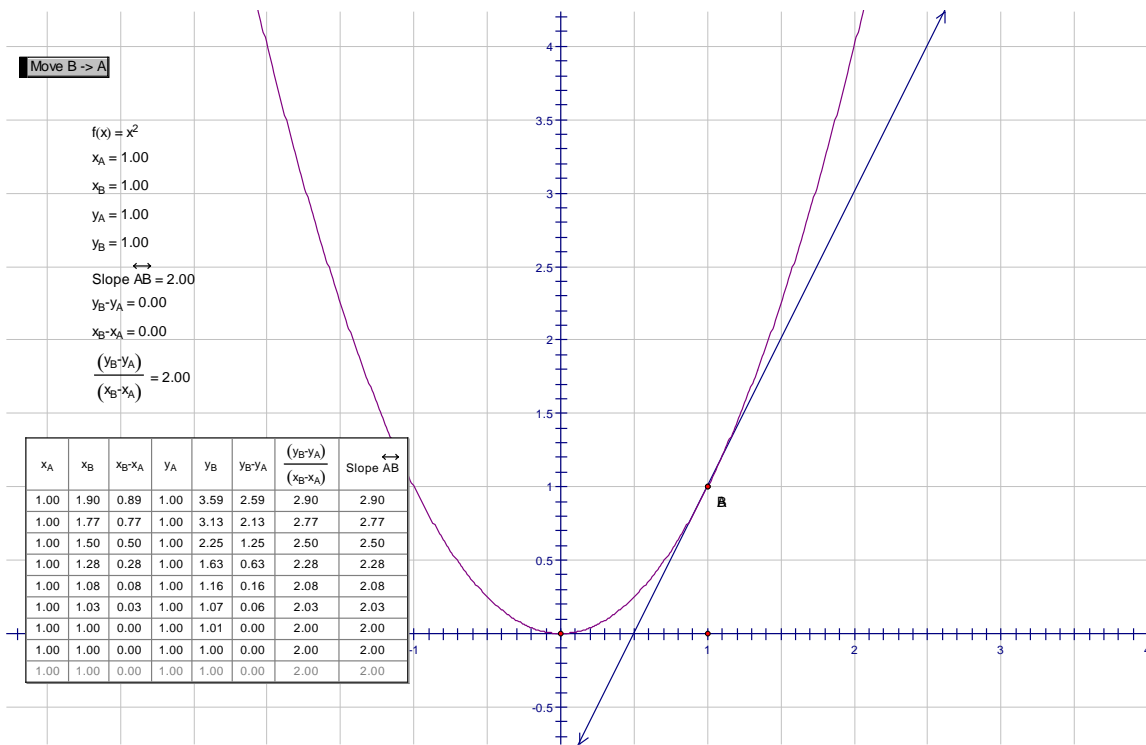
Figure B$_2$B2b: Tangent Line after Erasing Traces.

Step C6b: Discussion of the Tangent as the Limiting position of the Secant as B moves to A. Algebraic expression of the derivative as a limit of the differential quotient as B tends to A.
Can the Secant go beyond the Tangent at A?.............................YES/NO.
Explain your answer:...

Step C7:   a) What is the value of the gradient at (1.00;100)?...........................
b) What does the value mean or represent at this point?.................
....

Step C8:   a) What is the value of y_B - y_A on the table of values? ....

b) What is the value of $x_B - x_A$? …………………………………………

c) What does 0/0 mean? …………………………………………………

d) What is the answer for $(y_B - y_A)/(x_B - x_A)$ on your table of values?

e) Is this answer accurate?………………….………………..Yes/No.

d) Explain your answer………………………………………………

……………………………………………………………………………..

Step C9: Select **Show All Hidden** from the Display menu and right click the measurement/value of $(x_B - x_A)$ to obtain the Context menu.

Step C10: Select **Properties…..** in the Context menu and choose thousandths as your precision on the Value tab of the pop-up control panel.

Step C11: Repeat for increased precisions 'ten thousandths' and 'hundred thousandths'.

a) What do you notice?...

....

b)   What can you predict about the value $(y_B - y_A)/(x_B - x_A) = 0.00$?

....

Step C12: Increase the precision of this value to hundred thousandths as in steps 10 to 11.

Step C13: Increase the precision of values for $y_B - y_A$, and the slope of AB. What can you conclude?...

Step C14: Select **Undo Measurement** from the Edit menu repeatedly until you select **Undo Translate Point.**

Step C15: Select the table of values, right click and tick 'Track table values from the Table tab of the Properties control panel

Step C16: Right click the Construct Tangent Action button, select properties and adjust the speed to slow.

Step C17: Click the Construct Tangent Action Button and continue to double click it periodically as B moves until it reaches A. Your table of values changes until you get something like Figure 2b.
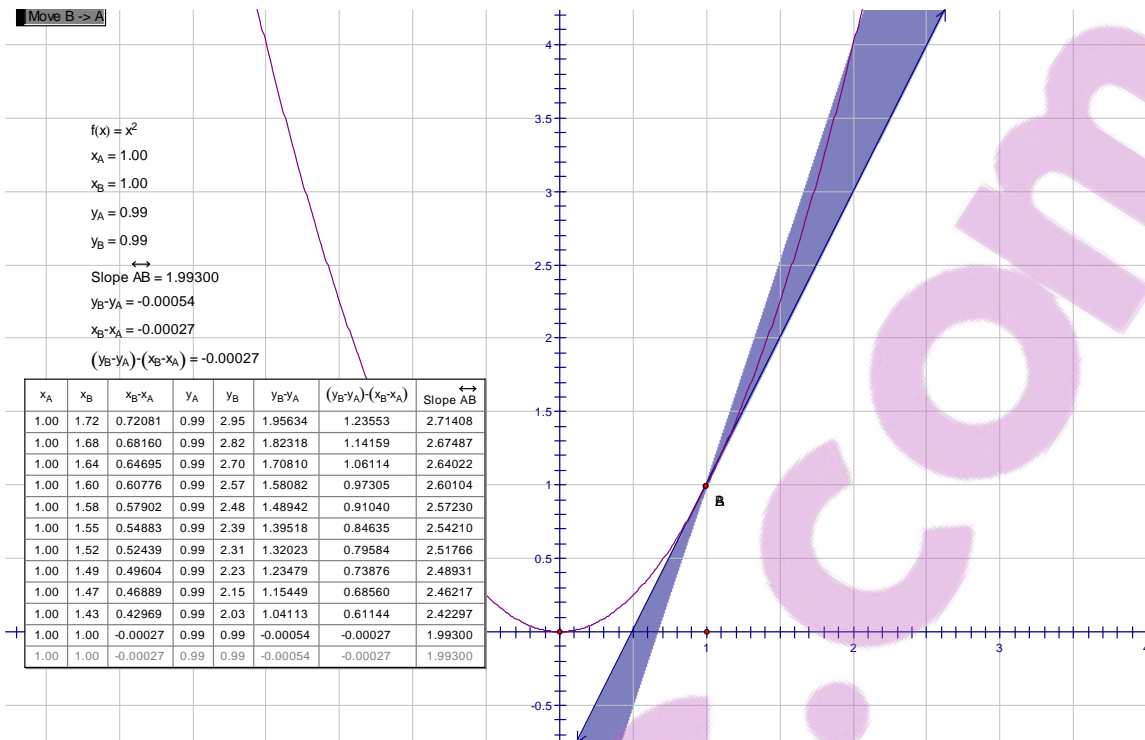
Figure B$_2$B2b: Display of changes in table of values.

Step C18: What can you say about the values of (x_B - x_A) as B moves to A?

Step C19: At this stage the derivative can be introduced as the limit of the differential quotient as *h* tends to zero. i.e. $\lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$.

Step C20: Save this sketch as Instantaneous Rate of Change.

Further Exploration

Step CFE1: Use the motion controller or drag to bring point B exactly on top of point A to the nearest one hundred thousandth i.e. so that x_B - x_A = 0.00000 What happens to the Tangent Line? (**It disappears**) Why? What is the value of the gradient? (**undefined)** Why? (0/0). (See Figure 2c)

237
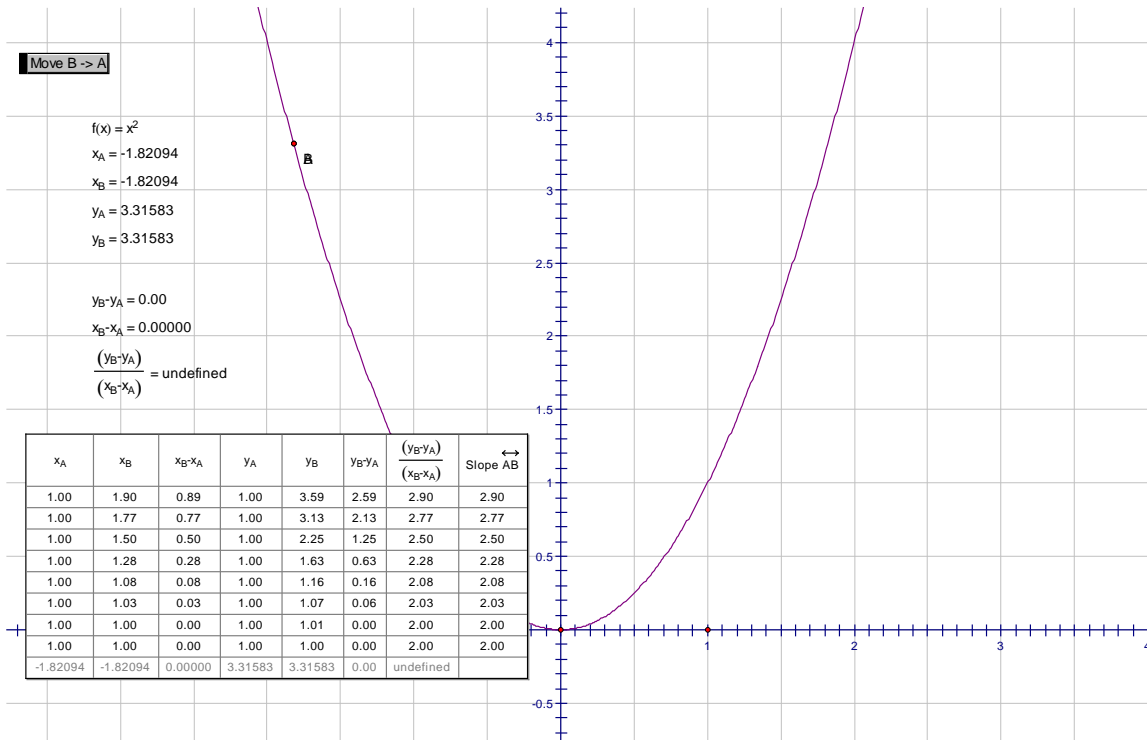
Figure B₂B2c: Mystery of the Disappearing Tangent

Step CFE2: How can the tangent be re-defined so that it does not disappear?

....
....
....

Activity D: Modeling t**he Derivative as the Rate of Change Function**

Example Problem Situations

A cup falls from the top of a cupboard, what is its velocity at any given moment of its fall?

Step D1:   Open the 'Instantaneous Rate of Change' page in the Rate of Change Sketch in Activity C.

Step D2:   Create a copy of this page and name it 'Rate of Change Function'.

Step D3:   Select the measurements of x_A and the slope of AB in that order.

Step D4:   Choose **Plot as (x,y)** from the Graph menu to plot the Rate of Change of the Tangent Gradient with respect to x i.e. the gradient function.

Step D5:   Choose **Trace Plotted Point** in the Display menu and select 'green' colour.

Step D6:   Label the plotted point as C and deselect the point.

Step D7:   Select the Tangent Line, and choose **Trace Line** from the Display menu to deselect tracing.

Step D8a: Choose **Animate Line** from the Display menu and observe the trace of point C. What shape is the line traced by C?

$f(x) = x^2$

$x_A = 2.20388$

$x_B = 2.20420$

$y_A = 4.85711$

$y_B = 4.85851$

$\text{Slope } \overleftrightarrow{AB} = 4.40809$

$y_B - y_A = 0.00141$

$x_B - x_A = 0.00032$

$\left(y_B - y_A\right) \div \left(x_B - x_A\right) = 0.00109$
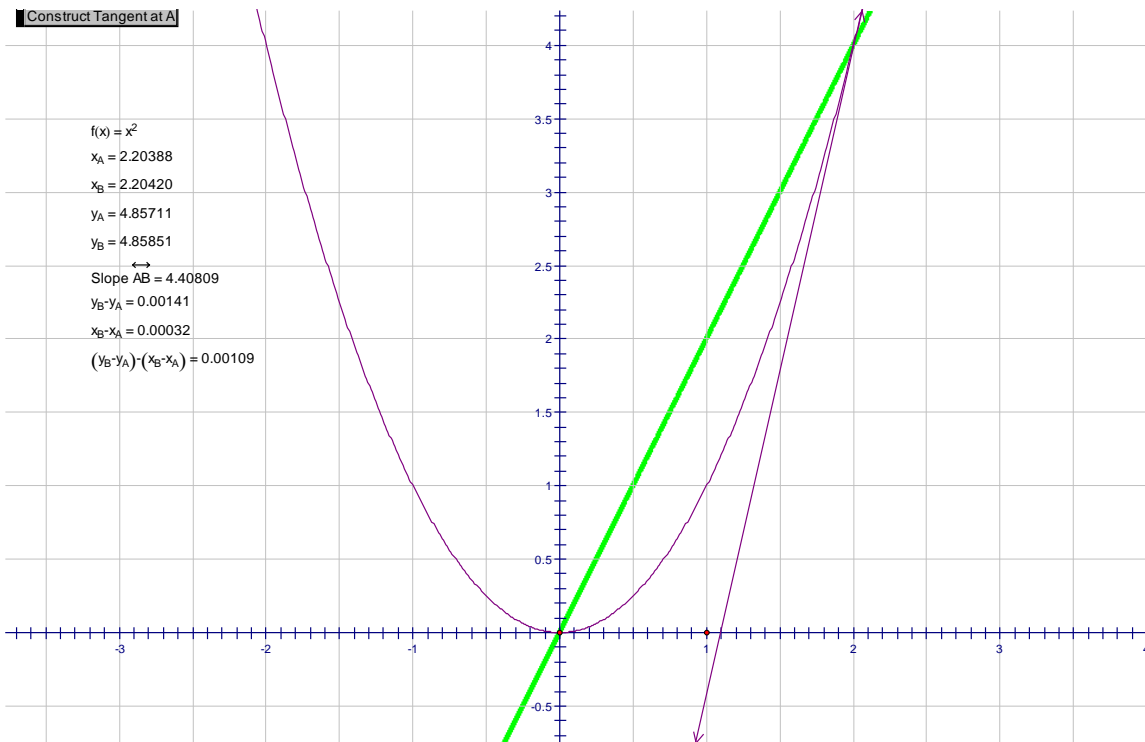
Figure B$_2$C3a: Gradient Function Plot

Step D8b: Click the values of x_A and the slope of AB. Choose **Tabulate** from the Graph menu and double click the table periodically as the animation of the Tangent Line progresses. (See Figure 3b).

$f(x) = x^2$

$x_A = 0.40481$

$x_B = 0.40546$

$y_A = 0.16387$

$y_B = 0.16440$

$\text{Slope } \overleftrightarrow{AB} = 0.81027$

Move A -> B

Animate Tangent

x_A	Slope $\overleftrightarrow{AB}$
-1.49278	-2.98520
-1.39972	-2.79908
-1.23498	-2.46960
-1.01364	-2.02694
-0.77471	-1.54906
-0.58481	-1.16928
-0.34713	-0.69392
-0.12580	-0.25125
0.01253	0.02541
0.18985	0.38005
0.33069	0.66174
0.50424	1.00883
0.75449	1.50934
0.93684	1.87403
1.10913	2.21861
1.28267	2.56570
1.51029	3.02093
1.81840	3.63714
1.99445	3.98926
0.40481	0.81027

$\overleftrightarrow{AB}$: y = 0.81x-0.16

x_A	$\overleftrightarrow{AB}$
2.13395	y = 4.27 x-4.55
1.93274	y = 3.87 x-3.74
1.72902	y = 3.46 x-2.99
1.32660	y = 2.65 x-1.76
0.93298	y = 1.87 x-0.87
0.50918	y = 1.02 x-0.26
0.29036	y = 0.58 x-0.08
0.08412	y = 0.17 x-0.01
-0.02654	y = -0.05 x-0.00
-0.20009	y = -0.40 x-0.04
-0.37499	y = -0.75 x-0.14
-0.61634	y = -1.23 x-0.38
-0.91690	y = -1.83 x-0.84
-1.04266	y = -2.08 x-1.09
-1.14704	y = -2.29 x-1.32
-1.27405	y = -2.55 x-1.62
0.40481	y = 0.81 x-0.16

Figure B$_2$C3b: The gradient function of y=x^2

Step D9: Compare the values of x$_A$ and with those of the slope on your table. What is the proportional relationship between these values?
……………………………………………………….…………………..

Step D10: Calculate the gradient of the path of C and find the equation it represents using the slope-intercept method.
a) How does this equation or function compare with your conjecture in Step 9?..
b) What does this function represent? ………………………………..
……………………………………………..…………………………..

Step D11: Choose **Erase Traces** from the Display menu.

Step D12. Select the function $f(x) = x^2$ on your sketch and choose **Derivative** from the Graph menu.
How does the given derivative compare with your answer in step 10?..

Step D13: Choose **Plot Function** from the Graph menu. How does your plotted derivative function fit with the trace of point C? (Figure…..)

Step D14: Select the tangent again and create an animation button and label it 'Animate Tangent'. (See Figure 3c)

Figure B₂C3b: The gradient function of y=x²

Within the figure:

Construct Tangent at A

Animate Tangent Line

$f(x) = x^2$ $f'(x) = 2 \cdot x$

$x_A = 0.38694$

$x_B = 0.38727$

$y_A = 0.14972$

$y_B = 0.14998$

Slope $\overleftrightarrow{AB} = 0.77421$

$y_B - y_A = 0.00026$

$x_B - x_A = 0.00033$

$\dfrac{(y_B - y_A)}{(x_B - x_A)} = 0.77421$

x_A	Slope $\overleftrightarrow{AB}$
2.08006	4.16043
2.04967	4.09965
2.00915	4.01862
1.96413	3.92858
1.91686	3.83404
1.86621	3.73274
1.75141	3.50314
1.65236	3.30505
1.56232	3.12497
1.44752	2.89536
1.36198	2.72429
1.24155	2.48343
0.38694	0.77421

Step D15: Use your Action Button in Step 14 to animate the tangent line again.(See Figure 3c)

Within the figure:

Construct Tangent at A

Animate Tangent Line

$f(x) = x^2$ $f'(x) = 2 \cdot x$

$x_A = -0.64838$

$x_B = -0.64805$

$y_A = 0.42040$

$y_B = 0.41997$

Slope $\overleftrightarrow{AB} = -1.29644$

$y_B - y_A = -0.00043$

$x_B - x_A = 0.00033$

$\dfrac{(y_B - y_A)}{(x_B - x_A)} = -1.29644$

x_A	Slope $\overleftrightarrow{AB}$
2.08006	4.16043
2.04967	4.09965
2.00915	4.01862
1.96413	3.92858
1.91686	3.83404
1.86621	3.73274
1.75141	3.50314
1.65236	3.30505
1.56232	3.12497
1.44752	2.89536
1.36198	2.72429
1.24155	2.48343
-0.64838	-1.29644

Figure B₂C3c: Comparing the trace of point C and the derivative function plot

Step D16: Does the Trace of point C confirm your conjectures/findings in Steps 10 and 12? ………………………………………………………………

Step D17: a) Expand the differential quotient $((x+h)^2 - x^2)/h$ and simplify it to obtain $2x+h$…… ………………………… ………………………..
…………………………………………………………………………..
…………………………………………………………………………..
b) What is the limit of this expression as h tends to zero?..........
………………………………………………………………………………
c) How does the limit compare with your answers in steps 10 and 12?
....

Step D18: Animate the Tangent Line and observe the smoothness of point C's movement.
a) Why does point C seem to wobble along?.................................
…………………………………………………………………………….
Discuss with a colleague……………………………………………….

Step D19: Repeat this Activity with the following functions: $f(x)=x^3$, $f(x) = x^4$, $f(x) = x^5$, $f(x) = x^6$, , $f(x) = x^1$.

a) What pattern can you observe between the coefficients and indices of a function and its derivative?......................................
………………………………………………………………………………

Step D20: Does your conjecture in step 19 work for negative and fractional indices?..

Real-World Problems

Problem D1. Let $y = x^2 + 1$
a) Find the average rate of change of y with respect to x over the interval (3,5)
b) Find the instantaneous rate of change of y with respect to x at the point $x = -4$.
c) Find the instantaneous rate of change of y with respect to x at a general point.

Problem D2 A car, initially at rest, begins to move along a straight road. The velocity increases steadily until suddenly the driver sees a concrete barrier in the road and applies the brakes sharply at time t_0. The car decelerates rapidly, but it is too late – the car crashes into the

barrier at time t_1 and instantaneously comes to rest. Sketch a position versus time curve that might represent the motion of the car.

Problem D3 A particle moves at constant velocity, what can you say about its position versus time curve?

Problem D4 The accompanying figure shows the position versus time curve for an elevator that moves upward a distance of 60 me and then discharges its passengers.

Figure Ex-4
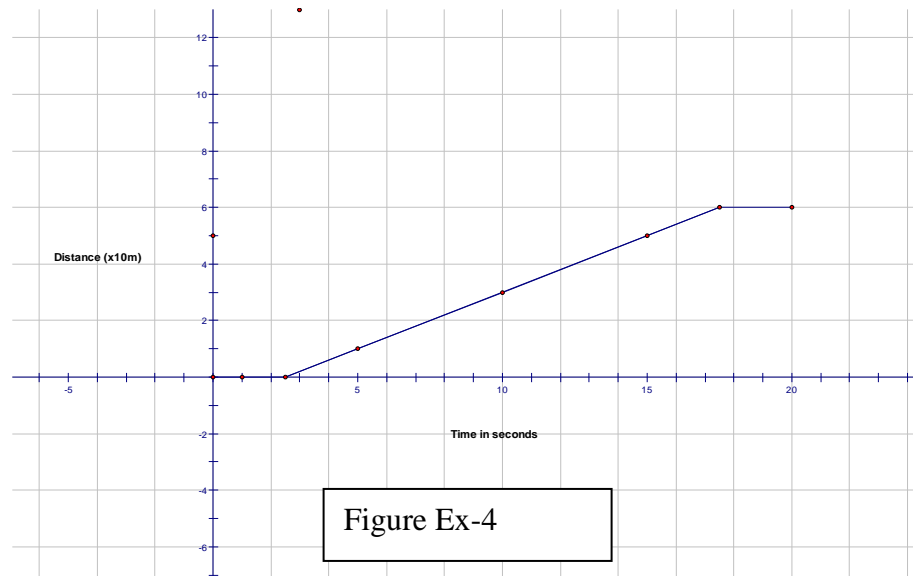
a) Estimate the instantaneous velocity of the elevator at $t = 10$ s.
b) Sketch a velocity versus time curve for the motion of the elevator for $0 \leq t \leq 20$.

Problem D5 A rock is dropped from a height of 57,6 m and falls toward Earth in a straight line. In t seconds the rock drops a distance of s = $9.8t^2$.

a) How many seconds after release does the rock hit the ground?
b) What is the average velocity of the rock during the time it is falling?
c) What is the average velocity of the rock for the first 3 s?
d) What is the instantaneous velocity of the rock when it hits the ground?

Problem D6 During the first 40s of a space shuttle flight, the Shuttle is propelled straight up so that in t seconds it reaches a height of $s = 5t^3$ m.

a) How high does the shuttle travel in 40 s?

b) What is the average velocity of the shuttle during the first 40 s?
c) What is the average velocity of the shuttle during the first 135 m of its flight.
d) What is the instantaneous velocity of the shuttle at the end of 40 s?

Problem D7 A rocket moves on a line away from its initial position of launch so that after t hours it is $s = 3t^2 + t$ kilometers from its initial position.
a) Find the average velocity of the rocket over the interval [1,3].
b) Find the instantaneous velocity at $t=1$.

Problem D8 A missile moves in a positive direction along a straight line so that after t minutes its distance is $s = 6m$ from the origin.

a) Find the average velocity of the missile over the interval [2,4].
b) Find the instantaneous velocity at $t=2$.

Problem D9 The accompanying figure shows the graph of the pressure p in atmospheres (atm) versus the volume V in litres (L) of 1 mole of an ideal gas at a constant temperature of 300 K (kelvins). Use the tangent lines shown in the figure to estimate the rate of change of pressure with respect to volume at the points where $V = 10$ L and $V = 25$ L.

Figure 3d Ex – 9.

245
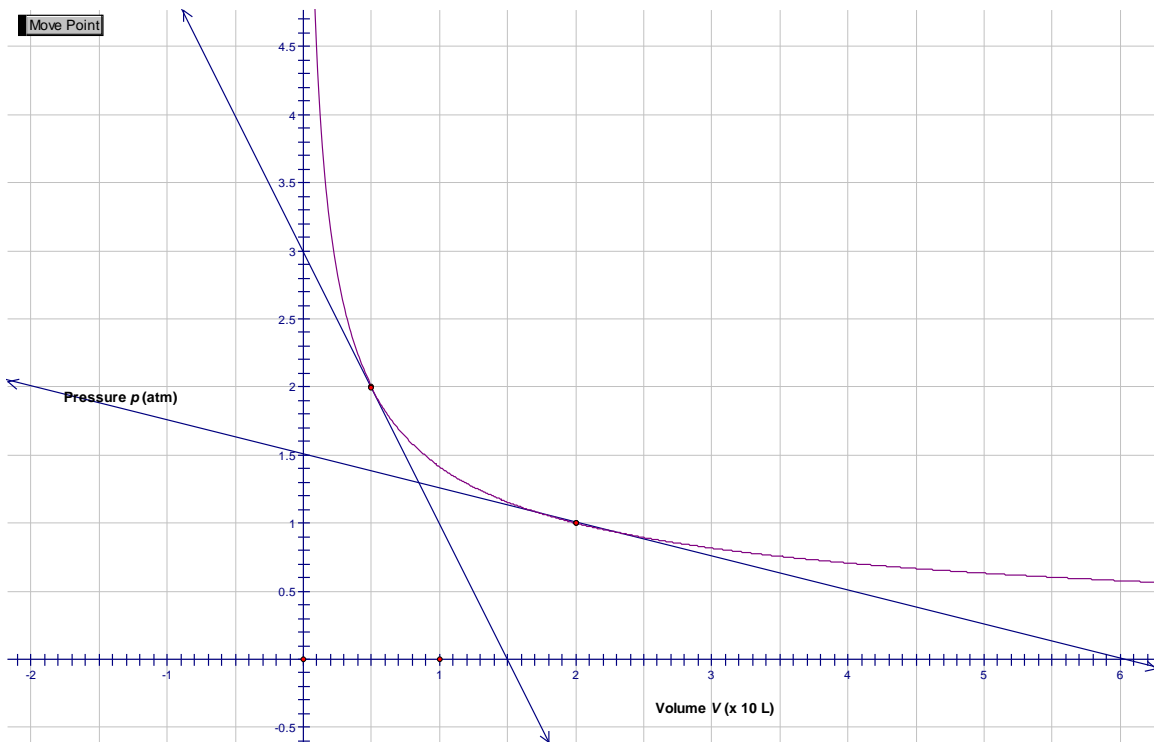
Modeling differentiability and optimization

   Definition: A function is differentiable at a point if its derivative exists at that point; a function is differentiable on an interval if it is differentiable at every point of within the interval.

Differentiability of a function at a point.

   Geometrically, the points of differentiability of f are the points where the curve y = f(x) has a tangent line, and the points of non-differentiability are the points where the curve does not have a tangent line

   Informally stated the most commonly encountered points of non-differentiability can be classified as:

   ➢ **Corners**
   ➢ **Points of vertical tangency**
   ➢ **Points of discontinuity**

   Figure B$_2$D1 illustrates each of these situations

a) Corner

b) Point of vertical tangency

c) Point of discontinuity
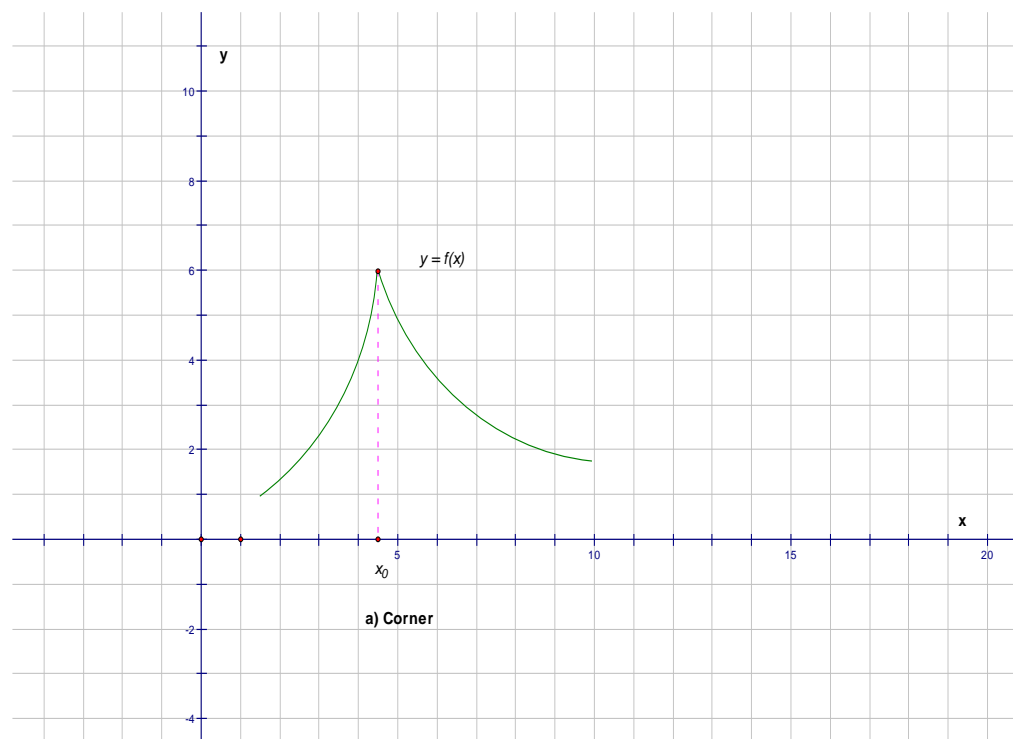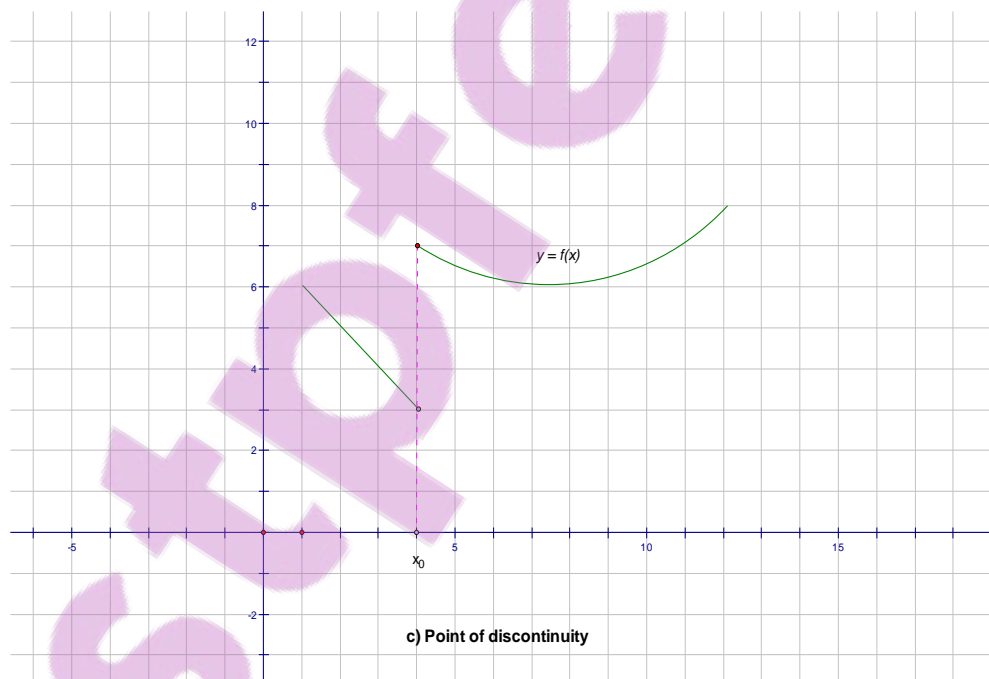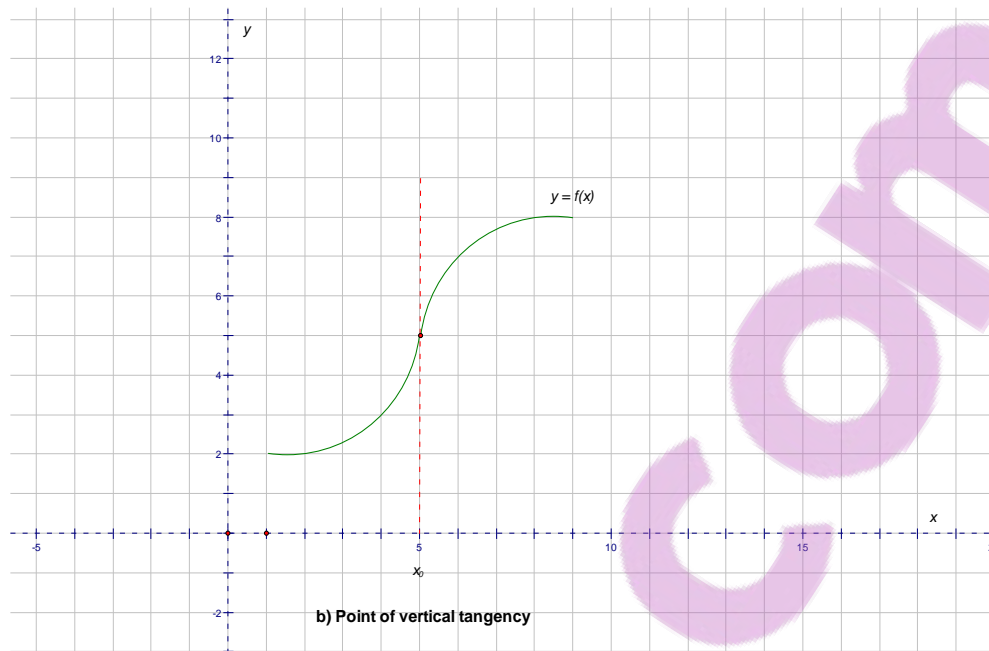
Figure B$_2$D1: Points of non-differentiability

Examples

Investigating the slope of tangents at non-differentiable points

Step E1: Open the Corner page of the Differentiability Sketch (figure B$_2$D1a) and Select the Corner point on the graph in Figure 1a.

Step E2: Choose **Label Point** from the Display menu and label the corner point P.

Step E3: Deselect point P and select part of the graph formed by the arc on the left of x_0.

Step E4: Choose **Point on Arc** from the Construct menu to construct point Q and select **Label Point** from the Display menu and label it Q.

Step E5: Select both points P and Q and choose **Line** from the Construct menu. Choose **Trace Line** from the Display menu.

Step E6: Deselect the line and select points Q and P in that order. Choose **Movement** from the **Action Buttons** sub-menu of the Edit menu.

Step E7: Use the action button in step 6 to move point Q to point P constructing a tangent at P. (See Figure B$_2$D2a)
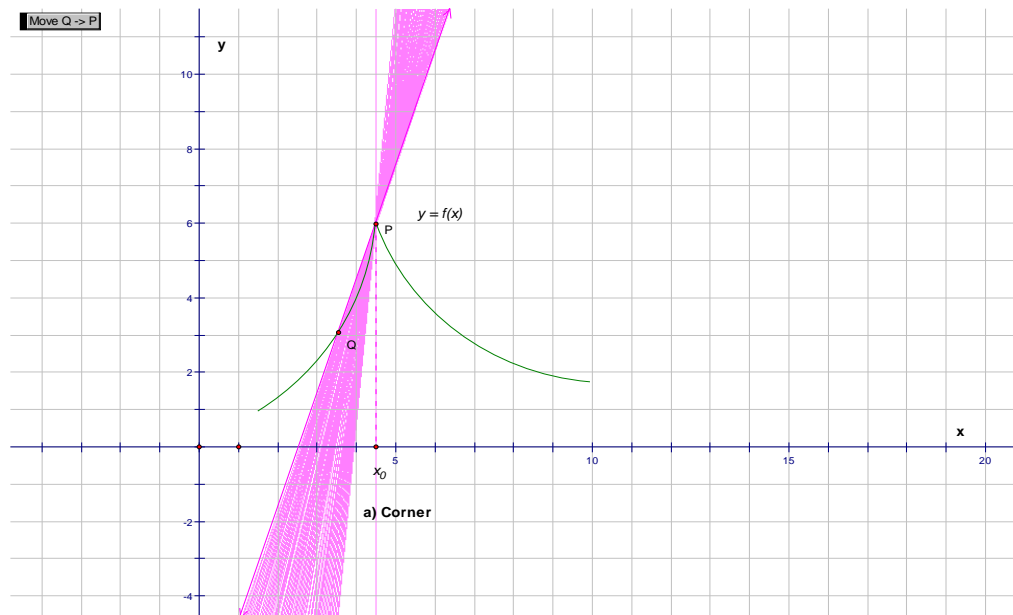
Figure B$_2$D2a: Tangent at Point P from the left

Step E8: Repeat steps 4 to 7 on the part of the graph on the right side of P. Label the point R and animate the secant line to produce the tangent from the right.(See Figure B$_2$D2b)
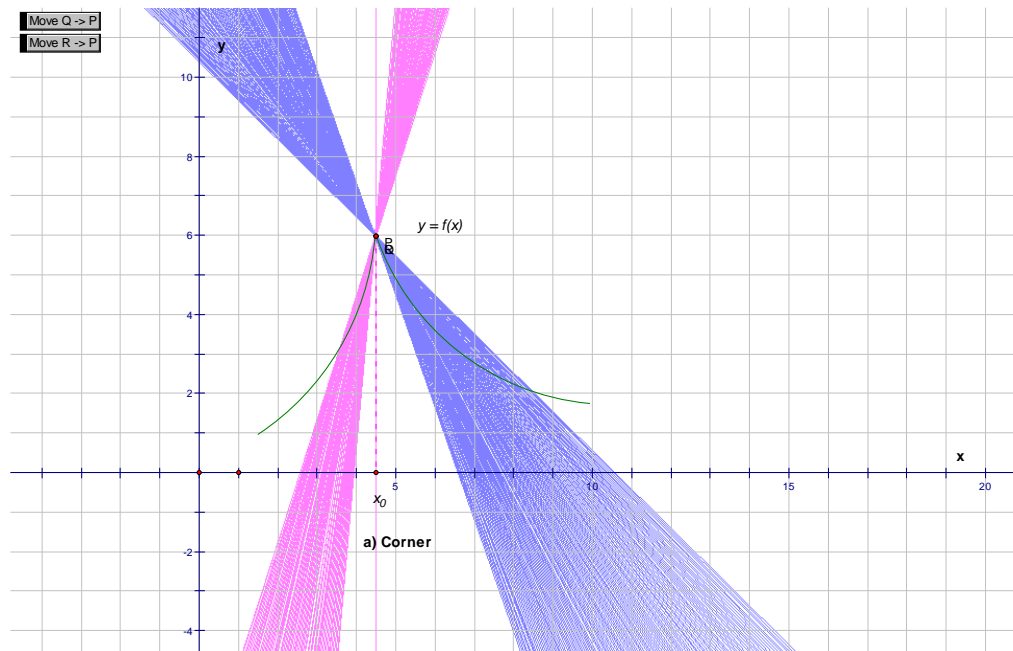
Figure B_2D2b: Tangent from the right of point P.

a) Do the left and right tangents coincide?......................Yes/No
b) Are the slopes of the tangents the same?..................Yes/No
c) What can you conclude about the left and right limits as x tends to x_0 limits?..
d) Does the derivative of this function exist at P? ………Yes/No
e) What about elsewhere?...Yes/No

Step E9: Repeat steps 4 to 7 for vertical tangency.
a) Where does the secant line terminate?................................
b) What is the slope of the tangent in this position?..................
…………………………...

b) Point of vertical tangency

b) Point of vertical tangency
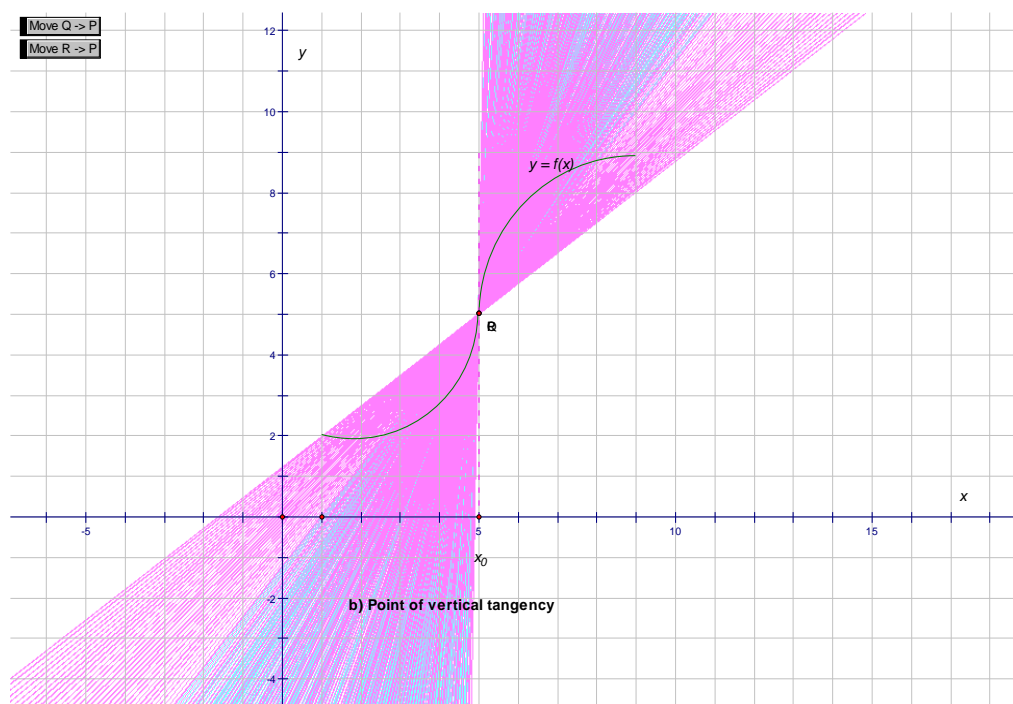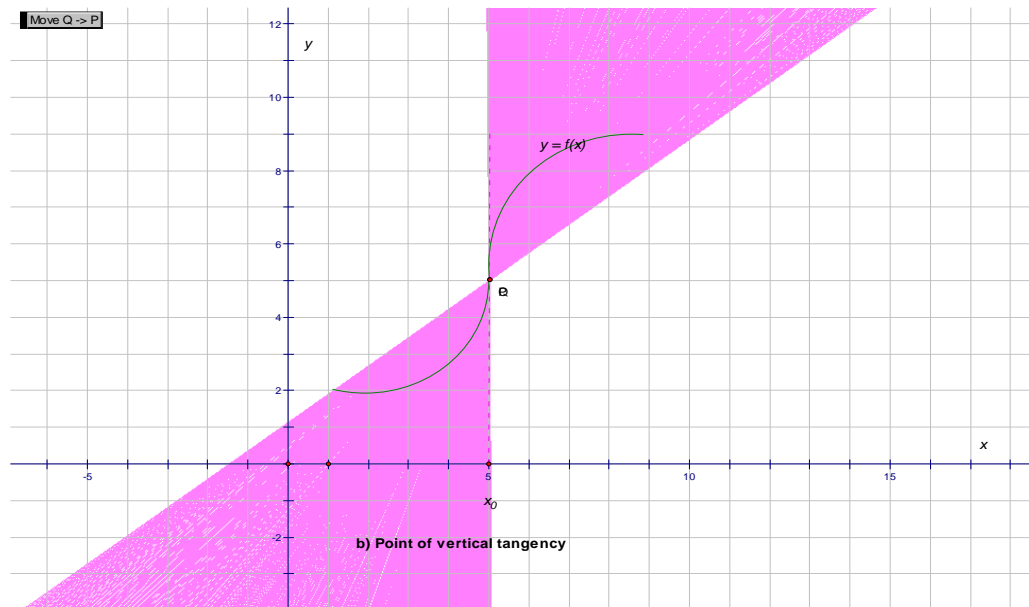
Step E10 Repeat steps 4 to 7 for point of discontinuity. Compare the slope of the tangent with that of the graph piece.

a) What can you conclude?...

b) What happens to the tangent line when Q reaches P?................
....

c) Discuss with a colleague and give a reason for your answer……
…………………………………………………………………………
…………………………………………………………………………

c) Point of discontinuity
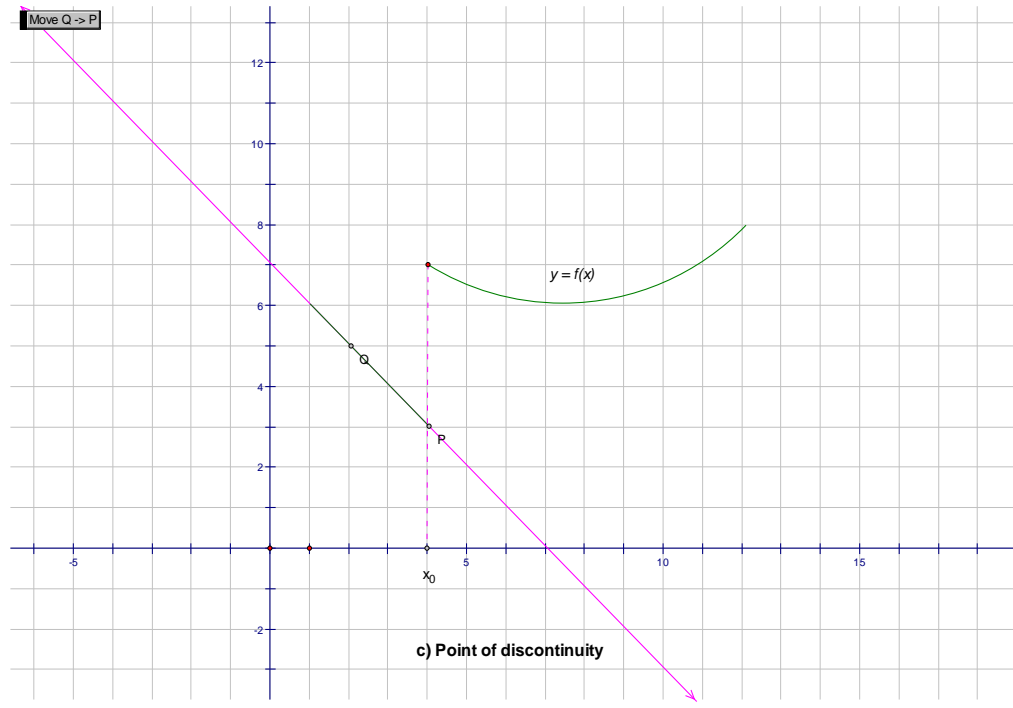
Does the graph have a tangent at point R from the right? Are the left and right tangents the same?...Yes/No
What can you conclude?..

$y = f(x)$

x_0

c) Point of discontinuity
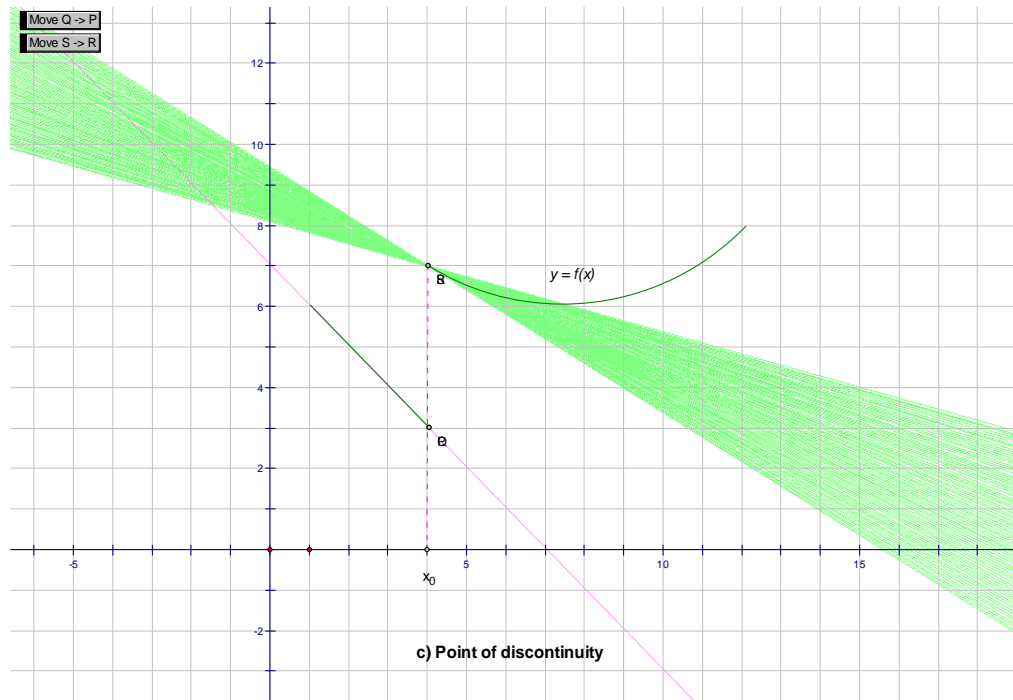
More Exercises

Draw the graphs of the following functions and investigate their differentiability at the corner point, and points of vertical tangency or discontinuity:

a) $y = |x|$, find and plot the derivative function

b) $y = |\sin x|$, use *Sketchpad* to find and plot the derivative of this function as well.

c) $y = (x^2 + x - 2)/(x-1)$

252

Activity F: More Practice in Modeling Functions and their Derivatives

Graph the following functions in *Sketchpad* and sketch their **derivatives** by hand

in the spaces provided **before** checking your answers in *Sketchpad*.

Problem E1. $y = x$, $y = x + 1$, $y = x + 5$, $y = x - 1$, $y = x - 3$,
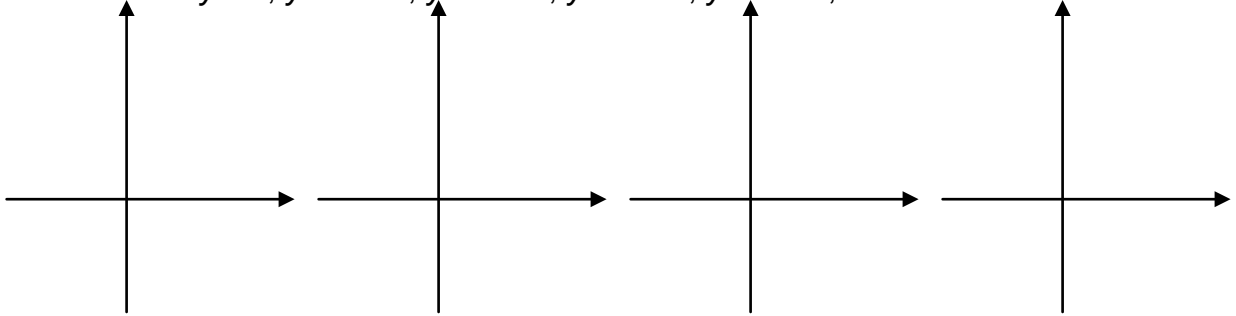
Describe what you notice and give reasons for your observations……..

………………………………………………………………………………

Problem E2. $y = x^2$, $y = x^2 + 1$, $y = x^2 + 5$, $y = x^2 -1$, $y = x^2 -5$,
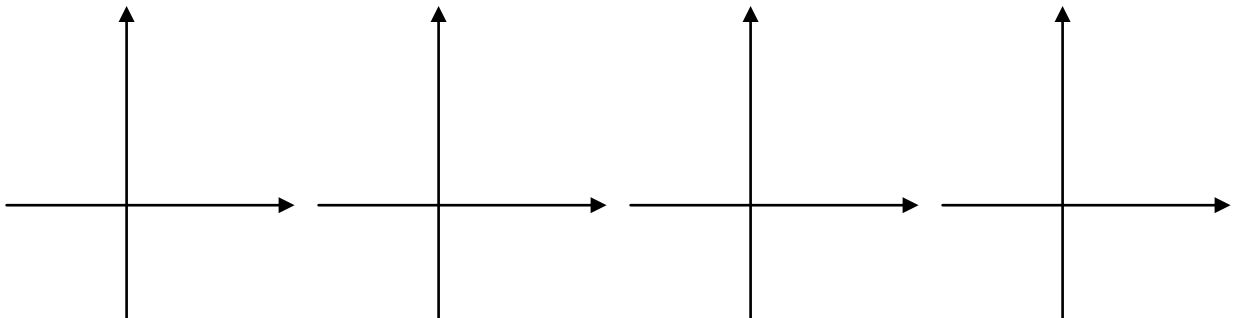
Describe what you notice and give reasons for your observations……...

…………………………………………………………………………………..

Problem E3. $y = x^2$, $y = x^2 - x$, $y = x^2 -2x$, $y = x^2 + x$, $y = x^2 + 2x$
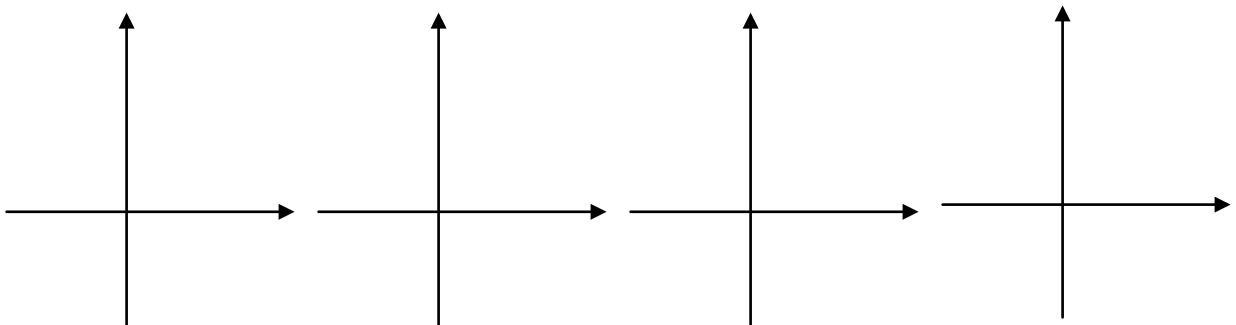
Describe what you notice and give reasons for your observations……….

……………………………….………………… ……… ..……………………… .

Problem E4. $y = x^3$, $y = x^3 - 1$, $y = x^3 + 2$, $y = x^3 + 2x$, $y = x^3 + x^2$
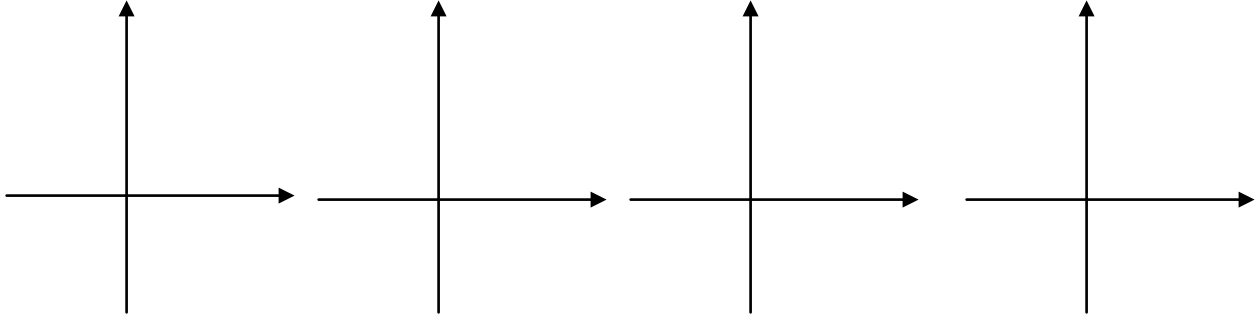
Describe what you notice and give reasons for your observations…..

……………………………………………………………………… .

Problem E5. Use *Sketchpad* to draw the graphs of the following functions. For

each of the graphs:

i) sketch the graph of the derivative on the axes provided,

ii) state whether the graph is differentiable or not at the turning point.

iii) give the value of the derivative (gradient) at the turning point, if not,

give a reason. (Use spaces provided)

a) $y = \sqrt{(|x|)}$
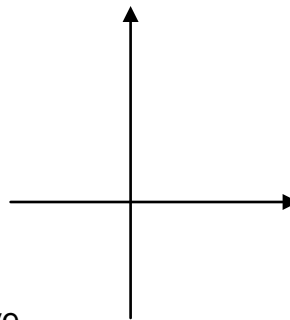
 i) graph of derivative

ii) differentiable/non-differentiable (delete the inapplicable)

iii) value of derivative at turning point (OR REASON) ………

……………………………………………………………….

b)   y = |x|
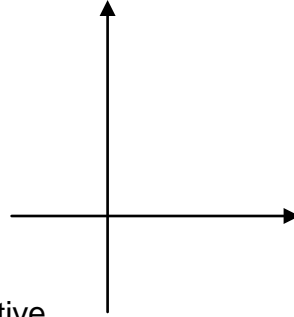
i) graph of derivative

ii) differentiable/non-differentiable (delete the inapplicable)

iii) value of derivative at turning point, OR REASON, for non-

differentiability……………………………………………………

…………………………………………………………………………..

c) y = $|x^3|$
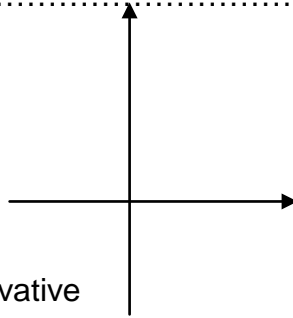
i)   graph of derivative

ii)   differentiable/non-differentiable at turning point (delete

inapplicable)

iii)   value of derivative at turning point, where applicable, OR

reason where not applicable………………………….……

………………………………………………………………….

d) y = sin$x$
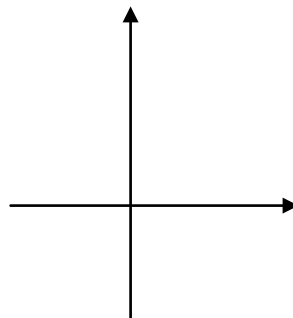
i)   graph of derivative

ii)   differentiable/non-differentiable at turning point (delete

the inapplicable)

iii)   value of derivative at turning point, where applicable, OR

REASON, where not applicable…………………………….

……………………………………………………………………

e) $y = e^x$
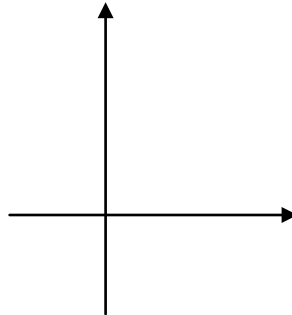
i)   graph of the derivative

ii)   differentiable/non-differentiable at turning point (delete

the inapplicable)

iii)   value of the derivative at turning point, where applicable,

OR REASON, where not applicable………..……………

……………………………………………………………….…..

Problem E6: Draw the derivatives of the following graphs on the same axes

provided.
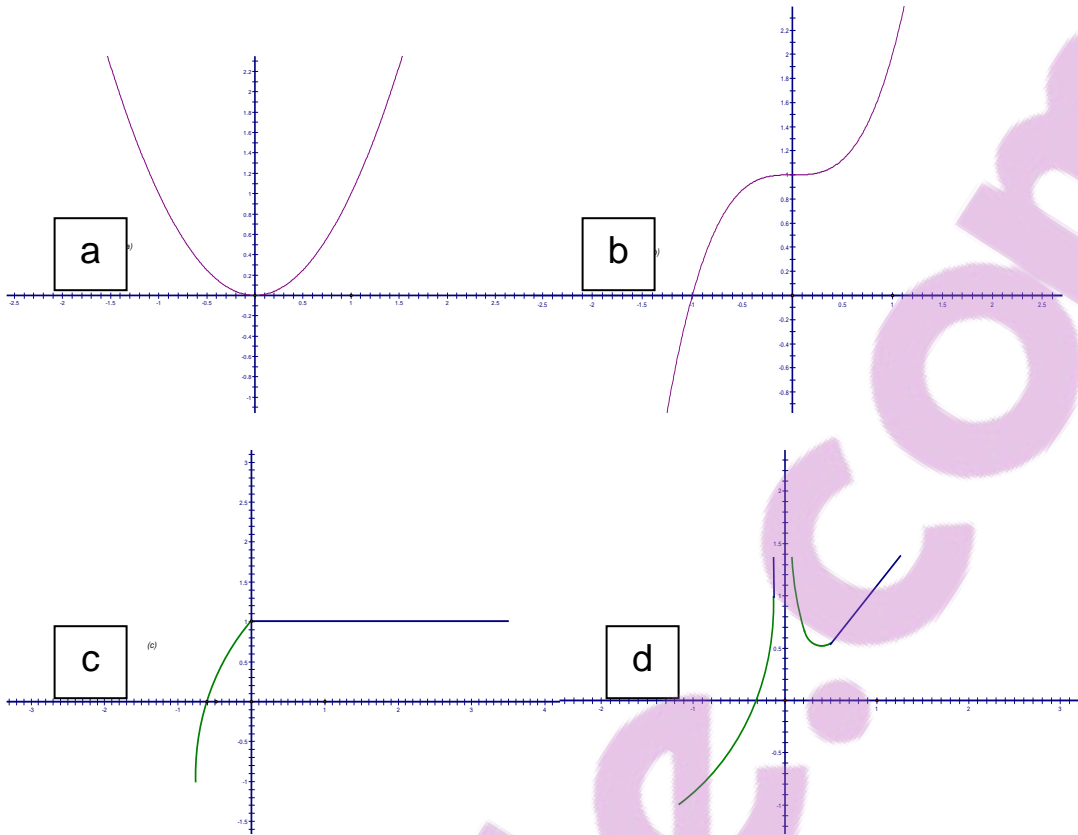
Figure B₂E1: Sketching derivatives of given functions

Appendix C:

T-test and Chi-Square calculations

<u>T-test calculation for dependent samples:</u>

$$t = \frac{\overline{D}}{\sqrt{\dfrac{\Sigma D^2 - \dfrac{(\Sigma D)^2}{N}}{N(N-1)}}} \text{ , where}$$

$\overline{D}$ is the mean difference for all pairs of scores,
ΣD^2 is the sum of the squares of the differences,
$(\Sigma D)^2$ is the square of the sum of the differences,
N is the number of pairs of scores, and
N - 1 is the degrees of freedom (one less than the number of pairs)

<u>T-test calculation for independent samples</u>

$$t = \frac{\overline{X}_1 - \overline{X}_2}{s_{\overline{x}_1 - \overline{x}_2}} \text{ , where}$$

t is the t-test statistic,
$\overline{X}_1$ is the posttest mean for the experimental group
$\overline{X}_2$ is the posttest mean for the control group and
$s_{\overline{x}_1 - \overline{x}_2}$ is the standard error of the difference in means given by

$$s_{\overline{x}_1 - \overline{x}_2} = s \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \text{ ,}$$

$$\text{where } s = \sqrt{\frac{\Sigma x_1^2 + \Sigma x_2^2}{df_1 + df_2}}$$

	EXPERIMENTAL	1	2	4	6	12	16	17	21	MEANS	∑D	∑D²	(∑D)²
Pilot Phase	PRETEST (x_1)	4	2	6	6	6	5	3	2	4.250			
	POSTTEST(x_2)	6	5	6	5	6	5	4	4	5.125	t=1.82485		
	$x_2 - x_1$	2	3	0	-1	0	0	1	2	0.875	7		49
	$(x_2 - x_1)^2$	4	9	0	1	0	0	1	4			19	
	EXPERIMENTAL	4	5	6	7	11	14			MEANS	∑D	∑D²	(∑D)²
Final Phase	PRETEST (x_1)	5	1	3	5	2	1			2.833			
	POSTTEST(x_2)	6	5	6	5	5	5			5.333	t=3.7268		
	$x_2 - x_1$	1	4	3	0	3	4			2.500	15		225
	$(x_2 - x_1)^2$	1	16	9	0	9	16					51	
	Table C1: Question 2: Gradient Concept *t*-test for dependent samples												

	EXPERIMENTAL	1	2	4	6	12	16	17		MEANS	∑x²
Pilot Phase	SCORES (x_1)	4	2	6	6	6	5	3		4.571	
	x_1^2	16	4	36	36	36	25	9			162
	CONTROL	3	5	7	10	11	13	18		t=0.0795	
	SCORES (x_2)	6	5	4	5	2	4	5		4.429	
	x_2^2	36	25	16	25	4	16	25			147
	RESPONDENT	4	5	6	7	11	14			MEANS	∑x²
Final Phase	EXP GROUP (x_1)	6	5	6	5	5	5			5.333	
	x_1^2	36	25	36	25	25	25				172
	RESPONDENT	1	8	9	12	13	15			t=1.0892	
	CONTROL (x_2)	5	5	4	2	3	2			3.500	
	x_2^2	25	25	16	4	9	4				83
	Table C2: Question 2: Gradient Concept t-test for independent samples										

	EXPERIMENTAL#	1	2	4	6	12	16	17	21	MEANS	∑D	∑D²	(∑D)²
Pilot Phase	PRETEST (x_1)	2	1	1	0	2	0	1	1	1.000			
	POSTTEST(x_2)	2	2	1	1	2	1	1	2	1.500	t=2.64575		
	$x_2 - x_1$	0	1	0	1	0	1	0	1	0.500	4		16
	$(x_2 - x_1)^2$	0	1	0	1	0	1	0	1			4	
Final Phase	EXPERIMENTAL#	4	5	6	7	11	14			MEANS	∑D	∑D²	(∑D)²
	PRETEST (x_1)	1	0	1	1	2	0			0.833			
	POSTTEST(x_2)	3	2	3	2	3	2			2.500	t=7.9057		
	$x_2 - x_1$	2	2	2	1	1	2			1.667	10		100
	$(x_2 - x_1)^2$	4	4	4	1	1	4					18	

Table C3: Question 3: T-test for dependent samples: Tangent Gradient as Limit

	EXPERIMENTAL #	1	2	4	6	12	16	17	MEANS	∑x²
Pilot Phase	SCORES (x_1)	2	2	1	1	2	1	1	1.429	
	x_1^2	4	4	1	1	4	1	1		16
	CONTROL #	3	5	7	10	11	13	18	t=0.6673	
	SCORES (x_2)	0	1	1	1	0	2	1	0.857	
	x_2^2	0	1	1	1	0	4	1		8
Final Phase	EXPERIMENTAL #	4	5	6	7	11	14		MEANS	∑x²
	EXP GROUP (x_1)	3	2	3	2	3	2		2.500	
	x_1^2	9	4	9	4	9	4			39
	CONTROL #	1	8	9	12	13	15		t=1.0325	
	CONTROL (x_2)	1	2	2	1	1	0		1.167	
	x_2^2	1	4	4	1	1	0			11

Table C4: Question 3: T-test for Independent Samples: Tangent Gradient as Limit

Question 4a) Graph of $y = x^2 - x$	Gradient at x = 0		Total
	Yes	No	
Experimental (N=8)	**8**	0	8
Control (N = 7)	**5**	2	7
Raw Totals	**13**	2	15

CALCULATION OF THE CHI- SQUARE VALUE

$X^2 = X_1^2 + X_2^2 + X_3^2 + X_4^2 = 0{,}1632 + 1{,}064 + 0{,}1890 + 1{,}2321 = 2{,}648$.

Degrees of freedom = (2-1)(2-1) = 1.

Table C5: Differentiability Chi-square test for posttest responses to the graph of $y = x^2 - x$

Question 4b) Differentiability of the Graph of $y = |x|$	Gradient at x = 0						Total
	Yes				**No**	nr	
	0	1	±1	other			
Experimental (N=8)	3	0	2	0	**3**	0	8
Control (N = 7)	0	1	1	0	**6**	0	7
Raw Totals	3	1	2	0	**9**	0	15

CALCULAITON OF THE CHI-SQUARE

$X^2 = X_1^2 + X_2^2 + X_3^2 + X_4^2 = 1{,}0125 + 0{,}0427 + 1{,}15586 + 0{,}77057 = 2{,}9432$.

Degrees of freedom = (2-1)(2-1) = 1.

Table C6: Differentiability Chi-square test for posttest responses to the graph of $y = |x|$

Question 4c) Differentiability of the Graph of $y = \sqrt{(|x|)}$	Gradient at x = 0									Total
	Yes					No				
	± ½	± 2	± 3	Other	Total	**U**	TP	Other	Total	
Experimental (N=8)	1	1	1	1	4	**3**	0	1	4	8
Control (N = 7)	0	0	0	1	1	**3**	2	1	6	7
Raw Totals	1	1	2	0	5	**6**	2	2	10	15

CALCULATION OF THE CHI-SQUARE

$X^2 = X_1^2 + X_2^2 + X_3^2 + X_4^2 = 0{,}6667 + 0{,}3333 + 0{,}7606 + 0{,}3802 = 2{,}1411$.

Degrees of freedom = (2-1)(2-1) = 1. <u>Key</u>: U = undefined , TP = turning point

Table C7: Differentiability Chi-square test for posttest responses to the graph of $y = \sqrt{(|x|)}$

Question 4d) Differentiability of the graph of $y = |x^3|$	Gradient at $x = 0$									Total
	Yes				No					
	0	U	Other	Total	Any	SL	MT		Total	
Experimental (N=8)	1	1	1	1	4	3	0	1	4	8
Control (N = 7)	0	0	0	1	1	3	2	1	6	7
Raw Totals	1	1	2	0	5	6	2	2	10	15

CALCULATION OF THE CHI-SQUARE

$$X^2 = X_1{}^2 + X_2{}^2 + X_3{}^2 + X_4{}^2 = 0,0833 + 0,16667 + 0,0962 + 0,1923 = 0,5385.$$

Degrees of freedom = (2-1)(2-1) = 1. Key: U = undefined , TP = turning point

Table C8: Differentiability Chi-square test for posttest responses to the graph of $y = |x^3|$

	EXPERIMENTAL	1	2	4	6	12	16	17	21	MEAN	$\sum$D	$\sum$D^2	$(\sum$D$)^2$
Pilot Posttest	PRETEST (x_1)	6	4	6	1	3	4	2	3	3.625			
	POSTTEST(x_2)	6	4	6	4	6	3	7	4	5.000		t=1.8825	
	x_2 - x_1	0	0	0	3	3	-1	5	1	1.375	11		121
	$(x_2 - x_1)^2$	0	0	0	9	9	1	25	1			45	
	EXPERIMENTAL	4	5	6	7	11	14			MEAN	$\sum$D	$\sum$D^2	$(\sum$D$)^2$
Final Posttest	PRETEST (x_1)	0	2	5	3	6	2			3.000			
	POSTTEST(x_2)	8	7	8	6	6	6			6.833		t=3.5571	
	x_2 - x_1	8	5	3	3	0	4			3.833	23		529
	$(x_2 - x_1)^2$	64	25	9	9	0	16					123	

Table C9: Question 4: Differentiability *t*-test for responses for dependent samples

	EXPERIMENTAL	1	2	4	6	12	16	17	21	MEANS	$\sum x^2$
Pilot Phase	SCORES (x_1)	6	4	6	4	6	3	7	4	5.000	
	x_1^2	36	16	36	16	36	9	49	16		214
	CONTROL #	**3**	**5**	**7**	**10**	**11**	**13**	**18**		t=0.4588	
	SCORES (x_2)	1	2	4	6	3	5	5		3.714	
	x_2^2	1	4	16	36	9	25	25			116
Final Phase	EXPERIMENTAL	**4**	**5**	**6**	**7**	**11**	**14**			MEANS	$\sum x^2$
	EXP GROUP (x_1)	8	7	8	6	6	6			6.833	
	x_1^2	64	49	64	36	36	36				285
	CONTROL #	**1**	**8**	**9**	**12**	**13**	**15**			t=1.5500	
	CONTROL (x_2)	1	5	2	3	4	0			2.500	
	x_2^2	1	25	4	9	16	0				55
Table C10: Question 4:Differentiability *t*-test for responses for independent samples											

The graph of $y = x^3$ and its derivative (Q.5)	**Correct**	Wrong					Total
	G3	G2	G4	nr	Total		
Experimental (N=8)	**7**	0	1	0	1		8
Control (N = 7)	**5**	0	1	1	2		7
Raw Totals	**12**	0	2	1	3		15

CALCULATION OF THE CHI-SQUARE
$X^2 = X_1^2 + X_2^2 + X_3^2 + X_4^2 = 0,05625 + 0,225 + 0,0647 + 0,2589 = 0,6048$.
Degrees of freedom = (2-1)(2-1) = 1.
Key: G2 = graph 2, G3 = graph 3, G4 = graph 4
Table C11: Question 5 posttest responses to the graph of $y = x^3$ and its derivative

	EXPERIMENTAL	4	5	6	7	11	14	MEANS	$\sum D$	$\sum D^2$	$(\sum D)^2$
Final Posttest	PRETEST (x_1)	6	5	4	8	6	4	5.500			
	POSTTEST(x_2)	13	10	18	13	14	14	13.667		t=6.0461	
	$x_2 - x_1$	7	5	14	5	8	10	8.167	49		2401
	$(x_2 - x_1)^2$	49	25	196	25	64	100			459	
Table C12: Comparison of dependent final phase samples for the Bottle Problem											

Pilot Phase Posttest	EXPERIMENTAL	1	2	4	6	12	16	17	21	MEANS	$\sum x^2$
	SCORES (x_1)	11	10	16	6	6	20	17	7	11.625	
	x_1^2	121	100	256	36	36	400	289	49		1287
	CONTROL	3	5	7	10	11	13	18		t=0.64113	
	SCORES (x_2)	11	4	9	5	13	10	0		7.429	
	x_2^2	121	16	81	25	169	100	0			512
Final Phase Posttest	EXPERIMENTAL	4	5	6	7	11	14			MEANS	$\sum x^2$
	EXP GROUP (x_1)	13	10	18	13	14	14			13.667	
	x_1^2	169	100	324	169	196	196				1154
	CONTROL	1	8	9	12	13	15			t=1.8433	
	CONTROL (x_2)	0	0	4	4	4	0			2.000	
	x_2^2	0	0	16	16	16	0				48
Table C13: Comparison of Independent Samples for the Bottle Problem											