Introduction

As the title suggests, the primary concern of this thesis is the study of two-sided residuation in the lattice of right topologizing filters on a ring.

A lattice ordered monoid is a monoid endowed with a lattice order that is compatible with the monoid operation in the sense that the latter distributes over finite joins. The systematic study of these structures, which has its origins in the 1930s paper of Ward and Dilworth [41], [9, 8] (see also [4]), was motivated by a classical prototype, the set of ideals of a commutative ring with identity. In this instance, the lattice structure derives from the usual partial order: meets correspond with intersections, joins with ideal sums, and the monoid operation with ideal multiplication. The lattice ordered monoid structure on the set of ideals of a ring, which we shall denote by $\mathrm{Id}\, R$, is not particular to commutative rings, however. Indeed, $\mathrm{Id}\, R$ is a lattice ordered monoid for all rings R (with identity). A key difference in the general case is that the monoid operation of ideal multiplication need not be commutative. The lattice ordered monoid $\mathrm{Id}\, R$ has the further desirable property that it is right and left residuated. This is to say, if I and J are ideals of R, then there is (a unique) largest ideal K of R, given by $K = \{r \in R : Jr \subseteq I\}$, such that $JK \subseteq I$. In this situation K is called the right residual of I by J and is denoted $J^{-1}I$. The left residual of I by J, denoted IJ^{-1}, is the largest ideal K of R such that $KJ \subseteq I$ and comprises $\{r \in R : rJ \subseteq I\}$.

Given its commutative ring theoretic antecedents, early work on lattice ordered monoids focused on the development of a commutative residuated theory. Residuated structures have since found application in other areas of mathematics, in particular, in the algebra of binary relations and the model theory of nonclassical logics.

Residuated lattice ordered monoids also arise in torsion theory, for the set of right topologizing filters $\mathrm{Fil}\, R_R$ on an arbitrary ring R (or equivalently, the collection of all hereditary pretorsion classes of right R-modules) is the order dual of a lattice ordered monoid. It is the study of lattice ordered monoids in this torsion theoretic context that shall be the setting of this thesis.

An important avenue of research in torsion theory explores the extent to which internal properties of a ring R and its category of right R-modules, are encoded in the structure $\operatorname{Fil} R_R$. The usefulness of $\operatorname{Fil} R$ as a tool for analysing R, lies in the fact that the former structure encodes at least as much information about the ring R as does the ideal lattice $\operatorname{Id} R$, for there is a canonical structure preserving embedding (that is in general not onto) of $\operatorname{Id} R$ into $[\operatorname{Fil} R_R]^{\mathrm{du}}$ that takes each $I \in \operatorname{Id} R$ onto the set of all right ideals of R containing I. A Condition placed on the larger structure $\operatorname{Fil} R_R$ imposes more stringent requirements on the underlying ring R than does the equivalent condition on the smaller $\operatorname{Id} R$. For example, a ring R for which $\operatorname{Id} R$ is trivial is, by definition, a simple ring. However, a ring R for which $\operatorname{Fil} R_R$ is trivial is much more than simple, it is necessarily simple artinian and thus isomorphic to a matrix ring over a division ring.

The point of departure for the investigation undertaken in this thesis, is the question: what rings R have the property that $[\operatorname{Fil} R_R]^{\mathrm{du}}$ is two-sided residuated, that is to say, is both left and right residuated? As results in this thesis show, the two-sided residuation condition manifests as a form of finiteness condition on the ring. For commutative rings R, a sufficient condition for $[\operatorname{Fil} R_R]^{\mathrm{du}}$ to be two-sided residuated is that R be noetherian. However, as results in Chapter 3 show, noetherianness is by no means necessary for two-sided residuation. The situation is less clear in the case of noncommutative rings. It is not known, for example, whether $[\operatorname{Fil} R_R]^{\mathrm{du}}$ is two-sided residuated in every (noncommutative) right noetherian ring R.

A detailed breakdown of the main results in Chapters 2 to 4 is provided in the abstract. Chapter 1 is introductory whilst Chapter 5 lists a number of problems for planned future research. We wish to point out that the paper [37] is a synthesis of results from Chapter 2, the first section of Chapter 3 and part of Chapter 4.

Convention on the numbering of results

Theorem 2.3 refers to Theorem 3 in Chapter 2. Theorems, Propositions, Lemmas, Corollaries, Examples and Remarks are numbered sequentially using the same counter.

Contents

Chapter 1

Preliminaries

The aims of this section are threefold, in the first place to describe the notational conventions used throughout this dissertation and secondly to collect in readily usable form a selection of mostly standard results from the theories of lattice ordered monoid, hereditary pretorsion classes, right topologizing filters and left residuation of right topologizing filters on rings which shall be used in the sequel. In the third place we include some new results with proofs which shall be used in proving some results in Chapter 3.

Except new results in this section we have chosen not to provide proofs for most standard results rather we provide reference from such standard books [31, 18, 17].

All of the Theorems, Propositions, Lemmas and Corollaries in this dissertation that have not been explicitly cited from another source represent original work.

1.1 Set theoretic conventions

The symbol $\subseteq$ denotes containment and $\subset$ proper containment for sets.

1.2 Rings and modules

Throughout this thesis R will denote an associative ring (not necessary commutative) with identity. By an ideal we always mean a two-sided ideal and the set of all two-sided ideals of a ring R will be denoted by $\operatorname{Id} R$. If I is an ideal of R we write $I \trianglelefteq R$.

The word module will mean a unitary right R-module and the category of unital right R-modules shall be denoted by $\text{Mod-}R$. We shall use R_R [resp. ${}_RR$] to denote the ring R considered as a right [resp. left] module over itself. If $N, M \in \text{Mod-}R$ we use the symbol $N \leq M$ to mean N is a submodule of M and $N \hookrightarrow M$ to mean N is embedded in M. We shall, on occasion, use the symbol $\mathcal{L}(M)$ to indicate the lattice of submodules of a right R-module M.

If X, Y are nonempty subsets of a right R-module M we define

$$Y^{-1}X = \{r \in R : Yr \subseteq X\}.$$

If $Y = \{y\}$ [resp. $X = \{x\}$] is a singleton we write $y^{-1}X$ [resp. $Y^{-1}x$] in place of $\{y\}^{-1}X$ [resp. $Y^{-1}\{x\}$].

If $\{A_i : i \in I\}$ is a family of abelian groups we write $\bigoplus_{i\in I} A_i$ for the direct sum and $\prod_{i\in I} A_i$ for the direct product where I is any index set. If there is no ambiguity and the index set I is understood, we simply write $\bigoplus A_i$ and $\prod A_i$ respectively. If $A_i = A$ for all $i \in I$, we write $A^{(I)}$ in place of $\bigoplus_{i\in I} A_i$ and A^I for $\prod_{i\in I} A_i$. If the indexed set $I = \{1, 2, \ldots, n\}$ is finite we shall denote the direct sum [resp. direct product] by $\bigoplus_{i=1}^n A_i$ [resp. $\prod_{i=1}^n A_i$].

If $M, N \in \text{Mod-}R$ we denote by $\operatorname{Hom}_R(M, N)$ the additive abelian group of all R-homomorphisms from M to N. In this situation, if X is any nonempty subset of M, we shall denote by $f[X]$ the image of X under f.

A right R-module M is simple if it has no proper nonzero submodules. The socle of an arbitrary right R-module M, denoted $\operatorname{soc}(M)$, is defined to be the sum of all simple submodules of M. We call M semisimple if $\operatorname{soc}(M) = M$. A ring R is simple semisimple if R_R is simple, or equivalently, ${}_RR$ is simple.

Recall that a right R-module M is said to be artinian [resp. noetherian] if its lattice of submodules satisfies the Descending Chain Condition (henceforth to be abbreviated DCC) [resp. Ascending Chain Condition (henceforth to be abbreviated ACC)]. We call the ring R right artinian [resp. right noetherian] if the module R_R is artinian [resp. noetherian].

A right R-module M is said to be an essential extension of a right R-module N if $N \leq M$ and every nonzero submodule of M intersects N nontrivially. In this situation we call N an essential submodule of M and write $N \leqslant_e M$. Note that M is said to be a maximal essential extension of N if M is an essential extension of N, and M has no proper essential extension.

A right R-module M is said to be N-injective with $N \in \text{Mod-}R$, if every right R-homomorphism from a submodule L of N to M can be lifted to an R-homomorphism from N to M. A right R-module M is called an injective module if M is N-injective for every $N \in \text{Mod-}R$. For every right R-module M there exists a minimal injective module containing M called the injective hull (envelope) of M. The injective hull of M is unique up to isomorphism and is denoted $E(M)$; it is furthermore a maximal essential extension of M.

Recall that an ideal P of a ring R is said to be a prime ideal of R, if $P \supseteq IJ$ implies $P \supseteq I$ or $P \supseteq J$ whenever $I, J \in \operatorname{Id} R$. We call the set of all prime ideals of a ring R, the spectrum of R and abbreviate it by $\operatorname{Spec} R$.
We call a ring R prime if the zero ideal of R is a prime ideal of R, this can be shown to be equivalent to the requirement $\forall a, b \in R$, $aRb = 0 \Rightarrow a = 0$ or $b = 0$ (see[7, p. 442-443]).

The intersection of all prime ideals of a ring R is called the prime radical of R, and is denoted by $\operatorname{rad} R$. If R is commutative, then $\operatorname{rad} R$ comprises the set of all nilpotent elements of R (see[7, p. 442-443]).

We call an ideal of a ring semiprime if and only if it is the intersection of prime ideals of the ring. We call a ring semiprime if the zero ideal is semiprime ideal. An equivalent definition for semiprime ideal is that, an ideal I in a ring R is called semiprime ideal if R/I is a semiprime ring. It is known that if I is an ideal of a ring R such that $I \subseteq \operatorname{rad} R$, then $\operatorname{rad}(R/I) = (\operatorname{rad} R)/I$ (see [7, Proposition

4, p.342]). Taking $I = \mathrm{rad}\, R$, we see that $\mathrm{rad}\,(R/\mathrm{rad}\, R) = 0$, thus $R/\mathrm{rad}\, R$ is semiprime ring.

1.3 Lattice ordered monoids

Lattices first arose in the work of Schröder and later Dedekind in 1890. Distributive and Boolean lattices in particular, arise naturally in many mathematical contexts. An elementary lattice theory is developed by Schröder in his book die Algebra der Logik. Dedekind, Birkhoff and Ore are other pioneers in the development of this subject.

By a *partially ordered set* (abbreviated poset) $\langle P, \leq \rangle$, we mean a nonempty set P together with a binary reflexive, antisymmetric and transitive relation $\leq$. We call $\leq$ the partial order (or just the order) on poset P.

If $\langle P, \leq \rangle$ is a poset, we shall denote by $\langle P, \leq \rangle^{\mathrm{du}}$ the order dual of $\langle P, \leq \rangle$. Recall that the partial order $\leq^{\mathrm{du}}$ on $\langle P, \leq \rangle^{\mathrm{du}}$ is defined by $x \leq^{\mathrm{du}} y$ if and only if $x \geq y$ for $x, y \in P$.

A *poset* $\langle P, \leq \rangle$ is said to be a *chain (or linearly ordered)* if $x \leq y$ or $y \leq x$ for all $x, y \in P$.

A lattice L is a partially ordered set in which every pair of elements a, b in L has both a greatest lower bound denoted $a \wedge b$, and a least upper bound denoted $a \vee b$. A lattice L is called bounded if L contains both a bottom element, usually denoted 0_L, and a top element, usually denoted 1_L. We call L a complete lattice if every nonempty (possibly infinite) subset X of L has a greatest lower bound, denoted $\bigwedge X$, and a least upper bound, denoted $\bigvee X$. Observe that a complete lattice is always bounded. It is clear that every chain is a lattice.

If L and L' are lattices, then a mapping $f : L \to L'$ is called a *lattice homomorphism* if f preserves (finite) meets and joins, that is to say, $f(a \wedge b) = f(a) \wedge f(b)$ and $f(a \vee b) = f(a) \vee f(b)$ for all $a, b \in L$. If L and L' are complete lattices, then we call f a complete lattice homomorphism if f preserves arbitrary meets and joins, that is to say, $f(\bigwedge X) = \bigwedge_{x \in X} f(x)$ and $f(\bigvee X) = \bigvee_{x \in X} f(x)$ for all nonempty subsets X of L.

An element c of a complete lattice L is said to be compact if $c \leq \bigvee X$, with X any nonempty subset of L, implies $c \leq \bigvee Y$ for some finite subset Y of X. A lattice is called algebraic (or compactly generated) if each of its elements is the join of compact elements. A nonzero element a in a bounded lattice L is said to be an atom [resp. coatom] if $b < a$ [resp. $b > a$] implies $b = 0_L$ [resp. $b = 1_L$]. A (bounded) lattice L is said to be atomic [resp. coatomic] if, for every $b \in L$, $b > 0_L$ [resp. $b < 1_L$], there exists an atom [resp. coatom] a such that $a \leq b$ [resp. $a \geq b$].

A lattice ordered monoid is a structure $\langle L; \leq, \cdot, e_L \rangle$ where:

(L1) $\langle L; \leq \rangle$ is a lattice.

(L2) $\langle L; \cdot, e_L \rangle$ is a monoid with identity element e_L.

(L3) The binary operation $\cdot$ respects the lattice $'\vee'$ join operation in the following sense:

$$a \cdot (b \vee c) = (a \cdot b) \vee (a \cdot c) \text{ and } (b \vee c) \cdot a = (b \cdot a) \vee (c \cdot a) \text{ for all } a, b, c \in L.$$

In the interests of brevity, we shall refer to L as a lattice ordered monoid in cases where the monoid and the lattice operations are understood and no ambiguity might arise from their suppression in the notation. Note that (L3) entails $\cdot$ will order preserving, that is, $a \leq b$ implies $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$. If L has a top element 1_L and this coincides with the monoid identity e_L, we say that the lattice ordered monoid is integral.

We say that a lattice ordered monoid L is *left residuated* if for every $a, b \in L$, there exists a largest $x \in L$ such that $x \cdot b \leq a$. In this situation we call x the *left residual* of a by b and denote it by ab^{-1}. Similarly, we say that L is *right residuated* if for every $a, b \in L$, there exists a largest element $x \in L$ such that $b.x \leq a$, called the *right residual* of a by b and denoted by $b^{-1}a$.

The following result is ring theoretic folklore.

Proposition 1.1 *Let R be any ring with identity. Then $\langle \mathrm{Id}\, R; +, \cap, \cdot, R \rangle$ is a complete, lattice ordered, left and right residuated, integral monoid, where the join is the operation $+$ of ideal addition, the meet is intersection $\cap$, and $\cdot$ is the monoid operation of ideal multiplication.*

Note that the ring R itself is the identity element with respect to ideal multiplication; it is also the largest member of $\mathrm{Id}\,R$ which explains why the lattice ordered monoid is integral. If $I, J \in \mathrm{Id}\,R$, then $IJ^{-1} = \{x \in R : xJ \subseteq I\}$ is the *left residual of I by J*, and $J^{-1}I = \{x \in R : Jx \subseteq I\}$ is the *right residual of I by J*.

Proposition 1.2 *[35, Proposition 5, p. 429] The following statements are equivalent for a complete lattice ordered monoid L:*

(a) L is right residuated;

(b) $a \cdot (\bigvee X) = \bigvee_{x \in X}(a \cdot x)$ for all $a \in L$ and $X \subseteq L$.

Remark 1.3 *It follows from the above that every lattice ordered monoid L satisfying the ACC is two-sided residuated. This may be inferred from Statement (b) of the previous proposition. Indeed, if L is a lattice ordered monoid satisfying the ACC and $X \subseteq L$, then $\bigvee X = \bigvee X'$ for some finite subset X' of X, but in every lattice ordered monoid, the monoid operation $\cdot$ distributes over finite meets.*

The following result sharpens [17, Proposition 4.15, p. 51].

Proposition 1.4 *Let L be a complete lattice ordered monoid, $a \in L$ and Y a nonempty subset of L. If the right residual $a^{-1}b$ exists for all $b \in Y$, then the right residual $a^{-1}(\bigwedge Y)$ exists and $a^{-1}(\bigwedge Y) = \bigwedge\{a^{-1}b : b \in Y\}$.*

Proof. Fix a in L and suppose $a^{-1}b$ exists for all $b \in Y$. Put $x_b = a^{-1}b$ for each $b \in Y$. Note that $a \cdot (\bigwedge_{b \in Y} x_b) \leq a \cdot x_b \leq b$ for all $b \in Y$, whence $a \cdot (\bigwedge_{b \in Y} x_b) \leq \bigwedge Y$. Now suppose $a \cdot c \leq \bigwedge Y$ for some $c \in L$. Then $a \cdot c \leq b$ for all $b \in Y$. Since $x_b = a^{-1}b$ is the largest element in L satisfying $a \cdot x_b \leq b$, we must have $c \leq x_b$ for all $b \in Y$. It follows that $c \leq \bigwedge_{b \in Y} x_b$. We conclude that $\bigwedge_{b \in Y} x_b$ is the largest element c in L satisfying $a \cdot c \leq \bigwedge Y$, that is, $\bigwedge_{b \in Y} x_b = \bigwedge\{a^{-1}b : b \in Y\}$, as required. ■

If $\{L_1, L_2, \ldots, L_n\}$ is any finite family of (complete) lattice ordered monoids, it is easily shown that the lattice and monoid structure on each L_i induces a canonical lattice and monoid structure for the cartesian product $\prod_{i=1}^{n} L_i = L_1 \times L_2 \times \cdots \times L_n$. Moreover, the defining identities for a lattice ordered monoid are passed from the L_i to $\prod_{i=1}^{n} L_i$. We thus have:

Theorem 1.5 *Let $\{L_1, L_2, \ldots, L_n\}$ be any finite family of (complete) lattice ordered monoids. Then $\prod_{i=1}^{n} L_i = L_1 \times L_2 \times \cdots \times L_n$ is canonically a (complete) lattice ordered monoid.*

1.4 Congruence relations on the lattice ordered monoids

Let $\langle L; \leq, \cdot, e_L\rangle$ and $\langle L'; \leq, \cdot, e_{L'}\rangle$ be lattice ordered monoids. Let $\varphi : L \to L'$ be a mapping that satisfies the following conditions:

$(\mathrm{H1})$ $\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$ for all $a, b \in L$;

$(\mathrm{H2})$ $\varphi(\bigvee X) = \bigvee_{x \in X} \varphi(x)$ for all nonempty $X \subseteq L$;

$(\mathrm{H3})$ $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$.

We shall call a map φ satisfying the above a *homomorphism of lattice ordered monoids*. (Note the lack of symmetry in conditions (H1) and (H2)).
Denote by $\equiv_\varphi$ the congruence on L induced by φ. Thus

$$a \equiv_\varphi b \Leftrightarrow \varphi(a) = \varphi(b) \text{ for all } a, b \in L.$$

Denote by $[a]_{\equiv_\varphi}$ the equivalence class of a. That is

$$[a]_{\equiv_\varphi} = \{b \in L : \varphi(a) = \varphi(b)\}.$$

As usual, we denote by $L/\equiv_\varphi$ the collection of all equivalence classes with respect to $\equiv_\varphi$, that is

$$L/\equiv_\varphi = \{[a]_{\equiv_\varphi} : a \in L\}.$$

Note that since φ is a homomorphism of lattice ordered monoids, $L/\equiv_\varphi$ inherits the same structure from L with meets, joins and the monoid operation defined in the obvious way. Since φ preserves arbitrary joins, each equivalence class $[a]_{\equiv_\varphi}$ contains a largest element, namely $\bigvee[a]_{\equiv_\varphi}$.
The map $a \mapsto \bigvee[a]_{\equiv_\varphi}$ constitute a nucleus on L. We remind the reader that a map σ from a lattice L to itself is called a nucleus if:

$(\mathrm{N1})$ σ is order preserving, i.e., $a \leq b$ implies $\sigma(a) \leq \sigma(b)$ for all $a, b \in L$;

$(\mathrm{N2})$ $\sigma(a \wedge b) = \sigma(a) \wedge \sigma(b)$ for all $a, b \in L$, (note that N2 is stronger than N1);

$(\mathrm{N3})$ σ is idempotent, meaning $\sigma(\sigma(a)) = \sigma(a)$ for all $a \in L$.

Note that some texts refer to a map σ satisfying the above conditions as a closure operator on L.
There is a canonical epimorphism of lattice ordered monoids from L to $L/\equiv_\varphi$ which takes a to $[a]_{\equiv_\varphi}$. A version of the Homomorphism Theorem tells us that the homomorphism $\varphi : L \to L'$ factors through the canonical epimorphism from L to $L/\equiv_\varphi$. That is; the following diagram commutes.

$$\begin{tikzcd} L \arrow[r, "\varphi"] \arrow[d, "canon."'] & L' \\ L/\equiv_\varphi \arrow[ur, "canon.\, embed."'] & \end{tikzcd}$$

The canonical monomorphism from $L/\equiv_\varphi$ to L' takes $[a]_{\equiv_\varphi}$ to $\varphi(a) \in L'$ for each $a \in L$. If φ is onto then the canonical embedding of $L/\equiv_\varphi$ into L' becomes an isomorphism.
If $\{\equiv_\delta : \delta \in \Delta\}$ is a family of congruences on L, then $\bigcap \equiv_\delta$ is a congruence on L and we have a canonical embedding of lattice ordered monoids given by:

$$L/\textstyle\bigcap_{\delta\in\Delta} \equiv_\delta \hookrightarrow \prod_{\delta\in\Delta}(L/\equiv_\delta)$$
$$[a]_{\bigcap_{\delta\in\Delta}\equiv_\delta} \mapsto \{[a]_{\equiv_\delta}\}_{\delta\in\Delta}.$$

1.5 Torsion Theory - hereditary pretorsion classes

This section provides the torsion theoretic background that is necessary for what follows. For further background and for the proof of the many unsubstantiated assertions made in this section, we refer the reader to the texts [18, 17, 31].

We shall use the following abbreviations, $\mathcal{A}$ being a nonempty class of right R-modules:

(A1) C$\mathcal{A}$ is the class of all direct sums of modules in $\mathcal{A}$;

(A2) H$\mathcal{A}$ is the class of all homomorphic images of modules in $\mathcal{A}$;

(A3) S$\mathcal{A}$ is the class of all submodules of modules in $\mathcal{A}$.

A nonempty class $\mathcal{T}$ of right R-modules is called *a hereditary pretorsion class* if it is closed under (arbitrary) direct sums, homomorphic images and submodules, that is $C\mathcal{T} \subseteq \mathcal{T}$, $H\mathcal{T} \subseteq \mathcal{T}$ and $S\mathcal{T} \subseteq \mathcal{T}$. If $\mathcal{A}$ is a nonempty class of right R-modules, then SHC$\mathcal{A}$ is a hereditary pretorsion class, indeed, it is the smallest such class containing $\mathcal{A}$. In this situation we say that $\mathcal{A}$ *subgenerates* the hereditary pretorsion class SHC$\mathcal{A}$. If $\mathcal{A} = \{M\}$ is a singleton, we say that M subgenerates SHC$\mathcal{A}$. The subclass HC$\mathcal{A}$ of SHC$\mathcal{A}$ is the smallest class of right R-modules containing $\mathcal{A}$ that is closed under direct sums and homomorphic images. In this situation we say that $\mathcal{A}$ *generates* HC$\mathcal{A}$.

Every hereditary pretorsion class $\mathcal{T}$ of right R-modules has a singleton generator. Indeed, using the fact that every $M \in \text{Mod-}R$ is canonically a homomorphic image of $\bigoplus_{x \in M} R/x^{-1}0$, it is easy to show that

$$\mathcal{T} = \mathsf{HC}\mathcal{C} = \mathsf{HC}\left\{\bigoplus \mathcal{C}\right\} \text{ where } \mathcal{C} \text{ is a representative set of cyclic modules in } \mathcal{T}. \tag{1.1}$$

We shall denote by $\mathrm{HP}R_R$ the set* of all hereditary pretorsion classes in $\text{Mod-}R$. It is easily checked that any intersection of hereditary pretorsion classes is again a hereditary pretorsion class. Thus $\mathrm{HP}R_R$ has the structure of a complete lattice with respect to inclusion if $X \subseteq \mathrm{HP}R_R$, then

*The fact that $\mathrm{HP}R_R$ is a set and not a proper class is a consequence of the existence of a bijection, to be described later in this section, between $\mathrm{HP}\,R_R$ and the set of all right topologizing filters on R.

$$\bigwedge X \stackrel{\text{def}}{=} \bigcap X, \text{ and}$$
$$\bigvee X \stackrel{\text{def}}{=} \mathsf{SHC}(\bigcup X).$$

If $\mathcal{T}$ is a hereditary pretorsion class of right R-modules and $M \in \text{Mod-}R$, then there is a (unique) largest submodule of M belonging to $\mathcal{T}$, denoted by $\mathcal{T}(M)$, and called the *$\mathcal{T}$-torsion submodule* of M. If $\mathcal{T}(M) = M$, or equivalently $M \in \mathcal{T}$, we say that M is *$\mathcal{T}$-torsion* and if $\mathcal{T}(M) = 0$ we say that M is *$\mathcal{T}$-torsion-free*.

We call a submodule N of $M \in \text{Mod-}R$, a *$\mathcal{T}$-dense submodule* [resp. *$\mathcal{T}$-pure submodule*] of M if M/N is $\mathcal{T}$-torsion [resp. $\mathcal{T}$-torsion-free].

The class of $\mathcal{T}$-torsion-free right R-modules can be shown to be closed under submodules, direct products, module extensions (this means if $N \leq M \in \text{Mod-}R$, then M will be $\mathcal{T}$-torsion-free whenever N and M/N are $\mathcal{T}$-torsion-free), and essential extensions see [31, Proposition 2.2, p. 140]. The last of these closure properties implies that the class of $\mathcal{T}$-torsion-free modules is closed under injective hulls for every module is an essential submodule of its injective hull.

1.6 Topologizing filters

A *right topologizing filter* on a ring R is a nonempty family $\mathfrak{F}$ of right ideals of a ring R that satisfies the following three conditions:

(F1) $A \in \mathfrak{F}$ implies $B \in \mathfrak{F}$ whenever B is a right ideal of R containing A;

(F2) $A, B \in \mathfrak{F}$ implies $A \cap B \in \mathfrak{F}$;

(F3) $A \in \mathfrak{F}$ and $r \in R$ implies $r^{-1}A \stackrel{\text{def}}{=} \{x \in R \,:\, rx \in A\} \in \mathfrak{F}$.

The set of all right topologizing filters on some fixed ring R, which we shall denote by $\text{Fil}\, R_R$; is closed under arbitrary intersections, and thus has the structure of a complete lattice with respect to inclusion.

The lattice join in $\mathrm{Fil}\, R_R$ has an internal description which we provide below. If $X \subseteq \mathrm{Fil}\, R_R$, then

$$\begin{aligned} &\bigwedge X = \bigcap X, \text{ and} \\ &\bigvee X = \{K \leqslant R_R : K \supseteq \bigcap X' \text{ for some finite subset } X' \text{ of } \bigcup X\}. \end{aligned} \tag{1.2}$$

The smallest element of $\mathrm{Fil}\, R_R$ is the singleton $\{R\}$ whilst the largest element is the family comprising all right ideals of R. It is clear from Property (F1) that a member of $\mathrm{Fil}\, R_R$ will coincide with the largest element of $\mathrm{Fil}\, R_R$ precisely if it contains the zero right ideal of R.

If $\mathcal{T} \in \mathrm{HP} R_R$, define

$$\mathfrak{F}_{\mathcal{T}} \stackrel{\text{def}}{=} \{K \leqslant R_R : R/K \in \mathcal{T}\} \tag{1.3}$$

and if $\mathfrak{F} \in \mathrm{Fil}\, R_R$, define

$$\mathcal{T}_{\mathfrak{F}} \stackrel{\text{def}}{=} \{M \in \text{Mod-}R \,:\, x^{-1}0 \in \mathfrak{F} \ \forall x \in M\}. \tag{1.4}$$

It is easily checked that $\mathfrak{F}_{\mathcal{T}}$ is a member of $\mathrm{Fil}\, R_R$ and $\mathcal{T}_{\mathfrak{F}}$ a member of $\mathrm{HP}\, R_R$. We shall call $\mathfrak{F}_{\mathcal{T}}$ the *right topologizing filter on R associated with $\mathcal{T}$* and $\mathcal{T}_{\mathfrak{F}}$ the *hereditary pretorsion class in Mod-R associated with $\mathfrak{F}$*.

It follows from (1.4) that if M is any right R-module, then

$$\mathcal{T}_{\mathfrak{F}}(M) = \{x \in M : x^{-1}0 \in \mathfrak{F}\}. \tag{1.5}$$

Equation (1.4) also has the consequence that

$$\begin{aligned} R/K \in \mathcal{T}_{\mathfrak{F}} \ &\Leftrightarrow \ r^{-1}K \in \mathfrak{F} \ \forall r \in R \\ &\Leftrightarrow \ K \in \mathfrak{F} \quad \text{(by Property (F3))}. \end{aligned} \tag{1.6}$$

Statements (1.3) and (1.6) imply that

$$\mathfrak{F}_{\mathcal{T}_{\mathfrak{F}}} = \mathfrak{F}. \tag{1.7}$$

The same statements also imply $R/K \in \mathcal{T}_{\mathfrak{F}_{\mathcal{T}}}$ if and only if $R/K \in \mathcal{T}$, so $\mathcal{T}_{\mathfrak{F}_{\mathcal{T}}}$ and $\mathcal{T}$ possess the same cyclic modules, whence

$$\mathcal{T}_{\mathfrak{F}_{\mathcal{T}}} = \mathcal{T} \tag{1.8}$$

in view of (1.1).

From statements (1.7) and (1.8) we deduce the following theorem.

Theorem 1.6 *The mappings $\mathfrak{F} \mapsto \mathcal{T}_{\mathfrak{F}}$ and $\mathcal{T} \mapsto \mathfrak{F}_{\mathcal{T}}$ are mutually inverse order preserving maps between $\mathrm{Fil}\, R_R$ and $\mathrm{HP} R_R$ and thus constitute mutually inverse lattice isomorphisms.*

Proposition 1.7 *[17, Proposition 2.16, p. 21] The following statements are equivalent for $\mathfrak{F} \in \mathrm{Fil}\, R_R$:*

(a) $\mathfrak{F}$ is compact;

(b) $\mathcal{T}_{\mathfrak{F}}$ is subgenerated by a finitely generated $M \in \mathrm{Mod}\text{-}R$;

(c) $\mathcal{T}_{\mathfrak{F}}$ is subgenerated by a cyclic $M \in \mathrm{Mod}\text{-}R$;

(d) $\mathfrak{F} = \mathfrak{F}_{\mathcal{T}_{\mathfrak{F}}} = \mathfrak{F}_{\mathsf{SHC}\{R/K\}}$ for some $K \leq R_R$.

Noting that

$$\begin{aligned}
\mathcal{T}_{\mathfrak{F}} &= \mathsf{HC}\{R/K \,:\, K \in \mathfrak{F}\} \text{ (by (1.1) and (1.6))} \\
&= \mathsf{SHC}\{R/K \,:\, K \in \mathfrak{F}\} \text{ (because } \mathcal{T}_{\mathfrak{F}} \text{ is closed under submodules)} \\
&= \bigvee_{K \in \mathfrak{F}} \mathsf{SHC}\{R/K\},
\end{aligned}$$

we have

$$\mathfrak{F} = \mathfrak{F}_{\mathcal{T}_{\mathfrak{F}}} = \bigvee_{K \in \mathfrak{F}} \mathfrak{F}_{\mathsf{SHC}\{R/K\}}. \tag{1.9}$$

Inasmuch as each topologizing filter $\mathfrak{F}_{\mathsf{SHC}\{R/K\}}$ appearing on the right-hand-side of Equation (1.9) above is compact by the previous proposition, we have thus proved:

Proposition 1.8 *[17, Proposition 2.17, p. 22] $\mathrm{Fil}\, R_R$ is an algebraic (i.e., compactly generated) lattice.*

A key component of the structure $\operatorname{Fil} R_R$ derives from a binary operation : defined by

$$\mathfrak{F} : \mathfrak{G} \stackrel{\text{def}}{=} \{K \leq R_R : \exists H \in \mathfrak{F} \text{ such that } K \subseteq H \text{ and } h^{-1}K \in \mathfrak{G}\ \forall h \in H\}.$$

Inasmuch as $H \in \mathfrak{F}$ if and only if $R/H \in \mathcal{T}_{\mathfrak{F}}$ (by Equation (1.6)) and $h^{-1}K \in \mathfrak{G}\ \forall h \in H$ if and only if $H/K \in \mathcal{T}_{\mathfrak{G}}$ (by Equation (1.4)), it follows that $K \in \mathfrak{F} : \mathfrak{G}$ if and only if there exists a short exact sequence

$$0 \to H/K \to R/K \to R/H \to 0$$

with $K \subseteq H \leq R_R$ such that $H/K \in \mathcal{T}_{\mathfrak{G}}$ and $R/H \in \mathcal{T}_{\mathfrak{F}}$. This can be generalized to: $M \in \mathcal{T}_{\mathfrak{F}:\mathfrak{G}}$ if and only if there exists a short exact sequence

$$0 \to N \to M \to M/N \to 0$$

such that $N \in \mathcal{T}_{\mathfrak{G}}$ and $M/N \in \mathcal{T}_{\mathfrak{F}}$. In other words, the hereditary pretorsion class associated with $\mathfrak{F} : \mathfrak{G}$ comprises all modules that are an extension of a module in the hereditary pretorsion class associated with $\mathfrak{G}$ by a module in the hereditary pretorsion class associated with $\mathfrak{F}$.

From the above exact sequence can be derived the identity

$$\mathcal{T}_{\mathfrak{F}}(M/\mathcal{T}_{\mathfrak{G}}(M)) = \mathcal{T}_{\mathfrak{F}:\mathfrak{G}}(M)/\mathcal{T}_{\mathfrak{G}}(M)$$

for all $M \in \text{Mod-}R$.

It is obvious from the definition of : that $\mathfrak{F} : \mathfrak{G} \supseteq \mathfrak{F}, \mathfrak{G}$ and so $\mathfrak{F} : \mathfrak{G} \supseteq \mathfrak{F} \vee \mathfrak{G}$. It is known [17, Proposition 3.3, p. 31] that if $\mathfrak{F}, \mathfrak{G} \in \operatorname{Fil} R_R$ and $A \in \mathfrak{F} B \in \mathfrak{G}$; then $AB \in \mathfrak{F} : \mathfrak{G}$.

Let $\mathfrak{F} \in \operatorname{Fil} R_R$. Inasmuch as $\mathcal{T}_{\mathfrak{F}}$ is closed under homomorphic images, it is easily seen that if $f \in \operatorname{Hom}_R(M, N)$, then $f[\mathcal{T}_{\mathfrak{F}}(M)] \subseteq \mathcal{T}_{\mathfrak{F}}(N)$. It is also clear from Equation (1.5) that if $L \leq M$, then $\mathcal{T}_{\mathfrak{F}}(L) = L \cap \mathcal{T}_{\mathfrak{F}}(M)$.

Notice that the smallest topologizing filter $\{R\}$ is an identity element with respect to :.

Theorem 1.9 *[17, Proposition 4.1, p. 43] If R is any ring then $\langle \operatorname{Fil} R_R; \vee, \wedge, :, \{R\}\rangle^{\mathrm{du}}$ is an integral, left residuated, complete lattice ordered monoid.*

The above theorem tells us, in particular, that the left residual $\mathfrak{F}\mathfrak{G}^{-1}$ of $\mathfrak{F}$ by $\mathfrak{G}$ exists for all $\mathfrak{F}, \mathfrak{G} \in \operatorname{Fil} R_R$. Since it is the order dual of $\operatorname{Fil} R_R$ that is left residuated, note that $\mathfrak{G}^{-1}\mathfrak{F}$ corresponds with the smallest element $\mathfrak{H} \in \operatorname{Fil} R_R$ satisfying $\mathfrak{G} : \mathfrak{H} \supseteq \mathfrak{F}$. As was noted in the introduction to this thesis, $[\operatorname{Fil} R_R]^{\mathrm{du}}$ suffers from the deficiency that it is not, in general, right residuated. A major theme of this thesis is the exploration of rings R that enjoy two-sided residuation.

If I is an ideal of ring R, then the family

$$\eta(I) \stackrel{\mathrm{def}}{=} \{K \leq R_R : K \supseteq I\}$$

is easily shown to constitute a right topologizing filter on R. Note that the hereditary pretorsion class $\mathcal{T}_{\eta(I)}$ associated with $\eta(I)$ coincides with $\{M \in \text{Mod-}R : MI = 0\}$ which is identifiable with the category of all right R/I-modules. Observe that $\eta(I)$ is a compact member of $\operatorname{Fil} R_R$, for $\mathcal{T}_{\eta(I)} = \mathsf{SHC}\{R/I\}$ (see Proposition 1.7). However, unless the ring R is commutative, not every compact member of $\operatorname{Fil} R_R$ has the form $\eta(I)$ for some $I \in \operatorname{Id} R$.

A right topologizing filter $\mathfrak{F}$ on a ring R is called jansian if it satisfies the equivalent conditions of the theorem below. The set of all jansian right topologizing filters on a ring R shall be denoted by $\operatorname{Jans} R_R$.

Theorem 1.10 *[17, Proposition 1.14 and Corollary 1.15, p. 9] The following conditions are equivalent for a right topologizing filter $\mathfrak{F}$ on a ring R:*

(a) $\mathfrak{F}$ is closed under arbitrary intersections;

(b) $\bigcap \mathfrak{F} \in \mathfrak{F}$;

(c) $\mathfrak{F} = \eta(I)$ for some ideal I of R;

(d) $\mathfrak{F} = \{A \leq R_R : A \supseteq M^{-1}0\}$ for some $M \in \text{Mod-}R$;

(e) $\mathcal{T}_{\mathfrak{F}}$ *is closed under arbitrary direct products.*

Observe that the smallest and largest elements of $\operatorname{Fil} R_R$ are jansian for they correspond with $\eta(R)$ and $\eta(0)$, respectively.

Recall that a right R-module M is said to be *finitely annihilated* if $M^{-1}0 = X^{-1}0$ for some finite subset X of M. This can be shown to be equivalent to the requirement that the right topologizing filter on R associated with SHC$\{M\}$ is jansian (see comments preceding Proposition 2.8).

We shall have need for the following theorem.

Theorem 1.11 *[3, Corollary 3.3, p. 25] The following conditions on a ring R are equivalent:*

(a) every $\mathfrak{F} \in \operatorname{Fil} R_R$ is jansian;

(b) every right R-module is finitely annihilated;

(c) R is right artinian.

It is easily shown that $\operatorname{Jans} R_R$ closed under arbitrary intersections, but it is not closed under arbitrary joins in general. Topologizing filters which are joins of jansian topologizing filters are easily characterized, however.

Let $\mathfrak{F} \in \operatorname{Fil} R_R$. A subset X of $\mathfrak{F}$ is said to be a *cofinal set* for $\mathfrak{F}$ if, given any $A \in \mathfrak{F}$, there exists $B \in X$ such that $A \supseteq B$. We call $\mathfrak{F}$ bounded if $\mathfrak{F}$ has a cofinal set consisting of ideals of R. Trivially, if R is commutative, then every $\mathfrak{F} \in \operatorname{Fil} R_R$ is bounded.

Theorem 1.12 *[17, Proposition 2.6, p. 17] A right topologizing filter $\mathfrak{F}$ on R is* bounded *if and only if it is the join of a nonempty set of* jansian topologizing filters.

The atoms of $\operatorname{Fil} R_R$ are easily described.

Theorem 1.13 *[17, Corollary 2.24, p. 24] Let S be any simple right R-module. If $\mathfrak{H}$ is the right topologizing filter on R associated with* SHC$\{S\}$*, then $\mathfrak{H}$ is an atom of $\operatorname{Fil} R_R$. Moreover, every atom of $\operatorname{Fil} R_R$ has the form $\mathfrak{F}_{\mathcal{T}}$ where $\mathcal{T} =$* SHC$\{S\}$ *for some simple right R-module S.*

Coatoms in $\mathrm{Fil}\, R_R$ have a less conspicuous form in general. However, if R is a (commutative) integral domain, it is easily checked that the family comprising all nonzero ideals of R is the unique coatom of $\mathrm{Fil}\, R$.

The following is a key result.

Theorem 1.14 *[17, Proposition 2.7, p. 17 and Proposition 3.4, p. 31] If R is any ring then the mapping from $\mathrm{Id}\, R$ to $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ defined by $I \mapsto \eta(I)$ is a one-to-one homomorphism in respect of the binary join, binary meet, and multiplication operations, that also preserves arbitrary joins. Thus there is an embedding of lattice ordered monoids of $\mathrm{Id}\, R$ into $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$.*

Remark 1.15 *Theorem 1.14 asserts that a ring R is right artinian precisely if every member of $\mathrm{Fil}\, R_R$ is jansian, that is to say, η, interpreted as a mapping from $\mathrm{Id}\, R$ into $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$, is onto and thus an isomorphism of lattice ordered monoids. Since $\mathrm{Id}\, R$ is two-sided residuated for all rings R (see Proposition 1.1), this means that $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ is two-sided residuated for all right artinian rings R. Rings that possess the two-sided residuated property in respect of their topologizing filters need not be artinain, however; as Theorem 2.12 shows. See also [34, Corollary 8, p. 91] which proves that for any commutative noetherian ring R, $\mathrm{Fil}\, R_R$ is commutative (meaning the operation $:$ is commutative). Inasmuch as $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ is left residuated for all rings R (Theorem 1.9), and the notions of left residuated and right residuated coincide in a commutative lattice ordered monoid, it follows that all commutative noetherian rings possess the two-sided residuation property.*

Taking $L = [\mathrm{Fil}\, R_R]^{\mathrm{du}}$ in Proposition 1.4 yields the next result. (Note that the joins in Proposition 1.16 below pertain to $\mathrm{Fil}\, R_R$ and thus correspond with meets in $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$.)

Proposition 1.16 *Suppose R is any ring, $\mathfrak{G} \in \mathrm{Fil}\, R_R$ and Y a nonempty subset of $\mathrm{Fil}\, R_R$. If the right residual $\mathfrak{G}^{-1}\mathfrak{F}$ exists in $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ for all $\mathfrak{F} \in Y$, then the right residual $\mathfrak{G}^{-1}(\bigvee Y)$ exists in $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ and*

$$\mathfrak{G}^{-1}(\bigvee Y) = \bigvee\{\mathfrak{G}^{-1}\mathfrak{F} : \mathfrak{F} \in Y\}.$$

Remark 1.17 *Insofar as* $\mathrm{Fil}\, R_R$ *is compactly generated by Proposition 1.8 it follows from Proposition 1.16 that* $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ *will be right residuated (and thus two-sided residuated in view of Theorem 1.9) if the right residual* $\mathfrak{G}^{-1}\mathfrak{F}$ *exists in* $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ *for all compact* $\mathfrak{F} \in \mathrm{Fil}\, R_R$*. This fact is needed in the proof of Corollary 2.2.*

We noted earlier in this section, that the hereditary pretorsion class associated with $\mathfrak{F} : \mathfrak{G}$ comprises all modules that are an extension of a module in the hereditary pretorsion class associated with $\mathfrak{G}$ by a module in the hereditary pretorsion class associated with $\mathfrak{F}$. In particular, taking $\mathfrak{F} = \mathfrak{G}$, we see that $\mathcal{T}_{\mathfrak{F}:\mathfrak{F}}$ is the class of all module extensions of modules in $\mathcal{T}_{\mathfrak{F}}$. Hence $\mathcal{T}_{\mathfrak{F}:\mathfrak{F}} = \mathcal{T}_{\mathfrak{F}}$ precisely if $\mathcal{T}_{\mathfrak{F}}$ is closed under module extensions. We have thus proved the equivalence of Statements (a) and (c) in the following theorem.

Theorem 1.18 *[31, Theorem 5.1, p. 146] and [17, p. 55] The following statements are equivalent for a right topologizing filter* $\mathfrak{F}$ *on* R*:*

(a) $\mathfrak{F}$ *is idempotent, i.e.,* $\mathfrak{F}^2 \stackrel{\mathrm{def}}{=} \mathfrak{F} : \mathfrak{F} = \mathfrak{F}$*;*

(b) If $I \leq R_R$ *and there exists some* $J \in \mathfrak{F}$ *such that* $a^{-1}I \in \mathfrak{F}\ \forall a \in J$*, then* $I \in \mathfrak{F}$*;*

(c) $\mathcal{T}_{\mathfrak{F}}$ *is closed under module extensions.*

A right topologizing filter $\mathfrak{F}$ on R satisfying the equivalent conditions of the above theorem, is called a *right Gabriel filter* on R. This name honours Gabriel who first introduced the notion in [14]. We denote the set of all right Gabriel filters on R by $\text{G-Fil}\, R_R$. It follows from Theorem 1.14 that if I is an ideal of R, then $\eta(I) \in \text{G-Fil}\, R_R$ if and only if I is idempotent. This means that both $\eta(0)$ and $\eta(R)$ are members of $\text{G-Fil}\, R_R$, so $\text{G-Fil}\, R_R$ is never empty. It is easy to show that $\text{G-Fil}\, R_R$ is closed under arbitrary intersections. Thus $\text{G-Fil}\, R_R$ is a meet-complete subsemilattice of $\mathrm{Fil}\, R_R$ and a complete lattice with respect to inclusion.

1.7 Change of rings

In this section we show how a ring homomorphism between two rings induces structure preserving maps between the ring's respective sets of topologizing filters. We derive a correspondence theorem in the process.

Proposition 1.19 *Let R and T be arbitrary rings and $\varphi : R \to T$ a ring homomorphism. Then the map $\varphi^* : \operatorname{Fil} T_T \to \operatorname{Fil} R_R$ given by*

$$\varphi^*(\mathfrak{F}) = \{K \leq R_R : K \supseteq \varphi^{-1}[L] \text{ for some } L \in \mathfrak{F}\}$$

is a complete lattice homomorphism, that is to say, φ^* preserves arbitrary meets and joins. Moreover, $\varphi^*(\mathfrak{F} : \mathfrak{G}) \subseteq \varphi^*(\mathfrak{F}) : \varphi^*(\mathfrak{G})$ for all $\mathfrak{F}, \mathfrak{G} \in \operatorname{Fil} T_T$.

Proof. That $\varphi^*(\mathfrak{F})$ is a right topologizing filter on R is easily established using the fact that $\varphi^{-1}[A \cap B] = \varphi^{-1}[A] \cap \varphi^{-1}[B]$ and $r^{-1}\varphi^{-1}[A] = \varphi^{-1}[\varphi(r)^{-1}A]$ for all $A, B \leq T_T$ and $r \in R$.

We first show that φ^* preserves arbitrary meets. To this end, let $\{\mathfrak{F}_\delta : \delta \in \Delta\}$ be a nonempty subset of $\operatorname{Fil} T_T$. Since φ^* is order preserving, the containment $\varphi^*(\bigcap_{\delta\in\Delta} \mathfrak{F}_\delta) \subseteq \bigcap_{\delta\in\Delta} \varphi^*(\mathfrak{F}_\delta)$ is clear. It remains to establish the reverse containment. Take $K \in \bigcap_{\delta\in\Delta} \varphi^*(\mathfrak{F}_\delta)$. Thus $K \in \varphi^*(\mathfrak{F}_\delta)$ for all $\delta \in \Delta$. Hence, there exists, for each $\delta \in \Delta$, a right ideal $L_\delta \in \mathfrak{F}_\delta$ such that $K \supseteq \varphi^{-1}[L_\delta]$. It follows that $K \supseteq \sum_{\delta\in\Delta} \varphi^{-1}[L_\delta] = \varphi^{-1}[\sum_{\delta\in\Delta} L_\delta]$. Since $\sum_{\delta\in\Delta} L_\delta \supseteq L_\delta$ for each $\delta \in \Delta$, it follows that $\sum_{\delta\in\Delta} L_\delta \in \bigcap_{\delta\in\Delta} \mathfrak{F}_\delta$, so $K \in \varphi^*(\bigcap_{\delta\in\Delta} \mathfrak{F}_\delta)$. Thus $\varphi^*(\bigcap_{\delta\in\Delta} \mathfrak{F}_\delta) \supseteq \bigcap_{\delta\in\Delta} \varphi^*(\mathfrak{F}_\delta)$, whence equality.

We now show that φ^* preserves arbitrary joins. The containment $\varphi^*(\bigvee_{\delta\in\Delta} \mathfrak{F}_\delta) \supseteq \bigvee_{\delta\in\Delta} \varphi^*(\mathfrak{F}_\delta)$ is clear. Take $K \in \varphi^*(\bigvee_{\delta\in\Delta} \mathfrak{F}_\delta)$. Then $K \supseteq \varphi^{-1}[L]$ for some $L \in \bigvee_{\delta\in\Delta} \mathfrak{F}_\delta$. It follows from our description of the join (see (1.2)) that there exists a finite subset Δ' of Δ and a right ideal $A_\delta \in \mathfrak{F}_\delta$ for each $\delta \in \Delta'$, such that $L \supseteq \bigcap_{\delta\in\Delta'} A_\delta$. Then $K \supseteq \varphi^{-1}[L] \supseteq \varphi^{-1}[\bigcap_{\delta\in\Delta'} A_\delta] = \bigcap_{\delta\in\Delta'} \varphi^{-1}[A_\delta]$. Clearly $\varphi^{-1}[A_\delta] \in \varphi^*(\mathfrak{F}_\delta)$ for all $\delta \in \Delta'$, so $\bigcap_{\delta\in\Delta'} \varphi^{-1}[A_\delta] \in \bigvee_{\delta\in\Delta} \varphi^*(\mathfrak{F}_\delta)$ by (1.2), whence $K \in \bigvee_{\delta\in\Delta} \varphi^*(\mathfrak{F}_\delta)$. The containment $\varphi^*(\bigvee_{\delta\in\Delta} \mathfrak{F}_\delta) \subseteq \bigvee_{\delta\in\Delta} \varphi^*(\mathfrak{F}_\delta)$ is thus shown, whence equality.

To complete the proof, it remains to show that $\varphi^*(\mathfrak{F}:\mathfrak{G}) \subseteq \varphi^*(\mathfrak{F}):\varphi^*(\mathfrak{G})$ for all $\mathfrak{F},\mathfrak{G} \in \mathrm{Fil}\, T_T$. Take $K \in \varphi^*(\mathfrak{F}:\mathfrak{G})$. Then $K \supseteq \varphi^{-1}[L]$ for some $L \in \mathfrak{F}:\mathfrak{G}$. Hence there exists $H \in \mathfrak{F}$ such that $H \supseteq L$ and $t^{-1}L \in \mathfrak{G}$ for all $t \in H$. Observe that

$$\varphi^{-1}[H] \in \varphi^*(\mathfrak{F}) \tag{1.10}$$

since $H \in \mathfrak{F}$. Take $r \in \varphi^{-1}[H]$. Inasmuch as

$$\begin{aligned} s \in r^{-1}\varphi^{-1}[L] \;&\Leftrightarrow\; rs \in \varphi^{-1}[L] \\ &\Leftrightarrow\; \varphi(rs) = \varphi(r)\varphi(s) \in L \\ &\Leftrightarrow\; \varphi(s) \in \varphi(r)^{-1}L \\ &\Leftrightarrow\; s \in \varphi^{-1}[\varphi(r)^{-1}L], \end{aligned}$$

we have $r^{-1}\varphi^{-1}[L] = \varphi^{-1}[\varphi(r)^{-1}L]$. Since $t^{-1}L \in \mathfrak{G}$ for all $t \in H$, it follows that $\varphi(r)^{-1}L \in \mathfrak{G}$ for all $r \in \varphi^{-1}[H]$, whence

$$r^{-1}\varphi^{-1}[L] = \varphi^{-1}[\varphi(r)^{-1}L] \in \varphi^*(\mathfrak{G}) \text{ for all } r \in \varphi^{-1}[H]. \tag{1.11}$$

Equations (1.10) and (1.11) imply that $\varphi^{-1}[L] \in \varphi^*(\mathfrak{F}):\varphi^*(\mathfrak{G})$, whence $K \in \varphi^*(\mathfrak{F}):\varphi^*(\mathfrak{G})$. We have thus shown that $\varphi^*(\mathfrak{F}:\mathfrak{G}) \subseteq \varphi^*(\mathfrak{F}):\varphi^*(\mathfrak{G})$. ■

We point out, with reference to the previous result, that φ^* is, in general, not a monoid homomorphism with respect to :.

Now let $I \in \mathrm{Id}\, R$ and $\pi : R \to R/I$ be the canonical ring epimorphism. Observe that in this situation, for each $\mathfrak{F} \in \mathrm{Fil}\,(R/I)_{R/I}$,

$$\begin{aligned} \pi^*(\mathfrak{F}) &= \{K \leq R_R : K \supseteq \pi^{-1}[L] \text{ for some } L \in \mathfrak{F}\} \\ &= \{K \leq R_R : K \supseteq I \text{ and } K/I \in \mathfrak{F}\}. \end{aligned} \tag{1.12}$$

We remind the reader that $\mathrm{Mod}\text{-}(R/I)$ may be interpreted as a subcategory of $\mathrm{Mod}\text{-}R$: if $M \in \mathrm{Mod}\text{-}(R/I)$ and $x \in M$, then

$$xr \overset{\text{def}}{=} x(r+I) \text{ for all } r \in R. \tag{1.13}$$

Now take $\mathfrak{F} \in \mathrm{Fil}\,(R/I)_{R/I}$ and let $M \in \mathrm{Mod}\text{-}(R/I)$. Then

$$\begin{aligned} x \in \mathcal{T}_{\mathfrak{F}}(M) &\Leftrightarrow \exists K \leq R_R \text{ such that } K \supseteq I,\ K/I \in \mathfrak{F} \text{ and } x(K/I) = 0 \\ &\Leftrightarrow \exists K \in \pi^*(\mathfrak{F}) \text{ such that } x(K/I) = 0 \text{ [by (1.12)]} \\ &\Leftrightarrow \exists K \in \pi^*(\mathfrak{F}) \text{ such that } xK = 0 \text{ [because, by (1.13), } xK = 0 \Leftrightarrow x(K/I) = 0] \\ &\Leftrightarrow x \in \mathcal{T}_{\pi^*(\mathfrak{F})}(M). \end{aligned}$$

We have thus shown that

$$\mathcal{T}_{\mathfrak{F}}(M) = \mathcal{T}_{\pi^*(\mathfrak{F})}(M) \tag{1.14}$$

for all $M \in \mathrm{Mod}\text{-}(R/I)$.

Proposition 1.20 *Let I be an ideal of arbitrary ring R and $\pi : R \to R/I$ the canonical ring epimorphism. Then*

$$\pi^*(\mathfrak{F} : \mathfrak{G}) = [\pi^*(\mathfrak{F}) : \pi^*(\mathfrak{G})] \cap \eta(I)$$

for all $\mathfrak{F}, \mathfrak{G} \in \mathrm{Fil}\,(R/I)_{R/I}$.

Proof. By Proposition 1.19, $\pi^*(\mathfrak{F} : \mathfrak{G}) \subseteq \pi^*(\mathfrak{F}) : \pi^*(\mathfrak{G})$. It is also clear from (1.12) that $\pi^*(\mathfrak{F}) \subseteq \{K \leq R_R : K \supseteq I\} = \eta(I)$ for every $\mathfrak{F} \in \mathrm{Fil}\,(R/I)_{R/I}$. Thus $\pi^*(\mathfrak{F} : \mathfrak{G}) \subseteq [\pi^*(\mathfrak{F}) : \pi^*(\mathfrak{G})] \cap \eta(I)$.

To establish the reverse containment, take $K \in [\pi^*(\mathfrak{F}) : \pi^*(\mathfrak{G})] \cap \eta(I)$. Then there exists $H \leq R_R$ such that $H \supseteq K$, R/H is $\mathcal{T}_{\pi^*(\mathfrak{F})}$-torsion and H/K is $\mathcal{T}_{\pi^*(\mathfrak{G})}$-torsion. Inasmuch as $K \in \eta(I)$, $H \supseteq K \supseteq I$. This means that the short exact sequence

$$0 \longrightarrow H/K \longrightarrow R/K \longrightarrow R/H \longrightarrow 0$$

in $\mathrm{Mod}\text{-}R$, induces the following short exact sequence in $\mathrm{Mod}\text{-}(R/I)$

$$0 \longrightarrow (H/I)/(K/I) \longrightarrow (R/I)/(K/I) \longrightarrow (R/I)/(H/I) \longrightarrow 0.$$

Since $H \in \pi^*(\mathfrak{F})$ (because R/H is $\mathcal{T}_{\pi^*(\mathfrak{F})}$-torsion), it follows from (1.12) that $H/I \in \mathfrak{F}$. Since $(H/I)/(K/I) \cong H/K$ is $\mathcal{T}_{\pi^*(\mathfrak{G})}$-torsion, it follows from (1.14) that $(H/I)/(K/I)$ is $\mathcal{T}_{\mathfrak{G}}$-torsion. We conclude that $K/I \in \mathfrak{F} : \mathfrak{G}$, so $K \in \pi^*(\mathfrak{F} : \mathfrak{G})$. We have thus shown that $[\pi^*(\mathfrak{F}) : \pi^*(\mathfrak{G})] \cap \eta(I) \subseteq \pi^*(\mathfrak{F} : \mathfrak{G})$, whence equality. ■

If I is an ideal of ring R, then, in general, the interval $[0, \eta(I)] \stackrel{\text{def}}{=} \{\mathfrak{F} \in \operatorname{Fil} R_R : \mathfrak{F} \subseteq \eta(I)\}$ of $\operatorname{Fil} R_R$ is not closed under the monoid operation :, for $\eta(I) : \eta(I) = \eta(I \cdot I) = \eta(I^2)$ (by Theorem 1.14) and $\eta(I^2)$ does not belong to $[0, \eta(I)]$ unless $I^2 = I$. This observation is, of course, consistent with our earlier observation that φ^* is, in general, not a monoid homomorphism with respect to :.

Let I be an ideal of arbitrary ring R. We define operation $:_I$ on $[0, \eta(I)]$ by

$$\mathfrak{F} :_I \mathfrak{G} \stackrel{\text{def}}{=} (\mathfrak{F} : \mathfrak{G}) \cap \eta(I)$$

for $\mathfrak{F}, \mathfrak{G} \in [0, \eta(I)]$.

Remark 1.21 *Note that if, in the above definition, the ideal I is idempotent, that is to say $I^2 = I$, then $[0, \eta(I)]$ will be closed under the operation :, in which case the operations : and $:_I$ coincide.*

In light of the previous definition and Proposition 1.20, we see that

$$\pi^*(\mathfrak{F} : \mathfrak{G}) = \pi^*(\mathfrak{F}) :_I \pi^*(\mathfrak{G}) \tag{1.15}$$

for all $\mathfrak{F}, \mathfrak{G} \in \operatorname{Fil}(R/I)_{R/I}$, which is to say, $\pi^* : \operatorname{Fil}(R/I)_{R/I} \to \langle [0, \eta(I)]; :_I \rangle$ is a monoid homomorphism.

As before, let I be an ideal of arbitrary ring R and $\pi : R \to R/I$ the canonical ring epimorphism. Define map $\pi_* : [0, \eta(I)] \to \operatorname{Fil}(R/I)_{R/I}$ by

$$\pi_*(\mathfrak{F}) \stackrel{\text{def}}{=} \{K/I : K \in \mathfrak{F}\}$$

for $\mathfrak{F} \in [0, \eta(I)]$. It is easily checked that $\pi_*(\mathfrak{F})$ is indeed a member of $\operatorname{Fil}(R/I)_{R/I}$ for every $\mathfrak{F} \in [0, \eta(I)]$.

Theorem 1.22 *(Correspondence Theorem) Let I be an ideal of arbitrary ring R and $\pi : R \to R/I$ the canonical ring epimorphism. Then π^* and π_* are mutually inverse complete lattice and monoid isomorphisms between $\mathrm{Fil}\,(R/I)_{R/I}$ and $\langle [0, \eta(I)]; :_I \rangle$.*

Proof. We have already proven that π^* is a complete lattice homomorphism by Proposition 1.19. It is, furthermore, a monoid homomorphism by (1.15). To complete the proof, it therefore suffices to show that π^* and π_* are mutually inverse bijections.

To this end, take $\mathfrak{F} \in \mathrm{Fil}\,(R/I)_{R/I}$. Let $K \leq R_R$ with $K \supseteq I$. Then

$$K/I \in (\pi_* \circ \pi^*)(\mathfrak{F}) = \pi_*(\pi^*(\mathfrak{F})) \Leftrightarrow K \in \pi^*(\mathfrak{F}) \Leftrightarrow K/I \in \mathfrak{F}.$$

Thus $(\pi_* \circ \pi^*)(\mathfrak{F}) = \mathfrak{F}$.

Now take $\mathfrak{G} \in [0, \eta(I)]$. Then

$$K \in (\pi^* \circ \pi_*)(\mathfrak{G}) = \pi^*(\pi_*(\mathfrak{G})) \Leftrightarrow K \supseteq I \text{ and } K/I \in \pi_*(\mathfrak{G}) \text{ [by (1.12)]} \Leftrightarrow K \in \mathfrak{G}.$$

Thus $(\pi^* \circ \pi_*)(\mathfrak{G}) = \mathfrak{G}$. We conclude that π^* and π_* are mutually inverse bijections, as required.

■

Inasmuch as $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ is a complete lattice ordered monoid for all rings R by Theorem 1.9, the following corollary to Theorem 1.22 is immediate.

Corollary 1.23 *Let I be an ideal of arbitrary ring R. Then $[\mathrm{Fil}\,(R/I)_{R/I}]^{\mathrm{du}}$ and $\langle [0, \eta(I)]; :_I \rangle^{\mathrm{du}}$ are isomorphic complete lattice ordered monoids.*

Let I be an ideal of arbitrary ring R. Consider the mapping from $\mathrm{Fil}\, R_R$ to $[0, \eta(I)]$ given by $\mathfrak{F} \mapsto \mathfrak{F} \cap \eta(I)$, $\mathfrak{F} \in \mathrm{Fil}\, R_R$. That this mapping is onto and preserves arbitrary meets is obvious. Moreover, if $\mathfrak{F}, \mathfrak{G} \in \mathrm{Fil}\, R_R$, then

$$\begin{aligned}
[\mathfrak{F}\cap\eta(I)] :_I [\mathfrak{G}\cap\eta(I)] &= ([\mathfrak{F}\cap\eta(I)] : [\mathfrak{G}\cap\eta(I)])\cap\eta(I)\\
&= [\mathfrak{F}:\mathfrak{G}]\cap[\mathfrak{F}:\eta(I)]\cap[\eta(I):\mathfrak{G}]\cap[\eta(I):\eta(I)])\cap\eta(I)\\
&= [\mathfrak{F}:\mathfrak{G}]\cap\eta(I) \quad \text{[because } \mathfrak{F}:\eta(I), \eta(I):\mathfrak{G} \text{ and } \eta(I):\eta(I) \text{ all contain } \eta(I)\text{]}.
\end{aligned}$$

We have thus proved the following proposition.

Proposition 1.24 *Let I be an ideal of arbitrary ring R. The mapping from $\mathrm{Fil}\,R_R$ to $\langle[0,\eta(I)];:_I\rangle$ given by $\mathfrak{F}\mapsto\mathfrak{F}\cap\eta(I)$, $\mathfrak{F}\in\mathrm{Fil}\,R_R$, is onto, preserves arbitrary meets, and is a monoid homomorphism.*

Theorem 1.25 *(Preservation Theorem) Let I be an ideal of arbitrary ring R.*

(a) If the monoid operation $:$ on $\mathrm{Fil}\,R_R$ is commutative, then so is the corresponding monoid operation on $\mathrm{Fil}\,(R/I)_{R/I}$.

(b) If every $\mathfrak{F}\in\mathrm{Fil}\,R_R$ is idempotent, that is to say, $\mathfrak{F}:\mathfrak{F}=\mathfrak{F}$, then the same is true of every member of $\mathrm{Fil}\,(R/I)_{R/I}$.

(c) If $[\mathrm{Fil}\,R_R]^{\mathrm{du}}$ is right residuated, then so is $[\mathrm{Fil}\,(R/I)_{R/I}]^{\mathrm{du}}$.

Proof. It follows from Theorem 1.22 and Proposition 1.24 that the composition of maps

$$\mathfrak{F}\mapsto\mathfrak{F}\cap\eta(I)\mapsto\pi_*(\mathfrak{F}\cap\eta(I))$$

from $\mathrm{Fil}\,R_R$ to $\mathrm{Fil}\,(R/I)_{R/I}$ is onto, preserves arbitrary meets, and is a monoid homomorphism. It follows that any property of $\mathrm{Fil}\,R_R$ that is characterizable in terms of an identity involving only meets and the monoid operation, is passed from $\mathrm{Fil}\,R_R$ to $\mathrm{Fil}\,(R/I)_{R/I}$.

Inasmuch as the properties for $\mathrm{Fil}\,R_R$ appearing in (a) and (b) are identities involving the monoid operation only, we may infer from the above explanation that the assertions made in (a) and (b) hold.

Regarding (c), Theorem 1.9 tells us that $\langle \mathrm{Fil}\, R_R; \vee, \wedge, :, \{R\}\rangle^{\mathrm{du}}$ is a complete lattice ordered monoid, and therefore by Proposition 1.2, the right residuation property is characterizable by means of the identity

$$\mathfrak{G} : \left(\bigcap_{\delta \in \Delta} \mathfrak{F}_\delta \right) = \bigcap_{\delta \in \Delta} (\mathfrak{G} : \mathfrak{F}_\delta), \quad \mathfrak{G} \in \mathrm{Fil}\, R_R,\ \{\mathfrak{F}_\delta : \delta \in \Delta\} \subseteq \mathrm{Fil}\, R_R.$$

Since the above identity clearly involves only meets and the monoid operation, it follows that assertion (c) holds. ■

Let $\{R_i : 1 \leq i \leq n\}$ be a finite family of rings and put $R = \prod_{i=1}^n R_i$. For each $i \in \{1, 2, \ldots, n\}$ let $\pi_i : R \to R_i$ be the canonical projection and put

$$I_i = \ker \pi_i = R_1 \times R_2 \times \cdots \times R_{i-1} \times 0 \times R_{i+1} \times \cdots \times R_n.$$

Observe that $R/I_i \cong R_i$ and that each I_i is an idempotent ideal of R. This latter fact means that the monoid operation $:_{I_i}$ on $[0, \eta(I_i)]$ coincides with $:$ for every $i \in \{1, 2, \ldots, n\}$, as explained in Remark 1.21.

We see from Corollary 1.23 that for each $i \in \{1, 2, \ldots, n\}$, $[0, \eta(I_i)]^{\mathrm{du}}$ is a complete lattice ordered monoid. Hence $\prod_{i=1}^n [0, \eta(I_i)]^{\mathrm{du}}$ is a complete lattice ordered monoid by Theorem 1.5.

Consider the map from $\mathrm{Fil}\, R_R$ to $\prod_{i=1}^n [0, \eta(I_i)]$ given by

$$\mathfrak{F} \longmapsto (\mathfrak{F} \cap \eta(I_1), \mathfrak{F} \cap \eta(I_2), \ldots, \mathfrak{F} \cap \eta(I_n)). \tag{1.16}$$

It follows from Proposition 1.24 that for each $i \in \{1, 2, \ldots, n\}$ the map from $\mathrm{Fil}\, R_R$ to $[0, \eta(I_i)]$ preserves arbitrary meets and is a homomorphism with respect to the monoid operation $:$. From this it follows that the map defined in (1.16) is a complete lattice and monoid homomorphism from $\mathrm{Fil}\, R_R$ to $\prod_{i=1}^n [0, \eta(I_i)]$. Our objective is to show that the map (1.16) is bijective and thus an isomorphism. To this end, we define a map from $\prod_{i=1}^n [0, \eta(I_i)]$ to $\mathrm{Fil}\, R_R$ given by

$$(\mathfrak{G}_1, \mathfrak{G}_2, \ldots, \mathfrak{G}_n) \longmapsto \bigvee_{i=1}^n \mathfrak{G}_i. \tag{1.17}$$

We claim that for each $j \in \{1, 2, \ldots, n\}$,

$$\left(\bigvee_{i=1}^{n} \mathfrak{G}_i\right) \cap \eta(I_j) = \mathfrak{G}_j. \tag{1.18}$$

For each $j \in \{1, 2, \ldots, n\}$, we certainly have $(\bigvee_{i=1}^{n} \mathfrak{G}_i) \cap \eta(I_j) \supseteq \mathfrak{G}_j$, for $\mathfrak{G}_j \subseteq \bigvee_{i=1}^{n} \mathfrak{G}_i$ and $\mathfrak{G}_j \in [0, \eta(I_j)]$, so $\mathfrak{G}_j \subseteq \eta(I_j)$. To establish the reverse containment, take $K \in (\bigvee_{i=1}^{n} \mathfrak{G}_i) \cap \eta(I_j)$. Since every right ideal of $R = \prod_{i=1}^{n} R_i$ has the form $K_1 \times K_2 \times \cdots \times K_n$ for suitable right ideals K_i of R_i, and since $K \in \eta(I_j)$ by hypothesis, we must have

$$K = R_1 \times R_2 \times \cdots \times R_{j-1} \times K_j \times R_{j+1} \times \cdots \times R_n \tag{1.19}$$

for some right ideal K_j of R_j. Since $K \in \bigvee_{i=1}^{n} \mathfrak{G}_i$, there exist $A_i \in \mathfrak{G}_i$ for each $i \in \{1, 2, \ldots, n\}$ such that $K \supseteq \bigcap_{i=1}^{n} A_i$ (see (1.2)). Take $i \in \{1, 2, \ldots, n\}$. Since $A_i \in \mathfrak{G}_i \subseteq \eta(I_i)$, A_i has the form

$$A_i = R_1 \times R_2 \times \cdots \times R_{i-1} \times B_i \times R_{i+1} \times \cdots \times R_n,$$

for some right ideal B_i of R_i. Then

$$K \supseteq \bigcap_{i=1}^{n} A_i = B_1 \times B_2 \times \cdots \times B_n.$$

The above, together with (1.19), imply that $K_j \supseteq B_j$, so

$$K \supseteq R_1 \times R_2 \times \cdots \times R_{j-1} \times B_j \times R_{j+1} \times \cdots \times R_n = A_j \in \mathfrak{G}_j.$$

We have thus shown that $(\bigvee_{i=1}^{n} \mathfrak{G}_i) \cap \eta(I_j) \subseteq \mathfrak{G}_j$, whence (1.18) holds.

We now show that for each $\mathfrak{F} \in \operatorname{Fil} R_R$,

$$\mathfrak{F} = \bigvee_{i=1}^{n} (\mathfrak{F} \cap \eta(I_i)). \tag{1.20}$$

The containment $\mathfrak{F} \supseteq \bigvee_{i=1}^{n} (\mathfrak{F} \cap \eta(I_i))$ is clear. To establish the reverse containment, take $K \in \mathfrak{F}$. Write $K = K_1 \times K_2 \times \cdots \times K_n$ with each K_i a right ideal of R_i. For each $i \in \{1, 2, \ldots, n\}$, put

$$A_i = R_1 \times R_2 \times \cdots \times R_{i-1} \times K_i \times R_{i+1} \times \cdots \times R_n.$$

Since $A_i \supseteq K$ and $A_i \supseteq I_i$ for each $i \in \{1, 2, \ldots, n\}$, we have $A_i \in \mathfrak{F} \cap \eta(I_i)$. It follows that $K = \bigcap_{i=1}^n A_i \in \bigvee_{i=1}^n (\mathfrak{F} \cap \eta(I_i))$. We have thus shown that $\mathfrak{F} \subseteq \bigvee_{i=1}^n (\mathfrak{F} \cap \eta(I_i))$, whence equality.

Identities (1.18) and (1.20) imply that the maps defined in (1.16) and (1.17) are inverses of each other. Thus $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ and $\prod_{i=1}^n [0, \eta(I_i)]^{\mathrm{du}}$ are isomorphic complete lattice ordered monoids. The Correspondence Theorem (Theorem 1.22) tells us that for each $i \in \{1, 2, \ldots, n\}$, $[\mathrm{Fil}\,(R/I_i)_{R/I_i}]^{\mathrm{du}}$ and $[0, \eta(I_i)]^{\mathrm{du}}$ are isomorphic complete lattice ordered monoids. Noting that $R/I_i \cong R_i$ for each $i \in \{1, 2, \ldots, n\}$, we have thus proved the following.

Theorem 1.26 *Let $\{R_i : 1 \leq i \leq n\}$ be a finite family of rings and put $R = \prod_{i=1}^n R_i$. Then $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ and $\prod_{i=1}^n [\mathrm{Fil}\, {R_i}_{R_i}]^{\mathrm{du}}$ are isomorphic complete lattice ordered monoids.*

Recall that rings R and S with identity are said to be *(right) Morita equivalent* if the categories of unital right R-modules and unital right S-modules are equivalent in the usual category theoretic sense. That is to say, there exist additive covariant functors $F : \mathrm{Mod}\text{-}R \to \mathrm{Mod}\text{-}S$ and $G : \mathrm{Mod}\text{-}S \to \mathrm{Mod}\text{-}R$ such that $FG \cong 1_{\mathrm{Mod}\text{-}S}$ and $GF \cong 1_{\mathrm{Mod}\text{-}R}$. We remind the reader that Morita equivalence is a left-right symmetric notion.

Theorem 1.27 below asserts that if R and S are Morita equivalent rings, then the structures $\mathrm{Fil}\, R_R$ and $\mathrm{Fil}\, S_S$ are isomorphic. We shall provide a very brief explanation of this fact. A detailed proof may be found in [33, Theorem 2, p. 102].

If F and G are functors between the categories $\mathrm{Mod}\text{-}R$ and $\mathrm{Mod}\text{-}S$ defined as above, then it is easily shown that they preserve and reflect monomorphisms, epimorphisms and direct sums. From this it may be deduced that the mapping from the set $\mathrm{HP}R_R$ of all hereditary pretorsion classes in $\mathrm{Mod}\text{-}R$ to the set $\mathrm{HP}S_S$, given by $\mathcal{T} \mapsto F[\mathcal{T}]$, is a bijection. Since there is a bijective correspondence between the sets $\mathrm{HP}R_R$ and $\mathrm{Fil}\, R_R$ for all rings R (see Theorem 1.6), the aforementioned bijection between $\mathrm{HP}R_R$ and $\mathrm{HP}S_S$ induces a bijection between $\mathrm{Fil}\, R_R$ and $\mathrm{Fil}\, S_S$, and this latter bijection can be shown to preserve both the lattice and monoid operations.

We thus have the following.

Theorem 1.27 *If R and S are Morita equivalent rings, then $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ and $[\mathrm{Fil}\, S_S]^{\mathrm{du}}$ are isomorphic complete lattice ordered monoids.*

Chapter 2

Two-sided residuation in the lattice ordered monoid of topologizing filters

2.1 Characterizations of two-sided residuation in $\mathrm{Fil}\,R_R$ for an arbitrary ring

In this chapter we exhibit a number of equivalent conditions which characterize, in module theoretic terms, when the right residual of a pair of elements in $[\mathrm{Fil}\,R_R]^{\mathrm{du}}$ exists.
A submodule U of $M \in \mathrm{Mod}\text{-}R$ is called a *hereditary pretorsion submodule* of M if $U = \mathcal{T}(M)$ for some hereditary pretorsion class $\mathcal{T}$ of $\mathrm{Mod}\text{-}R$, or equivalently, $U = \mathcal{T}_{\mathfrak{F}}(M)$ for some $\mathfrak{F} \in \mathrm{Fil}\,R_R$.

Theorem 2.1 *Let $\mathfrak{F}$ and $\mathfrak{G}$ be right topologizing filters on a ring R with associated hereditary pretorsion classes $\mathcal{T}_{\mathfrak{F}}$ and $\mathcal{T}_{\mathfrak{G}}$. Then the following statements are equivalent:*

(a) The right residual $\mathfrak{G}^{-1}\mathfrak{F}$ of $\mathfrak{F}$ by $\mathfrak{G}$ exists;

(b) There exists a subgenerator M of $\mathcal{T}_{\mathfrak{F}}$ such that the family of all $\mathcal{T}_{\mathfrak{G}}$-dense hereditary pretorsion submodules of M has a smallest member;

(c) For every subgenerator M of $\mathcal{T}_{\mathfrak{F}}$, the family of all $\mathcal{T}_{\mathfrak{G}}$-dense hereditary pretorsion submodules of M has a smallest member.

Proof. (c)$\Rightarrow$(b) is obvious.

(b)$\Rightarrow$(a) Let M be a subgenertaor for $\mathcal{T}_{\mathfrak{F}}$ satisfying (b). Denote by $M_{\mathfrak{G}}$ the smallest $\mathcal{T}_{\mathfrak{G}}$-dense hereditary pretorsion submodule of M. Put $\mathfrak{H} = \mathfrak{F}_{SHC\{M_{\mathfrak{G}}\}}$. We shall demonstrate that $\mathfrak{H} = \mathfrak{G}^{-1}\mathfrak{F}$. Consider the short exact sequence

$$0 \to M_{\mathfrak{G}} \to M \to M/M_{\mathfrak{G}} \to 0.$$

Since $M_{\mathfrak{G}} \in \mathcal{T}_{\mathfrak{H}}$ and $M/M_{\mathfrak{G}} \in \mathcal{T}_{\mathfrak{G}}$ (because $M_{\mathfrak{G}}$ is $\mathcal{T}_{\mathfrak{G}}$-dense in M), $M \in \mathcal{T}_{\mathfrak{G}:\mathfrak{H}}$. Since M is a subgenerator for $\mathcal{T}_{\mathfrak{F}}$, $\mathcal{T}_{\mathfrak{F}} \subseteq \mathcal{T}_{\mathfrak{G}:\mathfrak{H}}$, whence $\mathfrak{F} \subseteq \mathfrak{G} : \mathfrak{H}$.

Now suppose $\mathfrak{F} \subseteq \mathfrak{G} : \mathfrak{H}'$ with $\mathfrak{H}' \in \operatorname{Fil} R_R$. Since $M \in \mathcal{T}_{\mathfrak{F}} \subseteq \mathcal{T}_{\mathfrak{G}:\mathfrak{H}'}$, we must have $M/\mathcal{T}_{\mathfrak{H}'}(M) = \mathcal{T}_{\mathfrak{G}:\mathfrak{H}'}(M)/\mathcal{T}_{\mathfrak{H}'}(M) = \mathcal{T}_{\mathfrak{G}}(M/\mathcal{T}_{\mathfrak{H}'}(M)) \in \mathcal{T}_{\mathfrak{G}}$, so $\mathcal{T}_{\mathfrak{H}'}(M)$ is a $\mathcal{T}_{\mathfrak{G}}$-dense submodule of M; $\mathcal{T}_{\mathfrak{H}'}(M)$ is also, quite obviously, a hereditary pretorsion submodule of M. It follows from the minimality of $M_{\mathfrak{G}}$ that $M_{\mathfrak{G}} \subseteq \mathcal{T}_{\mathfrak{H}'}(M)$, so $M_{\mathfrak{G}} \in \mathcal{T}_{\mathfrak{H}'}$. Since $\mathfrak{H}$ is the smallest member of $\operatorname{Fil} R_R$ whose associated hereditary pretorsion class contains $M_{\mathfrak{G}}$, we must have $\mathfrak{H} \subseteq \mathfrak{H}'$. We conclude that $\mathfrak{H} = \mathfrak{G}^{-1}\mathfrak{F}$.

(a)$\Rightarrow$(c) Let M be an arbitrary subgenerator for $\mathcal{T}_{\mathfrak{F}}$. Put $\mathfrak{H} = \mathfrak{G}^{-1}\mathfrak{F}$. Note that $\mathfrak{G} : \mathfrak{H} \supseteq \mathfrak{F}$. Consider the short exact sequence

$$0 \to \mathcal{T}_{\mathfrak{H}}(M) \to M \to M/\mathcal{T}_{\mathfrak{H}}(M) \to 0.$$

Since $\mathfrak{G} : \mathfrak{H} \supseteq \mathfrak{F}$, $M \in \mathcal{T}_{\mathfrak{F}} \subseteq \mathcal{T}_{\mathfrak{G}:\mathfrak{H}}$. It follows that $M/\mathcal{T}_{\mathfrak{H}}(M) = \mathcal{T}_{\mathfrak{G}:\mathfrak{H}}(M)/\mathcal{T}_{\mathfrak{H}}(M) = \mathcal{T}_{\mathfrak{G}}(M/\mathcal{T}_{\mathfrak{H}}(M)) \in \mathcal{T}_{\mathfrak{G}}$, so $\mathcal{T}_{\mathfrak{H}}(M)$ is a $\mathcal{T}_{\mathfrak{G}}$-dense hereditary pretorsion submodule of M.

Let N be an arbitrary $\mathcal{T}_{\mathfrak{G}}$-dense hereditary pretorsion submodule of M. Then $N = \mathcal{T}_{\mathfrak{H}'}(M)$ for some $\mathfrak{H}' \in \operatorname{Fil} R_R$.

Consider the short exact sequence

$$0 \to N \to M \to M/N \to 0.$$

Since $N \in \mathcal{T}_{\mathfrak{H}'}$ and $M/N \in \mathcal{T}_{\mathfrak{G}}$, we must have $M \in \mathcal{T}_{\mathfrak{G}:\mathfrak{H}'}$, whence $\mathcal{T}_{\mathfrak{F}} = \mathsf{SHC}\{M\} \subseteq \mathcal{T}_{\mathfrak{G}:\mathfrak{H}'}$. This implies $\mathfrak{F} \subseteq \mathfrak{G} : \mathfrak{H}'$, so $\mathfrak{H} = \mathfrak{G}^{-1}\mathfrak{F} \subseteq \mathfrak{H}'$ and $\mathcal{T}_{\mathfrak{H}}(M) \subseteq \mathcal{T}_{\mathfrak{H}'}(M) = N$. We conclude that $\mathcal{T}_{\mathfrak{H}}(M)$ is the smallest $\mathcal{T}_{\mathfrak{G}}$-dense hereditary pretorsion submodule of M. $\blacksquare$

If $N \leq M \in \mathrm{Mod}\text{-}R$, standard theory tells us that the canonical epimorphism $\pi_N : M \to M/N$ induces a lattice isomorphism $\pi_N[\_]$ from the set of all submodules of M containing N to the set of all submodules of M/N, and that the mapping $\pi_N^{-1}[\_]$ constitutes an inverse isomorphism.

If $\mathfrak{G} \in \mathrm{Fil}\, R_R$, then the maps $\pi_N[\_]$ and $\pi_N^{-1}[\_]$ restrict to mutually inverse bijections between the sets $\mathbb{L}_{\mathfrak{G}}[N, M]$ of all $\mathcal{T}_{\mathfrak{G}}$-dense submodules of M containing N, and $\mathbb{L}_{\mathfrak{G}}[M/N]$ comprising all $\mathcal{T}_{\mathfrak{G}}$-dense submodules of M/N. To see this, observe that if $N \subseteq L \leq M$, then $(M/N)/(L/N) \cong M/L$, thus L will be $\mathcal{T}_{\mathfrak{G}}$-dense in M if and only if $L/N = \pi_N[L]$ is $\mathcal{T}_{\mathfrak{G}}$-dense in M/N.
We make use of these rudimentary observations, and adopt the notation used, in the next result.

Corollary 2.2 *Let $\mathfrak{F}$ and $\mathfrak{G}$ be right topologizing filters on a ring R. Suppose $\mathfrak{F}$ is compact so that $\mathfrak{F} = \mathfrak{F}_{\mathsf{SHC}\{R/A\}}$ for some $A \leq R_R$. The following statements are equivalent:*

(a) The right residual $\mathfrak{G}^{-1}\mathfrak{F}$ of $\mathfrak{F}$ by $\mathfrak{G}$ exists;

(b) The family of all $\mathcal{T}_{\mathfrak{G}}$-dense hereditary pretorsion submodule of R/A has a smallest member;

(c) The family $\{B \in \mathfrak{G} : B \supseteq A$ and B/A is a hereditary pretorsion submodules of $R/A\}$ has a smallest member.

Proof. Inasmuch as R/A is a subgenerator for $\mathcal{T}_{\mathfrak{F}}$, (a)$\Rightarrow$(b) follows from Theorem 2.1 ((a)$\Rightarrow$(c)), whilst (b)$\Rightarrow$(a) is a consequence of Theorem 2.1 ((b)$\Rightarrow$(a)).
(b)$\Leftrightarrow$(c) Put $X = \{B \in \mathfrak{G} : B \supseteq A$ and B/A is a hereditary pretorsion submodule of $R/A\}$ and let Y be the family of all $\mathcal{T}_{\mathfrak{G}}$-dense hereditary pretorsion submodules of R/A. Let $\pi_A : R_R \to R/A$ be the canonical epimorphism. Observe that $B \in X \subseteq \mathbb{L}_{\mathfrak{G}}[A, R_R]$ if and only if $\pi_A[B] \in Y \subseteq \mathbb{L}_{\mathfrak{G}}[R/A]$. This means that the map $\pi_A[\_]$ restricts to a lattice isomorphism from X to Y. The set X will thus possesses a smallest element, that is Statement (c) will hold, precisely if the set Y possess a smallest element, that is, Statement (b) holds. $\blacksquare$

Theorem 2.3 *Let $\mathfrak{F}$, $\mathfrak{G}$ be right topologizing filters on a ring R with associated hereditary pretorsion classes $\mathcal{T}_{\mathfrak{F}}$ and $\mathcal{T}_{\mathfrak{G}}$. The following statements are equivalent:*

(a) The right residual $\mathfrak{G}^{-1}\mathfrak{F}'$ of $\mathfrak{F}'$ by $\mathfrak{G}$ exists for all $\mathfrak{F}' \in \operatorname{Fil} R_R$ satisfying $\mathfrak{F}' \subseteq \mathfrak{F}$;

(b) For each $M \in \mathcal{T}_{\mathfrak{F}}$ the family of all $\mathcal{T}_{\mathfrak{G}}$-dense hereditary pretorsion submodules of M has a smallest member;

(c) For each finitely generated $M \in \mathcal{T}_{\mathfrak{F}}$ the family of all $\mathcal{T}_{\mathfrak{G}}$-dense hereditary pretorsion submodules of M has a smallest member;

(d) For each cyclic module $M \in \mathcal{T}_{\mathfrak{F}}$ the family of all $\mathcal{T}_{\mathfrak{G}}$-dense hereditary pretorsion submodules of M has a smallest member;

(e) For each $A \in \mathfrak{F}$ the family of all $\mathcal{T}_{\mathfrak{G}}$-dense hereditary pretorsion submodules of R/A has a smallest member;

(f) For each $A \in \mathfrak{F}$ the family $\{B \in \mathfrak{G} : B \supseteq A$ and B/A is a hereditary pretorsion submodule of $R/A\}$ has a smallest member.

Proof. (a)⇒(b) Take $M \in \mathcal{T}_{\mathfrak{F}}$ and put $\mathfrak{F}' = \mathfrak{F}_{\mathsf{SHC}}\{M\}$. Since M is a subgenerator for $\mathcal{T}_{\mathfrak{F}'}$ it follows from Theorem 2.1 ((a)⇒(c)), that the family of all $\mathcal{T}_{\mathfrak{G}}$-dense hereditary pretorsion submodules of M has a smallest member, as required.

(b)⇒(c)⇒(d) is obvious.

(d)⇔(e) is an immediate consequence of the fact that for a right R-module M, $M \in \mathcal{T}_{\mathfrak{F}}$ is cyclic if and only if $M \cong R/A$ for some $A \in \mathfrak{F}$ (see (1.6)).

(e)⇔(f) Take $A \in \mathfrak{F}$. Put $X = \{B \in \mathfrak{G} : B \supseteq A$ and B/A is a hereditary pretorsion submodule of $R/A\}$ and let Y be the family of all $\mathcal{T}_{\mathfrak{G}}$-dense hereditary pretorsion submodules of R/A. In the proof of Corollary 2.2 it was noted that X will have a smallest member precisely when Y does. The equivalence of (e) and (f) follows.

(e)⇒(a) Note first that if (e) holds in respect of $\mathfrak{F}, \mathfrak{G} \in \operatorname{Fil} R_R$, then it will hold for $\mathfrak{F}', \mathfrak{G}$ where $\mathfrak{F}'$ is any member of $\operatorname{Fil} R_R$ satisfying $\mathfrak{F}' \subseteq \mathfrak{F}$. It thus suffices to verify that (e) implies the existence of the right residual $\mathfrak{G}^{-1}\mathfrak{F}$ only.

To this end, note that (e) and Corollary 2.2 ((b)⇒(a)) imply that the right residual $\mathfrak{G}^{-1}\mathfrak{F}_{\mathsf{SHC}}\{R/A\}$ exists for all $A \in \mathfrak{F}$. Since

$$\mathfrak{F} = \bigvee_{A\in\mathfrak{F}} \mathfrak{F}_{\mathsf{SHC}}\{R/A\}$$

by (1.9), it follows from Proposition 1.16 that the right residual $\mathfrak{G}^{-1}\mathfrak{F}$ exists, as required. ∎

If $\mathcal{C}$ is a nonempty class of right R-modules, we shall call a right R-module P *projective with respect to* $\mathcal{C}$ or $\mathcal{C}$*-projective* if, given any epimorphism $\pi : A \to B$ with $A, B \in \mathcal{C}$ and any R-homomorphism $\alpha : P \to B$, there exists an R-homomorphism $\beta : P \to A$ which makes the diagram below with exact row commute.

$$\begin{tikzcd}
A \arrow[r, "\pi"] & B \arrow[r] & 0 \\
& P \arrow[u, "\alpha"'] \arrow[ul, dotted, "\beta"] &
\end{tikzcd}$$

Recall that a submodule N of a right R-module M is said to be *fully invariant* in M if $f[N] \subseteq N$ for every $f \in \mathrm{End}_R(M)$.

The following lemma is needed for Theorem 2.5.

Lemma 2.4 *Suppose right R-module P is projective with respect to the class* SHC$\{P\}$*. If U and V are fully invariant submodules of P, then*

$$U \subseteq V \ \textit{ iff } \mathsf{SHC}\{P/U\} \supseteq \mathsf{SHC}\{P/V\}.$$

Proof. Suppose $U \subseteq V$. Then P/V is an epimorphic image of P/U from which it follows that $P/V \in \mathsf{SHC}\{P/U\}$, whence $\mathsf{SHC}\{P/U\} \supseteq \mathsf{SHC}\{P/V\}$.

To establish the converse suppose $\mathsf{SHC}\{P/U\} \supseteq \mathsf{SHC}\{P/V\}$ so that $P/V \in \mathsf{SHC}\{P/U\}$. This entails the existence of an index set Δ, a right R-module M, an epimorphism $\psi : (P/U)^{(\Delta)} \to M$, and a monomorphism $\kappa : P/V \to M$. Let $\pi : P^{(\Delta)} \to (P/U)^{(\Delta)}$ and $\varphi : P \to P/V$ denote the canonical projections.

$$\begin{tikzcd}
(P/U)^{(\Delta)} \arrow[rr, two heads, "\psi"] & & M \\
P^{(\Delta)} \arrow[u, "\pi"] & P \arrow[l, dotted, "\alpha"] \arrow[r, two heads, "\varphi"] & P/V \arrow[u, hook, "\kappa"']
\end{tikzcd}$$

Since P is SHC$\{P\}$-projective and $P^{(\Delta)}$ and M both belong to SHC$\{P\}$, the homomorphism $\kappa\varphi : P \to M$ can be factored through the epimorphism $\psi\pi : P^{(\Delta)} \to M$ yielding an R-homomorphism $\alpha : P \to P^{(\Delta)}$ making the above diagram commute.

Since κ is monic, $\ker \kappa\varphi = \ker \varphi = V$. Since U is fully invariant in P, $\alpha[U] \subseteq U^{(\Delta)} = \ker \pi$, so $U \subseteq \ker \psi\pi\alpha = \ker \kappa\varphi = V$. Thus $U \subseteq V$, as required. ■

Theorem 2.5 *Let R be a ring for which $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ is two-sided residuated. If P is any finitely generated right R-module that is projective with respect to* SHC$\{P\}$*, then P satisfies the DCC on hereditary pretorsion submodules.*

Proof. Suppose $P \in \mathrm{Mod}\text{-}R$ is finitely generated and SHC$\{P\}$-projective. Let

$$U_1 \supseteq U_2 \supseteq \cdots$$

be a descending chain of hereditary pretorsion submodules of P. Inasmuch as each U_i is fully invariant in P, we obtain, in light of the previous lemma, the following ascending chain of hereditary pretorsion classes in $\mathrm{Mod}\text{-}R$.

$$\mathsf{SHC}\{P/U_1\} \subseteq \mathsf{SHC}\{P/U_2\} \subseteq \cdots$$

It is easily seen that $\mathcal{T} = \bigcup_{n\in\mathbb{N}} \mathsf{SHC}\{P/U_n\}$ is the join of the family $\{\mathsf{SHC}\{P/U_n\} : n \in \mathbb{N}\}$ in the lattice $\mathrm{HP}\, R_R$ of all hereditary pretorsion classes in $\mathrm{Mod}\text{-}R$. Let $\mathfrak{G} = \mathfrak{F}_{\mathcal{T}}$ be the right topologizing filter on R associated with $\mathcal{T}$. Since $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ is two-sided residuated, it follows from Theorem 2.3 ((a)⇒(b) or (a)⇒(c)), that P has a smallest $\mathcal{T}$-dense hereditary pretorsion submodule V, say. Since $P/V \in \mathcal{T}$, we must have SHC$\{P/V\} \subseteq \mathcal{T} = \bigcup_{n\in\mathbb{N}}$ SHC$\{P/U_n\}$. Since P and hence P/V is a finitely generated right R-module, SHC$\{P/V\}$ is a compact member of $\mathrm{HP}\, R_R$. It follows that SHC$\{P/V\} \subseteq$ SHC$\{P/U_n\}$ for some $n \in \mathbb{N}$. Inasmuch as V is a fully invariant submodule of P, the previous lemma entails $V \supseteq U_n$. Since each U_i is a $\mathcal{T}$-dense hereditary pretorsion submodule of P, it follows from the minimality of V that $V \subseteq U_i$ for all $i \in \mathbb{N}$. This implies $U_i = U_n$ for all $i \geq n$, so the descending chain $U_1 \supseteq U_2 \supseteq \cdots$ terminates. ■

The following corollary will be needed in the proof of Theorem 3.13.

Corollary 2.6 *If R is a ring for which $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ is two-sided residuated, then $(R/I)_R$ satisfies the DCC on hereditary pretorsion submodules for all proper ideals I of R.*

Proof. This is an immediate consequence of the previous theorem and the routine fact that for any ring R and proper ideal I of R, $(R/I)_R$ is projective with respect to SHC$\{(R/I)_R\}$. ■

Remark 2.7 *We do not know if the converse of the above corollary holds. Nor do we know, with reference to Theorem 2.5, whether $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ is forced to be two-sided residuated if every finitely generated and* SHC$\{P\}$*-projective P satisfies the DCC on hereditary pretorsion submodules. Certainly, as results in Chapter 3 show, if the ring R is commutative, then the two conditions of Corollary 2.6 are equivalent (see Theorem 3.13 ((b)⇔(h))).*

2.2 Right fully bounded noetherian rings

In Remark 1.15 we noted that if R is a commutative noetherian ring, then $\mathrm{Fil}\, R_R$ is commutative and from this can be drawn the easy inference that $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ is two-sided residuated, for $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ is left residuated for all rings R. Inasmuch as right fully bounded noetherian rings (these will be defined below) are a natural generalization of commutative noetherian rings and share many properties with their progenitor, it is natural to ask what properties relating to $\mathrm{Fil}\, R_R$ carry across from commutative noetherian rings to fully bounded noetherian rings. Commutativity of $\mathrm{Fil}\, R_R$ is too much to expect, however, for even in right artinain rings, and such rings are right fully bounded noetherian, ideal multiplication need not commute and since $\mathrm{Id}\, R$ embeds in $\mathrm{Fil}\, R_R$ (Theorem 1.14) this would imply the failure of commutativity in the larger $\mathrm{Fil}\, R_R$. A more insightful question is to ask whether the weaker two-sided residuation property holds for $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ in all right fully bounded noetherian rings R. The main theorem (Theorem 2.12) in this section answers this question in the affirmative.

Recall that a ring R is said to be right bounded if every essential right ideal of R contains a nonzero two-sided ideals that is essential in R_R. If R/P is right bounded for all prime ideals P of R, we say that R *is right fully bounded*.

Following the standard notational convention, we shall henceforth use the acronym right "FBN" to abbreviate "right fully bounded noetherian".

Finitely annihilated modules have a torsion theoretic characterization that we shall need to exploit: M is finitely annihilated if and only if

$$\mathsf{SHC}\{M\} = \mathsf{SHC}\{R/I\} = \mathcal{T}_{\eta(I)}, \text{where } \ I = M^{-1}0. \tag{2.1}$$

Following Gabriel [14] we say that a ring R satisfies Condition H (on the right) if every finitely generated right R-module is finitely annihilated.

Proposition 2.8 *[5, Proposition 7.6, p. 101 and Proposition 7.8, p. 102] The following statements are equivalent for a right noetherian ring R:*

(a) R is right fully bounded;

(b) R satisfies Condition H on the right.

We shall call a submodule U of a right R-module M an annihilator submodule if

$$U = \{x \in M : xI = 0\} \text{ for some } I \in \mathrm{Id}\, R.$$

Lemma 2.9 *Let R be a right FBN ring. If L is any finitely generated right R-module, then every pretorsion submodule of L is an annihilator submodule.*

Proof. Let $L \in \text{Mod-}R$ be finitely generated. Suppose $U = \mathcal{T}_{\mathfrak{F}}(L)$ is a hereditary pretorsion submodule of L where $\mathfrak{F} \in \operatorname{Fil} R_R$. Since R is right noetherian and L is finitely generated, so is U. By Proposition 2.8, R satisfies Condition H, so U is finitely annihilated. Put $I = U^{-1}0$. Since U is finitely annihilated, it follows from (2.1) that U is a subgenerator for $\mathcal{T}_{\eta(I)}$. Since U is in $\mathcal{T}_{\mathfrak{F}}$, we must have $\mathcal{T}_{\eta(I)} \subseteq \mathcal{T}_{\mathfrak{F}}$, whence $\mathcal{T}_{\eta(I)}(L) = \{x \in L : xI = 0\} \subseteq \mathcal{T}_{\mathfrak{F}}(L) = U$.
On the other hand, $\mathcal{T}_{\eta(I)} = \{x \in L : xI = 0\} \supseteq U$, so $U = \mathcal{T}_{\mathfrak{F}}(L) = \{x \in L : xI = 0\}$ is an annihilator submodule of L. ■

Lemma 2.10 *If a ring R satisfies the ACC on ideals, then every right R-module satisfies the DCC on annihilator submodules.*

Proof. Let $M \in \text{Mod-}R$. Denote by $\mathcal{L}(M)$ the lattice of all submodules of M. It is easily checked that the maps $\alpha : \operatorname{Id} R \to \mathcal{L}(M)$ and $\beta : \mathcal{L}(M) \to \operatorname{Id} R$ defined by $\alpha(I) = \{x \in M : xI = 0\}$ $(I \in \operatorname{Id} R)$ and $\beta(N) = N^{-1}0$ $(N \in \mathcal{L}(M))$ constitute an (antitone) Galois connection between the lattices IdR and $\mathcal{L}(M)$. It follows that if $\operatorname{Id} R$ satisfies the ACC, then the image of α, which comprises the set of all annihilator submodules of M, satisfies the DCC. ■

The two previous lemmas now yield:

Corollary 2.11 *Let R be right FBN ring. Then every finitely generated right R-module satisfies the DCC on hereditary pretorsion submodules.*

Theorem 2.12 *If R is a right FBN ring, then $[\operatorname{Fil} R_R]^{\mathrm{du}}$ is two-sided residuated.*

Proof. Let R be a right FBN ring. Take $\mathfrak{F}, \mathfrak{G} \in \operatorname{Fil} R_R$ and let $M \in \mathcal{T}_{\mathfrak{F}}$ be finitely generated. By the previous corollary, the family of all $\mathcal{T}_{\mathfrak{G}}$-dense hereditary pretorsion submodules of M has a minimal member. Since the families of $\mathcal{T}_{\mathfrak{G}}$-dense, and hereditary pretorsion, submodules of an arbitrary module are both closed under finite intersections, such a minimal member must be a (unique) smallest member. It follows from Theorem 2.3 ((c)⇒(a)) that the right residual $\mathfrak{G}^{-1}\mathfrak{F}'$ of $\mathfrak{F}'$ by $\mathfrak{G}$ exists for all $\mathfrak{F}' \in \mathrm{Fil}\, R_R$ with $\mathfrak{F}' \subseteq \mathfrak{F}$. In particular, the right residual $\mathfrak{G}^{-1}\mathfrak{F}$ of $\mathfrak{F}$ by $\mathfrak{G}$ exists. Since $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ is known to be left residuated for all rings R (Theorem 1.9), we conclude that $[[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ is two-sided residuated. ■

Remark 2.13 *A weakness of Theorem 2.12, is the absence of an example showing that no part of the FBN hypothesis may be dispensed with for the theorem's conclusion to remain valid. Certainly there are non-noetherian rings R for which $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ fails to be right residuated, as results in the next show. However, we have no example of a right noetherian ring R for which $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ is left but not right residuated.*

Chapter 3

Topologizing filters in commutative rings

3.1 Two-sided residuation in $\mathrm{Fil}\, R_R$

It is evident from the explanation provided in Remark 1.15 that if R is any ring for which (the monoid operation : on) $\mathrm{Fil}\, R_R$ is commutative, then $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ will be two-sided residuated. The first main result of this section (Theorem 3.7) shows that the converse is true whenever the ring R is commutative.

If R is a commutative ring, we may omit the subscript R in $\mathrm{Fil}\, R_R$ and write $\mathrm{Fil}\, R$ in its place. Let $\mathfrak{F} \in \mathrm{Fil}\, R_R$. We shall call a subset X of $\mathfrak{F}$ a *cofinal* set for $\mathfrak{F}$ if given any $A \in \mathfrak{F}$ there exists $B \in X$ such that $A \supseteq B$.

We shall make frequent use of the following result from [34, Lemma 3 and Remark 2, p. 90].

Lemma 3.1 *Let R be an arbitrary ring and $\mathfrak{F}, \mathfrak{G} \in \mathrm{Fil}\, R_R$. If $\{I_\gamma \ : \ \gamma \in \Gamma\}$ is a cofinal set of finitely generated right ideals for $\mathfrak{F}$ and $\{J_\theta \ : \ \theta \in \Theta\}$ is a cofinal set of (two-sided) ideals for $\mathfrak{G}$, then $\{I_\gamma J_\theta \ : \ \gamma \in \Gamma, \theta \in \Theta\}$ is a cofinal set for $\mathfrak{F} : \mathfrak{G}$.*

Lemma 3.2 *Let R be a commutative ring.*

(a) If $\mathfrak{F}, \mathfrak{G} \in \mathrm{Fil}\, R_R$ both possess cofinal sets comprising finitely generated ideals, then $\mathfrak{F} : \mathfrak{G} = \mathfrak{G} : \mathfrak{F}$.

(b) If I is a finitely generated ideal of R, then $\eta(I)$ is central in $\mathrm{Fil}\, R_R$, that is to say, $\eta(I) : \mathfrak{F} = \mathfrak{F} : \eta(I)$ for all $\mathfrak{F} \in \mathrm{Fil}\, R_R$.

Proof. (a) Suppose $\{I_\gamma : \gamma \in \Gamma\}$ and $\{J_\theta : \theta \in \Theta\}$ are cofinal sets of finitely generated ideals for $\mathfrak{F}$ and $\mathfrak{G}$, respectively. By Lemma 3.1, $\{I_\gamma J_\theta : \gamma \in \Gamma, \theta \in \Theta\}$ is a cofinal set for $\mathfrak{F} : \mathfrak{G}$.

Interchanging the roles of $\mathfrak{F}$ and $\mathfrak{G}$ in Lemma 3.1, we infer that $\{J_\theta I_\gamma : \gamma \in \Gamma, \theta \in \Theta\}$ is a cofinal set for $\mathfrak{G} : \mathfrak{F}$. Since R is commutative, $I_\gamma J_\theta = J_\theta I_\gamma$ for all $\gamma \in \Gamma, \theta \in \Theta$ from which it follows that $\mathfrak{F} : \mathfrak{G} = \mathfrak{G} : \mathfrak{F}$.

(b) Since $\{I\}$ is a cofinal set for $\eta(I)$ comprising a (single) finitely generated ideal, it follows from Lemma 3.1 that $\{IK : K \in \mathfrak{F}\}$ is a cofinal set for $\eta(I) : \mathfrak{F}$.

Now consider $\mathfrak{F} : \eta(I)$. By definition, $K \in \mathfrak{F} : \eta(I)$ if and only if there exists $B \in \mathfrak{F}$ such that $B \supseteq K$ and $B/K \in \mathcal{T}_{\eta(I)}$, whence $BI \subseteq K$. Thus $\{BI : B \in \mathfrak{F}\}$ is cofinal for $\mathfrak{F} : \eta(I)$. Since R is commutative, we have $IK = KI$ for all $K \in \mathfrak{F}$, so the sets $\{IK : K \in \mathfrak{F}\}$ and $\{BI : B \in \mathfrak{F}\}$ coincide. It follows that $\eta(I) : \mathfrak{F} = \mathfrak{F} : \eta(I)$. ∎

Proposition 3.3 *Let R be a commutative ring for which $[\mathrm{Fil}\, R]^{\mathrm{du}}$ is two-sided residuated. If I is an arbitrary ideal of R, then $\eta(I)$ is central in $\mathrm{Fil}\, R$.*

Proof. Let I be an arbitrary ideal of R. Since the map η takes arbitrary (possibly infinite) joins in $\mathrm{Id}\, R$ to meets in $\mathrm{Fil}\, R$ (see Theorem 1.14), it follows that

$$\eta(I) = \eta\left(\sum_{a \in I} aR\right) = \bigcap_{a \in I} \eta(aR).$$

For each $\mathfrak{F} \in \mathrm{Fil}\, R$ we have

$$\begin{aligned}
\eta(I) : \mathfrak{F} &= \left(\bigcap_{a\in I} \eta(aR)\right) : \mathfrak{F} \\
&= \bigcap_{a\in I} (\eta(aR) : \mathfrak{F}) && \text{[since } [\operatorname{Fil} R]^{\mathrm{du}} \text{ is left residuated, the monoid operation distributes over meets on the left - see dual of Proposition 1.2]} \\
&= \bigcap_{a\in I} (\mathfrak{F} : \eta(aR)) && \text{[since } aR \text{ is a finitely generated ideal, } \eta(aR) \text{ is central in } \operatorname{Fil} R \text{ by Lemma 3.2(b)]} \\
&= \mathfrak{F} : \left(\bigcap_{a\in I} \eta(aR)\right) && \text{[by hypothesis, } [\operatorname{Fil} R]^{\mathrm{du}} \text{ is right residuated, so the monoid operation distributes over meets on the right - see Proposition 1.2]} \\
&= \mathfrak{F} : \eta(I).
\end{aligned}$$

We conclude that $\eta(I)$ is central in $\operatorname{Fil} R$. ■

Let P be a poset. Recall that a subset X of P is said to be *downward [resp. upward] directed* if, given any pair of elements x_1, x_2 in X, there exists $y \in X$ such that $x_1 \geq y$ and $x_2 \geq y$ [resp. $x_1 \leq y$ and $x_2 \leq y$].

We omit the proof of the following routine result.

Proposition 3.4 *If X is any upward directed family in $\operatorname{Fil} R_R$, then $\bigcup X \in \operatorname{Fil} R_R$.*

The following is [17, Proposition 3.17(1)*, p. 39].

Proposition 3.5 *If $\mathfrak{F} \in \operatorname{Fil} R_R$ and X is any upward directed family in $\operatorname{Fil} R_R$, then $(\bigcup X) : \mathfrak{F} = \bigcup_{\mathfrak{G}\in X} \mathfrak{G} : \mathfrak{F}$.*

Proposition 3.6 *Let R be a commutative ring for which $[\operatorname{Fil} R]^{\mathrm{du}}$ is two-sided residuated. If $\mathfrak{F} \in \operatorname{Fil} R$ and X is any upward directed family in $\operatorname{Fil} R$, then $\mathfrak{F} : (\bigcup X) = \bigcup_{\mathfrak{G}\in X} \mathfrak{F} : \mathfrak{G}$.*

*This result assumes that the set X is a chain. However, the arguments used transfer mutatis mutandis to the more general case where X is upward directed.

Proof. Note first that since $\bigcup X \supseteq \mathfrak{G}\ \forall \mathfrak{G} \in X$, we have $\mathfrak{F} : (\bigcup X) \supseteq \bigcup_{\mathfrak{G}\in X} \mathfrak{F} : \mathfrak{G}$. Observe that the family $\{\eta(I) : I \in \mathfrak{F}\}$ is upward directed. This is because $\mathfrak{F}$ is a downward directed family in $\operatorname{Id} R$, and the map $\eta : \operatorname{Id} R \to \operatorname{Fil} R$ is order reversing . It follows from Proposition 3.5 that $\bigcup_{I\in\mathfrak{F}} \eta(I) \in \operatorname{Fil} R$, whence $\mathfrak{F} = \bigcup_{I\in\mathfrak{F}} \eta(I)$. Then

$$
\begin{aligned}
\mathfrak{F}:\left(\bigcup X\right) &= \left[\bigcup_{I\in\mathfrak{F}} \eta(I)\right] : \left(\bigcup X\right) \quad \text{[because } \mathfrak{F} = \textstyle\bigcup_{I\in\mathfrak{F}} \eta(I)\text{]} \\
&= \bigcup_{I\in\mathfrak{F}} \left[\eta(I) : \left(\bigcup X\right)\right] \quad \text{[by Proposition 3.5]} \\
&= \bigcup_{I\in\mathfrak{F}} \left[\left(\bigcup X\right) : \eta(I)\right] \quad \text{[because } \eta(I) \text{ is central by Proposition 3.3]} \\
&= \bigcup_{I\in\mathfrak{F}} \left[\bigcup_{\mathfrak{G}\in X} \mathfrak{G} : \eta(I)\right] \quad \text{[by Proposition 3.5]} \\
&= \bigcup_{\mathfrak{G}\in X} \left[\bigcup_{I\in\mathfrak{F}} \mathfrak{G} : \eta(I)\right] \\
&= \bigcup_{\mathfrak{G}\in X} \left[\bigcup_{I\in\mathfrak{F}} \eta(I) : \mathfrak{G}\right] \quad \text{[because } \eta(I) \text{ is central]} \\
&= \bigcup_{\mathfrak{G}\in X} [\mathfrak{F} : \mathfrak{G}] \quad \text{[by Proposition 3.5].}
\end{aligned}
$$

■

Theorem 3.7 *Let R be a commutative ring for which $[\operatorname{Fil} R]^{\mathrm{du}}$ is two-sided residuated. Then $\operatorname{Fil} R$ is commutative, that is to say, $\mathfrak{F} : \mathfrak{G} = \mathfrak{G} : \mathfrak{F}\ \forall \mathfrak{F}, \mathfrak{G} \in \operatorname{Fil} R$.*

Proof. Take $\mathfrak{F}, \mathfrak{G} \in \operatorname{Fil} R_R$ and write $\mathfrak{G} = \bigcup_{I\in\mathfrak{G}} \eta(I)$. Since $\mathfrak{G}$ is a downward directed family in $\operatorname{Id} R$, $\{\eta(I) : I \in \mathfrak{G}\}$ is an upward directed family in $\operatorname{Fil} R_R$. Then

$$
\begin{aligned}
\mathfrak{F} : \mathfrak{G} &= \mathfrak{F} : \left[\bigcup_{I \in \mathfrak{G}} \eta(I) \right] \\
&= \bigcup_{I \in \mathfrak{G}} [\mathfrak{F} : \eta(I)] \quad \text{[by Proposition 3.6]} \\
&= \bigcup_{I \in \mathfrak{G}} [\eta(I) : \mathfrak{F}] \quad \text{[by Proposition 3.3]} \\
&= \left[\bigcup_{I \in \mathfrak{G}} \eta(I) \right] : \mathfrak{F} \quad \text{[by Proposition 3.5]} \\
&= \mathfrak{G} : \mathfrak{F}, \text{ as required.}
\end{aligned}
$$

■

Recall that a member $\mathfrak{F}$ of $\mathrm{Fil}\, R_R$ is compact if and only if $\mathfrak{F} = \mathfrak{F}_{\mathsf{SHC}\{R/K\}}$ for some cyclic right R-module K, and that by (1.9) every $\mathfrak{F} \in \mathrm{Fil}\, R_R$ can be expressed as $\mathfrak{F} = \bigvee_{K \in \mathfrak{F}} \mathfrak{F}_{\mathsf{SHC}\{R/K\}}$. Recall also that a submodule N of $M \in \mathrm{Mod}\text{-}R$ is called a hereditary pretorsion submodule of M, if $N = \mathcal{T}_{\mathfrak{F}}(M)$ for some $\mathfrak{F} \in \mathrm{Fil}\, R_R$.

Proposition 3.8 *Let $M \in \mathrm{Mod}\text{-}R$. Suppose the family $X = \{N \leq M : N = \mathcal{T}_{\mathfrak{F}}(M)$ for some compact $\mathfrak{F} \in \mathrm{Fil}\, R_R\}$ satisfies the maximum condition, that is to say, every nonempty subset of X contains a maximal member, or equivalently, every ascending chain of objects in X, stabilizes. Then every hereditary pretorsion submodule of M is a member of X.*

Proof. Take $\mathfrak{F} \in \mathrm{Fil}\, R_R$ and write $\mathfrak{F} = \bigvee_{K \in \mathfrak{F}} \mathfrak{F}_{\mathsf{SHC}\{R/K\}}$. The map $K \mapsto \mathsf{SHC}\{R/K\}$ clearly constitutes an order reversing map from $\mathfrak{F}$ to the set of hereditary pretorsion classes in $\mathrm{Mod}\text{-}R$. Since $\mathfrak{F}$ is downward directed, the family $\{\mathsf{SHC}\{R/K\} : K \in \mathfrak{F}\}$ is upward directed. Thus $\{\mathfrak{F}_{\mathsf{SHC}\{R/K\}} : K \in \mathfrak{F}\}$ is an upward directed family in $\mathrm{Fil}\, R_R$. It follows from Proposition 3.4, that $\mathfrak{F} = \bigcup_{K \in \mathfrak{F}} \mathfrak{F}_{\mathsf{SHC}\{R/K\}}$. Hence $\mathcal{T}_{\mathfrak{F}}(M) = \bigcup_{K \in \mathfrak{F}} \mathcal{T}_{\mathfrak{F}_{\mathsf{SHC}\{R/K\}}}(M)$. Observe that since each $\mathfrak{F}_{\mathsf{SHC}\{R/K\}}$ is compact, $Y = \{\mathcal{T}_{\mathfrak{F}_{\mathsf{SHC}\{R/K\}}}(M) : K \in \mathfrak{F}\}$ is an upward directed subfamily of X. The hypothesis implies that Y must have a largest member. Thus $\mathcal{T}_{\mathfrak{F}}(M) = \mathcal{T}_{\mathfrak{F}_{\mathsf{SHC}\{R/K\}}}(M)$ for some $K \in \mathfrak{F}$, whence $\mathcal{T}_{\mathfrak{F}}(M) \in X$. ■

Note that if R is commutative, then for every ideal K of R, $\mathfrak{F}_{\mathsf{SHC}\{R/K\}} = \eta(K)$, whence $\mathcal{T}_{\mathfrak{F}_{\mathsf{SHC}\{R/K\}}}(M) = \mathcal{T}_{\eta(K)}(M) = \{x \in M : xK = 0\}$.

We have thus proved the following:

Proposition 3.9 *Let R be a commutative ring. The following statements are equivalent for a submodule N of right R-module M:*

(a) $N = \mathcal{T}_{\mathfrak{F}}(M)$ *for some compact* $\mathfrak{F} \in \mathrm{Fil}\, R_R$*;*

(b) N *is an annihilator submodule of* M*, that is to say,* $N = \{x \in M : xK = 0\}$ *for some ideal* K *of* R*.*

The preceding two propositions yield the following result:

Proposition 3.10 *Let R be a commutative ring and M a right R-module. If the family of all annihilator submodules of M satisfies the maximum condition, that is to say, M satisfies the ACC on annihilator submodules, then every hereditary pretorsion submodule of M is an annihilator submodule.*

The following theorem provides several non-torsion theoretic characterizations of the two-sided residuation property for a commutative ring.

Theorem 3.11 *The following statements are equivalent for a commutative ring R:*

(a) $\mathrm{Fil}\, R$ *is commutative;*

(b) $[\mathrm{Fil}\, R]^{\mathrm{du}}$ *is two-sided residuated;*

(c) The ring R/I satisfies the ACC on annihilator ideals for all proper ideals I of R;

(d) The ring R/I satisfies the DCC on annihilator ideals for all proper ideals I of R;

(e) $(R/I)_R$ *satisfies the ACC on annihilator submodules for all proper ideals I of R;*

(f) $(R/I)_R$ *satisfies the ACC on hereditary pretorsion submodules for all proper ideals I of R;*

(g) $(R/I)_R$ *satisfies the DCC on annihilator submodules for all proper ideals* I *of* R*;*

(h) $(R/I)_R$ *satisfies the DCC on hereditary pretorsion submodules for all proper ideals* I *of* R*.*

Proof. (a)⇔(b) is Theorem 3.7.

(c)⇔(d) For any commutative ring R, the map $K \mapsto K^{-1}0 = \{r \in R : Kr = 0\}$ constitutes a Galois connection on the set of annihilator ideals of R. Such a ring R will thus satisfy the ACC on annihilator ideals if and only if it satisfies the DCC on annihilator ideals. The equivalence of (c) and (d) follows.

(c)⇔(e) and (d)⇔(g) These equivalences are a consequence of the routine fact that the annihilator ideals of R/I, considered as a ring, coincide with the annihilator submodules of $(R/I)_R$.

(f)⇒(e) and (h)⇒(g) These equivalences are a consequence of the fact that for a module over an arbitrary ring, every annihilator submodule is a hereditary pretorsion submodule. Indeed, the annihilator submodules are precisely the hereditary pretorsion submodules with respect to jansian topologizing filters.

(e)⇒(f) This follows from Proposition 3.10, taking $M = (R/I)_R$ and noting that the hereditary pretorsion and annihilator submodules of M coincide under the assumption of (e).

(g)⇒(h) Since the chain of implications, (g)⇒(d)⇒(c)⇒(e) has already been established, we may assume that Statement (e) holds. As explained in the proof of (e)⇒(f) above, the hereditary pretorsion and annihilator submodules of $(R/I)_R$ coincide. The equivalence of (g) and (h) follows.

Thus far we have established the equivalences (a)⇔(b) and (c)⇔(d)⇔(e)⇔(f)⇔(g)⇔(h). To complete the proof, we shall show that (b)⇒(h) and (h)⇒(b).

(b)⇒(h) is Corollary 2.6.

(h)⇒(b) Let $\mathfrak{F}$ and $\mathfrak{G}$ be arbitrary members of $\operatorname{Fil} R$ and M an arbitrary nonzero cyclic right R-module belonging to $\mathcal{T}_{\mathfrak{F}}$. (If no such M exists, then $\mathfrak{F} = \{R\}$, in which case $\mathfrak{G}^{-1}\mathfrak{F} = \mathfrak{F}$ and there is nothing further to prove.) Since R is commutative, $M \cong (R/I)_R$ for some proper ideal I of R. Noting that the family of all $\mathcal{T}_{\mathfrak{G}}$-dense hereditary pretorsion submodules of M is closed under finite intersections, Statement (h) implies that the aforementioned family must have a smallest member. It follows from Theorem 2.3 ((d)⇒(a)) that the right residual $\mathfrak{G}^{-1}\mathfrak{F}$ exists. Since $\mathfrak{F}$ and $\mathfrak{G}$ were chosen arbitrarily, we may conclude that $[\mathrm{Fil}\, R]^{\mathrm{du}}$ is right and thus two-sided residuated. ■

Remark 3.12 *It is known [34, Proposition 31, p. 101] that if R is a commutative domain for which $\mathrm{Fil}\, R$ is commutative, then R satisfies the ACC on principal ideals. It is still unknown whether such a ring R satisfies the ACC on all ideals, that is to say, is noetherian.*

3.2 Semiartinian rings

A theorem of Shores [29, Theorem 6.1, p. 194] asserts that a commutative semiartinian ring R (to be defined below) for which both $\mathrm{soc}\,(R_R)$ and $\mathrm{soc}\,(R_R/\mathrm{soc}\,(R_R))$ are finitely generated as right R-modules, is necessarily artinian. Shores also constructs, for each cardinal $\aleph$, a commutative local semiartinian ring R with Jacobson radical $J(R)$ such that $\mathrm{soc}\,(R_R)$ has dimension 1 and $\mathrm{soc}\,(R_R/\mathrm{soc}\,(R_R))$ dimension $\aleph$ over the field $R/J(R)$. This example shows that, in isolation, the socle of a commutative semiartinian ring exercises no constraint on the 'length' of the ring. In contrast, the main result of this section, Theorem 3.17, shows that if R is a commutative semiartinian ring for which $\mathrm{Fil}\, R$ is commutative, then $\mathrm{soc}\,(R_R)$ will be finitely generated only if R is artinian.

Given a right R-module M, the (ascending) Loewy series of M is an ordinal-indexed family $\{\mathrm{soc}^\alpha(M)\}_\alpha$ of submodules of M defined recursively as follows:

$$\mathrm{soc}^0(M) = 0,$$

$$\mathrm{soc}^{\alpha+1}(M)/\mathrm{soc}^\alpha(M) = \mathrm{soc}\,(M/\mathrm{soc}^\alpha(M)), \text{ for ordinals } \alpha \geq 0,$$

$$\mathrm{soc}^\beta(M) = \textstyle\bigcup_{\alpha<\beta} \mathrm{soc}^\alpha(M), \text{ for limit ordinals } \beta.$$

Observe that $\mathrm{soc}^1(M)$ coincides with $\mathrm{soc}(M)$.

We call M *semiartinian*, or a *Loewy module* if $\mathrm{soc}^\alpha(M) = M$ for some ordinal α. In this situation, the smallest such ordinal α is referred to as the Loewy length of M. Note that if M is semiartinian and finitely generated then α is not a limit ordinal. It is known that $M \in \mathrm{Mod}\text{-}R$ is semiartinian if and only if every nonzero factor module of M has a nonzero socle [31, p. 182]. It follows from this equivalence that every artinian module is semiartinian. We call a ring R right semiartinian if the module R_R is semiartinian. For each ordinal α the module $\mathrm{soc}^{\alpha+1}(M)/\mathrm{soc}^{\alpha}(M)$ is called the α^{th} Loewy factor of M. The number of summands in a direct sum decomposition of $\mathrm{soc}^{\alpha+1}(M)/\mathrm{soc}^{\alpha}(M)$ into simples is an invariant of M, denoted $d^{\alpha}(M)$, and called the α^{th} Loewy invariant of M (see [29] for a more detailed exposition).

If $\mathcal{S}$ is any nonempty class of simple right R-modules and $M \in \mathrm{Mod}\text{-}R$, we define $\mathrm{soc}_{\mathcal{S}}(M)$ to be the sum of all simple submodules of M that are isomorphic to some member of $\mathcal{S}$. If $\mathcal{S} = \{S\}$ is a singleton, we write $\mathrm{soc}_S(M)$ in place of $\mathrm{soc}_{\{S\}}(M)$. The *(ascending) $\mathcal{S}$-Loewy series* $\{\mathrm{soc}^{\alpha}_{\mathcal{S}}(M)\}_{\alpha}$ of M is defined in a manner entirely analogous to $\{\mathrm{soc}^{\alpha}(M)\}_{\alpha}$.

If R is any ring and K a right ideal of R, then standard theory tells us that R will be right artinian if K_R and $(R/K)_R$ are both artinian. This result can be strengthened if additional conditions are placed on the ring R, as the following theorem shows.

Theorem 3.13 *[29, Theorem 6.1, p. 194] The following conditions are equivalent for a commutative semiartinian ring R:*

(a) R is artinian;

(b) R has an ideal K such that $\mathrm{soc}^2(K_R)$ and $(R/K)_R$ are artinian;

(c) The Loewy invariants d^0 and d^1 are finite, i.e., $\mathrm{soc}(R_R)$ and $\mathrm{soc}^2(R_R)/\mathrm{soc}(R_R) = \mathrm{soc}(R_R/\mathrm{soc}(R_R))$ are both finitely generated right R-modules.

Let R be a commutative ring such that $\mathrm{Fil}\, R$ is commutative and let P be a maximal ideal of R.

Define $\mathcal{S} = \{K \leq R_R : K \subseteq P$ and K is finitely generated $\}$. $\mathcal{S}$ is clearly an upward directed family of ideals of R. It follows that the family $\{K^{-1}0 : K \in \mathcal{S}\}$ is a downward directed family of annihilator ideals of R. Since $\mathrm{Fil}\, R_R$ is commutative R_R satisfies the DCC on annihilator submodules by Theorem 3.13. Thus the family $\{K^{-1}0 : K \in \mathcal{S}\}$ contains a minimal member; since the family is also downward directed, it must have a smallest member, say $(K^*)^{-1}0$ where $K^* \in \mathcal{S}$. We claim that $p^{-1}0 \supseteq (K^*)^{-1}0$ for all $p \in P$. Suppose not, that is $p^{-1}0 \not\supseteq (K^*)^{-1}0$ for some $p \in P$. Observe that $(pR + K^*)^{-1}0 = p^{-1}0 \cap (K^*)^{-1}0 \subset (K^*)^{-1}0$. Since K^* is finitely generated, so is $pR + K^*$, hence $pR + K^* \in \mathcal{S}$. The minimality of $(K^*)^{-1}0$ is thus contradicted and this establishes our claim. Since $p^{-1}0 \supseteq (K^*)^{-1}0$ for all $p \in P$, we must have $P^{-1}0 \supseteq (K^*)^{-1}0$. The reverse containment also holds since $K^* \subseteq P$, whence equality $P^{-1}0 = (K^*)^{-1}0$.

Write $K^* = \sum_{i=1}^{n} t_i R$ with $\{t_i : 1 \leq i \leq n\} \subseteq P$. Then $(K^*)^{-1}0 = \bigcap_{i=1}^{n} t_i^{-1}0$. We have a canonical embedding of right R-modules $R/P^{-1}0 = R/(K^*)^{-1}0 = R/\bigcap_{i=1}^{n} t_i^{-1}0 \hookrightarrow \prod_{i=1}^{n} t_i R$.

If Q is any ideal of R containing $P^{-1}0$, the above embedding restricts to an embedding

$$Q/P^{-1}0 \hookrightarrow \prod_{i=1}^{n} t_i Q.$$

Taking $Q = (P^2)^{-1}0$, we obtain $(P^2)^{-1}0/P^{-1}0 \hookrightarrow \prod_{i=1}^{n} t_i(P^2)^{-1}0$. For each $i \in \{1, 2, \cdots, n\}$ we have $t_i(P^2)^{-1}0 \subseteq P(P^2)^{-1}0 \subseteq P^{-1}0$. We thus have an embedding

$$(P^2)^{-1}0/P^{-1}0 \hookrightarrow \prod_{i=1}^{n} P^{-1}0. \tag{3.1}$$

For any integer $n \geq 0$, if $S = (R/P)_R$ we know that $\mathrm{soc}_{\mathrm{S}}^{\mathrm{n}}(R_R) = \{x \in R : xP^n = 0\} = (P^n)^{-1}0$. In particular, $\mathrm{soc}_{\mathrm{S}}(R_R) = P^{-1}0$ and $\mathrm{soc}_{\mathrm{S}}^{2}(R_R) = (P^2)^{-1}0$. Equation (3.1) can thus be written as follows:

$$\mathrm{soc}_{\mathrm{S}}^{2}(R_R)/\mathrm{soc}_{\mathrm{S}}(R_R) \hookrightarrow \prod_{i=1}^{n} \mathrm{soc}_{\mathrm{S}}(R_R). \tag{3.2}$$

We claim that for each $n \in \mathbb{N}$,

$$\mathrm{soc}_{\mathrm{S}}^{\mathrm{n+1}}(R_R)/\mathrm{soc}_{\mathrm{S}}^{\mathrm{n}}(R_R) \hookrightarrow \prod_{i=1}^{m} \mathrm{soc}_{\mathrm{S}}(R_R) \tag{3.3}$$

for some $m \in \mathbb{N}$. This can be proved using an inductive argument on n. The case $n = 1$ is (3.2). To establish the inductive step, suppose

$$\mathrm{soc}_{\mathrm{S}}^{\mathrm{k+1}}(R_R)/\mathrm{soc}_{\mathrm{S}}^{\mathrm{k}}(R_R) \hookrightarrow \prod_{i=1}^{l} \mathrm{soc}_{\mathrm{S}}(R_R) \tag{3.4}$$

for some $k, l \in \mathbb{N}$. Consider $\mathrm{soc}_{\mathrm{S}}^{\mathrm{k+2}}(R_R)/\mathrm{soc}_{\mathrm{S}}^{\mathrm{k+1}}(R_R)$. If $\mathrm{soc}_{\mathrm{S}}^{\mathrm{k+2}}(R_R) = \mathrm{soc}_{\mathrm{S}}^{\mathrm{k+1}}(R_R)$, there is clearly nothing to prove, so suppose $\mathrm{soc}_{\mathrm{S}}^{\mathrm{k+2}}(R_R) \supset \mathrm{soc}_{\mathrm{S}}^{\mathrm{k+1}}(R_R)$. Observe that $P \supseteq \mathrm{soc}_{\mathrm{S}}^{\mathrm{k+1}}(R_R)$, for otherwise $R = P + \mathrm{soc}_{\mathrm{S}}^{\mathrm{k+1}}(R_R)$, whence

$$\begin{aligned} P^{k+1} &= P^{k+2} + \mathrm{soc}_{\mathrm{S}}^{\mathrm{k+1}}(R_R)P^{k+1} \\ &= P^{k+2} + 0 \\ &= P^{k+2}, \end{aligned}$$

so $\mathrm{soc}_{\mathrm{S}}^{\mathrm{k+1}}(R_R) = (P^{k+1})^{-1}0 = (P^{k+2})^{-1}0 = \mathrm{soc}_{\mathrm{S}}^{\mathrm{k+2}}(R_R)$, a contradiction.
Putting $\bar{R} = R/\mathrm{soc}_{\mathrm{S}}^{\mathrm{k}}(R_R)$, we see that $\bar{R}$ is a nontrivial factor ring of R and that $\mathrm{Fil}\,\bar{R}$ is commutative by Theorem 1.25. Note also that S is canonically a simple right $\bar{R}$-module, since $S = R/P$ and $\mathrm{soc}_{\mathrm{S}}^{\mathrm{k}}(R_R) \subseteq \mathrm{soc}_{\mathrm{S}}^{\mathrm{k+1}}(R_R) \subseteq P$.
Applying (3.2) to the ring $\bar{R}$ and simple right $\bar{R}$-module S, we see that

$$\mathrm{soc}_{\mathrm{S}}^{2}(\bar{R}_{\bar{R}})/\mathrm{soc}_{\mathrm{S}}(\bar{R}_{\bar{R}}) \hookrightarrow \prod_{i=1}^{l'} \mathrm{soc}_{\mathrm{S}}(\bar{R}_{\bar{R}}), \tag{3.5}$$

for some $l' \in \mathbb{N}$. Now

$$\begin{aligned} \mathrm{soc}_{\mathrm{S}}^{2}(\bar{R}_{\bar{R}})/\mathrm{soc}_{\mathrm{S}}(\bar{R}_{\bar{R}}) &\cong [\mathrm{soc}_{\mathrm{S}}^{2}(R_R/\mathrm{soc}_{\mathrm{S}}^{\mathrm{k}}(R_R))]/[\mathrm{soc}_{\mathrm{S}}(R_R/\mathrm{soc}_{\mathrm{S}}^{\mathrm{k}}(R_R))] \\ &= [\mathrm{soc}_{\mathrm{S}}^{\mathrm{k+2}}(R_R)/\mathrm{soc}_{\mathrm{S}}^{\mathrm{k}}(R_R)]/[\mathrm{soc}_{\mathrm{S}}^{\mathrm{k+1}}(R_R)/\mathrm{soc}_{\mathrm{S}}^{\mathrm{k}}(R_R)] \\ &\cong \mathrm{soc}_{\mathrm{S}}^{\mathrm{k+2}}(R_R)/\mathrm{soc}_{\mathrm{S}}^{\mathrm{k+1}}(R_R) \end{aligned} \tag{3.6}$$

as right R-modules. Similarly,

$$\mathrm{soc}_{\mathrm{S}}(\bar{R}_{\bar{R}}) \cong \mathrm{soc}_{\mathrm{S}}^{\mathrm{k+1}}(R_R)/\mathrm{soc}_{\mathrm{S}}^{\mathrm{k}}(R_R), \tag{3.7}$$

as right R-modules. Then

$$\begin{aligned}
\mathrm{soc}_{\mathrm{S}}^{\mathrm{k+2}}(R_R)/\mathrm{soc}_{\mathrm{S}}^{\mathrm{k+1}}(R_R) &\cong \mathrm{soc}_{\mathrm{S}}^{2}(\bar{R}_{\bar{R}})/\mathrm{soc}_{\mathrm{S}}(\bar{R}_{\bar{R}}) \quad \text{[by 3.6]} \\
&\hookrightarrow \prod_{i=1}^{l'} \mathrm{soc}_{\mathrm{S}}(\bar{R}_{\bar{R}}) \quad \text{[by 3.5]} \\
&\cong \prod_{i=1}^{l'} [\mathrm{soc}_{\mathrm{S}}^{\mathrm{k+1}}(R_R)/\mathrm{soc}_{\mathrm{S}}^{\mathrm{k}}(R_R)] \quad \text{[by 3.7]} \\
&\hookrightarrow \prod_{i=1}^{l'} \left(\prod_{i=1}^{l} \mathrm{soc}_{\mathrm{S}}(R_R) \right) \quad \text{[by 3.4]} \\
&\cong \prod_{i=1}^{ll'} \mathrm{soc}_{\mathrm{S}}(R_R).
\end{aligned}$$

The inductive step is thus established.

The following result is an immediate consequence of the above analysis.

Proposition 3.14 *Let R be a commutative ring for which $\mathrm{Fil}\, R$ is commutative and let S be a simple right R-module. Then the following conditions are equivalent:*

(a) $\mathrm{soc}_{\mathrm{S}}(R_R)$ *is finitely generated;*

(b) $\mathrm{soc}_{\mathrm{S}}^{\mathrm{n}}(R_R)/\mathrm{soc}_{\mathrm{S}}^{\mathrm{n-1}}(R_R)$ *is finitely generated for all $n \in \mathbb{N}$;*

(c) $\mathrm{soc}_{\mathrm{S}}^{\mathrm{n}}(R_R)$ *is finitely generated for all $n \in \mathbb{N}$.*

Proposition 3.15 *[29, Theorem 4.1, p. 189] Let R be a commutative ring and M a semiartinian right R-module that contains only finitely many nonisomorphic simple submodules, that is to say, $\mathrm{soc}(M) = \mathrm{soc}_{\mathcal{S}}(M)$ for some finite nonempty family $\mathcal{S}$ of nonisomorphic simple right R-modules. Then for each ordinal α:*

(a) $\mathrm{soc}^{\alpha}(M) = \bigoplus_{S \in \mathcal{S}} \mathrm{soc}_{\mathrm{S}}^{\alpha}(M)$;

(b) $d^{\alpha}(M) = \sum_{S \in \mathcal{S}} d_S^{\alpha}(M)$.

Theorem 3.16 *The following conditions are equivalent for a commutative semiartinian ring R:*

(a) R *is artinian;*

(b) $\mathrm{soc}\,(R_R)$ *and* $\mathrm{soc}^2\,(R_R)$ *are both finitely generated;*

(c) $\mathrm{Fil}\,R$ *is commutative and* $\mathrm{soc}\,(R_R)$ *is finitely generated.*

Proof. (a)⇔(b) is Theorem 3.14.

(a)⇒(c) Suppose that (a) holds. The first part of (c) follows from [34, Proposition 2, p. 89]. The second part follows from the fact that every artinian ring is noetherian and hence $\mathrm{soc}\,(R_R)$ is finitely generated.

(c)⇒(b) Suppose that (c) holds. Let $\mathcal{S}$ be a representative family of simple submodules of R_R. Since $\mathrm{soc}\,(R_R)$ is finitely generated by hypothesis and $\mathrm{soc}_{\mathrm{S}}\,(R_R) \leq \mathrm{soc}\,(R_R)$ for each $S \in \mathcal{S}$, it follows that each $\mathrm{soc}_{\mathrm{S}}\,(R_R)$ is finitely generated. Hence by Proposition 3.14(c), $\mathrm{soc}^2_{\mathrm{S}}\,(R_R)$ is finitely generated for each $S \in \mathcal{S}$. Since $\mathrm{soc}^2\,(R_R) = \bigoplus_{S\in\mathcal{S}} \mathrm{soc}^2_{\mathrm{S}}(R_R)$ by Proposition 3.15(a) with $\mathcal{S}$ finite, it follows that $\mathrm{soc}^2\,(R_R)$ is finitely generated since it is a finite direct sum of finitely generated modules. The proof of the theorem is thus completed. ■

3.3 A class of examples of commutative semiartinian rings

We construct a family of examples which serves to delineate earlier theory.

For the remainder of this section F will denote a field, V and U F-spaces, and $\mu : V \times V \to U$ a symmetric F-bilinear map. For each pair of elements $(v, v') \in V \times V$, we shall denote $\mu(v, v')$ by $v \cdot v'$.

We endow the set $F \times V \times U$ with an F-algebra structure by taking addition to be natural and defining multiplication by:

$$(a_1, v_1, u_1) \cdot (a_2, v_2, u_2) \stackrel{\text{def}}{=} (a_1a_2, a_1v_2 + a_2v_1, a_1u_2 + v_1 \cdot v_2 + a_2u_1).$$

The above multiplication operation in $F \times V \times U$ may be represented in terms of suitable 3×3 matrices thus:

$$\begin{pmatrix} a_1 & v_1 & u_1 \\ 0 & a_1 & v_1 \\ 0 & 0 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & v_2 & u_2 \\ 0 & a_2 & v_2 \\ 0 & 0 & a_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 & a_1v_2 + a_2v_1 & a_1u_2 + v_1.v_2 + a_2u_1 \\ 0 & a_1a_2 & a_1v_2 + a_2v_1 \\ 0 & 0 & a_1a_2 \end{pmatrix}.$$

It is easily checked that the resulting structure, which we shall denote by $\langle F, V, U, \mu \rangle$ is indeed an F-algebra. Notice that the symmetry of the F-bilinear map μ guarantees that $\langle F, V, U, \mu \rangle$ is commutative.

If X and Y are F-subspaces of V and U, respectively, we define F-subspace $X^{-1}Y$ of V as follows:

$$\begin{aligned} X^{-1}Y &\stackrel{\text{def}}{=} \{v \in V : X \cdot v \subseteq Y\} \\ &= \{v \in V : x \cdot v \in Y \ \forall x \in X\} \end{aligned}$$

We shall call $V^{-1}0 = \{v \in V : V \cdot v = 0\}$ the *degenerate* part of V.

Proposition 3.17 *If V' is any F-subspace complement of $V^{-1}0$ in V, then $\langle F, V, U, \mu \rangle$ and $\langle F, V', V^{-1}0 \times U, \mu' \rangle$ are isomorphic F-algebras where μ' is the restriction of μ to V'.*

Proof. It is routine matter to check that the map from $\langle F, V, U, \mu \rangle$ to $\langle F, V', V^{-1}0 \times U, \mu' \rangle$ given by

$$(a, v' + w, u) \mapsto (a, v', (w, u))$$

where $a \in F, v' \in V, w \in V^{-1}0$ and $u \in U$, is the required isomorphism. ∎

It follows from the above result that no generality is lost when considering the ring $\langle F, V, U, \mu \rangle$, if it is assumed that the degenerate part of V is trivial, that is to say, $V^{-1}0 = 0$. We shall henceforth make this assumption.

The following result establishes a number of properties of $\langle F, V, U, \mu \rangle$. We omit the routine proof details.

Proposition 3.18 *Let $R = \langle F, V, U, \mu \rangle$. Then:*

(a) R *is a commutative local* F*-algebra with unique maximal proper ideal* $J(R) = \langle 0, V, U\rangle = \{(0, v, u) : v \in V, u \in U\}$.

(b) $\operatorname{soc}(R_R) = \langle 0, 0, U\rangle = \{(0, 0, u) : u \in U\}$ *and* $\operatorname{soc}^2(R_R) = \langle 0, V, U\rangle = J(R)$ *whence* $\operatorname{soc}^3(R_R) = R_R$.

Thus R_R *is semiartinian with Loewy length 3.*

(c) *If* I *is any ideal of* R *satisfying* $\operatorname{soc}(R_R) \subseteq I \subseteq J(R)$, *then* I *has the form* $I = \langle 0, W, U\rangle$ *for some* F*-subspace* W *of* V.

(d) *If* I' *is any ideal of* R *satisfying* $I' \subseteq \operatorname{soc}(R_R)$, *then* I' *has the form* $I' = \langle 0, 0, Y\rangle$ *for some* F*-subspace* Y *of* U.

We now describe all ideals of $R = \langle F, V, U, \mu\rangle$ that have the form $K^{-1}I = \{r \in R : Kr \subseteq I\}$ for some $I, K \in \operatorname{Id} R$. Our interest in such ideals stems from the fact that by Theorem 3.11 ((a)⇔(c)) a commutative ring R will be such that $\operatorname{Fil} R$ is commutative if and only if for each proper ideal I of R, the family of ideals $\{K^{-1}I : K \in \operatorname{Id} R\}$ satisfies the ACC.

Lemma 3.19 *The following assertions are equivalent for a proper nonzero ideal* A *of* $R = \langle F, V, U, \mu\rangle$*:*

(a) $A = K^{-1}I$ *for some* $I, K \in \operatorname{Id} R$*;*

(b) $A = \langle 0, X^{-1}Y, U\rangle$ *for some* F*-subspaces* X *and* Y *of* V *and* U *respectively.*

Proof. (b)⇒(a) Suppose A satisfies (b). Put $K = \langle 0, X, U\rangle$ and $I = \langle 0, 0, Y\rangle$. It is easily shown that I and K are ideals of R and that $K^{-1}I = \langle 0, X^{-1}Y, U\rangle = A$.

(a)⇒(b) Suppose $A = K^{-1}I$ with $I, K \in \operatorname{Id} R$. Note that A and K are proper ideals of R. The latter ideal is proper because A is nonzero by hypothesis. Since R is local by Proposition 3.18(a), $A, K \subseteq J(R)$. Putting $K' = K + \operatorname{soc}(R_R)$ we have

$$
\begin{aligned}
[K']^{-1}I &= K^{-1}I \cap \operatorname{soc}(R_R)^{-1}I \\
&\supseteq K^{-1}I \cap \operatorname{soc}(R_R)^{-1}0 \\
&= K^{-1}I \cap J(R) \\
&= K^{-1}I \quad \text{[because } A = K^{-1}I \subseteq \text{J (R)]}.
\end{aligned}
$$

The reverse containment $[K']^{-1}I \subseteq K^{-1}I$ clearly also holds since $K' \supseteq K$ whence equality $[K']^{-1}I = K^{-1}I$. It follows from the above that no generality is lost if we suppose that $\operatorname{soc}(R_R) \subseteq K \subseteq J(R)$.

It follows from the above, and Proposition 3.18(c), that

$$K = \langle 0, X, U \rangle \tag{3.8}$$

for some F-subspace X of V. Putting $I' = I \cap \operatorname{soc}(R_R)$ we have

$$
\begin{aligned}
K^{-1}I' &= K^{-1}I \cap K^{-1}\operatorname{soc}(R_R) \\
&\supseteq K^{-1}I \cap J(R)^{-1}\operatorname{soc}(R_R) \quad \text{[because } K \subseteq J(R)\text{]} \\
&\supseteq K^{-1}I \cap J(R) \quad \text{[because } J(R)^2 \subseteq \operatorname{soc}(R_R) \text{ by Proposition 3.18(b)]} \\
&= K^{-1}I \quad \text{[because } A = K^{-1}I \subseteq \text{J (R)]}.
\end{aligned}
$$

The reverse containment $K^{-1}I' \subseteq K^{-1}I$ clearly also holds since $I' \subseteq I$, whence equality $K^{-1}I' = K^{-1}I$.

It follows from the above that no generality is lost if we suppose that $I \subseteq \operatorname{soc}(R_R)$.

It follows from the above, and Proposition 3.18(d), that

$$I = \langle 0, 0, Y \rangle \tag{3.9}$$

for some F-subspace Y of U.

It is a simple exercise to show that if K and I are defined as in (3.8) and (3.9), then $A = K^{-1}I = \langle 0, X^{-1}Y, U \rangle$, as required. ■

The following theorem follows immediately from the previous lemma and Theorem 3.11 ((a)⇔(c)).

Theorem 3.20 *Let $R = \langle F, V, U, \mu\rangle$. The following statements are equivalent:*

(a) $\mathrm{Fil}\, R$ *is commutative;*

(b) For each proper ideal I of R, the family $\{K^{-1}I : K \in \mathrm{Id}\, R\}$ of ideals of R, satisfies the ACC;

(c) For each F-subspace Y of U, the family $\{X^{-1}Y : X$ is an F-subspace of $V\}$ of F-subspaces of V, satisfies the ACC.

The following corollary is also immediate.

Corollary 3.21 *Let $R = \langle F, V, U, \mu\rangle$. If $\dim_F V$ is finite, then $\mathrm{Fil}\, R$ is commutative.*

We draw the reader's attention to the fact that in the above corollary, there is no finiteness requirement on $\mathrm{soc}\,(R_R) = \langle 0, 0, U\rangle$ in order that $\mathrm{Fil}\, R$ be commutative. The consequence is a plentiful supply of commutative non-artinian semiartinian rings R for which $\mathrm{Fil}\, R$ is commutative.
In the next example we show that the sufficient condition of the previous corollary, is not a necessary condition for $\mathrm{Fil}\, R$ to be commutative.

Example 3.22 *We show that choices for V, U and μ can be made such that $\dim_F V$ and $\dim_F U$ are both infinite, but the ring $R = \langle F, V, U, \mu\rangle$ is such that $\mathrm{Fil}\, R$ is commutative.*
Let T be the commutative F-algebra defined by

$$T \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} a & w \\ 0 & a \end{pmatrix} : a \in F, w \in W \right\}$$

where W is any infinite dimensional F-space. Take $V = U = T$ with symmetric F-bilinear map $\mu : T \times T \to T$ the usual multiplication map on T. Observe that $\dim_F T$ is infinite because $\dim_F W$ is infinite. Let $R = \langle F, V, U, \mu\rangle = \langle F, T, T, \mu\rangle$. We use Theorem 3.20, to show that $\mathrm{Fil}\, R$ is commutative. To this end, let Y be an F-subspace of T. It is easily seen that Y may be written as

$$Y = \left\{ \begin{pmatrix} ax & sx \\ 0 & ax \end{pmatrix} : x \in F \right\} + \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix}$$

for some fixed $\begin{pmatrix} a & s \\ 0 & a \end{pmatrix} \in T$ *and* F*-subspace* Z *of* W*.*

A routine calculation shows that if X *is any* F*-subspace of* T*, then* $X^{-1}Y$ *has one of the following four forms:*

$$X^{-1}Y = \begin{cases} 0; \\ T; \\ \begin{pmatrix} 0 & W \\ 0 & 0 \end{pmatrix} = J(T); \\ \left\{ \begin{pmatrix} bx & tx \\ 0 & bx \end{pmatrix} : x \in F \right\} + \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} \text{ for some fixed } \begin{pmatrix} b & t \\ 0 & b \end{pmatrix} \in T. \end{cases}$$

It is clear from the above that the family $\{X^{-1}Y : X$ is an F-subspace of $T\}$ admits no strictly ascending chain of F-subspaces, so by Theorem 3.20 ((a)⇔(c)), $\mathrm{Fil}\, R$ is commutative.

Example 3.23 *In this example we choose* V*,* U *and* μ *such that* $\dim_F V$ *is infinite,* $\dim_F U = 1$*, but the ring* $R = \langle F, V, U, \mu\rangle$ *is such that* $\mathrm{Fil}\, R$ *is not commutative.*
Let $V = F^{(\Omega)}$*,* Ω *an infinite index set, and* $U = F$*. The map* $\mu : V \times V \to F$*, we take to be the usual inner product given by*

$$\mu(\{a_\lambda\}_{\lambda\in\Omega}, \{b_\lambda\}_{\lambda\in\Omega}) \stackrel{\text{def}}{=} \textstyle\sum_{\lambda\in\Omega} a_\lambda b_\lambda.$$

Take $Y = 0$ *and, for every nonempty subset* Ω' *of* Ω*, let* $X_{\Omega'} \stackrel{\text{def}}{=} F^{(\Omega')}$ *interpreted in the natural manner as an* F*-subspace of* $F^{(\Omega)}$*. Observe that* $X_{\Omega'}^{-1}0 = X_{\Omega\setminus\Omega'}$ *for every proper nonempty subset* Ω' *of* Ω*. Note further that any strictly descending chain* $\Omega_1 \supset \Omega_2 \supset \cdots$ *of subsets of* Ω *induces* *a strictly ascending chain* $X_{\Omega_1}^{-1}0 \subset X_{\Omega_2}^{-1}0 \subset \cdots$ *of* F*-subspaces of* V*. It follows that the family* $\{X^{-1}0 : X$ *is an* F*-subspace of* $V\}$ *does not satisfy the ACC, whence* $\mathrm{Fil}\, R$ *is not commutative by Theorem 3.20 ((a)⇔(c)).*

3.4 Localization in commutative rings

Basic terminology and concepts are taken from texts [2, 16, 7, 27]. Throughout this section R is a commutative ring with identity.

A nonempty subset S of a ring R is said to be a *multiplicative subset* of R if $1 \in S$ and S is closed under multiplication, i.e., $s_1 s_2 \in S$ whenever $s_1, s_2 \in S$.

Consider $R \times S$ and define a relation $\sim$ on $R \times S$ by $(r, s) \sim (r', s')$ if and only if there exists $u \in S$ such that $u(rs' - r's) = 0$. It is easy to check that $\sim$ is an equivalence relation on $R \times S$. We denote the equivalence class of a pair $(r, s) \in R \times S$ by $\frac{r}{s}$. The set

$$RS^{-1} \overset{\text{def}}{=} \{\tfrac{r}{s} : r \in R, s \in S\}$$

of all equivalence classes on $R \times S$ is called the *ring of fractions of* R *with respect to* S. It is routine to check that $\langle RS^{-1}, +, \cdot \rangle$ is a commutative ring with identity with respect to addition and multiplication defined by:

$$\frac{r}{s} + \frac{r'}{s'} \overset{\text{def}}{=} \frac{rs' + r's}{ss'} \text{ and } \frac{r}{s} \cdot \frac{r'}{s'} \overset{\text{def}}{=} \frac{rr'}{ss'}.$$

The ring RS^{-1} comes with a natural ring homomorphism $\varphi_S : R \to RS^{-1}$ given by $r \mapsto \frac{r}{1}$ $(r \in R)$. The following properties are easily established:

(a) $\varphi_S(t)$ is a unit of RS^{-1} for all $t \in S$;

(b) Every $a \in RS^{-1}$ is expressible in the form $a = \varphi_S(r)\varphi_S(t)^{-1}$ for some $r \in R$ and some $t \in S$;

(c) The kernel of φ_S is the set $\{r \in R : rs = 0$ for some $s \in S\}$.

Theorem 3.24 *[7, Theorem 1, p. 395] Let S be a multiplicative subset of a commutative ring R. Let A be a commutative ring and $f : R \to A$ a ring homomorphism such that $f(t)$ is a unit of A for all $t \in S$. Then there exists a unique ring homomorphism $g : RS^{-1} \to A$ such that $g \circ \varphi_S = f$, i.e., the diagram below commutes.*

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "\varphi_S"'] & A \\ RS^{-1} \arrow[ur, "g"'] & \end{tikzcd}$$

It is easily checked that for each ideal I of R,

$$IS^{-1} \stackrel{\text{def}}{=} \left\{ \frac{a}{s} \in RS^{-1} : a \in I, s \in S \right\}$$

is an ideal of RS^{-1}. We thus have a map from $\operatorname{Id} R$ to $\operatorname{Id} RS^{-1}$ given by $I \mapsto IS^{-1}$.

We include a proof of the following theorem in the absence of a suitable reference.

Theorem 3.25 *Let S be a multiplicative subset of a commutative ring R. Then:*

(a) The map from $\operatorname{Id} R$ to $\operatorname{Id} RS^{-1}$ given by $I \mapsto IS^{-1}$ is onto, that is to say, every ideal of RS^{-1} has the form IS^{-1} for some $I \in \operatorname{Id} R$.

(b) $(\bigcap_{i=1}^{n} I_i)S^{-1} = \bigcap_{i=1}^{n} I_i S^{-1}$ for every finite family $\{I_i : 1 \leq i \leq n\}$ of ideals of R.

(c) $(\sum \mathcal{I})S^{-1} = \sum_{I \in \mathcal{I}} IS^{-1}$ for every nonempty family $\mathcal{I}$ of ideals of R.

(d) $(I_1 I_2 \ldots I_n)S^{-1} = (I_1 S^{-1})(I_2 S^{-1}) \cdots (I_n S^{-1})$ for every finite family $\{I_i : 1 \leq i \leq n\}$ of ideals of R.

Proof. (a) Take $K \in \operatorname{Id} RS^{-1}$. It is easily seen that $\varphi_S^{-1}[K] = \{r \in R : \varphi_S(r) \in K\}$ is an ideal of R. We shall show that $(\varphi_S^{-1}[K])S^{-1} = K$. To this end, take $\frac{a}{t} \in K$. Then $\varphi_S(a) = \frac{a}{1} = \frac{a}{t} \cdot \frac{t}{1} \in K$ since K is an ideal of RS^{-1} and $\frac{a}{t} \in K$. Thus $a \in \varphi_S^{-1}[K]$, whence $\frac{a}{t} \in (\varphi_S^{-1}[K])S^{-1}$. This shows that $K(\varphi_S^{-1}[K])S^{-1}$.

To establish the reverse containment take $\dfrac{a}{t} \in (\varphi_S^{-1}[K])S^{-1}$ where $a \in \varphi_S^{-1}[K]$, $t \in S$. Inasmuch as $a \in \varphi_S^{-1}[K]$ we can see that $\dfrac{a}{1} = \varphi_S(a) \in K$. Since K is an ideal of $\operatorname{Id} RS^{-1}$ with $\dfrac{a}{1} \in K$, we must have $\dfrac{a}{t} = \dfrac{a}{1} \cdot \dfrac{1}{t} \in K$. This shows that $(\varphi_S^{-1}[K])S^{-1} \subseteq K$ and hence $(\varphi_S^{-1}[K])S^{-1} = K$.
(b) Suppose that $x \in (\bigcap_{i=1}^n I_i)S^{-1}$. Then $x = \dfrac{a}{s}$ for some $a \in \bigcap_{i=1}^n I_i$ and $s \in S$, whence $x = \dfrac{a}{s} \in I_iS^{-1}$ for all $i \in \{1, 2, \ldots, n\}$, so $x \in \bigcap_{i=1}^n I_iS^{-1}$. This shows that $(\bigcap_{i=1}^n I_i)S^{-1} \subseteq \bigcap_{i=1}^n I_iS^{-1}$.

On the other hand, if $y \in \bigcap_{i=1}^n I_iS^{-1}$, then

$$y = \frac{a_1}{s_1} = \frac{a_2}{s_2} = \cdots = \frac{a_n}{s_n}$$

for some $a_i \in I_i$ and $s_i \in S$ for $i \in \{1, 2, \ldots, n\}$. Take $i \in \{1, 2, \ldots, n\}$. Since $\dfrac{a_1}{s_1} = \dfrac{a_i}{s_i}$ there exists $u_i \in S$ such that $u_i(a_1s_i - a_is_1) = 0$, whence $u_ia_1s_i = u_ia_is_1$. Since I_i is an ideal of R and $a_i \in I$, we must have $u_ia_1s_i = u_ia_is_1 \in I_i$. Since R is commutative, the above entails $a = (u_1u_2 \ldots u_n)a_1(s_1s_2 \ldots s_n) \in \bigcap_{i=1}^n I_i$, and so

$$y = \frac{a_1}{s_1} = \frac{(u_1u_2 \ldots u_n)a_1(s_1s_2 \ldots s_n)}{(u_1u_2 \ldots u_n)s_1(s_1s_2 \ldots s_n)} = \frac{a}{t}$$

with $t = (u_1u_2 \ldots u_n)s_1(s_1s_2 \ldots s_n) \in S$. It follows that $y \in (\bigcap_{i=1}^n I_i)S^{-1}$. Thus $\bigcap_{i=1}^n I_iS^{-1} \subseteq (\bigcap_{i=1}^n I_i)S^{-1}$ and hence $(\bigcap_{i=1}^n I_i)S^{-1} = \bigcap_{i=1}^n I_iS^{-1}$.

(c) If $x \in (\sum \mathcal{I})S^{-1}$, then there exist a finite subfamily $\{I_i : 1 \leq i \leq n\}$ of $\mathcal{I}$, elements $a_i \in I_i$ for each $i \in \{1, 2, \ldots, n\}$, and $s \in S$, such that

$$x = \frac{\sum_{i=1}^n a_i}{s} = \sum_{i=1}^n \frac{a_i}{s}.$$

This implies

$$x = \sum_{i=1}^n \frac{a_i}{s} \in \sum_{i=1}^n (I_iS^{-1}) \subseteq \sum_{I \in \mathcal{I}} IS^{-1},$$

and so, $(\sum \mathcal{I})S^{-1} \subseteq \sum_{I \in \mathcal{I}} IS^{-1}$. To establish the reverse containment, take $y \in \sum_{I \in \mathcal{I}} IS^{-1}$, then there exist a finite subfamily $\{I_i : 1 \leq i \leq n\}$ of $\mathcal{I}$, elements $a_i \in I_i$ and $s_i \in S$ for each $i \in \{1, 2, \ldots, n\}$, such that

$$y = \sum_{i=1}^n \frac{a_i}{s_i}.$$

Define $b_1 = a_1 s_2 s_3 \dots s_n$, $b_i = a_i s_1 s_2 \cdots s_{i-1} s_{i+1} s_n$ for $i \in \{2, 3, \dots, n-1\}$ and $b_n = a_n s_1 s_2 \dots s_{n-1}$. Putting $t = s_1 s_2 \dots s_n$, we see that $b_i \in I_i$ and $\frac{a_i}{s_i} = \frac{b_i}{t}$ for each $i \in \{1, 2, \dots, n\}$. Hence

$$y = \sum_{i=1}^{n} \frac{a_i}{s_i} = \sum_{i=1}^{n} \frac{b_i}{t} = \frac{\sum_{i=1}^{n} b_i}{t} \in \left(\sum_{i=1}^{n} I_i \right) S^{-1} \subseteq \left(\sum \mathcal{I} \right) S^{-1}.$$

This implies $\sum_{I \in \mathcal{I}} IS^{-1} \subseteq (\sum \mathcal{I}) S^{-1}$, whence equality.

(d) Suppose $x \in (I_1 I_2 \cdots I_n) S^{-1}$. Then $x = \frac{a}{s}$ for some $a \in I_1 I_2 \cdots I_n$ and $s \in S$. Write $a = \sum_{i=1}^{k} a_{1i} a_{2i} \cdots a_{ni}$ with $a_{1i} \in I_1, a_{2i} \in I_2, \cdots, a_{ni} \in I_n$. Then

$$x = \sum_{i=1}^{k} \frac{a_{1i} a_{2i} \cdots a_{ni}}{s} = \sum_{i=1}^{k} \frac{a_{1i}}{1} \cdot \frac{a_{2i}}{1} \cdots \frac{a_{ni}}{s} \in I_1 S^{-1} I_2 S^{-1} \dots I_n S^{-1}.$$

Thus $(I_1 I_2 \dots I_n) S^{-1} \subseteq (I_1 S^{-1})(I_2 S^{-1}) \cdots (I_n S^{-1})$.

On the other hand, if $y \in I_1 S^{-1} I_2 S^{-1} \dots I_n S^{-1}$, then

$$y = \frac{a_1}{s_1} \cdot \frac{a_2}{s_2} \cdots \frac{a_n}{s_n} = \frac{a_1 a_2 \dots a_n}{s_1 s_2 \dots s_n}$$

for some $a_i \in I_i, s_i \in S$ for $i \in \{1, 2, \dots, n\}$.

This implies $y = \frac{a_1 a_2 \dots a_n}{s_1 s_2 \dots s_n} \in (I_1 I_2 \dots I_n) S^{-1}$ since $a_1 a_2 \dots a_n \in I_1 I_2 \dots I_n$ and $s_1 s_2 \dots s_n \in S$. Thus $(I_1 I_2 \dots I_n) S^{-1} \supseteq (I_1 S^{-1})(I_2 S^{-1}) \cdots (I_n S^{-1})$ and hence equality. ■

Remark 3.26 *The previous theorem tells us that if S is a multiplicative subset of a commutative ring R, then the mapping from $\operatorname{Id} R$ to $\operatorname{Id} RS^{-1}$ given by $I \mapsto IS^{-1}$, is an onto homomorphism of lattice ordered monoids.*

Let P be a prime ideal of a commutative ring R. It is easily seen that $S = R \setminus P$ is multiplicative subset of R. We shall write R_P in place of RS^{-1} and I_P in place of IS^{-1} for each $I \in \operatorname{Id} R$. Inasmuch as $IS^{-1} = RS^{-1}$ whenever $I \cap S \neq \emptyset$, it is easily seen that R_P is a local ring with unique maximal ideal P_P. We call R_P the localization of R at P.

Proposition 3.27 *[2, Corollary 3.13, p. 42] Let S be a multiplicative subset of a commutative ring R. Then the map $P \mapsto PS^{-1}$ is a bijection from the set of prime ideals P of R disjoint from S to the set of prime ideals of RS^{-1}.*

Let S be a multiplicative subset of a commutative ring R and let M be a right R-module. We define a relation $\sim$ on $M \times S$ by $(m.s) \sim (m', s')$ if and only if there exists $u \in S$ such that $(ms' - m's)u = 0$. It is easy to check that $\sim$ is an equivalence relation on $M \times S$. We denote the equivalence class of a pair (m,s) by $\dfrac{m}{s}$. The set

$$MS^{-1} \stackrel{\text{def}}{=} \{\frac{m}{s} : m \in M, s \in S\}$$

is called the module of fractions of M with respect to S; it has the structure of a right RS^{-1}-module with addition and scaler multiplication given by

$$\frac{m}{s} + \frac{m'}{s'} \stackrel{\text{def}}{=} \frac{ms' + m's}{ss'} \text{ and } \frac{m}{s} \cdot \frac{r}{t} = \frac{mr}{st}.$$

There is a canonical R-homomorphism $\varphi_S^M : M \to MS^{-1}$ given by $\varphi_S^M(m) = \dfrac{m}{1}$ for $m \in M$. Note that $\operatorname{Ker} \varphi_S^M = \{m \in M : ms = 0 \text{ for some } s \in S\}$.

Proposition 3.28 *[2, Corollary 3.4, p. 39] Let S be a multiplicative subset of a commutative ring R; N, P submodules of a right R-module M. Then the following statements hold.*

(a) $(N + P)S^{-1} = NS^{-1} + PS^{-1}$.

(b) $(N \cap P)S^{-1} = NS^{-1} \cap PS^{-1}$.

(c) $(M/N)S^{-1} \cong MS^{-1}/NS^{-1}$.

3.5 Topologizing filters on the ring of fractions RS^{-1}

Throughout this section R will denote a commutative ring with multiplicative subset S.

The mapping from $\operatorname{Id} R$ to $\operatorname{Id} RS^{-1}$ given by $I \mapsto IS^{-1}$ induces in turn a map from $\operatorname{Fil} R$ to

$\mathrm{Fil}\, RS^{-1}$. For each $\mathfrak{F} \in \mathrm{Fil}\, R$ define

$$\hat{\varphi}_S(\mathfrak{F}) \stackrel{\mathrm{def}}{=} \{AS^{-1} : A \in \mathfrak{F}\}. \tag{3.10}$$

Proposition 3.29 *Let S be a multiplicative subset of a commutative ring R. Then the following statements hold.*

(a) $\hat{\varphi}_S(\mathfrak{F})$ is a topologizing filter on RS^{-1} for all $\mathfrak{F} \in \mathrm{Fil}\, R$, so that $\hat{\varphi}_S$ is a mapping from $\mathrm{Fil}\, R$ to $\mathrm{Fil}\, RS^{-1}$.

(b) The mapping $\hat{\varphi}_S : \mathrm{Fil}\, R \to \mathrm{Fil}\, RS^{-1}$ is onto.

(c) $\hat{\varphi}_S$ preserves infinite meets.

(d) $\hat{\varphi}_S$ preserves finite joins.

Proof. (a) Suppose $\mathfrak{F} \in \mathrm{Fil}\, R$, $K \in \hat{\varphi}_S(\mathfrak{F})$ and J is an ideal of RS^{-1} such that $J \supseteq K$. We need to show that $J \in \hat{\varphi}_S(\mathfrak{F})$. Note that $K \in \hat{\varphi}_S(\mathfrak{F})$ implies that $K = AS^{-1}$ for some $A \in \mathfrak{F}$ by (3.10). Since J is an ideal of RS^{-1} it has the form $J = BS^{-1}$ for some ideal B of R by Theorem 3.25(a). Since $J \supseteq K$ we have $BS^{-1} \supseteq AS^{-1}$. Note that $(A+B)S^{-1} = AS^{-1} + BS^{-1} = BS^{-1}$ by Proposition 3.28(a). Since $A + B \supseteq A \in \mathfrak{F}$, it follows that $A + B \in \mathfrak{F}$, so $J = BS^{-1} = (A+B)S^{-1} \in \hat{\varphi}_S(\mathfrak{F})$, whence Condition (F1) is satisfied.

Take $K, H \in \hat{\varphi}_S(\mathfrak{F})$. Then $K = AS^{-1}$ and $H = BS^{-1}$ for some $A, B \in \mathfrak{F}$. By Theorem 3.25(b), $K \cap H = AS^{-1} \cap BS^{-1} = (A \cap B)S^{-1} \in \hat{\varphi}_S(\mathfrak{F})$ since $A \cap B \in \mathfrak{F}$. Hence (F2) is also satisfied.

To show that (F3) holds, take $K \in \hat{\varphi}_S(\mathfrak{F})$ and $x \in RS^{-1}$. Then $K = AS^{-1}$ for some $A \in \mathfrak{F}$ and $x = \dfrac{r}{s}$ for some $r \in R$ and $s \in S$. Take $\dfrac{q}{t} \in (r^{-1}A)S^{-1}$ with $q \in r^{-1}A$ and $t \in S$. Note that $x\left(\dfrac{q}{t}\right) = \dfrac{rq}{st} \in AS^{-1} = K$ because $rq \in A$ and $st \in S$. Thus $\dfrac{q}{t} \in x^{-1}K$ and so $(r^{-1}A)S^{-1} \subseteq x^{-1}K$. Since $r^{-1}A \in \mathfrak{F}$, we have $(r^{-1}A)S^{-1} \in \hat{\varphi}_S(\mathfrak{F})$. We have already shown that $\hat{\varphi}_S(\mathfrak{F})$ satisfies (F1), from which we infer that $x^{-1}K \in \hat{\varphi}_S(\mathfrak{F})$. This shows that Condition (F3) is also satisfied. Thus $\hat{\varphi}_S(\mathfrak{F})$ is a topologizing filter on RS^{-1}.

(b) Recall that the natural ring homomorphism $\varphi_S : R \to RS^{-1}$ induces a map $\varphi_S^* : \mathrm{Fil}\, RS^{-1} \to \mathrm{Fil}\, R$ given by $\varphi_S^*(\mathfrak{G}) \stackrel{\mathrm{def}}{=} \{I \leq R_R : I \supseteq \varphi_S^{-1}[L] \text{ for some } L \in \mathfrak{G}\}$. To show that $\hat{\varphi}_S$ is onto, it suffices to show that $(\hat{\varphi}_S \circ \varphi_S^*)(\mathfrak{G}) = \mathfrak{G}$ for all $\mathfrak{G} \in \mathrm{Fil}\, RS^{-1}$. Take $\mathfrak{G} \in \mathrm{Fil}\, RS^{-1}$. Then

$$\begin{aligned}
& K \in (\hat{\varphi}_S \circ \varphi_S^*)(\mathfrak{G}) = \hat{\varphi}_S(\varphi_S^*(\mathfrak{G})) \\
\Rightarrow\ & K = AS^{-1} \text{ for some } A \in \varphi_S^*(\mathfrak{G}) \text{ [by (3.10)]} \\
\Rightarrow\ & K \supseteq (\varphi_S^{-1}[L])S^{-1} \text{ for some } L \in \mathfrak{G} \text{ [by the definition of } \varphi_S^*] \\
\Rightarrow\ & K \supseteq L \text{ for some } L \in \mathfrak{G}.
\end{aligned}$$

The last of the above implications is a consequence of the identity

$$(\varphi_S^{-1}[L])S^{-1} = L \quad \forall L \in \mathrm{Id}\, RS^{-1}. \tag{3.11}$$

To see this, note that if $L \in \mathrm{Id}\, RS^{-1}$, then

$$\begin{aligned}
\frac{a}{s} \in L \ &\Leftrightarrow\ \varphi_S(a) = \frac{a}{1} \in L \text{ [because } \tfrac{a}{1} = \tfrac{a}{s} \cdot \tfrac{s}{1} \text{ with } \tfrac{s}{1} \text{ a unit of } RS^{-1}] \\
&\Leftrightarrow\ a \in \varphi_S^{-1}[L] \\
&\Leftrightarrow\ \frac{a}{s} \in (\varphi_S^{-1}[L])S^{-1}.
\end{aligned}$$

Since $K \supseteq L \in \mathfrak{G}$, we must have $K \in \mathfrak{G}$. Thus $(\hat{\varphi}_S \circ \varphi_S^*)(\mathfrak{G}) \subseteq \mathfrak{G}$.

On the other hand, if $K \in \mathfrak{G}$ then, by definition of φ_S^*, we must have $\varphi_S^{-1}[K] \in \varphi_S^*(\mathfrak{G})$. It follows from (3.10) that $(\varphi_S^{-1}[K])S^{-1} \in \hat{\varphi}_S(\varphi_S^*(\mathfrak{G}))$ and since $(\varphi_S^{-1}[K])S^{-1} = K$ [by (3.11)] it follows that $K \in \hat{\varphi}_S(\varphi_S^*(\mathfrak{G}))$ and so, $\mathfrak{G} \subseteq \hat{\varphi}_S(\varphi_S^*(\mathfrak{G}))$. Thus $(\hat{\varphi}_S \circ \varphi_S^*)(\mathfrak{G}) = \mathfrak{G}$, as required.

(c) We need to show that $\hat{\varphi}_S$ preserves infinite meets, i.e., $\hat{\varphi}_S(\bigcap_{\delta\in\Delta} \mathfrak{F}_\delta) = \bigcap_{\delta\in\Delta} \hat{\varphi}_S(\mathfrak{F}_\delta)$ for every family $\{\mathfrak{F}_\delta : \delta \in \Delta\} \subseteq \mathrm{Fil}\, R$. Clearly $\bigcap_{\delta\in\Delta} \mathfrak{F}_\delta \subseteq \mathfrak{F}_\delta$ for each $\delta \in \Delta$. Since $\hat{\varphi}_S$ is order preserving, we have $\hat{\varphi}_S(\bigcap_{\delta\in\Delta} \mathfrak{F}_\delta) \subseteq \hat{\varphi}_S(\mathfrak{F}_\delta)$ for each $\delta \in \Delta$. It follows that $\hat{\varphi}_S(\bigcap_{\delta\in\Delta} \mathfrak{F}_\delta) \subseteq \bigcap_{\delta\in\Delta} \hat{\varphi}_S(\mathfrak{F}_\delta)$.

To show the reverse containment, assume $K = AS^{-1} \in \bigcap_{\delta\in\Delta} \hat{\varphi}_S(\mathfrak{F}_\delta)$ with $A \in \mathrm{Id}\, R$. Then $K \in \hat{\varphi}_S(\mathfrak{F}_\delta)$ for each $\delta \in \Delta$, and so there exist $B_\delta \in \mathfrak{F}_\delta$ for each $\delta \in \Delta$ such that $AS^{-1} = B_\delta S^{-1}$.

Putting $B = \sum_{\delta\in\Delta} B_\delta$, we have $B \in \bigcap_{\delta\in\Delta} \mathfrak{F}_\delta$ and $BS^{-1} = (\sum_{\delta\in\Delta} B_\delta)S^{-1} = \sum_{\delta\in\Delta}(B_\delta S^{-1})$ [by Theorem 3.25(c)] which equals $\sum_{\delta\in\Delta} AS^{-1} = AS^{-1}$. Thus $K = AS^{-1} \in \hat{\varphi}_S(\bigcap_{\delta\in\Delta} \mathfrak{F}_\delta)$ [by (3.10)]. Hence $\hat{\varphi}_S$ preserves arbitrary (infinite) meets.

(d) Let $\{\mathfrak{F}_i : 1 \leq i \leq n\}$ be a finite subfamily of $\operatorname{Fil} R$. Clearly $\bigvee_{i=1}^n \mathfrak{F}_i \supseteq \mathfrak{F}_i$ for each $i \in \{1, 2, \ldots, n\}$ and this implies $\hat{\varphi}_S(\bigvee_{i=1}^n \mathfrak{F}_i) \supseteq \hat{\varphi}_S(\mathfrak{F}_i)$ for each $i \in \{1, 2, \ldots, n\}$. It follows that $\hat{\varphi}_S(\bigvee_{i=1}^n \mathfrak{F}_i) \supseteq \bigvee_{i=1}^n \hat{\varphi}_S(\mathfrak{F}_i)$.

On the other hand, take $K \in \hat{\varphi}_S(\bigvee_{i=1}^n \mathfrak{F}_i)$. Then $K = AS^{-1}$ for some $A \in \bigvee_{i=1}^n \mathfrak{F}_i$. It follows from our description of the join (see (1.2)) that there exist ideals $L_i \in \mathfrak{F}_i$ for each $i \in \{1, 2, \ldots, n\}$ such that $A \supseteq \bigcap_{i=1}^n L_i$. Then

$$\begin{aligned} K &= AS^{-1} \\ &\supseteq (\textstyle\bigcap_{i=1}^n L_i)S^{-1} \text{ [because } A \supseteq \textstyle\bigcap_{i=1}^n L_i] \\ &= \textstyle\bigcap_{i=1}^n L_i S^{-1} \text{ [by Theorem 3.25(b)]} \\ &\in \textstyle\bigvee_{i=1}^n \hat{\varphi}_S(\mathfrak{F}_i) \text{ [because } L_i S^{-1} \in \hat{\varphi}_S(\mathfrak{F}_i) \text{ for each } i \in \{1, 2, \ldots, n\}]. \end{aligned}$$

Thus $K \in \bigvee_{i=1}^n \hat{\varphi}_S(\mathfrak{F}_i)$ and so $\hat{\varphi}_S(\bigvee_{i=1}^n \mathfrak{F}_i) \subseteq \bigvee_{i=1}^n \hat{\varphi}_S(\mathfrak{F}_i)$, whence equality $\hat{\varphi}_S(\bigvee_{i=1}^n \mathfrak{F}_i) = \bigvee_{i=1}^n \hat{\varphi}_S(\mathfrak{F}_i)$. ■

Let S be a multiplicative subset of a commutative ring R. We define

$$\mathfrak{F}_S \stackrel{\text{def}}{=} \{A \leq R : A \cap S \neq \emptyset\}. \tag{3.12}$$

The following result is a special case of [31, Proposition VI.6.1, p. 148]. A more detailed proof may be found in [33, Proposition III.2.4, p. 132].

Proposition 3.30 *Let S be a multiplicative subset of a commutative ring R. Then $\mathfrak{F}_S$ is a (right) Gabriel topology on R.*

The following result shows that the Gabriel topology $\mathfrak{F}_S$ satisfies a weak form of centrality in $\operatorname{Fil} R$.

Proposition 3.31 *Let S be a multiplicative subset of a commutative ring R. Then*

$$\mathfrak{F}_S : \mathfrak{G} \subseteq \mathfrak{G} : \mathfrak{F}_S$$

for all $\mathfrak{G} \in \operatorname{Fil} R$.

Proof. Note that $\mathfrak{F}_S$ has cofinal set $\{tR : t \in S\}$ comprising principal (and thus finitely generated) ideals of R. This is because an ideal A of R is such that $A \cap S \neq \emptyset$ precisely if $t \in A$ for some $t \in S$, which is to say, $A \supseteq tR$ for some $t \in S$. It follows from Lemma 3.1, that $\{tK : t \in S, K \in \mathfrak{G}\}$ is a cofinal set for $\mathfrak{F}_S : \mathfrak{G}$. Take $t \in S$ and $K \in \mathfrak{G}$ and consider the short exact sequence

$$0 \to K/tK \to R/tK \to R/K \to 0.$$

Observe that the right R-module K/tK is annihilated by t and is thus $\mathcal{T}_{\mathfrak{F}_S}$-torsion. Since R/K is $\mathcal{T}_{\mathfrak{G}}$-torsion, it follows that R/tK is $\mathcal{T}_{\mathfrak{G}:\mathfrak{F}_S}$-torsion, whence $tK \in \mathfrak{G} : \mathfrak{F}_S$. Since the family $\{tK : t \in S, K \in \mathfrak{G}\}$ is cofinal in $\mathfrak{F}_S : \mathfrak{G}$, we conclude that $\mathfrak{F}_S : \mathfrak{G} \subseteq \mathfrak{G} : \mathfrak{F}_S$. ■

Corollary 3.32 *Let S be a multiplicative subset of a commutative ring R. Then*

$$\mathfrak{F}_S : \mathfrak{G} : \mathfrak{F}_S = \mathfrak{G} : \mathfrak{F}_S$$

for all $\mathfrak{G} \in \operatorname{Fil} R$.

Proof. Take $\mathfrak{G} \in \operatorname{Fil} R$. Certainly $\mathfrak{F}_S : \mathfrak{G} : \mathfrak{F}_S \supseteq \mathfrak{G} : \mathfrak{F}_S$. By the previous proposition,

$$\begin{aligned}
\mathfrak{F}_S : \mathfrak{G} : \mathfrak{F}_S &\subseteq (\mathfrak{G} : \mathfrak{F}_S) : \mathfrak{F}_S \\
&= \mathfrak{G} : (\mathfrak{F}_S : \mathfrak{F}_S) \\
&= \mathfrak{G} : \mathfrak{F}_S \text{ [because } \mathfrak{F}_S \text{ is a Gabrieal topology, so } \mathfrak{F}_S : \mathfrak{F}_S = \mathfrak{F}_S].
\end{aligned}$$

Thus $\mathfrak{F}_S : \mathfrak{G} : \mathfrak{F}_S = \mathfrak{G} : \mathfrak{F}_S$. ■

Lemma 3.33 *Let S be a multiplicative subset of a commutative ring R. The following statements are equivalent for a right R-module M:*

(a) $MS^{-1} = 0$;

(b) M *is a* $\mathcal{T}_{\mathfrak{F}_S}$*-torsion right* R*-module.*

Proof. (a)⇒(b) Suppose (a) holds. If $x \in M$, then $\dfrac{x}{1} = \dfrac{0}{1}$. This implies that there exists $t \in S$ such that $xt = 0$, so $x^{-1}0 \cap S \neq \emptyset$. Thus $x^{-1}0 \in \mathfrak{F}_S$ for all $x \in M$, hence M is $\mathcal{T}_{\mathfrak{F}_S}$-torsion.

(b)⇒(a) Take $\dfrac{x}{s} \in MS^{-1}$. Since M is $\mathcal{T}_{\mathfrak{F}_S}$-torsion, $x^{-1}0 \in \mathfrak{F}_S$, so $x^{-1}0 \cap S \neq \emptyset$. It follows that $xt = 0$ for some $t \in S$ which implies that $\dfrac{x}{s} = 0$. We conclude that $MS^{-1} = 0$. ■

We require the following generalisation of the previous lemma.

Lemma 3.34 *Let* S *be a multiplicative subset of a commutative ring* R*. Let* $\mathfrak{G} \in \operatorname{Fil} R$*. The following statements are equivalent for a right* R*-module* M*:*

(a) MS^{-1} *is a* $\mathcal{T}_{\hat{\varphi}_S(\mathfrak{G})}$*-torsion right* RS^{-1}*-module;*

(b) M *is a* $\mathcal{T}_{\mathfrak{G}:\mathfrak{F}_S}$*-torsion right* R*-module.*

Proof. (a)⇒(b) Take $x \in M$. Since $\dfrac{x}{1} \in MS^{-1}$ and MS^{-1} is $\mathcal{T}_{\hat{\varphi}_S(\mathfrak{G})}$-torsion, $\left(\dfrac{x}{1}\right) I = 0$ for some $I \in \hat{\varphi}_S(\mathfrak{G})$. Put $I = AS^{-1}$ with $A \in \mathfrak{G}$. Then $\left(\dfrac{x}{1}\right) I = (xA)S^{-1} = 0$. By the previous lemma, xA must be a $\mathcal{T}_{\mathfrak{F}_S}$-torsion R-submodule of M. Consider the short exact sequence

$$0 \to xA \to xR \to xR/xA \to 0. \tag{3.13}$$

Note that xR/xA is $\mathcal{T}_{\mathfrak{G}}$-torsion since it is an epimorphic image of R/A which is $\mathcal{T}_{\mathfrak{G}}$-torsion because $A \in \mathfrak{G}$. It follows from (3.13) that xR is $\mathcal{T}_{\mathfrak{G}:\mathfrak{F}_S}$-torsion, so $x \in \mathcal{T}_{\mathfrak{G}:\mathfrak{F}_S}(M)$. We conclude that M is $\mathcal{T}_{\mathfrak{G}:\mathfrak{F}_S}$-torsion.

(b)⇒(a) Take $\dfrac{x}{s} \in MS^{-1}$ with $x \in M$. Put $x^{-1}0 = A \in \operatorname{Id} R$. It is easily seen that $\left(\dfrac{x}{s}\right)(AS^{-1}) = 0$. Since M is $\mathcal{T}_{\mathfrak{G}:\mathfrak{F}_S}$-torsion, $A \in \mathfrak{G} : \mathfrak{F}_S$, so there exists $H \in \mathfrak{G}$ such that $H \supseteq A$ and H/A is $\mathfrak{F}_S$-torsion. Consider the short exact sequence

$$0 \to H/A \to R/A \to R/H \to 0$$

in $\mathrm{Mod}\text{-}R$. This induces the following short exact sequence in $\mathrm{Mod}\text{-}RS^{-1}$:

$$0 \to (H/A)S^{-1} \to (R/A)S^{-1} \to (R/H)S^{-1} \to 0.$$

Since H/A is $\mathfrak{F}_S$-torsion, it follows from the previous lemma that $(H/A)S^{-1} = 0$ which implies $(R/A)S^{-1}$ and $(R/H)S^{-1}$ are isomorphic right RS^{-1}-modules. By Proposition 3.28(c), $(R/H)S^{-1} \cong RS^{-1}/HS^{-1}$ and $(R/A)S^{-1} \cong RS^{-1}/AS^{-1}$. Since $H \in \mathfrak{G}$, $HS^{-1} \in \hat{\varphi}_S(\mathfrak{G})$ from which we infer that $RS^{-1}/HS^{-1} \cong RS^{-1}/AS^{-1}$ is $\mathcal{T}_{\hat{\varphi}_S(\mathfrak{G})}$-torsion, whence $AS^{-1} \in \hat{\varphi}_S(\mathfrak{G})$. Since $\left(\dfrac{x}{s}\right)(AS^{-1}) = 0$, $\dfrac{x}{s} \in \mathcal{T}_{\hat{\varphi}_S(\mathfrak{G})}(MS^{-1})$. Thus MS^{-1} is $\mathcal{T}_{\hat{\varphi}_S(\mathfrak{G})}$-torsion. ■

Proposition 3.35 *Let S be a multiplicative subset of a commutative ring R. Then:*

(a) $\hat{\varphi}_S(\mathfrak{F} : \mathfrak{F}_S) = \hat{\varphi}_S(\mathfrak{F}_S : \mathfrak{F}) = \hat{\varphi}_S(\mathfrak{F})$ *for all* $\mathfrak{F} \in \mathrm{Fil}\, R$.

(b) $\hat{\varphi}_S(\mathfrak{F} : \mathfrak{G}) = \hat{\varphi}_S(\mathfrak{F}) : \hat{\varphi}_S(\mathfrak{G})$ *for all* $\mathfrak{F}, \mathfrak{G} \in \mathrm{Fil}\, R$.

Proof. We first show that $\hat{\varphi}_S(\mathfrak{F} : \mathfrak{G}) \subseteq \hat{\varphi}_S(\mathfrak{F}) : \hat{\varphi}_S(\mathfrak{G})$ for all $\mathfrak{F}, \mathfrak{G} \in \mathrm{Fil}\, R$. Take $K \in \hat{\varphi}_S(\mathfrak{F} : \mathfrak{G})$. Then $K = AS^{-1}$ for some $A \in \mathfrak{F} : \mathfrak{G}$. There exists therefore $H \in \mathfrak{F}$ containing A such that H/A is $\mathcal{T}_{\mathfrak{G}}$-torsion. Since $H \in \mathfrak{F}$, $HS^{-1} \in \hat{\varphi}_S(\mathfrak{F})$ and so $RS^{-1}/HS^{-1} \cong (R/H)S^{-1}$ [by Proposition 3.28(c)] is $\mathcal{T}_{\hat{\varphi}_S(\mathfrak{F})}$-torsion. Note also that $HS^{-1}/AS^{-1} \cong (H/A)S^{-1}$ is $\mathcal{T}_{\hat{\varphi}_S(\mathfrak{G})}$-torsion since H/A is $\mathcal{T}_{\mathfrak{G}}$-torsion. Now consider the short exact sequence

$$0 \to HS^{-1}/AS^{-1} \to RS^{-1}/AS^{-1} \to RS^{-1}/HS^{-1} \to 0.$$

Since HS^{-1}/AS^{-1} is $\mathcal{T}_{\hat{\varphi}_S(\mathfrak{G})}$-torsion and RS^{-1}/HS^{-1} is $\mathcal{T}_{\hat{\varphi}_S(\mathfrak{F})}$-torsion, it follows that $K = AS^{-1} \in \hat{\varphi}_S(\mathfrak{F}) : \hat{\varphi}_S(\mathfrak{G})$, and hence $\hat{\varphi}_S(\mathfrak{F} : \mathfrak{G}) \subseteq \hat{\varphi}_S(\mathfrak{F}) : \hat{\varphi}_S(\mathfrak{G})$.

(a) Take $\mathfrak{F} \in \mathrm{Fil}\, R$. Certainly $\hat{\varphi}_S(\mathfrak{F} : \mathfrak{F}_S) \supseteq \hat{\varphi}_S(\mathfrak{F})$ since $\mathfrak{F} : \mathfrak{F}_S \supseteq \mathfrak{F}$. To establish the reverse containment, we observe that by the above argument, $\hat{\varphi}_S(\mathfrak{F} : \mathfrak{F}_S) \subseteq \hat{\varphi}_S(\mathfrak{F}) : \hat{\varphi}_S(\mathfrak{F}_S)$. But $\hat{\varphi}_S(\mathfrak{F}_S) = \{RS^{-1}\}$, for if $I \in \hat{\varphi}_S(\mathfrak{F}_S)$, then $I = AS^{-1}$ for some $A \in \mathfrak{F}_S$ and this means $A \cap S \neq \emptyset$, whence $I = AS^{-1} = RS^{-1}$. Since $\{RS^{-1}\}$ is the identity of $\mathrm{Fil}\, RS^{-1}$ with respect to the monoid operation, we obtain $\hat{\varphi}_S(\mathfrak{F} : \mathfrak{F}_S) \subseteq \hat{\varphi}_S(\mathfrak{F}) : \{RS^{-1}\} = \hat{\varphi}_S(\mathfrak{F})$. Thus $\hat{\varphi}_S(\mathfrak{F} : \mathfrak{F}_S) \subseteq \hat{\varphi}_S(\mathfrak{F})$, whence equality $\hat{\varphi}_S(\mathfrak{F} : \mathfrak{F}_S) = \hat{\varphi}_S(\mathfrak{F})$.

The proof that $\hat{\varphi}_S(\mathfrak{F}_S : \mathfrak{F}) = \hat{\varphi}_S(\mathfrak{F})$ is similar to the above.

(b) Take $\mathfrak{F}, \mathfrak{G} \in \operatorname{Fil} R$. It follows from the argument preceding the proof of (a) above that $\hat{\varphi}_S(\mathfrak{F} : \mathfrak{G}) \subseteq \hat{\varphi}_S(\mathfrak{F}) : \hat{\varphi}_S(\mathfrak{G})$. It remains to establish the reverse containment. To this end, take $K = AS^{-1} \in \hat{\varphi}_S(\mathfrak{F}) : \hat{\varphi}_S(\mathfrak{G})$ with $A \in \operatorname{Id} R$. There exists therefore an ideal $H \in \hat{\varphi}_S(\mathfrak{F})$ containing K such that H/K is $\mathcal{T}_{\hat{\varphi}_S(\mathfrak{G})}$-torsion. Since $H \in \hat{\varphi}_S(\mathfrak{F})$, $H = BS^{-1}$ for some $B \in \mathfrak{F}$. Noting that $(A+B)S^{-1} = AS^{-1} + BS^{-1} = BS^{-1}$ by Proposition 3.28(a), and that $A + B \in \mathfrak{F}$ because $A + B \supseteq B \in \mathfrak{F}$, we see that no generality is lost if we replace B with $A + B$ and assume that $A \subseteq B$. The short exact sequence

$$0 \to B/A \to R/A \to R/B \to 0 \tag{3.14}$$

in $\operatorname{Mod}\text{-}R$ induces the short exact sequence

$$0 \to (B/A)S^{-1} \to (R/A)S^{-1} \to (R/B)S^{-1} \to 0$$

in $\operatorname{Mod}\text{-}RS^{-1}$. Inasmuch as $H/K = BS^{-1}/AS^{-1} \cong (B/A)S^{-1}$ [by Proposition 3.28(c)] is $\mathcal{T}_{\hat{\varphi}_S(\mathfrak{G})}$-torsion, we infer from Lemma 3.34 that B/A is $\mathcal{T}_{\mathfrak{G}:\mathfrak{F}_S}$-torsion. It follows from (3.14) that R/A is $\mathcal{T}_{\mathfrak{F}:\mathfrak{G}:\mathfrak{F}_S}$-torsion, hence $A \in \mathfrak{F} : \mathfrak{G} : \mathfrak{F}_S$ and $K = AS^{-1} \in \hat{\varphi}_S(\mathfrak{F} : \mathfrak{G} : \mathfrak{F}_S)$. This shows that $\hat{\varphi}_S(\mathfrak{F}) : \hat{\varphi}_S(\mathfrak{G}) \subseteq \hat{\varphi}_S(\mathfrak{F} : \mathfrak{G} : \mathfrak{F}_S)$. The required containment follows noting that $\hat{\varphi}_S(\mathfrak{F} : \mathfrak{G} : \mathfrak{F}_S) = \hat{\varphi}_S(\mathfrak{F} : \mathfrak{G})$ by (a). ■

Propositions 3.29 and 3.35(b) now yield the following theorem.

Theorem 3.36 *Let S be a multiplicative subset of a commutative ring R. Then $\hat{\varphi}_S : [\operatorname{Fil} R]^{\mathrm{du}} \to [\operatorname{Fil} RS^{-1}]^{\mathrm{du}}$ is an onto homomorphism of lattice ordered monoids.*

Theorem 3.37 *(Preservation Theorem) Let S be a multiplicative subset of a commutative ring R. Then the following statements hold.*

(a) If $\operatorname{Fil} R$ is commutative then so is $\operatorname{Fil} RS^{-1}$.

(b) If every member of $\operatorname{Fil} R$ is idempotent then the same is true of every member of $\operatorname{Fil} RS^{-1}$.

Proof. Since the mapping $\hat{\varphi}_S : \mathrm{Fil}\, R \to \mathrm{Fil}\, RS^{-1}$ is onto [by Proposition 3.29(b)] and preserves the monoid operation [by Proposition 3.35(b)] any property of $\mathrm{Fil}\, R$ that is characterizable in terms of an identity involving only the monoid operation, is passed from $\mathrm{Fil}\, R$ to $\mathrm{Fil}\, RS^{-1}$. The result follows noting that (a) and (b) are such properties. ∎

Remark 3.38 *The value of Statement (b) of the previous theorem is tempered by the known fact (see [39, Theorem 2.1, p. 547]) that a commutative ring R for which every member of $\mathrm{Fil}\, R$ is idempotent, is necessarily semisimple and thus a finite product of fields.*

If S is a multiplicative subset of a commutative ring R, then Theorem 3.36 tells us that the mapping $\hat{\varphi}_S : [\mathrm{Fil}\, R]^{\mathrm{du}} \to [\mathrm{Fil}\, RS^{-1}]^{\mathrm{du}}$ is a homomorphism of lattice ordered monoids. Such a map $\hat{\varphi}_S$ gives rise to a canonical congruence relation $\equiv_{\hat{\varphi}_S}$ on $\mathrm{Fil}\, R$ defined by:

$$\mathfrak{F} \equiv_{\hat{\varphi}_S} \mathfrak{G} \;\Leftrightarrow\; \hat{\varphi}_S(\mathfrak{F}) = \hat{\varphi}_S(\mathfrak{G}). \tag{3.15}$$

Proposition 3.39 *Let S be a multiplicative subset of a commutative ring R. The following statements are equivalent for $\mathfrak{F}, \mathfrak{G} \in \mathrm{Fil}\, R$:*

(a) $\mathfrak{F} \equiv_{\hat{\varphi}_S} \mathfrak{G}$, i.e., $\hat{\varphi}_S(\mathfrak{F}) = \hat{\varphi}_S(\mathfrak{G})$;

(b) $\mathfrak{F} : \mathfrak{F}_S = \mathfrak{G} : \mathfrak{F}_S$.

Proof. (a)⇒(b) Take $M \in \mathrm{Mod}\text{-}R$. Then:

$$\begin{aligned}
M \text{ is } \mathcal{T}_{\mathfrak{F}:\mathfrak{F}_S}\text{-torsion} &\Leftrightarrow MS^{-1} \text{ is } \mathcal{T}_{\hat{\varphi}_S(\mathfrak{F})}\text{-torsion [by Lemma 3.34]} \\
&\Leftrightarrow MS^{-1} \text{ is } \mathcal{T}_{\hat{\varphi}_S(\mathfrak{G})}\text{-torsion [by (a)]} \\
&\Leftrightarrow M \text{ is } \mathcal{T}_{\mathfrak{G}:\mathfrak{F}_S}\text{-torsion [by Lemma 3.34].}
\end{aligned}$$

The above shows that the classes of $\mathcal{T}_{\mathfrak{F}:\mathfrak{F}_S}$-torsion and $\mathcal{T}_{\mathfrak{G}:\mathfrak{F}_S}$-torsion modules coincide from which we deduce that $\mathfrak{F} : \mathfrak{F}_S = \mathfrak{G} : \mathfrak{F}_S$.

(b)⇒(a) is an immediate consequence of Proposition 3.35(a). ∎

Lemma 3.40 *Let R be an arbitrary ring, $\mathfrak{F}, \mathfrak{G} \in \mathrm{Fil}\, R_R$ and $\mathfrak{H}_1, \mathfrak{H}_2 \in \mathrm{Fil}\, R_R$ with $\mathfrak{H}_1 \subseteq \mathfrak{H}_2$ and $\mathfrak{H}_2$ is idempotent (that is to say, $\mathfrak{H}_2 : \mathfrak{H}_2 = \mathfrak{H}_2$, i.e., $\mathfrak{H}_2$ is a right Gabriel topology on R). If $\mathfrak{F} : \mathfrak{H}_1 \subseteq \mathfrak{G} : \mathfrak{H}_1$, then $\mathfrak{F} : \mathfrak{H}_2 \subseteq \mathfrak{G} : \mathfrak{H}_2$.*

Proof. $\mathfrak{F} \subseteq \mathfrak{F} : \mathfrak{H}_1 \subseteq \mathfrak{G} : \mathfrak{H}_1 \subseteq \mathfrak{G} : \mathfrak{H}_2$ [because $\mathfrak{H}_1 \subseteq \mathfrak{H}_2$]. Therefore, $\mathfrak{F} : \mathfrak{H}_2 \subseteq (\mathfrak{G} : \mathfrak{H}_2) : \mathfrak{H}_2 = \mathfrak{G} : (\mathfrak{H}_2 : \mathfrak{H}_2) = \mathfrak{G} : \mathfrak{H}_2$ [because $\mathfrak{H}_2$ is idempotent]. ■

Theorem 3.41 *Let $\{S_\delta : \delta \in \Delta\}$ be a family of multiplicative subsets of a commutative ring R. Then the following statements are equivalent for $\mathfrak{F}, \mathfrak{G} \in \mathrm{Fil}\, R$:*

(a) $\mathfrak{F} \equiv_{\hat{\varphi}_{S_\delta}} \mathfrak{G}$ for all $\delta \in \Delta$;

(b) $\mathfrak{F}$ is congruent to $\mathfrak{G}$ with respect to the intersection of congruences $\bigcap_{\delta\in\Delta} \equiv_{\hat{\varphi}_{S_\delta}}$;

(c) $\mathfrak{F} : \mathfrak{F}_{S_\delta} = \mathfrak{G} : \mathfrak{F}_{S_\delta}$ for all $\delta \in \Delta$;

If, moreover, $\mathfrak{F}$ and $\mathfrak{G}$ commute with each $\mathfrak{F}_{S_\delta}$ and also with $\bigcap_{\delta\in\Delta} \mathfrak{F}_{S_\delta}$, that is to say, $\mathfrak{F} : \mathfrak{F}_{S_\delta} = \mathfrak{F}_{S_\delta} : \mathfrak{F}$, $\mathfrak{G} : \mathfrak{F}_{S_\delta} = \mathfrak{F}_{S_\delta} : \mathfrak{G}$ and also $\mathfrak{F} : (\bigcap_{\delta\in\Delta} \mathfrak{F}_{S_\delta}) = (\bigcap_{\delta\in\Delta} \mathfrak{F}_{S_\delta}) : \mathfrak{F}$ and $\mathfrak{G} : (\bigcap_{\delta\in\Delta} \mathfrak{F}_{S_\delta}) = (\bigcap_{\delta\in\Delta} \mathfrak{F}_{S_\delta}) : \mathfrak{G}$, then the above conditions are equivalent to:

(d) $\mathfrak{F} : (\bigcap_{\delta\in\Delta} \mathfrak{F}_{S_\delta}) = \mathfrak{G} : (\bigcap_{\delta\in\Delta} \mathfrak{F}_{S_\delta})$.

Proof. The equivalence of (a) and (b) is self-evident, whilst the equivalence of (a) and (c) is just Proposition 3.39.

(d)⇒(c) Since $\bigcap_{\delta\in\Delta} \mathfrak{F}_{S_\delta} \subseteq \mathfrak{F}_{S_\delta}$ for each $\delta \in \Delta$ and since each $\mathfrak{F}_{S_\delta}$ is idempotent by Proposition 3.30, it follows from the previous lemma that $\mathfrak{F} : \mathfrak{F}_{S_\delta} = \mathfrak{G} : \mathfrak{G}_{S_\delta}$ $\forall \delta \in \Delta$.

(c)⇒(d) Suppose (c) holds. Then

$$
\begin{aligned}
\mathfrak{F} : (\textstyle\bigcap_{\delta\in\Delta} \mathfrak{F}_{S_\delta}) &= (\textstyle\bigcap_{\delta\in\Delta} \mathfrak{F}_{S_\delta}) : \mathfrak{F} \text{ [because } \mathfrak{F} \text{ commutes with } \textstyle\bigcap_{\delta\in\Delta} \mathfrak{F}_{S_\delta}] \\
&= \textstyle\bigcap_{\delta\in\Delta} (\mathfrak{F}_{S_\delta} : \mathfrak{F}) \text{ [by the left analogue of Proposition 1.2, noting} \\
&\qquad\qquad \text{that } \mathrm{Fil}\, R \text{ is always left residuated].} \\
&= \textstyle\bigcap_{\delta\in\Delta} (\mathfrak{F} : \mathfrak{F}_{S_\delta}) \text{ [because } \mathfrak{F} \text{ commutes with each } \mathfrak{F}_{S_\delta}] \\
&= \textstyle\bigcap_{\delta\in\Delta} (\mathfrak{G} : \mathfrak{F}_{S_\delta}) \text{ [by (c)]} \\
&= \textstyle\bigcap_{\delta\in\Delta} (\mathfrak{F}_{S_\delta} : \mathfrak{G}) \text{ [because } \mathfrak{G} \text{ commutes with each } \mathfrak{F}_{S_\delta}] \\
&= (\textstyle\bigcap_{\delta\in\Delta} \mathfrak{F}_{S_\delta}) : \mathfrak{G} \text{ [by the left analogue of Proposition 1.2]} \\
&= \mathfrak{G} : (\textstyle\bigcap_{\delta\in\Delta} \mathfrak{F}_{S_\delta}) \text{ [because } \mathfrak{G} \text{ commutes with } \textstyle\bigcap_{\delta\in\Delta} \mathfrak{F}_{S_\delta}].
\end{aligned}
$$

■

If R is a commutative ring, we shall denote by $\mathrm{Spec_m}\, R$ the set of all maximal (proper) ideals of R. For each $P \in \mathrm{Spec_m}\, R$, define multiplicative subset S_P of R by

$$S_P \stackrel{\text{def}}{=} R \backslash P.$$

Lemma 3.42 *If R is any commutative ring, then $\bigcap\{\mathfrak{F}_{S_P} : P \in \mathrm{Spec_m}\, R\} = \{R\}$, the identity of $\mathrm{Fil}\, R$ with respect to the monoid operation.*

Proof. Let I be any proper ideal of R. Since I is proper, $I \subseteq P$ for some $P \in \mathrm{Spec_m}\, R$. This means that $I \cap S_P = \emptyset$, which is to say, $I \notin \mathfrak{F}_{S_P}$ and so $I \notin \bigcap\{\mathfrak{F}_{S_P} : P \in \mathrm{Spec_m}\, R\}$. We conclude that $\bigcap\{\mathfrak{F}_{S_P} : P \in \mathrm{Spec_m}\, R\} = \{R\}$. ■

Proposition 3.43 *Let R be a commutative ring for which $\mathrm{Fil}\, R$ is commutative. Then $\bigcap\{\equiv_{\hat{\varphi}_{S_P}} : P \in \mathrm{Spec_m}\, R\}$ is the identity congruence on $\mathrm{Fil}\, R$, that is, for all $\mathfrak{F}, \mathfrak{G} \in \mathrm{Fil}\, R$,*

$$\mathfrak{F} = \mathfrak{G} \Leftrightarrow \mathfrak{F} \equiv_{\hat{\varphi}_{S_P}} \mathfrak{G} \ \ \forall P \in \mathrm{Spec_m}\, R.$$

Proof. Taking the family $\{S_\delta : \delta \in \Delta\}$ of Theorem 3.41 to be $\{S_P : P \in \mathrm{Spec_m}\, R\}$, the result follows from Theorem 3.41 ((a)⇔(d)) and the previous lemma. ■

Remark 3.44 *We have no example to show that the previous proposition fails if the requirement that* $\mathrm{Fil}\, R$ *is commutative, is dispensed with.*

If R is a commutative ring for which $\mathrm{Fil}\, R$ is commutative, then the previous proposition yields the following subdirect decomposition:

$$[\mathrm{Fil}\, R]^{\mathrm{du}} \cong [\mathrm{Fil}\, R]^{\mathrm{du}} / \left(\bigcap_{P \in \mathrm{Spec_m}\, R} \equiv_{\hat{\varphi}_{S_P}} \right) \hookrightarrow \prod_{P \in \mathrm{Spec_m}\, R} ([\mathrm{Fil}\, R]^{\mathrm{du}} / \equiv_{\hat{\varphi}_{S_P}}) \cong \prod_{P \in \mathrm{Spec_m}\, R} [\mathrm{Fil}\, R_P]^{\mathrm{du}}.$$

With reference to the above sequence of mappings, recall that by Theorem 3.36, the mapping

$$\hat{\varphi}_{S_P} : [\mathrm{Fil}\, R]^{\mathrm{du}} \to [\mathrm{Fil}\, R_P]^{\mathrm{du}}$$
$$\mathfrak{F} \mapsto \hat{\varphi}_{S_P}(\mathfrak{F})$$

defines an onto homomorphism of lattice ordered monoids with $\equiv_{\hat{\varphi}_{S_P}}$ the congruence on $\mathrm{Fil}\, R$ induced by $\hat{\varphi}_{S_P}$ (see Section 1.4). It follows that

$$\mathrm{Fil}\, R / \equiv_{\hat{\varphi}_{S_P}} \cong \mathrm{Fil}\, R_P \text{ for each } P \in \mathrm{Spec_m}\, R.$$

The aforementioned subdirect decomposition thus takes $\mathfrak{F}$ in $\mathrm{Fil}\, R$ onto $\{\hat{\varphi}_{S_P}(\mathfrak{F})\}_{P \in \mathrm{Spec_m}\, R}$ in $\prod_{P \in \mathrm{Spec_m}\, R} \mathrm{Fil}\, R_P$.

If $\mathfrak{F} \in \mathrm{Fil}\, R$ is jansian, that is to say, $\mathfrak{F} = \eta(I)$ for some $I \in \mathrm{Id}\, R$, then it follows from [17, Proposition 3.13, p. 37], that $\mathfrak{F} : (\bigcap_{\delta \in \Delta} \mathfrak{F}_\delta) = \bigcap_{\delta \in \Delta} (\mathfrak{F} : \mathfrak{F}_\delta)$ for all families $\{\mathfrak{F}_\delta : \delta \in \Delta\}$ in $\mathrm{Fil}\, R$. Given any family $\{S_\delta : \delta \in \Delta\}$ of multiplicative subsets of R, we thus have, for jansian $\mathfrak{F}, \mathfrak{G} \in \mathrm{Fil}\, R$, the following:

$$\begin{aligned} \mathfrak{F} \equiv_{\hat{\varphi}_{S_\delta}} \mathfrak{G}\ \forall \delta \in \Delta \quad &\Leftrightarrow \quad \mathfrak{F} : \mathfrak{F}_{S_\delta} = \mathfrak{G} : \mathfrak{F}_{S_\delta}\ \forall \delta \in \Delta \text{ [by Theorem 3.41 ((a)}\Leftrightarrow\text{(c))]} \\ &\Rightarrow \quad \textstyle\bigcap_{\delta \in \Delta} (\mathfrak{F} : \mathfrak{F}_{S_\delta}) = \bigcap_{\delta \in \Delta} (\mathfrak{G} : \mathfrak{F}_{S_\delta}) \\ &\Leftrightarrow \quad \mathfrak{F} : (\textstyle\bigcap_{\delta \in \Delta} \mathfrak{F}_{S_\delta}) = \mathfrak{G} : (\bigcap_{\delta \in \Delta} \mathfrak{F}_{S_\delta}) \text{ [because } \mathfrak{F} \text{ and } \mathfrak{G} \text{ are jansian].} \end{aligned}$$

Taking the family $\{S_\delta : \delta \in \Delta\}$ to be $\{S_P : P \in \mathrm{Spec_m}\, R\}$, it follows from the above and Lemma 3.42 that

$$\mathfrak{F} \equiv_{\hat{\varphi}_{S_\delta}} \mathfrak{G} \ \ \forall \delta \in \Delta \Leftrightarrow \mathfrak{F} = \mathfrak{G}.$$

We have thus proved the following:

Corollary 3.45 *Let R be an arbitrary commutative ring. For all jansian $\mathfrak{F}, \mathfrak{G} \in \mathrm{Fil}\, R$, we have*

$$\mathfrak{F} = \mathfrak{G} \Leftrightarrow \mathfrak{F} \equiv_{\hat{\varphi}_{S_P}} \mathfrak{G} \ \ \forall P \in \mathrm{Spec_m}\, R.$$

Proposition 3.46 *Let S be a multiplicative subset of a commutative ring R. Then the following statements hold.*

(a) If $\mathfrak{F} \in \mathrm{Fil}\, R$ is jansian, then so is $\hat{\varphi}_S(\mathfrak{F})$.

(b) Every jansian $\mathfrak{G} \in \mathrm{Fil}\, RS^{-1}$ has the form $\hat{\varphi}_S(\mathfrak{F})$ for some jansian $\mathfrak{F} \in \mathrm{Fil}\, R$.

Proof. (a) Suppose $\mathfrak{F} \in \mathrm{Fil}\, R$ is jansian, i.e., $\mathfrak{F} = \eta(I)$ for some $I \in \mathrm{Id}\, R$. Then

$$\begin{aligned} \hat{\varphi}_S(\mathfrak{F}) = \hat{\varphi}_S(\eta(I)) &= \{AS^{-1} : A \in \eta(I)\} \\ &= \{AS^{-1} : A \supseteq I\} \\ &= \{AS^{-1} : AS^{-1} \supseteq IS^{-1}\} \\ &= \eta(IS^{-1}). \end{aligned}$$

Hence $\hat{\varphi}_S(\mathfrak{F})$ is a jansian member of $\mathrm{Fil}\, RS^{-1}$.

(b) If $\mathfrak{G}$ is an arbitrary jansian member of $\mathrm{Fil}\, RS^{-1}$, then $\mathfrak{G} = \eta(K)$ for some $K \in \mathrm{Id}\, RS^{-1}$. Put $K = IS^{-1}$ with $I \in \mathrm{Id}\, R$. Since the map $\hat{\varphi}_S$ from $\mathrm{Fil}\, R$ to $\mathrm{Fil}\, RS^{-1}$ is onto by Proposition 3.29(b), there exists $\mathfrak{F} \in \mathrm{Fil}\, R$ such that $\mathfrak{G} = \hat{\varphi}_S(\mathfrak{F})$. It is easy to verify that $\mathfrak{G} = \hat{\varphi}_S(\mathfrak{F}) = \eta(IS^{-1})$ if and only if $\mathfrak{F} = \eta(I)$, (i.e., $\mathfrak{F}$ is jansian in $\mathrm{Fil}\, R$) and hence the result follows. ■

The above result tells us that the homomorphism $\hat{\varphi}_S : [\mathrm{Fil}\, R]^{\mathrm{du}} \to [\mathrm{Fil}\, RS^{-1}]^{\mathrm{du}}$ restricts to a homomorphism from $[\mathrm{Jans}\, R]^{\mathrm{du}}$ onto $[\mathrm{Jans}\, RS^{-1}]^{\mathrm{du}}$. We thus obtain the following diagram of lattice ordered monoids and lattice ordered monoid homomorphisms.

$$\begin{tikzcd}
\mathfrak{F} \arrow[r, mapsto] & \{\hat{\varphi}_{S_P}\}_P \\[-2em]
{[\operatorname{Jans} R]^{\mathrm{du}}} \arrow[r, hook] & \prod_P [\operatorname{Jans} R_P]^{\mathrm{du}} \\
\operatorname{Id} R \arrow[u, "\eta_R"', "iso"] \arrow[r, hook] & \prod_P \operatorname{Id} R_P \arrow[u, "iso", "\prod_P \eta_{R_P}"'] \\[-2em]
I \arrow[r, mapsto] & \{I_P\}_P
\end{tikzcd}$$

[Note that in the above diagram, products are indexed by the set $\operatorname{Spec_m} R$.]

In the above diagram, we have appended the relevant ring as subscript to each of the canonical embedding maps η which take an ideal I onto the jansian topologizing filter $\eta(I)$. The map $\prod_P \eta_{R_P}$ is the canonical homomorphism induced by the family of homomorphisms $\{\eta_{R_P} : P \in \operatorname{Spec_m} R\}$.

3.6 An application of congruences on $\operatorname{Fil} R$

Recall that a commutative domain R is a *Prüfer domain* if and only if R_P is a valuation domain for all maximal ideals P of R. (The reader will find a brief introduction to valuation domains in the early pages of Chapter 4.)

Our main result in this section (Theorem 3.55) proves that a Prüfer domain R for which $\operatorname{Fil} R$ is commutative, is necessarily noetherian and thus a Dedekind domain. This result extends [34, Corollary 32, p. 102] which says that a valuation domain R for which $\operatorname{Fil} R$ is commutative, is noetherian and thus rank 1 discrete.

We start by extending results from Section 1 of Chapter 2. These result do not make the assumption that the ring R is commutative.

The following result follows by taking $A = 0$ in the hypothesis of Corollary 2.2, so that $\mathfrak{F} = \mathfrak{F}_{\mathsf{SHC}\{R/A\}} = \mathfrak{F}_{\mathsf{SHC}\{R\}} = 1 (1 = \eta(0))$.

Proposition 3.47 *The following statements are equivalent for a right topologizing filter $\mathfrak{G}$ on an arbitrary ring R:*

(a) The right residual $\mathfrak{G}^{-1}1$ of 1 by $\mathfrak{G}$ exists;

(b) The family of all $\mathcal{T}_{\mathfrak{G}}$-dense hereditary pretorsion submodules of R_R (that is the family of all hereditary pretorsion submodules of R_R that belong to $\mathfrak{G}$) has a smallest member.

Corollary 3.48 *Let R be an arbitrary ring for which $[\operatorname{Fil} R_R]^{\mathrm{du}}$ is two-sided residuated. Then for each right topologizing filter $\mathfrak{G}$ on R, the family of all hereditary pretorsion submodules of R_R that belong to $\mathfrak{G}$, has a smallest member.*

Corollary 3.49 *Let R be an arbitrary ring for which $[\operatorname{Fil} R_R]^{\mathrm{du}}$ is two-sided residuated. Then R_R satisfies the DCC on hereditary pretorsion submodules.*

Proof. Suppose $[\operatorname{Fil} R_R]^{\mathrm{du}}$ is two-sided residuated and that, contrary to the statement of the corollary, R_R admits a strictly descending chain of hereditary pretorsion submodules

$$U_1 \supset U_2 \supset \cdots .$$

Note that each U_n is an ideal of R. Define

$$\mathfrak{G} \overset{\text{def}}{=} \{K \leq R_R : K \supseteq U_n \text{ for some } n \in \mathbb{N}\}.$$

It is clear that $\mathfrak{G}$ is a bounded right topologizing filter on R. It follows from the previous corollary that the family comprising all hereditary pretorsion submodules of R_R that belong to $\mathfrak{G}$, has a smallest member, say K. Since each U_n is a hereditary pretorsion submodule of R_R belonging to $\mathfrak{G}$, it follows from the minimality of K that $U_n \supseteq K$ for all $n \in \mathbb{N}$. However, since $K \in \mathfrak{G}$, we must have $K \supseteq U_n$ for some $n \in \mathbb{N}$, an impossibility. ■

Recall that an ideal I of an arbitrary ring R is called a *right [resp. left] annihilator ideal* if $I = A^{-1}0 = \{r \in R : Ar = 0\}$ [resp. $I = 0A^{-1} = \{r \in R : rA = 0\}$] for some $A \in \operatorname{Id} R$.

Observe that every left annihilator ideal of R is a hereditary pretorsion submodule of R_R, indeed $0A^{-1} = \mathcal{T}_{\eta(A)}(R_R)$ for every $A \in \mathrm{Id}\, R$.

If R is an arbitrary ring, the maps $A \mapsto A^{-1}0$ and $A \mapsto 0A^{-1}$ represent a Galois connection between the sets of left annihilator ideals of R, and right annihilator ideals of R. Thus R will satisfy the DCC on left annihilator ideals of R, precisely if it satisfies the ACC on right annihilator ideals of R. We shall make use of this equivalence in the next result.

A theorem in [34, Theorem 19, p. 98] states that for an arbitrary ring R, if $\mathrm{Fil}\, R_R$ is commutative, then R satisfies the ACC on (right) annihilator ideals. As the following result shows, the theorem remains valid if the requirement that $\mathrm{Fil}\, R_R$ is commutative is weakened to $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ is two-sided residuated.

Theorem 3.50 *Let R be an arbitrary ring for which $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ is two-sided residuated. Then R satisfies the DCC on left annihilator ideals, and the ACC on right annihilator ideals.*

Proof. Since left annihilator ideals are hereditary pretorsion submodules of R_R, it follows from Corollary 3.49 that R satisfies the DCC on left annihilator ideals. This in turn, is equivalent to the ACC on right annihilator ideals as observed above. ∎

Remark 3.51 *If R is a semiprime ring, then the notions left annihilator ideal and right annihilator ideal coincide. This allows us to omit the prefixes left and right when referring to annihilator ideals in a semiprime ring.*

It is known that the following statements are equivalent for a semiprime ring R:

(a) R satisfies the ACC on annihilator ideals;

(b) R satisfies the DCC on annihilator ideals;

(c) R is a finite subdirect product of prime rings.

It is known that a commutative noetherian ring has finitely many minimal prime ideals [20, Corollary 3.14(a), p. 41]. Rings R for which $[\mathrm{Fil}\, R]^{\mathrm{du}}$ is two-sided residuated enjoy the same property as the next result shows.

Theorem 3.52 *If R is an arbitrary ring for which $[\mathrm{Fil}\, R]^{\mathrm{du}}$ is two-sided residuated, then R contains finitely many minimal prime ideals.*

Proof. Suppose that $[\mathrm{Fil}\, R]^{\mathrm{du}}$ is two-sided residuated. Recall that $\mathrm{rad}\, R$ denotes the prime radical of R (see the end of Section 1.2). Since the two-sided residuation property is passed to factor rings by Theorem 1.25, no generality is lost if we replace R by $R/\mathrm{rad}\,(R)$ and assume that R is semiprime.

It follows from Theorem 3.50 and the previous remark that R is a finite subdirect product of prime rings. Hence there are prime ideals $P_1, P_2, \cdots, P_n$ of R such that $\bigcap_{i=1}^{n} P_i = 0$. Let Q be a minimal prime ideal of R. Since $Q \supseteq \bigcap_{i=1}^{n} P_i$, the primeness of Q entails $Q \supseteq P_i$ for some $i \in \{1, 2, \cdots, n\}$. The minimality of Q implies $Q = P_i$. Thus every minimal prime ideal of R is a member of $\{P_i : 1 \leq i \leq n\}$. ■

Proposition 3.53 *Let R be a Prüfer domain for which $\mathrm{Fil}\, R$ is commutative. Then R_P is a (noetherian) rank 1 discrete valuation domain for every maximal ideal P of R.*

Proof. Take $P \in \mathrm{Spec_m}\, R$. It follows from Statement (a) of Theorem 3.37 (Preservation Theorem), that $\mathrm{Fil}\, R_P$ is commutative. Since R_P is a valuation domain for which $\mathrm{Fil}\, R_P$ is commutative, R_P must be noetherian, and thus rank 1 discrete, by [34, Corollary 32, p. 102]. ■

Proposition 3.54 *Let R be a Prüfer domain for which $\mathrm{Fil}\, R$ is commutative. Then every nonzero prime ideal of R is maximal.*

Proof. Suppose, contrary to the result, that R has nonzero prime ideal Q and maximal ideal P, such that $Q \subset P$. It follows from Proposition 3.27 that Q_P and P_P are prime ideals of R_P satisfying $0 \subset Q_P \subset R_P$. Thus R_P has rank greater than 1, and this contradicts Proposition 3.53. ■

We are now in a position to prove the main result of this section.

Theorem 3.55 *The following statements are equivalent for a Prüfer domain R:*

(a) R is noetherian and thus a Dedekind domain;

(b) $\operatorname{Fil} R$ is commutative.

Proof. (a)⇒(b) This is clear since $\operatorname{Fil} R$ is commutative in any commutative noetherian ring R by [34, Corollary 8, p. 91].

(b)⇒(a) To show that R is noetherian it suffices to show that R/I is noetherian for all nonzero ideals I of R (see for example [30, p. 172]). To this end, take I to be a nonzero ideal of R. Note that since all nonzero prime ideals of R are maximal by Proposition 3.54, no two nonzero prime ideals of R are comparable.

Let $\mathcal{P}$ be the set of all maximal ideals of R containing I. Take $P \in \mathcal{P}$ and consider P/I. We claim that P/I is a minimal prime ideal of the ring R/I. Certainly, P/I is a prime ideal of R/I. To establish minimality, suppose Q/I is a prime ideal of R/I contained in P/I with Q an ideal of R satisfying $I \subseteq Q \subseteq P$. Since Q/I is a prime ideal of R/I, Q is a prime ideal of R. However, as noted above, R contains no nonzero comparable prime ideals, hence $Q = P$ and so $Q/I = P/I$. This establishes our claim.

Since the commutativity of $\operatorname{Fil} R$ is passed from R to any factor ring of R by Theorem 1.25 we may infer from Theorem 3.52, that R/I has only finitely many minimal prime ideals. Thus $\{P/I : P \in \mathcal{P}\}$ is finite, whence $\mathcal{P}$ is finite.

Note that since $I \nsubseteq P$ for all $P \in (\operatorname{Spec_m} R) \setminus \mathcal{P}$, we have $I_P = R_P$ for all $P \in (\operatorname{Spec_m} R) \setminus \mathcal{P}$.

Consider the canonical embedding

$$\operatorname{Id} R \hookrightarrow \prod_{P \in \operatorname{Spec_m} R} \operatorname{Id} R_P$$
$$K \mapsto \{K_P\}_{P \in \operatorname{Spec_m} R}$$

Observe that the above embedding maps the interval $[I, R]$ in $\operatorname{Id} R$ into $\prod_{P \in \operatorname{Spec}_{\mathrm{m}} R} [I_P, R_P] \subseteq \prod_{P \in \operatorname{Spec}_{\mathrm{m}} R} \operatorname{Id} R_P$. Since $[I_P, R_P]$ is a singleton for all $P \in (\operatorname{Spec}_{\mathrm{m}} R) \setminus \mathcal{P}$, we have

$$\prod_{P \in \operatorname{Spec}_{\mathrm{m}} R} [I_P, R_P] \cong \prod_{P \in \mathcal{P}} [I_P, R_P].$$

Since I_P is a nonzero ideal of the rank 1 discrete valuation domain R_P, the interval $[I_P, R_P]$ is finite for all $P \in \mathcal{P}$. It follows that $\prod_{P \in \mathcal{P}} [I_P, R_P]$ is finite, so $[I, R]$ is finite. This clearly implies that the ring R/I is noetherian, as required. ■

Chapter 4

Valuation domains

Our main theorem in this section asserts that a valuation domain R enjoys two-sided residuation precisely if R is rank one discrete (this is equivalent to R having value group $\mathbb{Z}$ or that R is noetherian [11, Proposition 1.7(b), p. 3]). This result thus provides us with a host of examples of (commutative) rings which satisfy left-sided residuation only. For illustrative purposes we shall provide a detailed analysis of the structure of $\operatorname{Fil} R_R$ in the case of two such examples, these being the valuation domain R with value group $\mathbb{R}$ and the valuation domain R with value group $\mathbb{Z} \times \mathbb{Z}$.

We start with a brief summary of those aspects of rudimentary valuation theory that have a direct bearing on what follows. For further background, we refer the reader to the texts [10, 11].

Recall that a *linearly ordered abelian group* is a structure $\langle G, +, \leq \rangle$ where $\langle G, + \rangle$ is an (additive) abelian group and $\langle G, \leq \rangle$ a linearly ordered poset (i.e., a chain) satisfying:

$$g_1 \leq g_2 \text{ and } h_1 \leq h_2 \text{ imply } g_1 + h_1 \leq g_2 + h_2 \text{ for all } g_1, g_2, h_1, h_2 \in G.$$

The subset $G^+ \stackrel{\text{def}}{=} \{g \in G \,:\, g \geq 0_G\}$ is called the *positive cone* of G.

A *convex subgroup* of a linearly ordered abelian group G is a subgroup H of G with the property that whenever $h_1, h_2 \in H$ and $h_1 \leq t \leq h_2$, then $t \in H$. We say that G is *rank* n if n is the length of a longest chain of nonzero convex subgroups of G. It is known that G has rank 1 if and only if G is order isomorphic to an ordered subgroup of the additive group of reals $\mathbb{R}$.

If G is a rank n linearly ordered abelian group, we say G is *discrete* if G is order isomorphic to $\mathbb{Z}^n$ ordered lexicographically.

If G is a linearly ordered abelian group, we adjoin to G the symbol ∞, to be regarded as larger than every element of G, and set $g + \infty = \infty + g = \infty$ for all $g \in G$.

Let F be a field. A *valuation* on F is a map $v : F \to G \cup \{\infty\}$ (no generality is lost if we assume that v is onto) such that for all $a, b \in F$ the following conditions hold:

(V1) $v(a) = \infty$ if and only if $a = 0$;

(V2) $v(ab) = v(a) + v(b)$;

(V3) $v(a + b) \geq \min\{v(a), v(b)\}$.

In the above situation, the subring $R_v \stackrel{\text{def}}{=} \{a \in F : v(a) \geq 0_G\}$ of F is called the *valuation domain associated with* v and G the *value group* of R_v. Note that v restricts to a map from R_v onto $G^+ \cup \{\infty\}$.

Recall that a *(lattice) filter* on a lattice L is a nonempty subset X of L that is upward (meaning, if $x \leq y$ and $x \in X$, then $y \in X$) and closed under finite meets (meaning, if $x, y \in X$, then $x \wedge y \in X$). Observe that if the lattice L is linearly ordered, then the requirement that X be closed under finite meets is redundant.

If G is a linearly ordered abelian group, we shall denote by $\mathrm{Fil}\,(G^+ \cup \{\infty\})$ the set of all filters (or equivalently, upward subsets) on $G^+ \cup \{\infty\}$.

Let R be a valuation domain with value group G. There is a canonical correspondence between the ideals of R and the filters on $G^+ \cup \{\infty\}$ which we now describe. Define a map $v[\_] : \mathrm{Id}\, R \to \mathrm{Fil}\,(G^+ \cup \{\infty\})$ by

$$v[I] \stackrel{\text{def}}{=} \{v(a) : a \in I\} \quad (I \in \mathrm{Id}\, R),$$

and a map $v^{-1}[\_] : \mathrm{Fil}\,(G^+ \cup \{\infty\}) \to \mathrm{Id}\, R$ by

$$v^{-1}[X] \stackrel{\text{def}}{=} \{a \in R \,:\, v(a) \in X\} \quad (X \in \mathrm{Fil}\,(G^+ \cup \{\infty\})).$$

The maps $v[\_]$ and $v^{-1}[\_]$ are known ([11, Proposition 3.2, p. 11]) to constitute a pair of mutually inverse order preserving maps.

Since G^+ is linearly ordered, it follows that $\mathrm{Fil}\,(G^+ \cup \{\infty\})$ and thus $\mathrm{Id}\, R$ are linearly ordered by inclusion. On the other hand, if R is any commutative integral domain whose ideal lattice is linearly ordered by inclusion, then it is possible to construct a linearly ordered abelian group G and valuation map $v : Q \to G \cup \{\infty\}$ where Q denotes the field of fractions of R, such that $R = R_v$ (see [11, p. 11]).

If R is a valuation domain with value group G, we say that R has *rank* n [resp. is *rank* n *discrete*] if G has rank n [resp. is rank n discrete].

[34, Corollary 32, p. 102] asserts that if R is a valuation domain, then $\mathrm{Fil}\, R$ is commutative if and only if R is rank 1 discrete. Since the commutativity of $\mathrm{Fil}\, R$ and the two-sided residuation property coincide in commutative rings by Theorem 3.7, we obtain the following extension of [34, Corollary 32, p. 102].

Theorem 4.1 *The following statements are equivalent for a valuation domain R:*

(a) $\mathrm{Fil}\, R$ *is commutative;*

(b) $[\mathrm{Fil}\, R]^{\mathrm{du}}$ *is two-sided residuated;*

(c) R *is noetherian, i.e.,* R *is rank* 1 *discrete.*

The following result of A. M. Viola-Prioli and J. E. Viola-Prioli [38, Lemmas 5 and 6, p. 24] shows that in a ring whose right ideals are linearly ordered by inclusion, the right topologizing filters have a conspicuous form.

Theorem 4.2 *Let R be a ring whose right ideals are linearly ordered by inclusion and I an ideal of R. Then:*

$$\eta(I) = \{K \leq R_R : K \supseteq I\}, \text{ and}$$

$$\hat{\eta}(I) \stackrel{\text{def}}{=} \{K \leq R_R : K \supset I\}$$

are right topologizing filters on R. Moreover, every $\mathfrak{F} \in \operatorname{Fil} R_R$ has the form $\mathfrak{F} = \eta(I)$ or $\mathfrak{F} = \hat{\eta}(I)$ for some $I \in \operatorname{Id} R$.

Proposition 4.3 *Let R be a valuation domain. If $A = aR$ is a nonzero principal ideal of R, then every submodule of R/A is a hereditary pretorsion submodule.*

Proof. Take $0 \neq a \in R$ and put $A = aR$. Choose $B \leq R_R$ with $B \supseteq A$. We must show that B/A is a hereditary pretorsion submodule of R/A.

Since R is a valuation domain, it is easily seen that the family $\mathfrak{F} = \{K \leq R_R : K \supseteq b^{-1}aR$ for some nonzero $b \in B\}$* is a member of $\operatorname{Fil} R$. We shall demonstrate that $\mathcal{T}_{\mathfrak{F}}(R/A) = B/A$. To this end, note that for every $b \in B$,

$$b^{-1}A = \{r \in R : br \in A\} = \begin{cases} R & \text{if } b \in A, \\ b^{-1}aR \subset R & \text{if } b \notin A. \end{cases}$$

In both of the above cases we see that $b^{-1}A \in \mathfrak{F}$, so $b + A \in \mathcal{T}_{\mathfrak{F}}(R/A)$. This shows that $B/A \subseteq \mathcal{T}_{\mathfrak{F}}(R/A)$.

To establish the reverse containment, suppose $0 \neq c + A \in \mathcal{T}_{\mathfrak{F}}(R/A)$. Then $c^{-1}A = \{r \in R : cr \in A\} = c^{-1}aR \in \mathfrak{F}$. It follows from the definition of $\mathfrak{F}$ that $c^{-1}aR \supseteq b^{-1}aR$ for some nonzero $b \in B$, whence $baR \supseteq caR$, so $bR \supseteq cR$ (because $a \neq 0$), i.e., $c \in bR \subseteq B$. We conclude that $\mathcal{T}_{\mathfrak{F}}(R/A) \subseteq B/A$, whence equality. ■

*The symbol b^{-1} as it appears here in the definition of $\mathfrak{F}$, is intended to denote the inverse of b in the field of quotients Q of R. If $bR \supseteq aR$, then the element $b^{-1}a$ belongs to R in which case $b^{-1}aR$ corresponds with a principal ideal of R. If, on the other hand, $bR \subset aR$, then $b^{-1}aR$ is a cyclic R-submodule of Q_R properly containing R.

Remark 4.4 *The requirement in Proposition 4.3 that $A = aR$ be a nonzero principal ideal cannot be relaxed to an arbitrary nonzero ideal. Indeed, let R be a valuation domain of rank at least two (an example of such is a valuation domain with value group $\mathbb{Z} \times \mathbb{Z}$ ordered lexicographically) and P any nonzero non-maximal prime ideal of R. (If the value group of R is $G = \mathbb{Z} \times \mathbb{Z}$ and v is the corresponding valuation map, then an ideal meeting these specifications is given by $P = v^{-1}[X]$ where $X \in \mathrm{Fil}\,(G^+ \cup \{\infty\})$ is given by $X = \{(m, n) \in G^+ : m \geq 1\} \cup \{\infty\}$.) Since R/P is not a field, submodules of $(R/P)_R$ are plentiful. If, however, $\mathfrak{F} \in \mathrm{Fil}\,R$ and $0 \neq r + P \in \mathcal{T}_{\mathfrak{F}}(R/P)$, then $r^{-1}P \in \mathfrak{F}$. The primeness of P means that $P \in \mathfrak{F}$, whence $R/P \in \mathcal{T}_{\mathfrak{F}}$, i.e., $\mathcal{T}_{\mathfrak{F}}(R/P) = R/P$. Thus $(R/P)_R$ contains no proper nonzero hereditary pretorsion submodule.*

Proposition 4.5 *Let R be a valuation domain, I a nonzero proper ideal of R, and P a nonzero principal ideal of R contained in I. Put:*

$$\mathfrak{F} = \eta(P) = \{K \leq R_R \,:\, K \supseteq P\}, \text{ and}$$

$$\mathfrak{G} = \hat{\eta}(I) = \{K \leq R_R \,:\, K \supset I\}.$$

If the right residual $\mathfrak{G}^{-1}\mathfrak{F}$ exists, then I has a successor in the lattice of ideals of R.

Proof. Put $X = \{B \in \mathfrak{G} \,:\, B \supseteq P$ and B/P is a hereditary pretorsion submodule of $R/P\}$. Suppose the right residual $\mathfrak{G}^{-1}\mathfrak{F}$ exists. Since $\mathfrak{F} = \eta(P)$ is compact with R/P a cyclic subgenerator for $\mathcal{T}_{\mathfrak{F}}$, it follows from Corollary 2.2 ((a)⇒(c)) that X has a smallest member.

Observe that the requirement $B \supseteq P$ in the definition of X is redundant since $B \in \mathfrak{G}$ implies $B \supset I \supseteq P$. The requirement too that B/P be a hereditary pretorsion submodule of R/P is also redundant in the light of Proposition 4.3. We conclude that $X = \mathfrak{G}$. Thus $\mathfrak{G} = \hat{\eta}(I)$ has a smallest member, that is to say, I has a successor in $\mathrm{Id}\,R$. ∎

4.1 Illustrative examples

Example 4.6 *Valuation domain R with value group $\mathbb{Z} \times \mathbb{Z}$*

In this example, R shall denote a valuation domain with value group $\mathbb{Z}\times\mathbb{Z}$ ordered lexicographically, and valuation map $v : R \to (\mathbb{Z}\times\mathbb{Z})^+ \cup \{\infty\}$. Such a ring R is not noetherian, so Theorem 4.1 tells us that the monoid operation : on $\mathrm{Fil}\, R$ must be noncommutative and that $[\mathrm{Fil}\, R]^{\mathrm{du}}$ must fail to be right residuated.

It is easily checked that a nonzero ideal I of R is principal if and only if it has the form

$$I = I_{(m,n)} \stackrel{\text{def}}{=} \{r \in R \,:\, v(r) \geq (m,n) \text{ for some } (m,n) \in (\mathbb{Z}\times\mathbb{Z})^+\} \tag{4.1}$$

and is non-principal (that is to say, infinitely generated) if and only if it has the form

$$I = I_{(m,\infty)} \stackrel{\text{def}}{=} \bigcup_{n\in\mathbb{Z}} I_{(m,n)} \text{ for some } m \geq 1. \tag{4.2}$$

The multiplicative structure of $\mathrm{Id}\, R$ is captured in the following table.

Figure 4.1: Multiplication in $\mathrm{Id}\, R$

$\cdot$	$I_{(m,n)}$	$I_{(m,\infty)}$
$I_{(m',n')}$	$I_{(m'+m,n'+n)}$	$I_{(m'+m,\infty)}$
$I_{(m',\infty)}$	$I_{(m'+m,\infty)}$	$I_{(m'+m,\infty)}$

Adopting the notation of Proposition 4.5, it follows from this theorem that every proper member of $\mathrm{Fil}\, R$ corresponds with precisely one of the following three types[†]:

Type 1 : $\eta(I_{(m,n)}),\ (m,n) \in (\mathbb{Z}\times\mathbb{Z})^+$;

Type 2 : $\eta(I_{(m,\infty)}),\ m \geq 1$;

Type 3 : $\hat{\eta}(I_{(m,\infty)}),\ m \geq 1$.

[†]We shall include in Type 3 the topologizing filter $\hat{\eta}(0)$ comprising all nonzero ideals of R. Multiplication by $\hat{\eta}(0)$ is somewhat featureless since $\mathfrak{F} : \hat{\eta}(0) = \hat{\eta}(0) : \mathfrak{F} = \hat{\eta}(0)$ for all proper $\mathfrak{F} \in \mathrm{Fil}\, R$.

Observe that members of $\operatorname{Fil} R$ that have the form $\hat{\eta}(I_{(m,n)})$ do not correspond with an additional type for it is easily shown that $\hat{\eta}(I_{(m,n)}) = \eta(I_{(m,n-1)})$. Note also that

$$\eta(I_{(m,n)}) \supset \bigcap_{i \in \mathbb{Z}} \eta(I_{(m,i)}) = \eta(I_{(m,\infty)}) \supset \hat{\eta}(I_{(m,\infty)})$$

for all $m \geq 1$ and $n \in \mathbb{Z}$, and that $\eta(I_{(m,\infty)})$ is the successor of $\hat{\eta}(I_{(m,\infty)})$ in the lattice $\operatorname{Fil} R$.

What follows is an investigation of the multiplicative structure of $\operatorname{Fil} R$, the culmination of which is the table in Figure 4.2.

Figure 4.2: Multiplication in $\operatorname{Fil} R$

$:$	$\eta(I_{(m,n)})$	$\eta(I_{(m,\infty)})$	$\hat{\eta}(I_{(m,\infty)})$
$\eta(I_{(m',n')})$	$\eta(I_{(m'+m,n'+n)})$	$\eta(I_{(m'+m,\infty)})$	$\hat{\eta}(I_{(m'+m,\infty)})$
$\eta(I_{(m',\infty)})$	$\eta(I_{(m'+m,\infty)})$	$\eta(I_{(m'+m,\infty)})$	$\hat{\eta}(I_{(m'+m,\infty)})$
$\hat{\eta}(I_{(m',\infty)})$	$\hat{\eta}(I_{(m'+m,\infty)})$	$\eta(I_{(m'+m-1,\infty)})$	$\hat{\eta}(I_{(m'+m-1,\infty)})$

Observe that those entries lying in both the first two rows and two columns of the above table correspond with $\operatorname{Id} R$ seen as a substructure of $\operatorname{Fil} R$ via the embedding η (see Theorem 1.14).

Note that topologizing filters on R of Types 1 and 3 possess cofinal sets of principal ideals and thus commute in the light of Lemma 3.2(a).

We now compute the entries in the third row and column of the table in Figure 4.2.

Since $\{I_{(m'-1,i)} \ : \ i \in \mathbb{Z}\}$ is a cofinal set for $\hat{\eta}(I_{(m',\infty)})$, it follows from Lemma 3.1 that the set $\{I_{(m'-1,i)} I_{(m,n)} \ : \ i \in \mathbb{Z}\} = \{I_{(m'+m-1,i+n)} \ : \ i \in \mathbb{Z}\} = \{I_{(m'+m-1,i)} \ : \ i \in \mathbb{Z}\}$ is cofinal for $\hat{\eta}(I_{(m',\infty)}) : \eta(I_{(m,n)})$, whence

$$\hat{\eta}(I_{(m',\infty)}) : \eta(I_{(m,n)}) = \hat{\eta}(I_{(m'+m,\infty)}). \tag{4.3}$$

Interchanging the roles of (m', n') and (m, n) in (4.3) and using the fact that $\hat{\eta}(I_{(m,\infty)})$ and $\eta(I_{(m',n')})$ commute by Lemma 3.2 ((a) or (b)), we may infer that

$$\eta(I_{(m',n')}) : \hat{\eta}(I_{(m,\infty)}) = \hat{\eta}(I_{(m'+m,\infty)}). \tag{4.4}$$

A similar application of Lemma 3.1 yields

$$\hat{\eta}(I_{(m',\infty)}) : \eta(I_{(m,\infty)}) = \eta(I_{(m'+m-1,\infty)}) \tag{4.5}$$

and

$$\hat{\eta}(I_{(m',\infty)}) : \hat{\eta}(I_{(m,\infty)}) = \hat{\eta}(I_{(m'+m-1,\infty)}). \tag{4.6}$$

It remains to compute $\eta(I_{(m',\infty)}) : \hat{\eta}(I_{(m,\infty)})$. We point out that Lemma 3.1 is inapplicable here since $\eta(I_{(m',\infty)})$ has no cofinal set of finitely generated ideals.

Since the operation : is order preserving, it follows that

$$\begin{aligned} \eta(I_{(m',\infty)}) : \hat{\eta}(I_{(m,\infty)}) \ &\subseteq \ \eta(I_{(m',n')}) : \hat{\eta}(I_{(m,\infty)}) \\ &= \ \hat{\eta}(I_{(m'+m,\infty)}) \ \text{[by (4.4)]}. \end{aligned}$$

We now establish the reverse containment. Note that

$$\begin{aligned} K \in \eta(I_{(m',\infty)}) : \hat{\eta}(I_{(m,\infty)}) \ &\Leftrightarrow \ \exists H \in \eta(I_{(m',\infty)}) \text{ such that } H \supseteq K \text{ and } \forall a \in H,\ a^{-1}K \in \hat{\eta}(I_{(m,\infty)}) \\ &\Leftrightarrow \ \exists H \leq R_R \text{ such that } H \supseteq I_{(m',\infty)} + K \text{ and } \forall a \in H,\ a^{-1}K \in \hat{\eta}(I_{(m,\infty)}) \\ &\Leftrightarrow \ \forall a \in I_{(m',\infty)},\ a^{-1}K \in \hat{\eta}(I_{(m,\infty)}) \\ &\Leftrightarrow \ \forall a \in I_{(m',\infty)},\ \exists K_a \in \hat{\eta}(I_{(m,\infty)}) \text{ such that } aK_a \subseteq K. \end{aligned}$$

Consider now the principal ideal $K = I_{(m'+m-1,i)}$. Let a be an arbitrary element in $I_{(m',\infty)}$. Then $aR \subseteq I_{(m',n')}$ for some $n' \in \mathbb{Z}$. Putting $K_a = I_{(m-1,i-n')}$, we see that $K_a \in \hat{\eta}(I_{(m,\infty)})$ and $aK_a = aRK_a \subseteq I_{(m',n')}I_{(m-1,i-n')} = I_{(m'+m-1,i)} = K$. We conclude that $K \in \eta(I_{(m',\infty)}) : \hat{\eta}(I_{(m,\infty)})$. Thus $\{I_{(m'+m-1,i)} : i \in \mathbb{Z}\} \subseteq \eta(I_{(m',\infty)}) : \hat{\eta}(I_{(m,\infty)})$. Since $\{I_{(m'+m-1,i)} : i \in \mathbb{Z}\}$ is a cofinal set for $\hat{\eta}(I_{(m'+m,\infty)})$, the above shows that $\hat{\eta}(I_{(m'+m,\infty)}) \subseteq \eta(I_{(m',\infty)}) : \hat{\eta}(I_{(m,\infty)})$, whence the equality

$$\eta(I_{(m',\infty)}) : \hat{\eta}(I_{(m,\infty)}) = \hat{\eta}(I_{(m'+m,\infty)}). \tag{4.7}$$

Equations (4.3) - (4.7) complete the table in Figure 4.2.

The next two tables capture the residuated structure of $[\mathrm{Fil}\, R]^{\mathrm{du}}$. We again point out that whereas the left residual $\mathfrak{F}\mathfrak{G}^{-1}$ exists for all $\mathfrak{F}, \mathfrak{G} \in \mathrm{Fil}\, R$, the right residual $\mathfrak{G}^{-1}\mathfrak{F}$ must fail to exist for certain choices of $\mathfrak{F}$ and $\mathfrak{G}$ in the light of Theorem 4.1.

In what follows, little generality is lost if we assume $m > m'$. We leave all but two computations to the reader.

Figure 4.3: Left residuals in $[\mathrm{Fil}\, R]^{\mathrm{du}}$

$\mathfrak{F}\mathfrak{G}^{-1}$	$\eta(I_{(m,n)})$	$\eta(I_{(m,\infty)})$	$\hat{\eta}(I_{(m,\infty)})$ $\}\ \mathfrak{F}$
$\eta(I_{(m',n')})$	$\eta(I_{(m-m',n-n')})$	$\eta(I_{(m-m',\infty)})$	$\hat{\eta}(I_{(m-m',\infty)})$
$\eta(I_{(m',\infty)})$	$\eta(I_{(m-m'+1,\infty)})$	$\eta(I_{(m-m',\infty)})$	$\eta(I_{(m-m',\infty)})$
$\hat{\eta}(I_{(m',\infty)})$	$\eta(I_{(m-m'+1,\infty)})$	$\eta(I_{(m-m'+1,\infty)})$	$\eta(I_{(m-m',\infty)})$
$\underbrace{\qquad}_{\mathfrak{G}}$			

Taking $I = I_{(m',\infty)}$ and $P = I_{(m,n)}$ in Proposition 4.5 and noting that the ideal $I_{(m',\infty)}$ has no successor in $\mathrm{Id}\, R$, we see that the right residual $\hat{\eta}(I_{(m',\infty)})^{-1}\eta(I_{(m,n)})$ cannot exist.

Figure 4.4: Right residuals in $[\mathrm{Fil}\, R]^{\mathrm{du}}$

$\mathfrak{G}^{-1}\mathfrak{F}$	$\eta(I_{(m,n)})$	$\eta(I_{(m,\infty)})$	$\hat{\eta}(I_{(m,\infty)})$	$\Big\}\ \mathfrak{F}$
$\eta(I_{(m',n')})$	$\eta(I_{(m-m',n-n')})$	$\eta(I_{(m-m',\infty)})$	$\hat{\eta}(I_{(m-m',\infty)})$	
$\eta(I_{(m',\infty)})$	$\hat{\eta}(I_{(m-m'+1,\infty)})$	$\eta(I_{(m-m',\infty)})$	$\hat{\eta}(I_{(m-m',\infty)})$	
$\hat{\eta}(I_{(m',\infty)})$	does not exist	$\eta(I_{(m-m'+1,\infty)})$	does not exist	
$\underbrace{\qquad\qquad}_{\mathfrak{G}}$				

The right residual $\hat{\eta}(I_{(m',\infty)})^{-1}\hat{\eta}(I_{(m,\infty)})$ also fails to exist for $\hat{\eta}(I_{(m',\infty)}) : \eta(I_{(m-m',i)}) = \hat{\eta}(I_{(m,\infty)})$ for all $i \in \mathbb{Z}$, yet

$$\begin{aligned}
\hat{\eta}(I_{(m',\infty)}) : \left[\bigcap_{i\in\mathbb{Z}} \eta(I_{(m-m',i)})\right] &= \hat{\eta}(I_{(m',\infty)}) : \eta(I_{(m-m',\infty)}) \\
&= \eta(I_{(m-1,\infty)}) \\
&\not\supseteq \hat{\eta}(I_{(m,\infty)}).
\end{aligned}$$

Example 4.7 *Valuation domain with value group* $\mathbb{R}$

In this example R shall denote a valuation domain with value group $\mathbb{R}$ ordered in the usual fashion. Such a ring R is not noetherian, so Theorem 4.1 tells us that the monoid operation on $\mathrm{Fil}\, R$ must be noncommutative and that $[\mathrm{Fil}\, R]^{\mathrm{du}}$ must fail to be right residuated.

There are only two classes of nonzero ideals in R (see [10, Example 4.1, p. 68]) which we describe below.

The nonzero principal ideals of R have the form:

$$I_{[r,\infty)} \stackrel{\mathrm{def}}{=} \{x \in R : v(x) \geq r\} \text{ for some } r \in \mathbb{R}^{+}, \text{ while}$$

the non-principal ideals have the form:

$$I_{(r,\infty)} \stackrel{\text{def}}{=} \{x \in R : v(x) > r\} \text{ for some } r \in \mathbb{R}^+.$$

The multiplicative structure of $\operatorname{Id} R$ is captured in Figure 4.5.

Figure 4.5: Multiplication in $\operatorname{Id} R$

$\cdot$	$I_{(r,\infty)}$	$I_{[r,\infty)}$
$I_{(s,\infty)}$	$I_{(r+s,\infty)}$	$I_{(r+s,\infty)}$
$I_{[s,\infty)}$	$I_{(r+s,\infty)}$	$I_{[r+s,\infty)}$

Adopting the notation of Theorem 4.2, it follows from this theorem that every proper member of $\operatorname{Fil} R$ corresponds with precisely one of the following types (in each of the descriptions below, $r \in \mathbb{R}^+$):

$$\text{Type 1} : \eta(I_{(r,\infty)});$$

$$\text{Type 2} : \eta(I_{[r,\infty)});$$

$$\text{Type 3} : \hat{\eta}(I_{[r,\infty)}).$$

As was the case with Example 4.6, we note that members of $\operatorname{Fil} R$ that have the form $\hat{\eta}(I_{(r,\infty)})$ do not correspond with an additional type for it is easily shown that $\hat{\eta}(I_{(r,\infty)}) = \eta(I_{[r,\infty)})$. Note also that

$$\eta(I_{(r,\infty)}) \supset \eta(I_{[r,\infty)}) \supset \hat{\eta}(I_{[r,\infty)}).$$

Using procedures similar to those used in Example 4.6, we compute each of the entries in the multiplication table shown in Figure 4.6. We again point out that those entries lying in both the first two rows and two columns of the table correspond with $\operatorname{Id} R$ seen as a substructure of $\operatorname{Fil} R$

Figure 4.6: Multiplication in $\mathrm{Fil}\, R$

$:$	$\eta(I_{(r,\infty)})$	$\eta(I_{[r,\infty)})$	$\hat{\eta}(I_{[r,\infty)})$
$\eta(I_{(s,\infty)})$	$\eta(I_{(r+s,\infty)})$	$\eta(I_{(r+s,\infty)})$	$\eta(I_{(r+s,\infty)})$
$\eta(I_{[s,\infty)})$	$\eta(I_{(r+s,\infty)})$	$\eta(I_{[r+s,\infty)})$	$\hat{\eta}(I_{[r+s,\infty)})$
$\hat{\eta}(I_{[s,\infty)})$	$\hat{\eta}(I_{[r+s,\infty)})$	$\hat{\eta}(I_{[r+s,\infty)})$	$\hat{\eta}(I_{[r+s-\epsilon,\infty)})$

via the embedding η. Note that topologizing filters on R of Types 1 and 3 possess cofinal sets of principal ideals and thus commute in the light of Lemma 3.2(a).

Since the set $\{I_{[s-\epsilon,\infty)} : \epsilon > 0\}$ is a cofinal set of principal (and thus finitely generated) ideals for $\hat{\eta}(I_{[s,\infty)})$, it follows from Lemma 3.1 that $\{I_{[s-\epsilon,\infty)} I_{(r,\infty)} : \epsilon > 0\} = \{I_{(r+s-\epsilon,\infty)} : \epsilon > 0\}$ is a cofinal set for $\hat{\eta}(I_{[r+s,\infty)})$, whence

$$\hat{\eta}(I_{[s,\infty)}) : \eta(I_{(r,\infty)}) = \hat{\eta}(I_{[r+s,\infty)}). \tag{4.8}$$

A similar application of Lemma 3.1 yields the next three equations:

$$\eta(I_{[s,\infty)}) : \hat{\eta}(I_{[r,\infty)}) = \hat{\eta}(I_{[r+s,\infty)}). \tag{4.9}$$

$$\hat{\eta}(I_{[s,\infty)}) : \eta(I_{[r,\infty)}) = \hat{\eta}(I_{[r+s,\infty)}). \tag{4.10}$$

$$\hat{\eta}(I_{[s,\infty)}) : \hat{\eta}(I_{[r,\infty)}) = \hat{\eta}(I_{[r+s,\infty)}). \tag{4.11}$$

To compute $\eta(I_{(s,\infty)}) : \hat{\eta}(I_{[r,\infty)})$, we note first that Lemma 3.1 is not applicable since $\eta(I_{(s,\infty)})$ has no cofinal set of finitely generated ideals. We claim that $\eta(I_{(s,\infty)}) : \hat{\eta}(I_{[r,\infty)}) = \eta(I_{(r+s,\infty)})$. Applying the definition of the monoid operation :, we see that

$$\begin{aligned}
K \in \eta(I_{(s,\infty)}) : \hat{\eta}(I_{[r,\infty)}) &\Leftrightarrow \exists H \in \eta(I_{(s,\infty)}) \text{ such that } H \supseteq K \text{ and } \forall a \in H,\ a^{-1}K \in \hat{\eta}(I_{[r,\infty)}) \\
&\Leftrightarrow \exists H \leq R_R \text{ such that } H \supseteq I_{(s,\infty)} + K \text{ and } \forall a \in H,\ a^{-1}K \in \hat{\eta}(I_{[r,\infty)}) \\
&\Leftrightarrow \forall a \in I_{(s,\infty)},\ a^{-1}K \in \hat{\eta}(I_{[r,\infty)}) \\
&\Leftrightarrow \forall a \in I_{(s,\infty)},\ \exists K_a \in \hat{\eta}(I_{[r,\infty)}) \text{ (i.e., } K_a \supset I_{[r,\infty)}) \text{ such that } aK_a \subseteq K.
\end{aligned}$$

Take arbitrary $a \in I_{(s,\infty)}$ and put $v(a) = s'$ so that $aR = I_{[s',\infty)}$. Note that since $a \in I_{(s,\infty)}$, we must have $s' > s$. Put $\epsilon = s' - s > 0$ and consider the ideal $K_a \stackrel{\text{def}}{=} I_{[r-\frac{\epsilon}{2},\infty)} \supset I_{[r,\infty)}$. Then

$$aK_a = I_{[s',\infty)}I_{[r-\frac{\epsilon}{2},\infty)} = I_{[s'+r-\frac{\epsilon}{2},\infty)} = I_{[s+\epsilon+r-\frac{\epsilon}{2},\infty)} = I_{[s+r+\frac{\epsilon}{2},\infty)} \subseteq I_{(r+s,\infty)}.$$

This shows that $I_{(r+s,\infty)} \in \eta(I_{(s,\infty)}) : \hat{\eta}(I_{[r,\infty)})$, whence $\eta(I_{(r+s,\infty)}) \subseteq \eta(I_{(s,\infty)}) : \hat{\eta}(I_{[r,\infty)})$.

To establish equality, suppose ideal K is such that $K \notin \eta(I_{(r+s,\infty)})$. This means that $K \not\supseteq I_{(r+s,\infty)}$ and so $K \subset I_{(r+s,\infty)}$. This entails $K \subseteq I_{[r+s+\epsilon,\infty)}$ for some $\epsilon > 0$. Now choose $a \in R$ such that $v(a) = s + \frac{\epsilon}{2}$. Note that $aR = I_{[s+\frac{\epsilon}{2},\infty)}$. It is clear that there can exist no ideal K_a that meets both the requirements $K_a \supset I_{[r,\infty)}$ and $aK_a \subseteq I_{(r+s+\epsilon,\infty)}$. It follows from the above that $K \notin \eta(I_{(s,\infty)}) : \hat{\eta}(I_{[r,\infty)})$. We have thus shown that:

$$\eta(I_{[s,\infty)}) : \hat{\eta}(I_{[r,\infty)}) = \eta(I_{(r+s,\infty)}). \tag{4.12}$$

Equations (4.8) - (4.12) complete the multiplication table for $\operatorname{Fil} R$ shown in Figure 4.6.

The final two tables capture the residuated structure of $[\operatorname{Fil} R]^{\text{du}}$. We leave the calculation of all left residuals (see Figure 4.7) to the reader and provide details only in the single instance where right residuation fails (see Figure 4.8). Little generality is lost if we again make the assumption that $r > s$.

Consider the residual $\hat{\eta}(I_{[s,\infty)})^{-1}\eta(I_{(r,\infty)})$. For this residual to exist, we require a smallest $\mathfrak{H}$ in $\operatorname{Fil} R$ such that $\hat{\eta}(I_{[s,\infty)}) : \mathfrak{H} \supseteq \eta(I_{(r,\infty)})$. Note that for each $\epsilon > 0$, if $\mathfrak{H}$ is chosen to be $\hat{\eta}(I_{[r-s+\epsilon,\infty)})$, then we see that by (4.11), $\hat{\eta}(I_{[s,\infty)}) : \mathfrak{H} = \hat{\eta}(I_{[r+\epsilon,\infty)}) \supseteq \eta(I_{(r,\infty)})$. However, if $\mathfrak{H}$ is chosen to be $\bigcap_{\epsilon\in\mathbb{R}^+} \hat{\eta}(I_{[r-s+\epsilon,\infty)}) = \eta(I_{(r-s,\infty)})$, then by (4.8), $\hat{\eta}(I_{[s,\infty)}) : \mathfrak{H} = \hat{\eta}(I_{[r,\infty)}) \not\supseteq \eta(I_{[r,\infty)})$. We conclude that the right residual $\hat{\eta}(I_{[s,\infty)})^{-1}\eta(I_{(r,\infty)})$ does not exist.

Figure 4.7: Left residuals in $[\mathrm{Fil}\, R]^{\mathrm{du}}$

$\mathfrak{F}\mathfrak{G}^{-1}$	$\eta(I_{(r,\infty)})$	$\eta(I_{[r,\infty)})$	$\hat{\eta}(I_{[r,\infty)})$
$\eta(I_{(s,\infty)})$	$\eta(I_{(r-s,\infty)})$	$\eta(I_{(r-s,\infty)})$	$\hat{\eta}(I_{[r-s,\infty)})$
$\eta(I_{[s,\infty)})$	$\eta(I_{[r-s,\infty)})$	$\eta(I_{[r-s,\infty)})$	$\hat{\eta}(I_{[r-s,\infty)})$
$\hat{\eta}(I_{[s,\infty)})$	$\hat{\eta}(I_{[r-s,\infty)})$	$\hat{\eta}(I_{[r-s,\infty)})$	$\hat{\eta}(I_{[r-s,\infty)})$

$\}\ \mathfrak{F}$

$\underbrace{\qquad}_{\mathfrak{G}}$

Figure 4.8: Right residuals in $[\mathrm{Fil}\, R]^{\mathrm{du}}$

$\mathfrak{G}^{-1}\mathfrak{F}$	$\eta(I_{(r,\infty)})$	$\eta(I_{[r,\infty)})$	$\hat{\eta}(I_{[r,\infty)})$
$\eta(I_{(s,\infty)})$	$\eta(I_{(r-s,\infty)})$	$\eta(I_{(r-s,\infty)})$	$\hat{\eta}(I_{[r-s,\infty)})$
$\eta(I_{[s,\infty)})$	$\eta(I_{[r-s,\infty)})$	$\eta(I_{[r-s,\infty)})$	$\hat{\eta}(I_{[r-s,\infty)})$
$\hat{\eta}(I_{[s,\infty)})$	does not exist	$\hat{\eta}(I_{[r-s,\infty)})$	$\hat{\eta}(I_{[r-s,\infty)})$

$\}\ \mathfrak{F}$

$\underbrace{\qquad}_{\mathfrak{G}}$

Chapter 5

Open problems and planned future work

This chapter lists some open problems that need to be addressed in our future work. It is known [34, Example 30, p. 101] that there is a commutative non-noetherian semiprime ring R for which Fil$R$ is commutative. On the other hand, we show in Example 4.1 that there is a commutative semiprime non-noetherian ring R for which Fil$R$ is not commutative. This motivates the following questions:
(1) Let R be a commutative domain for which $\operatorname{Fil} R$ is commutative. Is R necessarily noetherian?
It is known [34, Proposition 31 and Corollary 32] that if R is a commutative domain such that $\operatorname{Fil} R$ is commutative, then R satisfies ACC on principal ideals. For a chain domain R $\operatorname{Fil} R$ is commutative if and only if R is noetherian. In connection with this we raise the following question:
(2) Let R be a commutative ring for which $\operatorname{Fil} R$ is commutative. Does R satisfy the ACC on prime ideals?
We point out that in Remark 2.13 a weakness of Theorem 2.3, is the absence of an example showing that no part of the FBN hypothesis may be dispensed with for the theorem's conclusion to remain valid. Certainly there are *non-noetherian rings* R for which $[\operatorname{Fil} R_R]^{\mathrm{du}}$ fails to be right residuated (for example rank 2 valuation domains). However, we have no example of a noetherian ring R for which $[\operatorname{Fil} R_R]^{\mathrm{du}}$ is right but not left residuated.
(3) Can we find an example of a noetherian ring R for which $[\operatorname{Fil} R_R]^{\mathrm{du}}$ is right but not left residuated?

(4) Extend results on two-sided residuation in Mod-R to the Wisbauer category $\sigma[M]$, for a right R-module M. Characterization and properties of hereditary pretorsion classes for M-ptors modules and M-artnian modules in $\sigma[M]$.

It is known [15] that every residuated lattice ordered monoid L can be embedded in a larger residuated lattice ordered monoid L^c which is L^c is obtained via the Dedekind-MacNeille completion. This prompts the following question.

(5) Is the two sided residuated property passed from L to its completion in the case where $L = [\mathrm{Fil}\, R_R]^{\mathrm{du}}$ for some ring R?

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Index

Nomenclature

$\subseteq$ set containment, 1

$\subset$ proper set containment, 1

$Id\,R$ the set of all two-sided ideals of R, 2

$I \trianglelefteq R$ I is a two-sided ideal of R, 2

R_R all regular right R-mdules, 2

${}_RR$ all regular left R-module, 2

$\text{Mod-} - R$ the category of right R-modules, 2

$\mathcal{L}(\mathcal{M})$ lattice of submodules of M in Mod-R, 2

$N \leq_e M$ N is an essential submodule of M, 3

$\operatorname{soc}(M)$ socle of M, 2

DCC Descending Chain Condition, 3

ACC Ascending Chain Condition, 3

$\bigoplus_{i\in I} A_i$ arbitrary direct sum of R_i, 2

$\prod_{i\in I} A_i$ arbitrary direct product R_i, 2

$\bigoplus_{i=1}^{n} A_i$ finite direct sum of R_i, 2

$A^{(I)}$ direct sum of I copies of A, 2

A^I direct product of I copies of A, 2

$\operatorname{soc} M$ socle of a module M, 2

$N \leq M$ N is a submodule of M, 2

$N \hookrightarrow M$ N is embedded in M, 2

$\operatorname{Spec} R$ The spectrum of R, 3

$rad\, R$ The radical of R, 3

$E(M)$ injective hull of M, 3

$\leq$ order relation, 4

$\geq$ dual of the order relation $\leq$, 4

$\bigwedge, \wedge$ meet, 4

$\bigvee, \vee$ join, 4

$(P, \leq)$ a poset, 4

e_L an identity element with respect to L, 5

$\cap, \bigcap$ intersection, 5

$c \leq \bigvee X$ c less or equal the join of X, 5

ab^{-1} left residual of a by b, 6

$b^{-1}a$ right residual of a by b, 6

$J^{-1}I$ right residual of I by J, 6

IJ^{-1} left residual of I by J, 6

$\prod_{i=1}^{n} L_i$ finite product of lattice ordered monoid, 7

$\equiv_\varphi$ congruence relation on a lattice induced by φ, 7

$[a]_{\equiv_\varphi}$ the equivalence class of a with respect to φ, 7

$L/\equiv_\varphi$ the collection of all equivalences with respect to $\equiv_\varphi$, 7

$\bigvee [a]_{\equiv_\varphi}$ the join of $[a]_{\equiv_\varphi}$, 8

$\{\equiv_\delta : \delta \in \Delta\}$ a family of congruences on lattice L, 8

$\mathcal{T}$ hereditary pretorsion class of right R-modules, 8

$\bigcup$ union, 20

$\mathcal{T}(M)$ the $\mathcal{T}$-torsion submodule of M, 9

HPR_R the set of all hereditary pretorsion classes in Mod-R, 9

M/N factor module, 10

$\mathfrak{F}$ filter, 10

$\stackrel{\text{def}}{=}$ defined as, 10

$FilR_R$ the set of right topologizing filters on a ring R, 10

$\mathfrak{F}_T$ right topologizing filter on R associated with $\mathcal{T}$, 11

$\mathcal{T}_{\mathfrak{F}}$ herditary pretorsion class in Mod-R associated with $\mathfrak{F}$, 11

$x^{-1}0$ the right annihilator of x by 0, 11

$\mathcal{T}_{\mathfrak{F}:\mathfrak{G}}$ the hereditary pretorsion class associated with $\mathfrak{F} : \mathfrak{G}$, 11

$\mathfrak{F} : \mathfrak{G}$ multiplication operation on $FilR_R$, 13

$\mathfrak{G}^{-1}\mathfrak{F}$ the right residual of $\mathfrak{F}$ by $\mathfrak{G}$, 14

$\mathfrak{F}\mathfrak{G}^{-1}$ the left residual of $\mathfrak{F}$ by $\mathfrak{G}$, 14

$[FilR_R]^{\text{du}}$ the dual order of $FilR_R$, 14

$\eta(I)$ smallest topologizing filters containing I, 14

$\mathfrak{G}^{-1}\bigvee Y$ the right residual of the join of Y by $\mathfrak{G}$, 16

$G\text{-Fil}\,R_R$ Gabriel filter, 17

$\varphi^{-1}(L)$ inverse image of L under a homomorphism φ, 18

$Fil(R/I)_{R/I}$ the right topologizing filter on R/I-module, 19

$:_I$ the multiplication operation $:$ on filters in $[0, \eta(I)]$, 21

$\{R_i : 1 \leq i \leq n\}$ finite family of rings, 24

$\prod_{i=1}^{n} R_i$ finite direct product of R_i, 24

$\prod_{i=1}^{n}[\text{Fil}\,R_{iR_i}]^{du}$ finite product of the order dual of $\text{Fil}\,R_{iR_i}$, 26

$GF \cong 1_{\text{Mod-}R}$ the identity map on $\text{Mod-}R$, 26

$GF \cong 1_{\text{Mod-}S}$ the identity map on $\text{Mod-}S$, 26

$\text{HP}S_S$ the hereditary pretorsion class of the category of right S-modules, 26

$M_{\mathfrak{G}}$ the smallest $\mathcal{T}_{\mathfrak{G}}$-dense hereditary pretorsion submodudle of M, 29

π_N canonical epimorphism from M to M/N, 30

$\pi[\_]$ lattice isomorphism between M and M/N, 30

$\pi^{-1}[\_]$ inverse isomorphism of $\pi[\_]$, 30

$\mathbb{L}_{\mathfrak{G}}[N, M]$ all $\mathcal{T}_{\mathfrak{G}}$-dense submodules of M containing N, 30

$g \circ \phi_S$ composition of ϕ_S with g, 57

$R \setminus P$ complement of R in P, 59

I_P the set of all prime ideals of R_P, 59

P_P the unique maximal ideal of R_P, 59

MS^{-1} module of fractions of M with respect to S, 60

$\operatorname{Ker} \varphi_S^M$ the kernel of the map φ_S^M, 60

$\hat{\varphi}_S$ a map from $\operatorname{Fil} \mathrm{R}$ to $\operatorname{Fil} RS^{-1}$, 61

φ_S^* a map from $\operatorname{Fil} RS^{-1}$ to $\operatorname{Fil} \mathrm{R}$, 61

$\mathfrak{F}_S$ topologizing filter associated with S, 64

G^+ positive cone of G, 88

∞ infinity, 79

v valuation map from a field F into $G \bigoplus \{\infty\}$, 80

R_v valuation domain associated with v, 80

$Fil(G^+ \cup \{\infty\})$ the set of all filters on $G^+ \bigoplus \{\infty\}$, 80

$v[\_]$ a map from $Id\, R$ into $Fil(G^+ \bigoplus \{\infty\})$, 80

$v^{-1}[\_]$ a map from $Fil(G^+ \bigoplus \{\infty\})$ into $Id\, R$, 81

$\hat{\eta}(I)$ all right ideals of R_R strictly containing I, 82

$\mathbb{Z} \times \mathbb{Z}$ a value group of a discrete rank two valuation domain R, 83

$(\mathbb{Z} \times \mathbb{Z})^+$ the positive cone of $\mathbb{Z} \times \mathbb{Z}$, 84

(m, n) an element of $(\mathbb{Z} \times \mathbb{Z})^+$, 84

$\mathbb{R}$ a value group of a a valuation domain R, 97

$\mathbb{R}^+$ the positive cone of $\mathbb{R}$, 97

ϵ epsilon representing small positive number, 94