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Existence of solutions for stochastic Navier-Stokes alpha and Leray alpha models of fluid turbulence and their relations to the stochastic Navier-Stokes equations

by

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DECLARATION

I, the undersigned, hereby declare that the thesis submitted herewith for the degree Philosophiae Doctor to the University of Pretoria contains my own, independent work and has not been submitted for any degree at any other university.

Name: Gabriel Deugoué

Date: August 2010

To my children: **Armand Sorel Tientcheu**

Marc Jorel Deugoué Tienga

Pierre Deugoué Ngantcheu

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Title Existence of solutions for stochastic Navier-Stokes alpha and Leray alpha models of fluid turbulence and their relations to the stochastic Navier-Stokes equations

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Department Mathematics and Applied Mathematics

Degree Philosophiae Doctor

Abstract

In this thesis, we investigate the stochastic three dimensional Navier-Stokes-α model and the stochastic three dimensional Leray-α model which arise in the modelling of turbulent flows of fluids.

We prove the existence of probabilistic weak solutions for the stochastic three dimensional Navier-Stokes-α model. Our model contains nonlinear forcing terms which do not satisfy the Lipschitz conditions. We also discuss the uniqueness. The proof of the existence combines the Galerkin approximation and the compactness method. We also study the asymptotic behavior of weak solutions to the stochastic three dimensional Navier-Stokes-α model as α approaches zero in the case of periodic box. Our result provides a new construction of the weak solutions for the stochastic three dimensional Navier-Stokes equations as approximations by sequences of solutions of the stochastic three dimensional Navier-Stokes-α model.

Finally, we prove the existence and uniqueness of strong solution to the stochastic three dimensional Leray-α equations under appropriate conditions on the data. This is achieved by means of the Galerkin approximation combines with the weak convergence methods. We also study the asymptotic behavior of the strong solution as alpha goes to zero. We show that a sequence of strong solution converges in appropriate topologies to weak solutions of the stochastic three dimensional Navier-Stokes equations.

All the results in this thesis are new and extend works done by several leading experts in both deterministic and stochastic models of fluid dynamics.

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Frequently used notation

a.e.	almost everywhere
a.s.	almost surely
$a \wedge b$	min(a,b)
$a \vee b$	max(a,b)
$\rightarrow$	strong convergence in the sense of functional analysis
$\rightharpoonup$	weak convergence in the sense of functional analysis
$\mathbb{N}$	set of positive integers
$\mathbb{R}$	set of real numbers
V'	dual space of the reflexive Banach space V
$\mathcal{B}(V)$	σ-algebra of all Borel measurables sets of V
$V^{\otimes m}$	the product of m copies of the set V
I_A	indicator function for the set A
$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$	filtered probability space
$\mathbb{E}$	mathematical expectation with respect to P
$\sigma(X_s : 0 \leq s \leq t)$	The smallest σ-field with respect to the random variable X_s is measurable for all $s \in [0, t]$
$\mathcal{D}(D)$	the set of infinitely differentiable functions with compact support in D
$\mathcal{L}(X, Y)$	space of linear bounded operator from the Banach space X to the Banach space Y
C([0,T]; X)	the space of all continuous function on $[0, T]$ with values in X
$\mathcal{D}'(0, T; X)$	the space of distributions on $]0, T[$ with values in X
σ^T	the transpose matrix of the matrix σ

Chapter 1

Introduction

1.1 The Navier-Stokes equations and turbulence

The flows of most commonly encountered fluids in nature and engineering applications are turbulent. Their prediction remains one of the greatest challenge in applied sciences. The mathematical theory of the Navier-Stokes equation is of fundamental importance to a deep understanding, prediction and control of turbulence in nature and in technological applications such as weather prediction, the dynamic of atmosphere, ocean and in aviation.

The classical three dimensional Navier-Stokes equations describe the time evolution of an incompressible fluid and are given by

$$\partial_t u(t) = \nu \Delta u(t) - (u(t).\nabla)u(t) + \nabla p(t) + f(t)$$

and

$$\text{div u(t)} = 0$$

where $u(t,x) = (u^1(t,x), u^2(t,x), u^3(t,x))$ represents the velocity field, ν is the viscosity constant, $p(t,x)$ denotes the pressure, and f is an external force field acting on the fluid.

It was stated in [36] that the Navier-Stokes equations capture the characteristic features of a turbulent flow (the distribution of eddy sizes, shapes, speeds, vorticity, circulation, nonlinear convection and viscous dissipation) and correctly predict how the cascade of turbulent kinetic energy and vorticity accelerate. Nevertheless, even with knowledge of the mathematical model, the problem of turbulence remains one of the last great unsolved problems of physics.

The existence of global in time regular solutions or the uniqueness of weak solutions are classical examples of persisting open problems of mathematical analysis [30]. The Clay Mathe-

matics Institute has called this one of the seven most important open problems in mathematics and offers a US$ 1,000,000 prize for a solution or a counter-example.
From the numerical point of view, only direct numerical simulation at moderate Reynolds numbers are possible. The direct numerical simulation for many physical applications with high Reynolds number flows is intractable even using state of the art numerical methods on the most advanced supercomputer available (see [31],[64]). Over the last decades, researchers have developed turbulence models as an attempt to solve this simulation barrier. The objective of turbulence models is to capture certain statistical features of the physical phenomenon of turbulence at computably low resolution by modelling the effect of the small scales in terms of the large scales.
In this thesis, we are going to study two such models: the Navier-Stokes-α model and the Leray-α model.

1.2 The Navier-Stokes-α model and the Leray-α model

1.2.1 Derivation and their relation to the turbulence

The study of the averaged motion of an incompressible fluid is motivated by the numerical inability in resolving small spatial scales. There are two approaches to modelling the averaged motion of an incompressible fluid. The first approach is the Reynolds averaging which suppose that the velocity of the fluid is a random variable which can be represented by the Reynolds decomposition

$$u(t,x) = U(t,x) + u'(t,x) \tag{1.1}$$

where $u'(t,x)$ denotes a random variable with mean value zero and $U(t,x)$ the average value of the velocity. The derivation of the averaged Navier-Stokes model is obtained by substituting the decomposition (1.1) into the Navier-Stokes equations and then averaging. This procedure produces the Reynolds averaged Navier-Stokes equations which are given by

$$\partial_t U + (U.\nabla)U + Div\, \overline{u' \otimes u'} = -gradP + \nu \Delta U. \tag{1.2}$$

The tensor $\overline{u' \otimes u'}$ is called the Reynolds stress which is given by

$$\overline{u' \otimes u'} = \nu_E(t, x, Def\ U).Def\ U,$$

where ν_E is the eddy viscosity and $Def\ U$ is the rate of deformation tensor defined as

$$Def\ U = \frac{1}{2}\left[\nabla U + (\nabla U)^T\right].$$

As a result of such an averaging, artificial viscosity is added into the system to remove energy which is contained in the small scales at which u' resides. Since it is still necessary to guess the form of ν_E, an improvement to the procedure of modelling the averaged motion of a fluid is needed.

The second approach to modelling the averaged motion of an incompressible fluid is the Lagrangien averaging. The Navier-Stokes-α model (also known in the literature as the viscous Camassa-Holm equations, or the Lagrangien averaged Navier-Stokes alpha model) is the first turbulence closure model produced by Lagrangien averaging, from which it derives its name. The inviscid Navier-Stokes-α (also called Euler-α) equations first appeared in [43] as a n-dimensional generalization of the one dimensional Camassa-Holm equations. The one dimensional Camassa-Holm equations describes shallow water with nonlinear dispersion and admits solitons solutions called "peakons"[9]. Holm, Marsden and Ratiu in [43] used variational asymptotics to obtain the Euler-α equations on all of $\mathbb{R}^n$, using an approximation of Hamilton's principle for the Euler's equations. Dissipation was added to the Euler-α equations to produce the Navier- Stokes-α equations. The extension of this approach to bounded domains was made in [55] by averaging over the set of solutions u^ϵ of the Euler equations with initial data u_0^ϵ in a phase- space ball of radius α. This extension was improved in [4]. The derivation of the Euler-α in [4] consists of expanding the original Lagrangian with respect to a perturbation parameter ϵ which is giving by

$$l(u^\epsilon) = \frac{1}{2}\int_D |u^\epsilon|^2 \, dx$$

where D is the space filled by the fluid, truncating the expansion to $O(\epsilon^2)$ terms and then taking average. The Euler-α equations is then derived by applying the Hamilton's principle to the averaged Lagrangian (See[4], for more details).

The study in ([15]-[17],[60], [38],[39],[18]) mentioned that there is a connection between the solutions of the Navier-Stokes-α model and turbulence. It was proved that the explicit steady analytical solutions of the Navier-Stokes-α model compare successfully with empirical and numerical experimental data for a wide range of Reynolds numbers in turbulent channel and pipe flows. The numerical study of the Navier-Stokes-α model in [60] shows that this model, indeed, captures most of the large scale features of a turbulent flow.

Pioneering investigation of this model were undertaken at the Los Alamos National Laboratory (USA).

Motivated by the remarkable performance of the Navier-Stokes-α model, the Leray-α model has been studied in [21] [42],[38],[39]. It was mentioned in [21] that by using this model as a closure model in turbulent channels and pipes, one obtains the same reduced system of equations as those produced by the Navier Stokes-α model, whose solutions give excellent agreement with empirical data for a wide range of large Reynolds numbers. Therefore the Leray-α has similar properties as the Navier-Stokes-α model. Other approximate α-models (Clark-α model, Modified Leray- α, Simplified Bardina-α model) for the three dimensional Navier-Stokes equations also show good agreement with empirical data. The Leray-α model is given by the following system of partial differential equations

$$\begin{cases} \partial_t v - \nu \Delta v + (u.\nabla)v = \nabla p + f, \\ \nabla.u = \nabla.v = 0, \\ v = u - \alpha^2 \Delta u, \\ u(x,0) = u_0(x). \end{cases} \tag{1.3}$$

Formally, the above system is the Navier-Stokes equations system when α=0, that is $u = v$. In order to study the question of existence of solutions to the Navier-Stokes equations, Leray considered in his pioneering work [50] a general regularization form of the Navier-Stokes equations in which the relationship between u and v in (1.3) is given by $u = \phi_\alpha * v$, where ϕ_α is an arbitrary smoothing kernel and $*$ denotes the convolution such that u converges to v, in some sense, as α tends to zero. In the particular case of system (1.3), the kernel ϕ_α is noting other than the Green's function associated with the Helmholtz operator $(I - \alpha^2\Delta)$. For this reason, system (1.3) is called the Leray-α model.

1.2.2 Previous analytical results: Deterministic and Stochastic.

In [34], the deterministic Cauchy problem for the three dimensional Navier-Stokes-α model subject to periodic boundary conditions was studied. The global existence and uniqueness of weak solutions were established, the regularity of weak solutions was proved and the global attractor for this model was constructed. Moreover, upper bounds for the dimension of the global attractor were found in terms of the relevant physical parameters. It has been also proved that the solutions of the Navier-Stokes-α model converge to certain solutions of the three dimensional Navier-Stokes equations as α approaches zero. These results were extended to the case of Dirichlet-type boundary conditions in [25]. The authors of [25] used a sequence of classical solutions in [54],to prove that this sequence converges in $C([0,T];H^1)$ to a H^1-weak solution of the Navier-Stokes-α model for all $T > 0$. They also proved the existence of a nonempty, compact, convex, and connected global attractor. The authors of [20] study the connection between the long-time dynamics of the three dimensional Navier-Stokes-α model and the three dimensional Navier-Stokes equations as α approaches zero. In particular, they showed that the trajectory attractor of the Navier-Stokes-α model converges to the trajectory attractor of the three dimensional Navier-Stokes system when α approaches zero. Similar results were proved in [19],[75],[21] for the Leray-α model.

The mathematical literature for the stochastic Navier-Stokes equations is extensive and dates back to early 1970's with the work of Bensoussan and Temam [3]. It is well known that there exists a probabilistic weak solution (also called martingale solution) for the stochastic three dimensional Navier-Stokes equations (see [32],[57] just to cite a few). But uniqueness is open. Brzeźniak and Peszat in [8], Mikulevicius and Rozovskii in [58], obtained the existence and uniqueness of a strong maximal local solution in W_p^1 with $p > 3$. Recently, Glatt and Ziane [40], Mikulevicius[59] have established the existence and uniqueness of local strong H^1-solution. Here the word "strong" means "strong" in the sense of the theory of stochastic differential equations; that is a complete probability space and a Wiener process are given in advance. All these results are global in two dimensions. Breckner [5] as well as Menaldi and Sritharan [56] established the existence and uniqueness of strong global L^2-solution for the two dimensional stochastic Navier-Stokes equations. The proof in [56] used the local monotonicity of the nonlinearity to obtain the solution.

The authors of [10] proved the existence and uniqueness of probabilistic strong solutions for the three dimensional stochastic Navier-Stokes-α model under the Lipschitz assumptions on the coefficients. The proof of the existence uses the Galerkin approximation and the weak convergence methods. The asymptotic behavior for the three dimensional stochastic Navier-Stokes-α model was proved in [11]. To the best of our knowledge, there is no systematic work for the three dimensional stochastic Leray-α model.

1.3 Main Results and Organization of the thesis

The aim of this thesis is twofold. Firstly, we study the existence of probabilistic weak solutions for the stochastic three dimensional Navier-Stokes-α model under continuity and linear growth conditions on the coefficients, extending the result of Caraballo, Real and Taniguchi in [10]. We also discuss the uniqueness and study the asymptotic behavior of weak solutions as α approaches zero in the case of periodic boundary conditions. Secondly, we establish the existence and uniqueness of the probabilistic strong solution for the three dimensional stochastic Leray-alpha model and study the asymptotic behavior of strong solution as α approaches zero.

The thesis contains five chapters (including the current one which deals with the introduction). In Chapter 2, we prove the existence of probabilistic weak solutions for the stochastic three dimensional Navier-Stokes-α model under weak assumptions on the coefficients. The proof is different from the one used in [10]. The techniques used here are the construction of the Galerkin approximation of the solutions, some a priori estimates which enable us to obtain some compactness properties of the probability measures generated by these solutions. The uniqueness result for the probabilistic weak solution is derived under strong assumptions. This uniqueness together with the famous Yamada-Watanabe theorem enable us to derive the existence of pathwise strong solution. This chapter has been the object of publication in Abstract and Applied Analysis.

In Chapter 3, we deal with the asymptotic behavior of probabilistic weak solutions of the stochastic three dimensional Navier-Stokes-α model as α approaches zero in the case of periodic boundary conditions. We approximate the solutions of the stochastic three dimensional Navier-Stokes equations by a sequence of weak solutions for the stochastic Navier-Stokes-α equations. For this, we study the tightness of probability measures induced by the weak solutions of the three dimensional Stochastic Navier-Stokes-α model. We prove that a sequence of solutions of the Navier-Stokes-α model converge in suitable topologies to weak solutions for the three dimensional stochastic Navier-Stokes equations. This provides us with another proof of existence of weak solutions for the stochastic Navier-Stokes equations. This chapter has been accepted for publication in Journal Mathematical Analysis and Applications.

Chapter 4 is devoted to the existence and uniqueness of a strong solution to the three dimensional stochastic Leray-α equations. Moreover, we study the asymptotic behavior of the strong solution as α goes to zero. We show that a sequence of strong solutions converges in appropriate topologies to weak solutions of the three dimensional stochastic Navier-Stokes equations. For the proof of the existence, we use the Galerkin method. The techniques applied are the properties of stopping times and some basic convergence principles from Functional Analysis. Another result is that the Galerkin approximation converges in mean square to the solution of the three dimensional stochastic Leray-α model. This chapter has been the object of publication in Boundary Value Problems.

The final chapter of the thesis contains an appendix with useful properties from functional and stochastic analysis. We included them for the convenience of the reader and because we often make use of them.

Chapter 2

On the Stochastic 3D Navier-Stokes-α Model

2.1 Introduction

In this chapter, we are interested in the study of probabilistic weak solutions of the $3D$ Navier-Stokes- α model (also known as the Lagrangien averaged Navier-Stokes-α model or the viscous Camassa-Holm equations) with homogeneous Dirichlet boundary conditions in a bounded domain in the case in which random perturbations appear. To be more precise, let D be a connected and bounded open subset of $\mathbb{R}^3$ with C^2 boundary ∂D, and let $T > 0$ be a final time. We denote by A the Stokes operator and consider the system

$$\begin{cases} \partial_t(u - \alpha\Delta u) + \nu(Au - \alpha\Delta(Au)) + (u.\nabla)(u - \alpha\Delta u) - \alpha(\nabla u)^T.\, \Delta u + \nabla p \\ = F(t,u) + G(t,u)\frac{dW}{dt}, \quad \text{in } D \times (0,T), \\ \nabla.u = 0, \quad \text{in } D \times (0,T), \\ u = 0, \quad Au = 0, \quad \text{on } \partial D \times (0,T), \\ u(0) = u_0, \quad \text{in } D, \end{cases} \tag{2.1}$$

where $u = (u_1, u_2, u_3)$ and p are unknown random fields on $D \times (0,T)$, representing respectively, the large-scale velocity and the pressure, in each point of $D \times (0,T)$. The constant $\nu > 0$ and $\alpha > 0$ are given, and represent, respectively, the kinematic viscosity of the fluid, and the square of the spatial scale at which fluid motion is filtered. The terms $F(t,u)$ and $G(t,u)\frac{dW}{dt}$

are external forces depending on u, where W is an $\mathbb{R}^m$-valued standard Wiener process. Finally u_0 is a given non random velocity field.

The deterministic version of (2.1), i.e. when $G = 0$ has been the object of intense investigations over the last years ([15]- [18], [33]). In view of many interesting futures, it was stated in [51] that the numerical study in [60], shows that this model, captures most of the large scale features of a turbulence flow, in particular those scales of motion larger than the length scale α, while the scales of motion smaller than alpha follow a faster decay of energy when compared with the energy of the Navier Stokes equations making it a more computable analytical large eddy simulation model of turbulence. Many important analytical results have been obtained in the deterministic case. In the case of periodic boundary conditions, Foias, Holm and Titi in [34] proved the global well posedness of H^1- weak solutions in dimension three. They also proved that the solutions of the Navier-Stokes-α equations converge to certain solutions of the Navier-Stokes equations as α approaches zero. Marsden and Shkoller in [54] proved the global well posedness of classical solutions in dimension three in the case of non-slip boundary condition. The authors of [25] proved the global in time existence, uniqueness and regularity of H^1- weak solutions , extending the result of Foias, Holm and Titi to the case of non-slip boundary data. The proof in [25] uses a sequence of classical solutions from [54] which is shown to converge to an H^1-weak solution of the Navier-Stokes-α equations. They also proved the existence of a nonempty compact, convex and connected global H^1-attractor in both two and three dimensions.

However, in order to consider a more realistic model of the problem, it is sensible to introduce some kind of noise in the equations. This may reflect, some environmental effects on the phenomena, some external random forces, etc. To the best of our knowledge, the existence and uniqueness of solutions of the problem (2.1) in the strong probabilistic sense has only been analyzed in [10] (see also [11],[12]) in the case of Lipschitz assumptions on F and G. The case of non Lipschitz assumptions on the coefficients F and G, is the main concern of the present chapter. This question has been opened till now. We merely assume continuity of $F(.,u)$ and $G(.,u)$ in u and some linear growth. In this case, the appropriate notion of solution is that of probabilistic weak solution also refered as martingale solution.

In this chapter, we shall establish the existence of probabilistic weak solutions for the problem (2.1) under appropriate conditions on the data. The approach used for the proof of our existence results is different from the one in [10]. To prove the existence, we use the Galerkin approximation method employing special bases, combined with some deep compactness theorems of probabilistic nature due to Prokhorov [65] and Skorokhod [68].

The chapter is organized as follows. In Section 2.2, we establish some properties of nonlinear term appearing in our equations. The rigorous statement of our problem as well as the main results are included in Section 2.3 and we show how our problem can be reformulated as an abstract stochastic model. Section 2.4 is devoted to the proof of our main results.

2.2 Properties of the nonlinear terms

Following [10], we establish some properties of the nonlinear term $(u.\nabla)(u-\alpha\Delta u)-\alpha(\nabla u)^T.\,\Delta u$ appearing in (2.1).

We denote by $(.,.)$ and $|.|$, respectively, the scalar product and associated norm in $(L^2(D))^3$, and by $(\nabla u, \nabla v)$ the scalar product in $((L^2(D))^3)^3$ of the gradients of u and v. We consider the scalar product in $(H_0^1(D))^3$ defined by

$$((u,v)) = (u,v) + \alpha(\nabla u, \nabla v), \qquad u, v \in (H_0^1(D))^3, \tag{2.2}$$

where its associated norm $\|.\|$ is, in fact, equivalent to the usual gradient norm. We denote by H the closure in $(L^2(D))^3$ of the set

$$\mathcal{V} = \{v \in (\mathcal{D}(D))^3 : \nabla.\, v = 0 \quad \text{in} \quad D\},$$

and by V the closure of $\mathcal{V}$ in $(H_0^1(D))^3$. Then H is a Hilbert space equipped with the inner product of $(L^2(D))^3$, and V is a Hilbert subspace of $(H_0^1(D))^3$.

Denote by A the Stokes operator, with domain $D(A) = (H^2(D))^3 \cap V$, defined by

$$Aw = -\mathcal{P}_1(\Delta w), \qquad w \in D(A),$$

where $\mathcal{P}_1$ is the projection operator from $(L^2(D))^3$ onto H. Recall that as ∂D is C^2, $|Aw|$ defines in $D(A)$ a norm which is equivalent to the $(H^2(D))^3$ norm, i.e. there exists a constant $c_1(D)$, depending only on the domain D, such that

$$\|w\|_{(H^2(D))^3} \le c_1(D)|Aw|, \qquad \forall w \in D(A), \tag{2.3}$$

and so $D(A)$ is a Hilbert space with respect to the scalar product

$$(v,w)_{D(A)} = (Av, Aw).$$

For $u \in D(A)$ and $v \in (L^2(D))^3$, we define $(u.\nabla)v$ as the element of $(H^{-1}(D))^3$ given by

$$\langle (u.\nabla)v, w\rangle_{-1} = \sum_{i,j=1}^{3} \langle \partial_i v_j, u_i w_j\rangle_{-1}, \qquad \forall w \in (H_0^1(D))^3 \tag{2.4}$$

where by $\langle u, v\rangle_{-1}$ we denote either the duality product between $(H^{-1}(D))^3$ and $(H_0^1(D))^3$ or between $H^{-1}(D)$ and $H_0^1(D)$.

(2.4) is meaningful, since $H^2(D) \subset L^\infty(D)$, and $H_0^1(D) \subset L^6(D)$, with continuous injections since dim D =3. This implies that $u_i w_j \in H_0^1(D)$, and there exists a constant $c_2(D) > 0$, depending only on D, such that

$$|\langle (u.\nabla)v, w\rangle_{-1}| \le c_2(D)|Au||v|\|w\|, \qquad \forall (u,v,w) \in D(A) \times (L^2(D))^3 \times (H_0^1(D))^3. \tag{2.5}$$

If $v \in (H^1(D))^3$, then the definition above coincides with the definition of $(u.\nabla)v$ as the vector function whose components are $\sum_{i=1}^3 u_i \partial_i v_j$, for $j = 1,2,3$. However, as it not known whether the solutions of the stochastic problem (2.1) have the same regularity as the deterministic case(we only can ensure H^2 instead of H^3), the present extension is necessary.

Now, if $u \in D(A)$, then $(\nabla u)^T \in (H^1(D))^{3\times 3} \subset (L^6(D))^{3\times 3}$, and consequently, for $v \in (L^2(D))^3$, we have that $(\nabla u)^T.\ v \in (L^{\frac{3}{2}}(D))^3 \subset (H^{-1}(D))^3$, with

$$\langle (\nabla u)^T.\ v, w\rangle_{-1} = \sum_{i,j=1}^{3} \int_D (\partial_j u_i) v_i w_j \, dx, \qquad \text{for all } \ w \in (H_0^1(D))^3.$$

It follows that there exists a constant $c_3(D)$, depending only on D, such that

$$|\langle (\nabla u)^T.\ v, w\rangle_{-1}| \le c_3(D)|Au||v|\|w\|, \qquad \text{for all } \ (u,v,w) \in D(A) \times (L^2(D))^3 \times (H_0^1(D))^3. \tag{2.6}$$

We have the following results

Proposition 1. *For all $(u, w) \in D(A) \times D(A)$ and all $v \in (L^2(D))^3$, it follows that*

$$\langle (u.\nabla)v, w\rangle_{-1} = -\langle (\nabla w)^T.\ v, u\rangle_{-1}. \tag{2.7}$$

Proof. If $(u, w) \in D(A) \times D(A)$, then for each $i, j = 1,2,3$, we have $u_i w_j \in H_0^1(D)$ and consequently

$$\begin{aligned}\langle \partial_i v_j, u_i w_j\rangle_{-1} &= -\int_D v_j \partial_i (u_i w_j)\, dx \\ &= -\int_D v_j w_j \partial_i u_i \, dx - \int_D v_j u_i \partial_i w_j \, dx\end{aligned}$$

using $\nabla.\ u = 0$, we have (2.7). $\square$

Consider now the trilinear form defined by

$$\begin{aligned} b^*(u,v,w) =& \langle (u.\nabla)v, w\rangle_{-1} + \langle (\nabla u)^T.v,\ w\rangle_{-1}, \\ & (u,v,w) \in D(A) \times (L^2(D))^3 \times (H_0^1(D))^3.\end{aligned}$$

Proposition 2. *The trilinear form b^* satisfies*

$$b^*(u,v,w) = -b^*(w,v,u), \qquad \forall (u,v,w) \in D(A) \times (L^2(D))^3 \times D(A), \tag{2.8}$$

and consequently,

$$b^*(u,v,u) = 0, \qquad \forall (u,v) \in D(A) \times (L^2(D))^3. \tag{2.9}$$

Moreover, there exists a constant $c(D) > 0$, depending only on D, such that

$$|b^*(u,v,w)| \leq c(D)|Au||v|\|w\|, \qquad \forall (u,v,w) \in D(A) \times (L^2(D))^3 \times (H_0^1(D))^3, \tag{2.10}$$

$$|b^*(u,v,w)| \leq c(D)\|u\||v||Aw| \qquad \forall (u,v,w) \in D(A) \times (L^2(D))^3 \times D(A). \tag{2.11}$$

Thus, in particular, b^* is continuous on $D(A) \times (L^2(D))^3 \times (H_0^1(D))^3$.

Proof. The proof is straightforward consequence of (2.5), (2.6), (2.7). See [10] □

2.3 Statement of the problem and the main result

We now introduce some probabilistic evolutions spaces.
Let $(\Omega, F, \{F_t\}_{0\leq t\leq T}, P)$ be a filtered probability space and let X be a Banach space.
For $r,\ q \geq 1$, we denote by

$$L^p(\Omega, F, P; L^r(0,T;X))$$

the space of functions $u = u(x,t,\omega)$ with values in X defined on $[0,T]\times\Omega$ and such that:

1) u is measurable with respect to (t,ω) and for almost all t, u is F_t measurable,

2)

$$\|u\|_{L^p(\Omega,F,P;L^r(0,T;X))} = \left[\mathbb{E}\left(\int_0^T \|u\|_X^r dt\right)^{\frac{p}{r}}\right]^{\frac{1}{r}} < \infty,$$

where $\mathbb{E}$ denote the mathematical expectation with respect to the probability measure P .

The space $L^p(\Omega, F, P; L^r(0,T;X))$ so defined is a Banach space.
When $r = \infty$, the norm in $L^p(\Omega, F, P; L^\infty(0,T;X))$ is given by

$$\|u\|_{L^p(\Omega,F,P;L^\infty(0,T;X))} = \left(\mathbb{E}\ ess \sup_{0\leq t\leq T} \|u\|_X^p\right)^{\frac{1}{p}}.$$

We make precise our assumptions on problem (2.1).

We start with the nonlinear functions F and G. We assume that:

$$\begin{aligned}
&F : (0,T) \times V \to (H^{-1}(D))^3, \quad \text{measurable}\\
&a.e.t,\ u \mapsto F(t,u) : \text{continuous from } V \text{ to } (H^{-1}(D))^3\\
&\|F(t,u)\|_{(H^{-1}(D))^3} \le C_1(1+\|u\|),
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
&G : (0,T) \times V \to \left((L^2(D))^3\right)^m, \quad \text{measurable}\\
&a.e.t,\ u \mapsto G(t,u) :\ \text{continuous from } V \text{ to } \left((L^2(D))^3\right)^m\\
&|G(t,u)|_{((L^2(D))^3)^m} \le C_2(1+\|u\|).
\end{aligned} \tag{2.13}$$

The constants C_1 and C_2 are independent of t and u.

We now define the concept of weak solution of the problem (2.1) namely

Definition 1. *By a weak solution of problem (2.1), we shall mean a system* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathcal{P}, W, u)$ *such that*

1) $(\Omega, \mathcal{F}, \mathcal{P})$*is a probability space,* $(\{\mathcal{F}_t\},\ 0 \le t \le T)$*is a filtration,*

2) W *is an m-dimensional* $\{\mathcal{F}_t\}$*standard Wiener process,*

3) $u(t)$ *is* $\mathcal{F}_t$ *adapted for all* $t \in [0,T]$
$u \in L^p(\Omega, \mathcal{F}, \mathcal{P}; L^2(0,T,D(A))) \cap L^p(\Omega, \mathcal{F}, \mathcal{P}; L^\infty(0,T,V))$ *for all* $1 \le p < \infty$,

4) For all $t \in [0,T]$*, the following equation holds* $\mathcal{P}$*- a.s.*

$$\begin{aligned}
((u(t),\Phi)) + \nu \int_0^t (u(s) + \alpha A u(s), A\Phi)ds + \int_0^t b^*(u(s), u(s) - \alpha\Delta u(s), \Phi)\, ds\\
= ((u_0,\Phi)) + \int_0^t \langle F(s,u(s)), \Phi\rangle_{-1}\, ds + \left(\int_0^t G(s,u(s))\, dW(s), \Phi\right)
\end{aligned} \tag{2.14}$$

for all $\Phi \in D(A)$.

Our main result is the following

Theorem 1. *(Existence) We assume that the above condition on* F *and* G *hold and* $u_0 \in V$*. Then there exists a weak solution* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathcal{P}, W, u)$ *of problem (2.1) in the sense of Definition 1.*

Moreover $u \in L^p(\Omega, \mathcal{F}, \mathcal{P}; C([0,T]; V))$*, and there exists* $\tilde{p} \in L^2(\Omega, \mathcal{F}_t, \mathcal{P}; H^{-1}(0,t; H^{-1}(D))$*, for* *all $t \in [0,T]$, such that $\mathcal{P}-a.s.$*

$$\begin{aligned}
&\partial_t(u-\alpha\Delta u)+\nu(Au-\alpha\Delta(Au))+(u.\nabla)(u-\alpha\Delta u)-\alpha(\nabla u)^T.\,\Delta u+\nabla\tilde{p}\\
&\qquad\qquad = F(t,u)+G(t,u)\frac{dW}{dt}, \quad in \quad (\mathcal{D}'((0,T)\times D))^3,\\
&\qquad\qquad\qquad\qquad \int_D \tilde{p}\,dx = 0, \quad in \quad \mathcal{D}'(0,T),
\end{aligned} \tag{2.15}$$

where $G(t,u)\frac{dW}{dt}$ denotes the time derivative of $\int_0^t G(s,u(s))\,dW_s$, that is, by definition

$$G(t,u)\frac{dW}{dt} = \partial_t\left(\int_0^{\cdot} G(s,u(s))\,dW_s\right), \quad in \quad \mathcal{D}'(0,T;(L^2(D))^3), \quad \mathcal{P}-a.s..$$

Corollary 1. *(Uniqueness) We assume that F and G are Lipschitz with respect to the second variable, $u_0 \in V$. Then there exist a unique weak solution of problem (2.1) in the sense of Definition 1.*

Moreover, two strong solutions on the same Brownian stochastic basis coincide a.s..

2.3.1 Abstract formulation of problem (2.1)

We are going to rewrite our model as an abstract problem.

We identify V with its topological dual V' and we have the Gelfand triple $D(A) \subset V \subset D(A)'$. We denote by $\langle .,. \rangle$ the duality product between $D(A)'$ and $D(A)$. We define

$$\langle \widetilde{A}u, v\rangle = \nu(Au,v)+\nu\alpha(Au,Av), \qquad u,v \in D(A).$$

It is clear that for all $v \in D(A)$,

$$2\langle \widetilde{A}u, v\rangle = 2\nu(Av,v)+2\nu\alpha(Av,Av) \geq 2\nu\alpha|Av|^2,$$

and, if we denote by λ_k and $w_k, k \geq 1$, the eigenvalues and their corresponding eigenfunctions associated to A, then

$$\langle \widetilde{A}w_k, v\rangle = \nu\lambda_k((w_k,v)).$$

Thus, taking

$$\widetilde{\alpha} = 2\nu\alpha,$$

we have:

(a) $\widetilde{A} \in \mathcal{L}(D(A), D(A)')$, such that

$$(a1) \quad \widetilde{A} \text{ is self-adjoint}$$

$$(a2) \quad \text{there exists } \widetilde{\alpha} > 0, \text{ such that}$$

$$2\langle \widetilde{A}v, v\rangle \geq \widetilde{\alpha}\|v\|^2_{D(A)} \quad for\ all\ v \in D(A). \tag{2.16}$$

Next, we define the operators $\widetilde{B}$ and $\widetilde{F}$

$$\langle \widetilde{B}(u,v), w\rangle = b^*(u, v - \alpha\Delta v, w), \quad (u,v,w) \in D(A) \times D(A) \times D(A),$$

$$((\widetilde{F}(t,u), w)) = \langle F(t,u), w\rangle_{-1}, \quad (u,w) \in V \times V. \tag{2.17}$$

Thus it is straightforward to check that if we take

$$\gamma = (1+\alpha)c_1(D)c(D),$$

then we obtain that

(b) $\widetilde{B} : D(A) \times D(A) \to D(A)'$ is a bilinear mapping such that

$$(b1) \quad \langle \widetilde{B}(u,v), u\rangle = 0 \quad \text{for all } u, v \in D(A), \tag{2.18}$$

$$(b2) \quad \|\widetilde{B}(u,v)\|_{D(A)'} \leq \gamma\|u\|\|v\|_{D(A)}, \quad \text{for all } u, v \in D(A) \times D(A), \tag{2.19}$$

$$(b3) \quad |\langle \widetilde{B}(u,v), w\rangle| \leq \gamma\|u\|_{D(A)}\|v\|_{D(A)}\|w\|, \quad \text{for all } u, v, w \in D(A). \tag{2.20}$$

The constants $c_1(D)$ and $c(D)$ are from (2.3) and (2.10).

(c) $\widetilde{F} : (0,T) \times V \to V$, measurable such that

$$(c1) \quad a.e.t,\ u \mapsto \widetilde{F}(t,u) : \text{continuous from } V \text{ to } V$$

$$(c2) \quad \|\widetilde{F}(t,u)\| \leq C_1(1 + \|u\|). \tag{2.21}$$

The constant C_1 is from (2.12).

Now, let I denote the identity operator in H, and define $\widetilde{G}(t,u)$ as

$$\widetilde{G}(t,u) = (I + \alpha A)^{-1} \circ \mathcal{P}_1 \circ G(t,u), \qquad u \in V. \tag{2.22}$$

$I + \alpha A$ is bijective from $D(A)$ onto H, and

$$(((I + \alpha A)^{-1} f, w)) = (f, w), \quad \text{for all f} \in H, w \in V.$$

Thus, for each $f \in H$,

$$\|(I+\alpha A)^{-1}f\|^2 = (f,u) \leq |f||u|,$$

where $u = (I+\alpha A)^{-1}f$;

that is $(u, w_k) + \alpha(Au, w_k) = (f, w_k),\ for\ all\ k \geq 1$. And

$$(1+\alpha\lambda_k)(u, w_k) = (f, w_k).$$

This implies

$$(u, w_k) = \frac{1}{(1+\alpha\lambda_k)}(f, w_k) \leq \frac{1}{1+\alpha\lambda_1}(f, w_k),$$

$$|u|^2 = \sum_{k=1}^{\infty}(u, w_k)^2 \leq \frac{1}{(1+\alpha\lambda_1)^2}\sum_{k=1}^{\infty}(f, w_k)^2 = \frac{1}{(1+\alpha\lambda_1)^2}|f|^2.$$

Therefore,

$$\|(I+\alpha A)^{-1}f\|^2 \leq \frac{1}{1+\alpha\lambda_1}|f|^2.$$

And consequently, taking

$$\widetilde{C} = \frac{C_2}{\sqrt{1+\alpha\lambda_1}}, \tag{2.23}$$

we see that $\widetilde{G}$ satisfies the following conditions:

(d) $\widetilde{G} : (0,T) \times V \to V^{\otimes m}$, measurable such that

$$\begin{aligned} &(d1) \quad a.e.t,\ u \mapsto \widetilde{G}(t,u) : \text{continuous from } V \text{ to } V^{\otimes m} \\ &(d2) \quad \|\widetilde{G}(t,u)\|_{V^{\otimes m}} \leq \widetilde{C}(1+\|u\|). \end{aligned} \tag{2.24}$$

Next, for all $(t, u, \Phi) \in (0,T) \times V \times D(A)$, we have

$$(G(t,u), \Phi) = ((I+\alpha A)\widetilde{G}(t,u), \Phi) = ((\widetilde{G}(t,u), \Phi)).$$

Furthermore, for all $u \in L^2(\Omega, \mathcal{F}, \mathcal{P}; L^\infty(0,T;V))$, $(t,\Phi) \in (0,T) \times D(A)$, we have

$$\begin{aligned} \left(\int_0^t G(s,u(s))\,dW(s), \Phi\right) &= \sum_{j=1}^{d}\int_0^t (G_j(s,u(s)), \Phi)\,dW_j(s) \\ &= \sum_{j=1}^{d}\int_0^t \left(\left(\widetilde{G}_j(s,u(s),\Phi)\right)\right)dW_j(s) \\ &= \left(\left(\int_0^t \widetilde{G}(s,u(s))\,dW(s), \Phi\right)\right). \end{aligned}$$

Consequently, we have the following version of the definition of a weak solution of problem (2.1) in the abstract setting as

Definition 2. *$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathcal{P}, W, u)$ is a probabilistic weak solution of problem (2.1) if it satisfies properties 1), 2), 3) of Definition 1 and*

4) For all $t \in [0,T]$, the following equation holds $\mathcal{P}$- a.s.

$$u(t)+\int_0^t \widetilde{A}u(s)\,ds + \int_0^t \widetilde{B}(u(s),u(s))\,ds$$
$$= u_0 + \int_0^t \widetilde{F}(s,u(s))\,ds + \int_0^t \widetilde{G}(s,u(s))\,dW(s), \tag{2.25}$$

as equality in $D(A)'$.

Remark 1. *The equation (2.25) implies that $u \in \mathcal{C}(0,T;D(A)')$ then u is weakly continuous in V (see[72], p.263) and the initial condition is meaningful.*

2.4 Proof of the main result

2.4.1 Proof of Theorem 1

We make use of the Galerkin approximation combined with the method of compactness.
We will split the proof into six steps.

Step 1 : Construction of an approximating sequence

As the injection $D(A) \hookrightarrow V$ is compact, we consider an orthonormal basis $\{e_j\}_{j=1,2,\dots}$in $D(A)$ which is orthogonal in V such that e_j are eigenfunctions of the spectral problem

$$(e_j, v)_{D(A)} = \beta_j((e_j, v)), \quad \text{for all } v \in D(A)$$

where $(.,.)_{D(A)}$ denotes the scalar product in $D(A)$. For each $N \in \mathbb{N}$, let V_N be the span of $\{e_1, ..., e_N\}$.
We consider the following stochastic ordinary differential equations in V_N

$$d((u^N, e_j)) + \Big(\langle \widetilde{A}u^N(t), e_j\rangle + \langle \widetilde{B}(u^N(t), u^N(t)), e_j\rangle\Big)\,dt \tag{2.26}$$
$$= ((\widetilde{F}(t,u^N(t)), e_j))\,dt + ((\widetilde{G}(t,u^N(t)), e_j))\,dW, \quad j = 1,2,...,N$$
$$u^N(0) = u_0^N,$$

where $u_0^N \in V_N$ and is chosen with the requirements that

$$u_0^N \to u_0 \quad \text{in V as } N \to \infty. \tag{2.27}$$

By the result in [69], this system has a probabilistic weak solution $(\Omega_N, \mathcal{F}_N, \{\mathcal{F}_t^N\}_{0\le t\le T}, P_N, W_N, u^N)$.
We have the following Fourier expansion

$$u^N(t) = \sum_{j=1}^{N} (u^N(t), e_j)_{D(A)} e_j = \sum_{j=1}^{N} \beta_j((u^N(t), e_j))e_j, \tag{2.28}$$

and

$$\|u^N(t)\|^2 = \sum_{j=1}^{N} \beta_j((u^N(t), e_j))^2.$$

Step 2. A priori estimates

Throughout C and $C_i (i = 1, ...)$ denote positive constants independent of N and α. The same symbol will be used for different constants.
We have

Lemma 1. *u^N satisfies the following a priori estimates*

$$\mathbb{E}_N \sup_{0\le t\le T} \|u^N(s)\|^2 + 2\widetilde{\alpha}\mathbb{E}_N \int_0^T \|u^N(s)\|^2_{D(A)}\, ds \le C_1, \tag{2.29}$$

where C_1 is a constant independent of N and α. $\mathbb{E}_N$ is the mathematical expectation with respect to the probability space $(\Omega_N, \mathcal{F}^N, P_N)$.

Proof. By Itô's formula, we obtain from (2.18) and (2.26) that

$$d\|u^N(t)\|^2 + 2\langle \widetilde{A}u^N(t), u^N(t)\rangle\, dt = \left[2((\widetilde{F}(t, u^N(t)), u^N(t))) + \sum_{j=1}^{N} \lambda_j((\widetilde{G}(t, u^N(t)), e_j))^2 \right] dt$$
$$+ 2((\widetilde{G}(t, u^N(t)), u^N(t)))\, dW_N. \tag{2.30}$$

Integrating (2.30) with respect to t, and using (2.16) and (2.21), we have

$$\|u^N(t)\|^2 + 2\widetilde{\alpha} \int_0^t \|u^N(s)\|^2_{D(A)}\, ds \le \|u_0^N\|^2 + C + C \int_0^t \|u^N(s)\|^2\, ds +$$
$$+ 2 \int_0^t ((\widetilde{G}(s, u^N(s)), u^N(s)))\, dW_N(s). \tag{2.31}$$

For each integer $n \ge 1$, consider the $\mathcal{F}_t^N$- stopping time λ_n^N defined by

$$\lambda_n^N = \inf \left\{ t \in [0,T]; \|u^N(t)\|^2 + \int_0^t \|u^N(s)\|^2_{D(A)}\, ds \ge n^2 \right\} \wedge T.$$

From (2.31), we have

$$\sup_{s\in[0,t\wedge\lambda_n^N]} \|u^N(s)\|^2 + 2\widetilde{\alpha} \int_0^{t\wedge\lambda_n^N} \|u^N(s)\|^2_{D(A)}\, ds \le \|u_0\|^2 + C + C \int_0^t \sup_{r\in[0,s\wedge\lambda_n^N]} \|u^N(r)\|^2\, ds$$
$$+ 2 \sup_{s\in[0,t\wedge\lambda_n^N]} \int_0^s ((\widetilde{G}(r, u^N(r)), u^N(r)))\, dW_N(r) \tag{2.32}$$

$\forall t \in [0,T]$ and all N , $n \geq 1$. Let us estimate the stochastic integral in this inequality. By Burkholder-Davis Gundy's inequality [45], we have

$$\begin{aligned}
&\mathbb{E}_N \sup_{s\in[0,t\wedge\lambda_n^N]} \left| \int_0^s ((\widetilde{G}(s,u^N(s)),u^N(s)))\,dW_N(s) \right| \\
&\leq C\mathbb{E}_N \left(\int_0^{t\wedge\lambda_n^N} ((\widetilde{G}(s,u^N(s)),u^N(s)))^2\,ds \right)^{\frac{1}{2}} \\
&\leq \epsilon\mathbb{E}_N \sup_{s\in[0,t\wedge\lambda_n^N]} \|u^N(s)\|^2 + C_\epsilon \mathbb{E}_N \int_0^{t\wedge\lambda_n^N} (1+\|u^N(s)\|^2)\,ds. \qquad (2.33)
\end{aligned}$$

Here we have used $H\ddot{o}lder's$ and Young's inequalities; ϵ is an arbitrary positive number.
Taking expectation in (2.32), and using (2.33), we obtain

$$\begin{aligned}
\mathbb{E}_N \sup_{s\in[0,t\wedge\lambda_n^N]} \|u^N(s)\|^2 + 2\widetilde{\alpha}\mathbb{E}_N \int_0^{t\wedge\lambda_n^N} \|u^N(s)\|^2_{D(A)}\,ds \leq \|u_0\|^2 + C + 2\epsilon\mathbb{E}_N \sup_{s\in[0,t\wedge\lambda_n^N]} \|u^N(s)\|^2 \\
+ C_\epsilon \int_0^t \mathbb{E}_N \sup_{r\in[0,s\wedge\lambda_n^N]} \|u^N(r)\|^2\,ds. \quad (2.34)
\end{aligned}$$

Using (2.34) together with appropriate choice of ϵ, we obtain

$$\mathbb{E}_N \sup_{s\in[0,t\wedge\lambda_n^N]} \|u^N(s)\|^2 + 2\widetilde{\alpha}\mathbb{E}_N \int_0^{t\wedge\lambda_n^N} \|u^N(s)\|^2_{D(A)}\,ds \leq \|u_0\|^2 + C + C\int_0^t \mathbb{E}_N \sup_{r\in[0,s\wedge\lambda_n^N]} \|u^N(r)\|^2\,ds.$$

Using Gronwall's lemma and the fact that $(\lambda_n^N)_n$ is increasing to T when n goes to ∞, it follows that

$$\mathbb{E}_N \sup_{0\leq t\leq T} \|u^N(s)\|^2 + 2\widetilde{\alpha}\mathbb{E}_N \int_0^T \|u^N(s)\|^2_{D(A)}\,ds \leq C_1$$

where C_1 is independent of N and α.

□

The following result is related to the higher-integrability of u^N.

Lemma 2. *We have*

$$\mathbb{E}_N \sup_{0\leq s\leq T} \|u^N(s)\|^p \leq C_p \qquad \textit{for all } 1 \leq p < \infty.$$

Proof. By Itô's formula, it follows from (2.30) that for $p \geq 4$, we have

$$\begin{aligned}
d\|u^N(t)\|^{\frac{p}{2}} = \frac{p}{2}\|u^N(t)\|^{\frac{p}{2}-2}\Big[-\langle \widetilde{A}u^N(t),u^N(t)\rangle - 2\langle \widetilde{B}(u^N(t),u^N(t)),u^N(t)\rangle + 2((\widetilde{F}(t,u^N(t)),u^N(t))) \\
+\frac{1}{2}\sum_{i=1}^N \lambda_i((\widetilde{G}(t,u^N(t)),e_i))^2 + \frac{p-4}{4}\frac{((\widetilde{G}(u^N(t),u^N(t))))^2}{\|u^N(t)\|^2}\Big]\,dt \\
+\frac{p}{2}\|u^N(t)\|^{\frac{p}{2}-2}((\widetilde{G}(t,u^N(t)),u^N(t)))\,dW_N.
\end{aligned}$$

Using the assumptions (2.18),(2.21),(2.24), it follows that

$$\sup_{0\le s\le t}\|u^N(s)\|^{\frac{p}{2}} \le \|u_0^N\|^{\frac{p}{2}} + C\int_0^t (1+\|u^N(s)\|^{\frac{p}{2}})\,ds + \frac{p}{2}\sup_{0\le s\le t}\left|\int_0^s \|u^N(s)\|^{\frac{p}{2}-2}((\widetilde{G}(s,u^N(s)),u^N(s)))\,dW_N\right|.$$

Squaring the both sides of this inequality and passing to mathematical expectation, we deduce from the martingale inequality that

$$\mathbb{E}_N \sup_{0\le s\le t}\|u^N(s)\|^p \le C\left(\|u_0^N\|^p + T + \mathbb{E}_N\int_0^t \|u^N(s)\|^p\,ds\right).$$

From the Gronwall's inequality, we deduce that

$$\mathbb{E}_N \sup_{0\le s\le t}\|u^N(s)\|^p \le C_p$$

for all $1\le p<\infty$. □

We also have

Lemma 3. *u^N satisfies*

$$\mathbb{E}_N\left(\int_0^T \|u^N(s)\|^2_{D(A)}\,ds\right)^p \le \frac{C_p}{\alpha^p}$$

for all $1\le p<\infty$.

Proof. From (2.31), we have

$$\widetilde{\alpha}^p\left(\int_0^T \|u^N(s)\|^2_{D(A)}\,ds\right)^p \le C_p\left(\|u_0^N\|^{2p} + 1 + \left(\int_0^T \|u^N(s)\|^2\,ds\right)^p\right) + C_p\sup_{t\in[0,T]}\left|\int_0^t \left(\left(\widetilde{G}(s,u^N(s)),u^N(s)\right)\right)dW_N\right|^p. \quad (2.35)$$

By Burkholder-Gundy's inequality, we have

$$\begin{aligned}
&\mathbb{E}_N\sup_{t\in[0,T]}\left|\int_0^t \left(\left(\widetilde{G}(s,u^N(s)),u^N(s)\right)\right)dW_N(s)\right|^p \\
&\le C_p\mathbb{E}_N\left(\int_0^T \left(\left(\widetilde{G}(s,u^N(s)),u^N(s)\right)\right)^2 ds\right)^{\frac{p}{2}} \\
&\le C_p\left(\mathbb{E}_N\sup_{t\in[0,T]}\|u^N(s)\|^{2p}\right) + C_pT. \qquad (2.36)
\end{aligned}$$

Thus from (2.35) and Lemma 2, we have

$$\mathbb{E}_N\left(\int_0^T \|u^N(s)\|^2_{D(A)}\,ds\right)^p \le \frac{C_p}{\alpha^p}.$$

□

Lemma 4. *We have*[1]

$$\mathbb{E}_N \sup_{0\le|\theta|\le\delta\le 1} \int_0^T \|u^N(t+\theta) - u^N(t)\|^2_{D(A)'}\, dt \le \frac{C\delta}{\alpha} + C\delta$$

Proof. We note that the functions $\{\beta_j e_j\}_{j=1,2,..}$ form an orthonormal basis in the dual $D(A)'$ of $D(A)$. Let P^N be the orthogonal projection of $D(A)'$ onto the span $\{\beta_1 e_1, ..., \beta_N e_N\}$ that is

$$P^N h = \sum_{j=1}^N \beta_j \langle h, e_j\rangle e_j.$$

Thus the equation (2.26) can be rewritten in an integral form as the equality between random variables with values in $D(A)'$ as

$$\begin{aligned} &u^N(t) + \int_0^t P^N \left(\widetilde{A}u^N(s) + \widetilde{B}(u^N(s), u^N(s)) - \widetilde{F}(s, u^N(s))\right) ds \\ &= u_0^N + \int_0^t P^N \widetilde{G}(s, u^N(s))\, dW_N. \end{aligned}$$

For any positive θ, we have

$$\begin{aligned} &\|u^N(t+\theta) - u^N(t)\|_{D(A)'} \\ &\le \left\| \int_t^{t+\theta} (\widetilde{A}u^N(s) + \widetilde{B}(u^N(s), u^N(s)) - \widetilde{F}(s, u^N(s)))\, ds \right\|_{D(A)'} + \left\| \int_t^{t+\theta} \widetilde{G}(s, u^N(s))\, dW_N \right\|_{D(A)'}. \end{aligned}$$

Taking the square and use the properties of $\widetilde{A}, \widetilde{B}$ and $\widetilde{F}$, we have

$$\begin{aligned} \|u^N(t+\theta) - u^N(t)\|^2_{D(A)'} &\le C\theta^2 + \\ &C \left(\int_t^{t+\theta} \|u^N(s)\|^2_{D(A)}\, ds \right)^2 + C \sup_{0\le t\le T} \|u^N(s)\|^2 \left(\int_t^{t+\theta} \|u^N(s)\|_{D(A)}\, ds \right)^2 \\ &\quad + C\theta^2 \sup_{0\le s\le T} \|u^N(s)\|^2 + \left\| \int_t^{t+\theta} \widetilde{G}(s, u^N(s))\, dW_N \right\|^2. \end{aligned}$$

For fixed δ, taking the supremum over $\theta \le \delta$, integrating with respect to t and taking the mathematical expectation, we have

$$\begin{aligned} \mathbb{E}_N \sup_{0\le\theta\le\delta} \int_0^T \|u^N(t+\theta) - u^N(t)\|^2_{D(A)'}\, dt &\le C\delta^2 + C\mathbb{E}_N \int_0^T \left(\int_t^{t+\delta} \|u^N(s)\|^2_{D(A)}\, ds \right)^2 dt \\ &+ C\mathbb{E}_N \sup_{0\le s\le T} \|u^N(s)\|^2 \int_0^T \left(\int_t^{t+\delta} \|u^N(s)\|_{D(A)}\, ds \right)^2 dt \\ &+ C\delta^2 \mathbb{E}_N \sup_{0\le s\le T} \|u^N(s)\|^2 + \mathbb{E}_N \int_0^T \sup_{0\le\theta\le\delta} \left\| \int_t^{t+\theta} \widetilde{G}(s, u^N(s))\, dW_N \right\|^2 dt. \quad (2.37) \end{aligned}$$

[1] u^N is extended by 0 outside [0,T]

We estimate the integrals in this inequality.

We have by $H\ddot{o}lder's$ inequality

$$I_1 = \mathbb{E}_N \sup_{0\le s\le T} \|u^N(s)\|^2 \int_0^T \left(\int_t^{t+\delta} \|u^N(s)\|_{D(A)}\, ds \right)^2 dt$$
$$\le \delta^2 \mathbb{E}_N \sup_{0\le s\le T} \|u^N(s)\|^2 \int_0^T \|u^N(s)\|_{D(A)}^2\, ds$$

Using the $H\ddot{o}lder's$ inequality and the estimates of Lemmas 2, 3, we have

$$I_1 \le \frac{C\delta^2}{\alpha}.$$

By martingale inequality, we have

$$I_2 = \mathbb{E}_N \int_0^T \sup_{0\le\theta\le\delta} \left\| \int_t^{t+\theta} \widetilde{G}(s, u^N(s))\, dW_N \right\|^2 dt$$
$$\le \mathbb{E}_N \int_0^T \left(\int_t^{t+\delta} \|\widetilde{G}(s, u^N(s))\|^2\, ds \right) dt.$$

Using the assumptions on $\widetilde{G}$ and the estimate of Lemma 2, we have

$$I_2 \le C\delta.$$

Collecting these results and proceeding similarly for the case $\theta < 0$, we obtain from (2.37) that

$$\mathbb{E}_N \sup_{0\le|\theta|\le\delta} \int_0^T \|u^N(t+\theta) - u^N(t)\|_{D(A)'}^2 \le \frac{C\delta}{\alpha} + C\delta$$

□

The following compactness results is from [2] and represents a variation of the compactness theorems in ([52] . Chap I, Section 5), and will be useful for us to prove the tightness property of Galerkin's solutions.

Proposition 3. *For any sequences of positives real ν_m, μ_m which tend to 0 as $m \to \infty$, the injection of*[2]

$$Y_{\mu_n,\nu_n} = \left\{ y \in L^2(0,T; D(A)) \cap L^\infty(0,T; V) | \sup_m \frac{1}{\nu_m} \sup_{|\theta|\le\mu_m} \left(\int_0^T \|y(t+\theta) - y(t)\|_{D(A)'}^2 \right)^{\frac{1}{2}} < \infty \right\}$$

in $L^2(0,T;V)$ is compact.

Proof. See Appendix, Proposition 6 with $B_0 = B_1 = D(A)$ and $B_2 = V$. □

[2] y is extended by 0 outside (0,T)

Furthermore Y_{μ_n,ν_n} is a Banach space with the norm

$$\|y\|_{Y_{\mu_n,\nu_n}} = ess \sup_{0\le t\le T} \|y(t)\| + \left(\int_0^T \|y(t)\|^2_{D(A)} dt\right)^{\frac{1}{2}} + \\ + \sup_n \frac{1}{\nu_n} \sup_{|\theta|\le\mu_n} \left(\int_0^T \|y(t+\theta)-y(t)\|^2_{D(A)'} dt\right)^{\frac{1}{2}}.$$

Alongside with Y_{μ_n,ν_n}, we also consider the space $X_{p,\mu_n,\nu_n}(1\le p<\infty)$ of random variables y defined on some probability space (we denote the expectation on that space by $\bar{\mathbb{E}}$) such that

$$\bar{\mathbb{E}}\, ess \sup_{0\le t\le T} \|y(t)\|^p < \infty; \bar{\mathbb{E}} \left(\int_0^T \|y(t)\|^2_{D(A)} dt\right)^{\frac{p}{2}} < \infty; \bar{\mathbb{E}} \sup_n \frac{1}{\nu_n} \sup_{|\theta|\le\mu_n} \int_0^T \|y(t+\theta)-y(t)\|^2_{D(A)'} dt < \infty.$$

Endowed with the norm

$$\|y\|_{X_{p,\nu_n,\mu_n}} = \left(\bar{\mathbb{E}}\, ess \sup_{0\le t\le T} \|y(t)\|^p\right)^{\frac{1}{p}} + \left(\bar{\mathbb{E}}\left(\int_0^T \|y(t)\|^2_{D(A)} dt\right)^{\frac{p}{2}}\right)^{\frac{2}{p}} + \\ + \bar{\mathbb{E}} \sup_n \frac{1}{\nu_n} \left(\sup_{|\theta|\le\mu_n} \int_0^T \|y(t+\theta)-y(t)\|^2_{D(A)'} dt\right)^{\frac{1}{2}}$$

X_{p,μ_n,ν_n} is a Banach space. The priori estimates of the preceding lemmas enable us to claim that for any $1\le p<\infty$ and for μ_n,ν_n such that the series $\sum_{n=1}^{\infty} \frac{\sqrt{\mu_n}}{\nu_n}$ converges, the sequence of Galerkin's solutions $\{u^N : N\in N\}$ is bounded in X_{p,μ_n,ν_n}.

Step 3. Tightness property of Galerkin's solutions

Now, we consider the set

$$S = C(0,T;\mathbb{R}^m)\times L^2(0,T;V),$$

and $\mathcal{B}(S)$ the σ-algebra of the Borel sets of S.

For each N , let ϕ_N be the map

$$\phi_N : \Omega_N \to S : \omega \mapsto (W_N(\omega,.), u^N(\omega,.)).$$

For each N, we introduce a probability measure Π_N on $(S,\mathcal{B}(S))$ by

$$\Pi_N(A) = P_N(\phi_N^{-1}(A))$$

for all $A\in\mathcal{B}(S)$. The main result of this subsection is

Theorem 2. *: The family of probability measures*$\{\Pi_N; N\in N\}$ *is tight in* $(S,\mathcal{B}(S))$.

Proof. For $\varepsilon > 0$, we should find the compact subsets

$$\Sigma_\varepsilon \subset C(0,T;R^m), Y_\varepsilon \subset L^2(0,T;V)$$

such that

$$P_N(\omega : W_N(\omega,.) \notin \Sigma_\varepsilon) \le \frac{\varepsilon}{2}, \tag{2.38}$$

$$P_N(\omega : u^N(\omega,.) \notin Y_\varepsilon) \le \frac{\varepsilon}{2}. \tag{2.39}$$

The quest for Σ_ε is made by taking account of some fact about the Wiener process such as the formula

$$\mathbb{E}_N|W_N(t_2) - W_N(t_1)|^{2j} = (2j-1)!(t_2-t_1)^j, j = 1, 2, ..$$

For a constant L_ε depending on ε to be chosen later and $n \in N$, we consider the set

$$\Sigma_\varepsilon = \{W(.) \in C(0,T;R^m) : \sup_{t_1,t_2\in[0,T],|t_2-t_1|\le\frac{1}{n^6}} n|W(t_2) - W(t_1)| \le L_\varepsilon\}.$$

Σ_ε is relatively compact in $C(0,T;R^m)$ by Arzsela- Ascoli's Theorem. Furthermore Σ_ε is closed in $C(0,T;R^m)$. Therefore Σ_ε is a compact subset of $C(0,T;R^m)$.
Making use of Markov's inequality:

$$\bar{P}(\omega : \xi(\omega) \ge \alpha) \le \frac{1}{\alpha^k}\bar{\mathbb{E}}\left[|\xi(\omega)|^k\right]$$

for a random variable ξ defined on some probability space $(\bar{\Omega}, \bar{F}, \bar{P})$ and positive real α and k, we get

$$\begin{aligned}
P_N(\omega : W_N(\omega,.) \notin \Sigma_\varepsilon) \le & P_N\left[\bigcup_n \{\omega : \sup_{t_1,t_2\in[0,T]:|t_2-t_1|<\frac{1}{n^6}} |W_N(t_2) - W_N(t_1)| > \frac{L_\varepsilon}{n}\}\right] \\
\le & \sum_{n=1}^{\infty}\sum_{i=0}^{n^6-1}\left(\frac{n}{L_\varepsilon}\right)^4 \mathbb{E}_N \sup_{\frac{iT}{n^6}\le t\le\frac{(i+1)T}{n^6}} |W_N(t) - W_N(iTn^{-6})|^4 \\
\le & c\sum_{n=1}^{\infty}\left(\frac{n}{L_\varepsilon}\right)^4 (Tn^{-6})^2 n^6 \\
= & \frac{c}{L_\varepsilon^4}\sum_{n=1}^{\infty}\frac{1}{n^2}
\end{aligned}$$

we choose

$$L_\varepsilon^4 = 2C\varepsilon^{-1}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

to get (2.38).

Next we choose Y_ε as a ball of radius M_ε in Y_{μ_n,ν_n} centered at zero and with μ_n, ν_n independent of ε, converging to zero and such that $\sum_n \frac{\sqrt{\mu_n}}{\nu_n}$ converges.

From Proposition 3, Y_ε is a compact subset of $L^2(0,T;V)$.

We have further

$$P_N(\omega : u^N(\omega,.) \notin Y_\varepsilon) \leq P_N(\omega : \|u^N\|_{Y_{\mu_n,\nu_n}} > M_\varepsilon) \leq \frac{1}{M_\varepsilon}\mathbb{E}_N\|u^N\|_{Y_{\mu_n,\nu_n}}$$
$$\leq \frac{c}{M_\varepsilon}$$

choosing $M_\varepsilon = 2c\varepsilon^{-1}$, we get (2.39).

From (2.38)and(2.39), we have

$$P_N(\omega : W_N(\omega,.) \in \Sigma_\varepsilon; u^N(\omega,.) \in Y_\varepsilon) \geq 1-\varepsilon$$

and this proves that

$$\Pi_N(\Sigma_\varepsilon \times Y_\varepsilon) \geq 1-\varepsilon \quad \forall\ N \in \mathbb{N}.$$

□

Step 4. Applications of Prokhorov and Skorokhod results

From the tightness property of $\{\Pi_N\}$ and Prokhorov's theorem (see Appendix C), we have that there exist a subsequence $\{\Pi_{N_j}\}$ and a probability measure Π such that $\Pi_{N_j} \to \Pi$ weakly. By Skorokhod's theorem (see Appendix C), there exist a probability space $(\Omega,\mathcal{F},P)$ and random variables $(W_{N_j}, u^{N_j}), (W,u)$ on $(\Omega,\mathcal{F},P)$ with values in S such that

$$\text{the law of } (W_{N_j}, u^{N_j})\ is\ \Pi_{N_j}$$

and

$$\text{the law of } (W,u)\ is\ \Pi$$

$$(W_{N_j}, u^{N_j}) \ \to\ (W,u) \text{ in } S, \quad P-a.s.. \tag{2.40}$$

Hence $\{W_{N_j}\}$ is a sequence of an m-dimensional standard Wiener process.

Let $\mathcal{F}^t = \sigma\{W(s), u(s), 0 \leq s \leq t\}$.

Arguing as in [1], we prove that $W(t)$ is an m-dimensional $\mathcal{F}^t$-standard Wiener process and the pair (W_{N_j}, u^{N_j}) satisfies the equation

$$u^{N_j}(t) + \nu\int_0^t P^{N_j}\widetilde{A}u^{N_j}(s)ds + \int_0^t P^{N_j}\widetilde{B}(u^{N_j}(s), u^{N_j}(s))ds =$$
$$\int_0^t P^{N_j}\widetilde{F}(s,u^{N_j}(s))ds + \int_0^t P^{N_j}\widetilde{G}(s,u^{N_j}(s))dW_{N_j} + u_0^{N_j}. \tag{2.41}$$

In fact to prove that $W(t)$ is an m-dimensional $\mathcal{F}^t$ standard Wiener process, it is sufficient to prove that for $\lambda \in \mathbb{R}^m$, $s \leq t$ and $i^2 = -1$

$$\mathbb{E}\left[exp\left\{i\lambda.(W(t) - W(s))\right\}|\mathcal{F}^s\right] = exp - \frac{|\lambda|^2}{2}(t-s), \tag{2.42}$$

where $\mathbb{E}$ denotes the mathematical expectation with respect to the probability space $(\Omega, \mathcal{F}, P)$. Let $\Lambda_s(b(.), z(.))$ be any continuous bounded functional on S which depends only on the restriction of $b(.), z(.)$ on $(0, s)$. To prove (2.42), it is sufficient to prove that:

$$\mathbb{E}\left[exp\left\{i\lambda.\left(W(t) - W(s)\right)\right\}\Lambda_s\left(W(.), u(.)\right)\right] = exp - \frac{|\lambda|^2}{2}(t-s)\mathbb{E}\Lambda_s\left(W(.), u(.)\right). \tag{2.43}$$

But

$$\mathbb{E}\left[exp\left\{i\lambda.\left(W_{N_j}(t) - W_{N_j}(s)\right)\right\}\Lambda_s\left(W_{N_j}(.), u^{N_j}(.)\right)\right] = exp - \frac{|\lambda|^2}{2}(t-s)\mathbb{E}\Lambda_s\left(W_{N_j}(.), u^{N_j}(.)\right)$$

since $\Lambda_s\left(W_{N_j}(.), u^{N_j}(.)\right)$ is independent of $W_{N_j}(t) - W_{N_j}(s)$ and W_{N_j} is a Wiener process.
In view of (2.40) and the continuity of Λ_t, we can pass to the limit in this equality and get (2.43).

Next, we need to prove that $\left(u^{N_j}, W_{N_j}\right)$ satisfies the equation

$$\begin{aligned} &u^{N_j}(t) + \nu\int_0^t P^{N_j}\widetilde{A}u^{N_j}(s)ds + \int_0^t P^{N_j}\widetilde{B}(u^{N_j}(s), u^{N_j}(s))ds \\ &= \int_0^t P^{N_j}\widetilde{F}(s, u^{N_j}(s))ds + \int_0^t P^{N_j}\widetilde{G}(s, u^{N_j}(s))dW_{N_j} + u_0^{N_j}. \end{aligned} \tag{2.44}$$

We set

$$\begin{aligned} \mathcal{E}_N(t) &= u^N(t) + \int_0^t P^N\left[\widetilde{A}u^N(s) + \widetilde{B}\left(u^N(s), u^N(s)\right)\right] ds \\ &- u_0^N - P^N\left[\int_0^t \widetilde{F}\left(s, u^N(s)\right)\, ds + \int_0^t \widetilde{G}\left(s, u^N(s)\right)\, dW_N(s)\right] \end{aligned}$$

and

$$X_N = \int_0^T \|\mathcal{E}_N(s)\|^2_{D(A)'}\, ds.$$

Obviously almost surely $X_N = 0$; hence in particular

$$\mathbb{E}_N\frac{X_N}{1+X_N} = 0$$

since $X_N \geq 0$ and

$$\mathbb{E}_N\frac{X_N}{1+X_N} \leq \mathbb{E}_N X_N.$$

Let

$$\mathcal{E}_{N_j}(t) = u^{N_j}(t) + \int_0^t P^{N_j}\left[\widetilde{A}u^{N_j}(s) + \widetilde{B}\left(u^{N_j}(s), u^{N_j}(s)\right)\right] ds$$
$$- u_0^{N_j} - P^{N_j}\left[\int_0^t \widetilde{F}\left(s, u^{N_j}(s)\right) ds + \int_0^t \widetilde{G}\left(s, u^{N_j}(s)\right) dW_{N_j}(s)\right]$$

and

$$Y_{N_j} = \int_0^T \|\mathcal{E}_{N_j}(s)\|^2_{D(A)'} ds.$$

Our claims will be proved if we can show that

$$\mathbb{E}\frac{Y_{N_j}}{1+Y_{N_j}} = 0. \tag{2.45}$$

An obstacle in the realisation of this goal is the fact that X_N is not a deterministic functional of u^N and $\bar{W}$ in view of the presence of the stochastic integral in X_N.

In order to circumvent that we introduce a regularization of $\widetilde{G}$ in t given by

$$\widetilde{G}^{\varepsilon}(u)(t) = \frac{1}{\varepsilon}\int_0^t exp\left[-\frac{t-s}{\varepsilon}\right] G\left(s, u(s)\right) ds.$$

We have the following properties of $\widetilde{G}^{\varepsilon}$:

$$\mathbb{E}\int_0^T \|\widetilde{G}^{\varepsilon}(u)(t)\|^2_{V^{\otimes m}} \le \mathbb{E}\int_0^T \|\widetilde{G}(t,u(t))\|^2_{V^{\otimes m}} dt \tag{2.46}$$

and

$$\widetilde{G}^{\varepsilon}(u)(.) \to \widetilde{G}(., u(.)) \quad in \quad L^2\left(\Omega, \mathcal{F}, P; L^2(0,T;V^{\otimes m})\right) \quad as \quad \varepsilon \to 0. \tag{2.47}$$

We denote by $X_{N,\varepsilon}$ and $Y_{N_j,\varepsilon}$ the analog of X_N and Y_{N_j} with $\widetilde{G}$ replaced by $\widetilde{G}^{\varepsilon}$.

Introduce the mapping

$$\Phi_{N,\varepsilon} : C(0,T;R^m) \times L^2(0,T;V) \to \mathbb{R}$$

given by

$$\Phi_{N,\varepsilon}(W_N; u^N) = \frac{X_{N,\varepsilon}}{1+X_{N,\varepsilon}}.$$

Owing to the definition of $X_{N,\varepsilon}$, $\Phi_{N,\varepsilon}$ is bounded and continuous on $C(0,T;R^m) \times L^2(0,T;V)$.

Let similarly

$$\Phi_{N_j,\varepsilon}\left(W_{N_j}, u^{N_j}\right) = \frac{Y_{N_j,\varepsilon}}{1+Y_{N_j,\varepsilon}}.$$

We have, using the conclusion of Prokhorov's theorem

$$\begin{aligned}\mathbb{E}\frac{Y_{N_j,\varepsilon}}{1+Y_{N_j,\varepsilon}} = \mathbb{E}\Phi_{N_j,\varepsilon}\left(W_{N_j}, u^{N_j}\right) &= \int_S \Phi_{N_j,\varepsilon}(\omega, x)\, d\Pi_{N_j} \\ &= \mathbb{E}_{N_j}\Phi_{N_j,\varepsilon}\left(W_{N_j}, u^{N_j}\right) \\ &= \mathbb{E}_{N_j}\frac{X_{N_j,\varepsilon}}{1+X_{N_j,\varepsilon}}.\end{aligned} \tag{2.48}$$

But

$$\mathbb{E}\frac{Y_{N_j}}{1+Y_{N_j}} - \mathbb{E}_{N_j}\frac{X_{N_j}}{1+X_{N_j}} = \mathbb{E}\left(\frac{Y_{N_j}}{1+Y_{N_j}} - \frac{Y_{N_j,\varepsilon}}{1+Y_{N_j,\varepsilon}}\right) + \mathbb{E}\frac{Y_{N_j,\varepsilon}}{1+Y_{N_j,\varepsilon}} - \mathbb{E}_{N_j}\frac{X_{N_j,\varepsilon}}{1+X_{N_j,\varepsilon}} + \mathbb{E}_{N_j}\left(\frac{X_{N_j,\varepsilon}}{1+X_{N_j,\varepsilon}} - \frac{X_{N_j}}{1+X_{N_j}}\right). \tag{2.49}$$

Moreover

$$\begin{aligned}\mathbb{E}\left|\frac{Y_{N_j}}{1+Y_{N_j}} - \frac{Y_{N_j,\varepsilon}}{1+Y_{N_j,\varepsilon}}\right| &= \mathbb{E}\left|\frac{Y_{N_j} - Y_{N_j,\varepsilon}}{(1+Y_{N_j})(1+Y_{N_j,\varepsilon})}\right| \\ &\leq \mathbb{E}\left|Y_{N_j} - Y_{N_j,\varepsilon}\right| \\ &\leq C\left(\mathbb{E}\int_0^T \|\widetilde{G}^{\varepsilon}(u^{N_j})(t) - \widetilde{G}(u^{N_j})(t)\|^2_{V^{\otimes m}}\, dt\right)^{\frac{1}{2}} \end{aligned}\tag{2.50}$$

and, similarly

$$\mathbb{E}_{N_j}\left|\frac{X_{N_j,\varepsilon}}{1+X_{N_j,\varepsilon}} - \frac{X_{N_j}}{1+X_{N_j}}\right| \leq C\left(\mathbb{E}_{N_j}\int_0^T \|\widetilde{G}^{\varepsilon}(u^{N_j})(t) - \widetilde{G}(u^{N_j})(t)\|^2_{V^{\otimes m}}\, dt\right)^{\frac{1}{2}}.$$

Combining these relations with (2.47)-(2.48) and letting $\varepsilon \to 0$, it follows from (2.49) that

$$\mathbb{E}\frac{Y_{N_j}}{1+Y_{N_j}} = \mathbb{E}_{N_j}\frac{X_{N_j}}{1+X_{N_j}} = 0.$$

This proves (2.45) and hence (2.44).

Step 5. Passage to the limit

From (2.41), it follows that u^{N_j} satisfies the results of the Lemmas 2, 3, 4. Therefore we have for $p \geq 1$ the a priori estimates

$$\mathbb{E}\sup_{0\leq t\leq T}\|u^{N_j(t)}\|^p \leq C;\ \mathbb{E}\left(\int_0^T \|u^{N_j}(t)\|^2_{D(A)}dt\right)^p \leq C;\ \mathbb{E}\sup_{0\leq\theta\leq\delta}\int_0^T \|u^{N_j}(t+\theta)-u^{N_j}\|^2_{D(A)'}dt \leq C(\alpha)\delta \tag{2.51}$$

thus modulo the extraction of a subsequence denoted again by u^{N_j} , we have

$$u^{N_j} \to u \textit{ weakly} * \textit{ in } L^p(\Omega,\mathcal{F},P;L^\infty(0,T;V))$$

$$u^{N_j} \to u \textit{ weakly in } L^p(\Omega,\mathcal{F},P;L^2(0,T;D(A)))$$

$$\mathbb{E}\sup_{0\leq t\leq T}\|u(t)\|^p \leq C\ ;\ \mathbb{E}\left(\int_0^T \|u(t)\|^2_{D(A)}dt\right)^p \leq C;$$

$$\mathbb{E}\sup_{0\leq\theta\leq\delta}\int_0^T \|u(t+\theta) - u(t)\|^2_{D(A)'}dt \leq \frac{C\delta}{\alpha}.$$

Combining (2.40) with the first estimate in (2.51) and Vitali's theorem, we have

$$u^{N_j} \to u \text{ strongly in } L^2(\Omega, \mathcal{F}, P; L^2(0,T,V)) \tag{2.52}$$

and thus modulo the extraction of a subsequence and for almost every (ω, t) with respect to the measure $dP \otimes dt$:

$$u^{N_j} \ \rightarrow \ u \text{ in } V.$$

This convergence together with the condition on $\widetilde{F}$, the first estimate in (2.51) and Vitali's theorem, give

$$\widetilde{F}(., u^{N_j}(.)) \to \widetilde{F}(., u(.)) \textit{ strongly in } L^2(\Omega, \mathcal{F}, P; L^2(0,T,V))$$

$$\int_0^t \widetilde{F}(s, u^{N_j(s)})ds \to \int_0^t \widetilde{F}(s, u(s))ds \textit{ strongly in } L^2(\Omega, \mathcal{F}, P; L^2(0,T,V))$$

As

$$u^{N_j} \ \rightarrow \ u \textit{ weakly in } L^2(\Omega, \mathcal{F}, P; L^2(0,T; D(A)))$$

then

$$\int_0^t \widetilde{A}u^{N_j}(s)\,ds \ \rightarrow \ \int_0^t \widetilde{A}u(s)\,ds \quad \textit{weakly in } L^2(\Omega, \mathcal{F}, P; L^2(0,T; D(A)'))$$

We also have

$$\int_0^t \widetilde{B}(u^{N_j}(s), u^{N_j}(s))ds \to \int_0^t \widetilde{B}(u(s), u(s))ds \ \ \textit{weakly} \ \ \textit{in } L^2(\Omega, \mathcal{F}, P; L^2(0,T; D(A)'))$$

In fact since $L^\infty(\Omega\times(0,T), dP\times dt; D(A))$ is dense in $L^2(\Omega, \mathcal{F}, P; L^2(0,T; D(A)))$, and $\widetilde{B}(u^{N_j}(s), u^{N_j}(s))$ is bounded in $L^2(\Omega, \mathcal{F}, P; L^2(0,T; D(A)'))$, it suffices to prove that $\forall \varphi \in L^\infty(\Omega \times (0,T), dP \times dt; D(A))$

$$\mathbb{E}\int_0^T \langle \widetilde{B}(u^{N_j}(s), u^{N_j}(s)), \varphi(s)\rangle_{D(A)'} ds \to \mathbb{E}\int_0^T \langle \widetilde{B}(u(s), u(s)), \varphi(s)\rangle_{D(A)'} ds.$$

Indeed, we have

$$\mathbb{E}\int_0^T \langle \widetilde{B}(u^{N_j}(s), u^{N_j}(s)) - \widetilde{B}(u(s), u(s)), \varphi(s)\rangle_{D(A)} ds =$$

$$\mathbb{E}\int_0^T \langle \widetilde{B}(u^{N_j}(s) - u(s), u^{N_j}(s)), \varphi(s)\rangle_{D(A)} ds + \mathbb{E}\int_0^T \langle \widetilde{B}(u(s), u^{N_j}(s) - u(s)), \varphi(s)\rangle_{D(A)} ds = I_{1j} + I_{2j}.$$

$$I_{1j} = \mathbb{E}\int_0^T \langle \widetilde{B}(u^{N_j}(s) - u(s), u^{N_j}(s)), \varphi(s)\rangle_{D(A)} ds$$

By the property (2.19) of $\widetilde{B}$, we have

$$I_{1j} \leq C\mathbb{E}\int_0^T \|u^{N_j}(s) - u(s)\| \|u^{N_j}(s)\|_{D(A)} |A\varphi(s)| ds.$$

Applying Cauchy-Schwarz's inequality, we get

$$I_{1j} \leq C_\varphi \left(\mathbb{E} \int_0^T \|u^{N_j}(s) - u(s)\|^2 ds \right)^{\frac{1}{2}} \left(\mathbb{E} \int_0^T \|u^{N_j}(s)\|^2_{D(A)} \, ds \right)^{\frac{1}{2}}.$$

Since

$$u^{N_j} \rightarrow u \text{ } strongly \text{ } in \text{ } L^2(\Omega, \mathcal{F}, P; L^2(0,T;V))$$

and u^{N_j} is bounded in $L^2(\Omega, \mathcal{F}, P; L^2(0,T;D(A)))$, we conclude that

$$I_{1j} \rightarrow 0 \text{ } as \text{ } j \rightarrow \infty.$$

$$I_{2j} = \mathbb{E} \int_0^T \langle \widetilde{B}(u(s), u^{N_j} - u(s)), \varphi(s) \rangle_{D(A)'} ds.$$

Again thanks to the property (2.20) of $\widetilde{B}$, as

$$u^{N_j} \rightarrow u \quad weakly \quad \text{in} \quad L^2(\Omega, \mathcal{F}, P; L^2(0,T;D(A)))$$

we obtain $I_{2j} \rightarrow 0$ as $j \rightarrow \infty$ since any strongly continuous linear operator is weakly continuous. We are now left with the proof of

$$\int_0^t \widetilde{G}(s, u^{N_j}(s)) dW_{N_j}(s) \rightarrow \int_0^t \widetilde{G}(s, u(s)) dW(s) \text{ } weakly \text{ } L^2(\Omega, \mathcal{F}, P; V).$$

We consider for that purpose the already introduced regularisation

$$\widetilde{G}^\varepsilon(u)(t) = \frac{1}{\varepsilon} \int_0^t exp(-\frac{t-s}{\varepsilon}) \widetilde{G}(s, u(s)) \, ds$$

which satisfies (2.46)-(2.47). Also for N_j fixed

$$\widetilde{G}^\varepsilon(u^{N_j})(.) \rightarrow \widetilde{G}(u^{N_j})(.) \quad in \quad L^2(\Omega, \mathcal{F}, P; L^2(0,T;V^{\otimes m})) \tag{2.53}$$

as $\varepsilon \rightarrow 0$. From the definition of $\widetilde{G}^\varepsilon$, we have

$$\begin{aligned} &\int_0^t \|\widetilde{G}^\varepsilon(u^{N_j})(s) - \widetilde{G}^\varepsilon(u)(s)\|^2_{V^{\otimes m}} \, ds \\ &\leq \int_0^t \|\widetilde{G}(u^{N_j})(s) - \widetilde{G}(u)(s)\|^2_{V^{\otimes m}} \, ds \\ &= \int_0^t \|\widetilde{G}(s, u^{N_j}(s)) - \widetilde{G}(s, u(s))\|^2_{V^{\otimes m}} \, ds. \end{aligned} \tag{2.54}$$

Next by integration by parts, we have

$$\begin{aligned} \int_0^t \widetilde{G}^\varepsilon(u^{N_j})(s) \, dW_{N_j}(s) =& \frac{1}{\varepsilon} W_{N_j}(T) \int_0^T \widetilde{G}(t, u^{N_j}(t)) e^{-\frac{T-t}{\varepsilon}} \, dt \\ & - \frac{1}{\varepsilon} \int_0^T W_{N_j}(t). \left(\widetilde{G}(t, u^{N_j}(t)) - \frac{1}{\varepsilon} \int_0^t \widetilde{G}(s, u^{N_j}(s)) e^{-\frac{t-s}{\varepsilon}} \, ds \right) dt. \end{aligned} \tag{2.55}$$

In view of (2.40), passing to the limit as $j \to \infty$ in (2.55), we get

$$\begin{aligned}\int_0^t \widetilde{G}^\varepsilon(u^{N_j})(s)\,dW_{N_j}(s) \longrightarrow & \frac{1}{\varepsilon}W(T)\int_0^T \widetilde{G}(t,u(t))e^{-\frac{T-t}{s}}\,dt \\ & -\frac{1}{\varepsilon}\int_0^T W(t)\left(\widetilde{G}(t,u(t)) - \frac{1}{\varepsilon}\int_0^t \widetilde{G}(s,u(s))e^{-\frac{t-s}{\varepsilon}}\,ds\right)\,dt \end{aligned} \tag{2.56}$$

for almost all (ω, t). The right hand side of (2.56) is equal to $\int_0^t \widetilde{G}^\varepsilon(u)(s)\,dW(s)$.

Thus

$$\int_0^t \widetilde{G}^\varepsilon(u^{N_j})(s)\,dW_{N_j}(s) \longrightarrow \int_0^t \widetilde{G}^\varepsilon(u)(s)\,dW(s) \tag{2.57}$$

for almost every (ω, t) as $j \to \infty$ for fixed ε.

Picking any element $\xi \in L^2(\Omega, \mathcal{F}, P; V)$, we have

$$\mathbb{E}\left(\left(\xi, \int_0^t \widetilde{G}^\varepsilon(u^{N_j})(s)\,dW_{N_j}(s)\right)\right) \longrightarrow \mathbb{E}\left(\left(\xi, \int_0^t \widetilde{G}^\varepsilon(u)(s)\,dW(s)\right)\right) \tag{2.58}$$

as $j \to \infty$, for fixed ε. This follows from (2.57) and Vitali's theorem.

In the other hand, we have

$$\mathbb{E}\left\|\int_0^t \widetilde{G}(u^{N_j})(s)\,dW_{N_j}(s)\right\|^2 \le \mathbb{E}\int_0^t \|\widetilde{G}(u^{N_j})(s)\|^2_{V^{\otimes m}}\,ds \le C,$$

thus there exists $\eta \in L^2(\Omega, \mathcal{F}, P; V)$ such that for all $h \in L^2(\Omega, \mathcal{F}, P; V)$

$$\mathbb{E}\left(\left(h, \int_0^t \widetilde{G}(u^{N_j})(s)\,dW_{N_j}(s)\right)\right) \longrightarrow \mathbb{E}\left((h,\eta)\right).$$

We show that

$$\eta = \int_0^t \widetilde{G}(u)(s)\,dW(s).$$

We have

$$\begin{aligned}&\mathbb{E}\left(\left(h, \int_0^t \widetilde{G}(u^{N_j})(s)\,dW_{N_j}(s)\right)\right) - \mathbb{E}\left(\left(h, \int_0^t \widetilde{G}(u)(s)\,dW(s)\right)\right) \\ &= I_1 + I_2 + I_3 \end{aligned} \tag{2.59}$$

where

$$\begin{aligned} I_1 &= \mathbb{E}\left(\left(h, \int_0^t \left(\widetilde{G}(u^{N_j})(s) - \widetilde{G}^\varepsilon(u^{N_j})(s)\right)\,dW_{N_j}(s)\right)\right) \\ I_2 &= \mathbb{E}\left(\left(h, \int_0^t \widetilde{G}^\varepsilon(u^{N_j})(s)\,dW_{N_j}(s) - \int_0^t \widetilde{G}^\varepsilon(u)(s)\,dW(s)\right)\right) \\ I_3 &= \mathbb{E}\left(\left(h, \int_0^t \left(\widetilde{G}^\varepsilon(u)(s) - \widetilde{G}(u)(s)\right)\,dW(s)\right)\right). \end{aligned}$$

By Cauchy-Schwarz's inequality and (2.54), we have

$$\begin{aligned} I_1 \leq & C\left(\mathbb{E}\|h\|^2\right)^{\frac{1}{2}}\left(\mathbb{E}\int_0^T \|\widetilde{G}(u^{N_j})(s)-\widetilde{G}(u)(s)\|^2_{V^{\otimes m}}\,ds\right)^{\frac{1}{2}} \\ & + C\left(\mathbb{E}\|h\|^2\right)^{\frac{1}{2}}\left(\mathbb{E}\int_0^T \|\widetilde{G}(u^{N_j})(s)-\widetilde{G}(u)(s)\|^2_{V^{\otimes m}}\,ds\right)^{\frac{1}{2}} \\ & + C\left(\mathbb{E}\|h\|^2\right)^{\frac{1}{2}}\left(\mathbb{E}\int_0^T \|\widetilde{G}^{\varepsilon}(u)(s)-\widetilde{G}(u)(s)\|^2_{V^{\otimes m}}\,ds\right)^{\frac{1}{2}}. \end{aligned} \tag{2.60}$$

Letting $j \to \infty$ and $\varepsilon \to 0$, and using (2.47) and (2.53), we have $I_1 \to 0$.

I_2 converges to zero by (2.58)

I_3 converges to zero by Cauchy- Schwarz's inequality and (2.47).

Collecting all the convergence results, we deduce that:

$$\begin{aligned} & u(t) + \nu\int_0^t \widetilde{A}u(s)ds + \int_0^t \widetilde{B}(u(s),u(s))ds \\ & = \int_0^t \widetilde{F}(s,u(s))ds + \int_0^t \widetilde{G}(s,u(s))dW(s) + u_0, \quad P-a.s. \end{aligned} \tag{2.61}$$

as equality in $D(A)'$.

We have $\widetilde{B}(u,u) \in L^2(\Omega,\mathcal{F},P;L^\infty(0,T;D(A)'))$, $\quad \widetilde{A}u-\widetilde{F}(t,u) \in L^2(\Omega,\mathcal{F},P;L^\infty(0,T;D(A)'))$, $\widetilde{G}(t,u) \in L^2(\Omega,\mathcal{F},P;L^\infty(0,T;V^{\otimes m}))$.

Thus, from the classical results in [47] (see also [61]), we deduce from (2.61) that u is $P-a.s.$ continuous with values in V.

Step 6. Existence of the pressure

Here, the proof follows the same line as in [10]. For the existence of the pressure, we use a generalization of the Rham's Theorem processes (see [49],Theorem 4.1,Remark 4.3). From (2.14), we have for all $v \in \mathcal{V}$,

$$\langle -\partial_t(u-\alpha\Delta u)-\nu(Au-\alpha\Delta(Au))-(u.\nabla)(u-\alpha\Delta u)+\alpha(\nabla u)^T.\,\Delta u+F(.,u)+G(.,u)\frac{dW}{dt},v\rangle_{(\mathcal{D}'(D))^3\times(\mathcal{D}(D))^3}=0.$$

We denote

$$h=-\partial_t(u-\alpha\Delta u)-\nu(Au-\alpha\Delta(Au))-(u.\nabla)(u-\alpha\Delta u)+\alpha(\nabla u)^T.\,\Delta u+F(.,u)+G(.,u)\frac{dW}{dt}.$$

We are going to prove that the regularity on u, implies that

$$h \in L^2(\Omega,\mathcal{F}_t,P;H^{-1}(0,t;(H^{-2}(D))^3).$$

By (2.5) and (2.6), we have as $u \in L^4(\Omega, \mathcal{F}, P; L^2(0,T; D(A)))$,

$$(u.\nabla(u - \alpha\Delta u)) + (\nabla u)^T.\ \Delta u \in L^2(\Omega, \mathcal{F}_t, P; L^1(0,t; (H^{-1}(D))^3)),$$

$$Au - \alpha\Delta(Au) \in L^4(\Omega, \mathcal{F}_t, P; L^2(0,t; (H^{-2}(D))^3)).$$

We also have

$$u - \alpha\Delta u \in L^4(\Omega, \mathcal{F}_t, P; L^2(0,t; (L^2(D))^3))$$

and

$$\partial_t(u - \alpha\Delta u) \in L^4(\Omega, \mathcal{F}_t, P; H^{-1}(0,t; (L^2(D))^3)) \quad \forall \quad t \in [0,T].$$

Again, as $u \in L^4(\Omega, \mathcal{F}, P; C([0,T]; V))$, then it follows that

$$F(t,u) \in L^4(\Omega, \mathcal{F}_t, P; L^2(0,t; (H^{-1}(D))^3))$$

$$G(t,u)\frac{dW}{dt} \in L^4(\Omega, \mathcal{F}_t, P; W^{-1,\infty}(0,t; (L^2(D))^3))$$

for all $t \in [0,T]$.
Then $h \in L^2(\Omega, \mathcal{F}_t, P; H^{-1}(0,t; (H^{-2}(D))^3)$, and

$$\langle h, v\rangle_{(\mathcal{D}'(D))^3 \times (\mathcal{D}(D))^3} = 0 \quad \text{for all} \quad \mathrm{v} \in \mathcal{V}.$$

Therefore, by a generalization of the Rham's Theorem processes [49], there exists a $\tilde{p} \in L^2(\Omega, \mathcal{F}_t, P; H^{-1}(0,t; (H^{-1}(D))^3)$ *such that* $P - a.s.$

$$\nabla\tilde{p} = h \quad \text{and} \quad \int_D \tilde{p}\, dx = 0.$$

This establishes (2.15) and completes the proof of Theorem 1.

2.4.2 Proof of Corollary 1

Proof. We going to prove the pathwise uniqueness which implies uniqueness of weak solutions. Let L_F and L_G be two real such that

$$\|F(t,u) - F(t,v)\|_{(H^{-1}(D))^3} \leq L_F\|u - v\|,$$

$$\|G(t,u) - G(t,v)\|_{((L^2(D))^3)^m} \leq L_G\|u - v\|.$$

Then $\widetilde{F}$ and $\widetilde{G}$ defined respectively by (2.17) and (2.22) satisfy

$$\|\widetilde{F}(t,u) - \widetilde{F}(t,v)\|_V \leq L_{\widetilde{F}}\|u - v\|,$$

$$\|\widetilde{G}(t,u)-\widetilde{G}(t,u)\|_{V^{\otimes m}}\leq L_{\widetilde{G}}\|u-v\|.$$

Let u_1 and u_2 two weak solutions of problem (2.1) defined on the same probability space together with the same Wiener process and starting from the same initial value u_0.
We denote $\bar{u}=u_1-u_2$. Take $\mu>0$ to be defined later and $\rho(t)=exp\left(-\mu\int_0^t\|u_2(s)\|^2_{D(A)}\,ds\right)$, $0\leq t\leq T$.
Applying Itô's formula to the real process $\rho(t)\|\bar{u}(t)\|^2$, we obtain from (2.16), (2.20), (2.21), (2.24) that,

$$\begin{aligned}\rho(t)\|\bar{u}(t)\|^2+\widetilde{\alpha}\int_0^t\rho(s)\|\bar{u}(s)\|^2_{D(A)}\,ds&\leq L^2_{\widetilde{G}}\int_0^t\rho(s)\|\bar{u}(s)\|^2\,ds\\&+2\widetilde{c}\int_0^t\rho(s)\|u_2(s)\|_{D(A)}\|\bar{u}(s)\|_{D(A)}\|\bar{u}(s)\|\,ds\\&+2L_{\widetilde{F}}\int_0^t\rho(s)\|\bar{u}(s)\|_{D(A)}\|\bar{u}(s)\|\,ds\\&+2\int_0^t\Big(\Big(\rho(s)(\widetilde{G}(s,u_1(s))-\widetilde{G}(s,u_2(s))),\bar{u}(s)\Big)\Big)\,dW(s)\\&-\mu\int_0^t\int_0^t\rho(s)\|u_2(s)\|^2_{D(A)}\|\bar{u}(s)\|^2\,ds.\end{aligned}\tag{2.62}$$

for all $t\in[0,T]$.
By Young's inequality, we have

$$2\widetilde{c}\|u_2(s)\|_{D(A)}\|\bar{u}(s)\|_{D(A)}\|\bar{u}(s)\|\leq\frac{\widetilde{\alpha}}{2}\|\bar{u}(s)\|^2_{D(A)}+\frac{2\widetilde{c}^2}{\widetilde{\alpha}}\|u_2(s)\|^2_{D(A)}\|\bar{u}(s)\|^2$$

and

$$2L_{\widetilde{F}}\|\bar{u}(s)\|_{D(A)}\|\bar{u}(s)\|\leq\frac{\widetilde{\alpha}}{2}\|\bar{u}(s)\|^2_{D(A)}+\frac{2L^2_{\widetilde{F}}}{\widetilde{\alpha}}\|\bar{u}(s)\|^2.$$

If we take $\mu=2\frac{\widetilde{c}^2}{\widetilde{\alpha}}$, we obtain from (2.62)

$$\rho(t)\|\bar{u}(t)\|^2\leq(L^2_{\widetilde{G}}+\frac{2L^2_{\widetilde{F}}}{\widetilde{\alpha}})\int_0^t\rho(s)\|\bar{u}(s)\|^2\,ds+2\int_0^t\Big(\Big(\rho(s)(\widetilde{G}(s,u_1(s))-\widetilde{G}(s,u_2(s)),\bar{u}(s))\Big)\Big)\,dW.\tag{2.63}$$

As $0<\rho(t)\leq1$, the expectation of the stochastic integral in (2.63) vanishes and

$$\mathbb{E}\rho(t)\|\bar{u}(t)\|^2\leq(L^2_{\widetilde{G}}+\frac{2L^2_{\widetilde{F}}}{\widetilde{\alpha}})\mathbb{E}\int_0^t\rho(s)\|\bar{u}(s)\|^2\,ds.$$

The Gronwall's lemma implies that $\bar{u}(t)=0$, P-a.s for all $t\in[0,T]$. And the corollary is proved. □

Remark 2. *Using the famous Yamada-Watanabe theorem [44] , Corollary 1 implies the existence of a unique strong solution of problem (2.1).*

Chapter 3

The Stochastic 3D Navier-Stokes-α model: α tends to 0

3.1 Introduction

The Navier-Stokes-α model are system of partial differential equations designed to capture the large scale dynamics of the incompressible Navier-Stokes equations. This model was developed in an effort to provide an efficient numerical simulation of the 3D turbulence and was used as a closure model for the Reynolds averaged Navier-Stokes. It was tested successfully against experimental measurements and direct simulations of turbulent channel and pipe flows. Several analytical and numerical results seem to confirm that the Navier-Stokes-α model gives good approximation in the study of many problems related to the turbulence flows.
In [34], the Cauchy problem for the deterministic 3D Navier-Stokes-α model with periodic boundary conditions was studied, the global existence, uniqueness and regularity of solutions were established. Furthermore, the relation between the solutions of the Navier-Stokes-α model and the solutions of the Navier-Stokes equations was proved as α approaches zero. In particular, the authors of [34] showed that there exists a subsequence of solutions of the 3D Navier-Stokes-α model that converges to one of the weak solutions of the 3D Navier-Stokes equations. Later the authors of [20] showed that the trajectory attractor of the Navier-Stokes-α model converges to the trajectory attractor of weak solutions of the 3D Navier-Stokes equations as α approaches zero.

In the stochastic case, the existence and uniqueness of strong solutions to the stochastic 3D

Navier-Stokes-α with Dirichlet boundary conditions started with the work of [10] under Lipschitz conditions on the forces. The existence result under more general assumptions on the data were established in the previous chapter; these results hold in the case of periodic boundary conditions with the needed notational changes. We shall make use of these existence results throughout. Other related problems on stochastic 3D Navier-Stokes-α can be found in [11] and [12]. However, in the stochastic case, there is no work on the convergence of solutions of the stochastic 3D Navier- Stokes-α model towards the solutions of the stochastic 3D Navier-Stokes equations as α approaches zero.

In this chapter, we investigate the approximation of the stochastic 3D Navier-Stokes equations by a sequence of solutions of the stochastic 3D Navier-Stokes-α model as α approaches zero. For this purpose, we study the weak compactness of weak solutions for the stochastic 3D Navier-Stokes-α model as α approaches zero. This is not derived directly from the priori estimates obtained in Chapter 2 because some explode when α approaches zero. One the main difficulties of this chapter lies in obtaining needed a priori estimates in which the constants are independent of α. One such estimate is the following

$$\mathbb{E}_\alpha \sup_{0\le|\theta|\le\delta\le 1} \int_0^T |u_\alpha(t+\theta) - u_\alpha(t)|^2_{D(A)'}\, dt \le C\delta$$

where C is a constant independent of α,$(\Omega_\alpha, \mathcal{F}_\alpha, \{\mathcal{F}_{\alpha t}\}_{0\le t\le T}, P_\alpha, W_\alpha, u_\alpha)$ is a probabilistic weak solutions of the Navier-Stokes-α model and E_α is the mathematical expectation with respect to P_α (see Definition 4 below). To do this, we adopt the method developed for the deterministic 3D Navier-Stokes-α model in [34]. In this method an important role is played by the operator $(I + \alpha^2 A)^{-1}$. Once the a priori estimates are secured, the next task is to obtain the tightness of the family of probability measures generated by the sequence $\{u_\alpha\}_{\alpha>0}$ which enables us to make use of Prokhorov and Skorokhod's compactness results. The last main issue is the passage to the limit which turns out to be rather complicated in view of the nature of the nonlinear terms involved in our model.

The question of asymptotic analysis of partial differential equations when some physical parameters converge to some limit has always been of great interest. Notable example is the vanishing viscosity question in Navier-Stokes equations which is still not fully solved when the problem is assigned Dirichlet boundary conditions for instance. We refer to [24], [41] [70], [71], and the several references in the last paper. In the stochastic case fewer investigation has been carried out; we refer to [6] for relevant investigation.

The chapter is organized as follows. In Section 3.2, we recall the definition of the weak solutions for the stochastic 3D Navier-Stokes equations and formulate the corresponding existence result. In Section 3.3, we consider the stochastic 3D Navier-Stokes-α model. We formulate the main properties of this model and the periodic boundary conditions version of the main result of the authors in [27]. Section 3.3 is the main core of the chapter. Here we obtain uniform a priori estimates for weak solutions $\{u_\alpha\}_{\alpha>0}$ of the stochastic 3D Navier-Stokes α model, we derive the results on the tigthness of the corresponding probability measures and perform the passage to the limit which establishes the convergence of u_α to the weak solutions of the stochastic 3D Navier-Stokes equations. This gives us another proof of the existence of weak solutions to the stochastic 3D Navier-Stokes equations.

3.2 The Stochastic 3D Navier-Stokes equations

Let $T>0$ be a final time and consider the following viscous stochastic 3D Navier-Stokes equations in the periodic box $\mathcal{T}=[0,L]^3$:

$$\begin{cases} du+[-\nu\Delta u+(u.\nabla)u+\nabla p]\,dt=F(t,u)dt+G(t,u)dW, \quad \text{in } \mathcal{T}\times(0,T), \\ \nabla.u=0, \quad \text{in } \mathcal{T}\times(0,T), \\ u(0)=u^0, \quad \text{in } \mathcal{T}, \\ u=u\,(x,t) \ \text{ is periodic in } x\in\mathcal{T},\ \int_{\mathcal{T}}u\,dx=0, \end{cases} \tag{3.1}$$

where $x=(x_1,x_2,x_3)$, $u=(u_1\,(x,t)\,,u_2\,(x,t)\,,u_3\,(x,t))$ and $p=p\,(x,t)$ are unknown random fields on $\mathcal{T}\times(0,T)$ representing, respectively, the velocity and the pressure, at each point of an incompressible viscous fluid with constant density filling the domain $\mathcal{T}$. The constant ν represent the kinematic viscosity of the fluid. The term $F(t,u)$ and $G(t,u)dW$ are external forces depending eventually on u, where W is an $\mathbf{R}^m$- valued standard Wiener process. Finally, u^0 is a given non random initial velocity field.

We denote by $C^\infty_{per}(\mathcal{T})^3$ the space of all $\mathcal{T}$-periodic C^∞ vector fields defined on $\mathcal{T}$. We set

$$\mathcal{V}=\{\Phi\in C^\infty_{per}(\mathcal{T})^3|\nabla.\Phi=0 \text{ and } \int_{\mathcal{T}}\Phi\,dx=0\}.$$

We denote by H and V the closure of the set $\mathcal{V}$ in the spaces $L^2(\mathcal{T})^3$ and $H^1(\mathcal{T})^3$ respectively. We endow H with the $L^2(\mathcal{T})^3$ scalar product $(.,.)$ and norm $|.|$. The space V is a Hilbert space for the scalar product $((u,v))_V=(u,v)+\alpha^2(\nabla u,\nabla v)$ where its associated norm, which is in fact equivalent to the usual gradient norm, will be denoted by $\|.\|_V$. For $u,v\in V$, we denote $((u,v)) = (\nabla u, \nabla v)$ and $\|u\| = |\nabla u|$. We denote by A the stokes operator with domain $D(A) = H^2(\mathcal{T})^3 \cap V$ and $\mathcal{P}_2 : L^2(\mathcal{T})^3 \to H$ the Leray operator. The operator A is an isomorphism from V to V' and $\langle Au, v\rangle = ((u,v))$ where $\langle .,.\rangle$ denotes the duality between V and V'.

We define the bilinear operator $B(u,v) : V \times V \to V'$ as

$$\langle B(u,v), z\rangle = \int_{\mathcal{T}} z(x).(u(x).\nabla)v(x)\,dx$$

for all $z \in V$. By the incompressibility condition, for all $u, v, z \in V$, we have

$$\langle B(u,v), v\rangle = 0,$$

$$\langle B(u,v), z\rangle = -\langle B(u,z), v\rangle.$$

Alongside the problem (3.1), given at the beginning of the section, we shall consider the abstract stochastic evolution equation which is formally obtained from (3.1) by projecting over the space of divergence free fields:

$$\begin{cases} du + \nu Au(t)dt + B(u(t), u(t))dt = F(t,u)dt + G(t,u)dW, \\ u(0) = u_0. \end{cases} \tag{3.2}$$

F and G are two nonlinear operators such that:

$$\begin{aligned} &F : (0,T) \times H \to H, \text{ measurable,} \\ &\text{a.e. } t, u \to F(t,u) : \text{continuous from } H \text{ to } H, \\ &|F(t,u)|_H \le C(1+|u|), \end{aligned} \tag{3.3}$$

$$\begin{aligned} &G : (0,T) \times H \to H^{\otimes m}, \text{ measurable,} \\ &\text{a.e. } t, u \to G(t,u) : \text{ continuous from } H \text{ to } H^{\otimes m} \\ &|G(t,u)|_{H^{\otimes m}} \le C(1+|u|). \end{aligned} \tag{3.4}$$

Finally, we define the concept of solution of the system (3.2) that we need, namely

Definition 3. *By a weak solution of the problem (3.2), we shall mean a system* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, P, W, u)$ *where*

1) $(\Omega, \mathcal{F}, P)$ *is a probability space,* $(\{\mathcal{F}_t\}, 0 \le t \le T)$ *is a filtration on* $(\Omega, \mathcal{F}, P)$*,*

2) W *is an* m*-dimensional* $\mathcal{F}_t$ *standard Wiener process,* *4)* $u \in L^p(\Omega, \mathcal{F}, P; L^2(0,T;V)) \cap L^p(\Omega, \mathcal{F}, P; L^\infty(0,T;H))$ *for all* $1 \leq p < \infty$*,*

5) the following equation holds almost surely

$$(u(t), \varphi) + \nu \int_0^t (u(s), A\varphi)\, ds + \int_0^t \langle B(u(s), u(s)), \varphi \rangle_{D(A)'}\, ds =$$
$$(u_0, \varphi) + \int_0^t (F(s, u(s)), \varphi)\, ds + \left(\int_0^t G(s, u(s))\, dW(s), \varphi \right)$$

for all $\varphi \in D(A)$*,* $t \in [0,T]$*.*

We recall the following existence result due to Bensoussan [2].

Theorem 3. *We assume that (3.3), (3.4) hold and* $u_0 \in H$*.Then problem (3.2) has a weak solution in the sense of Definition 3.*

Several other authors have dealt with the existence and pathwise uniqueness of solutions of the stochastic Navier-Stokes equations. We refer to [2], [7], [14], [32], [53], [73]; just to cite a few.

3.3 The stochastic 3D Navier-Stokes-α model

We consider the family $\{(u_\alpha, p_\alpha)\}_{\alpha>0}$ of solutions of the stochastic 3D Navier-Stokes-α model in the periodic box $\mathcal{T} = [0, L]^3$:

$$\begin{cases} d(u_\alpha - \alpha^2 \Delta u_\alpha) + \left[-\nu \Delta (u_\alpha - \alpha^2 \Delta u_\alpha) - (u_\alpha \times (\nabla \times (u_\alpha - \alpha^2 \Delta u_\alpha))) + \nabla p_\alpha \right] dt = \\ F(t, u_\alpha) dt + G(t, u_\alpha)\, dW \quad in \quad \mathcal{T} \times (0,T), \\ \nabla . u_\alpha = 0 \quad in \quad \mathcal{T} \times (0,T), \\ u_\alpha(0) = u_0 \quad in \quad \mathcal{T}, \\ u_\alpha = u_\alpha\,(t, x) \text{ is periodic in } x \in \mathcal{T},\ \int_{\mathcal{T}} u_\alpha\, dx = 0. \end{cases} \tag{3.5}$$

In system (3.5): α is a fixed positive parameter called "the sub-grid (filter) length scale" of the model, $a \times b$ is the vector product in $\mathbf{R}^3$. Observe that when $\alpha = 0$, system (3.5) coincides with the stochastic 3D Navier-Stokes equations.

We denote for $u, v \in V$,

$$\widetilde{B}(u, v) = -\mathcal{P}(u \times (\nabla \times v)).$$

Recall that

$$\widetilde{B}(u, u) = B(u, u).$$

For convenience, we summarize in the next lemma some results from [34] that we shall need here.

Lemma 5. *i) The operator $\widetilde{B}$ can be extended continuously from $V \times V$ with values in V', and in particular it satisfies*

$$|\langle \widetilde{B}(u,v), w\rangle_{V'}| \leq c|u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}\|v\|\|w\|$$

$$|\langle \widetilde{B}(u,v), w\rangle_{V'}| \leq c\|u\|\|v\||w|^{\frac{1}{2}}\|w\|^{\frac{1}{2}}$$

for all $u, v \in V$. Moreover

$$\langle \widetilde{B}(u,v), w\rangle_{V'} = -\langle \widetilde{B}(w,v), u\rangle_{V'}$$

and in particular,

$$\langle \widetilde{B}(u,v), u\rangle_{V'} = 0$$

for all $u, v \in V$.

ii) Furthermore, we have

$$|\langle \widetilde{B}(u,v), w\rangle_{D(A)'}| \leq c(|u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}|v||Aw| + \|u\||v|\|w\|^{\frac{1}{2}}|Aw|^{\frac{1}{2}})$$

for all $u \in V, v \in H, w \in D(A)$.

$$|\langle \widetilde{B}(u,v), w\rangle_{D(A)'}| \leq c\|u\||v||Aw|$$

for all $u \in V, v \in H, w \in D(A)$.
In particular

$$|\widetilde{B}(u,v)|_{D(A)'} \leq c\|u\||v|$$

for all $u \in V, v \in H$.

Alongside the system (3.5), we shall consider the abstract stochastic evolution equation which is formally obtained from (3.5) by projecting on the space of divergence free fields:

$$\begin{cases} d(u_\alpha + \alpha^2 A u_\alpha) + \left[\nu A(u_\alpha + \alpha^2 A u_\alpha) + \widetilde{B}(u_\alpha, u_\alpha + \alpha^2 A u_\alpha)\right] dt = \\ F(t, u_\alpha)dt + G(t, u_\alpha)\, dW, \\ u_\alpha(0) = u_0. \end{cases} \tag{3.6}$$

We shall define the concept of solution of the system (3.6)

Definition 4. *By a weak solution of problem (3.6), we shall mean a system* $(\Omega_\alpha, \mathcal{F}_\alpha, \{\mathcal{F}_{\alpha t}\}_{0\le t\le T}, P_\alpha, W_\alpha, u_\alpha)$ *where*

1) $(\Omega_\alpha, \mathcal{F}_\alpha, P_\alpha)$ *is a probability space,* $\mathcal{F}_{\alpha t}$ *is a filtration on* $(\Omega_\alpha, \mathcal{F}_\alpha, P_\alpha)$*,*

2) W_α *is an* m*-dimensional* $\mathcal{F}_{\alpha t}$ *standard Wiener process,*

3) a.e. t*,* $u_\alpha(t)$ *is* $\mathcal{F}_{\alpha t}$ *measurable,*

4) $u_\alpha \in L^p(\Omega_\alpha, \mathcal{F}_\alpha, P_\alpha; L^2(0,T; D(A))) \cap L^p(\Omega_\alpha, \mathcal{F}_\alpha, P_\alpha; L^\infty(0,T;V))$ *for* $1 \le p < \infty$

5) The following equation holds almost surely

$$((u_\alpha(t),\varphi))_V + \nu\int_0^t (u_\alpha(s)+\alpha^2 A u_\alpha(s), A\varphi)\,ds + \int_0^t \langle \widetilde{B}(u_\alpha(s), u_\alpha(s)+\alpha^2 A u_\alpha(s)),\varphi\rangle_{D(A)'}\,ds = ((u^0,\varphi))_V + \int_0^t (F(s,u_\alpha(s)),\varphi)\,ds + \left(\int_0^t G(s,u_\alpha(s))\,dW_\alpha(s),\varphi\right)$$

for all $\varphi \in D(A)$*,* $t \in [0,T]$*.*

Theorem 4. *We assume that (3.3), (3.4) hold and* $u_0 \in V$*. Then problem (3.6) has a weak solution* $(\Omega_\alpha, \mathcal{F}, \{\mathcal{F}_{\alpha t}\}_{0\le t\le T}, P_\alpha, W_\alpha, u_\alpha)$ *in the sense of Definition 4.*
u_α *satisfies the following inequalities:*

$$a)\ E_\alpha \sup_{0\le t\le T} \|u_\alpha(t)\|_V^p \le C_{p,1},$$

$$b)\ E_\alpha \left(\int_0^T |u_\alpha(t)|^2_{D(A)}\,dt\right)^p \le C_{p,2}(\alpha),$$

$$c)\ E_\alpha \sup_{0\le|\theta|\le\delta\le 1} \int_0^T |u_\alpha(t+\theta)-u_\alpha(t)|^2_{D(A)'}\,dt \le C_3(\alpha)\delta,$$

$$d)\ 2\nu E_\alpha \int_0^T (\|u_\alpha(s)\|^2 + \alpha^2|Au_\alpha(s)|^2)\,ds \le C_4,$$

$$e)\ E_\alpha \left(\int_0^T (\|u_\alpha(s)\|^2 + \alpha^2|Au_\alpha(s)|^2)\,ds\right)^p \le C_{p,5},$$

where the constants $C_{p,1}, C_4$ *and* $C_{p,5}$ *are independent of* α *and* $1 \le p < \infty$*. The constants* $C_3(\alpha)$ *and* $C_{p,2}(\alpha)$ *tend to* ∞ *when* α *tends to zero.* E_α *denotes the expectation with respect to* P_α *and* $1 \le p < \infty$*.*

The proof of this theorem was the object of the preceding chapter. It was based on a careful blending of the Galerkin approximation scheme together with deep compactness results of both analytic and probabilistic nature. We note that the existence of strong solution for a similar model under stronger conditions (Lipschtzity of the functions F and G) was obtained in [10].

Remark 3. *We note that the constants in the right-hand sides of estimates a), d) and e) of Theorem 4 are independent of α. This fact plays the key role in the proof of the convergence of solutions of the 3D stochastic Navier-Stokes-α model to the solutions of the 3D stochastic Navier-Stokes system as α approaches zero. Since the existence involved fixed α the explosion of $C_3(\alpha)$ as α approaches zero was not an obstacle in the proof of Theorem 4.*

3.4 Asymptotic behavior of the stochastic 3D Navier-Stokes-α model.

3.4.1 Weak compactness of weak solutions for the stochastic 3D Navier-Stokes-α model.

In the section, we prove the tightness of weak solutions of the stochastic 3D Navier-Stokes- α model as α approaches zero. The crucial point of the proof is to show that the following estimate holds:

$$E_\alpha \sup_{0\le|\theta|\le\delta\le 1} \int_0^T |u_\alpha(t+\theta) - u_\alpha(t)|^2_{D(A)'}\, dt \le C\delta,$$

where C is a constant independent of α. Thereby we sharply improve the estimate c) in Theorem 4. This will require skillful technics and is the object of the following

Lemma 6. *Let u_α be a weak solution for the stochastic 3D Navier-Stokes-α model. We have*

$$E_\alpha \sup_{0\le|\theta|\le\delta\le 1} \int_0^T |u_\alpha(t+\theta) - u_\alpha(t)|^2_{D(A)'}\, dt \le C\delta$$

where C is a constant independent of α; here we extend u_α by 0 outside $[0,T]$.

Proof. We have

$$d(I+\alpha^2 A)u_\alpha + \nu A(u_\alpha + \alpha^2 A u_\alpha)dt + \widetilde{B}(u_\alpha, u_\alpha + \alpha^2 A u_\alpha)\, dt = F(t,u_\alpha)\, dt + G(t,u_\alpha)\, dW. \quad (3.7)$$

We recall that $I + \alpha^2 A$ is an isomorphism from $D(A)$ to H and

$$\|(I+\alpha^2 A)^{-1}\|_{\mathcal{L}(H,H)} \le 1.$$

From (3.7), we have

$$du_\alpha + \nu A u_\alpha dt + (I+\alpha^2 A)^{-1}\widetilde{B}(u_\alpha, v_\alpha)dt = (I+\alpha^2 A)^{-1}F(t,u_\alpha)dt + (I+\alpha^2 A)^{-1}G(t,u_\alpha)\, dW,$$

where $v_\alpha = u_\alpha + \alpha^2 A u_\alpha$.

Owing to the fact that $D(A)' = D\left(A^{-1}\right)$ we have

$$
\begin{aligned}
&|A^{-1}(u_\alpha(t+\theta) - u_\alpha(t))| \qquad (3.8)\\
&\leq \int_t^{t+\theta} \left(|A^{-1}(I+\alpha^2 A)^{-1} F(\tau, u_\alpha(\tau))| + \nu |u_\alpha(\tau)| + |A^{-1}(I+\alpha^2 A)^{-1} \widetilde{B}(u_\alpha(\tau), v_\alpha(\tau))|\right) d\tau + \\
&\left|\int_t^{t+\theta} A^{-1}(I+\alpha^2 A)^{-1} G(\tau, u_\alpha(\tau))\, dW(\tau)\right|.
\end{aligned}
$$

We estimate the first two terms of the right hand side of (3.8)

$$
\begin{aligned}
&|A^{-1}(I+\alpha^2 A)^{-1} \widetilde{B}(u_\alpha(\tau), v_\alpha(\tau))| \\
&\leq |A^{-1} \widetilde{B}(u_\alpha(\tau), v_\alpha(\tau))| \leq C |v_\alpha(\tau)| \|u_\alpha(\tau)\| \\
&\leq C \|u_\alpha(\tau)\| \left(|u_\alpha(\tau)| + \alpha^2 |A u_\alpha(\tau)|\right) \\
&\leq C \{|u_\alpha(\tau)| \|u_\alpha(\tau)\| + \alpha \|u_\alpha(\tau)\| \alpha |A u_\alpha(\tau)|\} \\
&\leq C \left(|u_\alpha(\tau)|^2 + \alpha^2 \|u_\alpha(\tau)\|^2\right)^{\frac{1}{2}} \left(\|u_\alpha(\tau)\|^2 + \alpha^2 |A u_\alpha(\tau)|^2\right)^{\frac{1}{2}},
\end{aligned}
$$

where we have used Lemma 5 (part ii) and Cauchy's inequality.

On the other hand, we have

$$
|A^{-1}(I+\alpha^2 A)^{-1} F(\tau, u_\alpha(\tau))| \leq |A^{-1} F(\tau, u_\alpha(\tau))| \leq C(1 + |u_\alpha(\tau)|).
$$

Squaring the both sides of (3.8) and using the above estimates, we have

$$
\begin{aligned}
&|A^{-1}(u_\alpha(t+\theta) - u_\alpha(t))|^2 \leq C\theta^2 + C_1 \left(\int_t^{t+\theta} |u_\alpha(\tau)|\, d\tau\right)^2 \qquad (3.9)\\
&+ \nu^2 \left(\int_t^{t+\theta} |u_\alpha(\tau)|\, d\tau\right)^2 + C_4 \sup_{\tau\in[0,T]} \left(|u_\alpha(\tau)|^2 + \alpha^2 \|u_\alpha(\tau)\|^2\right) \left(\int_t^{t+\theta} \left(\|u_\alpha(\tau)\|^2 + \alpha^2 |A u_\alpha(\tau)|^2\right)^{\frac{1}{2}} d\tau\right)^2 \\
&+ \left|\int_t^{t+\theta} A^{-1}(I+\alpha^2 A)^{-1} G(\tau, u_\alpha(\tau))\, dW(\tau)\right|^2.
\end{aligned}
$$

For fixed δ, taking the supremum over $\theta \leq \delta$ yields

$$
\begin{aligned}
&\sup_{0\leq\theta\leq\delta} |A^{-1}(u_\alpha(t+\theta) - u_\alpha(t))|^2 \leq C\delta^2 + TC_1\delta^2 \sup_{\tau\in[0,T]} |u_\alpha(\tau)|^2 \\
&+ C_4 \sup_{\tau\in[0,T]} \left(|u_\alpha(\tau)|^2 + \alpha^2 \|u_\alpha(\tau)\|^2\right) \left(\int_t^{t+\delta} \left(\|u_\alpha(\tau)\|^2 + \alpha^2 |A u_\alpha(\tau)|^2\right)^{\frac{1}{2}} d\tau\right)^2 \\
&+ \sup_{0\leq\theta\leq\delta} \left|\int_t^{t+\theta} A^{-1}(I+\alpha^2 A)^{-1} G(\tau, u_\alpha(\tau))\, dW(\tau)\right|^2.
\end{aligned}
$$

Integrating over $t \in [0,T]$ and taking the mathematical expectation, we deduce

$$E_\alpha \sup_{0\le\theta\le\delta} \int_0^T |A^{-1}(u_\alpha(t+\theta)-u_\alpha(t))|^2\,dt \le C\delta^2 + TC\delta^2 E_\alpha \sup_{\tau\in[0,T]} |u_\alpha(\tau)|^2 +$$
$$C_4 E_\alpha \sup_{\tau\in[0,T]} \|u_\alpha(\tau)\|_V^2 \int_0^T \left(\int_t^{t+\delta} \left(\|u_\alpha(\tau)\|^2+\alpha^2|Au_\alpha(\tau)|^2\right)^{\frac12} d\tau\right)^2 dt +$$
$$E_\alpha \int_0^T \sup_{0\le\theta\le\delta} \left| \int_t^{t+\theta} A^{-1}(I+\alpha^2 A)^{-1} G(\tau, u_\alpha(\tau))\,dW(\tau)\right|^2 dt.$$

By Hölder's inequality, we have

$$E_\alpha \sup_{\tau\in[0,T]} \|u_\alpha(\tau)\|_V^2 \int_0^T \left(\int_t^{t+\delta} \left(\|u_\alpha(\tau)\|^2+\alpha^2|Au_\alpha(\tau)|^2\right)^{\frac12} d\tau\right)^2 dt$$
$$\le \delta^2 E_\alpha \sup_{\tau\in[0,T]} \|u_\alpha(\tau)\|_V^2 \int_0^T \left(\|u_\alpha(\tau)\|^2+\alpha^2|Au_\alpha(\tau)|^2\right) d\tau$$
$$\le \delta^2 \left(E_\alpha \sup_{\tau\in[0,T]} \|u_\alpha(\tau)\|_V^4\right)^{\frac12} \left(E_\alpha \left(\int_0^T \left(\|u_\alpha(\tau)\|^2+\alpha^2|Au_\alpha(\tau)|^2\right) d\tau\right)^2\right)^{\frac12}.$$

Using the estimates a) and e) of Theorem 4, we have

$$E_\alpha \sup_{\tau\in[0,T]} \|u_\alpha(\tau)\|_V^2 \int_0^T \left(\int_t^{t+\delta} \left(\|u_\alpha(\tau)\|^2+\alpha^2|Au_\alpha(\tau)|^2\right)^{\frac12} d\tau\right)^2 dt \le C\delta^2$$

where C is a constant independent of α.

Next, using the martingale inequality, we have

$$E_\alpha \int_0^T \sup_{0\le\theta\le\delta} \left| \int_t^{t+\theta} A^{-1}(I+\alpha^2 A)^{-1} G(s, u_\alpha(s))\,dW(s)\right|^2 dt$$
$$\le E_\alpha \int_0^T \left(\int_t^{t+\delta} |A^{-1}(I+\alpha^2 A)^{-1} G(s,u_\alpha(s))|^2\,ds\right) dt$$
$$\le CE_\alpha \int_0^T \left(\int_t^{t+\delta} (1+|u_\alpha(s)|^2)\,ds\right) dt$$
$$\le C\delta.$$

Collecting the two last estimates, we finally obtain

$$E_\alpha \sup_{0\le|\theta|\le\delta\le1} \int_0^T |u_\alpha(t+\theta)-u_\alpha(t)|^2_{D(A)'}\,dt \le C\delta,$$

where C is a constant independent of α. □

Remark 4. *From estimate a) of Theorem 4, we also have*

$$E_\alpha \sup_{t\in[0,T]} |u_\alpha(t)|^p \le C_{p,1}.$$

Using the Poincaré's inequality and the estimate e) of Theorem 4, we also have

$$E_\alpha\left(\int_0^T \|u_\alpha(s)\|_V^2\,ds\right)^p \leq CC_{p,5}$$

where C is a constant independent of α and $1 \leq p < \infty$.

The following compactness result plays a crucial role in the proof of the tightness of the probability measures generated by the sequence $\{u_\alpha\}_{\alpha>0}$.

Lemma 7. *Let ν_n and μ_n two sequences of positives real numbers which tend to 0 as $n \to \infty$. The injection of* [1]

$$\mathcal{D} = \left\{ q \in L^\infty(0,T;H) \cap L^2(0,T;V); \sup_n \frac{1}{\nu_n} \sup_{|\theta|\leq\mu_n} \left(\int_0^T |q(t+\theta)-q(t)|_{D(A)'}^2\,dt\right)^{\frac{1}{2}} < \infty \right\}$$

in $L^2(0,T;H)$ is compact.

Proof. See Appendix A, Proposition 6. Take $B_0 = D(A), B_1 = V, B_2 = H$. □

$\mathcal{D}_{\mu_n,\nu_n}$ is a Banach space with the norm

$$\|y\|_{\mathcal{D}_{\mu_n,\nu_n}} = ess\sup_{0\leq t\leq T}|y(t)| + \left(\int_0^T \|y(t)\|_V^2\,dt\right)^{\frac{1}{2}} + \sup_n \frac{1}{\nu_n}\sup_{|\theta|\leq\mu_n}\left(\int_0^T |y(t+\theta)-y(t)|_{D(A)'}^2\,dt\right)^{\frac{1}{2}}.$$

Alongside $\|y\|_{\mathcal{D}_{\mu_n,\nu_n}}$, we also consider the space Z_{p,μ_n,ν_n} $(1 \leq p < \infty)$ of random variables y such that

$$E_\alpha ess\sup_{0\leq t\leq T}|y(t)|^p < \infty; \quad E_\alpha\left(\int_0^T \|y(t)\|_V^2 dt\right)^{\frac{p}{2}} < \infty; \quad E_\alpha \sup_n \frac{1}{\nu_n}\sup_{|\theta|\leq\mu_n}\int_0^T |y(t+\theta)-y(t)|_{D(A)'}^2 dt < \infty.$$

Endowed with the norm

$$\|y\|_{Z_{p,\nu_n,\mu_n}} = \left(E_\alpha ess\sup_{0\leq t\leq T}|y(t)|^p\right)^{\frac{1}{p}} + \left(E_\alpha\left(\int_0^T \|y(t)\|_V^2 dt\right)^{\frac{p}{2}}\right) +$$
$$+ E_\alpha \sup_n \frac{1}{\nu_n}\left(\sup_{|\theta|\leq\mu_n}\int_0^T |y(t+\theta)-y(t)|_{D(A)'}^2 dt\right)^{\frac{1}{2}}$$

Z_{p,μ_n,ν_n} is a Banach space.
Combining the estimates a); e) of Theorem 4 and the estimate of Lemma 6, we have

Proposition 4. *For any real $p \geq 1$ and for any sequences μ_n, ν_n converging to zero such that the series $\sum_{n=1}^\infty \frac{\sqrt{\mu_n}}{\nu_n}$ converges, the sequence $(u_\alpha)_{\alpha>0}$ is bounded in Z_{p,μ_n,ν_n} uniformly in α for all n.*

[1] q is extended by 0 outside (0,T)

Next we define

$$S_1 = C(0,T;\mathbb{R}^m) \times L^2(0,T;H)$$

equipped with the Borel σ-algebra $\mathcal{B}(S_1)$.

For $\alpha > 0$, let

$$\Phi_\alpha : \Omega_\alpha \to S_1 : \omega \rightarrowtail (W_\alpha(\omega,.), u_\alpha(\omega,.)) .$$

For each $\alpha > 0$, we introduce the probability measure Π_α on $(S_1, \mathcal{B}(S_1))$ by

$$\Pi_\alpha(A) = P_\alpha(\Phi_\alpha^{-1}(A)),$$

where $A \in \mathcal{B}(S_1)$.

We have

Theorem 5. *The family of probability measures $\{\Pi_\alpha; \alpha > 0\}$ is tight in $(S_1, \mathcal{B}(S_1))$.*

Proof. For $\varepsilon > 0$, we should find the compact subsets

$$\Sigma_\varepsilon^{'} \subset C(0,T;R^m);\; Y_\varepsilon^{'} \subset L^2(0,T;H)$$

such that

$$P_\alpha(\omega : W_\alpha(\omega,.) \notin \Sigma_\varepsilon^{'}) \leq \frac{\varepsilon}{2}, for\ all\ \alpha > 0, \tag{3.10}$$

$$P_\alpha(\omega : u_\alpha(\omega,.) \notin Y_\varepsilon^{'}) \leq \frac{\varepsilon}{2}, for\ all\ \alpha > 0. \tag{3.11}$$

The quest for $\Sigma_\varepsilon^{'}$ is made by taking into account some facts about the Wiener process such as the formula

$$E_\alpha|W_\alpha(t_2) - W_\alpha(t_1)|^{2j} = (2j-1)!(t_2-t_1)^j, j = 1,2,.. \tag{3.12}$$

for all $\alpha > 0$.

For a constant L_ε depending on ε to be chosen later and $n \in \mathbf{N}$, we consider the set

$$\Sigma_\varepsilon^{'} = \{W(.) \in C(0,T;\mathbf{R}^m) : \sup_{t_1,t_2 \in [0,T], |t_2 - t_1| \leq \frac{1}{n^6}} n|W(t_2) - W(t_1)| \leq L_\varepsilon\}$$

$\Sigma_\varepsilon^{'}$ is relatively compact in $C(0,T;R^m)$ by Arzsela- Ascoli's Theorem. Furthermore $\Sigma_\varepsilon^{'}$ is closed in $C(0,T;R^m)$. Therefore $\Sigma_\varepsilon^{'}$ is a compact subset of $C(0,T;R^m)$.

Making use of Markov's inequality:

$$P(\omega : \xi(\omega) \geq \beta) \leq \frac{1}{\beta^k} E\left[|\xi(\omega)|^k\right]$$

for a random variable ξ on *any probability space*(Ω, F, P) and positives numbers α and k, we get

$$\begin{aligned}P_\alpha(\omega : W_\alpha(\omega,.) \notin \Sigma_\varepsilon') \leq & P_\alpha\left[\bigcup_n \{\omega : \sup_{t_1,t_2\in[0,T]:|t_2-t_1|<\frac{1}{n^6}} |W_\alpha(t_2) - W_\alpha(t_1)| > \frac{L_\varepsilon}{n}\}\right] \\ \leq & \sum_{n=1}^{\infty}\sum_{i=0}^{n^6-1}\left(\frac{n}{L_\varepsilon}\right)^4 E_\alpha \sup_{\frac{iT}{n^6}\leq t\leq\frac{(i+1)T}{n^6}} |W_\alpha(t) - W_\alpha(iTn^{-6})|^4 \\ \leq & c\sum_{n=1}^{\infty}\left(\frac{n}{L_\varepsilon}\right)^4 (Tn^{-6})^2 n^6 \\ = & \frac{c}{L_\varepsilon^4}\sum_{n=1}^{\infty}\frac{1}{n^2}\end{aligned}$$

where we have used (3.12) and the constant c is independent of α.
We choose

$$L_\varepsilon^4 = \frac{1}{2c\varepsilon}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{-1}$$

to get (3.10). Here we note that the independence in α is due to the fact that the right hand side of (3.12) is independent of α.
Next, we choose Y_ε' as a ball of radius M_ε in $\mathcal{D}_{\mu_n,\nu_n}$ centered at zero and with μ_n, ν_n independent of ε, converging to zero and such that the series $\sum_{n=1}^{\infty}\frac{\sqrt{\mu_n}}{\nu_n}$ converges. From Lemma 7, Y_ε' is a compact subset of $L^2(0,T;H)$.
We have further

$$\begin{aligned}P_\alpha(\omega : u_\alpha(\omega,.) \notin Y_\varepsilon') \leq & P_\alpha(\omega : \|u_\alpha\|_{\mathcal{D}_{\mu_n,\nu_n}} > M_\varepsilon) \\ \leq & \frac{1}{M_\varepsilon}E_\alpha\|u_\alpha\|_{\mathcal{D}_{\mu_n,\nu_n}} \\ \leq & \frac{1}{M_\varepsilon}\|u_\alpha\|_{Z_{1,\mu_n,\nu_n}} \\ \leq & \frac{C}{M_\varepsilon}\end{aligned}$$

where C is a constant independent of α (see Proposition 4 for more details).
Choosing $M_\varepsilon = 2C\varepsilon^{-1}$, we get (3.11).
From (3.10) *and* (3.11), we have

$$P_\alpha(\omega : W_\alpha(\omega,.) \in \Sigma_\varepsilon'; u_\alpha(\omega,.) \in Y_\varepsilon') \geq 1 - \varepsilon$$

for all $\alpha > 0$ and this proves that

$$\Pi_\alpha(\Sigma_\varepsilon' \times Y_\varepsilon') \geq 1 - \varepsilon$$

for all $\alpha > 0$. This completes the proof of the tightness of $\{\Pi_\alpha; \alpha > 0\}$ in $(S_1, \mathcal{B}(S_1))$. $\square$

3.4.2 Approximation of the stochastic 3D Navier-Stokes equations

In this section, we prove that the weak solutions of the stochastic 3D Navier-Stokes equations is obtained from a sequence of solutions of the stochastic 3D Navier-Stokes α model as α approaches zero.

Application of Prokhorov's and Skorokhod's results

From the tightness property of $\{\Pi_\alpha; \alpha > 0\}$ and Prokhorov's theorem , we have that there exists a subsequence $\{\Pi_{\alpha_j}\}$ and a measure Π such that $\Pi_{\alpha_j} \to \Pi$ weakly. By Skorokhod's theorem , there exist a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ and random variables $(\widetilde{W}_{\alpha_j}, \tilde{u}_{\alpha_j})$, $(\widetilde{W}, \tilde{u})$ on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ with values in S_1 such that:

$$\text{the law of } (\widetilde{W}_{\alpha_j}, \tilde{u}_{\alpha_j}) \text{ is } \Pi_{\alpha_j},$$

$$\text{the law of } (\widetilde{W}, \tilde{u}) \text{ is } \Pi,$$

$$(\widetilde{W}_{\alpha_j}, \tilde{u}_{\alpha_j}) \to (\widetilde{W}, \tilde{u}) \quad \text{in} \quad S_1, \quad \bar{P} - a.s.. \tag{3.13}$$

Hence $\{\widetilde{W}_{\alpha_j}\}$ is a sequence of an m-dimensional standard Wiener process.
Let

$$\bar{\mathcal{F}}_t = \sigma\{\widetilde{W}(s), \tilde{u}(s) : s \le t\}.$$

Arguing as in the proof of Theorem 1, step 5, Chapter 2 , we can prove that $\widetilde{W}$ is an m-dimensional $\bar{\mathcal{F}}_t$-standard Wiener process and the pair $(\widetilde{W}_{\alpha_j}, \tilde{u}_{\alpha_j})$ satisfies

$$\begin{aligned} & (\tilde{v}_{\alpha_j}(t), \varphi) + \nu \int_0^t (\tilde{v}_{\alpha_j}(s), A\varphi)\, ds + \int_0^t \langle \widetilde{B}(\tilde{u}_{\alpha_j}(s), \tilde{v}_{\alpha_j}(s)), \varphi \rangle_{D(A)'}\, ds \\ = \; & (u_0 + \alpha_j^2 A u_0, \varphi) + \int_0^t (F(s, \tilde{u}_{\alpha_j}(s)), \varphi)\, ds + \left(\int_0^t G(s, \tilde{u}_{\alpha_j}(s))\, d\widetilde{W}_{\alpha_j}(s), \varphi \right), \end{aligned} \tag{3.14}$$

for all $\varphi \in D(A)$, where

$$\tilde{v}_{\alpha_j}(s) = \tilde{u}_{\alpha_j}(s) + \alpha_j^2 A \tilde{u}_{\alpha_j}(s).$$

The main result of this chapter is

Theorem 6. *Assume that (3.3) - (3.4) hold, and $u_0 \in V$. Then there is a subsequence of $\tilde{u}_{\alpha_j}$, $\tilde{v}_{\alpha_j}$ (obtained above) such that as $\alpha_j \to 0$, we have :*

$$\tilde{u}_{\alpha_j} \to \tilde{u} \quad \textit{strongly in} \quad L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^2(0,T;H)),$$

$$\tilde{u}_{\alpha_j} \to \tilde{u} \quad \textit{weakly in} \quad L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^\infty(0,T;H)),$$

$$\tilde{u}_{\alpha_j} \to \tilde{u} \quad \textit{weakly in} \quad L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^2(0,T;V)),$$

$$\tilde{v}_{\alpha_j} \to \tilde{u} \quad \textit{strongly in} \quad L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^2(0,T;V')),$$

$$\tilde{v}_{\alpha_j} \to \tilde{u} \quad \textit{weakly in} \quad L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^2(0,T;H)),$$

where $(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t)_{t\in[0,T]}, \bar{P}, \widetilde{W}, \tilde{u})$ *is a weak solution for the* 3*D stochastic Navier-Stokes equations with the initial value* $u(0) = u_0$.

Proof. From (3.14), it follows that $\tilde{u}_{\alpha_j}$ satisfies the estimates

$$\widetilde{\mathbb{E}} \sup_{0\le s\le T} |\tilde{u}_{\alpha_j}(s)|^p \le C_{p,1}; \qquad \widetilde{\mathbb{E}} \int_0^T \|\tilde{u}_{\alpha_j}(s)\|_V^2 \, ds \le CC_{p,5}$$

$$\widetilde{\mathbb{E}} \sup_{0\le s\le T} \|\tilde{u}_{\alpha_j}(s)\|_V^p \le C_{p,1},$$

$$\widetilde{\mathbb{E}} \sup_{0\le\theta\le\delta} \int_0^T |\tilde{u}_{\alpha_j}(t+\theta) - \tilde{u}_{\alpha_j}(t)|_{D(A)'}^2 \, dt \le C\delta$$

$$\widetilde{\mathbb{E}} \left(\int_0^T \left(\|\tilde{u}_{\alpha_j}(s)\|^2 + \alpha^2 |A\tilde{u}_{\alpha_j}(s)|^2 \right) \, ds \right)^p \le C_{p,5}$$

$$\widetilde{\mathbb{E}} \sup_{0\le s\le T} \|\tilde{u}_{\alpha_j}(s)\|_V^2 + 2\nu \widetilde{\mathbb{E}} \int_0^T \left(\|\tilde{u}_{\alpha_j}(s)\|^2 + \alpha_j^2 |A\tilde{u}_{\alpha_j}(s)|^2 \right) \, ds \le C_4,$$

where $\widetilde{\mathbb{E}}$ denote the mathematical expectation with respect to the probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$. Thus modulo the extraction of a subsequence denoted again $\tilde{u}_{\alpha_j}$, we have

$$\tilde{u}_{\alpha_j} \to \tilde{u} \quad \text{weakly in} \quad L^p(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^\infty(0,T;H)),$$

$$\tilde{u}_{\alpha_j} \to \tilde{u} \quad \text{weakly in} \quad L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^2(0,T;V)),$$

$$\tilde{v}_{\alpha_j} \to \tilde{v} \quad \text{weakly in} \quad L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^2(0,T;H)),$$

and

$$\widetilde{\mathbb{E}} \sup_{0\le s\le T} |\tilde{u}(s)|^p \le C_{p,1}; \qquad \widetilde{\mathbb{E}} \int_0^T \|\tilde{u}\|_V^2 \, ds \le CC_{p,5}$$

$$\widetilde{\mathbb{E}} \sup_{0\le\theta\le\delta} \int_0^T |\tilde{u}(t+\theta) - \tilde{u}(t)|_{D(A)'}^2 \, dt \le C\delta.$$

By Vitali's theorem and (3.13), we have

$$\tilde{u}_{\alpha_j} \to \tilde{u} \quad \text{in} \quad L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^2(0,T;H)). \tag{3.15}$$

Thus modulo the extraction of a new subsequence and for almost every (ω, t) with respect to the measure $d\bar{P} \otimes dt$

$$u_{\alpha_j} \to \tilde{u} \text{ in } H.$$

This convergence together with the conditions on F and Vitali's theorem, imply

$$\int_0^t F(s,\tilde{u}_{\alpha_j}(s))\,ds \to \int_0^t F(s,\tilde{u}(s))\,ds \text{ in } L^2(\bar{\Omega},\bar{\mathcal{F}},\bar{P};L^2(0,T;D(A)'))$$

Arguing as in the proof of Theorem 1, step 5, Chapter 3 we can prove that

$$\int_0^t G(s,\tilde{u}_{\alpha_j}(s))\,d\widetilde{W}_{\alpha_j}(s) \to \int_0^t G(s,\tilde{u}(s))\,d\widetilde{W}(s) \text{ weakly star in } L^2(\bar{\Omega},\bar{\mathcal{F}},\bar{P};L^\infty(0,T;D(A)')).$$

We also have

$$\widetilde{\mathbb{E}}\int_0^T |A^{-\frac{1}{2}}(\tilde{v}_{\alpha_j}(t)-\tilde{u}_{\alpha_j}(t))|^2\,dt = \alpha_j^2\widetilde{\mathbb{E}}\int_0^T \alpha_j^2\|\tilde{u}_{\alpha_j}(t)\|^2\,dt.$$

We then deduce that

$$\tilde{v}_{\alpha_j} \to \tilde{u} \text{ in } L^2(\bar{\Omega},\bar{\mathcal{F}},\bar{P};L^2(0,T;V')) \tag{3.16}$$

and $\tilde{v}(t)=\tilde{u}(t)$ a.e. in $d\bar{P}\times dt$ since $\widetilde{E}\int_0^T \alpha_j^2|A\tilde{u}_{\alpha_j}(t)|^2\,dt$ is bounded uniformly in α_j. Arguing as in [20], we are going to prove that

$$\int_0^t \widetilde{B}(\tilde{u}_{\alpha_j}(s),\tilde{v}_{\alpha_j}(s))\,ds \to \int_0^t B(\tilde{u}(s),\tilde{u}(s))\,ds \text{ weakly in } L^\beta(\bar{\Omega},\bar{\mathcal{F}},\bar{P};L^\beta(0,T;D(A)'))$$

for some $1<\beta<2$.

Indeed, it suffices to prove that

$$\widetilde{B}(\tilde{u}_{\alpha_j},\tilde{v}_{\alpha_j}) \to \widetilde{B}(\tilde{u},\tilde{u}) = B(\tilde{u},\tilde{u}) \text{ weakly in } L^\beta(\bar{\Omega},\bar{\mathcal{F}},\bar{P};L^\beta(0,T;D(A)'))$$

for some $1<\beta<2$.

We recall that

$$\tilde{v}_{\alpha_j} = \tilde{u}_{\alpha_j} + \alpha_j^2 A\tilde{u}_{\alpha_j}$$

and

$$\begin{aligned}\widetilde{B}(\tilde{u}_{\alpha_j},\tilde{v}_{\alpha_j}) &= \widetilde{B}(\tilde{u}_{\alpha_j},\tilde{u}_{\alpha_j}) + \alpha_j^2\widetilde{B}(\tilde{u}_{\alpha_j},A\tilde{u}_{\alpha_j})\\ &= B(\tilde{u}_{\alpha_j},\tilde{u}_{\alpha_j}) + \alpha_j^2\widetilde{B}(\tilde{u}_{\alpha_j},A\tilde{u}_{\alpha_j}).\end{aligned}$$

We are going to prove that for $1<\beta<2$,

$$\alpha_j^2\widetilde{B}(\tilde{u}_{\alpha_j},A\tilde{u}_{\alpha_j}) \to 0 \text{ strongly in } L^\beta(\bar{\Omega},\bar{\mathcal{F}},\bar{P};L^\beta(0,T;D(A)')) \text{ as } \alpha_j \to 0, \tag{3.17}$$

and

$$B(\tilde{u}_{\alpha_j},\tilde{u}_{\alpha_j}) \to B(\tilde{u},\tilde{u}) \text{ weakly in } L^2(\bar{\Omega},\bar{\mathcal{F}},\bar{P};L^2(0,T;D(A)')) \text{ as } \alpha_j \to 0.$$

We start with (3.17). By part ii) of Lemma 5, we have

$$\|\alpha_j^2\widetilde{B}(\tilde{u}_{\alpha_j},A\tilde{u}_{\alpha_j})\|_{D(A)'} \le C\alpha_j^2\|\tilde{u}_{\alpha_j}\||A\tilde{u}_{\alpha_j}|.$$

Fixing an arbitrary $\beta, 1 < \beta < 2$, we obtain the following chain of inequalities

$$\begin{aligned}
&\int_0^T \|\alpha_j^2 \widetilde{B}(\tilde{u}_{\alpha_j}(t), A\tilde{u}_{\alpha_j}(t)\|^\beta_{D(A)'}\, dt \qquad (3.18)\\
&\leq C^\beta \alpha_j^{2\beta} \int_0^T \|\tilde{u}_{\alpha_j}(t)\|^\beta |A\tilde{u}_{\alpha_j}(t)|^\beta \, dt\\
&\leq C^\beta \alpha_j^{2\beta} \left(\sup_{0\leq t\leq T} \|\tilde{u}_{\alpha_j}(t)\|^\gamma\right) \int_0^T \|\tilde{u}_{\alpha_j}(t)\|^{\beta-\gamma} |A\tilde{u}_{\alpha_j}(t)|^\beta \, dt\\
&\leq C^\beta \alpha_j^{2\beta} \left(\sup_{0\leq t\leq T} \|\tilde{u}_{\alpha_j}(t)\|^\gamma\right) \left[\int_0^T \|\tilde{u}_{\alpha_j}(t)\|^{q(\beta-\gamma)}\, dt\right]^{\frac{1}{q}} \left[\int_0^T |A\tilde{u}_{\alpha_j}(t)|^{p\beta}\, dt\right]^{\frac{1}{p}}
\end{aligned}$$

where γ is an arbitrary number such that $0 < \gamma < \beta$, and, in (3.18), we have applied the Hölder's inequality with $\frac{1}{p} + \frac{1}{q} = 1$ (these numbers will be determined later on).
Continuing the chain of inequalities after, we have

$$\begin{aligned}
&\int_0^T \|\alpha_j^2 \widetilde{B}(\tilde{u}_{\alpha_j}(t), A\tilde{u}_{\alpha_j}(t)\|^\beta_{D(A)'}\, dt \qquad (3.19)\\
&\leq C^\beta \alpha_j^{2\beta} \left(\sup_{0\leq t\leq T} \|\tilde{u}_{\alpha_j}(t)\|^2\right)^{\frac{\gamma}{2}} \left[\int_0^T \|\tilde{u}_{\alpha_j}(t)\|^{q(\beta-\gamma)}\, dt\right]^{\frac{1}{q}} \left[\int_0^T |A\tilde{u}_{\alpha_j}(t)|^{p\beta}\, dt\right]^{\frac{1}{p}}.
\end{aligned}$$

We now set $p = \frac{2}{\beta}, q = \frac{2}{2-\beta}$. Let the number γ satisfies the equation $q(\beta - \gamma) = 2$, that is,

$$\frac{2}{2-\beta}(\beta - \gamma) = 2 \iff \gamma = 2(\beta - 1).$$

We see that the inequality $0 < \gamma < \beta$ holds since

$$\gamma = 2(\beta - 1) < \beta \iff \beta < 2.$$

Replacing such p, q, and γ into (3.19), we obtain the following estimate:

$$\begin{aligned}
&\int_0^T \|\alpha_j^2 \widetilde{B}(\tilde{u}_{\alpha_j}(t), A\tilde{u}_{\alpha_j}(t))\|^\beta_{D(A)'}\, dt \qquad (3.20)\\
&\leq C^\beta \alpha_j^{2-\beta} \left(\sup_{0\leq t\leq T} \alpha_j^2 \|\tilde{u}_{\alpha_j}(t)\|^2\right)^{\beta-1} \left[\int_0^T \|\tilde{u}_{\alpha_j}(t)\|^2\, dt\right]^{\frac{2-\beta}{2}} \left[\int_0^T \alpha_j^2 |A\tilde{u}_{\alpha_j}(t)|^2\, dt\right]^{\frac{\beta}{2}}.
\end{aligned}$$

Taking the mathematical expectation in (3.20) and using Hölder′s inequality, we have

$$\begin{aligned}
&\widetilde{\mathbb{E}} \int_0^T \|\alpha_j^2 \widetilde{B}(\tilde{u}_{\alpha_j}, A\tilde{u}_{\alpha_j}(t))\|^\beta_{D(A)'}\, dt\\
&\leq C^\beta \alpha_j^{2-\beta} \left[\widetilde{\mathbb{E}}\left(\sup_{0\leq t\leq T} \alpha_j^2 \|\tilde{u}_{\alpha_j}(t)\|^2\right)\right]^{\beta-1} \left(\widetilde{\mathbb{E}}\left[\int_0^T \|\tilde{u}_{\alpha_j}(t)\|^2\, dt\right]\right)^{\frac{2-\beta}{2}} \left[\widetilde{\mathbb{E}}\left(\int_0^T \alpha_j^2 |A\tilde{u}_{\alpha_j}(t)|^2\, dt\right)^{\frac{\beta}{2-\beta}}\right]^{\frac{2-\beta}{2}}.
\end{aligned}$$

Using the estimates on $\tilde{u}_{\alpha_j}$ at the beginning of the proof, we then have

$$\widetilde{\mathbb{E}} \int_0^T \|\alpha_j^2 \widetilde{B}(\tilde{u}_{\alpha_j}(t), A\tilde{u}_{\alpha_j}(t))\|^\beta_{D(A)'}\, dt \leq C\alpha_j^{2-\beta},\ 1 < \beta < 2.$$

Therefore, as $\alpha_j \to 0$ the term

$$\alpha_j^2 \widetilde{B}(\tilde{u}_{\alpha_j}, A\tilde{u}_{\alpha_j}) \to 0 \;\text{ strongly in } L^\beta(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^\beta(0,T; D(A)')).$$

We readily have from (3.15)

$$B(\tilde{u}_{\alpha_j}, \tilde{u}_{\alpha_j}) \to B(\tilde{u}, \tilde{u}) \text{ weakly } \text{ in } \; L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^2(0,T; D(A)')).$$

Collecting all the convergence results, we then have from (3.14) that

$$(\tilde{u}(t), \varphi) + \nu \int_0^t (\tilde{u}(s), A\varphi)\, ds + \int_0^t \langle B(\tilde{u}(s), \tilde{u}(s)), \varphi \rangle_{D(A)'}\, ds = \\ (u_0, \varphi) + \int_0^t (F(s, \tilde{u}(s)), \varphi)\, ds + \left(\int_0^t G(s, \tilde{u}(s))\, d\widetilde{W}(s), \varphi \right)$$

for all $\varphi \in D(A)$. This concludes the proof of Theorem 6. $\square$

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Chapter 4

Strong solution for the 3D Stochastic Leray-α Model

4.1 Introduction

It is computationally expensive to perform reliable direct numerical simulation of the Navier-Stokes equations for high Reynolds number flows due to the wide range of scales of motion that need to be resolved. The use of numerical models allows researchers to simulate turbulent flows using smaller computational resources. In this chapter, we study a particular sub-grid scale turbulence model known as the Leray-alpha model (Leray-α). This model together with the Navier-Stokes-α model which was considered in the previous chapters, are strong contenders for deep understanding of turbulence in Newtonian fluids.

We are interested in the study of the probabilistic strong solutions of the $3D$ Leray-α equations, subject to space periodic boundary conditions, in the case in which random perturbations appear. To be more precise, let $\mathcal{T} = [0, L]^3$, $T > 0$, and consider the system

$$\begin{cases} d(u - \alpha^2 \Delta u) + \left[-\nu\Delta(u - \alpha^2\Delta u) - u.\nabla(u - \alpha^2 \Delta u) + \nabla p\right] dt = \\ \qquad\qquad F(t, u)dt + G(t, u)dW \quad \text{in} \quad (0, T) \times \mathcal{T}, \\ \nabla.u = 0 \quad \text{in} \quad (0, T) \times \mathcal{T}, \\ u(t, x) \quad \text{is} \quad \text{periodic} \quad \text{in} \quad x \quad \text{and} \quad \int_{\mathcal{T}} u\, dx = 0, \\ u(0) = u_0 \quad \text{in} \quad \mathcal{T}, \end{cases} \tag{4.1}$$

where $u = (u_1, u_2, u_3)$ and p are unknown random fields on $[0, T] \times \mathcal{T}$, representing, respectively, the velocity and the pressure, at each point of $[0, T] \times \mathcal{T}$, of an incompressible viscous fluid with constant density filling the domain $\mathcal{T}$. The constant $\nu > 0$ and α represent, respectively, the kinematic viscosity of the fluid and spatial scale at which fluid motion is filtered. The terms $F(t,u)$ and $G(t,u)dW$ are external forces depending on u, where W is an R^m-valued standard Wiener process. Finally, u_0 is a given random initial velocity field.

The deterministic version of (4.1), i.e. when $G = 0$, has been the object of intense investigation over the last years. The initial motivation was to find a closure model for the 3D turbulence averaged Reynolds model; for more details we refer to [21] and the references therein. A key interest in the model is the fact that it serves as a good approximation of the 3D Navier-Stokes equations. It is readily seen that when $\alpha = 0$, the problem reduces to the usual 3D Navier-Stokes equations. Many important results have been obtained in the deterministic case. More precisely, the global well-posedness of weak solutions for the deterministic Leray-α equations has been established in [75] and also their relation with Navier-Stokes equations as α approaches zero. The global attractor was constructed in [19] and [21].

The addition of white noise driven terms to the basic governing equations for a physical system is natural for both practical and theoretical applications. For example, these stochastically forced terms can be used to account for numerical and empirical uncertainties and thus provide a means to study the robustness of a basic model. Specifically in the context of fluids, complex phenomena related to turbulence may also be produced by stochastic perturbations. For instance, in the recent work of Mikulevicius and Rozovskii [57], such terms are shown to arise from basic physical principals. To the best of our knowledge, there is no systematic work for the 3D stochastic Leray-α model.

In this chapter, we shall prove the existence and uniqueness of strong solutions to our stochastic Leray-α equations under appropriate conditions on the data, by approximating it by means of the Galerkin method (see Theorem 7). Here, the word "strong" means "strong" in the sense of the theory of stochastic differential equations, assuming that the stochastic processes are defined on a complete probability space and the Wiener process is given in advance. Since we consider the strong solution of the stochastic Leray-alpha equations, we do not need to use the techniques considered in the case of weak solutions as we did in the previous chapters of the present thesis. The techniques applied in this chapter use in particular the properties of stopping times and some basic convergence principles from functional analysis (see [76],[5]). An important result, which cannot be proved in the case of weak solutions, is that the Galerkin approximation converge in mean square to the solution of the stochastic Leray-α equations (see Theorem 8). We can prove such result by using the property of higher order moments for the solution. Moreover, as in the deterministic case [75], we study the asymptotic behavior of the solutions as α approaches 0. More precisely, we show that a subsequence of solutions in question converges to a probabilistic weak solution for the 3D stochastic Navier-Stokes equations (see Theorem 9). This is reminiscent of the vanishing viscosity method; see for instance [6], [67].

This chapter is organized as follows. In Section 4.2, we formulate the problem and state the first result on the existence and uniqueness of strong solutions for the 3D stochastic Leray-α model. In Section 4.3, we introduce the Galerkin approximation of our problem and derive crucial a priori estimates for its solutions. Section 4.4 is devoted to the proof of the existence and uniqueness of strong solutions for the 3D stochastic Leray-α model. In Section 4.5, We prove the convergence result of Theorem 8. In Section 4.6, we study the asymptotic behavior of the strong solutions for the 3D stochastic Leray-α model as α approaches 0.

4.2 Statement of the problem and the first main result

Let $\mathcal{T} = [0, L]^3$. We denote by $C^\infty_{per}(\mathcal{T})^3$ the space of all $\mathcal{T}$-periodic C^∞ vector fields defined on $\mathcal{T}$. We set

$$\mathcal{V} = \{\Phi \in C^\infty_{per}(\mathcal{T})^3 / \int_{\mathcal{T}} \Phi \, dx = 0; \nabla.\Phi = 0\}.$$

We denote by H and V the closure of the set $\mathcal{V}$ in the spaces $L^2(\mathcal{T})^3$ and $H^1(\mathcal{T})^3$ respectively. Then H is a Hilbert space equipped with the inner product of $L^2(\mathcal{T})^3$. V is Hilbert space equipped with inner product of $H^1(\mathcal{T})^3$. We denote by $(.,.)$ and $|.|$ the inner product and norm in H. The inner product and norm in V are denoted by $((.,.))$ and $\|.\|$, respectively. Let $A = -\mathcal{P}\Delta$ be the Stokes operator with domain $D(A) = H^2(\mathcal{T})^3 \cap V$, where $\mathcal{P} : L^2(\mathcal{T})^3 \to H$ is the Leray projector. A is an isomorphism from V to V'(the dual space of V) with compact inverse, hence A has eigenvalues$\{\lambda_k\}_{k=1}^\infty, i.e., \frac{4\pi^2}{L^2} = \lambda_1 \le \lambda_2 \le ... \le \lambda_n \to \infty (n \to \infty)$ and corresponding eigenfunctions $\{w_k\}_{k=1}^\infty$ which form an orthonormal basis of H such that $Aw_k = \lambda_k w_k$.
We also have

$$\langle Av, v\rangle_{V'} \ge \beta \|v\|^2 \tag{4.2}$$

for all $v \in V$, where $\beta > 0$ and $\langle ., .\rangle_{V'}$ denotes the duality between V and V'.
Following the notations common in the study of Navier-Stokes equations, we set

$$B(u, v) = \mathcal{P}(u.\nabla)v \quad \text{for all } u, v \in V.$$

Then (see [72],[35],[24])

$$\langle B(u,v),v\rangle_{V'} = 0 \qquad \text{for all } u,v \in V. \tag{4.3}$$

$$\langle B(u,v),w\rangle_{V'} = -\langle B(u,w),v\rangle_{V'} \qquad \text{for all } u,v,w \in V. \tag{4.4}$$

$$|(B(u,v),w)| \leq C|Au|\|v\||w|, \qquad \text{for all } u \in D(A),\ v \in V,\ w \in H. \tag{4.5}$$

$$|\langle B(u,v),w\rangle_{D(A)'} \leq C|u|\|v\||Aw|, \qquad \text{for all } u \in H,\ v \in V,\ w \in D(A). \tag{4.6}$$

$$|\langle B(u,v),w\rangle_{V'}| \leq C|u|^{\frac{1}{4}}\|u\|^{\frac{3}{4}}|v|^{\frac{1}{4}}\|v\|^{\frac{3}{4}}\|w\|, \qquad \text{for all } u \in V,\ v \in V,\ w \in V. \tag{4.7}$$

$$|(B(u,v),w)| \leq C|u|^{\frac{1}{4}}\|u\|^{\frac{3}{4}}\|v\|^{\frac{1}{4}}|Av|^{\frac{3}{4}}|w|, \qquad \text{for all } u \in V,\ v \in D(A),\ w \in \mathrm{H}. \tag{4.8}$$

Let $(\Omega,\mathcal{F},P)$ be a complete probability space and $\{\mathcal{F}_t\}_{0\leq t\leq T}$ an increasing and right-continuous family of sub σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$. Let W be an R^m-valued Wiener process on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{0\leq t\leq T},P)$.$\mathbb{E}$ denotes the mathematical expectation with respect to the probability measure P.
The function spaces used in this chapter are denoted as in the previous chapters.

We make precise our assumptions on F and G. We suppose that F and G are measurable Lipschitz mappings from $\Omega\times(0,T)\times H$ into H and from $\Omega\times(0,T)\times H$ into $H^{\otimes m}$, respectively. Namely, assume that, for all $u,v \in H$, $F(.,u)$ and $G(.,u)$ are $\mathcal{F}_t$-adapted, and $dP\times dt-a.e.$ in $dP\times dt$

$$|F(t,u)-F(t,v)|_H \leq L_F|u-v|, \tag{4.9}$$

$$F(t,0)=0, \tag{4.10}$$

$$|G(t,u)-G(t,v)|_{H^{\otimes m}} \leq L_G|u-v| \tag{4.11}$$

$$G(t,0)=0. \tag{4.12}$$

Finally, we assume that $u_0 \in L^2(\Omega,\mathcal{F}_0,P;D(A))$.

Remark 5. *The condition (4.10) is given only to simplify the calculations. It can be omitted; in which case one could use the estimate*

$$|F(t,u)|^2 \leq 2L_F^2|u|^2 + 2|F(t,0)|^2$$

that follows from the Lipschitz condition. The same remark applies to G.

Alongside problem (4.1), we shall consider the equivalent abstract stochastic evolution equation

$$\begin{cases} d(u+\alpha^2 Au) + [\nu A(u+\alpha^2 Au) + B(u, u+\alpha^2 Au)]\, dt = F(t,u)dt + G(t,u)dW, \\ u(0) = u_0. \end{cases} \tag{4.13}$$

We now define the concept of strong solution of the problem (4.13); namely:

Definition 5. *By a strong solution of problem (4.13), we mean a stochastic process u such that*

1) $u(t)$ is $\mathcal{F}_t$ adapted for all $t \in [0,T]$,

2) $u \in L^p(\Omega, \mathcal{F}, P; L^2(0,T; D(A^{\frac{3}{2}}))) \cap L^p(\Omega, \mathcal{F}, P; L^\infty(0,T, D(A)))$ for all $1 \le p < \infty$,

3) u is weakly continuous with values in $D(A)$,

4) P-a.s., the following integral equation holds

$$\left(u(t)+\alpha^2 Au(t), \Phi\right) + \nu \int_0^t \left(u(s)+\alpha^2 Au(s), A\Phi\right) ds + \int_0^t \left(B(u(s), u(s)+\alpha^2 Au(s)), \Phi\right) ds = \left(u_0+\alpha^2 Au_0, \Phi\right) + \int_0^t \left(F(s,u(s), \Phi\right) ds + \int_0^t \left(G(s,u(s)), \Phi\right) dW(s)$$

for all $\Phi \in \mathcal{V}$, and $t \in [0,T]$.

Our first result is the following

Theorem 7. *(**Existence and uniqueness**) Suppose the hypotheses (4.9)-(4.12) hold, and $u_0 \in L^2(\Omega, \mathcal{F}_0, P; D(A))$. Then problem (4.13) has a solution in the sense of Definition 5. The solution is unique almost surely and has in $D(A)$ almost surely continuous trajectories.*

We also prove that the sequence (u_n) of our Galerkin approximation (see (4.14) below) approximates the solution u of the $3D$ stochastic Leray-α model in mean square.
This is the object of the second result of this chapter.

Theorem 8. *(**Convergence results**) Under the hypotheses of Theorem 7, the following convergences hold:*

$$\mathbb{E}\int_0^t \|u_n(s)-u(s)\|^2_{D(A^{\frac{3}{2}})}\, ds \to 0$$

as $n \to \infty$ and

$$\mathbb{E}\|u_n(t)-u(t)\|^2_{D(A)} \to 0$$

as $n \to \infty$, for all $t \in [0,T]$.

4.3 Galerkin approximation and a priori estimates

We now introduce the Galerkin scheme associated with the original equation (4.13) and establish some uniform estimates.

4.3.1 The approximate equation

Let $\{w_j\}_{j=1}^{\infty}$ be an orthonormal basis of H consisting of eigenfunctions of the operator A. Denote $H_n = span\{w_1, ..., w_n\}$ and let P_n be the L^2-orthogonal projection from H onto H_n.
We look for a sequence $(u_n(t))_n$ of solutions in H_n of the following initial value problem

$$\begin{cases} d\, v_n + [\nu A v_n + P_n B(u_n, v_n)]\; dt = P_n F(t, u_n)dt + P_n G(t, u_n)dW \\ u_n(0) = P_n u_0 \\ v_n = u_n + \alpha^2 A u_n. \end{cases} \tag{4.14}$$

As in Chapter 2, there is a unique continuous $(\mathcal{F}_t)$-adapted process $u_n(t) \in L^2(\Omega, \mathcal{F}, P; L^2(0, T; H_n))$ solutions of problem (4.14) (see [44], [46],[63]). The conditions in Chapter 2 did not guaranty uniqueness. The local Lipschitzity and the linear growth of the nonlinearity provide global unique solution.
We next establish some uniform estimates on u_n and v_n.

4.3.2 A priori estimates

Throughout this section $C,\ C_i(i = 1, ...)$ denote positive constants independent of n and α.

Lemma 8. *u_n and v_n satisfy the following a priori estimates:*

$$\begin{aligned} &\mathbb{E} \sup_{0\le s\le T} |v_n(s)|^2 + 4\nu\beta\mathbb{E}\int_0^T \|v_n(s)\|^2\, ds \le C_1, \\ &\mathbb{E} \sup_{0\le s\le T} |u_n(s)|^2 \le C_2 \qquad ; \qquad \mathbb{E} \sup_{0\le s\le T} \|u_n(s)\|^2 < \frac{C_3}{2\alpha^2}, \\ &\mathbb{E} \sup_{0\le s\le T} |Au_n(s)|^2 \le \frac{C_4}{\alpha^4} \qquad ; \qquad \mathbb{E}\int_0^T \|u_n(s)\|^2\, ds \le C_5, \\ &\mathbb{E}\int_0^T |Au_n(s)|^2\, ds \le \frac{C_6}{2\alpha^2} \qquad ; \qquad \mathbb{E}\int_0^T |A^{\frac{3}{2}} u_n(s)|^2\, ds \le \frac{C_7}{\alpha^4},. \end{aligned} \tag{4.15}$$

Proof. To prove Lemma 8, it suffices to establish the first inequality and use the fact that

$$|v_n|^2 = |u_n + \alpha^2 A u_n|^2 = |u_n|^2 + 2\alpha^2\|u_n\|^2 + \alpha^4 |Au_n|^2,$$

$$\|v_n\|^2 = \|u_n\|^2 + 2\alpha^2|Au_n|^2 + \alpha^4|A^{\frac{3}{2}}u_n|^2.$$

By Itô's formula, we have from (4.14)

$$\begin{aligned} & d|v_n(t)|^2 + 2\left[\nu\langle Av_n, v_n\rangle_{V'} + \langle B(u_n, v_n), v_n\rangle_{V'}\right] dt \\ = & \left((2F(t,u_n), v_n) + |P_nG(t,u_n)|^2\right) dt + 2\left(G(t,u_n), v_n\right) dW. \end{aligned} \tag{4.16}$$

But then, taking into account (4.3), (4.2) and the fact that

$$(F(s,u_n(s)), v_n(s)) \leq C(1+|v_n(s)|^2),$$

$$|P_nG(s,u_n(s))|^2 \leq C(1+|v_n(s)|^2),$$

we deduce from (4.16) that

$$|v_n(t)|^2 + 2\nu\beta\int_0^t \|v_n(s)\|^2\,ds \leq |v_n(0)|^2 + C_2T + C_3\int_0^t |v_n(s)|^2\,ds + 2\int_0^t (G(s,u_n(s)), v_n(s))\;dW(s). \tag{4.17}$$

For each integer $N>0$, consider the $\mathcal{F}_t$-stopping time τ_N defined by

$$\tau_N = \inf\{t : |v_n(t)|^2 \geq N^2\} \wedge T.$$

It follows from (4.17) that

$$\begin{aligned} & \sup_{s\in[0,t\wedge\tau_N}|v_n(s)|^2 + 2\nu\beta\int_0^{t\wedge\tau_N}\|v_n(s)\|^2\,ds \\ \leq\; & |v_n(0)|^2 + C_8T + C_9\int_0^{t\wedge\tau_N}|v_n(s)|^2\,ds + 2\sup_{s\in[0,t\wedge\tau_N}\left|\int_0^s (G(s,u_n(s)), v_n(s))\;dW(s)\right| \end{aligned} \tag{4.18}$$

for all $t\in(0,T)$ and all N, $n\geq 1$. Taking expectation in (4.18) , by Doob's inequality it follows that

$$\begin{aligned} & \mathbb{E}\sup_{s\in[0,t\wedge\tau_N]}\int_0^s (G(s,u_n(s)), v_n(s))\;dW(s) \\ & \leq 3E\left(\int_0^{t\wedge\tau_N}(G(s,u_n(s)), v_n(s))^2\;ds\right)^{\frac{1}{2}} \\ & \leq 3\mathbb{E}\left(\int_0^{t\wedge\tau_N}|G(s,u_n(s))|^2|v_n(s)|^2\,ds\right)^{\frac{1}{2}} \\ & \leq \frac{1}{2}\mathbb{E}\sup_{0\leq s\leq t\wedge\tau_N}|v_n(s)|^2 + C_{10}T + C_{11}\mathbb{E}\int_0^{t\wedge\tau_N}|v_n(s)|^2\,ds. \end{aligned}$$

Next using Gronwall's lemma, we have that there exists a constant C_1 depending on T, C such that, for all $n\geq 1$

$$\mathbb{E}\sup_{0\leq s\leq T}|v_n(s)|^2 + 4\nu\beta\mathbb{E}\int_0^T\|v_n(s)\|^2\,ds \leq C_1.$$

□

The following result is related to the higher integrability of u_n and v_n

Lemma 9. *We have*

$$\mathbb{E} \sup_{0\le s\le T} |v_n(s)|^p \le C_p \quad ; \quad \mathbb{E} \sup_{0\le s\le T} |u_n(s)|^p \le C_p, \tag{4.19}$$

$$\mathbb{E} \sup_{0\le s\le T} \|u_n(s)\|^p \le \frac{C_p}{\alpha^p}, \tag{4.20}$$

$$\mathbb{E} \sup_{0\le s\le T} |u_n(s)|^p_{D(A)} \le \frac{C_p}{\alpha^{2p}}, \tag{4.21}$$

for all $1 \le p < \infty$.

Proof. By Itô's formula, we have for $4 \le p < \infty$

$$\begin{aligned} & d|v_n(t)|^{\frac{p}{2}} \\ = \ & \frac{p}{2}|v_n(t)|^{\frac{p}{2}-2}\left(-\nu\langle Av_n, v_n\rangle_{V'} - \langle B(u_n,v_n),v_n\rangle_{V'} + (F(t,u_n),v_n) + \frac{p-4}{4}\frac{(G(t,u_n),v_n)^2}{|v_n(t)|^2}\right) dt \\ & + \frac{p}{2}|v_n(t)|^{\frac{p}{2}-2}(G(t,u_n),v_n)\, dW. \end{aligned} \tag{4.22}$$

By (4.17), (4.17) and Young's inequality, we have

$$|v_n(s)|^{\frac{p}{2}-2}(F(t,u_n),v_n) \le C(1+|v_n(s)|^{\frac{p}{2}})$$

and

$$\frac{(G(s,u_n),v_n)^2}{|v_n(s)|^2} \le C(1+|v_n(s)|^2),$$

Taking account of these inequalities together with (4.22) and (4.3), we get

$$|v_n(t)|^{\frac{p}{2}} \le |v_n(0)|^{\frac{p}{2}} + C\int_0^t (1+|v_n(s)|^{\frac{p}{2}})\, ds + \frac{p}{2}\int_0^t |v_n(s)|^{\frac{p}{2}-2}(G(s,u_n(s)),v_n(s))\, dW(s). \tag{4.23}$$

Taking the supremum, the square and the mathematical expectation in (4.23), and owing to the martingale inequality we have

$$\begin{aligned} & \mathbb{E} \sup_{0\le s\le T} \left| \int_0^s |v_n(s)|^{\frac{p}{2}-2}(G(s,u_n(s)),v_n(s))\, dW(s)\right|^2 \\ & \le 4\mathbb{E}\int_0^T |v_n(s)|^{p-4}(G(s,u_n(s)),v_n(s))^2\, ds \\ & \le 4C\ \mathbb{E}\int_0^T (1+|v_n(s)|^p)\, ds. \end{aligned}$$

Applying Gronwall's lemma, it follows that there exists a constant C_p, such that

$$\mathbb{E} \sup_{0\le s\le T} |v_n(s)|^p \le C_p$$

for all $p \geq 4$. This being proved for any $p \geq 4$, it is subsequently true for any $1 \leq p < \infty$. Other inequalities are deduced from the relation

$$|v_n(s)|^2 = |u_n(s)|^2 + 2\alpha^2\|u_n(s)\|^2 + \alpha^4|Au_n(s)|^2.$$

□

We also have

Lemma 10.
$$\mathbb{E}\left(\int_0^T \|v_n(s)\|^2\, ds\right)^p \leq C_p \quad \textit{for } 1 \leq p < \infty.$$

Proof. From (4.17), we have

$$(2\nu\beta)^p \left(\int_0^T \|v_n(s)\|^2\, ds\right)^p \leq C_p\left(|v_n(0)|^{2p} + T^p + \left(\int_0^T |v_n(s)|^2\, ds\right)^p\right) + C_p' \sup_{t\in[0,T]}\left|\int_0^t (G(s,u_n(s)), v_n(s))\; dW(s)\right|^p. \quad (4.24)$$

By Burkholder-Gundy's inequality, we have

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t (G(s,u_n(s)), v_n(s))\; dW(s)\right|^p \\
&\leq C_{p1}\mathbb{E}\left(\int_0^T (G(s,u_n(s)), v_n(s))^2\; ds\right)^{\frac{p}{2}} \\
&\leq C_{p2}\mathbb{E}\left(\int_0^T \left(1 + |v_n(s)|^{2p}\right)\; ds\right) \\
&\leq C_{p2}T + C_{p2}T\mathbb{E}\sup_{s\in[0,T]}|v_n(s)|^{2p}.
\end{aligned}$$

Taking the mathematical expectation in (4.24) and using the first inequality of Lemma 9, we have the inequality sought. □

4.4 Proof of Theorem 7

4.4.1 Existence

With the uniform estimates on the solution of the Galerkin approximation in hand, we proceed to identify a limit u. This stochastic process is shown to satisfy a stochastic partial differential equations (see (4.32)) with unknown terms corresponding to the nonlinear portions of the equation. Next, using the properties of stopping times and some basic convergence principles from functional analysis, we identify the unknown portions.

We will split the proof of the existence into two steps.

Step1:Passage to the limit

Lemma 11. *Under the hypotheses of Theorem 7, there exist adapted processes u, B^*, F^* and G^* with the regularity:*

$$u \in L^p(\Omega, \mathcal{F}, P; L^2(0,T; D(A^{\frac{3}{2}}))) \cap L^p(\Omega, \mathcal{F}, P; L^\infty(0,T; D(A))), \tag{4.25}$$

$$v \in L^p(\Omega, \mathcal{F}, P; L^2(0,T; V)), \tag{4.26}$$

$$v \in C(0,T; H) \ a.s., \tag{4.27}$$

$$u \in C(0,T; D(A)) \ a.s., \tag{4.28}$$

and

$$B^* \in L^2(\Omega, \mathcal{F}, P; L^2(0,T: V')), \tag{4.29}$$

$$F^* \in L^2(\Omega, \mathcal{F}, P; L^2(0,T; H)), \tag{4.30}$$

$$G^* \in L^2(\Omega, \mathcal{F}, P; L^2(0,T; H^{\otimes m})), \tag{4.31}$$

such that u, B^*, F^* and G^* satisfy:

$$v(t) + \nu \int_0^t Av(s)\,ds + \int_0^t B^*(s)\,ds = v(0) + \int_0^t F^*(s)\,ds + \int_0^t G^*(s)\,dW(s) \tag{4.32}$$

where $v(t) = u(t) + \alpha^2 Au(t)$ and $1 \le p < \infty$.

Proof. Using (4.7) and Hölder's inequality, we have

$$\mathbb{E}\int_0^T \|P_n B(u_n(t), v_n(t)\|_{V'}^2 \le C\left(\mathbb{E}\sup_{t\in[0,T]}\|u_n(t)\|^4\right)^{\frac{1}{2}}\left(\mathbb{E}\left(\int_0^T \|v_n(t)\|^2\,dt\right)^2\right)^{\frac{1}{2}}. \tag{4.33}$$

The later quantity is uniformly bounded as a consequence of Lemmas 9, 10. From (4.33), we can deduce that the sequence $P_nB(u_n, v_n)$ is bounded in $L^2(\Omega, \mathcal{F}, P; L^2(0,T; V'))$. On the other hand, from Lemmas 8, 9, 10 and the Lipschitz conditions on F and G, we have that the sequence u_n is bounded in $L^p\left(\Omega, \mathcal{F}, P; L^2(0,T; D(A^{\frac{3}{2}})\right) \cap L^p\left(\Omega, \mathcal{F}, P; L^\infty(0,T; D(A)\right)$, the sequence v_n is bounded in $L^2(\Omega, \mathcal{F}, P; L^2(0,T; V)) \cap L^2(\Omega, \mathcal{F}, P; L^\infty(0,T; H))$, the sequence $v_n(0)$ is bounded in $L^2(\Omega, \mathcal{F}_0, P; H)$, the sequence $u_n(0)$ is bounded in $L^2(\Omega, \mathcal{F}_0, P; D(A))$, the sequence $P_nF(t, u_n)$ is bounded in $L^2(\Omega, \mathcal{F}, P; L^2(0,T; H))$ and $P_nG(t, u_n)$ is bounded in $L^2(\Omega, \mathcal{F}, P; L^2(0,T; H^{\otimes m}))$. Thus with Alaoglu's theorem, we can ensure that there exists a subsequence $\{u_{n'}\} \subset \{u_n\}$, and

the functions $u \in L^p(\Omega, \mathcal{F}, P; L^2(0,T; D(A^{\frac{3}{2}}))) \cap L^p(\Omega, \mathcal{F}, P; L^\infty(0,T; D(A)))$,
$v \in L^2(\Omega, \mathcal{F}, P; L^2(0,T; V)) \cap L^2(\Omega, \mathcal{F}, P; L^\infty(0,T; H))$, $B^* \in L^2(\Omega, \mathcal{F}, P; L^2(0,T; V'))$,
$F^* \in L^2(\Omega, \mathcal{F}, P; L^2(0,T; H))$, $\rho_1 \in L^2(\Omega, \mathcal{F}_0, H)$, $\rho_2 \in L^2(\Omega, \mathcal{F}_0, D(A))$ and
$G^* \in L^2(\Omega, \mathcal{F}, P; L^2(0,T; H^{\otimes m}))$ such that:

$$u_{n'} \rightharpoonup u \text{ in } L^p(\Omega, \mathcal{F}, P; L^2(0,T; D(A^{\frac{3}{2}}))) \cap L^p(\Omega, \mathcal{F}, P; L^\infty(0,T; D(A))), \tag{4.34}$$

$$v_{n'} \rightharpoonup v \text{ in } L^2(\Omega, \mathcal{F}, P; L^2(0,T; V)), \tag{4.35}$$

$$P_{n'}B(u_{n'}, v_{n'}) \rightharpoonup B^* \text{ in } L^2(\Omega, \mathcal{F}, P; L^2(0,T; V')), \tag{4.36}$$

$$P_{n'}F(t, u_{n'}) \rightharpoonup F^* \text{ in } L^2(\Omega, \mathcal{F}, P; L^2(0,T; H)), \tag{4.37}$$

$$v_{n'}(0) \rightharpoonup \rho_1 \text{ in } L^2(\Omega, \mathcal{F}_0, H)$$

$$u_{n'}(0) \rightharpoonup \rho_2 \text{ in } L^2(\Omega, \mathcal{F}_0, D(A))$$

$$P_{n'}G(t, u_{n'}) \rightharpoonup G^* \text{ in } L^2(\Omega, \mathcal{F}, P; L^2(0,T; H^{\otimes m})). \tag{4.38}$$

Using the weak convergence above, we obtain from (4.14)

$$v(t) + \nu \int_0^t Av(s)\,ds + \int_0^t B^*(s)\,ds = v_0 + \int_0^t F^*(s)\,ds + \int_0^t G^*(s)\,dW(s) \tag{4.39}$$

for all $t \in [0,T]$, where $v(t) = u(t) + \alpha^2 Au(t)$ and $v_0 = u_0 + \alpha^2 Au_0$.
Referring then to the results of [46],[61],[47], we find that v has modification such that $v \in C(0,T; H)$ a.s. which implies that u has modification in $C(0,T; D(A))$ a.s.. □

Step 2 :Proof of $B^* = B(u,v)$, $F^* = F(t,u)$ and $G^* = G(t,u)$

For simplicity let us denote by $\{u_n\}$ the subsequence $\{u_{n'}\}$.
Let $(X(t))_{t\in[0,T]}$ be a process in the space $L^2(\Omega, \mathcal{F}, P; L^2(0,T; V))$. Using the properties of A and of its eigenvectors $\{w_1, w_2, ...\}$($\lambda_1, \lambda_2, ...$ are the corresponding eigenvalues), we have

$$\|P_n X(t)\| \le \|X(t)\|; \quad |P_n X(t)| \le |X(t)|; \quad |X(t) - P_n X(t)| \le |X(t)| \tag{4.40}$$

$$\begin{aligned} \beta\|X(t) - P_n X(t)\|^2 &\le \langle AX(t) - AP_n X(t), X(t) - P_n X(t)\rangle_{V'} \\ &= \sum_{i=n}^{i=\infty} \lambda_i (X(t), w_i)^2 \\ &\le \langle AX(t), X(t)\rangle_{V'} \\ &\le C\|X(t)\|^2. \end{aligned}$$

Hence for $dP \times dt$ a.e. $(w,t) \in \Omega \times [0,T]$, we have

$$\lim_{n\to\infty} \|X(w,t) - P_n X(w,t)\|^2 = 0.$$

By the Lebesgue dominated convergence theorem, it follows that

$$\lim_{n\to\infty} \int_0^T \|X(t) - P_n X(t)\|^2 \, dt = 0,$$

$$\lim_{n\to\infty} \mathbb{E} \int_0^T \|X(t) - P_n X(t)\|^2 \, dt = 0,$$

and

$$\lim_{n\to\infty} \mathbb{E}\|X(t) - P_n X(t)\|^2 = 0. \tag{4.41}$$

Applying this result to $X = v \in L^2(\Omega, \mathcal{F}, P; L^2(0,T;V))$ or $X = u$, we have

$$P_n v \to \ v \text{ in } L^2(\Omega, \mathcal{F}, P; L^2(0,T;V)), \tag{4.42}$$

$$P_n u \to \ u \text{ in } L^2(\Omega, \mathcal{F}, P; L^2(0,T;V)). \tag{4.43}$$

With a candidate solution in hand, it remains to show that

$$B^* = B(u,v), F^* = F(t,u), G^* = G(t,u).$$

In the next lemma, we compare v and the sequence $v_n = u_n + \alpha^2 A u_n$, at least up to a stopping time $\tau_m \uparrow T$ a.s.; this is sufficient to deduce the existence result. Here, we are adapting techniques used in [5] and in [10].

Let $m \in \mathbf{N}^*$, consider the $\mathcal{F}_t$- stopping time τ_m defined by

$$\tau_m = \inf\{t; |v(t)|^2 + \int_0^t \|v(s)\|^2 \, ds \geq m^2\} \wedge T.$$

Notice that τ_m is increasing as a function of m and moreover $\tau_m \to T$ a.s. as m tends to ∞.

Lemma 12. *we have*

$$\lim_{n\to\infty} \mathbb{E} \int_0^{\tau_m} \|v_n(s) - v(s)\|^2 \, ds = 0$$

Proof. Using (4.42), it suffices to prove that

$$\lim_{n\to\infty} \mathbb{E} \int_0^{\tau_m} \|P_n v(s) - v_n(s)\|^2 \, ds = 0.$$

Using equations (4.14) and (4.39), the difference of $P_n v$ and v_n satisfies the relation

$$d(P_n v - v_n) + [\nu A(P_n v - v_n) + P_n B^* - P_n B(u_n, v_n)] \, dt = P_n(F^* - F(t,u_n))dt + P_n(G^* - G(t,u_n)) \, dW.$$

Let $\sigma(t) = exp\{-n_1 t - n_2 \int_0^t \|v(s)\|^2\, ds\}, 0 \leq t \leq T$, with n_1 and n_2 positive constants to be fixed later.

Applying Itô's formula to the process $\sigma(t)|P_n v - v_n|^2$, we have

$$\sigma(t)|P_n v(t) - v_n(t)|^2 + 2\beta\nu \int_0^t \sigma(t)\|P_n v(s) - v_n(s)\|^2\, ds \leq 2\int_0^t \sigma(s)\langle B^*(s) - B(u_n(s), v_n(s)), P_n v(s) - v_s\rangle_{V'}\, ds$$
$$+ 2\int_0^t \sigma(s)(F^*(s) - F(s, u_n(s)), P_n v(s) - v_n(s))\, ds + 2\int_0^t \sigma(s)|P_n(G^*(s) - G(s, u_n(s)))|^2\, ds$$
$$2\int_0^t \sigma(s)(G^*(s) - G(s, u_n(s)), P_n v(s) - v_n(s))\, dW - n_1 \int_0^t \sigma(s)|P_n v(s) - v_n(s)|^2\, ds$$
$$- n_2 \int_0^t \sigma(s)\|v(s)\|^2 |P_n v(s) - v_n(s)|^2\, ds. \quad (4.44)$$

We are going to estimate the first three terms of the right hand side of (4.44).

For the first term, using the cancellation property (4.3) and (4.7), we have

$$\langle B^* - B(u_n, v_n), P_n v - v_n\rangle_{V'} \quad (4.45)$$
$$= \langle B^*, P_n v - v_n\rangle_{V'} + \langle B(u_n - P_n u, P_n v), v_n - P_n v\rangle_{V'} + \langle B(P_n u, P_n v), v_n - P_n v\rangle_{V'}$$
$$\leq \langle B^*, P_n v - v_n\rangle_{V'} + C|u_n - P_n u|^{\frac{1}{4}}\|u_n - P_n u\|^{\frac{3}{4}}|P_n v|^{\frac{1}{4}}\|P_n v\|^{\frac{3}{4}}\|v_n - P_n v\| + \langle B(P_n u, P_n v), v_n - P_n v\rangle_{V'}$$
$$\leq \langle B^*, P_n v - v_n\rangle_{V'} + \frac{C}{2\beta}\|v\|^2|v_n - P_n v|^2 + \frac{\beta}{2}\|v_n - P_n v\|^2 + \langle B(P_n u, P_n v), v_n - P_n v\rangle_{V'}.$$

For the term involving F^* and F, using the Lipschitz condition on F, we have

$$2(F^* - F(t, u_n), P_n v - v_n) \quad (4.46)$$
$$\leq 2(F^* - F(t, u), P_n v - v_n) + 2(F(t, u) - F(t, P_n u), P_n v - v_n) + 2L_F|P_n u - u_n||P_n v - v_n|$$
$$\leq 2(F^* - F(t, u), P_n v - v_n) + 2(F(t, u) - F(t, P_n u), P_n v - v_n) + 2CL_F|P_n v - v_n|^2.$$

For the term involving G^* and G, using the Lipschitz conditions on G, we have

$$|P_n(G^* - G(t, u_n))|^2 \quad (4.47)$$
$$\leq 2L_G^2|P_n u - u_n|^2 + 2L_G^2|u - P_n u|^2 + 2(G^* - G(t, u), P_n(G^* - G(t, u_n))) - |P_n(G^* - G(t, u))|^2$$
$$\leq 2L_G^2|P_n v - v_n|^2 + 2L_G^2|u - P_n u|^2 + 2(G^* - G(t, u), P_n(G^* - G(t, u_n))) - |P_n(G^* - G(t, u))|^2.$$

Taking into account (4.45)-(4.47), we obtain from (4.44) that

$$\begin{aligned}
&\sigma(t)|P_nv(t)-v_n(t)|^2+2\beta\int_0^t\sigma(s)\|P_nv(s)-v_n(s)\|^2\,ds+2\int_0^t\sigma(s)|P_n(G^*(s)-G(s,u(s)))|^2\,ds\\
&\leq 2\int_0^t\sigma(s)\langle B^*(s),P_nv(s)-v_n(s)\rangle_{V'}\,ds+\frac{C}{\beta}\int_0^t\sigma(s)\|v(s)\|^2|v_n(s)-P_nv(s)|^2\,ds+\\
&\qquad\beta\int_0^t\sigma(s)\|P_nv(s)-v_n(s)\|^2\,ds\\
&+2\int_0^t\sigma(s)\langle B(P_nu(s),P_nv(s)),v_n(s)-P_nv(s)\rangle_{V'}\,ds+4CL_F\int_0^t\sigma(s)|P_nv(s)-v_n(s)|^2\,ds\\
&+4\int_0^t\sigma(s)(F^*(s)-F(s,u(s)),P_nv(s)-v_n(s))\,ds+4\int_0^t\sigma(s)(F(s,u(s))-F(s,P_nu(s)),P_nv(s)-v_n(s))\,ds\\
&+4L_G^2\int_0^t\sigma(s)|P_nv(s)-v_n(s)|^2\,ds+4L_G^2\int_0^t\sigma(s)|u(s)-P_nu(s)|^2\,ds\\
&+4\int_0^t\sigma(s)(G^*(s)-G(s,u(s)),P_n(G^*(s)-G(s,u(s))))\,ds\\
&-n_1\int_0^t\sigma(s)|P_nv(s)-v_n(s)|^2\,ds-n_2\int_0^t\sigma(s)\|v(s)\|^2|P_nv(s)-v_n(s)|^2\\
&+2\int_0^t\sigma(s)(G^*(s)-G(s,u_n(s)),P_nv(s)-v_n(s))\,dW.
\end{aligned}\tag{4.48}$$

Therefore, if we take $n_1=4CL_F+4L_G^2$ and $n_2=\frac{C}{\beta\nu}$, we obtain from (4.48)

$$\begin{aligned}
&\mathbb{E}\sigma(\tau_m)|P_nv(\tau_m)-v_n(\tau_m)|^2+\frac{3\beta\nu}{2}\mathbb{E}\int_0^{\tau_m}\sigma(s)\|P_nv(s)-v_n(s)\|^2\,ds+2\mathbb{E}\int_0^{\tau_m}\sigma(s)|P_n(G^*(s)-G(s,u(s)))|^2\,ds\\
&\leq 2\mathbb{E}\int_0^{\tau_m}\sigma(s)\langle B^*(s),P_nv(s)-v_n(s)\rangle_{V'}\,ds++2\mathbb{E}\int_0^{\tau_m}\sigma(s)\langle B(P_nu(s),P_nv(s)),v_n(s)-P_nv(s)\rangle_{V'}\,ds\\
&+4\mathbb{E}\int_0^{\tau_m}\sigma(s)(F^*(s)-F(s,u(s)),P_nv(s)-v_n(s))\,ds\\
&+4\mathbb{E}\int_0^{\tau_m}\sigma(s)(F(s,u(s))-F(s,P_nu(s)),P_nv(s)-v_n(s))\,ds\\
&+4L_G^2\mathbb{E}\int_0^{\tau_m}\sigma(s)|u(s)-P_nu(s)|^2\,ds+4\mathbb{E}\int_0^{\tau_m}\sigma(s)(G^*(s)-G(s,u(s)),P_n(G^*(s)-G(s,u(s))))\,ds.
\end{aligned}\tag{4.49}$$

Next, we are going to prove the convergence to zero of each term on the right hand side of (4.49). Here we use some basic convergence principles from functional analysis (see Appendix C). For the first two terms, we have

$$\begin{aligned}
&\mathbb{E}\int_0^{\tau_m}\sigma(s)\langle B(P_nu(s),P_nv(s))-B^*(s),v_n(s)-P_nv(s)\rangle_{V'}\,ds=\\
&\qquad\mathbb{E}\int_0^{\tau_m}\sigma(s)\langle B(P_nu(s),P_nv(s))-B(u(s),v(s)),v_n(s)-P_nv(s)\rangle_{V'}\,ds\\
&\qquad+\mathbb{E}\int_0^{\tau_m}\sigma(s)\langle B(u(s),v(s))-B^*(s),v_n(s)-P_nv(s)\rangle_{V'}\,ds.
\end{aligned}\tag{4.50}$$

From the properties of B, we have

$$\begin{aligned}
&\|B(P_n u, P_n v) - B(u,v)\|_{V'} \\
&\leq \|B(P_n u - u, P_n v)\|_{V'} + \|B(u, P_n v - v)\|_{V'} \\
&\leq (\|P_n u - u\| \|P_n v\| + \|u\| \|P_n v - v\|).
\end{aligned}$$

We have from (4.42) and (4.43)

$$\|I_{[0,\tau_m]}\sigma(t)B(P_n u, P_n v) - B(u,v)\|_{V'} \to 0, as\ n \to \infty, dt \times dP - a.e. \tag{4.51}$$

$$\|I_{[0,\tau_m]}\sigma(t)(B(P_n u, P_n v) - B(u,v))\|_{V'} \leq C\|u(t)\|\|v(t)\| \in L^2(\Omega, \mathcal{F}, P; L^2(0,T;\mathbb{R})). \tag{4.52}$$

where $I_[0, \tau_m]$ is the indicator of the interval $[0, \tau_m]$. Using (4.35) and (4.42), we have

$$v_n - P_n v \rightharpoonup 0 \text{ in } L^2(\Omega, \mathcal{F}, P; L^2(0,T;V)). \tag{4.53}$$

Applying the results of weak convergence (see Appendix B), it follows from (4.51)-(4.53) that

$$\lim_{n\to\infty} E \int_0^{\tau_m} \sigma(s)\langle B(P_n u, P_n v) - B(u,v), v_n(s) - P_n v(s)\rangle_{V'}\, ds = 0. \tag{4.54}$$

Also as $I_{[0,\tau_m]}\sigma(t)B(u,v) - B^* \in L^2(\Omega, \mathcal{F}, P; L^2(0,T;V'))$, we have from (4.53)

$$\lim_{n\to\infty} E \int_0^{\tau_m} \sigma(s)\langle B(u(s), v(s)) - B^*(s), v_n(s) - P_n v(s)\rangle_{V'}\, ds = 0. \tag{4.55}$$

On the other hand, from (4.43), the Lipschitz conditions on F, G and the fact that $v_n - P_n v \rightharpoonup 0$ in $L^2(\Omega, \mathcal{F}, P; L^2(0,T;H))$, we have

$$\lim_{n\to\infty} E \int_0^{\tau_m} \sigma(s)\left(G(s,u(s)) - G(s, P_n u(s)), v_n(s) - P_n v(s)\right) ds = 0, \tag{4.56}$$

$$\lim_{n\to\infty} E \int_0^{\tau_m} \sigma(s)\left(F(s,u(s)) - F(s, P_n u(s)), v_n(s) - P_n v(s)\right) ds = 0. \tag{4.57}$$

Again from (4.53) and the fact that

$$F^* - F(t,u) \in L^2(\Omega, \mathcal{F}, P; L^2(0,T;H)),$$

$$G^* - G(t,u) \in L^2(\Omega, \mathcal{F}, P; L^2(0,T;H^{\otimes m})),$$

we have

$$\lim_{n\to\infty} \mathbb{E} \int_0^{\tau_m} \sigma(s)\left(F^*(s) - F(s,u(s)), v_n(s) - P_n v(s)\right) ds = 0, \tag{4.58}$$

$$\lim_{n\to\infty} \mathbb{E} \int_0^{\tau_m} \sigma(s)\left(G^*(s) - G(s,u(s)), v_n(s) - P_n v(s)\right) ds = 0. \tag{4.59}$$

As

$$P_n(G^* - G(t, u_n)) \rightharpoonup 0 \text{ in } L^2(\Omega, \mathcal{F}, P; L^2(0, T; H^{\otimes m})),$$

we also have

$$\lim_{n\to\infty} \mathbb{E} \int_0^{\tau_m} \sigma(s) \left(G^*(s) - G(s, u(s)), P_n(G^*(s) - G(s, u_n(s)))\right) ds = 0. \tag{4.60}$$

From (4.54)-(4.60), and the fact that

$$exp(-n_1T - n_2m) \leq I_{[0,\tau_m]\sigma(t)} \leq 1,$$

we obtain from (4.49),

$$\lim_{n\to\infty} \mathbb{E} \left(|P_n v(\tau_m) - v_n(\tau_m)|^2\right) = 0, \tag{4.61}$$

$$\lim_{n\to\infty} E \int_0^{\tau_m} \|P_n v(s) - v_n(s)\|^2 \, ds = 0, \tag{4.62}$$

$$\mathbb{E} \int_0^{\tau_m} |G^*(s) - G(s, u(s))|^2 \, ds = 0. \tag{4.63}$$

Now from (4.63) and the fact that the sequence τ_m tend to T, we have

$$G^*(t) = G(t, u(t))$$

as elements of the space $L^2(\Omega, \mathcal{F}, P; L^2(0, T; H^{\otimes m}))$.

Also observe that (4.61) and (4.42) imply that

$$v_n I_{[0,\tau_m]} \to v I_{[0,\tau_m]} \; in \; L^2(\Omega, \mathcal{F}, P; L^2(0, T; V)), \tag{4.64}$$

where $I_{[0,\tau_m]}$ is the indicator function of $[0, \tau_m]$. Let $w \in V$. We have the following estimate from B

$$\begin{aligned}
&|\langle B(u, v) - P_n B(u_n, v_n), w\rangle_{V'}| \\
&\leq |\langle B(u, v) - B(u_n, v_n), w\rangle_{V'}| + |\langle (I - P_n)B(u_n, v_n), w\rangle_{V'}| \\
&\leq C(\|u - u_n\|\|v\| + \|v_n - v\|\|v_n\|)\|w\| + C\|(I - P_n)w\|\|u_n\|\|v_n\|.
\end{aligned} \tag{4.65}$$

Thus from (4.65) and using Hölder's inequality, we have

$$\begin{aligned}
&\mathbb{E} \int_0^{\tau_m} \langle B(u(s), v(s)) - P_n B(u_n(s), v_n(s)), w\rangle_{V'} \, ds \\
&\quad \leq C \left(\mathbb{E} \int_0^{\tau_m} \|u(s) - u_n(s)\|^2 \, ds\right)^{\frac{1}{2}} \left(\mathbb{E} \int_0^{T} \|v(s)\|^2 \, ds\right)^{\frac{1}{2}} \\
&\quad + \left(\mathbb{E} \int_0^{\tau_m} \|v_n(s) - v(s)\|^2 \, ds\right)^{\frac{1}{2}} \left(\mathbb{E} \int_0^{T} \|v_n(s)\|^2 \, ds\right)^{\frac{1}{2}} \\
&\quad + C\|(I - P_n)w\| \left(\mathbb{E} \int_0^{T} \|u_n(s)\|^2 \, ds\right)^{\frac{1}{2}} \left(\mathbb{E} \int_0^{T} \|v_n(s)\|^2 \, ds\right)^{\frac{1}{2}}.
\end{aligned} \tag{4.66}$$

Consequently, by (4.64) and (4.66), we have

$$\lim_{n\to\infty} \mathbb{E}\int_0^{\tau_m} \langle B(u(s),v(s)) - P_n B(u_n(s),v_n(s)), w\rangle_{V'}\, ds = 0. \tag{4.67}$$

Taking into account (4.36), it follows from (4.67) that

$$\mathbb{E}\int_0^{\tau_m} \langle B(u(s),v(s)) - B^*(s), z(s)\rangle_{V'}\, ds = 0 \tag{4.68}$$

for all $z \in \mathcal{D}_V(\Omega\times[0,T])$, where $\mathcal{D}_V(\Omega\times[0,T])$ is the set of functions $\psi \in L^\infty(\Omega,\mathcal{F},P;L^\infty(0,T;V))$ with

$$\psi = w\phi,\ \ \phi \in L^\infty(\Omega\times[0,T];\mathbb{R})\ \mathit{and}\ w \in V.$$

Therefore, as τ_m tends to T and $\mathcal{D}_V(\Omega\times[0,T])$ is dense in $L^2(\Omega,\mathcal{F},P;L^2(0,T;V))$, we obtain from (4.68) that $B(u(t),v(t)) = B^*(t)$ as elements of the space $L^2(\Omega,\mathcal{F},P;L^2(0,T;V'))$. Analogously, using the Lipschitz condition on F and (4.64), we have $F(t,u(t)) = F^*(t)$ as elements of the space $L^2(\Omega,\mathcal{F},P;L^2(0,T;H))$. And the existence result follows. □

4.4.2 Uniqueness

Let u_1 and u_2 two solutions of problem (4.13), which have in $D(A)$ almost surely continuous trajectories with the same initial data u_0. Denote

$$v_1 = u_1 + \alpha^2 A u_1;\quad v_2 = u_2 + \alpha^2 A u_2,$$

$$v = v_1 - v_2;\quad u = u_1 - u_2.$$

By Itô's formula, we have

$$\begin{aligned} & |v(t)|^2 + 2\int_0^t \langle Av(s), v(s)\rangle_{V'} + 2\int_0^t \langle B(u_1(s),v_1(s)) - B(u_2(s),v_2(s)), v(s)\rangle_{V'} \qquad (4.69)\\ = \;& 2\int_0^t (F(s,u_1(s)) - F(s,u_2(s)), v(s))\, ds + 2\int_0^t (G(s,u_1(s)) - G(s,u_2(s)), v(s))\, ds \\ & + \int_0^t |G(s,u_1(s)) - G(s,u_2(s))|^2_{H^{\otimes m}}\, ds. \end{aligned}$$

Take $\lambda > 0$ to be fixed later and define

$$\sigma(t) = exp\{-\frac{b}{\beta}\int_0^t \|v_1(s)\|^2\, ds - \lambda t\}.$$

Applying Itô's formula to the real-valued process $\sigma(t)|v(t)|^2$, we obtain from (4.69)

$$
\begin{aligned}
&\sigma(t)|v(t)|^2 + 2\beta\nu \int_0^t \sigma(s)\|v(s)\|^2\,ds \qquad (4.70)\\
\le\;& 2\int_0^t \sigma(s)\langle B(u(s), v_1(s)), v(s)\rangle_{V'}\,ds + 2\int_0^t \sigma(s)(F(s,u_1(s)) - F(s,u_2(s)), v(s))\,ds\\
&+2\int_0^t \sigma(s)(G(s,u_1(s)) - G(s,u_2(s)), v(s))\,dW(s) + \int_0^t \sigma(s)|G(s,u_1(s)) - G(s,u_2(s))|^2_{H^{\otimes m}}\,ds\\
&-\int_0^t \frac{b}{\beta}\|v_1(s)\|^2|v(s)|^2\sigma(s)\,ds - \int_0^t \lambda\sigma(s)|v(s)|^2\,ds.
\end{aligned}
$$

But from (4.7), we have

$$
\begin{aligned}
&\langle B(u(s), v_1(s)), v(s)\rangle_{V'}\\
&\le C|u(s)|^{\frac14}\|u(s)\|^{\frac34}\|v_1(s)\|^{\frac34}\|v(s)\|\\
&\le C|v(s)|^{\frac14}|v(s)|^{\frac34}\|v_1(s)\|\|v(s)\|\\
&\le \frac{C}{2\nu\beta}\|v_1(s)\|^2|v(s)|^2 + \frac{\beta\nu}{2}\|v(s)\|^2,
\end{aligned}
$$

and from the conditions on F and G, we have

$$(F(s,u_1(s)) - F(s,u_2(s)), v(s)) \le L_F|v(s)|^2,$$

$$|G(s,u_1(s)) - G(s,u_2(s))|_{H^{\otimes m}} \le L_G|v(s)|.$$

We then obtain from (4.70)

$$
\begin{aligned}
&\sigma(t)|v(t)|^2 + 2\beta\nu \int_0^t \sigma(s)\|v(s)\|^2\,ds \qquad (4.71)\\
\le\;& \frac{\widetilde{C}}{\beta}\int_0^t \sigma(s)\|v_1(s)\|^2|v(s)|^2\,ds + \frac{\nu\beta}{2}\int_0^t \sigma(s)\|v(s)\|^2\,ds + 2\,L_F\int_0^t \sigma(s)|v(s)|^2\,ds\\
&+2\int_0^t \sigma(s)(G(s,u_1(s)) - G(s,u_2(s)), v(s))\,dW(s) + L_G^2\int_0^t \sigma(s)|v(s)|^2\,ds\\
&-\int_0^t \frac{b}{\beta}\|v_1(s)\|^2|v(s)|^2\sigma(s)\,ds - \int_0^t \lambda\sigma(s)|v(s)|^2\,ds.
\end{aligned}
$$

Taking $\lambda = L_G^2$ and $b = \widetilde{C}$, we obtain from (4.71)

$$
\begin{aligned}
&\sigma(t)|v(t)|^2 + \frac{3\nu\beta}{2}\int_0^t \sigma(s)\|v(s)\|^2\,ds \qquad (4.72)\\
\le\;& 2L_F\int_0^t \sigma(s)|v(s)|^2\,ds + 2\int_0^t \sigma(s)(G(s,u_1(s)) - G(s,u_2(s)), v(s))\,dW(s)
\end{aligned}
$$

for all $t \in [0,T]$.

As $0 < \sigma(t) \le 1$, the expectation of the stochastic integral in (4.72) vanishes, and

$$\mathbb{E}\sigma(t)|v(t)|^2 \le 2L_G\mathbb{E}\int_0^t \sigma(s)|v(s)|^2\,ds,$$

for all $t \in [0,T]$. The Gronwall's Lemma implies that

$$|v(t)| = 0, \qquad P-a.s. \text{ for all } t \in [0,T],$$

in particular

$$u(t) = 0, \qquad P-a.s. \text{ for all } t \in [0,T].$$

This completes the proof of the uniqueness.

4.5 Proof of Theorem 8

To prove the convergence result of Theorem 8, we need Lemma 18 from [5]. We recall the proof in the appendix.

It follows from (4.62) and (4.42) that

$$\lim_{n\to\infty} \mathbb{E} \int_0^{\tau_m} \|v_n(t) - v(t)\|^2 \, dt = 0. \tag{4.73}$$

Also from (4.61) and (4.41), we have

$$\lim_{n\to\infty} \mathbb{E}|v_n(\tau_m) - v(\tau_m)|^2 = 0. \tag{4.74}$$

Applying Lemma 18 to $Q_n(t) = \int_0^t \|v_n(s) - v(s)\|^2 \, ds$ and $\sigma_m = \tau_m$, and taking into account the estimate of v_n in Lemmas 9,10, (4.73) and the uniqueness of v (or u), one obtains that the whole sequence v_n defined in (4.14) satisfies

$$\lim_{n\to\infty} \mathbb{E} \int_0^t \|v_n(s) - v(s)\|^2 \, ds = 0$$

for all $t \in [0,T]$. Next, using the expression of v_n and v, we deduce that

$$\lim_{n\to\infty} \mathbb{E} \int_0^t \|u_n(s) - u(s)\|^2_{D(A^{\frac{3}{2}})} \, ds = 0.$$

Analogously, applying Lemma 18 to $Q_n(t) = |v_n(t) - v(t)|^2$ and $\sigma_m = \tau_m$, and taking into account (4.74), the uniqueness of u and the estimate of v_n in Lemmas 9,10 we have that the whole sequence v_n defined by (4.14) satisfies $\lim_{n\to\infty} \mathbb{E}|v_n(t) - v(t)|^2 = 0$. Using the expression of v_n and v, we have $\lim_{n\to\infty} \mathbb{E}\|u_n(t) - u(t)\|^2_{D(A)} = 0$ for all $t \in [0,T]$. This complete the proof of Theorem 8.

4.6 Asymptotic behavior of strong solutions for the 3D stochastic Leray-α model as α approaches zero

The purpose of this section is to study the asymptotic behavior of strong solutions $\{u_\alpha\}_{\alpha>0}$ for the 3D stochastic Leray-α model as α goes to zero. This will be carried out by investigating the weak compactness of these strong solutions as α approaches zero. One of the crucial point is to show that

$$\mathbb{E}\sup_{0\le|\theta|\le\delta\le 1}\int_0^T |u_\alpha(t+\theta)-u_\alpha(t)|^2_{D(A)'}\,dt \le C\delta,$$

where C is a constant independent of α. To do this, we adopt the method developed for the deterministic 3D Leray-α equations [75]. In this method, an important role is played by the operator $(I+\alpha^2A)^{-1}$. Here our line of investigation follows Chapters 2 and 3 of this present thesis.

4.6.1 Tightness of strong solutions for the 3D stochastic Leray-α equations

In this subsection, we prove the tightness of strong solutions of the $3D$ stochastic Leray-α equations as α approaches zero. The main result of this subsection is the following lemma

Lemma 13. *Suppose the hypotheses (4.9)-(4.12) hold, and $u_0 \in D(A)$ and non random. Let u_α be a strong solution for the $3D$ stochastic Leray-α equations. We have* [1]

$$\mathbb{E}\sup_{0\le|\theta|\le\delta\le 1}\int_0^T |u_\alpha(t+\theta)-u_\alpha(t)|^2_{D(A)'}\,dt \le C\delta,$$

where C is a constant independent of α.

Proof. We recall that $D(A)' = D\left(A^{-1}\right)$.
From (4.13), we have

$$d(I+\alpha^2A)u_\alpha + \nu A(u_\alpha+\alpha^2Au_\alpha)dt + B(u_\alpha, u_\alpha+\alpha^2Au_\alpha)\,dt = F(t,u_\alpha)\,dt + G(t,u_\alpha)\,dW. \quad (4.75)$$

Note that $I+\alpha^2A$ is an isomorphism from $D(A)$ onto H and

$$\|(I+\alpha^2A)^{-1}\|_{\mathcal{L}(H,H)} \le 1.$$

From (4.75), we have

$$du_\alpha + \nu Au_\alpha dt + (I+\alpha^2A)^{-1}B(u_\alpha, v_\alpha)dt = (I+\alpha^2A)^{-1}F(t,u_\alpha)dt + (I+\alpha^2A)^{-1}G(t,u_\alpha)\,dW,$$

[1] u_α is extended by 0 outside (0,T)

where $v_\alpha = u_\alpha + \alpha^2 A u_\alpha$.

We deduce that

$$
\begin{aligned}
&|A^{-1}(u_\alpha(t+\theta) - u_\alpha(t))| \qquad (4.76)\\
&\int_t^{t+\theta} \left(|A^{-1}(I+\alpha^2 A)^{-1} F(\tau, u_\alpha(\tau))| + \nu |u_\alpha(\tau)| + |A^{-1}(I+\alpha^2 A)^{-1} B(u_\alpha(\tau), v_\alpha(\tau))|\right) d\tau \\
&+ \left| \int_t^{t+\theta} A^{-1}(I+\alpha^2 A)^{-1} G(\tau, u_\alpha(\tau))\, dW(\tau) \right|.
\end{aligned}
$$

We estimate the first terms of the left hand side of (4.76) using (4.6) and the Lipschitz condition on F

$$
\begin{aligned}
|A^{-1}(I+\alpha^2 A)^{-1} B(u_\alpha(\tau), v_\alpha(\tau))| &\le |A^{-1} B(u_\alpha(\tau), v_\alpha(\tau))| \\
&\le C |u_\alpha(\tau)| \|v_\alpha(\tau)\|,
\end{aligned}
$$

$$
|A^{-1}(I+\alpha^2 A)^{-1} F(\tau, u_\alpha(\tau))| \le |A^{-1} F(\tau, u_\alpha(\tau))| \le C(1 + |u_\alpha(\tau)|).
$$

Collecting the above inequalities and taking the square in (4.76), we have

$$
\begin{aligned}
|A^{-1}(u_\alpha(t+\theta) - u_\alpha(t))|^2 &\le C\theta^2 + C_1 \left(\int_t^{t+\theta} |u_\alpha(\tau)|\, d\tau \right)^2 \\
&+ \nu^2 \left(\int_t^{t+\theta} |u_\alpha(\tau)|\, d\tau \right)^2 + C \left(\int_t^{t+\theta} |u_\alpha(\tau)| \|v_\alpha(\tau)\|\, d\tau \right)^2 \\
&+ \left| \int_t^{t+\theta} A^{-1}(I+\alpha^2 A)^{-1} G(\tau, u_\alpha(\tau))\, dW(\tau) \right|^2.
\end{aligned}
$$

For fixed δ, taking the supremum over $\theta \le \delta$ yields

$$
\begin{aligned}
\sup_{0 \le \theta \le \delta} |A^{-1}(u_\alpha(t+\theta) - u_\alpha(t))|^2 &\le C\delta^2 + TC_1 \delta^2 \sup_{\tau \in [0,T]} |u_\alpha(\tau)|^2 \\
&+ C_4 \sup_{\tau \in [0,T]} |u_\alpha(\tau)|^2 \left(\int_t^{t+\delta} \|v_\alpha(\tau)\|\, d\tau \right)^2 \\
&+ \sup_{0 \le \theta \le \delta} \left| \int_t^{t+\theta} A^{-1}(I+\alpha^2 A)^{-1} G(\tau, u_\alpha(\tau))\, dW(\tau) \right|^2.
\end{aligned}
$$

For t, we integrate with respect to t over the interval $[\delta, T-\delta]$ and take the expectation. We deduce then

$$
\begin{aligned}
\mathbb{E} \sup_{0 \le \theta \le \delta} \int_0^T |A^{-1}(u_\alpha(t+\theta) - u_\alpha(t))|^2\, dt &\le C\delta^2 + TC\delta^2 \mathbb{E} \sup_{\tau \in [0,T]} |u_\alpha(\tau)|^2 \\
&+ C_4 \mathbb{E} \sup_{\tau \in [0,T]} |u_\alpha(\tau)|^2 \int_0^T \left(\int_t^{t+\delta} \|v_\alpha(\tau)\|\, d\tau \right)^2 dt \\
&+ \mathbb{E} \int_0^T \sup_{0 \le \theta \le \delta} \left| \int_t^{t+\theta} A^{-1}(I+\alpha^2 A)^{-1} G(\tau, u_\alpha(\tau))\, dW(\tau) \right|^2 dt.
\end{aligned}
$$

By Hölder′s inequality, we have

$$
\begin{aligned}
&\mathbb{E} \sup_{\tau\in[0,T]} |u_\alpha(\tau)|^2 \int_0^T \left(\int_t^{t+\delta} \|v_\alpha(\tau)\| \, d\tau \right)^2 dt \\
&\leq \delta^2 \mathbb{E} \sup_{\tau\in[0,T]} |u_\alpha(\tau)|^2 \int_0^T \|v_\alpha(\tau)\|^2 \, d\tau \\
&\leq \delta^2 \left(\mathbb{E} \sup_{\tau\in[0,T]} |u_\alpha(\tau)|^4 \right)^{\frac{1}{2}} \left[\mathbb{E} \left(\int_0^T \|v_\alpha(\tau)\|^2 \, d\tau \right)^2 \right]^{\frac{1}{2}}.
\end{aligned}
$$

Using the estimates of Lemmas 8, 9, 10, we obtain

$$
\mathbb{E} \sup_{\tau\in[0,T]} |u_\alpha(\tau)|^2 \int_0^T \left(\int_t^{t+\delta} \|v_\alpha(\tau)\| \, d\tau \right)^2 dt \leq C\delta^2
$$

where C is a constant independent of α.

Next, using martingale inequality, we have

$$
\begin{aligned}
&\mathbb{E} \int_0^T \sup_{0\leq\theta\leq\delta} \left| \int_t^{t+\theta} A^{-1}(I+\alpha^2 A)^{-1} G(s, u_\alpha(s)) \, dW(s) \right|^2 dt \\
&\leq \mathbb{E} \int_0^T \left(\int_t^{t+\delta} |A^{-1}(I+\alpha^2 A)^{-1} G(s, u_\alpha(s))|^2 \, ds \right) dt \\
&\leq C\mathbb{E} \int_0^T \left(\int_t^{t+\delta} (1 + |u_\alpha(s)|^2) \, ds \right) dt \\
&\leq C\delta.
\end{aligned}
$$

Collecting the above results, we finally obtain

$$
\mathbb{E} \sup_{0\leq\theta\leq\delta\leq 1} \int_0^T |u_\alpha(t+\theta) - u_\alpha(t)|^2_{D(A)'} \, dt \leq C\delta,
$$

where C is a constant independent of α. □

Remark 6. *From Lemma 9, we have*

$$
\mathbb{E} \sup_{t\in[0,T]} |u_\alpha(t)|^p \leq C_p.
$$

Also from Lemma 8, we have

$$
\mathbb{E} \int_0^T \|u_\alpha(s)\|^2 \, ds \leq C,
$$

where C is constant independent of α.

From the estimate of Lemma 13 and Remark 6, we derive the following Lemma which will be useful to prove the tightness of u_α.

Lemma 14. *Let ν_n and μ_n two sequences of positives real number which tend to 0 as $n \to \infty$. The injection of*

$$\mathcal{D} = \left\{ q \in L^\infty(0,T;H) \cap L^2(0,T;V); \sup_n \frac{1}{\nu_n} \sup_{|\theta| \le \mu_n} \left(\int_0^T |q(t+\theta) - q(t)|^2_{D(A)'}\, dt \right)^{\frac{1}{2}} < \infty \right\}$$

in $L^2(0,T;H)$ is compact.

Proof. See Appendix A, Proposition 6. Take $B_0 = D(A), B_1 = V, B_2 = H$. □

We define

$$S_2 = C(0,T;R^m) \times L^2(0,T;H)$$

equipped with its Borel σ-algebra $\mathcal{B}(S_2)$.

For $\alpha \in (0,1)$, let

$$\Phi_\alpha : \Omega \to S_2 : \omega \rightarrowtail (W(\omega,.), u_\alpha(\omega,.))\,.$$

For each $\alpha \in (0,1)$, we introduce a probability measure Π_α on $(S_2, \mathcal{B}(S_2))$ by

$$\Pi_\alpha(A) = P(\Phi_\alpha^{-1}(A))$$

where $A \in B(S)$.

In the next proposition, using the preceding Lemma, we can prove the tightness of Π_α. Its proof follows the same lines as in the proof of Theorem 5, Chapter 2.

Proposition 5. *The family of probability measures $\{\Pi_\alpha; \alpha \in (0,1)\}$ is tight in $(S_2, \mathcal{B}(S_2))$.*

4.6.2 Approximation of the stochastic 3D Navier-Stokes equations

In this section, we prove that the probabilistic weak solutions of the stochastic 3D Navier-Stokes equations is obtained by a sequence of solutions of the $3D$ stochastic Leray-α model as α approaches zero. The result also gives us a new construction of the weak solutions for the 3D stochastic Navier-Stokes equations.

Application of Prokhorov's and Skorokhod's results

From the tightness property of $\{\Pi_\alpha; 0 < \alpha \le 1\}$ and Prokhorov's theorem, we have that there exists a subsequence $\{\Pi_{\alpha_j}\}$ and a probability measure Π such that $\Pi_{\alpha_j} \to \Pi$ weakly. By Skorokhod's theorem, there exist a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ and random variables $(\widetilde{W}_{\alpha_j}, \tilde{u}_{\alpha_j})$, $(\widetilde{W}, \tilde{u})$ on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ with values in S_2 such that:

$$\text{the law of } (\widetilde{W}_{\alpha_j}, \tilde{u}_{\alpha_j}) \text{ is } \Pi_{\alpha_j},$$

$$\text{the law of } (\widetilde{W}, \tilde{u}) \text{ is } \Pi,$$

$$(\widetilde{W}_{\alpha_j}, \tilde{u}_{\alpha_j}) \to (\widetilde{W}, \tilde{u}) \quad \text{in} \quad S_2 \quad \bar{P} - a.s.. \tag{4.77}$$

Hence $\{\widetilde{W}_{\alpha_j}\}$ is a sequence of an m-dimensional standard Wiener process.

Let

$$\bar{\mathcal{F}}_t = \sigma\{\widetilde{W}(s), \tilde{u}(s) : s \le t\}.$$

Arguing as in Chapter 2, we can prove that $\widetilde{W}$ is an m-dimensional $\bar{\mathcal{F}}_t$ standard Wiener process and the pair $(\widetilde{W}_{\alpha_j}, \tilde{u}_{\alpha_j})$ satisfies

$$\begin{aligned} & (\tilde{v}_{\alpha_j}(t), \Phi) + \nu \int_0^t (\tilde{v}_{\alpha_j}(s), A\Phi)\, ds + \int_0^t B(\tilde{u}_{\alpha_j}(s), \tilde{v}_{\alpha_j}(s), \Phi)\, ds \qquad (4.78) \\ = \; & (u_0 + \alpha_j^2 A u_0, \Phi) + \int_0^t (F(s, \tilde{u}_{\alpha_j}(s)), \Phi)\, ds + \left(\int_0^t G(s, \tilde{u}_{\alpha_j}(s))\, d\widetilde{W}_{\alpha_j}(s), \Phi \right), \end{aligned}$$

for all $\Phi \in \mathcal{V}$, where

$$\tilde{v}_{\alpha_j}(s) = \tilde{u}_{\alpha_j}(s) + \alpha_j^2 A \tilde{u}_{\alpha_j}(s).$$

The main result of this section is the following theorem

Theorem 9. *Suppose the hypotheses (4.9)-(4.12) hold, and $u_0 \in D(A)$. Then there is a subsequence of $\tilde{u}_{\alpha_j}$ denoted by the same symbol such that as $\alpha_j \to 0$, we have:*

$$\tilde{u}_{\alpha_j} \to \tilde{u} \quad \textit{strongly in} \quad L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^2(0,T;H)),$$

$$\tilde{u}_{\alpha_j} \to \tilde{u} \quad \textit{weakly in} \quad L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^2(0,T;V)),$$

$$\tilde{v}_{\alpha_j} \to \tilde{u} \quad \textit{strongly in} \quad L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^2(0,T;H)),$$

where $(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t)_{t\in[0,T]}, \bar{P}, \widetilde{W}, \tilde{u})$ is a weak solution for the 3D stochastic Navier-Stokes equations with the initial value $u(0) = u_0$. (See [2] for the definition of weak solution of the 3D stochastic Navier-Stokes equations).

Proof. From (4.78), it follows that $\tilde{u}_{\alpha_j}$ satisfies the estimates

$$\widetilde{\mathbb{E}} \sup_{0\le s\le T} |\tilde{u}_{\alpha_j}(s)|^p \le C_p; \tag{4.79}$$

$$\widetilde{\mathbb{E}} \sup_{0\le s\le T} |\tilde{v}_{\alpha_j}(s)|^p \le C_p,$$

$$\widetilde{\mathbb{E}} \sup_{0\le \theta\le \delta} \int_0^T |\tilde{u}_{\alpha_j}(t+\theta) - \tilde{u}_{\alpha_j}(t)|^2_{D(A)'}\, dt \le C\delta,$$

$$\widetilde{\mathbb{E}} \left(\int_0^T \|\tilde{v}_{\alpha_j}(s)\|^2\, ds \right) \le C_p,$$

$$\widetilde{\mathbb{E}} \sup_{0\le s\le T} \|\tilde{v}_{\alpha_j}(s)\|^2 + 4\nu\beta \widetilde{\mathbb{E}} \int_0^T \|\tilde{v}_{\alpha_j}(s)\|^2\, ds \le C_1, \tag{4.80}$$

where $\widetilde{\mathbb{E}}$ denote the mathematical expectation with respect to the probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$. Thus modulo the extraction of a subsequence denoted again $\tilde{u}_{\alpha_j}$ (with the corresponding $\tilde{v}_{\alpha_j}$), there exists two stochastic processes $\tilde{u}$, $\tilde{v}$ such that

$$\begin{aligned} \tilde{u}_{\alpha_j} &\rightharpoonup \tilde{u} \quad \text{in} \quad L^p(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^\infty(0,T;H)), \\ \tilde{u}_{\alpha_j} &\rightharpoonup \tilde{u} \quad \text{in} \quad L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^2(0,T;V)), \\ \tilde{v}_{\alpha_j} &\rightharpoonup \tilde{v} \quad \text{in} \quad L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^2(0,T;V)), \end{aligned} \tag{4.81}$$

and

$$\widetilde{\mathbb{E}} \sup_{0 \le s \le T} |\tilde{u}(s)|^p \le C_p; \qquad \widetilde{\mathbb{E}} \int_0^T \|\tilde{u}(s)\|_V^2 \, ds \le C,$$
$$\widetilde{\mathbb{E}} \sup_{0 \le \theta \le \delta} \int_\delta^{T-\delta} |\tilde{u}(t+\theta) - \tilde{u}(t)|^2_{D(A)'} \, dt \le C\delta.$$

By (4.77), the estimate (4.79) and Vitali's Theorem, we have

$$\tilde{u}_{\alpha_j} \to \tilde{u} \quad \text{in} \quad L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^2(0,T;H)). \tag{4.82}$$

Thus modulo the extraction of a new subsequence and almost every (ω, t) with respect to the measure $d\bar{P} \otimes dt$

$$\tilde{u}_{\alpha_j} \to \tilde{u} \quad \text{in} \quad H.$$

Taking into account (4.82) and the Lipschitz condition on F, we have

$$\int_0^t F(s, \tilde{u}_{\alpha_j}(s)) \, ds \to \int_0^t F(s, \tilde{u}(s)) \, ds \text{ in } \quad L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^2(0,T;H))$$

Arguing as in Chapter 2, we can prove that

$$\int_0^t G(s, \tilde{u}_{\alpha_j}(s)) \, d\widetilde{W}_{\alpha_j}(s) \to \int_0^t G(s, \tilde{u}(s)) \, d\widetilde{W}(s) \quad \text{in } L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^\infty(0,T;D(A)')) \quad \text{weakly star.}$$

We also have

$$\widetilde{\mathbb{E}} \int_0^T |\tilde{v}_{\alpha_j}(t) - \tilde{u}_{\alpha_j}(t))|^2 \, dt = \alpha_j^2 \widetilde{\mathbb{E}} \int_0^T \alpha_j^2 |A\tilde{u}_{\alpha_j}(t)|^2 \, dt.$$

We then deduce that

$$\tilde{v}_{\alpha_j} \to \tilde{u} \text{ in } L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^2(0,T;H)) \tag{4.83}$$

since by the estimate (4.80), we have

$$\widetilde{\mathbb{E}} \int_0^T \alpha_j^2 |A\tilde{u}_{\alpha_j}(t)|^2 \, dt \tag{4.84}$$

is bounded uniformly in α_j.

From (4.81) and (4.83), we have $\tilde{v}(t) = \tilde{u}(t)$ a.e. in $d\bar{P} \times dt$.

We are going to prove that

$$\int_0^t B(\tilde{u}_{\alpha_j}(s), \tilde{v}_{\alpha_j}(s))\, ds \rightharpoonup \int_0^t B(\tilde{u}(s), \tilde{u}(s))\, ds \quad \text{in } L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^2(0,T; D(A)')).$$

Indeed, let $\Phi \in \mathcal{V}$. From (4.4),(4.6) and (4.8), we have

$$\begin{aligned}
&\int_0^t \langle B(\tilde{u}_{\alpha_j}(s), \tilde{v}_{\alpha_j}(s)), \Phi\rangle_{D(A)'} - \langle B(\tilde{u}(s), \tilde{u}(s)), \Phi\rangle_{D(A)'}\, ds \\
&= \int_0^t \langle B(\tilde{u}_{\alpha_j}(s) - \tilde{u}(s), \tilde{v}_{\alpha_j}(s)), \Phi\rangle_{D(A)'} ds + \int_0^t \langle B(\tilde{u}(s), \tilde{v}_{\alpha_j}(s) - \tilde{u}(s)), \Phi\rangle_{D(A)'}\, ds \\
&= \int_0^t \langle B(\tilde{u}_{\alpha_j}(s) - u(s), \tilde{v}_{\alpha_j}(s)), \Phi\rangle_{D(A)'} ds - \int_0^t (B(\tilde{u}(s), \Phi), \tilde{v}_{\alpha_j}(s) - \tilde{u}(s))\, ds \\
&\leq C \int_0^t |\tilde{u}_{\alpha_j}(s) - \tilde{u}(s)| \|\tilde{v}_{\alpha_j}(s)\| |A\Phi|\, ds + C \int_0^t \|\tilde{u}(s)\| |A\Phi| |\tilde{v}_{\alpha_j}(s) - \tilde{u}(s)|\, ds.
\end{aligned}$$

Further, by Hölder's inequality

$$\begin{aligned}
&\widetilde{\mathbb{E}}\left(\int_0^t \langle B(\tilde{u}_{\alpha_j}(s), \tilde{v}_{\alpha_j}(s)), \Phi\rangle_{D(A)'} - \langle B(\tilde{u}(s), \tilde{u}(s)), \Phi\rangle_{D(A)'}\, ds\right) \qquad (4.85) \\
\leq\;& C|A\Phi| \left(\widetilde{\mathbb{E}} \int_0^t |\tilde{u}_{\alpha_j}(s) - \tilde{u}(s)|^2\, ds\right)^{\frac{1}{2}} \left(\widetilde{\mathbb{E}} \int_0^t \|\tilde{v}_{\alpha_j}(s)\|^2\, ds\right)^{\frac{1}{2}} \\
&+ C|A\Phi| \left(\widetilde{\mathbb{E}} \int_0^t \|\tilde{u}(s)\|^2\, ds\right)^{\frac{1}{2}} \left(\widetilde{\mathbb{E}} \int_0^t |\tilde{v}_{\alpha_j}(s) - \tilde{u}(s)|^2\, ds\right)^{\frac{1}{2}}.
\end{aligned}$$

It then follows from (4.82), (4.83), (4.85)

$$\int_0^t B(\tilde{u}_{\alpha_j}(s), \tilde{v}_{\alpha_j}(s))\, ds \rightharpoonup \int_0^t B(\tilde{u}(s), \tilde{u}(s))\, ds \quad \text{in } L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^2(0,T; D(A)')).$$

Collecting all the convergence results and passing to the limit in (4.78) to obtain

$$\begin{aligned}
(\tilde{u}(t), \Phi) + \nu \int_0^t (\tilde{u}(s), A\Phi)\, ds + \int_0^t \langle B(\tilde{u}(s), \tilde{u}(s)), \Phi\rangle_{D(A)'}\, ds = (u_0, \Phi) \\
+ \int_0^t (F(s, \tilde{u}(s)), \Phi)\, ds + \int_0^t (G(s, \tilde{u}(s)), \Phi)\, d\widetilde{W}(s).
\end{aligned}$$

This completes the proof of Theorem 9. $\square$

Chapter 5

Appendices

Here, we summarize some important results that were used in the previous chapters. For the convenience, we recall some proofs

Appendix A : A compactness result

The following compactness result as been useful in the proof of tightness of a family of probability measures.

Let B_0, B_1, B_2 be three separables Hilbert spaces such that

$$B_0 \subset B_1 \subset B_2, \tag{5.1}$$

each space being densely embedded in the next one with the continuous injection. We assume, moreover that

$$\text{the injection of} \quad B_1 \to B_2 \quad \text{is compact.} \tag{5.2}$$

We also identify B_2 to its dual.

Proposition 6. *([1], Proposition 3.4; p.274) For any sequences of positives real numbers μ_n, ν_n which tend to zero as n tends ∞, the injection of* [1]

$$\mathcal{Z} = \left\{ q \in L^2(0,T;B_1) \cap L^\infty(0,T;B_2) : \sup_n \frac{1}{\nu_n} \sup_{|\theta| \le \mu_n} \left(\int_0^T \|q(t+\theta) - q(t)\|^2_{B_0'} \, dt \right)^{\frac{1}{2}} < \infty \right\}$$

in $L^2(0,T;B_2)$ is compact.

[1] q is extended by 0 outside (0,T)

To prove this proposition, we need the following lemma

Lemma 15. *(see [52], p.59) Under the hypotheses (5.1) and (5.2), we have* $\forall \eta > 0$, *there exists* C_η *such that*

$$\|v\|_{B_2} \leq \eta \|v\|_{B_1} + C_\eta \|v\|_{B_0'} \tag{5.3}$$

for all $v \in B_1$.

Proof. of Lemma 15
Suppose that (5.3) is false. Then for all $n > 0$, there exists $v_n \in B_1$ and $C_n \to \infty$ such that

$$\|v_n\|_{B_2} > n\|v_n\|_{B_1} + C_n\|v_n\|_{B_0'}.$$

Let

$$w_n = \frac{v_n}{\|v_n\|_{B_1}}.$$

Then

$$\|w_n\|_{B_2} > n + C_n\|w_n\|_{B_0'} \tag{5.4}$$

and $\quad \|w_n\|_{B_2} \leq C\|w_n\|_{B_1} \leq C$.
(5.4) implies that

$$\|w_n\|_{B_0'} \to 0. \tag{5.5}$$

But $\|w_n\|_{B_1} = 1$ and the injection of $B_1 \to B_2$ is compact, then we can extract a subsequence of (w_n) denoted again by (w_n) which converges strongly in B_2 and from (5.5) this subsequence converges to 0, and we have $\|w_n\|_{B_2} \to 0$ which contradict (5.4). And we have the proof of Lemma 15. □

Now, we give the proof of Proposition 6.

Proof. Let (z_k) be a sequence in $\mathcal{Z}$. We shall show that we can extract a subsequence which converges strongly in $L^2(0,T;B_2)$.
We can extract a subsequence still denoted (z_k) and z such that

$$z_k \rightharpoonup z \quad \text{in} \quad L^2(0,T;B_1) \quad \textit{weakly} \tag{5.6}$$

and

$$z_k \rightharpoonup z \quad \text{in} \quad L^\infty(0,T;B_2) \quad \textit{weak}-\textit{star}.$$

Since the injection of $B_1 \to B_2$ is compact, from Lemma 15, for any $\varepsilon > 0$, there is a constant $C(\varepsilon)$ such that

$$\|\varphi\|_{B_2}^2 \leq \varepsilon\|\varphi\|_{B_1}^2 + C(\varepsilon)\|\varphi\|_{B_0'}^2$$

for all $\varphi \in B_1$.

Hence

$$\int_0^T \|z_k(t) - z(t)\|_{B_2}^2 \, dt \leq \varepsilon \int_0^T \|z_k(t) - z(t)\|_{B_1}^2 \, dt + C(\varepsilon) \int_0^T \|z_k(t) - z(t)\|_{B_0'}^2 \, dt$$
$$\leq C\varepsilon + C(\varepsilon) \int_0^T \|z_k(t) - z(t)\|_{B_0'}^2 \, dt. \tag{5.7}$$

We have used the fact that $\int_0^T \|z_k(t) - z(t)\|_{B_1}^2 \, dt$ is bounded. Therefore, to prove that the left hand side of (5.7) tends to 0, it is sufficient to prove that

$$\int_0^T \|z_k(t) - z(t)\|_{B_0'}^2 \, dt \to 0. \tag{5.8}$$

Consider a function $\psi \in C_0^\infty(\mathbb{R}), \psi \geq 0, \int_{-\infty}^{+\infty} \psi(t) \, dt = 1, supp(\psi) = [-1, 1]$ and the mollifier

$$R_\varepsilon u(t) = \frac{1}{\varepsilon} \int_{-\infty}^{+\infty} \psi(\frac{t-s}{\varepsilon}) u(s) \, ds$$
$$= - \int_{-1}^{1} u(t - \varepsilon s) \psi(s) \, ds. \tag{5.9}$$

Pick any function $u \in L^2(0, T; B_0')$ which we extend by 0 outside (0,T). We have

$$\int_0^T \|R_\varepsilon u(t) - u(t)\|_{B_0^*}^2 \, dt \leq C \int_{-1}^{1} \left[\int_0^T \|u(t - \varepsilon s) - u(t)\|_{B_0'}^2 \, dt \right] \, ds.$$

We apply it with $u = \widetilde{z_k} = z_k - z$ and $\varepsilon = \mu_n$, we have

$$\int_0^T \|R_{\mu_n} \widetilde{z_k}(t) - \widetilde{z_k}(t)\|_{B_0'}^2 \, dt \leq C \int_{-1}^{1} \left[\int_0^T \|\widetilde{z_k}(t - \mu_n s) - \widetilde{z_k}(t)\|_{B_0'}^2 \, dt \right] \, ds$$
$$\leq 2C\nu_n^2 M. \tag{5.10}$$

From (5.6), we have

$$R_{\mu_n} \widetilde{z_k}(t) = \frac{1}{\mu_n} \int_0^T \psi(\frac{t-s}{\mu_n}) \widetilde{z_k}(s) \, ds \rightharpoonup 0$$

as $k \to \infty$ in B_1 weakly for any $n \geq 1, t \in [0, T]$.

Since the injection of $B_1 \to B_2$ is compact, the injection of $B_1 \to B_0'$ is also compact. Therefore, we have

$$R_{\mu_n} \widetilde{z_k}(t) \to 0$$

as $k \to \infty$ in B_0' strongly.

Then

$$\|R_{\mu_n} \widetilde{z_k}(t)\|_{B_0'}^2 \leq C_n.$$

where C_n is a constant independent of k. We have

$$R_{\mu_n} \widetilde{z_k} \to 0 \tag{5.11}$$

as $k \to \infty$ in $L^2(0,T;B_0')$ strongly.

(5.10) and (5.11) imply (5.8). This complete the proof of Proposition 6.

$\square$

Appendix B: Basic convergence results

For the convenience of the reader, we recall some basic convergence results.

Proposition 7. *([76],Proposition 10.13, P.480) Let (x_n) be a sequence in a Banach space S. Then the following assertions hold:*

i) If S is reflexive and (x_n) is bounded in S, then (x_n) has a weakly convergent subsequence. If, in addition, every weakly convergent subsequence of (x_n) has the same limit $x \in S$, then (x_n) converges weakly to x.

ii) If every subsequence of (x_n) has a subsequence which converges strongly to the same limit $x \in S$, then $x_n \to x$.

Proposition 8. *([76], Proposition 21.27, P.261) Let X_1 and X_2 be Banach spaces and $L : X_1 \to X_2$ be a continuous linear operator. If (x_n) is a sequence in X_1 such that $x_n \rightharpoonup x$ in X_1, then $L(x_n) \rightharpoonup L(x)$.*

Proposition 9. *([76], Proposition 21.23, P.258) Let X be a Banach space*

i) Then, it follows from

$$\begin{aligned} &u_n \rightharpoonup u \quad in \quad X \quad as \quad n \to \infty \\ &f_n \to f \quad in \quad X' \quad as \quad n \to \infty \end{aligned} \tag{5.12}$$

that $\langle f_n, u_n \rangle \to \langle f, u \rangle$ as $n \to \infty$.

ii) If X is reflexive, then it follows from

$$\begin{aligned} &u_n \to u \quad in \quad X \quad as \quad n \to \infty \\ &f_n \rightharpoonup f \quad in \quad X' \quad as \quad n \to \infty \end{aligned} \tag{5.13}$$

that $\langle f_n, u_n \rangle \to \langle f, u \rangle$ as $n \to \infty$.

iii) If (u_n) is bounded in X and if there exists $u \in X$ and a dense set D in X' such that

$$\langle f, u_n \rangle \to \langle f, u \rangle$$

as $n \to \infty$ *for all* $f \in D$*, then*

$$u_n \rightharpoonup u \text{ in } X$$

as $n \to \infty$.

Appendix C: Probabilistic background

We have used in this thesis two deep compactness results due to Prokhorov and Skorokhod. In order to formulate these results, we need the concept of tightness of probability measures. Let E be a separable Banach space and let $\mathcal{B}(E)$ be its Borel σ-field.

Definition 6. *A family of probability measures* $\mathbb{P}$ *on* $(E, \mathcal{B}(E))$ *is tight if for arbitrary* $\varepsilon > 0$*, there exists a compact set* $K_\varepsilon \subset E$ *such that*

$$\mu(K_\varepsilon) \geq 1 - \varepsilon$$

for all $\mu \in \mathbb{P}$.
A sequence of measures $\{\mu_n\}$ *on* $(E, \mathcal{B}(E))$ *is weakly convergent to a measure* μ *if for all continuous and bounded functions* Ψ *on* E*,*

$$\lim_{n\to\infty} \int_E \Psi(x)\mu_n(dx) = \int_E \Psi(x)\mu(dx).$$

The following result due to Prokhorov (see [65]) shows that the tightness property is a compactness criterion.

Lemma 16. *A sequence of measures* $\{\mu_n\}$ *on* $(E, \mathcal{B}(E))$ *is tight if and only if it is relatively compact, that is there exists a subsequence* $\{\mu_{n_k}\}$ *which weakly converges to a probability measure* μ.

Skorokhod proved in [68] the next result which relates the weak convergence of probability measures to that of almost everywhere convergence of random variables.

Lemma 17. *For an arbitrary sequence of probability measures* $\{\mu_n\}$ *on* $(E, \mathcal{B}(E))$ *weakly convergent to a probability measure* μ*, there exists a probability space* $(\Omega, \mathcal{F}, P)$ *and random variables* $X, X_1, ..., X_n, ...$ *with values in* E *such that the probability law of* X_n *is* μ_n*, the probability law of* X *is* μ *and* $\lim_{n\to\infty} X_n = X, P - a.s.$.

Recent account of Prokhorov's and Skorokhod's results can be found in [26].
We used the following result from [5] to prove the convergence of our Galerkin schemes (4.14) introduced in Chapter 4. For the reader's convenience, we recall the proof.

Lemma 18. *Let (Q_n) be a sequence of real-valued processes from the space $L^2(\Omega, \mathcal{F}, P; L^2(0,T;\mathbb{R}))$. Let $(\tau_M)_M$ and τ be $\mathcal{F}_t$- stopping times such that*

$$\lim_{M\to\infty} P(\tau_M < \tau) = 0.$$

We also assume that for each fixed M, we have

$$\lim_{n\to\infty} E|Q_n(\tau_M)| = 0$$

and there exists a positive constant C independent of n such that

$$\sup_{n\in\mathbb{N}} E|Q_n(\tau)|^2 < C.$$

Then

$$\lim_{n\to\infty} E|Q_n(\tau)| = 0.$$

Proof. Let $\varepsilon, \delta > 0$. There exists $M_0 \in \mathbb{N}$ such that

$$P(\tau_{M_0} < \tau) \leq \frac{\varepsilon}{2}.$$

By the hypothesis it follows that for this M_0, we have

$$\lim_{n\to\infty} E|Q_n(\tau_{M_0})| = 0.$$

Consequently, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{\delta}E|Q_n(\tau_{M_0})| \leq \frac{\varepsilon}{2}$$

for all $n \geq n_0$. We write

$$\begin{aligned} P\left(|Q_n(\tau)| \geq \delta\right) &\leq \frac{\varepsilon}{2} + P\left(|Q_n(\tau_{M_0})| \geq \delta\right) \\ &\leq \frac{\varepsilon}{2} + \frac{1}{\delta}E|Q_n(\tau_{M_0})| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned} \tag{5.14}$$

for all $n \geq n_0$. Hence, for all $\delta > 0$ we get

$$\lim_{n\to\infty} P\left(|Q_n(\tau)| \geq \delta\right) = 0.$$

Therefore, the sequence $(|Q_n(\tau)|)$ converges in probability to zero. From the hypothesis, it follows that this sequence is uniformly integrable (with respect to $\omega \in \Omega$). Hence it converges also in mean to zero that is

$$\lim_{n\to\infty} E|Q_n(\tau)| = 0.$$

□

Appendix D : Uniform integrability and Vitali's theorem

Let $(\Omega, \mathcal{F}, P)$ be a probability space

Definition 7. *A family $\{f_j\}_{j\in J}$ of real measurable functions f_j on Ω is called uniformly integrable if*

$$\lim_{M\to\infty}\left(\sup_{j\in J}\left\{\int_{\{|f_j|>M\}}|f_j|dP\right\}\right)=0.$$

One of the most useful tests for uniform integrability is obtained by using the following concept:

Definition 8. *A function ψ :$[0,\infty)\to[0,\infty)$ is called a uniform integrability test function if ψ is increasing, convex and $\lim_{x\to\infty}\frac{\psi(x)}{x}=\infty$.*

So for example $\psi(x)=x^p$ is a uniform integrability test function if $p>1$.
The justification for the name in the preceding definition is the following:

Theorem 10. *The family $\{f_j\}_{j\in J}$ is uniformly integrable if and only if there is a uniform integrability test function ψ such that*

$$\sup_{j\in J}\left\{\int_\Omega \psi(|f_j|)dP\right\}<\infty.$$

One major reason for the usefulness of uniform integrability is the following result (Vitali's theorem), which may be regarded as the generalization of the Lebesgue convergence theorem in integration theory. Its proof can be found in [37]

Theorem 11. *(Vitali's theorem)*
Suppose $(f_k)_{k=1}^{k=\infty}$ is a sequence of real integrables functions on Ω . Let f be a real function on Ω such that

$$f_k\to f \text{ in probability.}$$

Then the following are equivalent :

1) (f_k) is uniformly integrable.

2) $f\in L^1(\Omega,\mathcal{F},P;\mathbb{R})$ and $f_k\to f$ in $L^1(\Omega,\mathcal{F},P;\mathbb{R})$.

Conclusion

We proved the existence of probabilistic weak solutions for the stochastic 3D Navier-Stokes-α model under non Lipschitz conditions on the coefficients. We also studied the asymptotic behavior of weak solutions to the stochastic 3D Navier-Stokes-α model as α approaches zero in the case of periodic boundary conditions.

Furthermore, we showed the existence and uniqueness of strong solution to the stochastic 3D Leray-α equations. We also investigated the asymptotic behavior of the strong solution as α approaches zero.

In [23], Millet and Chueshov proved the large deviation principle for small multiplicative noise for a class of abstract nonlinear stochastic models, which covers the 2D Navier-Stokes equations, the 2D Magneto-hydrodynamic models, the 2D Magnetic Bénard problem, the 3D Leray-α model and some shells models of turbulence. However, this abstract nonlinear stochastic models does not cover the case of 3D Navier-Stokes-α model. In our future work, we intend to study the large deviation and the long time dynamic (random global attractor and ergodicity) of this model.

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