Constructive treatment of reaction-diffusion and Volterra integral equations for the SIS epidemiological model

by

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DECLARATION

I, the undersigned, declare that the thesis which I hereby submit for the degree Philosophiae Doctor at the University of Pretoria, is my own, independent work and has not previously been submitted by me for a degree at this or any other tertiary institution.

Signature:

Name: Yibeltal Adane Terefe

Date: May 2015

To my wife and my children

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Absract

We design and investigate the reliability of various nonstandard finite difference (NSFD) schemes for the SIS epidemiological model in three different settings.

For the classical SIS model, we construct two new NSFD schemes which faithfully replicate the property of the continuous model of having the parameter $\mathcal{R}_0$, the basic reproduction number, as a threshold to determine the stability properties of equilibrium points: the disease-free equilibrium (DFE) is globally asymptotically stable (GAS) when $\mathcal{R}_0 \leq 1$; it is unstable when $\mathcal{R}_0 > 1$ and there appears a unique GAS endemic equilibrium (EE) in this case. These schemes also preserve the positivity and boundedness properties of solutions of the classical SIS model. The schemes are further used to derive NSFD schemes for the SIS-diffusion model which constitutes the second setting of the study. The designed NSFD schemes are dynamically consistent with the global asymptotic stability of the disease-free equilibrium for $\mathcal{R}_0 \leq 1$ and the instability of the disease-free equilibrium for $\mathcal{R}_0 > 1$. In the latter case, the schemes replicate the global asymptotic stability of the endemic equilibrium. Positivity and boundedness properties of solutions of the SIS-diffusion model are also preserved by the NSFD schemes.

In a third step, the classical SIS model is extended into a SIS-Volterra integral equation model in which the contact rate is a function of fraction of infective individuals and allows a distributed period of infectivity. The qualitative analysis is now based on two threshold parameters $\mathcal{R}_0^c \leq 1 \leq \mathcal{R}_0^m$. The system can undergo the backward bifurcation phenomenon as follows. The DFE is the only equilibrium and it is GAS when $\mathcal{R}_0 < \mathcal{R}_0^c$; there exists only one EE, which is GAS when $\mathcal{R}_0 > \mathcal{R}_0^m$ with the DFE being unstable when $\mathcal{R}_0 > 1$; for $\mathcal{R}_0^c < \mathcal{R}_0 < 1$, the DFE is locally asymptotically stable (LAS) and coexists with at least one LAS endemic equilibrium. We design a NSFD scheme and prove theoretically and computationally that it preserves the above-stated stability properties of equilibria as well as positivity and boundedness of the solutions of the continuous model.

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Chapter 1

Introduction

Infectious diseases constitute a major threat to the development of the world. According to the World Health Organization (WHO) reports [88], more than 8.7 million people die every year because of infectious diseases such as Malaria, Dengue, Tuberculosis, HIV/AIDS, Ebola, Cancer and Diabetes. The situation in developing regions such as Africa, where new (eg. Ebola and Chikungugya) and old form (Cholera and Human African Trypanosomiasis (Sleeping Sickness)) of diseases emerge and re-emerge, is worse due to the lack of sustained health prevention and treatment programmes.

Clearly, health and clinical research are important to discover control strategies (eg. vaccination and antibiotics) of the diseases. In practice, the epidemiology and the dynamics of the diseases are complex. That is when research in mathematics comes in. Starting from the pioneer work of D. Bernoulli about smallpox [27], mathematical models have become important tools to analyze the spread, transmission mechanisms and control strategies of infectious diseases. Models are designed to capture the essence of various interactions allowing the outcome of the process to be more fully understood and to make predictions that can influence health policy [66].

Apart from Malthus law, i.e., the exponential growth in population [86], the other two main principles in mathematical epidemiology were formulated much later namely, early in the nineties. These are: the mass action principle [34, 74] and the threshold theory [43]. Many researchers have been interested in designing mathematical models to study infectious diseases. In the process of mathematical modeling of a disease, the population under consideration is divided into disjoint classes or compartments whose sizes change with time. Mainly, these compartments are labelled as M (infants with passive immunity), S (susceptible individuals), E (exposed individuals), I (infective individuals) and R (recovered individuals). The choice of which compartment to include in a model depends on the nature of the disease being modeled. Based on the flow patterns between compartments, several models have been developed with acronyms such as MSEIR, MSEIRS, SEIR, SEIRS, SIR, SIRS, SIS, etc. (See [37]).

The focus of this thesis is on theoretical and numerical analysis of model of infectious diseases represented by SIS-type epidemiological models. The SIS model, constructed initially by Kermack and McKendrick [43], is one of the simplest models for diseases without immunity after recovery. Several diseases (eg. gonorrhea, influenza) transmitted by direct contact are represented by the SIS model [24, 38]. We have opted for this simple model in order to obtain more explicit and precise results. Actually, this choice is not a restriction as such because the qualitative properties of the SIS system are extended to most compartmental models [13, 14, 20, 24, 26].

The SIS model will be investigated under the following three settings:

1. The classical SIS system of differential equations in which the number of adequate contacts per infective in unit time to transfer the infection is assumed to be constant.

2. The SIS-diffusion model whereby a diffusion term is added to the classical SIS model in order to capture the spatial spread of the disease from the defining reaction-diffusion partial differential equation.

3. The SIS-Volterra integral equation model whereby the classical SIS model is extended into a Volterra integral equation or integro-differential equation with the aim of capturing multiple exposure to infectious individuals, behavioural changes in the population and the distribution of period of infectivity of infective individuals [79].

In each of these settings, we do a comprehensive study of the model that includes well-posedness on a biologically feasible region, its qualitative properties in terms of the stability of the equilibria and its numerical approximation by dynamically consistent schemes. More precisely,

I. We establish the existence and uniqueness of positive and bounded solutions of the models in (1), (2) and (3) above in suitable function spaces.

II. We give precise stability properties of the equilibria of the systems with an emphasis on the global asymptotic stability of the disease-free equilibrium and of the endemic equilibrium. Furthermore, the existence of backward bifurcation or coexistence of the locally asymptotically stable disease-free equilibrium with at least one locally asymptotically stable endemic equilibrium is discussed for the SIS-Volterra integral equation model mentioned in (3).

III. We design and analyse nonstandard finite difference schemes which replicate the qualitative properties of the continuous models.

It should be noted that the continuous model under consideration cannot be completely solved by analytical techniques. Thus, the relevance of the numerical schemes proposed in this thesis as they are reliable in providing some useful insights on the solutions of the continuous models.

Different deep mathematical tools are used to establish the results in this thesis.

From the theoretical point of view, we have the Banach fixed-point theorem or contraction principle [87] which serves to prove existence and uniqueness of solutions. This is coupled with techniques such as the integrating factor, the Gronwall inequality and the comparison theorem [75] to show the positivity and boundedness of solutions. The linearization process, including the Hartman-Grobman theorem [76] is used to establish local stability of equilibria of dynamical systems, while LaSalle Invariance Principle [48] is applied to obtain their global asymptotic stability. For the SIS-diffusion model, the stability property is a result of the spectral theory of the Sturm-Liouville problem corresponding to its linearization and the energy method [5]. In all cases, a threshold condition defined through the basic reproduction number $\mathcal{R}_0$ is involved with $\mathcal{R}_0 = 1$ being a bifurcation point.

From the constructive point of view, we use the NSFD method based on two of Mickens' rules [61], as singled out in [8]:

- The standard denominator Δt of the discrete derivative is replaced by a more complex function;
- Nonlinear terms must be approximated in a nonlocal way.

To be more explicit about these two important rules, we note that in the classical SIS model, the derivative $\dfrac{dS}{dt}(t)$, for instance, can be approximated by

$$\frac{S^{k+1}-S^k}{\phi(\Delta t)} \quad \text{instead of} \quad \frac{S^{k+1}-S^k}{\Delta t},$$

where the complex denominator function $\phi(\Delta t)$ is expected to capture the feature of the continuous model under consideration and $S^k \simeq S(t_k)$ for $t_k = k\Delta t$ with $k \in \mathbb{N}$. Furthermore, in the SIS-diffusion model, instead of $\dfrac{S_n^k I_n^k}{S_n^k + I_n^k}$, the nonlinear term

$$\frac{S(x_n,t_k)I(x_n,t_k)}{S(x_n,t_k)+I(x_n,t_k)} \quad \text{is approximated by} \quad \frac{S_n^{k+1}I_n^k}{S_n^{k+1}+I_n^k} \quad \text{or} \quad \frac{S_n^{k+1}(I_{n+1}^k + I_n^k + I_{n-1}^k)}{3(S_n^{k+1}+I_n^{k+1})},$$

for $x_n = n\Delta x$ with $n \in \mathbb{Z}$, in order to replicate positivity of solutions and conservation law.

Another set of tools for the constructive part are the Bolzano-Weierstrass theorem [73], the Jury conditions [59] and the discrete energy method [5], which are used to prove the global asymptotic stability of fixed points.

It should be noted that the design and analysis of NSFD schemes as reliable numerical methods for epidemiological models have not yet been sufficiently investigated. Existing works include [6, 7, 23, 31, 70]. Further discussions and comments as to how our findings fit in the literature will be done within the relevant chapters and/or sections of the thesis.

Some results of the thesis are published in [55, 57] and were presented at international conferences [56].

We now describe briefly each chapter of the thesis. Chapter 2 is devoted to the classical SIS model in standard incidence formulation. It is shown that this model is a continuous dynamical system on a biologically feasible region of the positive cone. It is further shown that the disease-free equilibrium is globally asymptotically stable whenever the basic reproduction number $\mathcal{R}_0 \leq 1$, while this equilibrium becomes unstable and a unique globally asymptotically stable endemic equilibrium is born when $\mathcal{R}_0 > 1$. On the constructive side, NSFD schemes which replicate all these properties of the continuous model are designed and analyzed. Apart from constructing for the first time a nonstandard Runge-Kutta method of order four, the novelty of this chapter is to prove theoretically and computationally, the global asymptotic stability of fixed points of the NSFD schemes. Note that most existing NSFD schemes for the SIS model deal with the local property of elementary stability [6, 7, 55, 77].

Chapter 3 is concerned with the extension of the classical SIS model into the SIS-diffusion (partial differential) equation model that governs the spatial spread of an epidemic. This chapter includes a short introduction to reaction-diffusion equations which play a key role in mathematical epidemiology [75]. The existence of a unique and biologically meaningful solution is proved. Necessary and sufficient conditions for the existence of traveling wave solutions are investigated. It is also shown by the energy method that the disease-free equilibrium is globally asymptotically stable when the basic reproduction number $\mathcal{R}_0$ is less than or equal to unity, while it is unstable for $\mathcal{R}_0 > 1$ and there exists a globally asymptotically stable endemic equilibrium in this case. Once again, the chapter ends with the design and the theoretical and computational analysis of various NSFD schemes. Unlike the schemes in [22, 23], which preserve only the positivity and boundedness properties of solutions of the continuous model, the new schemes developed in this chapter also replicate the conservation law and the global stability of equilibria.

Chapter 4 discusses the extension of the classical SIS model into the SIS-Volterra integral equation model with a contact rate as a function of fraction of total number of infective individuals. The continuous model is taken from [79]. The existence of a unique positive solution is established. Two threshold quantities $\mathcal{R}_0^c \leq \mathcal{R}_0^m$ are found apart from the usual critical value $\mathcal{R}_0 = 1$. The existence of the backward bifurcation (i.e. the existence of multiple locally asymptotically stable equilibria) for $\mathcal{R}_0 \in (\mathcal{R}_0^c, 1)$ is shown, while the disease-free equilibrium is globally asymptotically stable for $\mathcal{R}_0 < \mathcal{R}_0^c$ and unstable for $\mathcal{R}_0 > 1$. The existence of a unique globally asymptotically stable endemic equilibrium is proved for $\mathcal{R}_0 > \mathcal{R}_0^m$. In accordance with the methodology of the previous chapters, a NSFD scheme is constructed for the SIS-Volterra integral equation model. It is shown theoretically and computationally that the NSFD scheme is dynamically consistent with the continuous model. The NSFD scheme presented in this chapter is new. This is in fact the first time that the nonstandard approach is rigorously applied to Volterra integral equations, despite the restrictive situation in [42].

In the last chapter, we provide concluding remarks on our findings and on possible future research directions.

Chapter 2

SIS Epidemiological Model

2.1 Introduction

In this chapter, we give the detailed discussion about the classical SIS model. It is the simplest epidemiological model to study the transmission of a disease in a population when individuals move from susceptible (S) class to the infective (I) class and then back to the susceptible class upon recovery. Proposed by Kermack and McKendrick [43] in 1927 to study the transmission dynamics of encephalitis and gonorrhea, the model has got a lot of attention in order to have an insight on different diseases in a population [24, 36, 38, 46, 50, 82].

The study of the SIS model from the numerical analysis point of view, which is also part of this chapter, is even more relevant because most models of interest cannot be solved completely by analytical techniques. In the numerical methods, we have two objectives. First, we design two NSFD schemes which replicate the essential dynamics of the continuous model. Second, we propose a higher order nonstandard finite difference scheme, an approach which to the authors' best knowledge has not been considered in the literature. Most results of this chapter are published in [55]. However, some of the results are improved here. For instance, we prove the global asymptotic stability of the endemic equilibrium point.

The rest of the chapter is arranged in the following order. The SIS model is formulated in the next section. Its quantitative and qualitative properties are discussed in Section 2.3. In Section 2.4, the nonstandard numerical schemes are designed and proved theoretically and computationally that these schemes replicate the properties of the continuous model.

Section 2.5 is devoted to the discussion of a higher order Runge-Kutta approximation for the classical SIS model. Numerical experiments to support the theoretical results are given.

2.2 SIS model

This thesis is exclusively devoted to the SIS epidemiological model in various settings. To establish the background, we start in this chapter with the SIS model as a system of ordinary differential equations. We assume that a population of size $N \equiv N(t)$ consists of only two disjoint compartments, susceptible $S \equiv S(t)$ and infective $I \equiv I(t)$:

$$S + I = N. \tag{2.2.1}$$

The interaction between the susceptible and the infective individuals occurs according to the following biological assumptions:

1. There is
   - a proportional natural death rate μ,
   - a constant number of births μK per unit time with K being the carrying capacity or the maximum possible population size,
   - a fraction γ of infective individuals recovering with no immunity against reinfection.
2. An average member of the population makes contact sufficient to transmit infection with β others per unit time. Consequently, the number of individuals leaving the susceptible compartment to the infective one per unit time is $\frac{\beta SI}{N}$.

These assumptions are shown on Fig. 2.1 and they translate into the following system of ordinary differential equations:

$$\frac{dS}{dt} = \mu K - \frac{\beta IS}{N} - \mu S + \gamma I \tag{2.2.2}$$

$$\frac{dI}{dt} = \frac{\beta IS}{N} - (\mu + \gamma)I. \tag{2.2.3}$$

The system is appended with the following initial conditions:

$$S(0) = S_0 \quad \text{and} \quad I(0) = I_0. \tag{2.2.4}$$

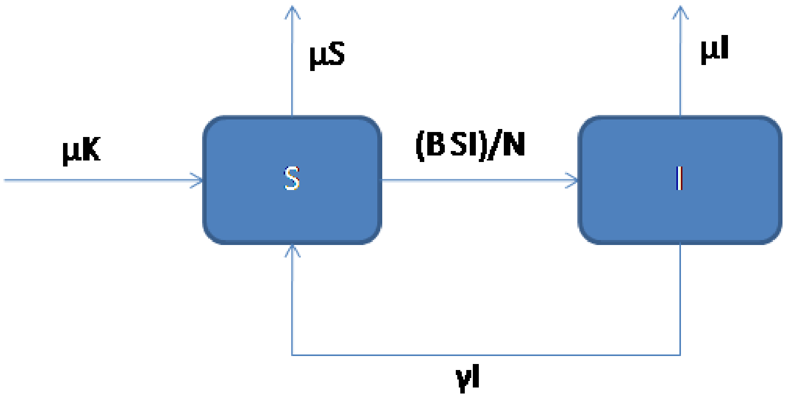

Figure 2.1: Flow chart for SIS model.

Remark 2.2.1. *Assumption (2) above refers to the so-called standard incidence formulation, which is to be contrasted with the mass action incidence formulation that would lead to βSI as number of new infective individuals per unit time. Extensions where the contact rate β is a function of the total population N are investigated in [77]. In Chapter 4, we will consider the case when the contact rate β depends on the fraction of infective individuals I.*

In its more general form, the SIS model also includes a disease induced death rate α, which changes Eq. (2.2.3) into

$$\frac{dI}{dt} = \frac{\beta SI}{N} - (\mu + \alpha + \gamma)I.$$

However, throughout this thesis, we assume that $\alpha = 0$.

Adding (2.2.2) and (2.2.3), yields the conservation law

$$\frac{dN}{dt} = \mu(K - N), \tag{2.2.5}$$

which for initial condition N_0 has exact solution

$$N(t) = K - (K - N_0)e^{-\mu t} \tag{2.2.6}$$

such that

$$0 \leq N(t) \leq K, \quad \text{whenever} \quad 0 \leq N_0 \leq K. \tag{2.2.7}$$

2.3 Quantitative and qualitative analysis

We start with the well-posedness result, which is stated as follows:

Theorem 2.3.1. *The SIS model (2.2.2)-(2.2.3) is a dissipative dynamical system on the biologically feasible region*

$$\Omega = \{(S, I) \in \mathbb{R}^2_+ : \; 0 \leq S + I = N \leq K\}.$$

Proof: We want to show that for $S_0 \geq 0, \; I_0 \geq 0$ with $S_0 + I_0 = N_0 > 0$, the system (2.2.2)-(2.2.3) possesses at all time $t \geq 0$ a unique solution $S(t)$ and $I(t)$ which lies in the region Ω.

In a first step, we show that any solution of (2.2.2)-(2.2.3) corresponding to nonnegative initial conditions S_0 and I_0 such that $S_0 + I_0 = N_0 > 0$ is nonnegative.

The situation is straightforward for the infective individuals because by (2.2.1), Eq. (2.2.3) becomes the Bernoulli equation

$$\frac{dI}{dt} = (\beta - \mu - \gamma)I - \frac{\beta}{N}I^2, \tag{2.3.1}$$

which has a solution

$$I(t) = \frac{e^{(\beta-\mu-\gamma)t}}{I_0 + \beta\int_0^t \frac{e^{(\beta-\mu-\gamma)u}}{N}du} \geq 0 \quad \forall t > 0. \tag{2.3.2}$$

In view of (2.3.2), the equation (2.2.2) of the susceptible individuals implies that

$$\frac{dS}{dt} + \left(\frac{\beta I}{N} + \mu\right) S \geq \mu K,$$

which yields with

$$P(t) = \frac{\beta I}{N} + \mu \quad \text{and} \quad \rho(t) = e^{\int_0^t P(u)du} > 0,$$

$$S(t) \geq e^{-\int_0^t P(u)du}\left(S_0 + \mu K \int_0^t P(u)du\right) \geq 0.$$

In a second step, we use the a priori estimate (2.2.7) together with the fact that the right-hand side of (2.2.2)-(2.2.3) is a locally Lipschitz function in the arguments S and I.

Combining these two steps, we conclude from a well-known result (see Theorem 2.1.5 in [76]) that (2.2.2)-(2.2.3) defines a dynamical system on Ω. $\square$

Equation (2.2.3) of infective individuals can be written as

$$\frac{dI}{dt} = (\mu+\gamma)\left(\mathcal{R}_0\frac{S}{N}-1\right)I, \tag{2.3.3}$$

$$\text{where} \quad \mathcal{R}_0 = \frac{\beta}{\mu+\gamma}. \tag{2.3.4}$$

A close look at (2.3.3) shows two interesting cases based on the values of $\mathcal{R}_0$. Firstly, if $\mathcal{R}_0 < 1$, it follows from (2.3.3) that, for the solution (2.2.2)-(2.2.3), we have

$$\frac{dI}{dt} \leq 0,$$

so that I is decreasing. Hence, we expect the disease to die out.

Secondly, if $\mathcal{R}_0 > 1$ and provided that there are sufficiently enough susceptible individuals $S(t)$, we have

$$\frac{dI}{dt} \geq 0$$

so that I is increasing. Thus, there will be an endemic situation. Indeed if $S(t) \in \left(\frac{K}{\mathcal{R}_0}, K\right)$, then we have from (2.3.3)

$$\begin{aligned}\mathcal{R}_0\frac{S}{N}-1 &> \mathcal{R}_0\frac{K/\mathcal{R}_0}{N}-1\\ &= \frac{K}{N}-1\\ &\geq 0.\end{aligned}$$

The number $\mathcal{R}_0$ given in (2.3.4) plays an important role in the study of the epidemiological model (2.2.2)-(2.2.3). It is the so-called basic reproduction number defined as follows.

Definition 2.3.2. *[19] The basic reproduction number is the average number of secondary cases produced by a single infective individual in a completely susceptible population.*

Remark 2.3.3. *The expression (2.3.4) for $\mathcal{R}_0$ is in line with Definition 2.3.2. Indeed, the number of susceptible individuals that one infective will infect for the duration*

$$\frac{1}{\mu+\gamma}$$

of its life in the infective class is

$$\mathcal{R}_0 = (\textit{contact rate}) \times (\textit{period of being infective in the infective class}). \tag{2.3.5}$$

Formula (2.3.4) or (2.3.5) is a byproduct of the so-called next generation matrix method [26], which is used for the calculation of $\mathcal{R}_0$ for more general epidemiological models.

The relevance of $\mathcal{R}_0$, mentioned above in the qualitative analysis of the model (2.2.2)-(2.2.3) is made more explicit in the rest of this chapter.

By using $\mathcal{R}_0$, Eq. (2.3.4), the SIS model (2.2.2)-(2.2.3) is equivalent to the scalar equation

$$\frac{dI}{dt} = \beta\left(1 - \frac{1}{\mathcal{R}_0}\right)\left(1 - \frac{I}{N(1-\frac{1}{\mathcal{R}_0})}\right)I. \tag{2.3.6}$$

An equilibrium solution of $E = (S, I)$ of the system (2.2.2)-(2.2.3) is given by the algebraic equations:

$$\begin{aligned} \mu K - \frac{\beta SI}{N} - \mu S + \gamma I &= 0 & (2.3.7)\\ \frac{\beta SI}{N} - (\mu+\gamma)I &= 0. & (2.3.8) \end{aligned}$$

Solving (2.3.7)-(2.3.8), we obtain a trivial solution

$$E_0 = (\bar{S}, \bar{I}) = (K, 0), \tag{2.3.9}$$

which is called the disease-free equilibrium point, as well as a nontrivial solution

$$E_\infty = (S_\infty, I_\infty) = \left(\frac{K}{\mathcal{R}_0}, K(1 - \frac{1}{\mathcal{R}_0})\right) \quad \text{for} \quad \mathcal{R}_0 > 1, \tag{2.3.10}$$

which is called the endemic equilibrium point.

Notice that the endemic equilibrium point (2.3.10) satisfies (2.3.11) and (2.3.12):

$$\begin{aligned} \mu K &= \mu(S_\infty + I_\infty), & (2.3.11)\\ \frac{\beta S_\infty}{S_\infty + I_\infty} &= \mu + \gamma. & (2.3.12) \end{aligned}$$

In the rest of this section, we determine the stability properties of the equilibrium points.

Theorem 2.3.4. *The disease-free equilibrium E_0 of the system (2.2.2)-(2.2.3) is locally asymptotically stable (LAS) when $\mathcal{R}_0 < 1$ and unstable for $\mathcal{R}_0 > 1$.*

Proof: The result is obtained by Hartman-Grobman Theorem [76]. The Jacobian matrix of the right-side of (2.2.2)-(2.2.3) at E_0 is

$$J(E_0) = \begin{pmatrix} -\mu & -\beta+\gamma \\ 0 & \beta - (\mu+\gamma) \end{pmatrix}, \tag{2.3.13}$$

which has eigenvalues $r_1 = -\mu < 0$ and $r_2 = \beta - (\mu + \gamma)$. In view of the expression of $\mathcal{R}_0$ in (2.3.4), it is clear that the equilibrium point E_0 is hyperbolic for $\mathcal{R}_0 \neq 1$ and $r_2 < 0$ if and only if $\mathcal{R}_0 < 1$. $\square$

Theorem 2.3.5. *For the dissipative dynamical system (2.2.2)-(2.2.3), the disease-free equilibrium is globally asymptotically stable (GAS) if* $\mathcal{R}_0 \leq 1$.

Proof: To prove the global asymptotic stability of E_0, we use LaSalle Invariance Principle [48], which amounts to checking three main steps that we list below for convenience and further use in the thesis.

1. *Construction of a Lyapunov function*:

   We consider the function

$$V : \Omega \subset \mathbb{R}^2 \to \mathbb{R}, \qquad V(S, I) = I. \tag{2.3.14}$$

   It is clear that V is positive definite i.e. $V(E_0) = 0$ and $V(E) > 0$ for $E_0 \neq E \in \Omega$. Let $(S, I) \in \Omega$. Denote by $\mathbf{f}(S, I)$ the vector-function in the right-side of (2.2.2)-(2.2.3) and by $\dot{V}$ the derivative along the trajectories (or the directional derivative of V in the direction of $\mathbf{f}(S, I)$). Then we have

$$\begin{aligned}
\dot{V} &= \nabla V.f(S, I) \\
&= (0, 1).f(S, I) \\
&= \left(\frac{\beta S}{N} - \mu - \gamma\right)I, \quad \text{by} \quad (2.2.3) \\
&= (\mu + \gamma)\left(\frac{\mathcal{R}_0 S}{N} - 1\right) I \quad \text{by} \quad (2.3.4) \qquad (2.3.15)\\
&\leq (\mu + \gamma)(\mathcal{R}_0 - 1)I \quad \text{since} \quad S \leq N. \qquad (2.3.16)
\end{aligned}$$

   Thus, $\dot{V} \leq 0$ on Ω if $\mathcal{R}_0 \leq 1$. Hence, in this case, V is a (linear) Lyapunov function for E_0 on Ω.

   In the particular case when $(S(t), I(t)) \in \Omega$ is a solution of (2.2.2)-(2.2.3) and $\mathcal{R}_0 \leq 1$, we have $\dot{V} = \frac{dV}{dt} = \frac{dI}{dt}$. It follows then from (2.3.15) that

$$\frac{dI}{dt}(t) = 0 \quad \forall t \geq 0 \quad \Leftrightarrow I(t) = 0 \quad \forall t \geq 0. \tag{2.3.17}$$

2. *Computation of $\mathcal{M}$, the largest invariant set contained in $\mathcal{E} = \{(S, I) \in \Omega : \dot{V} = 0\}$*:

   We claim that $\mathcal{M} = \{E_0\}$ whenever $\mathcal{R}_0 \leq 1$. To show this, let $\mathcal{B} \subset \mathcal{E}$ be an invariant set. Take $(S_0, I_0) \in \mathcal{B}$. Then the solution $(S(t), I(t))$ of (2.2.2)-(2.2.3) with initial condition (S_0, I_0) is in $\mathcal{B} \subseteq \mathcal{E}$, for $t \geq 0$. Using (2.3.17), we have

$I(t) = 0$ for all $t \geq 0$. The solution of (2.2.2) and (2.2.4) is then $S(t) = K - (K - S_0)e^{-\mu t}$ such that $S_0 = K$ because the contrary of this would imply that the disease-free equilibrium $(K, 0)$ is locally asymptotically stable for $\mathcal{R}_0 > 1$. This implies $(S(t), I(t)) = (K, 0) \in \mathcal{B}$ and $\mathcal{B} \subseteq \{E_0\}$. Therefore, $\mathcal{M} = \{E_0\}$.

3. *E_0 is globally asymptotically stable for the system (2.2.2)-(2.2.3) restricted to $\mathcal{M}$*:

   This is obvious because $\mathcal{M} = \{E_0\}$.

   Combining items (1), (2) and (3) above, we conclude by LaSalle Invariance Principle that E_0 is globally asymptotically stable on Ω for $\mathcal{R}_0 \leq 1$. $\square$

The fact stated in Theorem 2.3.4 that the disease-free equilibrium is unstable when $\mathcal{R}_0 > 1$ is made more precise in the next result.

Theorem 2.3.6. *The endemic equilibrium E_∞ of (2.2.2)-(2.2.3) is globally asymptotically stable in the interior of Ω if $\mathcal{R}_0 > 1$.*

Proof: We will apply again LaSalle Invariance Principle following the three steps mentioned in the proof of Theorem 2.3.5.

First, we construct a Lyapunov function on the set $\Omega^* = \{(S, I) \in \Omega : S, I > 0\}$. We consider as in [80] the real-valued function V of class C^∞ defined by

$$\begin{aligned} V(S,I) &= (S - S_\infty) + (I - I_\infty) - (S_\infty + I_\infty) \ln\left(\frac{S+I}{S_\infty + I_\infty}\right) \\ &+ \frac{2\mu}{\beta I_\infty}(S_\infty + I_\infty)\left(I - I_\infty - I_\infty \ln\left(\frac{I}{I_\infty}\right)\right). \end{aligned} \tag{2.3.18}$$

We have

$$V(E_\infty) = 0,$$

$$\frac{\partial V}{\partial S} = 1 - \frac{S_\infty + I_\infty}{S+I}, \qquad \frac{\partial V}{\partial I} = 1 - \frac{S_\infty + I_\infty}{S+I} + \frac{2\mu}{\beta I_\infty}(S_\infty + I_\infty)(1 - \frac{I_\infty}{I}),$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{S_\infty + I_\infty}{(S+I)^2}, \qquad \frac{\partial^2 V}{\partial I^2} = \frac{S_\infty + I_\infty}{(S+I)^2} + \frac{2\mu I_\infty (S_\infty + I_\infty)}{\beta I^2} \quad \text{and} \quad \frac{\partial^2 V}{\partial S \partial I} = \frac{S_\infty + I_\infty}{(S+I)^2}.$$

We observe that E_∞ is a critical point of the function V i.e. $\frac{\partial V}{\partial S}(E_\infty) = 0 = \frac{\partial V}{\partial I}(E_\infty)$. Furthermore,

$$D(E_\infty) := \frac{\partial^2 V}{\partial S^2}(E_\infty)\frac{\partial^2 V}{\partial I^2}(E_\infty) - \frac{\partial^2 V}{\partial S \partial I}(E_\infty) > 0 \quad \text{and} \quad \frac{\partial^2 V}{\partial S^2}(E_\infty) > 0.$$

Thus, by the condition for the existence of the extreme values for a function of several variables, V attains its global minimum at E_∞: $V(E_\infty) \leq V(E), \quad \forall E \in \Omega^*$.

Computing the derivative of V along the solutions of (2.2.2)-(2.2.3), we obtain

$$\begin{aligned}
\dot{V} &= \frac{[(S+I)-(S_\infty+I_\infty)]}{S+I}(\mu K - \mu(S+I)) \\
&+ \frac{2\mu}{\beta I_\infty}(S_\infty+I_\infty)\frac{(I-I_\infty)}{I}\left(\frac{\beta IS}{S+I}-(\mu+\gamma)I\right).
\end{aligned}$$

Using (2.3.11)-(2.3.12), we have

$$\begin{aligned}
\dot{V} &= \frac{[(S-S_\infty)+(I-I_\infty)]}{S+I}(\mu(S_\infty+I_\infty)-\mu(S+I)) \\
&+ \frac{2\mu}{\beta I_\infty}(S_\infty+I_\infty)(I-I_\infty)\left(\frac{\beta S}{S+I}-\frac{\beta S_\infty}{S_\infty+I_\infty}\right) \\
&= \frac{[(S-S_\infty)+(I-I_\infty)]}{S+I}(-\mu(S-S_\infty)-\mu(I-I_\infty)) \\
&+ \frac{2\mu}{I_\infty}(S_\infty+I_\infty)(I-I_\infty)\left(\frac{S}{S+I}-\frac{S_\infty}{S_\infty+I_\infty}\right).
\end{aligned}$$

By using the relation

$$\frac{S}{S+I}-\frac{S_\infty}{S_\infty+I_\infty} = \frac{I_\infty(S-S_\infty)-S_\infty(I-I_\infty)}{(S+I)(S_\infty+I_\infty)},$$

we obtain

$$\begin{aligned}
\dot{V} &= \frac{[(S-S_\infty)+(I-I_\infty)]}{S+I}(-\mu(S-S_\infty)-\mu(I-I_\infty)) \\
&+ \frac{2\mu}{I_\infty}(S_\infty+I_\infty)(I-I_\infty)\left(\frac{I_\infty(S-S_\infty)-S_\infty(I-I_\infty)}{(S+I)(S_\infty+I_\infty)}\right).
\end{aligned}$$

Further simplifications give

$$\dot{V} = \frac{-\mu(S-S_\infty)^2}{S+I} - \left(\mu + 2\mu\frac{S_\infty}{I_\infty}\right)\frac{(I-I_\infty)^2}{S+I}. \tag{2.3.19}$$

Clearly, $\dot{V}(E_\infty)=0$ and $\dot{V}(E)<0$ for $E \neq E_\infty$, which show that V is a Lyapunov function of Volterra type on Ω^*.

The second and the third steps are straightforward since from (2.3.19)

$$\mathcal{M} = \mathcal{E} = \{(S,I) : \frac{dV}{dt}(S,I) = 0\} = \{E_\infty\}.$$

Therefore, the endemic equilibrium E_∞ is globally asymptotically stable in Ω^*. $\square$

Remark 2.3.7. *For $\mathcal{R}_0 > 1$, the local asymptotic stability of the endemic equilibrium of the SIS model (2.2.2)-(2.2.3) can be obtained by the linearization procedure. The Jacobian matrix of the right-side of (2.2.2)-(2.2.3) at E_∞ is*

$$\begin{aligned} J(E_\infty) &= \begin{pmatrix} -\beta(1-\frac{1}{\mathcal{R}_0})-\mu & -\frac{\beta}{\mathcal{R}_0}+\gamma \\ \beta(1-\frac{1}{\mathcal{R}_0}) & \frac{\beta}{\mathcal{R}_0}-(\mu+\gamma) \end{pmatrix}, \\ &= \begin{pmatrix} -\beta(1-\frac{1}{\mathcal{R}_0})-\mu & -\mu \\ \beta(1-\frac{1}{\mathcal{R}_0}) & 0 \end{pmatrix}, \qquad \textit{by} \quad (2.3.4). \end{aligned}$$

It is clear that the endemic equilibrium E_∞ is hyperbolic for $\mathcal{R}_0 \neq 1$ and the trace of the matrix $J(E_\infty)$ is $-\beta(1-\frac{1}{\mathcal{R}_0})-\mu < 0$ for $\mathcal{R}_0 > 1$. For the eigenvalues of $J(E_\infty)$ to have negative real parts, the determinant should be positive i.e. $\mu\beta(1-\frac{1}{\mathcal{R}_0}) > 0$. This is true when $\mathcal{R}_0 > 1$. Hence, the endemic equilibrium point E_∞ in (2.3.10) is locally asymptotically stable for $\mathcal{R}_0 > 1$.

2.4 Nonstandard finite difference schemes

In this section, we elaborate the proofs of some of the results published in [55]. We design nonstandard finite difference (NSFD) numerical schemes that are dynamically consistent with the properties of the SIS model (2.2.2)-(2.2.5). The time variable $t \in [0, +\infty)$ is discretized by the grid points $t_k = k\triangle t$, where $k = 0, 1, 2, \ldots$ and $\triangle t > 0$ being the time step size that we shall sometimes denote by h.

Following the strategy in [64], it is important to start with the conservation law (2.2.5) and its exact solution in (2.2.6).

At the discrete time $t = t_{k+1} = (k+1)\Delta t$, the exact solution (2.2.6) is

$$N(t_{k+1}) = K - (K - N_0)e^{-\mu t_{k+1}}, \tag{2.4.1}$$

which by the semi-group property of the evolution operator for differential equations becomes

$$N(t_{k+1}) = K - (K - N(t_k))e^{-\mu\Delta t}. \tag{2.4.2}$$

Subtracting $N(t_k)$ from both sides of Eq. (2.4.2), we get

$$N(t_{k+1}) - N(t_k) = (K - N(t_k))(1 - e^{-\mu\Delta t}). \tag{2.4.3}$$

Hence, Eq. (2.4.3) is equivalent to

$$\frac{N(t_{k+1}) - N(t_k)}{(1 - e^{-\mu\Delta t})/\mu} = \mu(K - N(t_k)). \tag{2.4.4}$$

Setting $N^k := N(t_k)$, Eq. (2.4.4) gives the exact scheme

$$\frac{N^{k+1} - N^k}{(1 - e^{-\mu\Delta t})/\mu} = \mu(K - N^k). \tag{2.4.5}$$

Apart from the forward Euler type scheme (2.4.5), we have its backward counterpart

$$\frac{N^{k+1} - N^k}{(e^{\mu\Delta t} - 1)/\mu} = \mu(K - N^{k+1}). \tag{2.4.6}$$

In the exact schemes (2.4.5) and (2.4.6), the traditional denominator of discrete derivatives is replaced by complex functions that satisfy the asymptotic relation (2.4.8) below. However, in view of our future needs, we choose the denominator function

$$\phi(h) = \frac{1 - e^{-(\mu+\gamma)h}}{\mu + \gamma}, \tag{2.4.7}$$

which is such that

$$\phi(h) = h + \mathcal{O}(h^2), \quad \text{where} \quad h = \Delta t. \tag{2.4.8}$$

The denominator function ϕ being fixed, we consider for the SIS model (2.2.2)-(2.2.3) two NSFD schemes. The first NSFD scheme reads as follows:

$$\frac{S^{k+1} - S^k}{\phi} = \mu K - \frac{\beta S^{k+1} I^k}{S^{k+1} + I^k} - \mu S^{k+1} + \gamma I^k \tag{2.4.9}$$

$$\frac{I^{k+1} - I^k}{\phi} = \frac{\beta S^{k+1} I^k}{S^{k+1} + I^k} - \mu I^{k+1} - \gamma I^k. \tag{2.4.10}$$

It leads to the conservation law

$$\frac{N^{k+1} - N^k}{\phi} = \mu(K - N^{k+1}) \quad \text{or} \quad N^{k+1} = \frac{\mu\phi K + N^k}{1 + \mu\phi}, \tag{2.4.11}$$

which is similar to the implicit exact scheme given in Eq. (2.4.6).

The discrete conservation law related to the second NSFD scheme is inspired by the explicit exact scheme (2.4.5) and reads as follows:

$$\frac{N^{k+1} - N^k}{\phi} = \mu(K - N^k) \quad \text{or} \quad N^{k+1} = \mu\phi K + (1 - \mu\phi)N^k. \tag{2.4.12}$$

The second NSFD scheme is

$$\begin{aligned}\frac{S^{k+1}-S^k}{\phi} &= \mu K - \frac{\beta S^{k+1} I^k}{N^{k+1}} - \mu S^k + \gamma I^k &(2.4.13)\\ \frac{I^{k+1}-I^k}{\phi} &= \frac{\beta S^{k+1} I^k}{N^{k+1}} - (\mu+\gamma) I^k, &(2.4.14)\end{aligned}$$

where N^{k+1} is obtained from (2.4.12).

By using

$$S^{k+1} + I^{k+1} = N^{k+1}, \qquad (2.4.15)$$

the system (2.4.13)-(2.4.14) is equivalent to the scheme

$$\frac{I^{k+1}-I^k}{\phi} = \beta(1-\frac{1}{\mathcal{R}_0})\left(1-\frac{I^{k+1}}{N^{k+1}(1-\frac{1}{\mathcal{R}_0})}\right) I^k, \qquad (2.4.16)$$

which is inspired by the exact scheme of the logistic equation (2.3.6) when $N = K =$ constant [61].

The implementation of the second NSFD scheme is straightforward because (2.4.13)-(2.4.14) can be written as

$$\begin{aligned}S^{k+1} &= \frac{\mu\phi(K-S^k) + S^k + \gamma\phi I^k}{1+\frac{\beta\mu\phi I^k}{N^{k+1}}} &(2.4.17)\\ I^{k+1} &= \left(\frac{\beta\phi S^{k+1}}{N^{k+1}} + 1 - (\mu+\gamma)\phi\right) I^k &(2.4.18)\end{aligned}$$

and its equivalent form (2.4.16) is also equivalent to

$$I^{k+1} = \frac{N^{k+1}(1-\frac{1}{\mathcal{R}_0})\left(\beta(1-\frac{1}{\mathcal{R}_0})\phi+1\right) I^k}{N^{k+1}(1-\frac{1}{\mathcal{R}_0}) + \beta(1-\frac{1}{\mathcal{R}_0})\phi I^k}, \qquad (2.4.19)$$

which shows that $I^{k+1} \geq 0$ for $\mathcal{R}_0 > 1$.

On the contrary, more attention is needed for the implementation of the first NSFD scheme (2.4.9)-(2.4.10). This is achieved by using the Gauss-Seidel cycle as described below.

Assume that $S^k \geq 0$ and $I^k \geq 0$ are known and put

$$\begin{aligned}A &= 1+\mu\phi &(2.4.20)\\ B &= (1+(\beta+\mu-\gamma)\phi) I^k - (S^k + \mu\phi K) &(2.4.21)\\ C &= -\left(S^k + \mu\phi K + \gamma\phi I^k\right) I^k. &(2.4.22)\end{aligned}$$

Then Eq. (2.4.9) is equivalent to the following quadratic equation in S^{k+1}:

$$A(S^{k+1})^2 + BS^{k+1} + C = 0. \tag{2.4.23}$$

The unique positive root of (2.4.23) is given by

$$S^{k+1} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \tag{2.4.24}$$

whereas (2.4.10) yields

$$I^{k+1} = \frac{\left(\frac{\beta}{S^{k+1}+I^k}\phi S^{k+1} + (1-\gamma\phi)\right) I^k}{1+\mu\phi}, \tag{2.4.25}$$

where

$$1-\gamma\phi > 0 \quad \text{and} \quad 1-(\mu+\gamma)\phi > 0 \quad \text{by} \quad (2.4.7). \tag{2.4.26}$$

Theorem 2.4.1. *The NSFD schemes (2.4.9)-(2.4.10) and (2.4.13)-(2.4.14) are dynamically consistent with the property of the SIS model stated in Theorem 2.3.1. More precisely, the NSFD schemes are dynamical systems on*

$$\Omega = \{(S, I) \in \mathbb{R}^2_+ : 0 \leq S + I = N \leq K\}.$$

Proof: Let $(S^k, I^k) \in \Omega$ for $k \in \mathbb{N}$. From Eq. (2.4.17)-(2.4.18) and (2.4.24)-(2.4.25), we infer that $S^{k+1} \geq 0$ and $I^{k+1} \geq 0$. By using the second relations in Eq. (2.4.11)-(2.4.12) and Eq. (2.4.15), we have $(S^{k+1}, I^{k+1}) \in \Omega$. We conclude by mathematical induction that the NSFD schemes are dynamical systems on Ω. □

Next we determine the fixed points of the two NSFD schemes. For the first NSFD scheme (2.4.9)-(2.4.10), we consider its equivalent form (2.4.24)-(2.4.25) with the second relation in Eq. (2.4.11) and we set

$$\frac{-B + \sqrt{B^2 - 4AC}}{2A} = S, \tag{2.4.27}$$

$$\frac{\left(\frac{\beta}{S+I}\phi S + (1-\gamma\phi)\right) I}{1+\mu\phi} = I, \tag{2.4.28}$$

$$\frac{\mu\phi K + N}{1+\mu\phi} = N, \tag{2.4.29}$$

where B and C in (2.4.27) are evaluated in terms of S and I from (2.4.21) and (2.4.22). From Eq. (2.4.29), we get $N = K$. Eq. (2.4.28) leads to two cases.

The first case is when $I = 0$. Then $C = 0, \quad B = -(S + \mu\phi K)$ by (2.4.21) and (2.4.22). Solving the equation in (2.4.27) for S, we get $S = K$. Thus, we obtain the fixed point $(K, 0)$, which coincides with the equilibrium point E_0 given in (2.3.9).

The second case is $I > 0$ so that

$$\frac{\frac{\beta}{N}\phi S + (1 - \gamma\phi)}{1 + \mu\phi} = 1, \quad \text{where} \quad N = K.$$

It follows from the definition of $\mathcal{R}_0$ in (2.3.4) that $S = \frac{K}{\mathcal{R}_0}$. In view of (2.4.15), we have

$$I = K\left(1 - \frac{1}{\mathcal{R}_0}\right) > 0, \quad \text{for} \quad \mathcal{R}_0 > 1,$$

and thus the (endemic) fixed point coincides with the endemic equilibrium point E_∞ given in (2.3.10). Hence, we have proved that the first NSFD scheme (2.4.9)-(2.4.10) preserves the equilibrium points of the continuous model (2.2.2)-(2.2.3).

Since a similar reasoning to the one used above works for the second NSFD scheme, we have established the following result.

Theorem 2.4.2. *The equilibrium points E_0 and E_∞ in (2.3.9) and (2.3.10) are the only fixed points of the NSFD schemes (2.4.9)-(2.4.10) or (2.4.24)-(2.4.25) and (2.4.13)-(2.4.14) or (2.4.17)-(2.4.18), respectively.*

In the remaining part of this section, we want to check that the NSFD schemes replicate the stability properties of the equilibrium points of the continuous model (2.2.2)-(2.2.3).

Theorem 2.4.3. *The NSFD scheme (2.4.9)-(2.4.10) is dynamically consistent with Theorem 2.3.4. More precisely, the disease-free fixed point E_0 in (2.3.9) is locally asymptotically stable for $\mathcal{R}_0 < 1$ and unstable when $\mathcal{R}_0 > 1$.*

Proof: We use Hartman-Grobman theorem for discrete dynamical systems ([76]), which as we will see shortly applies $\mathcal{R}_0 \neq 1$. The Jacobian matrix of the system (2.4.9)-(2.4.10) at the disease-free fixed point E_0 is

$$J(E_0) = \begin{pmatrix} \frac{1}{1+\mu\phi} & \frac{-(\beta-\gamma)\phi}{1+\mu\phi} \\ & \\ 0 & \frac{1+(\beta-\gamma)\phi}{1+\mu\phi} \end{pmatrix}.$$

Using Eq. (2.3.4), the characteristic equation can be written as

$$det(rI - J) = \left(r - \frac{1}{1+\mu\phi} \right) \left(r - \frac{1 + (\mathcal{R}_0\mu + (\mathcal{R}_0 - 1)\gamma)\phi}{1+\mu\phi} \right) = 0.$$

This shows that

$$r_1 = \frac{1}{1+\mu\phi} \quad \text{and} \quad r_2 = \frac{1 + (\mathcal{R}_0\mu + (\mathcal{R}_0 - 1)\gamma)\phi}{1+\mu\phi}$$

are simple eigenvalues of $J(E_0)$. Clearly, the modulus of r_1 is less than one and the disease-free fixed point is hyperbolic for $\mathcal{R}_0 \neq 1$. The modulus of r_2 is less than one when $\mathcal{R}_0 < 1$ and greater than one when $\mathcal{R}_0 > 1$. Hence, E_0 is locally asymptotically stable for $\mathcal{R}_0 < 1$ and unstable for $\mathcal{R}_0 > 1$. $\square$

Theorem 2.4.4. *The NSFD scheme (2.4.9)-(2.4.10) is dynamically consistent with Theorem 2.3.5, i.e. for $\mathcal{R}_0 \leq 1$, the disease-free fixed point E_0 in (2.3.9) is globally asymptotically stable on Ω.*

Proof: To prove the global asymptotic stability of E_0, we use the LaSalle Invariance Principle [7] for difference equations. The three main steps of the principle are checked below.

1. *Construction of a Lyapunov function*:

   We consider the positive definite function V given in (2.3.14). Let $F : \Omega \subset \mathbb{R}^2 \to \Omega \subset \mathbb{R}^2$ be the vector function defined by the right-side of (2.4.24)-(2.4.25). That is, for $X = (S, I) \in \Omega$, we have $\mathrm{F}(X) = (f_1(X), f_2(X))$, where:

   - $f_1(X) = \dfrac{-B + \sqrt{B^2 - 4AC}}{2A} > 0$, A is given in (2.4.20), $B \equiv B(X)$ and $C \equiv C(X)$ are obtained by using (2.4.21) and (2.4.22) respectively,
   - $f_2(X) = \dfrac{\left[\frac{\beta\phi f_1(X)}{f_1(X)+I} + 1 - \gamma\phi\right] I}{1+\mu\phi}$.

Fix $X \in \Omega$. By definition

$$
\begin{aligned}
V[\mathrm{F}(X))] &= f_2(X) \\
&= \frac{\left(\frac{\beta\phi f_1(X)}{f_1(X)+I} + 1 - \gamma\phi\right) I}{1+\mu\phi} \\
&= \frac{\left(\frac{\mathcal{R}_0(\mu+\gamma)\phi f_1(X)}{f_1(X)+I} + 1 - \gamma\phi\right) I}{1+\mu\phi} \quad \text{by} \quad (2.3.4) \qquad (2.4.30) \\
&\leq \frac{(1+\mathcal{R}_0\mu\phi)\, I}{1+\mu\phi}, \quad \text{since} \quad \frac{f_1(X)}{f_1(X)+I} \leq 1 \quad \text{and} \quad \mathcal{R}_0 \leq 1 \\
&\leq I, \quad \text{since} \quad \mathcal{R}_0 \leq 1 \\
&= V(X) \quad \text{by} \quad (2.3.14).
\end{aligned}
$$

Hence, $\triangle V(X) := V(\mathrm{F}(X))) - V(X) \leq 0$ for $\mathcal{R}_0 \leq 1$. Thus, V is a Lyapunov function for the disease-free fixed point E_0 on Ω.

2. *Computation of $\mathcal{M}$, the largest invariant set contained in $\mathcal{E} = \{X \in \Omega : V(\mathrm{F}(X)) = V(X)\}$:*

   We claim that $\mathcal{M} = \{E_0\}$ whenever $\mathcal{R}_0 \leq 1$. Indeed from (2.4.30), it follows that for $\mathcal{R}_0 \leq 1$ and $X \in \Omega$

$$V[F(X)] = V(x) \Leftrightarrow I = 0.$$

   Thus, if $X \in \mathcal{M}$, then $X = (S, 0)$. Now plugging this X into the equation (2.4.24) and solving for S, we have $S = K$. This proves the claim.

3. *E_0 is globally asymptotically stable for the system (2.4.9)-(2.4.10) restricted to $\mathcal{M}$*: This is true because $\mathcal{M} = \{E_0\}$.

   By combining items (1), (2) and (3) given above, we infer from LaSalle Invariance Principle for discrete dynamical systems that the disease-free fixed point E_0 is globally asymptotically stable on Ω for $\mathcal{R} \leq 1$. $\square$.

To study the stability properties of the endemic fixed point E_∞ of the first NSFD scheme (2.4.9)-(2.4.10), we need the following notation:

Let a function $\varphi : \mathbb{R} \mapsto \mathbb{R}$ satisfy the asymptotic relation (2.4.8) such that

$$0 < \varphi(h) < 1 \quad \text{for} \quad h > 0. \qquad (2.4.31)$$

Let r_1 and r_2 be the eigenvalues of the Jacobian matrices of the right-side of (2.2.2)-(2.2.3) at the disease-free equilibrium and endemic equilibrium. Let us define a denominator function

$$\phi(h) = \frac{\varphi(qh)}{q}, \quad \text{where} \quad q \geq \max\{|r_1|, |r_2|\}. \tag{2.4.32}$$

Then we have the following result.

Theorem 2.4.5. *For $\mathcal{R}_0 > 1$, the NSFD scheme (2.4.9)-(2.4.10) is elementary stable whenever $\phi(h)$ is chosen according to equations (2.4.31) and (2.4.32).*

Proof: To prove this theorem, we apply the technique in [7] as exploited in [77] for the SIR model with general contact rate. The elementary stability of a discrete scheme, amounts to checking two main facts:

1. The NSFD scheme (2.4.9)-(2.4.10) has only E_0 and E_∞ as fixed points.
2. These fixed points preserve the stability of the continuous system (2.2.2)-(2.2.3) when applied to its linearized system.

Part 1 is shown in Theorem 2.4.2.

Regarding part 2, it is convenient to write the system (2.2.2)-(2.2.3) in matrix form. To this end, we introduce the vector notation $X = (S, I)$. Then our system (2.2.2)-(2.2.3) read

$$\dot{X} = A(X)X + \mathbb{F}, \tag{2.4.33}$$

where

$$A(X) = \begin{pmatrix} \frac{-\beta I}{N} - \mu & \gamma \\ \frac{\beta I}{N} & -(\mu + \gamma) \end{pmatrix} \quad \text{and} \quad \mathbb{F} = \begin{pmatrix} \mu K \\ 0 \end{pmatrix}.$$

Likewise with $X^{k+1} = (S^{k+1}, I^{k+1})$, the NSFD scheme (2.4.9)-(2.4.10) can be written as

$$\frac{X^{k+1} - X^k}{\phi} = B(S^{k+1}, S^k, I^k)X^{k+1} + \mathcal{F}(I^k), \tag{2.4.34}$$

where

$$B(S^{k+1}, S^k, I^k) = \begin{pmatrix} \frac{-\beta I^k}{S^{k+1}+I^k} - \mu & 0 \\ \frac{\beta I^k}{S^{k+1}+I^k} & -\mu \end{pmatrix} \quad \text{and} \quad \mathcal{F}(I^k) = \begin{pmatrix} \mu K + \gamma I^k \\ -\gamma I^k \end{pmatrix}.$$

The linear approximation of the continuous model about the endemic equilibrium point E_∞ is

$$\dot{Y} = J(E_\infty)Y, \quad Y = X - E_\infty, \tag{2.4.35}$$

where $J(E_\infty)$ is the corresponding Jacobian matrix of the right-side of (2.2.2)-(2.2.3) at E_∞. The NSFD scheme (2.4.34) applied to (2.4.35) yields

$$\frac{Y^{k+1} - Y^k}{\phi} = J(E_\infty)Y^{k+1} \quad \text{with} \quad Y^k = X^k - E_\infty, \tag{2.4.36}$$

or equivalently

$$Y^{k+1} = [I - \phi J(E_\infty)]^{-1} Y^k.$$

Notice that here $(I - \phi I(E_\infty))$ is non-singular matrix by definition of $\phi(h)$ in (2.4.32). Notice also that the eigenvalues r_1 and r_2 of $J(E_\infty)$ are complete. Thus, the matrix $J(E_\infty)$ is diagonalizable and we have

$$P^{-1}JP = diag(r_1, r_2),$$

where the columns of P are eigenvectors of $J(E_\infty)$. If we substitute the dependent variable by

$$Z = P^{-1}Y$$

and

$$Z^k = P^{-1}Y^k$$

then equations (2.4.35) and (2.4.36) become

$$\dot{Z} = diag(r_1, r_2)Z$$

and

$$Z^{k+1} = diag(\frac{1}{1 - \phi r_1}, \frac{1}{1 - \phi r_2})Z^k,$$

respectively. Since by Theorem 2.3.6, E_∞ is globally asymptotically stable for (2.4.35), the real parts of the eigenvalues r_1 and r_2 are negative. Hence, for the spectral radius of the matrix $(I - \phi J(E\infty))^{-1}$, we have:

$$\begin{aligned}
\rho(I - \phi J(E_\infty))^{-1} &= \max\{\frac{1}{|1 - \phi r_1|}, \frac{1}{|1 - \phi r_2|}\} \\
&= \max\{\frac{1}{\sqrt{1 - 2(Re r_1)\phi + \phi^2 |r_1|^2}}, \frac{1}{\sqrt{1 - 2(Re r_2)\phi + \phi^2 |r_2|^2}}\} \\
&< 1.
\end{aligned}$$

This shows that E_∞ is locally asymptotically stable for the given NSFD scheme.

As for the Jacobian matrix $J(E_0)$ at the disease-free equilibrium, a similar argument to the above shows that

$$\begin{aligned}\rho(I-\phi J(E_0))^{-1} &= \max\{\frac{1}{1+\mu\phi}, \frac{1}{|1-\phi r_2|}\}\\ &> 1,\end{aligned}$$

because $r_1 = -\mu$ and $r_2 = \beta - (\mu+\gamma) > 0$ for $\mathcal{R}_0 > 1$ (see (2.3.13)). Hence, for $\mathcal{R}_0 > 1$, the disease-free fixed point E_0 is unstable . $\square$

Theorem 2.4.6. *For the NSFD scheme (2.4.9)-(2.4.10), the endemic fixed point E_∞ is globally asymptotically stable if $\mathcal{R}_0 > 1$.*

Proof: By Theorem 2.4.5, E_∞ is LAS for $\mathcal{R}_0 > 1$. Thus, there exists $\delta > 0$ such that

$$|X^0 - E_\infty| \le \delta \Rightarrow \lim_{k\to\infty} |X^k - E_\infty| = 0, \tag{2.4.37}$$

where $X^k = (S^k, I^k) \in \Omega$ is the sequence defined recursively by the NSFD scheme (2.4.9)-(2.4.10) from $X^0 = (S^0, I^0) \in \Omega$. Let us now initiate the sequences $X^k = (S^k, I^k)$ from an arbitrary X^0. We want to show that X^k tends to E_∞ as $k \to \infty$. As the sequence X^k is bounded, Bolzano-Weierstrass theorem implies that there exists a subsequence $(X^{k_j})_{j\ge 0}$ of X^k, which is convergent. Clearly, from (2.4.24) and (2.4.25), we have

$$\lim_{j\to\infty} X^{k_j} = E_\infty.$$

Thus, there exists $j_0 \in \mathbb{N}$ such that

$$j \ge j_0 \Rightarrow |X^{k_j} - E_\infty| < \delta.$$

In particular

$$|X^{k_{j_0}} - E_\infty| < \delta$$

and thus by (2.4.37)

$$\lim_{\substack{k\to\infty\\ k\ge k_{j_0}}} |X^k - E_\infty| = \lim_{k\to\infty} |X^k - E_\infty| = 0.$$

This completes the proof. $\square$

Let us now turn to the qualitative analysis of the second NSFD scheme.

Theorem 2.4.7. *The disease-free fixed point E_0 of the second NSFD scheme (2.4.13)-(2.4.14) or (2.4.17)-(2.4.18) is locally asymptotically stable for $\mathcal{R}_0 < 1$ and unstable for $\mathcal{R}_0 > 1$.*

Proof: This theorem is proved by using Hartman-Grobman linearization theorem [76]. The Jacobian matrix of the system (2.4.17)-(2.4.18) at $E_0 = (K, 0)$ is

$$J(E_0) = \begin{pmatrix} 1 - \mu\phi & \gamma\phi - \mathcal{R}_0(\mu + \gamma)\mu\phi \\ 0 & 1 + (\mathcal{R}_0 - 1)(\mu + \gamma)\phi \end{pmatrix}.$$

The characteristic equation

$$det(rI - J) = (r - (1 - \mu\phi))\,(r - (1 + (\mathcal{R}_0 - 1)(\mu + \gamma)\phi) = 0,$$

has roots

$$r_1 = 1 - \mu\phi \quad \text{and} \quad r_2 = 1 + (\mathcal{R}_0 - 1)(\mu + \gamma)\phi,$$

which in view of (2.4.26) show that $0 < r_1 < 1$. E_0 is a hyperbolic fixed point for $\mathcal{R}_0 \neq 1$. Moreover, from the same Eq. (2.4.26), we have $0 < r_2 < 1$ for $\mathcal{R}_0 < 1$ and $r_2 > 1$ when $\mathcal{R}_0 > 1$. Thus, E_0 is locally asymptotically stable for $\mathcal{R}_0 < 1$ and unstable for $\mathcal{R}_0 > 1$. □

Theorem 2.4.8. *The disease-free fixed point E_0 of the second NSFD scheme (2.4.13)-(2.4.14) or (2.4.17)-(2.4.18) is globally asymptotically stable for $\mathcal{R}_0 \leq 1$.*

Proof: This theorem is proved in a similar manner as Theorem 2.4.4 by using the same Lyapunov function V in (2.3.14). However, the vector-function $F = (f_1, f_2) : \Omega \to \Omega$ to be considered from the right-side of (2.4.17)-(2.4.18) and the second equation of (2.4.12) is defined for $X = (S, I) \in \Omega$ by

$$f_1(X) = \frac{\mu\phi(K - S) + S + \gamma\phi I}{1 + [\beta\mu\phi I/\,(\mu\phi K + (1 - \mu\phi)(S + I))]}$$

and

$$f_2(X) = \left(\frac{\beta\phi f_1(X)}{\mu\phi K + (1 - \mu\phi)(S + I)} + 1 - (\mu + \gamma)\phi\right) I.$$

From the relation (2.4.26), it follows that $f_1(X) \leq \mu\phi K + (1 - \mu\phi)(S + I)$. Thus, the first step of the proof of Theorem 2.4.4 works for the case under consideration. The last two steps of that theorem remain the same. This completes the proof of the theorem. □

Theorem 2.4.9. *Under the condition $\mathcal{R}_0 > 1$, the NSFD scheme (2.4.13)-(2.4.14) or (2.4.16) is elementary stable.*

Proof: Let $\mathcal{R}_0 > 1$. It is clear that the equilibrium points $\bar{E}_0 = (\bar{N}, \bar{I}) = (K, 0)$ and $\bar{E}_\infty = (\bar{N}_\infty, \bar{I}_\infty) = (K, K(1 - \frac{1}{\mathcal{R}_0}))$ of the continuous model (2.2.5) and (2.3.6) are the only fixed points of the discrete scheme (2.4.12) and (2.4.16). The fact that the fixed points preserve the instability of equilibrium point $\bar{E}_0$ and the local asymptotic stability of the equilibrium point $\bar{E}_\infty$ is once again obtained by Hartman-Grobman linearization theorem [76]. □

With the local asymptotic stability of $\bar{E}_\infty$ and Bolzano-Weierstrass theorem, we obtain the following result as in the proof Theorem 2.4.6.

Theorem 2.4.10. *For $\mathcal{R}_0 > 1$, the endemic fixed-point $\bar{E}_\infty$ of the NSFD scheme (2.4.12) and (2.4.16) is globally asymptotically stable.*

Having dealt theoretically and constructively with the susceptible and infective compartments, we now record below the connections between the continuous and the discrete conservation laws.

Theorem 2.4.11. *Both NSFD schemes (2.4.11) and (2.4.12) are dynamically consistent with respect to positivity, boundedness and monotonicity of the exact solution of the conservation law. Furthermore, they preserve the global asymptotic stability of the equilibrium solution $\bar{N} = K$.*

Proof: The positivity and boundedness $0 \leq N^k \leq K$ follows directly by mathematical induction from (2.4.11) and (2.4.12). Moreover, from (2.4.11) and mathematical induction, we get

$$K - N^{k+1} = \frac{K - N^k}{1 + \mu\phi} \leq K - N^k \quad \text{and} \quad K - N^{k+1} = \frac{1}{(1 + \mu\phi)^{k+1}}(K - N^0).$$

Thus the sequence (N^k) is increasing and converges to K as $k \to \infty$.

The conclusion for the scheme (2.4.12) follows in a similar manner because

$$K - N^{k+1} = (1 - \mu\phi)(K - N^k) \leq K - N^k \quad \text{and} \quad K - N^{k+1} = (1 - \mu\phi)^{k+1}(K - N^0).$$

This completes the proof of the theorem. □

We conclude this section by providing numerical simulations to support the established theoretical results.

Let $K = 250$, $\mu = 0.2$ and $\gamma = 0.3$. With initial conditions $S_{01} = 90$, $S_{02} = 80$, $I_{01} = 10$ and $I_{02} = 20$, Theorem 2.4.4 and Theorem 2.4.8 are illustrated in Fig 2.2 and Fig. 2.4 for $\mathcal{R}_0 = 0.6$, respectively. For $\mathcal{R}_0 = 1.2$, Theorem 2.4.6 is displayed in Fig 2.3 (with the same initial conditions given above) and Theorem 2.4.10 is demonstrated in Fig. 2.5. Moreover, Fig 2.6 and Fig 2.7 illustrate the results mentioned in Theorem 2.4.11. The discrete solution for $I(t)$ given by (2.4.16) at $N = K =$ constant is presented in Table 2.1 for different values of h.

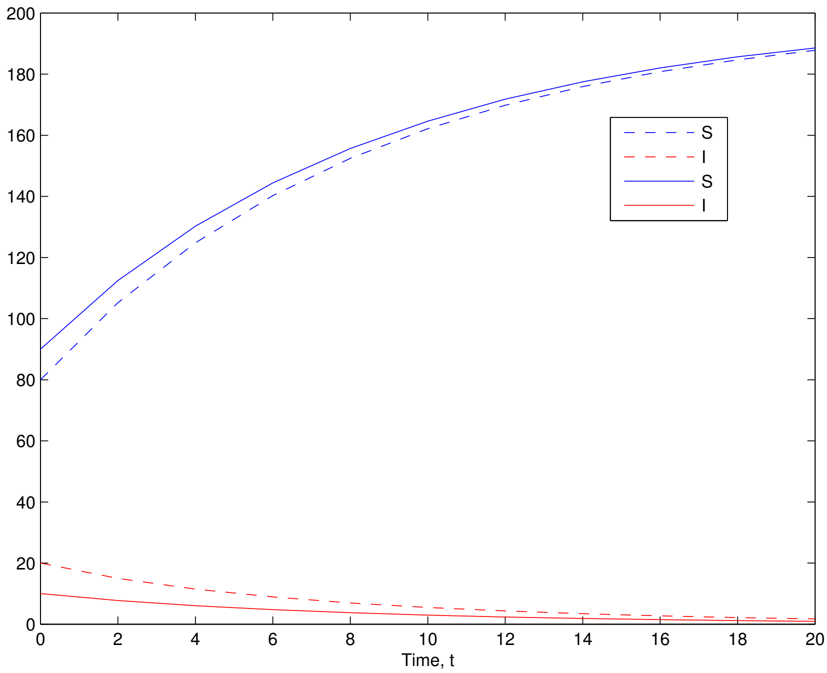

Figure 2.2: GAS of the disease-free fixed point for the NSFD scheme (2.4.9)-(2.4.10) with $\mathcal{R}_0 = 0.6$.

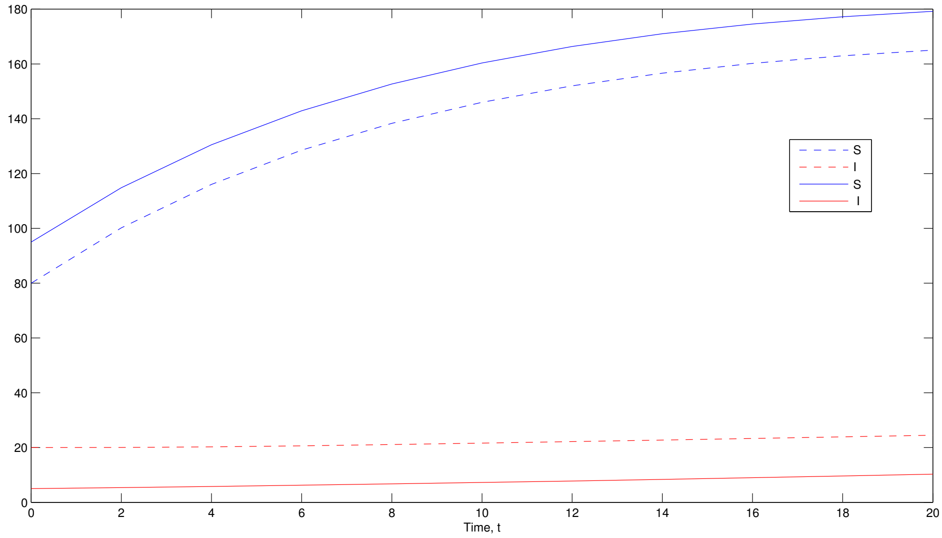

Figure 2.3: GAS of the endemic fixed point for the NSFD scheme (2.4.9)-(2.4.10) with $\mathcal{R}_0 = 1.2$.

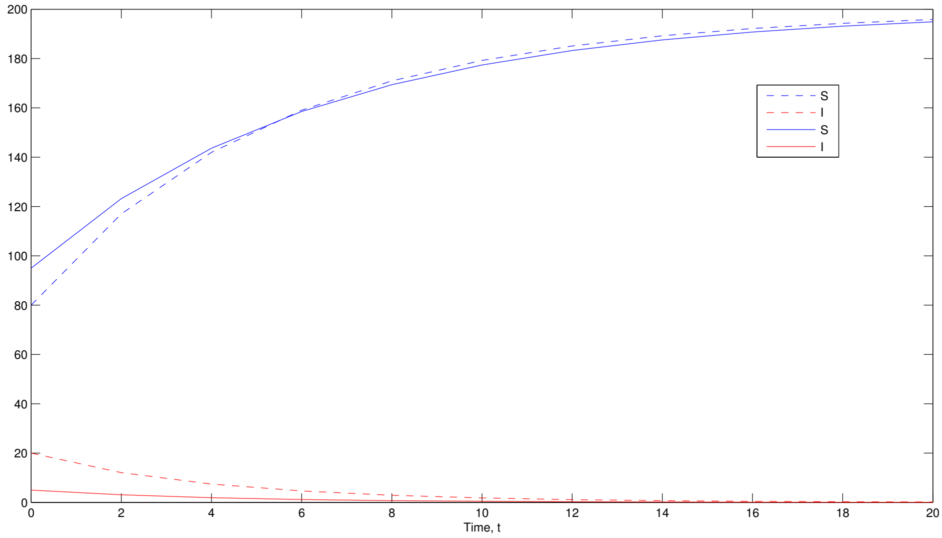

Figure 2.4: GAS of the disease-free fixed point for the NSFD scheme (2.4.13)-(2.4.14) with $\mathcal{R}_0 = 0.6$.

2.5 Nonstandard Runge-Kutta method for SIS model

The classical explicit n-stage Runge-Kutta method to determine the discrete solutions (x^k) at the time $t_k = kh$ for the solution of the initial value problem

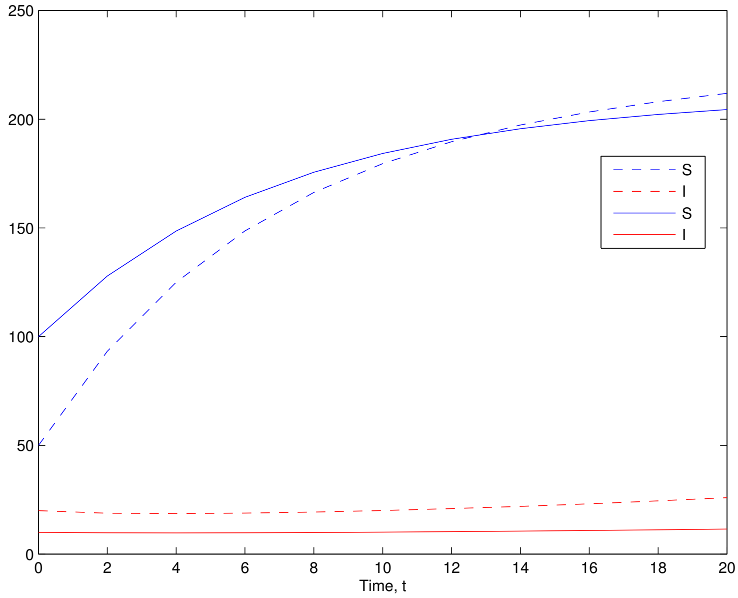

Figure 2.5: GAS of the endemic fixed point for the NSFD scheme (2.4.13)-(2.4.14) with $S_{01} = 100,\ S_{02} = 50,\ I_{01} = 10,\ I_{02} = 20$ and $\mathcal{R}_0 = 1.2$.

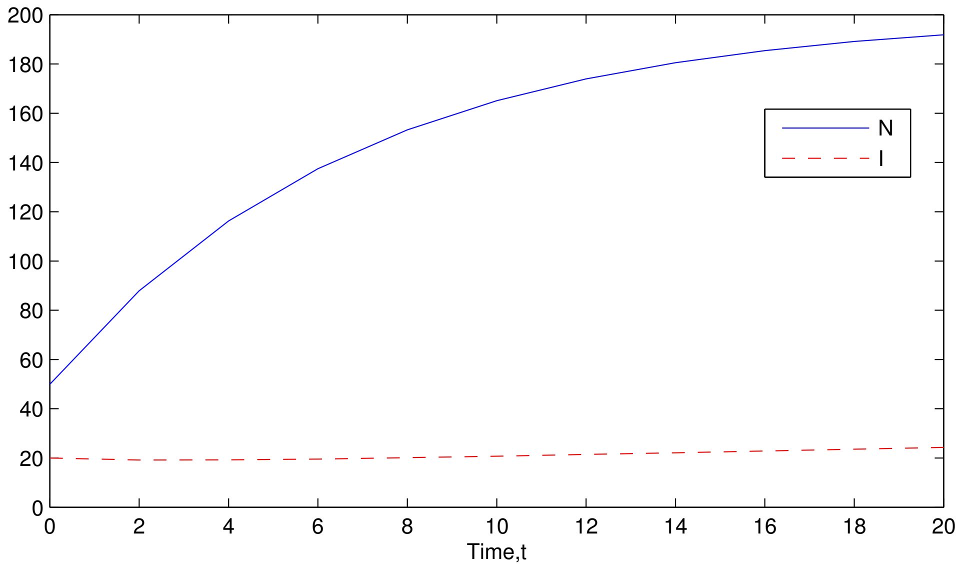

Figure 2.6: Positivity and boundedness of discrete solutions of the NSFD scheme (2.4.12) with $N_0 = 50,\ I_0 = 20$ and $\mathcal{R}_0 = 1.2$

$$\frac{dx}{dt} = f(x), \qquad x(0) = x_0 \tag{2.5.1}$$

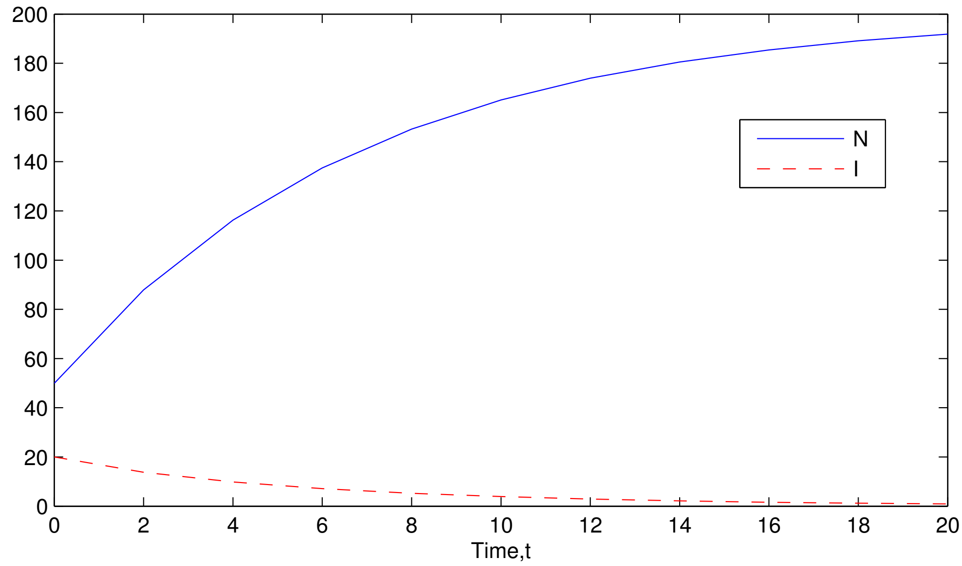

Figure 2.7: Positivity and bondedness of the NSFD scheme (2.4.12) with $N_0 = 50,\ I_0 = 20$ and $\mathcal{R}_0 = 0.6$.

Table 2.1: NSFD scheme (2.4.16) with $c = 2$ and $\mathcal{R}_0 = 3.25$

$Time$ t	$NSFD$ $scheme$	$Exact$ $solution$	$Error : h = 0.25$ $\phi = 0.2487$	$Error : h = 0.5$ $\phi = 0.4621$
$0.0000e+00$	$1.0000e+02$	$1.0000e+02$	$0.0000e+00$	$0.0000e+00$
$5.0000e+00$	$1.3753e+02$	$1.3787e+02$	$3.4169e-01$	$1.0362e+00$
$1.0000e+01$	$1.3845e+02$	$1.3845e+02$	$9.9151e-03$	$4.4197e-02$
$1.5000e+01$	$1.3846e+02$	$1.3846e+02$	$2.1713e-04$	$1.4954e-03$
$2.0000e+01$	$1.3846e+02$	$1.3846e+02$	$4.2943e-06$	$4.7623e-05$
$2.5000e+01$	$1.3846e+02$	$1.3846e+02$	$8.0825e-08$	$1.4867e-06$

is defined by ([47])

$$x^{k+1} = x^k + h \sum_{i=1}^{n} b_i K_i, \tag{2.5.2}$$

where the discrete derivative K_i evaluated at the intermediate times $t_k + c_i h$ are given by

$$K_i = f\left(x^k + h\sum_{j=1}^{i} a_{ij}K_j\right), \quad i = 1, 2, \ldots, n; \qquad c_i = \sum_{j=1}^{i} a_{ij}, \tag{2.5.3}$$

$(a_{ij})_{1\leq i,j\leq n}$ is a lower triangular matrix with zero diagonal and the weights (b_i) satisfy the consistency condition

$$\sum_{i=1}^{n} b_i = 1. \tag{2.5.4}$$

We restrict our study to $1 \leq n \leq 4$ because for this case it is well-known that the explicit n-stage Runge-Kutta method is of order n. More precisely, the local truncation error T^{k+1} is given by

$$T^{k+1} := x(t_{k+1}) - x(t_k) - h\sum_{i=1}^{n} b_i K_i = \mathcal{O}(h^{n+1}), \tag{2.5.5}$$

where the exact solution $x(t_k)$ is used in the definition of K_i given in (2.5.3).

On the other hand, we assume that (2.5.1) has a finite number of equilibrium points $\bar{x}$ which are all hyperbolic. Consider the linearized differential equation

$$\frac{dz}{dt} = Jz, \qquad z := x - \bar{x}, \tag{2.5.6}$$

where J, the Jacobian matrix of f at each equilibrium point $\bar{x}$ of (2.5.1), is supposed to be a diagonalizable $m \times m$ matrix with eigenvalues λ_l:

$$Q^{-1}JQ = \Lambda := diag(\lambda_1, \lambda_2, ..., \lambda_m). \tag{2.5.7}$$

For x_0 near each equilibrium point $\bar{x}$, the solution $x(t)$ of (2.5.1) behaves like the solution

$$z(t) = e^{tJ}z_0, \tag{2.5.8}$$

of (2.5.6) and has the asymptotic behavior

$$\lim_{t\to\infty} \|z(t)\| = 0 \tag{2.5.9}$$

if and only if all the eigenvalues λ_l have negative real parts.

It can be shown that the Runge-Kutta method applied to (2.5.6) reads

$$z^{k+1} = R(hJ)z^k, \tag{2.5.10}$$

where [47]

$$R(hJ) := Q\ diag[R(\lambda_l h)]Q^{-1} \quad \text{and} \quad R(\lambda_l h) = \sum_{j=0}^{n} \frac{(\lambda_l h)^j}{j!}. \tag{2.5.11}$$

Hence, (z^k) replicates the property (2.5.9) or $z^k \to 0$ as $k \to \infty$ if and only if

$$|R(\lambda_l h)| < 1 \quad \forall l, \tag{2.5.12}$$

which is known as the property of absolute stability of the Runge-Kutta method for each value $\lambda_l h$.

The necessity of modifying the n-stage Runge-Kutta method comes here. Consider a function $\varphi : \mathbb{R} \to \mathbb{R}_+$ such that $0 < \varphi(z) < 1$ for $z > 0$ and

$$\varphi(z) = z + \mathcal{O}(z^{n+1}). \tag{2.5.13}$$

For instance, we can take

$$\varphi(z) = \frac{z}{1 + c\, z^n}, \tag{2.5.14}$$

where c is a suitable positive constant.

We consider the function

$$\phi(h) = \frac{\varphi(qh)}{q}, \tag{2.5.15}$$

where $q \geq |\lambda|$ and λ traces all the eigenvalues of all the Jacobian matrices at the equilibrium points $\bar{x}$ for (2.5.1). Our nonstandard (NS) Runge-Kutta method is given by

$$x^{k+1} = x^k + \phi(h) \sum_{i=1}^{n} b_i K_i, \tag{2.5.16}$$

where all the K_i's are defined as in (2.5.3) but with $\phi(h)$ in place of h.

From the above construction, we have the following result:

Theorem 2.5.1. *For $1 \leq n \leq 4$, the NS Runge-Kutta method (2.5.16) is of order n. Furthermore, any equilibrium point $\bar{x}$ of (2.5.1) is a fixed-point of the scheme (2.5.16) in such a way that the stability property of $\bar{x}$ is preserved. More precisely, for any $h > 0$ and for $\|x - \bar{x}\|$ small, the discrete solution (x^k) given by (2.5.16) satisfies $\lim_{k\to\infty} \|x^k - \bar{x}\| = 0$ if $Re\lambda_l < 0$ for all λ_l whereas $\|x^k - \bar{x}\| \to \infty$ if there exists at least one eigenvalue λ_l with $Re\lambda_l > 0$.*

Proof: The order of the scheme results from the relation (2.5.13). The convergence or divergence as $k \to \infty$ is based on the representations analogous to (2.5.10) and (2.5.11):

$$x^{k+1} = R(\lambda\phi(h))x^k \quad \text{and} \quad R(\lambda\phi(h)) = \sum_{j=0}^{n} \frac{(\lambda\phi(h))^j}{j!}.$$

Remark 2.5.2. *The stronger asymptotic condition (2.5.14)-(2.5.15) is imposed in place of the usual condition (2.4.8) in order to guarantee high order of convergence. The NS Runge-Kutta scheme (2.5.16) is not elementary stable in the sense of [8] because it can have spurious fixed-points. The classical second order Runge-Kutta method applied to the delay equation* $\dot{x} = -x$ *exhibits spurious fixed-point [9].*

We investigate the performance of the NS Runge-Kutta method (2.5.16) on the SIS model in (2.2.2)-(2.2.3) or (2.3.6). We start with the case when the total population is constant: $N = K$. Then, the system (2.2.2)-(2.2.3) is the logistic equation

$$\frac{dI}{dt} = \beta(1 - \frac{1}{\mathcal{R}_0})\left(1 - \frac{I}{K(1-\frac{1}{\mathcal{R}_0})}\right) I =: f(I), \tag{2.5.17}$$

the exact solution of which is known to be

$$I(t) = \frac{K(1-\frac{1}{\mathcal{R}_0})I_0}{I_0 + \left(K(1-\frac{1}{\mathcal{R}_0}) - I_0\right) e^{-\beta(1-\frac{1}{\mathcal{R}_0})t}}. \tag{2.5.18}$$

For $0 \le I_0 \le K$ and $\mathcal{R}_0 > 0$, from Eq. (2.5.18), we obtain $0 \le I(t) \le K$. To be more specific, we consider the nonstandard analogue of the classical fourth order Runge-Kutta method $(n = 4)$. In this case, the choice (2.5.15)-(2.5.16) yields:

$$\phi(h) = \frac{h}{1 + c[|\beta(1-\frac{1}{\mathcal{R}_0})|h]^4}. \tag{2.5.19}$$

The NS Runge-Kutta scheme is

$$\frac{I^{k+1} - I^k}{\phi(h)} = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4), \tag{2.5.20}$$

where

$$l_1 = f(I^k), \;\; l_2 = f\left(I^k + \frac{1}{2}\phi(h)l_1\right), \;\; l_3 = f\left(I^k + \frac{1}{2}\phi(h)l_2\right) \;\; \text{and} \;\; l_4 = f\left(I^k + \phi(h)l_3\right).$$

The second case of the interest is when N is not constant. Then (2.2.2)-(2.2.5) is equivalent to

$$\frac{dI}{dt} = \beta(1 - \frac{1}{\mathcal{R}_0})\left(1 - \frac{I}{N(1-\frac{1}{\mathcal{R}_0})}\right) I =: f(I, N), \tag{2.5.21}$$

$$\frac{dN}{dt} = \mu(K - N) =: g(I, N). \tag{2.5.22}$$

The eigenvalues of the Jacobian matrices at the equilibrium points being $\beta(\frac{1}{\mathcal{R}_0}-1)$ and $-\mu$, we take $q=|\beta(\frac{1}{\mathcal{R}_0}-1)|+\mu$ so that (2.5.14) and (2.5.15) yield

$$\phi(h)=\frac{h}{1+c[(|\beta(\frac{1}{\mathcal{R}_0}-1)|+\mu)h]^4}. \tag{2.5.23}$$

Then the nonstandard Runge-Kutta takes the form

$$\frac{N^{k+1}-N^k}{\phi(h)}=\frac{1}{6}(k_1+2k_2+2k_3+k_4) \quad \text{and} \quad \frac{I^{k+1}-I^k}{\phi(h)}=\frac{1}{6}(l_1+2l_2+2l_3+l_4), \tag{2.5.24}$$

where

$$\begin{aligned}
k_1 &= \mu(K-N^k), & l_1 &= f(I^k,N^k)\\
k_2 &= \mu\left(K-N^k-\frac{1}{2}\phi(h)k_1\right), & l_2 &= f\left(I^k+\frac{1}{2}\phi(h)l_1, N^k+\frac{1}{2}\phi(h)k_1\right)\\
k_3 &= \mu\left(K-N^k-\frac{1}{2}\phi(h)k_2\right), & l_3 &= f\left(I^k+\frac{1}{2}\phi(h)l_2, N^k+\frac{1}{2}\phi(h)k_2\right)\\
k_4 &= \mu\left(K-N^k-\phi(h)k_3\right), & l_4 &= f\left(I^k+\phi(h)l_3, N^k+\phi(h)k_3\right).
\end{aligned}$$

The discrete solutions for $I(t)$ given by (2.5.20) are presented in Table 2.2 and Table 2.3 for different values of h. The results show that reducing the step size h by half results in reduction of the error by a factor of sixteen, showing a fourth order of convergence. This is in agreement with Theorem 2.5.1. The excellent performance of the nonstandard Runge-Kutta method is further illustrated in Fig 2.8, which shows the global asymptotic stability (GAS) of the disease-free equilibrium (DFE) ($\mathcal{R}_0<1$) as well as in Fig 2.9 which displays the local asymptotic stability (LAS) of the endemic equilibrium (EE) and the instability of the DFE ($\mathcal{R}_0>1$). Note also that all discrete solutions are positive as should be, whereas the standard Runge-Kutta method could produce negative solutions as shown in Fig 2.12.

We now illustrate the performance of the NS Runge-Kutta method (2.5.24) in the case when N is not constant. The discrete solution for $I(t)$ is shown in Fig 2.10, which supports the existence of a locally asymptotically stable EE and the instability of the DFE when $\mathcal{R}_0>1$.

In the case when N is not constant, the exact solution of (2.5.22) is known. Since this equation is decoupled from the other, we can plug its solution in (2.5.21) and then use the Runge-Kutta method for this scalar equation. It should also be noted that the solution N of (2.5.22) tends to K as $t\to\infty$. Thus, Eq. (2.5.21) is asymptotically the same as Eq. (2.5.17). This fact is apparent on comparing the profiles of the solutions in Fig 2.9 and Fig

Table 2.2: NS Runge-Kutta method with $c = 10$, $K = 200$, $\mathcal{R}_0 = 0.75$

$Time$ t	$Exact$ $scheme$	NS-RK $solution$	$Error:$ $h = 0.5$ $\phi = 0.2487$	NS-RK $solution$	$Error:$ $h = 0.25$ $\phi = 0.4621$
$0.0000e+00$	$8.0000e+01$	$8.0000e+01$	$0.0000e+00$	$8.0000e+01$	$0.0000e+00$
$2.0000e+01$	$5.3135e+00$	$5.3142e+00$	$-7.2150e-04$	$5.3136e+00$	$-4.5089e-05$
$4.0000e+01$	$6.7274e-01$	$6.7291e-01$	$-1.7050e-04$	$6.7275e-01$	$-1.0655e-05$
$6.0000e+01$	$9.0258e-02$	$9.0292e-02$	$-3.3987e-05$	$9.0261e-02$	$-2.1239e-06$
$8.0000e+01$	$1.2201e-02$	$1.2207e-02$	$-6.1162e-06$	$1.2201e-02$	$-3.8218e-07$
$1.0000e+02$	$1.6509e-03$	$1.6520e-03$	$-1.0341e-06$	$1.6510e-03$	$-6.4616e-08$

Table 2.3: NS Runge-Kutta method with $c = 2$, $\mathcal{R}_0 = 3.25$

$Time$ t	$Exact$ $scheme$	NS-RK $solution$	$Error:$ $h = 0.5$	NS-RK $solution$	$Error:$ $h = 0.25$
$0.0000e+00$	$8.0000e+01$	$8.0000e+01$	$0.0000e+00$	$8.0000e+01$	$0.0000e+00$
$4.0000e+00$	$1.0556e+02$	$1.0514e+02$	$4.1069e-01$	$1.0553e+02$	$2.5775e-02$
$8.0000e+00$	$1.1033e+02$	$1.1020e+02$	$1.2655e-01$	$1.1032e+02$	$7.6427e-03$
$1.2000e+01$	$1.1100e+02$	$1.1098e+02$	$2.7063e-02$	$1.1100e+02$	$1.5747e-03$
$1.6000e+01$	$1.1110e+02$	$1.1109e+02$	$5.0887e-03$	$1.1110e+02$	$2.8534e-04$
$2.0000e+01$	$1.1111e+02$	$1.1111e+02$	$8.9602e-04$	$1.1111e+02$	$4.8401e-05$

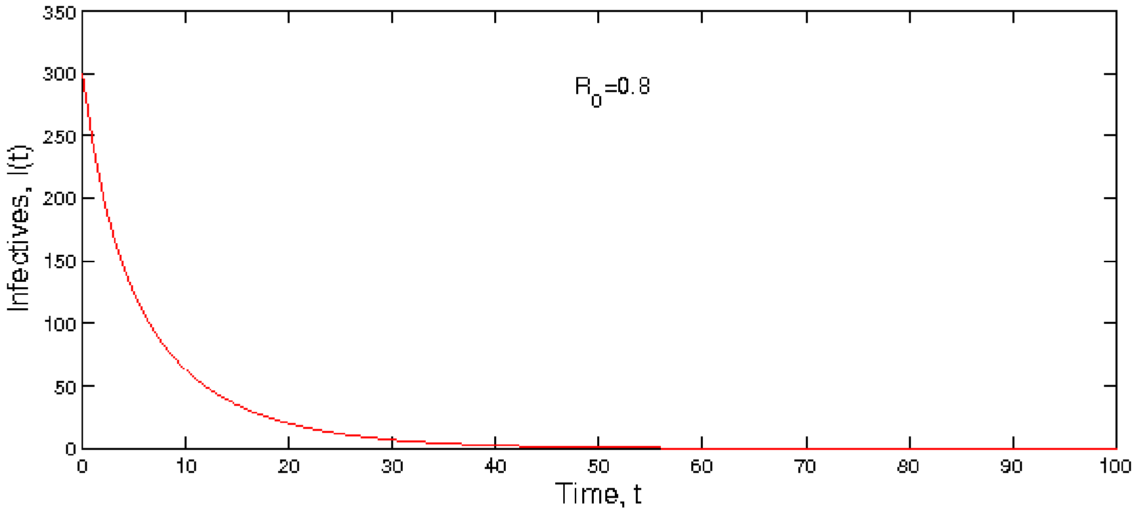

Figure 2.8: NS Runge-Kutta scheme (2.5.20) with $I_0 = 300$ for $\mathcal{R}_0 < 1$.

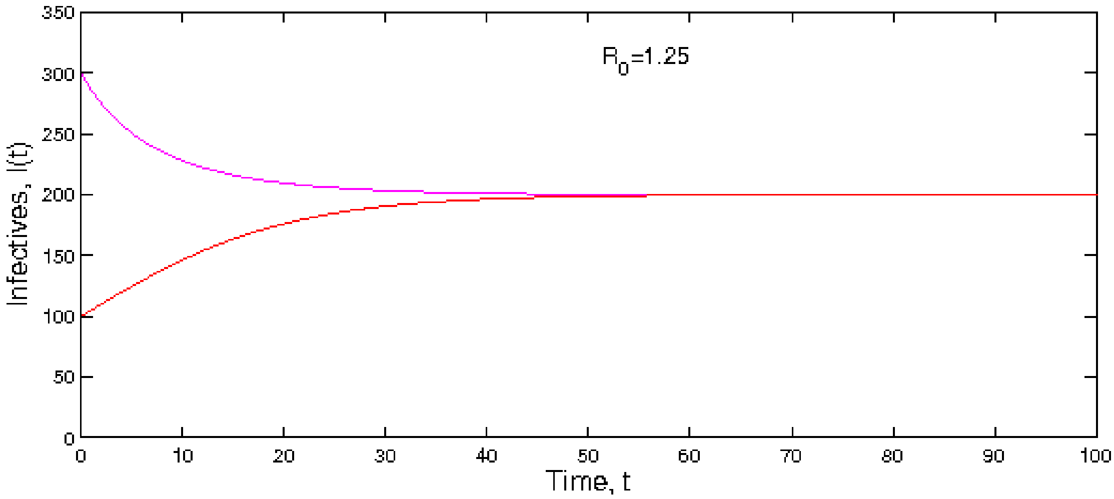

Figure 2.9: NS Runge-Kutta scheme (2.5.20) with $I_0 = 100$, $I_0 = 300$ for $\mathcal{R}_0 > 1$.

2.10 when $\mathcal{R}_0 > 1$. For $\mathcal{R}_0 < 1$, the profile of the solutions are given in Fig 2.8 and Fig 2.11.

For comparison, the standard Runge-Kutta method is shown in Fig 2.12 and Fig 2.13. It is apparent that this method is not dynamically consistent with positivity of solutions and the property of the endemic equilibrium point.

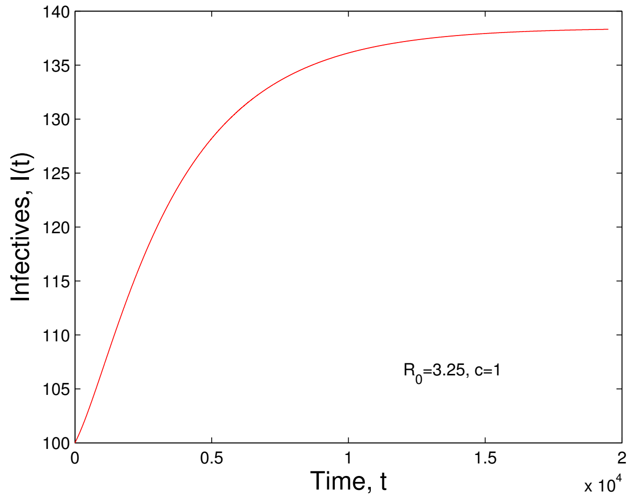

Figure 2.10: NS Runge-Kutta scheme for (2.5.21) with $K = 200$ for $\mathcal{R}_0 > 1$.

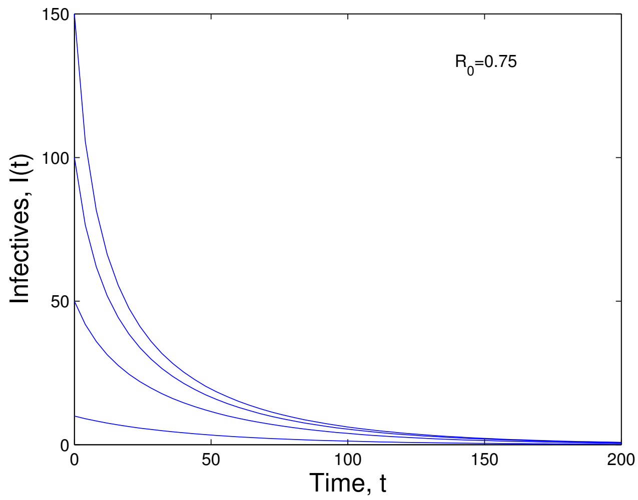

Figure 2.11: NS Runge-Kutta scheme for (2.5.21) with $c = 2$ and $K = 200$ for $\mathcal{R}_0 < 1$.

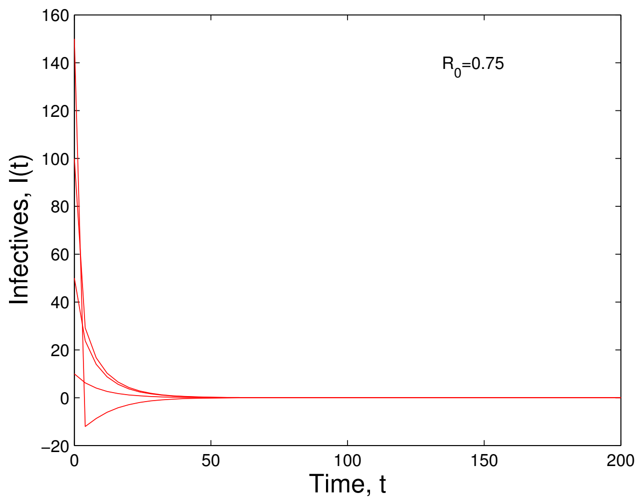

Figure 2.12: Standard Runge-Kutta scheme for (2.5.21) with $K = 200$ for $\mathcal{R}_0 < 1$.

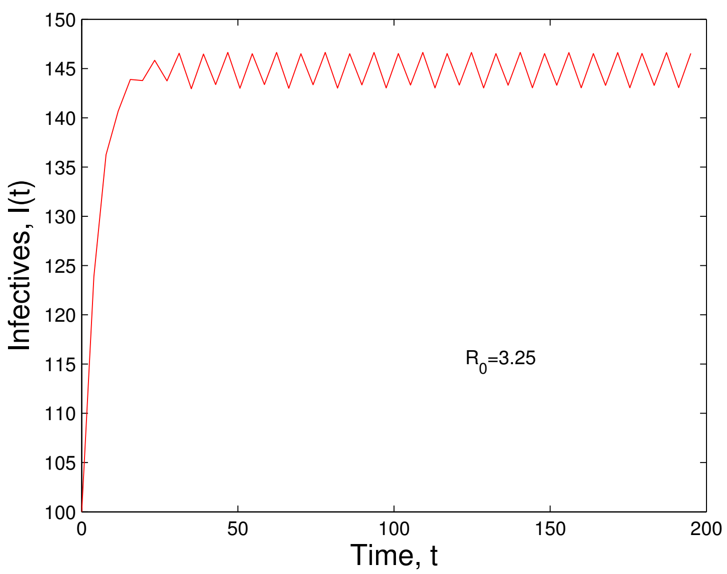

Figure 2.13: Standard Runge-Kutta scheme for (2.5.21) with $K = 200$ for $\mathcal{R}_0 > 1$.

Chapter 3

SIS-Diffusion Epidemiological Model

3.1 Introduction

Epidemiological models in which the density functions are spatially homogeneous and their evolution is described by partial differential equations were proposed by Kermack and McKendric [44] in 1932 and have since then received a lot of attention [1, 2, 28, 32, 53, 67, 69, 83, 85]. In this chapter, we extend the SIS epidemiological model investigated in the previous chapter into a reaction-diffusion system that we call SIS-diffusion epidemiological model.

Since the SIS-diffusion system cannot be solved completely by analytical techniques, our main focus is on the numerical approximation of this model. We propose two NSFD schemes that preserve the dynamics of the continuous model. The results presented here constitute an improvement on the local properties in the book chapter [55] in that the NSFD schemes are dynamically consistent with the global stability properties of the equilibria.

The rest of the chapter is organized as follows. The SIS-diffusion model is formulated in Section 3.2, which is followed by the quantitative and qualitative analysis of the model in Section 3.3. NSFD schemes that preserve the global asymptotic properties of the disease-free equilibrium and of the endemic equilibrium are investigated in Section 3.4. Numerical examples that support the theory are gathered at the end of Section 3.4.

3.2 SIS-diffusion model

In this section, we consider the SIS model with diffusion in the space region Ω. For every point (x, t), the notation $S \equiv S(x, t)$ and $I \equiv I(x, t)$ are used for the densities of susceptible and infected individuals at location $x \in \Omega$ and time $t \geq 0$. The total population at (x, t) is $N \equiv N(x, t)$:

$$N(x, t) = S(x, t) + I(x, t). \tag{3.2.1}$$

We assume that the dispersion is completely random and has the same structural properties. For the evolution in time of the disease and its spread in space, we consider the following model with standard incidence:

$$\frac{\partial S}{\partial t} = \mu K - \frac{\beta IS}{N} - \mu S + \gamma I + D\frac{\partial^2 S}{\partial x^2} \quad \text{in} \quad \Omega \times (0, \infty) \tag{3.2.2}$$

$$\frac{\partial I}{\partial t} = \frac{\beta IS}{N} - (\mu + \gamma)I + D\frac{\partial^2 I}{\partial x^2}, \quad \text{in} \quad \Omega \times (0, \infty), \tag{3.2.3}$$

where the number $D > 0$ is the diffusion-coefficient. The system (3.2.2)-(3.2.3) is appended with initial conditions

$$S(x, 0) = S_0(x) \geq 0, \quad I(x, 0) = I_0(x) \geq 0, \quad x \in \Omega. \tag{3.2.4}$$

The region Ω is generally assumed to be the whole space $(-\infty, \infty)$. However, we will sometimes assume Ω to be a bounded interval. In the latter case, we complete the system (3.2.2)-(3.2.3) by the Dirichlet or the Neumann boundary conditions i.e.

$$S = 0 = I \quad \text{on} \quad \partial\Omega \times (0, \infty), \quad \text{or} \quad \frac{\partial S}{\partial \nu} = 0 = \frac{\partial I}{\partial \nu} \quad \text{on} \quad x \in \partial\Omega \times (0, \infty), \tag{3.2.5}$$

where ν is unit outward normal vector.

By adding (3.2.2) and (3.2.3), we have the conservation law:

$$\frac{\partial N}{\partial t} = \mu(K - N) + D\frac{\partial^2 N}{\partial x^2}. \tag{3.2.6}$$

Thus, with (3.2.6) in mind, the SIS-diffusion system (3.2.2)-(3.2.3) is equivalent to the following scalar reaction-diffusion equation in the dependent variable I:

$$\frac{\partial I}{\partial t} = \beta(1 - \frac{1}{\mathcal{R}_0})\left(1 - \frac{I}{N(1 - \frac{1}{\mathcal{R}_0})}\right) I + D\frac{\partial^2 I}{\partial x^2}. \tag{3.2.7}$$

3.3 Quantitative and qualitative analysis

Theorem 3.3.1. *Assume that $N_0 : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $0 \leq N_0(x) \leq K$. Then, Eq. (3.2.6) admit unique solutions $N : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ that satisfy the condition $0 \leq N(x,t) \leq K$ with initial condition*

$$N(x, 0) = N_0(x). \tag{3.3.1}$$

Proof: For the linear differential equation (3.2.6), it is well-known (see eg. [30]) that its unique solution is

$$N(x,t) = K - e^{-\mu t} \int_{-\infty}^{\infty} \frac{(K - N_0(y))}{\sqrt{4D\pi t}} \exp\left(\frac{-(x-y)^2}{4Dt}\right) dy, \tag{3.3.2}$$

in which the fundamental solution

$$\mathcal{K}(x,t) = \begin{cases} \frac{1}{\sqrt{4D\pi t}} \exp\left(\frac{-x^2}{4tD}\right), & \text{if } \quad x \in \mathbb{R}, \ t > 0, \\ 0, & \text{if } \ x \in \mathbb{R}, \ t < 0, \end{cases}$$

of the heat equation

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2},$$

satisfies the condition

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi Dt}} \exp\left(\frac{-x^2}{4Dt}\right) dx = 1. \tag{3.3.3}$$

From Eq. (3.3.2), it is clear that $0 \leq N(x,t) \leq K$. □

Once the solution of (3.2.6) is obtained as indicated in (3.3.2), we use it to prove the well-posedness of (3.2.7) appended with the initial condition

$$I(x, 0) = I_0(x). \tag{3.3.4}$$

Theorem 3.3.2. *Assume that $N_0 : \mathbb{R} \to \mathbb{R}$ and $I_0 : \mathbb{R} \to \mathbb{R}$ are continuous functions such that $0 \leq I_0(x) \leq N_0(x) \leq K$. Then, Eq. (3.2.7) and (3.3.4) admit a unique solution $I : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ that satisfies the condition $0 \leq I(x,t) \leq N(x,t) \leq K$ and which has the integral representation*

$$I(x,\ t) = \int_{-\infty}^{\infty} \mathcal{K}(x-y,t) I_0(y) dy + \int_0^t \int_{-\infty}^{\infty} \mathcal{K}(x-y, t-s) f(I(y,s)) dy ds. \tag{3.3.5}$$

Proof: We introduce the differential operator $\mathcal{L}$ defined by

$$\mathcal{L}(I) = \frac{\partial I}{\partial t} - D\frac{\partial^2 I}{\partial x^2} - \beta(1 - \frac{1}{\mathcal{R}_0})\left(1 - \frac{I}{N(1-\frac{1}{\mathcal{R}_0})}\right) I.$$

Let I be a solution of the problem (3.2.7) with initial condition $0 \leq I_0(x) \leq N_0(x)$.

It is clear that

$$\mathcal{L}(0) = \mathcal{L}(I) = 0.$$

On the other hand, we have

$$\begin{aligned}
\mathcal{L}(N) &= \frac{\partial N}{\partial t} - D\frac{\partial^2 N}{\partial x^2} - \beta(1 - \frac{1}{\mathcal{R}_0})\left(1 - \frac{1}{(1-\frac{1}{\mathcal{R}_0})}\right) N \\
&= \mu\,(K - N) - \beta\left(1 - \frac{1}{\mathcal{R}_0}\right)\left(1 - \frac{1}{(1-\frac{1}{\mathcal{R}_0})}\right) N \qquad \text{by} \quad (3.2.6) \\
&> 0, \qquad \text{since} \quad 0 < N \leq K.
\end{aligned}$$

Thus, we have

$$\mathcal{L}(0) \leq \mathcal{L}(I) \leq \mathcal{L}(N) \quad \text{on} \quad \mathbb{R} \times (0, \infty)$$

and

$$0 \leq I_0(x) \leq N_0(x) \quad \text{on} \quad \mathbb{R},$$

which show that the null function 0 is a sub-solution of (3.2.7) whereas the function N is a super-solution of the same equation [32].

By the comparison theorem (see [53, 75]), we have

$$0 \leq I(x,t) \leq N(x,t) \quad \text{on} \quad \mathbb{R} \times (0, \infty). \qquad (3.3.6)$$

In view of the fact that $N \leq K$, the condition (3.3.6) means that any solution of (3.2.7) with initial data $I_0(x)$ satisfies a priori boundedness estimate. Since the function

$$f(I) := \beta(1 - \frac{1}{\mathcal{R}_0})\left(1 - \frac{I}{N(1-\frac{1}{\mathcal{R}_0})}\right) I \qquad (3.3.7)$$

is locally Lipschitz (being of class C^∞), it follows from the same references [53, 75] that the problem (3.2.6)-(3.2.7) admits a unique solution I such that $0 \leq I(x,t) \leq N(x,t)$ and satisfies (3.3.5). $\square$

The next step is the qualitative analysis of the model (3.2.2)-(3.2.3), which on using (3.2.6), is equivalent to (3.2.7). For convenience, we first recast the model (3.2.7) in the following form:

$$\frac{\partial I}{\partial t} = f(I) + D\frac{\partial^2 I}{\partial x^2} \tag{3.3.8}$$

and recall some definitions and tools from [21, 75].

Definition 3.3.3. *A constant* $\bar{I} \in \mathbb{R}$ *such that*

$$f(\bar{I}) + D\frac{\partial^2 I}{\partial x^2}|_{I=\bar{I}} = 0 \tag{3.3.9}$$

is called an equilibrium solution of the reaction-diffusion equation (3.3.8).

Definition 3.3.4. *Let* X *be the Lesbegue space* $L_\infty(0,b)$ *or* $L_2(0,b)$ *with norm denoted by* $\|.\|_X$. *An equilibrium solution* $\bar{I}$ *of (3.3.8) is said to be stable (with respect to the* X *topology) if for every* $\epsilon > 0$, *there exists* $\delta > 0$ *such that for any initial condition* $I_0 : \mathbb{R} \to \mathbb{R}$ *satisfying* $\|I_0 - \bar{I}\|_X < \delta$, *we have* $\|I(.,t) - \bar{I}\|_X < \epsilon$ *for all* $t > 0$, *where* $I(x,t)$ *is the solution of (3.3.8). In addition, if*

$$\lim_{t\to\infty} \|I(.,t) - \bar{I}\|_X = 0 \tag{3.3.10}$$

for $\|I_0 - \bar{I}\|_X$ *sufficiently small, then* $\bar{I}$ *is called locally asymptotically stable. If (3.3.10) is true for any initial condition* $I_0(x)$, *then* $\bar{I}$ *is called globally asymptotically stable. If* $\bar{I}$ *is not stable, it is said to be unstable.*

In practice, the local stability is established by the linearization process described below, assuming now that Ω is a bounded interval $(0,b)$.

Let $\bar{I}$ be an equilibrium solution of (3.3.8). By Taylor expansion about $\bar{I}$ and by (3.3.9), we have

$$\begin{aligned} f(I) &= f(\bar{I}) + f'(\bar{I})(I-\bar{I}) + \frac{1}{2}f''(\hat{I})(I-\bar{I})^2, \\ &= -D\frac{\partial^2 I}{\partial x^2}|_{I=\bar{I}} + f'(\bar{I})(I-\bar{I}) + \frac{1}{2}f''(\hat{I})(I-\bar{I})^2, \end{aligned}$$

for some number $\hat{I} \in (0,b)$. Under the conditions when the nonlinear term can be ignored in the previous relation, the nonlinear reaction-diffusion equation (3.3.8) can be approximated by the linear equation

$$\frac{\partial \Psi}{\partial t} = D\frac{\partial^2 \Psi}{\partial x^2} + f'(\bar{I})\Psi, \tag{3.3.11}$$

where $\Psi = I - \bar{I}$. By separation of variables, i.e $\Psi(x,t) = \omega(t)\phi(x)$, Eq. (3.3.11) leads to the equation

$$\frac{\omega'(t)}{\omega(t)} = \frac{D\phi''(x)}{\phi(x)} + f'(\bar{I}) = M = \text{constant},$$

which is equivalent to the system

$$\omega'(t) = M\omega(t) \tag{3.3.12}$$

$$D\phi''(x) = (M - f'(\bar{I}))\phi(x), \tag{3.3.13}$$

where M is independent of x and t. We append (3.3.13) with the Dirichlet or Neumann boundary conditions

$$\phi(0) = \phi(b) = 0 \quad \text{or} \quad \phi'(0) = \phi'(b) = 0 \tag{3.3.14}$$

to make it a Sturm-Liouville problem. The stability of equilibrium solutions is determined in the next theorem.

Theorem 3.3.5. *Suppose that $\bar{I}$ is an equilibrium solution of (3.3.8). If $f'(\bar{I}) < \frac{\pi^2 D}{b^2}$, then $\bar{I}$ is L_2-locally asymptotically stable; if there exists an $n \in \mathbb{N}$ such that $f'(\bar{I}) > \frac{n^2\pi^2 D}{b^2}$, then $\bar{I}$ is an unstable equilibrium solution.*

Proof: We prove the theorem for the Dirichlet boundary condition (3.3.14), the situation being similar for the Neumann boundary condition. It is well-known that

$$\Phi_n(x) = \frac{2}{b}\sin\left(\frac{n\pi}{b}x\right), \quad n = 1,\ 2,\ \ldots$$

constitute the eigenfunctions of the Sturm-Liouville problem (3.3.13)-(3.3.14) with associated eigenvalues

$$M_n = f'(\bar{I}) - \frac{n^2\pi^2 D}{b^2}. \tag{3.3.15}$$

Equally, it is known that the sequence $(\phi_n)_{n\geq 1}$ is a Hilbert basis of the space $L_2(0,b)$. Therefore, any solution $I(x) = I(x,t) \in \mathbb{R} \times (0,b)$ admits the Fourier expansion

$$I(x,t) = \sum_{n=1}^{\infty} c_n e^{M_n t} \sin\left(\frac{n\pi}{b}x\right), \quad x \in (0,b), \quad t > 0, \tag{3.3.16}$$

where $c_n = \frac{2}{b}\int_0^b I_0(x)\sin\left(\frac{n\pi x}{b}\right)dx$. From Eq. (3.3.16) and the expression of the L_2-norm of $I(x,t)$, it follows that $\bar{I}$ is locally asymptotically stable if $M_n < 0$ or $f'(\bar{I}) < \frac{\pi^2 D}{b^2} \leq \frac{n^2\pi^2 D}{b^2}$ for all $n \in \mathbb{N}$ and unstable if there exists $n \in \mathbb{N}$ such that $M_n > 0$ or $f'(\bar{I}) > \frac{n^2\pi^2 D}{b^2}$. $\square$

Remark 3.3.6. *If the series in (3.3.16) converges point wise to $I(x,t)$, as it is the case when $I(x,t)$ is periodic function, then the equilibrium $\bar{I}$ is L_∞- locally asymptotically stable.*

Let us now come back to the model (3.2.7). Since the equilibrium solution of the conservation law (3.2.6) is $\bar{N} = K$, the associated infective components of the equilibrium solutions of the model (3.2.7) are

$$\bar{I}_0 = 0 \quad \text{and} \quad \bar{I}_\infty = K(1 - \frac{1}{\mathcal{R}_0}) \quad \text{when} \quad \mathcal{R}_0 > 1.$$

Correspondingly, we have

$$\bar{S}_0 = K \quad \text{and} \quad \bar{S}_\infty = \frac{K}{\mathcal{R}_0}$$

for the susceptible components.

Eq. (3.3.2) shows clearly that the equilibrium solution $\bar{N} = K$ of the conservation law is globally asymptotically stable. Regarding the equilibrium of the model itself, we have the following result.

Theorem 3.3.7. *For the model (3.2.7), the disease-free equilibrium solution $\bar{I}_0 = 0$ is locally asymptotically stable for $\mathcal{R}_0 < 1$ and unstable when $\mathcal{R}_0 > 1$, whereas the endemic equilibrium solution $\bar{I}_\infty = K(1 - \frac{1}{\mathcal{R}_0})$ is locally asymptotically stable in this case.*

Proof: We apply Theorem 3.3.5. With $f(I)$ given in (3.3.7), we have

$$f'(0) = \beta(1 - \frac{1}{\mathcal{R}_0}) \tag{3.3.17}$$

and

$$f'\left(K(1 - \frac{1}{\mathcal{R}_0})\right) = -\beta(1 - \frac{1}{\mathcal{R}_0}). \tag{3.3.18}$$

By using Eq. (3.3.15) and (3.3.17), we have

$$M_n = \beta(1 - \frac{1}{\mathcal{R}_0}) - \frac{n^2\pi^2 D}{b^2},$$

for the equilibrium solution $\bar{I} = 0$. Then for $\mathcal{R}_0 < 1$ and $n \in \mathbb{N}$, $M_n < 0$ while for $\mathcal{R}_0 > 1$, we have $M_n > 0$ for at least one $n \in \mathbb{N}$. Thus, $\bar{I}_0 = 0$ is locally asymptotically stable for $\mathcal{R}_0 < 1$ and unstable for $\mathcal{R}_0 > 1$. In a similar way, Eq. (3.3.15) and (3.3.18) applied to the equilibrium $\bar{I}_\infty$ yields

$$M_n = -\beta(1 - \frac{1}{\mathcal{R}_0}) - \frac{n^2\pi^2 D}{b^2}.$$

Thus $\bar{I}_\infty$ is locally asymptotically stable for $\mathcal{R}_0 > 1$. $\square$

Theorem 3.3.7 is improved in Theorem 3.3.8 and Theorem 3.3.9.

Theorem 3.3.8. *For $\mathcal{R}_0 \leq 1$, the disease-free equilibrium $\bar{I}_0$ is $L_2(0,b)$ globally asymptotically stable.*

Proof: We use the energy method. Let $N = N(x,t)$ and $I = I(x,t)$ be solutions of (3.2.6) and (3.2.7) under the assumptions made in Theorem 3.3.2 such that $I(0,t) = I(b,t) = 0$ and $N(0,t) = N(b,t) = 0$. Multiplying (3.2.7) by $I(x,t)$ and integrating with respect to $x \in (0,b)$, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|I(.,t)\|^2_{L_2(0,b)} &= \int_0^b \frac{\partial I(x,t)}{\partial t} I(x,t)dx \\
&= \beta\left(1-\frac{1}{\mathcal{R}_0}\right)\int_0^b \left(1-\frac{I(x,t)}{N(x,t)\left(1-\frac{1}{\mathcal{R}_0}\right)}\right)[I(x,t)]^2 dx \\
&+ D\int_0^b I(x,t)\frac{\partial^2 I(x,t)}{\partial x^2}dx.
\end{aligned}$$

By Green's formula, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|I(.,t)\|^2_{L_2(0,b)} &= \beta\left(1-\frac{1}{\mathcal{R}_0}\right)\int_0^b [I(x,t)]^2 dx - D\int_0^b \left(\frac{\partial I(x,t)}{\partial x}\right)^2 dx \\
&- \beta\left(1-\frac{1}{\mathcal{R}_0}\right)\int_0^b \frac{[I(x,t)]^3}{N\left(1-\frac{1}{\mathcal{R}_0}\right)}dx.
\end{aligned}$$

Using Poincar$\acute{e}$-Friedrichs inequality in the Sobolev space $H_0^1(0,b)$ that contains $I(.,t)$ [21], we have for some constant $\lambda_1 > 0$,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|I(.,t)\|^2_{L_2(0,b)} &\leq \beta\left(1-\frac{1}{\mathcal{R}_0}\right)\|I(.,t)\|^2_{L_2(0,b)} - \frac{D}{\lambda_1}\|I(.,t)\|^2_{L_2(0,b)} \\
&- \beta\left(1-\frac{1}{\mathcal{R}_0}\right)\int_0^b \frac{[I(x,t)]^3}{N(x,t)\left(1-\frac{1}{\mathcal{R}_0}\right)}dx \\
&\leq \left[\beta\left(1-\frac{1}{\mathcal{R}_0}\right)-\frac{D}{\lambda_1}\right]\|I(.,t)\|^2_{L_2(0,b)}, \qquad \text{as} \quad \mathcal{R}_0 \leq 1.
\end{aligned}$$

Thus,

$$\frac{d}{dt}\|I(.,t)\|^2_{L_2(0,b)} \leq 2\left[\beta\left(1-\frac{1}{\mathcal{R}_0}\right)-\frac{D}{\lambda_1}\right]\|I(.,t)\|^2_{L_2(0,b)} \quad \text{for} \quad \mathcal{R}_0 \leq 1.$$

Gronwall inequality yields

$$\|I(.,t)\|^2_{L_2(0,b)} \leq \|I_0(.)\|^2_{L_2(0,b)} \exp\left(\left[\beta\left(1-\frac{1}{\mathcal{R}_0}\right)-\frac{D}{\lambda_1}\right]t\right),$$

which shows that

$$\|I(.,t)\|^2_{L_2(0,b)} \to 0 \qquad \text{as} \qquad t \to \infty. \qquad \square$$

Theorem 3.3.9. *For $\mathcal{R}_0 > 1$, the endemic equilibrium $\bar{I}_\infty$ is $L_2(0,b)$ globally asymptotically stable.*

Proof: The proof is similar to that of Theorem 3.3.8, with the necessary adjustments. Let $N = N(x,t)$ and $I = I(x,t)$ be the involved functions and solutions. Since N given in (3.3.2) converges to K as $t \to \infty$, Eq. (3.2.7) behaves like equation

$$\frac{\partial}{\partial t} I(x,t) = \beta \left(1 - \frac{1}{\mathcal{R}_0}\right) \left(1 - \frac{I(x,t)}{K\left(1 - \frac{1}{\mathcal{R}_0}\right)}\right) I(x,t) + D \frac{\partial^2}{\partial x^2} I(x,t),$$

for large values of t. It is convenient to write this equation in the following equivalent form:

$$\frac{\partial}{\partial t}\left(I(x,t) - \bar{I}_\infty\right) = -\frac{\beta}{K}\left[I(x,t) - \bar{I}_\infty\right] I(x,t) + D \frac{\partial^2}{\partial x^2}\left(I(x,t) - \bar{I}_\infty\right). \quad (3.3.19)$$

We obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|I(.,t) - \bar{I}_\infty\|^2_{L_2(0,b)} &= -\frac{\beta}{K}\int_0^b \left(I(x,t) - \bar{I}_\infty\right)^2 I(x,t) dx \\ &+ D\int_0^b \left(I(x,t) - \bar{I}_\infty\right) \frac{\partial^2\left(I(x,t) - \bar{I}_\infty\right)}{\partial x^2} dx \\ &\leq D\int_0^b \left(I(x,t) - \bar{I}_\infty\right) \frac{\partial^2\left(I(x,t) - \bar{I}_\infty\right)}{\partial x^2} dx, \end{aligned}$$

after we multiply (3.3.19) by $I(x,t) - \bar{I}_\infty$ and integrate with respect to $x \in (0,b)$. By Green's formula and Poincar$\acute{e}$-Friedrichs inequality for some constant $\lambda_2 > 0$ ([21]), we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|I(.,t) - \bar{I}_\infty\|^2_{L_2(0,b)} &\leq -D\int_0^b \left(\frac{\partial\left(I(x,t) - \bar{I}_\infty\right)}{\partial x}\right)^2 dx \\ &\leq -\frac{D}{\lambda_2}\|I(.,t) - \bar{I}_\infty\|^2_{L_2(0,b)}. \end{aligned}$$

By using Gronwall inequality, we obtain

$$\|I(.,t) - \bar{I}_\infty\|^2_{L_2(0,b)} \leq \|I_0(.) - \bar{I}_\infty\|^2_{L_2(0,b)} \exp\left(-\frac{D}{\lambda_2}t\right),$$

which shows that

$$\|I(.,t) - \bar{I}_\infty\|^2_{L_2(0,b)} \to 0 \quad \text{as} \quad t \to \infty. \qquad \square$$

Remark 3.3.10. *The global asymptotic stability of the disease-free and endemic equilibria established in Theorems 3.3.8 and 3.3.9 can be proved by using the Lyapunov direct method in which the Lyapunov functionals for the reaction-diffusion partial differential equation (3.2.7) is constructed from the Lyapunov function used for the SIS-ODE (2.3.6) [3, 35, 60].*

Often, the disease is spread in space as a wave with speed c. It is therefore legitimate to seek for a traveling wave solution (TWS) in the form

$$I(x,t) = W(z), \qquad z = x - ct \tag{3.3.20}$$

such that

$$\lim_{z\to\pm\infty} W(z) = \text{constant}. \tag{3.3.21}$$

We have the following result.

Theorem 3.3.11. *Assume that $\mathcal{R}_0 > 1$. Then Eq. (3.2.7) with initial condition $0 \leq I_0(x) \leq N_0(x)$ admits a traveling solution if N is identically equal to the carrying capacity K. Conversely, if $N = K$ and under the condition $c \geq 2\sqrt{(\mu+\gamma)D(\mathcal{R}_0 - 1)}$ for the speed, there exists a traveling wave solution $z \rightsquigarrow W(z)$, which is a monotone decreasing function with horizontal asymptotes $W = 0$ and $W = K\left(1 - \frac{1}{\mathcal{R}_0}\right)$.*

Proof: Assume that (3.2.7) with initial condition $0 \leq I_0(x) \leq N_0(x)$ has a TWS. This implies that Eq. (3.2.6) with initial condition in (3.3.1) also admits a TWS of the form

$$N(x,t) = M(z), \quad z = x - ct. \tag{3.3.22}$$

We get

$$DM'' + cM' + \mu(K - M) = 0,$$

by substituting (3.3.22) in (3.2.6). The general solution of this second order ODE is given by

$$M(z) = c_1 e^{r_1 z} + c_2 e^{r_2 z} + K,$$

where

$$r_1 = \frac{-c - \sqrt{c^2 + 4\mu D}}{2D} < 0,$$
$$r_2 = \frac{-c + \sqrt{c^2 + 4\mu D}}{2D} > 0,$$

$c_1 \in \mathbb{R}$ and $c_2 \in \mathbb{R}$. The analogue condition of (3.3.21) applied to $M(z)$ implies that

$$c_1 = c_2 = 0.$$

Hence, $M(z) = K$ and $N(x,t) = K$, as announced.

Conversely, assume that $N = K$. We want to show that (3.2.7) with initial condition $0 \leq I_0(x) \leq N_0(x)$ admits a TWS. We adapt to our case the idea in [53, 66]. Let a traveling wave solution of the form (3.3.20) be such that $W(z)$ is bounded and nonnegative. For $\mathcal{R}_0 > 1$, the equilibrium solutions $\bar{I}_0 = 0$ and $\bar{I}_\infty = K(1 - \frac{1}{\mathcal{R}_0}) > 0$ of (3.2.7) are unstable and stable, respectively. Therefore, we look for a TWS that satisfies

$$0 \leq W(z) \leq K\left(1 - \frac{1}{\mathcal{R}_0}\right)$$

with Eq, (3.3.21) reading particularly as

$$\lim_{z\to-\infty} W(z) = K\left(1 - \frac{1}{\mathcal{R}_0}\right) \quad \text{and} \quad \lim_{z\to\infty} W(z) = 0. \tag{3.3.23}$$

From Eq. (3.3.20) and by using the chain rule, we have

$$\frac{\partial I}{\partial t} = -c\frac{dW}{dz} = -cW' \quad \text{and} \quad \frac{\partial^2 I}{\partial x^2} = \frac{d^2 W}{dz^2} = W''.$$

Substitution of these terms in Eq. (3.2.7) gives the following equation for the wave W and its speed $c > 0$:

$$DW'' + cW' + (\lambda - \mu - \gamma)\left(1 - \frac{\mathcal{R}_0 W}{K(\mathcal{R}_0 - 1)}\right) W = 0. \tag{3.3.24}$$

Let

$$W' = V.$$

Then Eq. (3.3.24) is equivalent to first-order system of differential equations

$$W' = V =: f(W,V) \tag{3.3.25}$$
$$V' = \frac{-c}{D}V - \frac{(\mu+\gamma)}{D}(\mathcal{R}_0 - 1)\left(1 - \frac{\mathcal{R}_0 W}{K(\mathcal{R}_0 - 1)}\right) W =: g(W,V). \tag{3.3.26}$$

The equilibrium points of the system (3.3.25)-(3.3.26) are

$$P = (0,0) \quad \text{and} \quad Q = \left(K(1 - \frac{1}{\mathcal{R}_0}), 0\right).$$

The Jacobian matrix of the system at (W,V) being

$$J(W,V) = \begin{pmatrix} 0 & 1 \\ -\frac{(\mu+\gamma)}{D}(\mathcal{R}_0 - 1)\left(1 - \frac{2W\mathcal{R}_0}{K(\mathcal{R}_0-1)}\right) & -\frac{c}{D} \end{pmatrix},$$

we have

$$J(P) = \begin{pmatrix} 0 & 1 \\ -\frac{(\mu+\gamma)}{D}(\mathcal{R}_0 - 1) & -\frac{c}{D} \end{pmatrix},$$

with eigenvalues

$$r_{1,2} = \frac{-c \pm \sqrt{c^2 - 4(\mu+\gamma)D(\mathcal{R}_0 - 1)}}{2D}, \tag{3.3.27}$$

and

$$J(Q) = \begin{pmatrix} 0 & 1 \\ \frac{(\mu+\gamma)}{D}(\mathcal{R}_0 - 1) & -\frac{c}{D} \end{pmatrix},$$

with eigenvalues

$$r_{3,4} = \frac{-c \pm \sqrt{c^2 + 4(\mu+\gamma)D(\mathcal{R}_0 - 1)}}{2D}. \tag{3.3.28}$$

For $c \neq 0$, we infer from (3.3.27) and (3.3.28) that the eigenvalues have nonzero real parts. Thus the equilibrium points are hyperbolic, which allows us to use Hartman-Grobman theorem. It follows that the equilibrium P is locally asymptotically stable if $c \geq 2\sqrt{(\mu+\gamma)D(\mathcal{R}_0 - 1)}$, because the eigenvalues are negative, whereas it is a stable spiral if $c < 2\sqrt{(\mu+\gamma)D(\mathcal{R}_0 - 1)}$, because the eigenvalues are complex numbers with negative real parts.

Furthermore, the equilibrium Q is a saddle point, because the eigenvalues are real numbers of opposite signs. Therefore, $c \geq 2\sqrt{(\mu+\gamma)D(\mathcal{R}_0 - 1)}$ is the right wave speed for the required TWS. Let c be such a speed. As from now on, we work for convenience in the W-V plane.

Let E_3 and E_4 be the unstable and stable eigenspaces (associated with the eigenvalues $r_3 > 0$ and $r_4 < 0$, respectively which are straight lines in this case) (see Fig 3.1). Then the trajectories on E_3 leave Q whereas those on E_4 are attracted by Q.

Let U be a simply connected subset of the positive sector $W \geq 0,\ V \geq 0$ such that U is compact, invariant and contains the stable equilibrium point $P = (0,0)$ and the saddle point $Q = (K(1 - \frac{1}{\mathcal{R}_0}), 0)$. We observe that for the system (3.3.25)-(3.3.26)

$$\frac{\partial f}{\partial W} + \frac{\partial g}{\partial V} = -c < 0$$

and by Bendixson's criterion, we are guaranteed the system has no closed orbit lying entirely in U. Moreover, by Poincaré - Bendixon Theorem (see [84]), there exists a unique trajectory S called separatrix which coincides with E_3 near Q and such that its omega limit, $\omega(S) = P$ and its alpha limit, $\alpha(S) = Q$.

Moreover, any point $R \in U \backslash S$ is the initial point of a unique trajectory that does not intersect with S and is attracted by P. Transposed to the $z - W$ axes, the above–constructed unique trajectory S constitutes the traveling wave solution $W(z)$ which is a decreasing function and satisfies (3.3.23) (see Fig 3.2). □

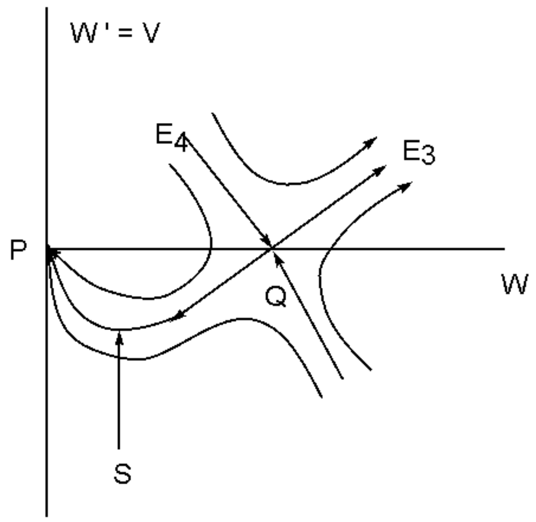

Figure 3.1: Phase portrait for (3.3.25)-(3.3.26) when $c \geq 2\sqrt{(\mu+\gamma)D(\mathcal{R}_0 - 1)}$.

Remark 3.3.12. *In the particular case when $N = K$ so that (3.2.7) is the Fisher equation, an explicit expression of the TWS is available in [15] and reads as follows*

$$I(x,t) = \frac{K(\mathcal{R}_0 - 1)}{\mathcal{R}_0[1 + d\exp(z)]^2},$$

where

$$z = \frac{\sqrt{(\mu+\gamma)D(\mathcal{R}_0 - 1)}}{\sqrt{6}}x - \frac{5}{6}((\mu+\gamma)D(\mathcal{R}_0 - 1)t$$

and $d =$ *constant.*

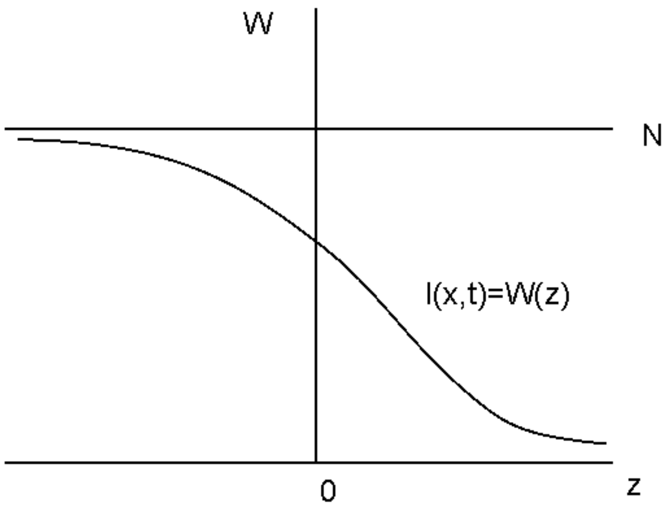

Figure 3.2: Traveling wave solution for $c \geq 2\sqrt{(\mu+\gamma)D(\mathcal{R}_0-1)}$.

3.4 Nonstandard finite difference schemes

In this section, we design numerical schemes that are dynamically consistent with the properties of the SIS-diffusion model (3.2.2)-(3.2.6). One of the main strategies towards the construction of the schemes is that Mickens' rule [8, 61] on the nonlocal approximation of nonlinear terms is implemented. We assume that the model (3.2.2)-(3.2.6) has a space variable x on a bounded interval $[0, b]$ for $b \in \mathbb{R}$ and a time variable $t \in [0, \infty)$. Let $\triangle x = b/N$ be the space step size for a positive integer N and $\triangle t$ be the time step size. Given a function $u = u(x, t)$, the notation u_n^k means an approximation of $u(x, t)$ at $x = x_n$ and $t = t_k$, where $x_n = n\Delta x$ and $t_k = k\Delta t$ for $n \in \{0, 1, 2, \ldots, N\}$ and $k \in \mathbb{N}$.

Based on Section 2.4., we design two NSFD schemes for the SIS-diffusion model (3.2.2)-(3.2.6).

The first NSFD scheme is

$$\frac{S_n^{k+1}-S_n^k}{\phi} = \mu K - \frac{\beta S_n^{k+1} I_n^k}{S_n^{k+1}+I_n^k} - \mu S_n^{k+1} + \gamma I_n^k + D\frac{S_{n+1}^k - 2S_n^k + S_{n-1}^k}{\psi^2} \tag{3.4.1}$$

$$\frac{I_n^{k+1}-I_n^k}{\phi} = \frac{\beta S_n^{k+1} I_n^k}{S_n^{k+1}+I_n^k} - \mu I_n^{k+1} - \gamma I_n^k + D\frac{I_{n+1}^k - 2I_n^k + I_{n-1}^k}{\psi^2}, \tag{3.4.2}$$

where $\phi(\Delta t)$ is given in (2.4.7) and

$$\psi(\Delta x) = \begin{cases} \frac{2D}{\sqrt{\beta D(1-\frac{1}{\mathcal{R}_0})}} \sin\left(\frac{\sqrt{\beta D(1-\frac{1}{\mathcal{R}_0})}}{2D}\Delta x\right), & \text{if} \quad \mathcal{R}_0 > 1 \\ \Delta x, & \text{if} \quad \mathcal{R}_0 = 1 \\ \frac{2D}{\sqrt{\beta D(\frac{1}{\mathcal{R}_0}-1)}} \sinh\left(\frac{\sqrt{\beta D(\frac{1}{\mathcal{R}_0}-1)}}{2D}\Delta x\right), & \text{if} \quad \mathcal{R}_0 < 1. \end{cases} \tag{3.4.3}$$

The sum of Eq. (3.4.1) and Eq. (3.4.2) leads to the NSFD scheme of the conservation law (3.2.6):

$$\frac{N_n^{k+1} - N_n^k}{\phi(\Delta t)} = \mu(K - N_n^{k+1}) + D\frac{N_{n+1}^k - 2N_n^k + N_{n-1}^k}{\psi^2(\Delta x)}. \tag{3.4.4}$$

The denominator function $\psi(\Delta x)$ to be used here and after comes from the exact scheme in [61] of the second order equation

$$D\frac{d^2 I}{dx^2} + \beta(1 - \frac{1}{\mathcal{R}_0})I = 0,$$

which is obtained from the stationary equation of (3.2.7) by ignoring its nonlinear part. Using

$$\frac{N_n^{k+1} - N_n^k}{\phi(\Delta t)} = \mu\left(K - \frac{N_{n+1}^k + N_n^k + N_{n-1}^k}{3}\right) + D\frac{N_{n+1}^k - 2N_n^k + N_{n-1}^k}{\psi^2(\Delta x)}, \tag{3.4.5}$$

as NSFD scheme for the conservation law (3.2.6), the second NSFD scheme for the SIS-diffusion model (3.2.6)-(3.2.7) of interest to us is

$$\begin{aligned} \frac{S_n^{k+1} - S_n^k}{\phi(\Delta t)} &= \mu K - \frac{\beta S_n^{k+1}}{N_n^{k+1}}\left(\frac{I_{n+1}^k + I_n^k + I_{n-1}^k}{3}\right) - \mu\left(\frac{S_{n+1}^k + S_n^k + S_{n-1}^k}{3}\right) \\ &+ \gamma\left(\frac{I_{n+1}^k + I_n^k + I_{n-1}^k}{3}\right) + D\frac{S_{n+1}^k - 2S_n^k + S_{n-1}^k}{\psi^2(\Delta x)} \end{aligned} \tag{3.4.6}$$

and

$$\begin{aligned} \frac{I_n^{k+1} - I_n^k}{\phi(\Delta t)} &= \beta(1 - \frac{1}{\mathcal{R}_0})\left(1 - \frac{I_n^{k+1}}{N_n^{k+1}(1 - \frac{1}{\mathcal{R}_0})}\right)\left(\frac{I_{n+1}^k + I_n^k + I_{n-1}^k}{3}\right) \\ &+ D\frac{I_{n+1}^k - 2I_n^k + I_{n-1}^k}{\psi^2(\Delta x)}. \end{aligned} \tag{3.4.7}$$

The implementation of the two NSFD schemes is an issue of interest. For the first NSFD scheme, we rearrange Eq. (3.4.1) to get a quadratic equation in S_n^{k+1}:

$$A(S_n^{k+1})^2 + BS_n^{k+1} + C = 0, \tag{3.4.8}$$

where

$$\begin{aligned}
A &= 1+\mu\phi \geq 0,\\
B &= (1+(\beta+\mu-\gamma)\phi)\, I_n^k - (S_n^k+\mu\phi K) - \frac{\phi D}{\psi^2}\left(S_{n+1}^k - 2S_n^k + S_{n-1}^k\right)\\
\text{and} \qquad C &= -\left[S_n^k + \gamma\phi I_n^k + \mu\phi K + \frac{\phi D}{\psi^2}\left(S_{n+1}^k - 2S_n^k + S_{n-1}^k\right)\right] I_n^k.
\end{aligned}$$

In what follows, we make the assumption

$$\frac{\phi}{\psi^2} = \frac{1}{2D} \tag{3.4.9}$$

between the step sizes so that the term S_n^k cancels in the expressions of B and C.

Assuming that $S_n^k \geq 0$ and $I_n^k \geq 0$, then the unique positive solution of Eq. (3.4.8) is given by

$$S_n^{k+1} = \frac{-B+\sqrt{B^2-4AC}}{2A}. \tag{3.4.10}$$

Moreover, (3.4.2) and (3.4.4) rearranged into

$$I_n^{k+1} = \frac{\left(\frac{\beta\phi S_n^{k+1}}{S_n^{k+1}+I_n^k} + 1 - \gamma\phi\right) I_n^k + \frac{\phi D}{\psi^2}\left(I_{n+1}^k - 2I_n^k + I_{n-1}^k\right)}{1+\mu\phi} \tag{3.4.11}$$

and

$$N_n^{k+1} = \frac{\mu\phi K + \frac{\phi D}{\psi^2}(N_{n+1}^k + N_{n-1}^k) + (1-\frac{2\phi D}{\psi^2})N_n^k}{1+\mu\phi} \tag{3.4.12}$$

respectively, show the positivity of I_n^{k+1} and N_n^{k+1} under the condition (2.4.26).

For the implementation of the second NSFD scheme, (3.4.5), (3.4.6) and (3.4.7) can be written as

$$N_n^{k+1} = \mu\phi K + \left(\frac{\phi D}{\psi^2} - \frac{\mu\phi}{3}\right)\left(N_{n+1}^k + N_{n-1}^k\right) + \left(1 - \frac{\mu\phi}{3} - \frac{2\phi D}{\psi^2}\right) N_n^k, \tag{3.4.13}$$

$$S_n^{k+1} = \frac{\mu\phi\left(K - \frac{S_{n+1}^k+S_n^k+S_{n-1}^k}{3}\right) + \gamma\phi\left(\frac{I_{n+1}^k+I_n^k+I_{n-1}^k}{3}\right) + \frac{\phi D}{\psi^2}\left(S_{n+1}^k - 2S_n^k + S_{n-1}^k\right) + S_n^k}{1 + \frac{\beta\phi}{N_n^{k+1}}\left(\frac{I_{n+1}^k+I_n^k+I_{n-1}^k}{3}\right)} \tag{3.4.14}$$

and

$$I_n^{k+1} = N_n^{k+1}(1-\frac{1}{\mathcal{R}_0})\frac{\frac{\phi D}{\psi^2}(I_{n+1}^k + I_{n-1}^k) + \left(1-\frac{2\phi D}{\psi^2}\right) I_n^k + \phi\beta(1-\frac{1}{\mathcal{R}_0})\left(\frac{I_{n+1}^k+I_n^k+I_{n-1}^k}{3}\right)}{N_n^{k+1}(1-\frac{1}{\mathcal{R}_0}) + \phi\beta(1-\frac{1}{\mathcal{R}_0})\left(\frac{I_{n+1}^k+I_n^k+I_{n-1}^k}{3}\right)}. \tag{3.4.15}$$

Remark 3.4.1. *Note that in the two constructed NSFD schemes, Mickens' rules [61] are reinforced as highlighted in [8]: the nonlinear terms are approximated in a nonlocal way, and the usual standard denominators Δt and Δx of the discrete derivatives are replaced by the complex denominator functions $\phi(\Delta t)$ and $\psi(\Delta x)$, respectively.*

The next two results confirm the relevance of functional relations between step sizes in order for the NSFD schemes to preserve the positivity and boundedness properties.

Theorem 3.4.2. *Under the functional relation*

$$\frac{\phi}{\psi^2} = \frac{1}{3D} \tag{3.4.16}$$

between the step sizes, the NSFD scheme (3.4.5) replicates the positivity and boundedness properties of the exact solution of (3.2.6):

$$0 \leq N_n^k \leq K \Rightarrow 0 \leq N_n^{k+1} \leq K.$$

The same conclusion holds for the NSFD scheme (3.4.4) under the condition (3.4.9).

Proof: Assume that $0 \leq N_n^k \leq K$ for all $k \in \mathbb{N}$ and $n \in \mathbb{Z}$. By using (3.4.16) and (2.4.26), the NSFD scheme (3.4.13) is reduced into

$$\begin{aligned} N_n^{k+1} &= \mu\phi K + \frac{(1-\mu\phi)}{3}\left(N_{n+1}^k + N_n^k + N_{n-1}^k\right) \\ &\leq \mu\phi K + (1-\mu\phi)K, \quad \text{since} \quad N_{n+1}^k + N_n^k + N_{n-1}^k \leq 3K \\ &= K. \end{aligned}$$

Equally for the NSFD scheme (3.4.4) or (3.4.12), we have by (3.4.9)

$$N_n^{k+1} = \frac{\mu\phi K + \frac{1}{2}\left(N_{n+1}^k + N_{n-1}^k\right)}{1+\mu\phi} \leq K. \qquad \square$$

Theorem 3.4.3. *Under the conditions of Theorem 3.4.2, the NSFD schemes (3.4.1)-(3.4.2) and (3.4.5)-(3.4.7) replicates positivity and boundedness properties of the exact solution stated in Theorem 3.3.1 in the following specific manner:*

$$0 \leq I_n^k \leq N_n^k \leq K \Rightarrow 0 \leq I_n^{k+1} \leq N_n^{k+1} \leq K.$$

Proof: Assume that $0 \leq I_n^k,\ S_n^k \leq N_n^k \leq K$ for all $k \in \mathbb{N}$ and $n \in \mathbb{Z}$. By using (3.4.9) and (2.4.26), from Theorem 3.4.2 and (3.4.10)-(3.4.11), we obtain $N_n^{k+1} \geq 0,\ S_n^{k+1} \geq 0$

and $I_n^{k+1} \geq 0$. Thus, from the time-space analogue of (2.4.15), we have $0 \leq I_n^{k+1},\ S_n^{k+1} \leq N_n^{k+1} \leq K$. Similarly, further implementation of (3.4.16) reduce equations (3.4.14) and (3.4.15) into

$$S_n^{k+1} = \frac{\mu\phi\left(K - \frac{S_{n+1}^k + S_n^k + S_{n-1}^k}{3}\right) + \gamma\phi\left(\frac{I_{n+1}^k + I_n^k + I_{n-1}^k}{3}\right) + \frac{1}{3}\left(S_{n+1}^k + S_n^k + S_{n-1}^k\right)}{1 + \frac{\beta\phi}{N_n^{k+1}}\left(\frac{I_{n+1}^k + I_n^k + I_{n-1}^k}{3}\right)}, \tag{3.4.17}$$

and

$$I_n^{k+1} = N_n^{k+1}(1 - \frac{1}{\mathcal{R}_0})\frac{\frac{1}{3}(I_{n+1}^k + I_n^k + I_{n-1}^k) + \phi\beta(1 - \frac{1}{\mathcal{R}_0})\left(\frac{I_{n+1}^k + I_n^k + I_{n-1}^k}{3}\right)}{N_n^{k+1}(1 - \frac{1}{\mathcal{R}_0}) + \phi\beta(1 - \frac{1}{\mathcal{R}_0})\left(\frac{I_{n+1}^k + I_n^k + I_{n-1}^k}{3}\right)} \tag{3.4.18}$$

respectively. Theorem 3.4.2 and equations (3.4.17)-(3.4.18) imply that $N_n^{k+1} \geq 0,\quad S_n^{k+1} \geq 0$ and $I_n^{k+1} \geq 0$ whenever $\mathcal{R}_0 > 1$. If $\mathcal{R}_0 < 1$, then these positivity results hold true under condition (2.4.26). Hence, using time-space analogue of (2.4.15), we infer that $0 \leq I_n^{k+1},\ S_n^{k+1} \leq N_n^{k+1} \leq K.$ $\square$.

Next, we want to determine the fixed points of the two NSFD schemes. The constant $\bar{N} = K$ is the only fixed point of the NSFD schemes (3.4.4) (or (3.4.12)) and (3.4.5) (or (3.4.13)). It is easy to check that

$$\bar{E}_0 = (K, 0) \quad \text{and} \quad \bar{E}_\infty = \left(\frac{K}{\mathcal{R}_0}, K(1 - \frac{1}{\mathcal{R}_0})\right)$$

are fixed points of the first NSFD scheme (3.4.1)-(3.4.2) (equivalent to (3.4.10)-(3.4.11)) and of the second NSFD scheme (3.4.6)-(3.4.7) (equivalent to (3.4.14)-(3.4.15)). Thus, the two designed NSFD schemes preserve the constant equilibrium solutions of the continuous SIS-diffusion model (3.2.2)-(3.2.6).

In the remaining part of this section, we want to check the dynamic consistency of the NSFD schemes with the stability properties of the equilibrium points of the continuous model (3.2.2)-(3.2.6). To this end and in order to use the discrete energy method, we recall some notation and results given in [5] regarding the space

$$l_2 \equiv l_2^N = \{\mathbf{u} = \{u_n\}_{n=0}^N,\ u_n \in \mathbb{R}\}, \tag{3.4.19}$$

which approximate the Lebesgue space $L_2(0, b)$. For $\mathbf{u} = (u_n) \in l_2$ and $\mathbf{v} = (v_n) \in l_2$, we introduce the norm

$$\|\mathbf{u}\|_{l_2}^2 = \Delta x \sum_{n=0}^{N} |u_n|^2. \tag{3.4.20}$$

The following three relations will be useful:

$$2\sum_{n=0}^{N}(u_n - v_n)u_n = \|\mathbf{u}\|_{l_2}^2 - \|\mathbf{v}\|_{l_2}^2 + \|\mathbf{u}-\mathbf{v}\|_{l_2}^2, \quad \text{for} \quad \mathbf{u},\ \mathbf{v} \in l_2; \tag{3.4.21}$$

$$\sum_{n=0}^{N} u_n D^+_{n,\triangle x} v = -\sum_{n=1}^{N+1} v_n D^-_{n,\triangle x} u \quad \text{for} \quad \mathbf{u},\ \mathbf{v} \in l_2 \ \text{ with } \ v_{N+1} = v_0 = 0, \tag{3.4.22}$$

where

$$D^+_{n,\triangle x} v = \frac{1}{\triangle x}(v_{n+1} - v_n), \ \text{ and } \quad D^-_{n,\triangle x} u = \frac{1}{\triangle x}(u_n - u_{n-1}) \tag{3.4.23}$$

are the forward and backward difference operators, respectively. Equation (3.4.22) is the discrete Green or integration by parts formula.

$$\|u\|_{l_2}^2 \le C\sum_{n=0}^{N}(D^+_{n,\triangle x} u)^2, \quad u \in l_2 \ \text{ with } \ u_N = u_0 = 0 \tag{3.4.24}$$

for some constant $C > 0$ not depending on u. This is the discrete Poincar$\acute{e}$-Friedrichs inequality.

Remark 3.4.4. *In the nonstandard context, we replace $\triangle x$ by $\psi(\triangle x)$ and obtain the operators $D^+_{n,\psi(\triangle x)}$ and $D^-_{n,\psi(\triangle x)}$ for which the relations (3.4.21), (3.4.22) and (3.4.24) are valid.*

In what follows, it is implicitly assumed that the arguments satisfy the boundary conditions that lead to the analogues (3.4.22) and (3.4.24).

Theorem 3.4.5. *The fixed point $\bar{N} = K$ of the first NSFD scheme (3.4.4) or (3.4.12) for the conservation law is l_2 globally asymptotically stable.*

Proof: We use the discrete energy method. Notice that Eq. (3.4.4) can be written as

$$\frac{(N_n^{k+1} - K) - (N_n^k - K)}{\phi} = \mu(K - N_n^{k+1}) + DD^+_{n,\psi}D^-_{n,\psi}(N^k - K).$$

Multiplying both sides by $N_n^{k+1} - K$ and summing through $n = 0$ to $n = N$, we get

$$\begin{aligned}\sum_{n=0}^{N}\frac{(N_n^{k+1} - K) - (N_n^k - K)}{\phi}.(N_n^{k+1} - K) &= -\mu\sum_{n=0}^{N}(K - N_n^{k+1})^2 \\ &+ D\sum_{n=0}^{N}\left(D^+_{n,\psi}D^-_{n,\psi}(N^k - K)\right).(N_n^{k+1} - K).\end{aligned}$$

By applying Eq. (3.4.21) and the Green formula (3.4.22), we have

$$\begin{aligned}\frac{1}{2\phi}\left[\|N^{k+1}-K\|_{l_2}^2-\|N^k-K\|_{l_2}^2+\|N^{k+1}-N^k\|_{l_2}^2\right] &= -\mu\|N^{k+1}-K\|_{l_2}^2\\ &- CD\sum_{n=0}^{N}\left[D_{n,\psi}^{+}(N^{k+1}-K)\right]^2,\end{aligned}$$

for $C>0$. Then by using the discrete Poincaré-Friedrichs inequality (3.4.24), we obtain

$$\begin{aligned}\frac{1}{2\phi}\left[\|N^{k+1}-K\|_{l_2}^2-\|N^k-K\|_{l_2}^2+\|N^{k+1}-N^k\|_{l_2}^2\right] &\leq -\mu\|N^{k+1}-K\|_{l_2}^2\\ &- \frac{D}{\psi^2 C}\|N^{k+1}-K\|_{l_2}^2.\end{aligned} \tag{3.4.25}$$

Further simplifications of Eq. (3.4.25) lead to

$$\begin{aligned}\frac{1}{2\phi}\|N^{k+1}-K\|_{l_2}^2 &\leq \frac{1}{2\phi}\|N^k-K\|_{l_2}^2-\frac{1}{2\phi}\|N^{k+1}-N^k\|_{l_2}^2-\mu\|N^{k+1}-K\|_{l_2}^2\\ &\leq \frac{1}{2\phi}\|N^k-K\|_{l_2}^2-\frac{1}{2\phi}\left[\|N^{k+1}-K\|_{l_2}-\|N^k-K\|_{l_2}\right]^2-\mu\|N^{k+1}-K\|_{l_2}^2,\\ &\leq \frac{1}{2\phi}\|N^k-K\|_{l_2}^2-\frac{1}{2\phi}\|N^{k+1}-K\|_{l_2}^2+\frac{1}{\phi}\|N^{k+1}-K\|_{l_2}\|N^k-K\|_{l_2}\\ &- \frac{1}{2\phi}\|N^k-K\|_{l_2}^2-\mu\|N^{k+1}-K\|_{l_2}^2\\ &= -(\frac{1}{2\phi}+\mu)\|N^{k+1}-K\|_{l_2}^2+\frac{1}{\phi}\|N^{k+1}-K\|_{l_2}\|N^k-K\|_{l_2}.\end{aligned}$$

This implies that

$$\left(\frac{1}{\phi}+\mu\right)\|N^{k+1}-K\|_{l_2}^2 \leq \frac{1}{\phi}\|N^{k+1}-K\|_{l_2}\|N^k-K\|_{l_2}.$$

Thus,

$$\|N^{k+1}-K\|_{l_2} \leq (1+\mu\phi)^{-1}\|N^k-K\|_{l_2}$$

and by induction,

$$\|N^k-K\|_{l_2} \leq (1+\mu\phi)^{-k}\|N^0-K\|_{l_2}.$$

Hence, $\|N^k-K\|_{l_2}\to 0$ as $k\to\infty$. This completes the proof. □

Theorem 3.4.6. *For the second NSFD scheme (3.4.5) or (3.4.13) of the conservation law, the fixed point $\bar{N}=K$ is l_2 globally asymptotically stable.*

Proof: By using the functional relation (3.4.16), Eq.(3.4.13) is reduced into

$$\begin{aligned}N_n^{k+1} &= \mu\phi K+\left(\frac{1}{3}-\frac{\mu\phi}{3}\right)(N_{n+1}^k+N_{n-1}^k)+\left(1-\frac{\mu\phi}{3}-\frac{2}{3}\right)N_n^k\\ &= \mu\phi K+(1-\mu\phi)\left(\frac{N_{n+1}^k+N_n^k+N_{n-1}^k}{3}\right).\end{aligned}$$

Thus,

$$N_{\max}^{k+1} = \mu\phi K + (1-\mu\phi)N_{\max}^{k}, \quad \text{where} \quad N_{\max}^{k} := \max_{n} N_n^k,$$

and

$$\begin{aligned} N_{\max}^{k+1} - K &= \mu\phi K + (1-\mu\phi)N_{\max}^{k} - K \\ &= (1-\mu\phi)(N_{\max}^{k} - K). \end{aligned}$$

By induction,

$$N_{\max}^{k} - K = (1-\mu\phi)^k (N_{max}^{0} - K).$$

Hence, the sequence $(N_{\max}^{k} - K)_{k\geq 0}$ converges to zero. This implies that $(N_n^k - K)_{k\geq 0} \to 0$ as $k \to \infty$ in l_2, because the $\max$ norm is equivalent to the l_2 norm. $\square$

Theorem 3.4.7. *The fixed point $\bar{I}_0 = 0$ of the first NSFD scheme (3.4.2) or (3.4.11) is l_2 globally asymptotically stable for $\mathcal{R}_0 < 1$.*

Proof: By applying Eq. (3.4.9) and the definition of $\mathcal{R}_0$ (see (2.3.4)), Eq (3.4.11) is reduced into

$$\begin{aligned} I_n^{k+1} &= \frac{\left(\frac{\mathcal{R}_0(\mu+\gamma)\phi S_n^{k+1}}{S_n^{k+1}+I_n^k} - \gamma\phi\right) I_n^k + \frac{1}{2}\left(I_{n+1}^k + I_{n-1}^k\right)}{1+\mu\phi} \\ &\leq \frac{\left[\mathcal{R}_0(\mu+\gamma)\phi - \gamma\phi\right] I_n^k + \frac{1}{2}\left(I_{n+1}^k + I_{n-1}^k\right)}{1+\mu\phi} \\ &= \frac{\left[\mathcal{R}_0\mu\phi + (\mathcal{R}_0 - 1)\gamma\phi\right] I_n^k + \frac{1}{2}\left(I_{n+1}^k + I_{n-1}^k\right)}{1+\mu\phi} \\ &\leq \frac{\mathcal{R}_o\mu\phi I_n^k + \frac{1}{2}\left(I_{n+1}^k + I_{n-1}^k\right)}{1+\mu\phi}, \quad \text{since} \quad \mathcal{R}_0 \leq 1. \end{aligned}$$

Thus,

$$I_{\max}^{k+1} \leq \frac{(1+\mathcal{R}_0\mu\phi)}{1+\mu\phi} I_{\max}^{k}.$$

With $\mathcal{R}_0 < 1$, the contraction mapping principle implies that the sequence $(I_{\max}^k)_{k\geq 0}$ converges to zero as $k \to \infty$. This means that $(I_n^k)_{k\geq 0}$ converges to zero in l_2. $\square$

Theorem 3.4.8. *The fixed point $\bar{I}_\infty = K(1 - \frac{1}{\mathcal{R}_0})$ of the first NSFD scheme (3.4.2) or (3.4.11) is l_2 globally asymptotically stable for $\mathcal{R}_0 > 1$.*

Proof: Since N_n^k given in (3.4.4) or (3.4.12) converges to K as $k \to \infty$ (see Theorem 3.4.5), and $I_n^k \approx I_n^{k+1}$ for k large, Eq. (3.4.2) behaves like equation

$$\frac{I_n^{k+1} - I_n^k}{\phi} = \frac{\beta(K - I_n^{k+1})I_n^k}{K} - (\mu + \gamma)I_n^k + D\frac{I_{n+1}^k - 2I_n^k + I_{n-1}^k}{\psi^2}.$$

Straightforward manipulations show that

$$\begin{aligned}
\frac{I_n^{k+1} - I_n^k}{\phi} &= \beta\left(1 - \frac{(\mu+\gamma)}{\beta} - \frac{I_n^{k+1}}{K}\right) I_n^k + D\frac{I_{n+1}^k - 2I_n^k + I_{n-1}^k}{\psi^2} \\
&= \beta\left(1 - \frac{1}{\mathcal{R}_0}\right)\left(1 - \frac{I_n^{k+1}}{K\left(1 - \frac{1}{\mathcal{R}_0}\right)}\right) I_n^k + D\frac{I_{n+1}^k - 2I_n^k + I_{n-1}^k}{\psi^2}, \\
&\qquad \text{by} \quad (2.3.4) \\
&= -\frac{\beta}{K}\left(I_n^{k+1} - \bar{I}_\infty\right) I_n^k + D\frac{I_{n+1}^k - 2I_n^k + I_{n-1}^k}{\psi^2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{(I_n^{k+1} - \bar{I}_\infty) - (I_n^k - \bar{I}_\infty)}{\phi} &= -\frac{\beta}{K}\left(I_n^{k+1} - \bar{I}_\infty\right) I_n^k + D\frac{(I_{n+1}^k - \bar{I}_\infty) - 2(I_n^k - \bar{I}_\infty) + (I_{n-1}^k - \bar{I}_\infty)}{\psi^2} \\
&= -\frac{\beta}{K}\left(I_n^{k+1} - \bar{I}_\infty\right) I_n^k + DD_{n,\psi}^+ D_{n,\psi}^-(I^k - \bar{I}_\infty). \qquad (3.4.26)
\end{aligned}$$

We proceed the proof as we did in the proof of Theorem 3.4.5 by multiplying Eq. (3.4.26) with $I_n^{k+1} - \bar{I}_\infty$ and summing through $n = 0$ to $n = N$ to get

$$\frac{1}{2\phi}\left[\|I^{k+1} - \bar{I}_\infty\|_{l_2}^2 - \|I^k - \bar{I}_\infty\|_{l_2}^2 + \|(I^{k+1} - I^k\|_{l_2}^2\right] \leq \frac{-D}{\psi^2 C}\|I^{k+1} - \bar{I}_\infty\|^2, \qquad \text{where} \quad C > 0.$$

Further rearrangement of this inequality (using Eq. (3.4.9)) gives

$$\left(1 + \frac{1}{2C}\right)\|I^{k+1} - \bar{I}_\infty\|_{l_2}^2 \leq \|I^{k+1} - \bar{I}_\infty\|_{l_2}\|I^k - \bar{I}_\infty\|_{l_2},$$

or by induction,

$$\|I^k - \bar{I}_\infty\|_{l_2} \leq \left(1 + \frac{1}{2C}\right)^{-k}\|I^0 - \bar{I}_\infty\|_{l_2}.$$

Thus, for $\mathcal{R}_0 > 1$ and $\bar{I}_\infty = K(1 - \frac{1}{\mathcal{R}_0})$,

$$\|I^k - \bar{I}_\infty\|_{l_2} \to 0 \qquad \text{as} \qquad k \to \infty.$$

This completes the proof of the theorem. □

Remark 3.4.9. *From Theorem 3.4.7 and 3.4.8, it follows that* $\bar{I}_0 = 0$ *is unstable when* $\mathcal{R}_0 > 1$.

Theorem 3.4.10. *For the second NSFD scheme (3.4.7) or (3.4.15), the fixed point $\bar{I}_0 = 0$ is l_2 globally asymptotically stable for $\mathcal{R}_0 < 1$.*

Proof: Since N_n^k given in (3.4.13) converges to K as $k \to \infty$, by using (3.4.16)), Eq. (3.4.15) behaves like

$$\begin{aligned}
I_n^{k+1} &= K\frac{\frac{1}{3}\left(I_{n+1}^k + I_n^k + I_{n-1}^k\right) + \phi\beta(1-\frac{1}{\mathcal{R}_0})\left(\frac{I_{n+1}^k+I_n^k+I_{n-1}^k}{3}\right)}{K + \phi\beta\left(\frac{I_{n+1}^k+I_n^k+I_{n-1}^k}{3}\right)}, \\
&\leq K\frac{\frac{1}{3}\left(I_{n+1}^k + I_n^k + I_{n-1}^k\right) + \phi\beta(1-\frac{1}{\mathcal{R}_0})\left(\frac{I_{n+1}^k+I_n^k+I_{n-1}^k}{3}\right)}{K} \\
&\leq (1+\phi(\mu+\gamma)(\mathcal{R}_0-1))\left(\frac{I_{n+1}^k + I_n^k + I_{n-1}^k}{3}\right).
\end{aligned}$$

Thus,

$$I_{\max}^{k+1} \leq [1+\phi(\mu+\gamma)(\mathcal{R}_0-1)]\, I_{\max}^k,$$

since $0 < 1+\phi(\mu+\gamma)(\mathcal{R}_0-1) < 1$ for $\mathcal{R}_0 < 1$. Hence, the sequence $(I_{\max}^k)_{k\geq 0}$ converges to zero as $k \to \infty$. □

Theorem 3.4.11. *The fixed point $\bar{I}_\infty = K(1-\frac{1}{\mathcal{R}_0})$ of the second NSFD scheme (3.4.7) or (3.4.15) is l_2 globally asymptotically stable for $\mathcal{R}_0 > 1$.*

Proof: The proof of this theorem is similar to that of Theorem 3.4.8. As $k \to \infty$, $N_n^k \to K$ (see Eq. (3.4.13)) and Eq. (3.4.7) behaves like equation

$$\begin{aligned}
\frac{I_n^{k+1} - I_n^k}{\phi(\Delta t)} &= \beta(1-\frac{1}{\mathcal{R}_0})\left(1 - \frac{I_n^{k+1}}{K(1-\frac{1}{\mathcal{R}_0})}\right)\left(\frac{I_{n+1}^k + I_n^k + I_{n-1}^k}{3}\right) \\
&+ D\frac{I_{n+1}^k - 2I_n^k + I_{n-1}^k}{(\psi(\Delta x))^2} \\
&= -\frac{\beta}{K}\left(I_n^{k+1} - \bar{I}_\infty\right)\left(\frac{I_{n+1}^k + I_n^k + I_{n-1}^k}{3}\right) + D\frac{I_{n+1}^k - 2I_n^k + I_{n-1}^k}{(\psi(\Delta x))^2}.
\end{aligned}$$

Further rearrangements of this equation lead into

$$\begin{aligned}
\sum_{n=0}^{N}\frac{(I_n^{k+1}-\bar{I}_\infty)-(I_n^k-\bar{I}_\infty)}{\phi}.(I_n^{k+1}-\bar{I}_\infty) &= -\frac{\beta}{K}\sum_{n=0}^{N}(I_n^{k+1}-\bar{I}_\infty)^2\left(\frac{I_{n+1}^k + I_n^k + I_{n-1}^k}{3}\right) \\
&+ D\sum_{n=0}^{N} D_{n,\psi}^{+}D_{n,\psi}^{-}(I^k-\bar{I}_\infty).(I_n^{k+1}-\bar{I}_\infty), \\
&\leq D\sum_{n=0}^{N} D_{n,\psi}^{+}D_{n,\psi}^{-}(I^k-\bar{I}_\infty).(I_n^{k+1}-\bar{I}_\infty). \quad (3.4.27)
\end{aligned}$$

Applying Eq. (3.4.21)-(3.4.22) and (3.4.24) on Eq. (3.4.27) gives

$$\|I^k - \bar{I}_\infty\|_{l_2} \leq \left(1 + \frac{1}{2C}\right)^{-k} \|I^0 - \bar{I}_\infty\|_{l_2}.$$

Thus,

$$\|I^k - \bar{I}_\infty\|_{l_2} \to 0 \quad \text{as} \quad k \to \infty. \qquad \square$$

Remark 3.4.12. *It follows from Theorem 3.4.10 and 3.4.11 that $\bar{I}_0 = 0$ is unstable when $\mathcal{R}_0 > 1$.*

The last result in this section is to check Theorem 3.4.3 for traveling wave solution when $N = K$ is constant. In this case, we have the next theorem where the condition (3.4.16) between step sizes is more relaxed.

Theorem 3.4.13. *Under the functional relation (3.4.9), the NSFD scheme (3.4.5)-(3.4.7) replicates the positivity and boundedness properties of the exact solution as follows:*

$$0 \leq I_n^k \leq K(1 - \frac{1}{\mathcal{R}_0}) \Rightarrow 0 \leq I_n^{k+1} \leq K(1 - \frac{1}{\mathcal{R}_0}), \quad (3.4.28)$$

and

$$K(1 - \frac{1}{\mathcal{R}_0}) \leq I_n^k \leq K \Rightarrow K(1 - \frac{1}{\mathcal{R}_0}) \leq I_n^{k+1} \leq K. \quad (3.4.29)$$

Proof: In view of (3.4.9), Eq. (3.4.15) becomes

$$I_n^{k+1} = K(1 - \frac{1}{\mathcal{R}_0}) \frac{\frac{1}{2}(I_{n+1}^k + I_{n-1}^k) + \phi\beta(1 - \frac{1}{\mathcal{R}_0}) \left(\frac{I_{n+1}^k + I_n^k + I_{n-1}^k}{3}\right)}{K(1 - \frac{1}{\mathcal{R}_0}) + \phi\beta(1 - \frac{1}{\mathcal{R}_0}) \left(\frac{I_{n+1}^k + I_n^k + I_{n-1}^k}{3}\right)}. \quad (3.4.30)$$

If $0 \leq I_n^k \leq K(1 - \frac{1}{\mathcal{R}_0})$ so that

$$\frac{I_{n+1}^k + I_{n-1}^k}{2} \leq K(1 - \frac{1}{\mathcal{R}_0}), \quad (3.4.31)$$

we have from (3.4.31)

$$\frac{\frac{1}{2}\left(I_{n+1}^k + I_{n-1}^k\right) + \phi\beta(1 - \frac{1}{\mathcal{R}_0}) \left(\frac{I_{n+1}^k + I_n^k + I_{n-1}^k}{3}\right)}{K(1 - \frac{1}{\mathcal{R}_0}) + \phi\beta(1 - \frac{1}{\mathcal{R}_0}) \left(\frac{I_{n+1}^k + I_n^k + I_{n-1}^k}{3}\right)} \leq 1,$$

which in view of (3.4.30) gives (3.4.28).

Assume now that $K(1-\frac{1}{\mathcal{R}_0}) \le I_n^k \le K$ so that $K(1-\frac{1}{\mathcal{R}_0}) \le \frac{1}{2}(I_{n+1}^k + I_{n-1}^k) \le K$. Then

$$
\begin{aligned}
K(1-\frac{1}{\mathcal{R}_0}) &= K(1-\frac{1}{\mathcal{R}_0})\frac{K(1-\frac{1}{\mathcal{R}_0}) + \phi\beta(1-\frac{1}{\mathcal{R}_0})\left(\frac{I_{n+1}^k+I_n^k+I_{n-1}^k}{3}\right)}{K(1-\frac{1}{\mathcal{R}_0}) + \phi\beta(1-\frac{1}{\mathcal{R}_0})\left(\frac{I_{n+1}^k+I_n^k+I_{n-1}^k}{3}\right)} \\
&\le K(1-\frac{1}{\mathcal{R}_0})\frac{\frac{1}{2}(I_{n+1}^k + I_{n-1}^k) + \phi\beta(1-\frac{1}{\mathcal{R}_0})\left(\frac{I_{n+1}^k+I_n^k+I_{n-1}^k}{3}\right)}{K(1-\frac{1}{\mathcal{R}_0}) + \phi\beta(1-\frac{1}{\mathcal{R}_0})\left(\frac{I_{n+1}^k+I_n^k+I_{n-1}^k}{3}\right)} = I_n^{k+1} \le K
\end{aligned}
$$

by (3.4.30). $\square$

To conclude this chapter, we consider some numerical experiments. In all these simulations, we take the values $K = 100$, $\mu = 0.2$ and $\gamma = 0.2$ so that β is the parameter that makes $\mathcal{R}_0$ varies. If $\beta = 1.24$, then $\mathcal{R}_0 = 3.1$ and $\mathcal{R}_0 = 0.6$ when $\beta = 0.24$. For different initial data, the excellent performance of the first scheme (3.4.1)-(3.4.2) and the second scheme (3.4.6)-(3.4.7) are displayed in Fig 3.3-3.6 and in Fig 3.7-3.10, respectively. By taking initial conditions $N_0(x) = 20 + 10\sin(2\pi x/5)$ and $I_0(x) = 10 + 10\sin(2\pi x/5)$, we illustrate the dynamical consistency of the NSFD schemes (3.4.4) and (3.4.5) with respect to positivity and boundedness, as stated in Theorem 3.4.2 (Fig 3.11 a) and Theorem 3.4.13 (3.11 b), respectively.

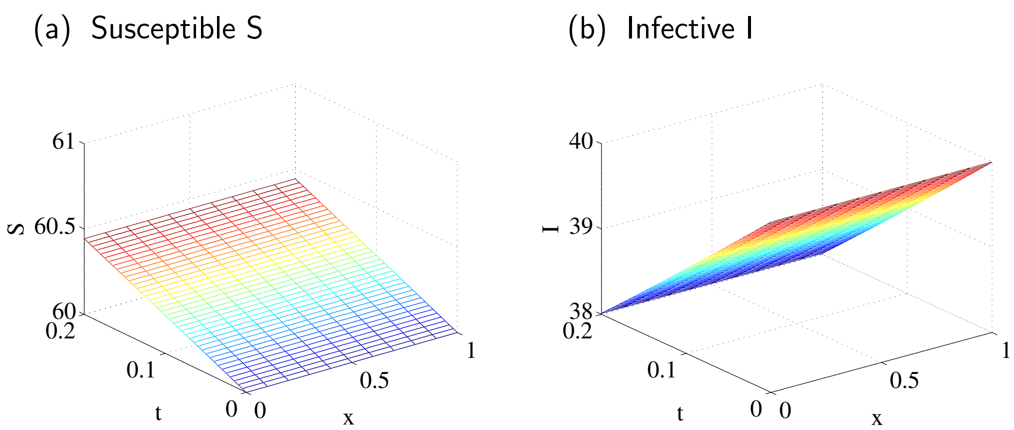

Figure 3.3: GAS of the disease-free fixed point for the first scheme (3.4.1)-(3.4.2) with $S^0(x) = 60,\quad I^0(x) = 40$ and $\mathcal{R}_0 = 0.6$.

(a) Susceptible S (b) Infective I

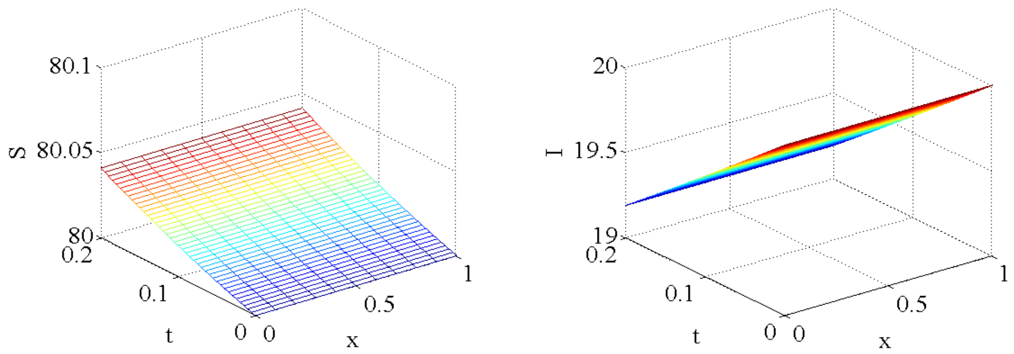

Figure 3.4: GAS of the disease-free fixed point for the first scheme (3.4.1)-(3.4.2) with $S^0(x) = 80, \quad I^0(x) = 20$ and $\mathcal{R}_0 = 0.6$.

(a) Susceptible S (b) Infective I

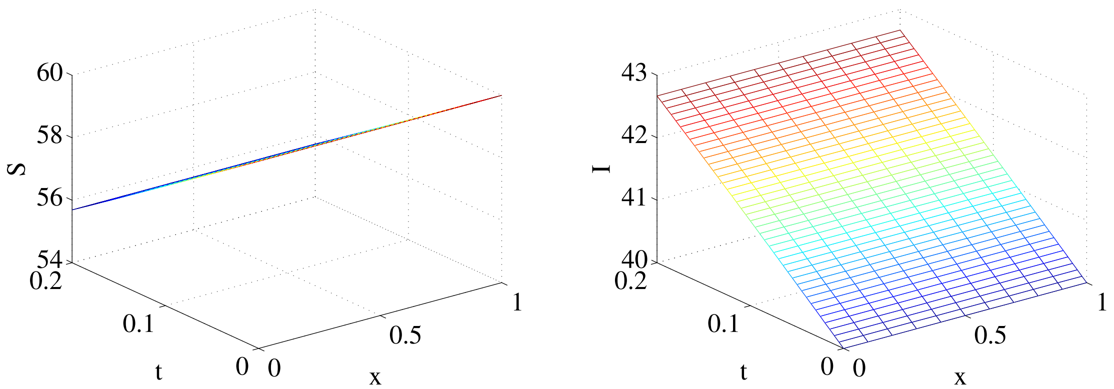

Figure 3.5: GAS of the endemic fixed point for the first scheme (3.4.1)-(3.4.2) with $S^0(x) = 60, \quad I^0(x) = 40$ and $\mathcal{R}_0 = 3.1$.

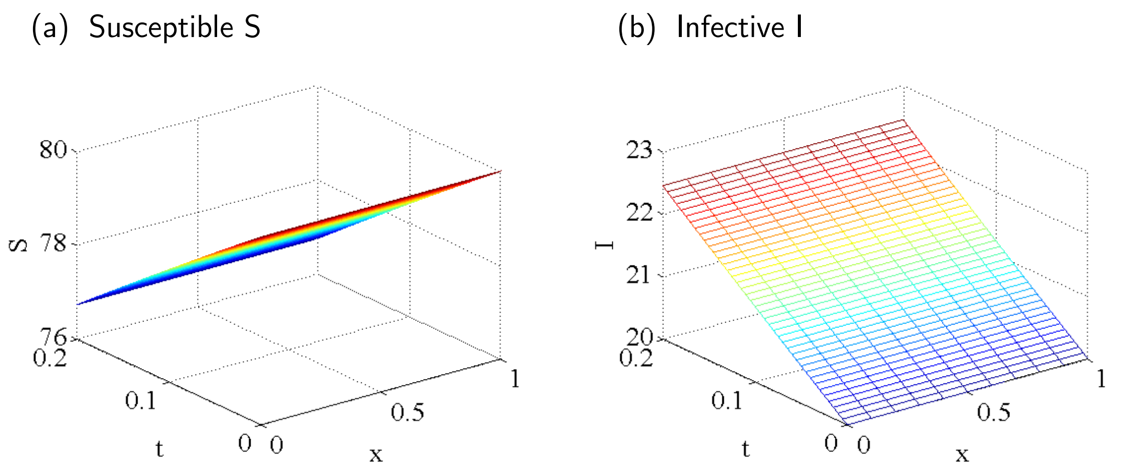

Figure 3.6: GAS of the endemic fixed point for the first scheme (3.4.1)-(3.4.2) with $S^0(x) = 80, \quad I^0(x) = 20$ and $\mathcal{R}_0 = 3.1$.

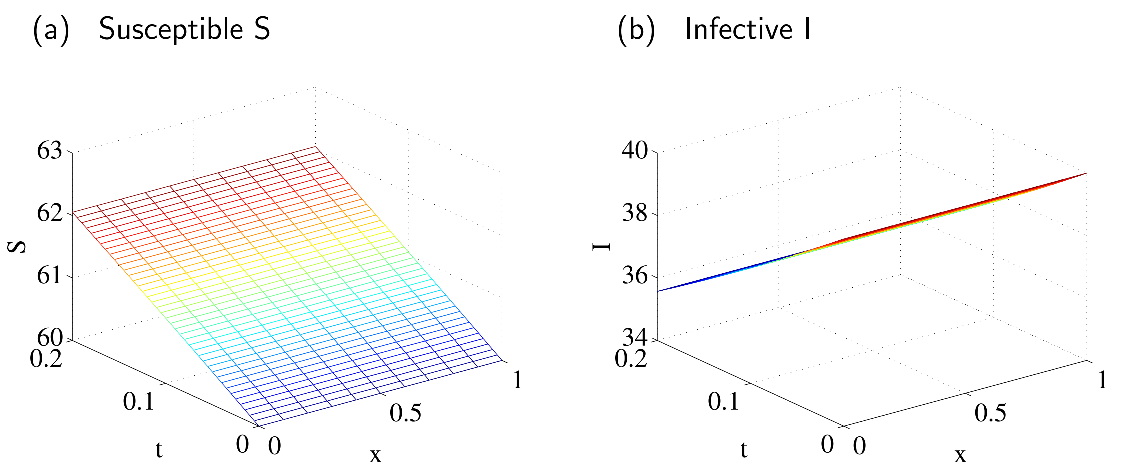

Figure 3.7: GAS of the disease-free fixed point for the second scheme (3.4.6)-(3.4.7) with $S^0(x) = 60, \quad I^0(x) = 40$ and $\mathcal{R}_0 = 0.6$.

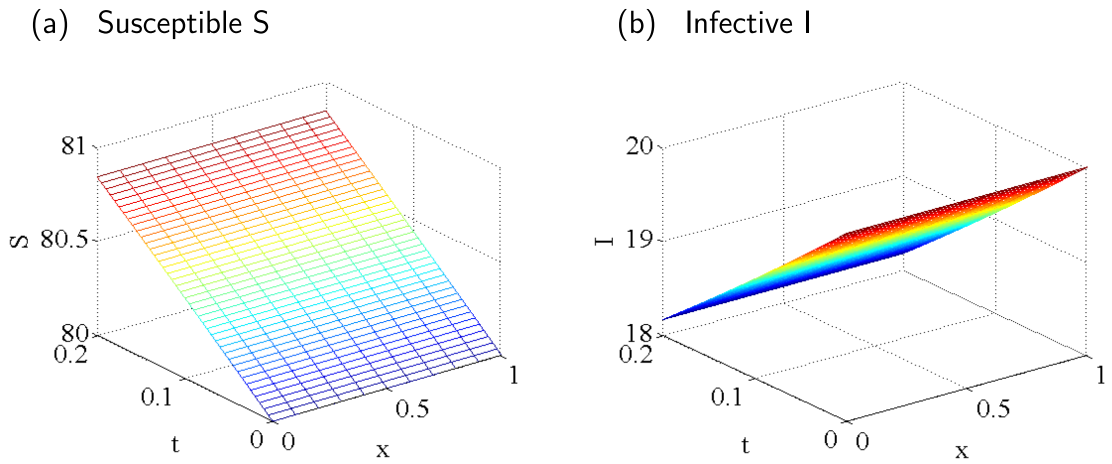

Figure 3.8: GAS of the disease-free fixed point for the second scheme (3.4.6)-(3.4.7) with $S^0(x) = 80, \quad I^0(x) = 20$ and $\mathcal{R}_0 = 0.6$.

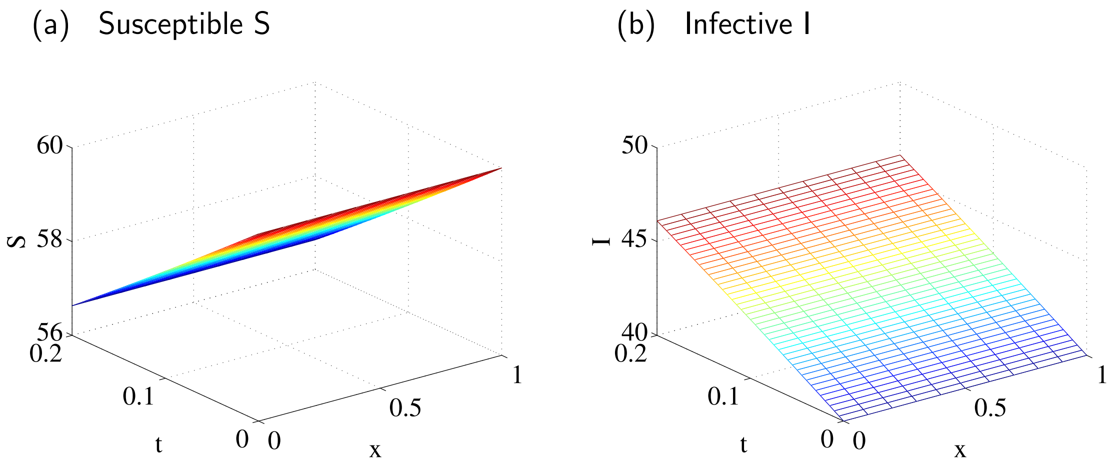

Figure 3.9: GAS of the endemic fixed point for the second scheme (3.4.6)-(3.4.7) with $S^0(x) = 60, \quad I^0(x) = 40$ and $\mathcal{R}_0 = 3.1$.

(a) Susceptible S (b) Infective I

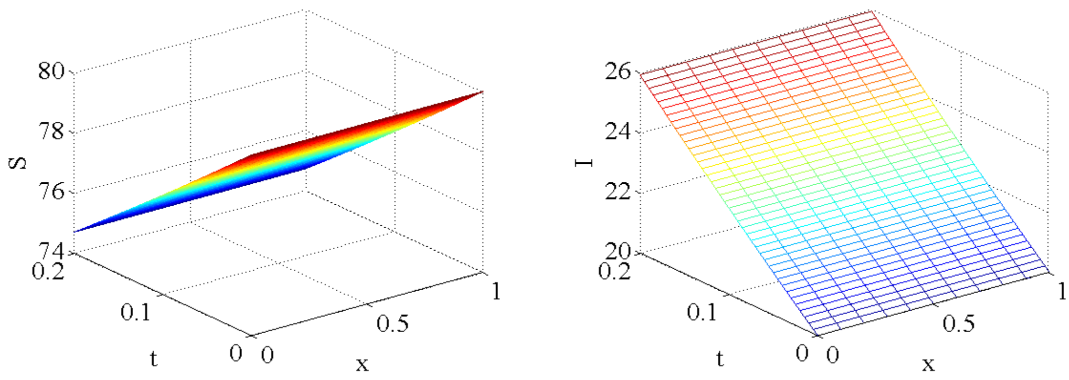

Figure 3.10: GAS of the endemic fixed point for the second scheme (3.4.6)-(3.4.7) with $S^0(x) = 80, \quad I^0(x) = 20$ and $\mathcal{R}_0 = 3.1$.

(a) NSFD scheme (3.4.4) (b) NSFD scheme (3.4.7)

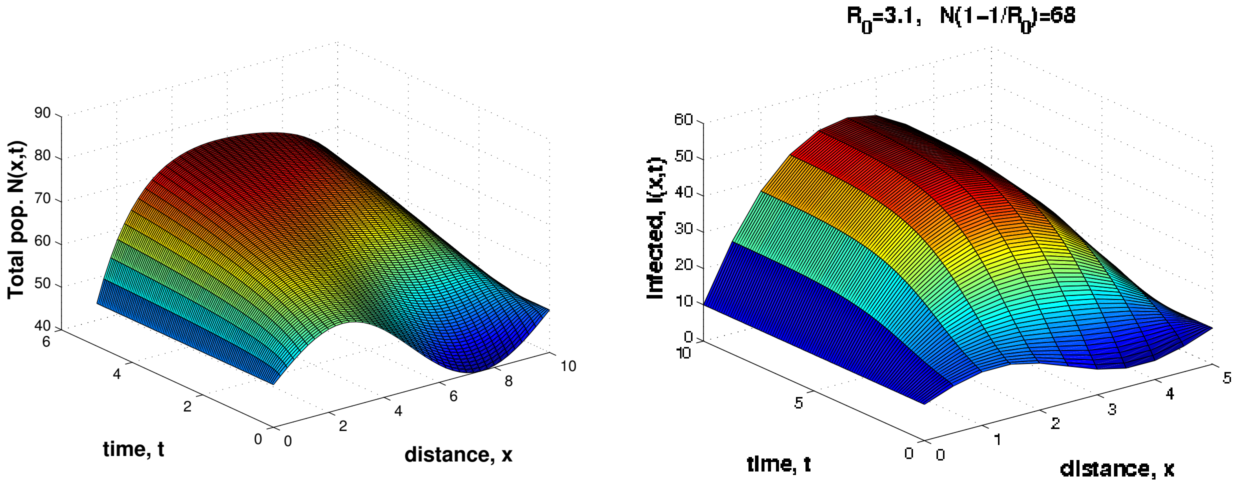

Figure 3.11: Positivity and boundedness of solutions.

Chapter 4

SIS-Volterra Integral Equation Model

4.1 Introduction

In chapter 3, the classical SIS model was extended to a reaction-diffusion partial differential equations to govern the spread of disease in space. In this chapter, we extend it to a Volterra integral equation of the second kind in order to incorporate the period of infectivity in the model. In this new setting, it is proved that the SIS-Volterra integral equation can exhibit the backward bifurcation phenomenon, whereby the locally stable disease-free equilibrium coexists with a locally stable endemic equilibrium whenever the basic reproduction number is less than unity. We also design a dynamically consistent NSFD scheme for the continuous SIS-Volterra integral equation based on Mickens' rules on complex denominator functions of discrete derivatives and nonlocal approximation of nonlinear terms [8, 61]. The results are published in [57].

This chapter is arranged in the following form. The detailed formulation of the continuous SIS-Volterra integral equation model is given in the next section. The quantitative and qualitative analysis of the continuous model is provided in Section 4.3. In Section 4.4, we design a nonstandard finite difference scheme and prove theoretically and computationally that it is dynamically consistent with the properties of the continuous model.

4.2 Model formulation

In this section, we consider an extension of the classical SIS model into the SIS-Volterra integral equation (SIS-VIE) model. The extension is based on the paper [79]. We will elaborate on and clarify some of the concepts that are used whenever necessary.

As a motivation, we reconsider the classical SIS model investigated in Chapter 2. However, we consider here its mass action incidence formulation, which reads

$$\frac{dS}{dt} = \mu - \lambda IS - \mu S + \gamma I, \quad S(0) = S_0 > 0, \tag{4.2.1}$$

$$\frac{dI}{dt} = \lambda IS - (\mu + \gamma)I, \quad I(0) = I_0 \geq 0, \tag{4.2.2}$$

where we recall that $\lambda > 0,\ \mu > 0$ and $\gamma > 0$ are the contact, the recruitment (or the natural death) and the recovery rates, respectively. In this chapter, we assume that $S = S(t)$ and $I = I(t)$ are fractions of susceptible and infective individuals at time t so that

$$S + I = 1. \tag{4.2.3}$$

In view of (4.2.3), the system (4.2.1)-(4.2.2) is equivalent to the scalar equation

$$\frac{dI}{dt} = \lambda I(1 - I) - (\mu + \gamma)I, \quad I(0) = I_0 \geq 0, \tag{4.2.4}$$

which takes the following equivalent form in terms of the basic reproduction number

$$\mathcal{R}_0 = \frac{\lambda}{\mu + \gamma} : \tag{4.2.5}$$

$$\frac{dI}{dt} = \lambda(1 - I)I\left(1 - \frac{1}{\mathcal{R}_0(1 - I)}\right). \tag{4.2.6}$$

By the fundamental theorem of calculus, Eq. (4.2.6) is equivalent to the integral equation

$$I(t) = I_0 + \int_0^t \lambda\left[1 - I(u)\right] I(u)\left(1 - \frac{1}{\mathcal{R}_0(1 - I(u))}\right) du. \tag{4.2.7}$$

In what follows, it is essential to write the equivalent equations (4.2.6) and (4.2.7) in an alternative (much richer) form. To this end, we introduce the initial function $I_0(t)$ defined by $I_0(t) = I_0 e^{-(\mu+\gamma)t}$ for $0 \leq t < \infty$. Consequently, $\frac{I_0(t)}{I_0} = e^{-(\mu+\gamma)t}$ is the probability for one infective individual to be infected through time t (and thus to infect others), given that this individual was infective at $t = 0$, and to leave the infectious class either by death or by returning to the class of susceptible individuals. The factor $P(t) = e^{-\gamma t}$ of the probability $e^{-(\mu+\gamma)t}$ is a similar probability for one infective individual, with the difference that the individual only leaves this class by returning to the susceptible class. Then the classical SIS-model (4.2.4) or (4.2.6) is equivalent to the following Volterra integral equation (VIE):

$$I(t) = I_0(t) + \int_0^t \lambda I(u)\,[1 - I(u)]\, I(u) P(t-u) e^{-\mu(t-u)} du. \tag{4.2.8}$$

The model (4.2.8) offers the opportunity to extend the classical SIS model by considering two realistic aspects.

The first aspect is to require that λ depend on the fraction of infective individuals I. There are many life situations which support this. Typical examples are saturation, multiple exposures to infectious individuals and behavioral changes in the population due to the increase of infective individuals. In the Table 4.1, we have gathered illustrative examples taken from the literature.

Contact rate $\lambda(I)$	Interpretation	Reference
$\beta(1+\nu I),\ \beta > 0\ \nu > 0$	Double exposure to infection	[79]
$\beta(1+\nu I^{p-1}),\ 1 < p < 2$ and $p > 2$	Due to crowding, multiple pathways to infections and change of behavior in the population	[24]
$\dfrac{\kappa I^{p-1}(1-I)^{q-1}}{1+mI^{p-1}},\ m,\ \kappa > 0$ and $p,\ q \geq 1$	A saturated contact rate	[52]
$\kappa I^{p-1}(1-I)^{q-1},\ \kappa > 0\ p > 1,\ q \geq 1$	Multiple exposures to infection	[79]

Table 4.1: Infective-dependent contact rates.

The second aspect is about the distribution of the infectivity period which in (4.2.8) is limited to the exponential setting $P(t) = e^{-\gamma t}$.

Based on these two facts, we make the following assumptions:

The contact rate $\lambda \equiv \lambda(I)$ is such that $\lambda(I) > 0$ on $(0,1)$, $\lambda(0) \geq 0, \lambda(I)$ is continuous and the rate of infection $\lambda(I)I(1-I)$ has a continuous derivative on its domain; (4.2.9)

The function $P(t) \geq 0$ is decreasing, differentiable for $t \geq 0$ and satisfies $P(0^+) = 1$; (4.2.10)

The function $I_0(t) \geq 0$ is decreasing, differentiable for $t \geq 0$ and satisfies $\lim\limits_{t\to\infty} I_0(t) = 0.$ (4.2.11)

The fraction $I(t)$ of infective individuals that are in the infective class at time $t > 0$ is given by the SIS-VIE ([79])

$$I(t) = I_0(t) + \int_0^t \lambda\left[I(u)\right] I(u)\left[1 - I(u)\right] P(t-u)e^{-\mu(t-u)}du. \tag{4.2.12}$$

Note that the integral in (4.2.12) sums the individuals that entered the infective class at time $u \geq 0$ and have remained infective through to time t. The expression $\lambda\left(I(u)\right)\left[1 - I(u)\right] I(u)$ represents the individuals that enter the infective class at time $u \geq 0$.

Moreover, it should be noted that the presence of the variable t in the integrand makes the Volterra integral equation (4.2.12) fundamentally different from the classical SIS model (4.2.4). Indeed, if we formally differentiate Eq. (4.2.12), with respect to the time variable t, we obtain

$$\begin{aligned}\frac{dI}{dt}(t) &= \frac{dI_0}{dt}(t) + \lambda\left[I(t)\right] I(t)\left[1 - I(t)\right] \\ &+ \int_0^t \lambda\left[I(u)\right] I(u)\left[1 - I(u)\right]\left(\frac{dP(t-u)}{dt} - \mu P(t-u)\right)e^{-\mu(t-u)}du.\end{aligned} \tag{4.2.13}$$

Thus, the Volterra integral equation (4.2.12) is rather equivalent to the integro-differential equation (4.2.13), which is more complicated than the ODE (4.2.4) (a comprehensive study of integro-differential equations is done in [17, 81]).

Remark 4.2.1. *In accordance with standard definition in statistics [4], the mean time an individual remains infective or the life expectancy of an individual or the expected value of the function $P(u)$ is given by the convergent integral*

$$\tau = \int_0^\infty P(u)e^{-\mu u}du. \tag{4.2.14}$$

Note that the improper integral in (4.2.14) is indeed convergent due to the assumption (4.2.10) made on the bounded function $P(t)$. Furthermore, in the language of ecologists [4], the function $t \rightsquigarrow e^{-\mu t}$ or more generally, the function $t \rightsquigarrow \exp\left(-\int_0^t [\mu(s)]ds\right)$, when the death rate μ is not constant is called the "survival function probability".

Remark 4.2.2. *From the condition (4.2.3), the integral equation (4.2.12) in I leads to the following integral equation in S:*

$$S(t) = S_0(t) - \int_0^t \lambda[1 - S(u)]S(u)[1 - S(u)]P(t-u)e^{-\mu(t-u)}du, \tag{4.2.15}$$

where $S_0(t) = 1 - I_0(t)$.

4.3 Quantitative and qualitative analysis

We start this section with the well-posedness of the Volterra integral equation (4.2.12).

Theorem 4.3.1. *Assume that the initial function $I_0(t)$ in (4.2.11) is such that $0 \leq I_0(t) \leq 1$. Then the SIS-VIE model (4.2.12) admits a unique solution $I : [0,\ \infty) \to \mathbb{R}$, which is a continuous function satisfying the condition $0 \leq I(t) \leq 1$ on $[0,\ \infty)$. If in addition the datum $I_0(t)$ is differentiable on $[0,\ \infty)$, then the solution $I(t)$ is differentiable on the same interval.*

Proof: Unlike [79], we provide a detail and complete proof which involves the following steps: local solution, global solution and positivity and boundedness of solution.

To show the existence of a unique local solution, we introduce the space C_k of real-valued continuous functions from $[0,\ T]$ equipped with the Banach structure defined by the norm

$$\|v\|_{C_k} = \sup_{t\in[0,\ T]} e^{-kt}|v(t)|,$$

where $k > 0$ and T will be determined shortly.

Define G a subset of C_k by

$$\begin{aligned} G &= \{I \in C_k :\ |I(t) - I_0(t)| \leq I_0(t) \quad \forall\, t \in [0,\ T]\} \\ &= \{I \in C_k :\ 0 \leq I(t) \leq 2I_0(t) \quad \forall\, t \in [0,\ T]\}. \end{aligned}$$

Then the set G is nonempty, because $I = 0 \in G$. It is also a closed subset of C_k, because any convergent sequence in G has its limit in G.

On G, we define the operator Φ by

$$(\Phi I)(t) = I_0(t) + \int_0^t \lambda\,[I(u)]\, I(u)\,[1 - I(u)]\, P(t-u) e^{-\mu(t-u)} du.$$

Then, we have for $I \in G$ and $0 \leq t \leq T$,

$$\begin{aligned} |(\Phi I)(t) - I_0(t)| &= \left|\int_0^t \lambda\,[I(u)]\, I(u)\,[1 - I(u)]\, P(t-u) e^{-\mu(t-u)} du\right|, \\ &\leq t \sup_{s\in[0,\ 1]} [1 + |\lambda(s)(1-s)|]\, 2 \sup_{u\in[0,\ t]} I_0(u), \\ &\leq T \sup_{s\in[0,\ 1]} [1 + |\lambda(s)(1-s)|]\, 2 \sup_{u\in[0,\ T]} I_0(u). \end{aligned}$$

We therefore take

$$T = \frac{1}{2\sup_{s\in[0,\ 1]}[1 + |\lambda(s)(1-s)|]},$$

which implies that

$$|(\Phi I)(t) - I_0(t)| \leq \sup_{u\in[0,\,T]} I_0(u)$$

and so Φ operates from G into G.

On the other hand, putting

$$g(I) = \lambda(I)I(1-I),$$

we have, for $I_1, I_2 \in G$,

$$\begin{aligned} |(\Phi I_1)(t) - (\Phi I_2)(t)| &= \left| \int_0^t \left[g(I_1(u)) - g(I_2(u))\right] P(t-u)e^{-\mu(t-u)}du \right| \\ &\leq \int_0^t |g(I_1(u)) - g(I_2(u))|\, du. \end{aligned}$$

Since g is Lipschitz on $[0,2]$ in view of the assumption (4.2.9) with Lipschitz constant denoted by L_G, we have

$$\begin{aligned} |(\Phi I_1)(t) - (\Phi I_2)(t)| &\leq L_G \int_0^t |I_1(u) - I_2(u)|du, \\ &= L_G \int_0^t e^{kt}e^{-kt}|I_1(u) - I_2(u)|du \\ &\leq L_G\|I_1 - I_2\|_{C_k} \int_0^t e^{kt}du \\ &\leq \frac{L_G}{k}\|I_1 - I_2\|_{C_k}e^{kt}. \end{aligned}$$

This gives

$$\|\Phi I_1 - \Phi I_2\|_{C_k} \leq \frac{L_G}{k}\|I_1 - I_2\|_{C_k}.$$

For the choice $k > L_G$, the operator Φ is a contraction. By Banach contraction principle, Φ has a unique fixed $I \in G$, which is precisely the unique local solution of the integral equation

$$I(t) = I_0(t) + \int_0^t \lambda\left[I(u)\right] I(u) \left[1 - I(u)\right] P(t-u)e^{-\mu(t-u)}du, \qquad 0 \leq t \leq T.$$

Notice that the local solution satisfies the boundedness property $0 \leq I(t) \leq 2$, for $t \in [0, T]$.

In order to prove the existence and uniqueness of global solution, we define

$$T_m = mT, \quad m = 0,\;\; 1,\;\; 2,\;\; \ldots\;\; .$$

What we did earlier can be rephrased as follows: There exists a continuous function $I_1 : [0,\ T_1] \to \mathbb{R}$, which is the unique solution of the Volterra integral equation (4.2.12) for $0 \leq t \leq T_1$:

$$I_1(t) = h_0(t) + \int_0^t g(t,u,I_1(u))P(t-u)e^{-\mu(t-u)}du, \qquad h_0(t) = I_0(t), \quad 0 \leq t \leq T_1. \tag{4.3.1}$$

We want to obtain a solution I_2 of the Volterra integral equation (4.2.12) for $0 \leq t \leq T_2$ such that

$$I_2(t) = I_1(t) \quad \text{for} \quad 0 \leq t \leq T_1.$$

We look first at the restriction of $I_2(t)$ to $[T_1, T_2]$, which we denote by $I_2^*(t)$:

$$I_2^* := I_2|_{[T_1,\, T_2]}.$$

By translation, we have for $0 \leq t \leq T_1$:

$$\begin{aligned} I_2^*(t+T_1) &= h_0(t+T_1) + \int_0^{t+T_1} g(I_2(u))P(t+T_1-u)e^{-\mu(t+T_1-u)}du \\ I_2^*(t+T_1) &= h_0(t+T_1) + \int_0^{T_1} g(I_1(u))P(t+T_1-u)e^{-\mu(t+T_1-u)}du \\ &+ \int_{T_1}^{t+T_1} g(I_2^*(u))P(t+T_1-u)e^{-\mu(t+T_1-u)}du. \end{aligned} \tag{4.3.2}$$

Thus, for $T_1 \leq t \leq T_2$, we have

$$I_2^*(t) = h_1(t) + \int_{T_1}^t g(I_2^*(u))P(t-u)e^{-\mu(t-u)}du, \tag{4.3.3}$$

where

$$h_1(t) = h_0(t) + \int_0^{T_1} g(I_1(u))P(t-u)e^{-\mu(t-u)}du. \tag{4.3.4}$$

By following the same procedure we used on $[0, T_1]$, we can define a contraction operator $\Phi : C_k([T_1,\ T_2];\ \mathbb{R}) \to C_k([T_1,\ T_2];\ \mathbb{R})$ whose fixed point $I_2^* \in C_k([T_1,\ T_2];\ \mathbb{R})$ is the unique solution of the integral equation (4.3.3). The required solution of the Volterra integral equation (4.2.12) for $0 \leq t \leq T_2$ is then defined by

$$I_2 = I_1 \cup I_2^* \quad \text{i.e.,} \quad I_2(t) = \begin{cases} I_1(t), & \text{if} \quad 0 \leq t \leq T_1 \\ I_2^*(t), & \text{if} \quad T_1 \leq t \leq T_2. \end{cases}$$

Assume by induction that two sequences $(h_0,\ h_1,\ ...,\ h_{m-1})$ and $(I_1,\ I_2,\ ...,\ I_m)$ of continuous functions have been constructed such that for $T_{i-1} \leq t \leq T_i$, $i = 2,\ 3,\ ...,m$, the following holds:

$$h_{i-1}(t) = h_{i-2}(t) + \int_0^{T_{i-1}} g(I_{i-1}(u)P(t-u)e^{-\mu(t-u)}du,$$

and

$$I_i(t) = \begin{cases} I_{i-1}(t), & \text{if} \quad 0 \leq t \leq T_{i-1} \\ I_i^*(t), & \text{if} \quad T_{i-1} \leq t \leq T_i, \end{cases}$$

where, I_i^* is the unique solution of the integral equation

$$I_i^*(t) = h_{i-1}(t) + \int_{T_{i-1}}^t g(I_i^*(u))P(t-u)e^{-\mu(t-u)}du,$$

and I_i is the unique solution of the Volterra integral equation (4.2.12) for $0 \leq t \leq T_i$.

To construct the functions h_m, I_{m+1}^* and I_{m+1}, we follow the steps we used to construct I_2 from I_1. More precisely, we want to obtain the solution I_{m+1} of the Volterra integral equation (4.2.12) for $0 \leq t \leq T_{m+1}$, such that

$$I_{m+1}(t) = I_m(t), \quad \text{for} \quad 0 \leq t \leq T_m.$$

The restriction of I_{m+1} to $[T_m, T_{m+1}]$ is denoted by

$$I_{m+1}^* := I_{m+1}|_{[T_m,\, T_{m+1}]}.$$

By translation, we have for $0 \leq t \leq T_1$,

$$\begin{aligned} I_{m+1}^*(t+T_m) &= h_0(t+T_m) + \int_0^{t+T_m} g(I_{m+1}(u))P(t+T_m-u)e^{-\mu(t+T_m-u)}du \\ I_{m+1}^*(t+T_m) &= h_0(t+T_m) + \int_0^{T_m} g(I_m(u))P(t+T_m-u)e^{-\mu(t+T_m-u)}du \\ &+ \int_{T_m}^{t+T_m} g(I_{m+1}^*(u))P(t+T_1-u)e^{-\mu(t+T_1-u)}du. \end{aligned} \tag{4.3.5}$$

Thus, for $t \in [T_m, T_{m+1}]$, we have

$$I_{m+1}^*(t) = h_m(t) + \int_{T_m}^t g(I_{m+1}^*(u))P(t-u)e^{-\mu(t-u)}du, \tag{4.3.6}$$

where

$$h_m(t) = h_{m-1}(t) + \int_0^{T_m} g(I_m(u))P(t-u)e^{-\mu(t-u)}du. \tag{4.3.7}$$

In a similar way, we can define a contraction operator

$$\Phi : C_k([T_m,\ T_{m+1}];\ \mathbb{R}) \to C_k([T_m,\ T_{m+1}];\ \mathbb{R})$$

whose fixed point $I^*_{m+1} \in C_k([T_m,\ T_{m+1}];\ \mathbb{R})$ is the unique solution of the integral equation (4.3.6). The required solution I_{m+1} of the Volterra integral equation (4.2.12) for $0 \le t \le T_{m+1}$ is then given by

$$I_{m+1} = I_m \cup I^*_{m+1} \quad \text{i.e.,} \quad I_{m+1}(t) = \begin{cases} I_m(t), & \text{if} \quad 0 \le t \le T_m \\ I^*_{m+1}(t), & \text{if} \quad T_m \le t \le T_{m+1}, \end{cases}$$

Hence, we have by induction constructed a sequence $(I_m)_{m\geq 1}$ of continuous functions $I_m : [0,\ T_m] \to \mathbb{R}$ that are the unique solutions of the Volterra integral equation (4.2.12) and that satisfy the compatibility condition

$$I_{m+1}|_{[0,\ T_m]} = I_m.$$

Since

$$\bigcup_{m\geq 0} [T_m,\ T_{m+1}] = [0,\ \infty),$$

the function

$$I(t) := \bigcup_{m\geq 1} I_m(t) : [0,\ \infty) \to \mathbb{R}$$

is the unique solution of (4.2.12). Hence, the integral equation (4.2.12) admits a global solution $I(t)$ which is continuous and satisfies the boundedness property

$$0 \le I(t) \le 2 \qquad \forall\, t \in [0,\ \infty).$$

The sharpest upper bound of the solution $I(t)$ is obtained by duality. Indeed, it follows using the previous reasoning that, there exists a unique continuous solution $S : [0,\ \infty) \to [0,\ \ 2]$ of the integral equation (4.2.15). In view of (4.2.3), we necessarily have $0 \le I(t),\ S(t) \le 1$.

When I_0 is differentiable, then the solution I is differentiable with its derivative being given by (4.2.13), because the integral equation is a continuous function in u and the $t \rightsquigarrow P(t-u)e^{-\mu(t-u)}$ is differentiable. $\square$

Remark 4.3.2. *By the assumption (4.2.11), we have $I_0(t) = 0$ for all $t \ge 0$ if $I_0(0) = 0$. In this case, the unique solution of the integral equation (4.2.12) is the null function $I \equiv 0$. This solution is referred to here and after as the disease-free equilibrium. A nontrivial solution will then be obtained whenever $I_0(0) > 0$.*

4.3.1 Equilibrium solutions

In order to do the qualitative analysis of the integro-differential equation (4.2.13), it is essential, by analogy with the classical SIS model (4.2.1)-(4.2.2) or (4.2.7), to rewrite (4.2.12) in such a way that the parameter that characterizes the dynamics of this system is incorporated. The said characteristic parameter is the basic reproduction number of the model. Using the formal definition of the basic reproduction number, we have

$$\mathcal{R}_0 = \lambda(0)\tau, \tag{4.3.8}$$

where τ given in (4.2.14) is the period of infectivity for a single infective individual, while $\lambda(0)$ assumed to be greater than 0, is the adequate contact per unit time made by a single infective.

For convenience, we also introduce the following notation in order to simplify the integrand in (4.2.12) through an adjusted force of infection $f(I)$ and Kernel $\tilde{P}(t-u)$:

$$\begin{aligned} f(I) &= \frac{1}{\lambda(0)}\lambda(I)(1-I), \qquad 0 \leq I \leq 1; && (4.3.9)\\ \tilde{P}(t) &= \frac{1}{\tau}P(t)e^{-\mu t}, \qquad t \geq 0. && (4.3.10) \end{aligned}$$

The Volterra integral equation (4.2.12) can then be written in the form

$$I(t) = I_0(t) + \mathcal{R}_0 \int_0^t I(u) f(I(u)) \tilde{P}(t-u)du. \tag{4.3.11}$$

Equation (4.2.12) or (4.3.11) falls in the general category of the Volterra integral equation of the form

$$x(t) = g(t) + \int_0^t a(t-u)F(x(u))du, \quad t \geq 0, \tag{4.3.12}$$

where the function g is of class $C[0,\infty)$, the convolution kernel a is such that the function $u \rightsquigarrow a(t-u)$, is of class $L^1(0,t)$ for $t > 0$ and $F \in C^1(\mathbb{R})$.

Let us now recall some tools and definitions about the qualitative properties of (4.3.12). We follow [17].

Definition 4.3.3. *A number $x^* \in \mathbb{R}$ is an equilibrium solution of the Volterra integral equation (4.3.12), if the constant function $x(t) = x^*$ for $t \geq 0$ is its solution.*

Definition 4.3.4. *Let x^* be an equilibrium solution of Eq. (4.3.12).*

1. *The equilibrium x^* is (Lyapunov) stable if, for each $\epsilon > 0$ and $t_0 \geq 0$, there exists $\delta \equiv \delta(\epsilon,\ t_0) > 0$ such that for any g satisfying*

$$|g(t) - x^*| < \delta \quad \textit{for} \quad t \in [0, t_0], \quad \textit{we have} \quad |x(t, g) - x^*| < \epsilon$$

for all $t \geq t_0$, where the function $x(t) \equiv x(t, g)$ is solution of the Volterra integral equation (4.3.12).

2. *The equilibrium x^* is uniformly stable if, for each $\epsilon > 0$, there exists $\delta \equiv \delta(\epsilon) > 0$ such that for $t_0 \geq 0$ and g satisfying*

$$|g(t) - x^*| < \delta \quad \textit{for} \quad t \in [0, t_0], \quad \textit{we have} \quad |x(t, g) - x^*| < \epsilon$$

for all $t \geq t_0$.

3. *The equilibrium x^* is locally asymptotically stable if it is stable and if for each $t_0 \geq 0$, there exists $\eta \equiv \eta(t_0) > 0$ such that for g satisfying*

$$|g(t) - x^*| < \eta \quad \textit{for} \quad t \in [0, t_0], \quad \textit{we have} \quad \lim_{t \to \infty} x(t, g) = x^*.$$

If the limit holds for any initial function g, x^* is said to be globally asymptotically stable.

4. *The equilibrium x^* is uniformly asymptotically stable, if it is uniformly stable and if there exists a number $\eta > 0$ (independent of t_0) such that, for $t_0 \geq 0$, g satisfying*

$$|g(t) - x^*| < \eta \quad \textit{for} \quad t \in [0, t_0], \quad \textit{we have} \quad \lim_{t \to \infty} x(t, g) = x^*.$$

Our interest is in comparing an equilibrium solution x^* with any other solution $x(t)$ of (4.3.12) when $t \to \infty$. We proceed by linearization process of our Volterra integral equation [25, 65].

Considering the linear approximation of the function F about x^*, i.e.,

$$F(x(u)) \simeq F(x^*) + \frac{dF}{dx}(x^*)(x(u) - x^*),$$

the integral equation (4.3.12) is approximated by the linear equation

$$y(t) = \int_0^t a(t - u)Jy(u)du, \tag{4.3.13}$$

where $J = \frac{dF}{dx}(x^*)$ and $y = x - x^*$.

The connection between the behaviors as $t \to \infty$ of the solutions of (4.3.12) and (4.3.13) is given in the next result [65].

Lemma 4.3.5. *In addition to assumptions made above about a and F, we assume that*

$$J \neq 0, \tag{4.3.14}$$

$$1 - \int_0^\infty e^{-su}a(u)du \neq 0 \quad \textit{for any} \quad s \in \mathcal{C} \quad \textit{with} \quad Res \geq 0. \tag{4.3.15}$$

1. *If any solution $y(t)$ of Eq. (4.3.13) converges to zero as $t \to \infty$, then each solution $x(t)$ of (4.3.12) converges to x^* as $t \to \infty$.*

2. *If the initial function $g(t) \to 0$ as $t \to \infty$, then $y(t) \to 0$ as $t \to \infty$.*

Remark 4.3.6. *The condition (4.3.15) is equivalent to the fact that the resolvent kernel of the linear equation (4.3.13) is of class $L^1(0, \infty)$. In view of this, part 1 of Lemma 4.3.5 results from the Lebesgue Dominated Convergence Theorem. Part 1 of the lemma is the analogue of Hartman-Grobman Theorem.*

Another result that describes the behavior at infinity of the solutions of (4.3.12) reads as follows [12]:

Lemma 4.3.7. *Apart from the conditions stated in Lemma 4.3.5 about g, a and F in Eq. (4.3.12), we assume that $\lim_{t\to\infty} g(t)$ exists, a is non-negative and non-increasing on $[0, \infty)$ and there is no interval on which $F'(x)\int_0^\infty a(u)du = 1$. Then for every bounded solution $x(t)$ of Eq. (4.3.12),*

$$\lim_{t\to\infty} x(t) = x^*, \quad \textit{where } \; x^* \; \textit{ satisfies the relation } \quad x^* = \lim_{t\to\infty} g(t) + F(x^*)\int_0^\infty a(u)du.$$

Remark 4.3.8. *Lemma 4.3.5 and Lemma 4.3.7 are in fact stability results.*

Let us now come back to the SIS-VIE model (4.3.11).

Proposition 4.3.9. *A real number $I^* \in [0, 1]$ is an equilibrium solution of the Volterra integral equation (4.2.12) or (4.3.11) if and only if*

$$I^* = \mathcal{R}_0 I^* f(I^*). \tag{4.3.16}$$

Proof: Assume first that $I(t) = I^*$ is a constant solution to Eq. (4.3.11) with the corresponding initial function denoted by $I_0(t) = I_0(t; I^*)$. Then we have

$$I^* = I_0(t; I^*) + \mathcal{R}_0 I^* f(I^*) \int_0^t \tilde{P}(t-u)du. \tag{4.3.17}$$

If we consider the limit of both sides of (4.3.17) as t goes to infinity, we obtain the expression

$$I^* = \mathcal{R}_0 I^* f(I^*), \qquad 0 \leq I^* \leq 1,$$

in view of the assumption (4.2.11) and of the formula

$$\int_0^\infty \tilde{P}(u)du = 1 \tag{4.3.18}$$

due to (4.2.14) and (4.3.10).

Conversely, assume that I^* is a constant solution of (4.3.16). Consider the function $I_0(t; I^*)$ defined by

$$I_0(t; I^*) = I^* - \mathcal{R}_0 I^* f(I^*) \int_0^t \tilde{P}(t-u)du.$$

Then this function meets the requirement (4.2.11) due to (4.3.16) and (4.3.18). Furthermore, we have

$$I^* = I_0(t; I^*) + \mathcal{R}_0 I^* f(I^*) \int_0^t \tilde{P}(t-u)du,$$

which shows that I^* is a solution of (4.3.11). $\square$

Remark 4.3.10. *Note that $I^* = 0$ always solves (4.3.16) and is, as mentioned earlier in Remark 4.3.2, referred to as the disease-free equilibrium. Any $I^* \in (0,1)$ that solves (4.3.16) is called an endemic equilibrium; we denote an endemic equilibrium by I_e. In this case, (4.3.16) leads to*

$$\mathcal{R}_0 f(I_e) = 1, \tag{4.3.19}$$

which is equivalent to saying that in the I-y axes, the graph of the function $y = f(I)$ intersects with the horizontal line $y = \frac{1}{\mathcal{R}_0}$.

The last comment in Remark 4.3.10 regarding the role of the horizontal line $y = \frac{1}{\mathcal{R}_0}$ in the existence of endemic equilibria leads us to do some analysis on the extremum values of the function $y = f(I)$. Being continuous on the compact interval $[0, 1]$, the function $I \rightsquigarrow f(I)$ admits a (global) maximum M and a minimum m values. Notice that from the explicit expression of f in (4.3.9), we have

$$0 = m = f(1).$$

However, M is in general not explicitly known. Taking into account the fact that $f(1) = 0$, the global minimum, we exclude the right end-point 1 to obtain the set

$$A = \{0\} \cup \{\, c \in (0,\ 1) \ :\ f'(c) = 0\}, \tag{4.3.20}$$

for which we have

$$\max_{I \in [0,\ 1]} f(I) = \max_{I \in A} f(I) =: \frac{1}{\mathcal{R}_0^c}. \tag{4.3.21}$$

In addition to (4.3.21), we consider the local minimum $\min_{I \in A} f(I)$, which is not the global minimum $\min_{I \in A \cup \{1\}} f(I) = 0$, since $1 \notin A$. We put

$$\min_{I \in A} f(I) =: \frac{1}{\mathcal{R}_0^m}, \tag{4.3.22}$$

which is well defined since $f(0) = 1$. Notice that the definition of $\mathcal{R}_0^m$ and $\mathcal{R}_0^c$ imply that

$$\mathcal{R}_0^m \geq 1 \geq \mathcal{R}_0^c. \tag{4.3.23}$$

Furthermore, we have the relation

$$0 \leq \mathcal{R}_0^c \leq \mathcal{R}_0^m. \tag{4.3.24}$$

The material accumulated so far regarding the extremum values of the function f enables us to specify the possible intersections between $y = f(I)$ and $y = \frac{1}{\mathcal{R}_0}$ in terms of the following result:

Theorem 4.3.11. *For the disease transmission model (4.3.11), with assumptions (4.2.9)-(4.2.11), we have the following facts:*

1. *The constant $I^* = 0$ is always an equilibrium (disease-free equilibrium);*

2. *There is no endemic equilibrium I_e if $\mathcal{R}_0 < \mathcal{R}_0^c$;*

3. *There exists at least one endemic equilibrium I_e if $\mathcal{R}_0 > \mathcal{R}_0^c$;*

4. *There exists exactly one endemic equilibrium* I_e *if* $\mathcal{R}_0 > \mathcal{R}_0^m$.

Proof:

1. On setting $I_0(t) = 0$, it is clear that the constant function $I(t) = I^* = 0$ is the unique solution of the Volterra integral equation (4.3.11).

2. Assume that $\mathcal{R}_0 < \mathcal{R}_0^c$. Then by Eq. (4.3.21), we have $f(I) \leq \max_{c\in A} f(c) < \frac{1}{\mathcal{R}_0}$ for all $I \in [0, 1]$. Thus, there is no intersection between the graphs of $y = f(I)$ and $y = \frac{1}{\mathcal{R}_0}$ (see Fig 4.1 a).

3. If $\mathcal{R}_0 > \mathcal{R}_0^c$, then $\max_{I\in A} f(I) > \frac{1}{\mathcal{R}_0}$ by Eq. (4.3.21). Since $f(1) = 0 < \frac{1}{\mathcal{R}_0} < \max_{I\in A} f(I)$ and $f(I)$ is continuous on $[0, 1]$, the intermediate value theorem guarantees the existence of at least one $I_e \in (0,\ 1)$ such that $f(I_e) = \frac{1}{\mathcal{R}_0}$ (see Fig 4.1b).

4. Assume that $\mathcal{R}_0 > \mathcal{R}_0^m$, which implies that $f(0) = 1$. Since $\mathcal{R}_0^m \geq \mathcal{R}_0^c$, we infer from part (3) above that there exists at least one endemic equilibrium $I_e \in (0,\ 1)$.

   We claim that the endemic equilibrium is unique. To this end, we assume by contradiction that there exist two endemic equilibria $I_{e_1},\ I_{e_2} \in (0,\ 1)$ such that $I_{e_1} < I_{e_2}$. Since $f(I_{e_1}) = f(I_{e_2}) = \frac{1}{\mathcal{R}_0}$ and the continuous function $f : [I_{e_1},\ I_{e_2}] \to \mathbb{R}$ is differentiable on $(I_{e_1},\ I_{e_2})$, it follows from Rolle's theorem that there exists $c \in (I_{e_1},\ I_{e_2})$ such that $f'(c) = 0$, which means that $c \in A$. Then we have two cases. The first case is when $f(c) < \frac{1}{\mathcal{R}_0}$. i.e $f(c)$ is a local minimum. This is impossible in view of the fact that the smallest local minimum satisfies $\min_{I\in A} f(I) > \frac{1}{\mathcal{R}_0}$ (see Fig 4.2a). We are then left with the second case, which read as $f(c) > \frac{1}{\mathcal{R}_0}$. i.e $f(c)$ is a local maximum (see Fig 4.2b). Thus $f(c) \geq \min_{I\in A} f(I) > \frac{1}{\mathcal{R}_0} > 0$. Under these circumstances, there exists a point $I_{e_3} \in (0,\ 1) \cap A$ with $I_{e_3} \neq c$ and $I_{e_3} \neq r$, where $f(r) = \min_{I\in A} f(I),\quad r \in (0,\ 1)$, such that $f(I_{e_3}) < \frac{1}{\mathcal{R}_0}$ is a local minimum. This contradicts the fact that $f(r)$ is the smallest local minimum. Therefore, there exists a unique endemic equilibrium in this case. $\square$

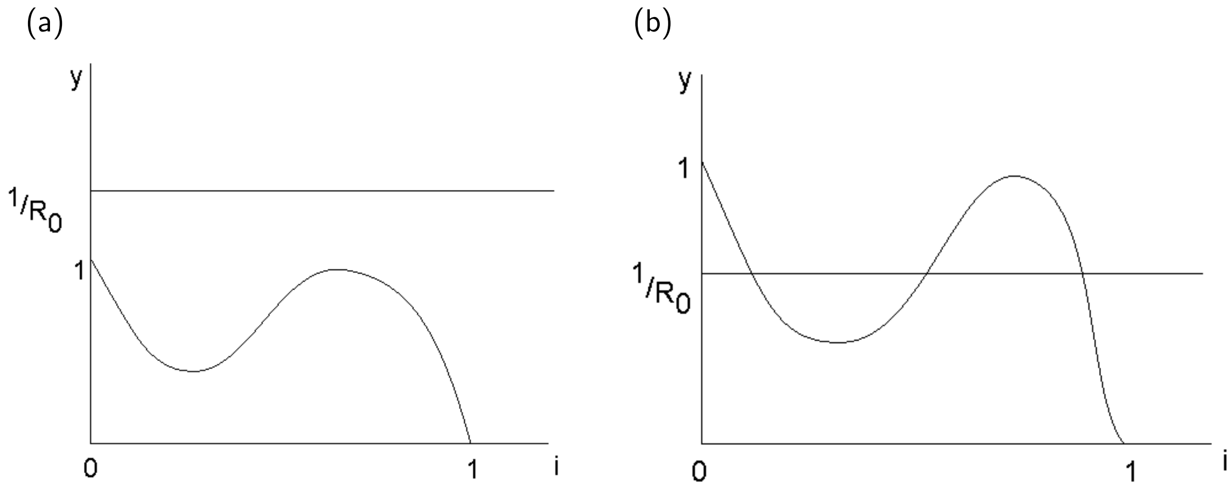

Figure 4.1: Non–existence (a) and existence (b) of endemic equilibrium

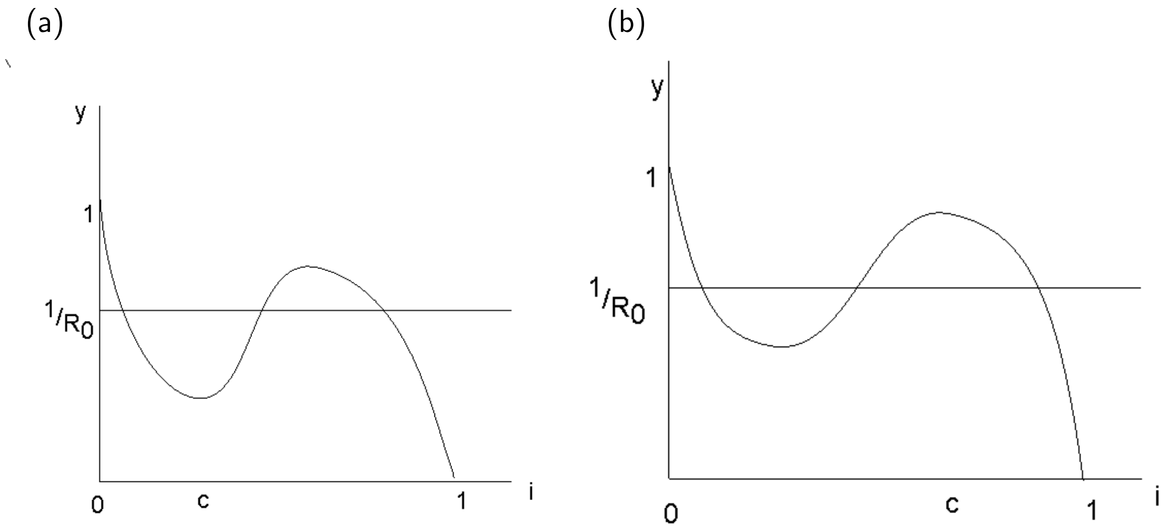

Figure 4.2: Existence of unique endemic equilibrium

Remark 4.3.12. *A few comments are in order regarding Theorem 4.3.11 as compared to the classical SIS model for which we have* $\mathcal{R}_0^m = \mathcal{R}_0^c$*. When* $\mathcal{R}_0 > \mathcal{R}_0^m$*, the fact that Theorem 4.3.11 guarantees the existence of a unique endemic equilibrium* I_e *agrees with the situation of the classical SIS model and other classical epidemiological models. By analogy with this classical case, we expect the disease-free equilibrium to be unstable, while the unique endemic equilibrium* I_e *is globally asymptotically stable. Similarly for* $\mathcal{R}_0 < \mathcal{R}_0^c \leq 1$*, the fact that the disease-free equilibrium is the only equilibrium is in line with the classical* *case and so we expect this equilibrium to be globally asymptotically stable in this case. The situation $\mathcal{R}_0^c < \mathcal{R}_0 < 1$ which, according to Theorem 4.3.11 could lead to multiple endemic equilibria, is a major difference between the classical SIS model and the one considered here. This is the case when the backward bifurcation phenomena can occur for such a simple model. These expectations on the stability and bifurcation of the equilibria constitute the focus of the next subsection.*

4.3.2 Stability and bifurcation analysis

Following the process outlined above, the linearized equation of (4.3.11) about an equilibrium I_e is

$$y(t) = R_{I_e} \int_0^t y(u)\tilde{P}(t-u)du, \tag{4.3.25}$$

where

$$R_{I_e} = \frac{d(\mathcal{R}_0 I f(I))}{dI}|_{I=I_e} = R_0 \left(f(I_e) + I_e f'(I_e)\right), \tag{4.3.26}$$

while the analogue of (4.3.15) is equivalent to the characteristic equation

$$1 = R_{I_e} \int_0^\infty e^{-zu}\tilde{P}(u)du, \tag{4.3.27}$$

for $z \in \mathcal{C}$ such that $Rez < 0$. In view of Lemma 4.3.5, the equilibrium I_e will be locally asymptotically stable or unstable according to the condition (4.3.27) holds true or not.

The next two lemmas are precisely devoted to investigate the properties of the roots of Eq. (4.3.27).

Lemma 4.3.13. *Assume that the function $a(u) \geq 0$ is non-increasing and $\int_0^\infty a(u)du < \infty$. Then*

$$\int_0^\infty a(u)\sin(yu)du = 0 \tag{4.3.28}$$

if and only if either $y = 0$ or $a(u)$ is constant on every subinterval $\frac{2\pi n}{y} < u < \frac{2\pi(n+1)}{y}$ where n is non-negative integer.

Proof: We elaborate on the proof in [79]. For $y = 0$, the proof is trivial. Therefore, we assume that $y \neq 0$ and we want to show that Eq. (4.3.28) is true if and only if $a(u)$ is constant on each subinterval $\frac{2\pi n}{y} < u < \frac{2\pi(n+1)}{y}$.

It is reasonable to consider only $y > 0$, because sine is an odd function: $\sin(-yu) = -\sin(yu)$ and the proof for $y < 0$ will follow in similar way. If we consider the subinterval $I_n = [\frac{2\pi n}{y}, \frac{2\pi(n+1)}{y}]$, then the integral given in Eq. (4.3.28) is the sum of the infinite sequence

$$a_n = \int_{\frac{2\pi n}{y}}^{\frac{2\pi(n+1)}{y}} a(u)\sin(yu)du,$$

which is convergent, since $a(u)$ is non-negative and integrable. Thus

$$\begin{aligned} a_n &= \int_{\frac{2\pi n}{y}}^{\frac{2\pi n}{y}+\frac{\pi}{y}} a(u)\sin(yu)du + \int_{\frac{2\pi n}{y}+\frac{\pi}{y}}^{\frac{2\pi(n+1)}{y}} a(u)\sin(yu)du, \\ &= \int_{\frac{2\pi n}{y}}^{\frac{2\pi n}{y}+\frac{\pi}{y}} a(u)\sin(yu)du + \int_{\frac{2\pi n}{y}}^{\frac{2\pi n}{y}+\frac{\pi}{y}} a(u+\frac{\pi}{y})\sin(y(u+\frac{\pi}{y}))du, \\ &= \int_{\frac{2\pi n}{y}}^{\frac{2\pi n}{y}+\frac{\pi}{y}} a(u)\sin(yu)du - \int_{\frac{2\pi n}{y}}^{\frac{2\pi n}{y}+\frac{\pi}{y}} a(u+\frac{\pi}{y})\sin(yu)du, \\ &\quad \text{by the property of sine function} \\ &= \int_{\frac{2\pi n}{y}}^{\frac{2\pi}{y}(n+\frac{1}{2})} (a(u)-a(u+\frac{\pi}{y}))\sin(yu)du \geq 0. \end{aligned} \tag{4.3.29}$$

Hence, Eq. (4.3.28) holds for each n, if and only if $a_n = 0$. From Eq. (4.3.29), $a_n = 0$ if and only if $a(u+\frac{\pi}{y}) = a(u)$ in the interval $\frac{2\pi n}{y} < u < \frac{2\pi}{y}(n+\frac{1}{2})$. Since $a(u)$ is non-increasing and non-negative, we infer that $a(u)$ is constant on the larger subinterval $\frac{2\pi n}{y} < u < \frac{2\pi(n+1)}{y}$ for each non-negative integer n. □

Lemma 4.3.14. *For Eq. (4.3.27), any non-real root has negative real part and there is at most one real root. Moreover, if it exists, this real root is unique, simple and positive if $R_{I_e} > 1$, negative if $R_{I_e} < 1$ and zero if $R_{I_e} = 1$.*

Proof: Once again, we provide details on the proof in [79]. For $x,\ y \in \mathbb{R}$, let $z = x+iy$ be a root of Eq. (4.3.27). Then by using Euler's formula, x and y satisfy

$$1 = R_{I_e}\int_0^\infty e^{-xu}[\cos(yu) - i\sin(yu)]\tilde{P}(u)du.$$

By comparison of the corresponding coefficients, we obtain two equations:

$$0 = R_{I_e}\int_0^\infty e^{-xu}\tilde{P}(u)\sin(yu)du, \tag{4.3.30}$$

$$1 = R_{I_e}\int_0^\infty e^{-xu}\tilde{P}(u)\cos(yu)du. \tag{4.3.31}$$

First, consider a root with non-negative real part, $(i.e., \ x \geq 0)$. If $R_{I_e} = 0$, then Eq. (4.3.31) gives $1 = 0$, which is a contradiction. Hence, $R_{I_e} \neq 0$. By applying Lemma 4.3.13 on Eq. (4.3.30), we get $y = 0$ or $e^{-xu}\tilde{P}(u)$ is constant on every subinterval $\frac{2\pi n}{y} < u < \frac{2\pi(n+1)}{y}$. In the latter case, Eq. (4.3.31) has no root, because the improper integral is zero. This leads to a contradiction with the left side. Hence, we get $x \geq 0$ and $y = 0$.

Now let us consider a real root x $(y = 0)$ in Eq. (4.3.27) and we get

$$\begin{aligned} 1 &= R_{I_e}\int_0^\infty e^{-xu}\tilde{P}(u)du, \qquad (4.3.32)\\ &= R_{I_e}\frac{\int_0^\infty e^{-(x+\mu)u}P(u)du}{\int_0^\infty e^{-\mu u}P(u)du}, \quad \text{using} \quad Eq.\,(4.2.14) \ \text{and} \ (4.3.10) \end{aligned}$$

For $\mu > 0$, if $x + \mu \geq 0$, then by assumption (4.2.10), the integral Eq. (4.3.32) is positive, bounded and decreasing in x. Hence, the root of Eq. (4.3.32), if it exists, is unique and simple. Since $P(u)$ is positive,

- there is no real root if $R_{I_e} \leq 0$,
- $x > 0$ if $R_{I_e} > 1$,
- $x < 0$ if $0 < R_{I_e} < 1$, and
- $x = 0$ if $R_{I_e} = 1$. This completes the proof. □

As a consequence of Lemma 4.3.14, we have the following stability result.

Theorem 4.3.15. *Suppose that assumptions (4.2.9)-(4.2.11) are satisfied for the model of Eq. (4.3.11). Then*

1. *the disease-free equilibrium $I_e = 0$ is locally asymptotically stable when $\mathcal{R}_0 < 1$ and unstable when $\mathcal{R}_0 > 1$.*
2. *an endemic equilibrium $I_e > 0$ is locally asymptotically stable if $f'(I_e) < 0$ or $\lambda'(I_e) < \frac{\lambda(I_e)}{1-I_e}$ and unstable if $f'(I_e) > 0$ or $\lambda'(I_e) > \frac{\lambda(I_e)}{1-I_e}$.*

Proof:

1. If $I_e = 0$, we have from (4.3.26), $\mathcal{R}_{I_e} = \mathcal{R}_0$. Thus, Lemma 4.3.14 implies the local stability of the disease-free equilibrium when $\mathcal{R}_0 < 1$ and its instability if $\mathcal{R}_0 > 1$.

2. Note that

$$f'(I_e) = \frac{\lambda'(I_e)(1 - I_e) - \lambda(I_e)}{\lambda(0)}, \tag{4.3.33}$$

in Eq. (4.3.26) for an endemic equilibrium $I_e > 0$. Note also from (4.3.19) that

$$R_{I_e} = 1 + \mathcal{R}_0 I_e f'(I_e).$$

Thus,

$$(R_{I_e} - 1) = \mathcal{R}_0 I_e f'(I_e).$$

Since $\mathcal{R}_0$ and I_e are positive,

$$sign(R_{I_e} - 1) = sign(f'(I_e)).$$

By using Lemma 4.3.14, I_e is locally asymptotically stable if $0 < R_{I_e} < 1$ or $f'(I_e) < 0$ or $\lambda'(I_e) < \frac{\lambda(I_e)}{1-I_e}$. It is unstable if $R_{I_e} > 1$ or $f'(I_e) > 0$ or $\lambda'(I_e) > \frac{\lambda(I_e)}{1-I_e}$. □

Remark 4.3.16. *Theorem 4.3.15 enables us to supplement the existence of endemic equilibria investigated in the proof of Theorem 4.3.11, via the intersection of the functions $y = f(I)$ and $y = \frac{1}{\mathcal{R}_0}$ in the I-y axis by the stability properties of these equilibria. More precisely, decreasing branch of $f(I)$ corresponds to locally asymptotically stable endemic equilibria, while an increase branch corresponds to unstable endemic equilibria (see Fig 4.2). Furthermore, Theorem 4.3.15 combined with Theorem 4.3.11 imply the following important facts:*

1. *The value 1 of the parameter $\mathcal{R}_0$ is a bifurcation point.*

2. *If $\mathcal{R}_0^c = 1$, then $\mathcal{R}_0 = 1$ is a forward bifurcation as for the classical SIS model.*

3. *If $\mathcal{R}_0^c < 1$, there exists a backward bifurcation whenever $\lambda'(0) \neq \lambda(0)$ or $f'(0) \neq 0$. In this case, the initial direction of the bifurcation at $\mathcal{R}_0 = 1$ is determined by the sign of $f'(0)$ as follows:*

   a. *If $f'(0) > 0$, then a branch of unstable and stable endemic equilibria is born from $(\mathcal{R}_0^{-1}, I) = (1, 0)$ in the interval $\mathcal{R}_0^c < \mathcal{R}_0 < 1$ with the branch of unstable equilibria having slope $f'(0) > 0$ at $(\mathcal{R}_0^{-1}, I) = (1, 0)$, while the branch of stable equilibria $(\mathcal{R}_0^{-1}, I_e)$ has slope $f'(I_e) < 0$.*

b. If $f'(0) < 0$, then a branch of stable, unstable and stable endemic equilibria is born from $(\mathcal{R}_0^{-1}, I) = (1, 0)$ with the following locations:

- *The branch of the first stable endemic equilibria is for $1 < \mathcal{R}_0 < \mathcal{R}_0^m$ and it has slope $f'(0) < 0$ at the point $(1, 0)$.*
- *The branch of unstable endemic equilibria is for $\mathcal{R}_0^c < \mathcal{R}_0 < \mathcal{R}_0^m$ and it has slope $f'(I_e) > 0$ at $(\mathcal{R}_0^{-1}, I_e)$.*
- *The branch of the second stable endemic equilibria is for $\mathcal{R}_0^c < \mathcal{R}_0 < 1$ and has slope $f'(I_e) < 0$ at $(\mathcal{R}_0^{-1}, I_e)$.*

Some of the locally asymptotically stable equilibrium points considered in Theorem 4.3.15 and Theorem 4.3.11 are globally asymptotically stable as specified in the next theorem.

Theorem 4.3.17. *For the SIS-VIE model (4.3.11),*

1. *the disease-free equilibrium is globally asymptotically stable for $\mathcal{R}_0 < \mathcal{R}_0^c$,*
2. *the unique endemic equilibrium, I_e, guaranteed by Theorem 4.3.11, is globally asymptotically stable for $\mathcal{R}_0 > \mathcal{R}_0^m$.*

Proof: Notice from Theorem 4.3.1 that all solutions of Eq. (4.3.11) are bounded. By using Lemma 4.3.7, $I^* := \lim_{t\to\infty} I(t)$ exists and it satisfies Eq. (4.3.16), i.e.

$$\lim_{t\to\infty} I(t) = I^*. \tag{4.3.34}$$

1. If $\mathcal{R}_0 < \mathcal{R}_0^c$, then by Theorem 4.3.11, the disease-free equilibrium is the only point which satisfies Eq. (4.3.16) and it satisfies (4.3.34). Hence, the disease free equilibrium is globally asymptotically stable.
2. If $\mathcal{R}_0 > \mathcal{R}_0^m$, then there are only two equilibrium points $I^* = 0$ and $I^* = I_e \in (0, 1)$. Since the disease-free equilibrium is unstable for $\mathcal{R}_0 > \mathcal{R}_0^m > 1$ (see Theorem 4.3.15), then only the unique endemic equilibrium I_e satisfies the relation (4.3.34). Thus I_e is globally asymptotically stable. □

4.4 Nonstandard finite difference scheme

In this section, we want to design a NSFD scheme which is dynamically consistent with respect to the properties of the continuous SIS-VIE model Eq. (4.2.12) investigated in the previous sections. The results of this section are published in the paper [57]. We elaborate on their presentation and provide more detailed proofs.

4.4.1 Construction of the scheme

Let $I(t)$ be the solution of SIS-VIE (4.2.12). At the discrete time $t_{k+1} = (k+1)\Delta t$, where Δt is the step size, we obtain

$$\begin{aligned} I(t_{k+1}) &= I_0(t_{k+1}) + \int_0^{t_{k+1}} \lambda[I(u)]I(u)[1-I(u)]P(t-u)e^{-\mu(t_{k+1}-u)}du \\ &= I_0(t_{k+1}) + \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} \lambda[I(u)]I(u)[1-I(u)]P(t_{k+1}-u)e^{-\mu(t_{k+1}-u)}du. \end{aligned}$$

By applying the mean-value theorem for integrals to each sub-interval, there exists $c_i \in [t_i, t_{i+1}]$ such that

$$I(t_{k+1}) = I_0(t_{k+1}) + \sum_{i=0}^{k} \lambda[I(c_i)]I(c_i)[1-I(c_i)]P(t_{k+1}-c_i) \int_{t_i}^{t_{i+1}} e^{-\mu(t_{k+1}-u)}du.$$

In order to approximate the nonlinear term $I(u)\,[1-I(u)]$ in a nonlocal way, we make the choice $c_i = t_i$ and $c_i = t_{i+1}$ to approximate $I(u)$ and $(1-I(u))P(t_{k+1}-u)$ respectively. We then have approximately

$$I(t_{k+1}) \simeq I_0(t_{k+1}) + \sum_{i=0}^{k} \lambda[I(t_i)]I(t_i)[1-I(t_{i+1})]P(t_{k+1}-t_{i+1}) \left[\frac{e^{-\mu(t_{k+1}-t_{i+1})} - e^{-\mu(t_{k+1}-t_i)}}{\mu}\right].$$

Further simplifications give

$$\begin{aligned} I(t_{k+1}) &\simeq I_0(t_{k+1}) + \sum_{i=0}^{k} \lambda[I(t_i)]I(t_i)[1-I(t_{i+1})]P[(k-i)\Delta t] \left[\frac{e^{-\mu(k-i)\Delta t} - e^{-\mu(k+1-i)\Delta t}}{\mu}\right] \\ &= I_0(t_{k+1}) + \sum_{i=0}^{k} \lambda[I(t_i)]I(t_i)[1-I(t_{i+1})]P[(k-i)\Delta t]e^{-\mu(k-i)\Delta t}\phi(\Delta t), \end{aligned} \tag{4.4.1}$$

where

$$\phi \equiv \phi(\Delta t) = \frac{1-e^{-\mu\Delta t}}{\mu}. \tag{4.4.2}$$

Denoting by I^i an approximation of $I(t_i)$ in (4.4.1), we obtain the following NSFD scheme for (4.2.12):

$$I^{k+1} = I_0(t_{k+1}) + \sum_{i=0}^{k} \lambda(I^i) I^i [1 - I^{i+1}] P[(k-i)\Delta t] e^{-\mu(k-i)\Delta t} \phi(\Delta t). \tag{4.4.3}$$

In what follows, it will be necessary to write the (4.4.3) in the three equivalent forms. Firstly, by rearranging Eq. (4.4.3), we have

$$I^{k+1} = \frac{I_0(t_{k+1}) + \sum_{i=0}^{k-1} \lambda(I^i) I^i [1 - I^{i+1}] P[(k-i)\Delta t] e^{-\mu(k-i)\Delta t} \phi(\Delta t) + \lambda(I^k) I^k \phi(\Delta t)}{1 + \lambda(I^k) I^k \phi(\Delta t)}. \tag{4.4.4}$$

Secondly, subtraction from (4.4.3) of I^k yields

$$\begin{aligned} \frac{I^{k+1} - I^k}{\phi(\Delta t)} &= \frac{I_0(t_{k+1}) - I_0(t_k)}{\phi(\Delta t)} \\ &+ \sum_{i=0}^{k-1} \lambda(I^i) I^i [1 - I^{i+1}] \left[P[(k-i)\Delta t] e^{-\mu(k-i)\Delta t} - P[(k-1-i)\Delta t] e^{-\mu(k-1-i)\Delta t} \right] \\ &+ \lambda(I^k) I^k [1 - I^{k+1}]. \end{aligned} \tag{4.4.5}$$

The equivalent formulation (4.4.5) is an approximation of the Volterra integro-differential equation (4.2.13), with the sum expected to be an approximation of the sum in (4.2.13). This formulation, (4.4.5), motivates the fact that (4.4.4) is called a NSFD scheme in the sense of [8, 61]. Indeed, the usual denominator of the discrete derivative is replaced by the complex denominator function $\phi(\Delta t)$, while the nonlocal terms are approximated in a nonlocal way.

The third formulation involves the basic reproduction number $\mathcal{R}_0$ and read as

$$I^{k+1} = I_0(t_{k+1}) + \mathcal{R}_0 \phi(\Delta t) \sum_{i=0}^{k} I^i f^i \tilde{P}[(k-i)\Delta t], \tag{4.4.6}$$

or

$$I^{k+1} = \frac{I_0(t_{k+1}) + \mathcal{R}_0 \phi(\Delta t) \sum_{i=0}^{k-1} I^i f^i \tilde{P}[(k-i)\Delta t] + I^k \lambda(I^k) \phi(\Delta t)}{1 + I^k \lambda(I^k) \phi(\Delta t)}, \tag{4.4.7}$$

where

$$f(I^i) \simeq f^i = \frac{\lambda(I^i)[1 - I^{i+1}]}{\lambda(0)}. \tag{4.4.8}$$

Theorem 4.4.1. *Assume that λ and the solution I are such that the integrand in (4.2.12) is of function of C^2 with derivatives being bounded. Then the NSFD scheme (4.4.1) is consistent with the SIS-VIE (4.2.12) in the sense that*

$$\lim_{\substack{k \to \infty \\ \Delta t \to 0 \\ k\Delta t = t}} |\tau_{k+1}| = 0,$$

where

$$\tau_{k+1} : \;=\; \int_0^{t_{k+1}} \lambda\left[I(u)\right] I(u)\left[1-I(u)\right] P(t-u)e^{-\mu(t-u)}du \\ -\; \sum_{i=0}^{k} \lambda[I(t_i)]I(t_i)[1-I(t_{i+1})]P(t_{k+1}-t_{i+1})e^{-\mu(t_{k+1}-t_{i+1})}\phi(\Delta t). \quad (4.4.9)$$

More precisely, the NSFD scheme is of order 1, i.e., $\tau_{k+1} = \mathcal{O}(\Delta t)$.

Proof: By adding and subtracting terms, we have

$$\Sigma_{i=0}^{k}\lambda[I(t_i)]I(t_i)[1-I(t_{i+1})]P(t_{k+1}-t_{i+1})e^{-\mu(t_{k+1}-t_{i+1})}\phi(\Delta t) = \mathcal{J}\Delta t \\ +\, \Sigma_{i=0}^{k}\lambda[I(t_i)]I(t_i)[1-I(t_{i+1})]P(t_{k+1}-t_{i+1})e^{-\mu(t_{k+1}-t_{i+1})}\phi(\Delta t) \\ -\, \Sigma_{i=0}^{k}\lambda[I(t_i)]I(t_i)[1-I(t_i)]P(t_{k+1}-t_i)e^{-\mu(t_{k+1}-t_i)}\phi(\Delta t),$$

where

$$\mathcal{J} = \Sigma_{i=0}^{k}\lambda[I(t_i)]I(t_i)[1-I(t_i)]P(t_{k+1}-t_i)e^{-\mu(t_{k+1}-t_i)}\frac{\phi(\Delta t)}{\Delta t}.$$

By using the mean-value theorem for derivatives, we get

$$\sum_{i=0}^{k} \lambda[I(t_i)]I(t_i)[1-I(t_{i+1})]P(t_{k+1}-t_{i+1})e^{-\mu(t_{k+1}-t_{i+1})}\phi(\Delta t) = \mathcal{J}\Delta t \\ +\, \sum_{i=0}^{k} \lambda[I(t_i)]I(t_i)g'(z_i)\Delta t\phi(\Delta t),$$

for some $z_i \in (t_i, t_{i+1})$, where

$$g(x) = (1-I(x))P(t_{k+1}-x)e^{-(t_{k+1}-x)} \quad \text{for} \quad x \in [t_i, t_{i+1}].$$

Then from Eq. (4.4.9), we have

$$|\tau_{k+1}| \le |\int_0^{t_{k+1}} \lambda\left[I(u)\right] I(u)\left[1-I(u)\right] P(t-u)e^{-\mu(t-u)}du - \mathcal{J}\Delta t| + M\sum_{i=0}^{k}\lambda[I(t_i)]I(t_i)\phi(\Delta t)\Delta t,$$

where

$$M = \sup_{z_i\in[t_i,t_{i+1}]} |g'(z_i)| \;\; \text{for} \;\; i = 0,\ 1,\ \ldots, k.$$

Hence,

$$|\tau_{k+1}| \;\; \le \;\; |\int_0^{t_{k+1}} \lambda\left[I(u)\right] I(u)\left[1-I(u)\right] P(t-u)e^{-\mu(t-u)}du - \mathcal{J}\Delta t| + M\left(\int_0^t \lambda(I(u))I(u)du\right)\Delta t.$$

Observing that $\phi(\Delta t) = \Delta t + \mathcal{O}(\Delta t^2)$, the term $\mathcal{J}\Delta t$ is the approximation of the integral

$$\int_0^{t_{k+1}} \lambda\left[I(u)\right] I(u)\left[1-I(u)\right] P(t-u)e^{-\mu(t-u)}du$$

by the rectangle formula with error in $\mathcal{O}(\Delta t^2)$ ([71]). This completes the proof. □

4.4.2 Qualitative properties

The first qualitative property of the NSFD scheme (4.4.3) is given in the next theorem.

Theorem 4.4.2. *The NSFD scheme (4.4.3) is dynamically consistent with respect to positivity and boundedness of the solution of (4.2.12):*

$$0 \leq I_0(t) \leq 1 \implies 0 \leq I^k \leq 1 \quad \forall k.$$

Proof: For $0 \leq I_0(t) \leq 1$, Eq. (4.4.4) yields $0 \leq I^k, \quad \forall k$. From (4.2.3), we have

$$S^k + I^k = 1, \quad \forall k. \tag{4.4.10}$$

Thus, the discrete susceptible obtained from the NSFD scheme (4.4.3) are given by the relation

$$S^{k+1} = S_0(t_{k+1}) - \phi(\Delta t)\sum_{i=0}^{k} \lambda(1 - S^i)[1 - S^i]S^{i+1}P[(k-i)\Delta t]e^{-\mu(k-i)\Delta t} \tag{4.4.11}$$

or

$$S^{k+1} = \frac{S_0(t_{k+1}) - \phi(\Delta t)\sum_{i=0}^{k-1} \lambda(1 - S^i)[1 - S^i]S^{i+1}P[(k-i)\Delta t]e^{-\mu(k-i)\Delta t}}{1 + \lambda(1 - S^k)[1 - S^k]\phi(\Delta t)}. \tag{4.4.12}$$

We apply the principle of mathematical induction on k. We assume that the claim is true for k i.e., $0 \leq I^k \leq 1$ and thus $0 \leq S^k \leq 1$ in view of (4.4.10). We want to show that $0 \leq I^{k+1} \leq 1$. From Eq. (4.4.12), we have

$$\begin{aligned} S^{k+1} &\geq \frac{S_0(t_k) - \phi(\Delta t)\sum_{i=0}^{k-1} \lambda(1 - S^i)[1 - S^i]S^{i+1}P[(k-i)\Delta t]e^{-\mu(k-i)\Delta t}}{1 + \lambda(1 - S^k)[1 - S^k]\phi(\Delta t)} \\ &\quad \text{because} \quad S_0(t_{k+1}) \geq S_0(t_k). \\ &\geq \frac{S^k}{1 + \lambda(1 - S^k)[1 - S^k]\phi(\Delta t)} \quad \text{by using} \quad Eq.\ (4.4.11) \\ &\geq 0, \quad \text{because} \quad S^k \geq 0. \end{aligned}$$

Using again Eq. (4.4.10), we have $0 \leq I^{k+1} \leq 1$. □

The second series of qualitative properties that we will investigate is related to fixed points. Given the specific nature of the definitions and tools needed, we spend some time to outline these concepts following [18, 28].

The NSFD scheme (4.4.3) or (4.4.6) has a general structure of a difference equation of order $m + 1$:

$$x^{k+1} = f(x^k, x^{k-1}, \ldots, x^{k-m}), \qquad k = 0, 1, 2, \ldots \tag{4.4.13}$$

where $f : U^{m+1} \to U$ is a C^1-function, U is an interval of real numbers and $U^{m+1} = U \times U \times \cdots \times U$ ($m+1$ times).

Definition 4.4.3. *A number $x^* \in \mathbb{R}$ is a fixed point of the difference equation in (4.4.13) if $f(x^*, x^*, \ldots, x^*) = x^*$.*

Definition 4.4.4. *Let x^* be a fixed point of (4.4.13). Then x^* is said to be*

1. *stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$|x^{-m} - x^*| + |x^{-m+1} - x^*| + \cdots + |x^0 - x^*| < \delta \quad \textit{implies} \quad |x^k - x^*| < \epsilon \quad \textit{for all} \quad k \geq 0,$$

   where here and after $\{x^k\}_{k=-m}^{\infty}$ represents any solution of (4.4.13).

2. *locally asymptotically stable if x^* is stable and in addition there exists $\gamma > 0$ such that*

$$|x^{-m} - x^*| + |x^{-m+1} - x^*| + \cdots + |x^0 - x^*| < \gamma \quad \textit{implies} \quad \lim_{k \to \infty} x^k = x^*.$$

3. *globally asymptotically stable if x^* is stable and*

$$\lim_{k \to \infty} x^k = x^*$$

   for any choice of the initial guess.

4. *unstable if it is not stable.*

Applying Definition 4.4.4 to check the stability properties of a fixed point is not easy. The usual way is to replace Eq. (4.4.13) by its linearized form

$$y^{k+1} = -\sum_{i=0}^{k} a_{k-i} y^i, \tag{4.4.14}$$

where

$$a_{k-i} = -\frac{\partial f}{\partial x^i}(x^*, x^*, \ldots, x^*) \quad \text{not all zero and} \quad y^i = x^i - x^* \quad \text{for} \quad i = 0,\ 1, \cdots, m.$$

The characteristic polynomial of (4.4.14) is

$$r^{k+1} = -\sum_{i=0}^{k} a_{k-i} r^i. \tag{4.4.15}$$

Theorem 4.4.5. *Let x^* be a fixed point of Eq. (4.4.13). If all roots of the characteristic equation (4.4.15) satisfy $|r| < 1$ then x^* is locally asymptotically stable. If at least one root satisfies $|r| > 1$ then it is unstable.*

A sufficient condition for stability property of a fixed point for Eq. (4.4.14) as stated in Theorem 4.4.5 is given in terms of the coefficients of the characteristic polynomial as follows (see [59]).

Theorem 4.4.6. *Assume that*

$$1 > a_0 > a_1 > \cdots > a_{k-1} > a_k > 0.$$

Then the roots r of the characteristic polynomial satisfy the stability condition

$$|r| < 1.$$

Finally, we want to recall a result in [28] for the stability of a difference equation of the form

$$x^{k+1} = ax^k + \sum_{i=0}^{k} b_{k-i} x^i, \qquad (4.4.16)$$

where a and b_{k-i} are real numbers.

Theorem 4.4.7. *Assume that b_k in Eq. (4.4.16) doesn't change sign for $k \geq 1$.*

1. *If*

$$|a| + |\sum_{k=0}^{\infty} b_k| < 1,$$

then the fixed point $x^* = 0$ of (4.4.16) is locally asymptotically stable.

2. *The fixed point $x^* = 0$ of (4.4.16) is not locally asymptotically stable if any one of the following conditions holds:*

   i. $a + \sum_{k=0}^{\infty} b_k \geq 1$.

   ii. $a + \sum_{k=0}^{\infty} b_k < -1$ *and* $b_k > 0$ *for some* $k \geq 1$.

   iii. $a + \sum_{k=0}^{\infty} b_k < -1$ *and* $b_k < 0$ *for some* $k \geq 1$ *and* $\sum_{k=0}^{\infty} b_k$ *is sufficiently small.*

We are now in a position to deal with the stability of the fixed points of Eq. (4.4.6) or (4.4.7).

Proposition 4.4.8. *The NSFD scheme (4.4.6) is dynamically consistent with Proposition 4.3.9. More precisely, $I^* \in [0,1]$ is a fixed point if and only if it satisfies*

$$I = f(I)I\mathcal{R}_0. \tag{4.4.17}$$

Proof: To start the proof, we fix a time t and represent it by $t = k\Delta t$ for various values of k and Δt. Let $I^k = I^*$ be a fixed point of the NSFD scheme (4.4.6) and let $I_0(t_{k+1}) = I_0(t_{k+1}; I^*)$ be the corresponding initial condition. Then

$$\lim_{\substack{k\to\infty \\ \Delta t\to 0 \\ t=k\Delta t}} I_0(t_{k+1}; I^*) = I_0(t; I^*)$$

and

$$\mathcal{R}_0 I^* f(I^*) \lim_{\substack{k\to\infty \\ \Delta t\to 0 \\ t=k\Delta t}} \sum_{i=0}^{k} \tilde{P}[(k-i)\Delta t]\phi(\Delta t) = \mathcal{R}_0 I^* f(I^*) \int_0^t \tilde{P}(u)du.$$

Thus,

$$I^* = I_0(t; I^*) + \mathcal{R}_0 I^* f(I^*) \int_0^t \tilde{P}(u)du.$$

Letting $t \to \infty$ in both sides gives (4.4.17) for $I = I^*$.

Conversely, let I^* solve (4.4.17). Define

$$\begin{aligned} I_0(t_{k+1}; I^*) &= I^* - \mathcal{R}_0 I^* f(I^*) \sum_{i=0}^{k} \tilde{P}[(k-i)\Delta t]\phi(\Delta t) \\ &= I^* - I^* \sum_{i=0}^{k} \tilde{P}[(k-i)\Delta t]\phi(\Delta t). \quad \text{by} \quad (4.4.17) \end{aligned}$$

Then we have

$$\lim_{\substack{k\to\infty \\ \Delta t\to 0 \\ t=k\Delta t}} I_0(t_{k+1}; I^*) = I^* - I^* \int_0^t \tilde{P}(t-u)du.$$

If we take limit of both sides as $t \to \infty$, we get $I^* - I^* = 0$, because $\int_0^\infty \tilde{P}(u)du = 1$. Therefore, I^* is a fixed point of the NSFD scheme (4.4.6). □

Remark 4.4.9. *Note that $I^* = 0$ always solve (4.4.17) and is called the disease-free fixed point. A nonzero solution I^* of (4.4.17) satisfies*

$$\mathcal{R}_0 f(I^*) = 1$$

and is called the endemic fixed point (we denote an endemic fixed point by I_e). Hence, Remark 4.3.10 regarding the existence of multiple endemic equilibrium applies to the existence of multiple endemic fixed points.

The number of fixed points depends on the parameters $\mathcal{R}_0$, $\mathcal{R}_0^c$ and $\mathcal{R}_0^m$, as specified in the next theorem.

Theorem 4.4.10. *The NSFD scheme (4.4.3) or (4.4.6) is dynamically consistent with respect to Theorem 4.3.11. More precisely,*

1. *the disease-free equilibrium point is the disease-free fixed point,*
2. *there is no endemic fixed point for $\mathcal{R}_0 < \mathcal{R}_0^c$,*
3. *there exists at least one endemic fixed point for $\mathcal{R}_0 > \mathcal{R}_0^c$,*
4. *there exists exactly one endemic fixed point for $\mathcal{R}_0 > \mathcal{R}_0^m$.*

Proof: In view of the existence/uniqueness of equilibrium points of the continuous model given in Theorem 4.3.11, Theorem 4.4.10 will be proved if we show that I^* is a fixed point of the NSFD scheme (4.4.3) or (4.4.6) if and only if I^* satisfies Eq. (4.4.17). This is precisely what was done in the proof of Proposition 4.4.8. □

So far, we have been using the complex denominator function $\phi(\Delta t)$ in (4.4.2) that involves the parameter μ of the continuous model. However, it is well-known that numerical schemes may involve additional parameters if they are to capture the dynamics of the continuous model. For this reason, we modify in the rest of this chapter, the denominator function as

$$\phi(\Delta t) = \frac{1 - e^{-q\Delta t}}{q}, \tag{4.4.18}$$

where

$$q \geq \max\left\{\mu,\ \max_{I\in[0,1]} \lambda(I),\ 2\max_{I\in[0,1]} \lambda^2(I)\right\}. \tag{4.4.19}$$

The main motivation for the choice in (4.4.18) and (4.4.19) follows from the investigation of the stability properties of the fixed points in (4.4.6), which we consider now. We first write the NSFD scheme (4.4.7) in the equivalent form

$$I^{k+1} = \frac{I_0(t_{k+1})}{1 + I^k\lambda(I^k)\phi(\Delta t)} + G(I^0, I^1, \cdots, I^k), \tag{4.4.20}$$

where

$$G(I^0, I^1, \cdots, I^k) = \frac{\mathcal{R}_0 \sum_{i=0}^{k-1} I^i f^i \tilde{P}[(k-i)\Delta t]\phi(\Delta t) + I^k \lambda(I^k)\phi(\Delta t)}{1 + I^k \lambda(I^k)\phi(\Delta t)}. \qquad (4.4.21)$$

Let I^* be a fixed point of Eq. (4.4.20). Then we have

$$I^* = \frac{I_0(t_{k+1}; I^*)}{1 + I^* \lambda(I^*)\phi(\Delta t)} + G(I^*, I^*, \cdots, I^*), \qquad (4.4.22)$$

where $I_0(t_{k+1}; I^*)$ is the initial value associated with I^*. On setting

$$Z^i = I^i - I^*, \quad \text{for} \quad i = 0, 1, \ldots, k,$$

linearization of Eq. (4.4.20) about I^* gives

$$Z^{k+1} = -\sum_{i=0}^{k} a_{k-i} Z^i, \qquad (4.4.23)$$

where

$$a_k = -\frac{\partial G}{\partial I^0}(I^*) = \frac{-\phi[\lambda(I^*) + I^*\lambda'(I^*)](1 - I^*)P(k\Delta t)}{1 + I^*\lambda(I^*)\phi}, \qquad (4.4.24)$$

$$a_{k-i} = -\frac{\partial G}{\partial I^i}(I^*) = \frac{-\phi\left([\lambda(I^*) + I^*\lambda'(I^*)](1 - I^*)P[(k-i)\Delta t] - I^*\lambda(I^*)P[(k-i+1)\Delta t]\right)}{1 + I^*\lambda(I^*)\phi},$$
$$\text{for } i = 1,\ 2,\ \cdots,\ k-1; \qquad (4.4.25)$$

$$a_0 = -\frac{\partial G}{\partial I^k}(I^*) = \frac{-\phi\left[\lambda(I^*)[-I^*P(\Delta t) + 1 - I^*P(\Delta t)\lambda(I^*)\phi] + A\right]}{(1 + I^*\lambda(I^*)\phi)^2} \qquad (4.4.26)$$

and

$$A = I^*\lambda'(I^*) - I^*\lambda(I^*)P(\Delta t)I^*\lambda'(I^*)(1 - I^*)\phi.$$

The characteristic polynomial of (4.4.23), with complex argument r, is

$$r^{k+1} = -\sum_{i=0}^{k} a_{k-i} r^i. \qquad (4.4.27)$$

For the fixed point to be locally asymptotically stable, i.e., $|r| < 1$ for all roots of Eq. (4.4.27), we use a version of Jury conditions stated in Theorem 4.4.6 as follows:

$$1 > a_0 > a_1 > \cdots > a_{k-1} > a_k > 0. \qquad (4.4.28)$$

Under the assumption

$$\lambda'(I^*) < \frac{-\lambda(I^*)}{I^*}, \tag{4.4.29}$$

which we make in what follows for an endemic equilibrium which is locally asymptotically stable for the continuous model, Eq. (4.4.24) leads to

$$a_k > 0.$$

By definition of $(a_i)_{i=1}^{k-2}$ in (4.4.25) and the decreasing property of the function P (see (4.2.10)), it follows from (4.4.29) that

$$a_{k-i} < a_{k-(i+1)}, \qquad \text{for} \quad i = 1,\ 2,\ \cdots, k-2.$$

Notice that $a_k < a_{k-1}$ by (4.4.24)-(4.4.25) and (4.4.29).

Let us show that

$$a_0 > a_1.$$

This will be subjected to the condition

$$\phi \leq \frac{1}{max_{I\in[0,1]}\lambda(I)}, \tag{4.4.30}$$

which we explain now. By using (4.4.29) and (4.4.30), a_0 in (4.4.26) is positive. As the denominator of a_0 is always positive, we expect to have positive numerator i.e.,

$$-I^*\lambda(I^*)P(\Delta t)+\lambda(I^*)+I^*\lambda'(I^*)-I^*\lambda(I^*)P(\Delta t)\lambda(I^*)\phi-I^*\lambda(I^*)P(\Delta t)I^*\lambda'(I^*)(1-I^*)\phi < 0.$$

or

$$I^*\lambda(I^*)P(\Delta t)(1+\lambda(I^*)\phi) - [\lambda(I^*) + I^*\lambda'(I^*) - I^*\lambda(I^*)P(\Delta t)I^*\lambda'(I^*)(1-I^*)\phi] > 0.$$

To this end, we further assume that

$$\lambda(I^*) + I^*\lambda'(I^*)[1 - \lambda(I^*)P(\Delta t)I^*(1-I^*)\phi] < 0,$$

which in view of $\lambda(I^*) > 0$ and $\lambda'(I^*) < 0$, is equivalent to

$$\phi < \frac{1}{\lambda(I^*)P(\Delta t)I^*(1-I^*)}.$$

We therefore choose the denominator function ϕ as stated in (4.4.30). Thus using (4.4.30) and (4.4.25)-(4.4.26), we have $a_0 > a_1$ as desired.

Finally, under the condition

$$\lambda'(I^*) < -1 - \lambda(I^*), \tag{4.4.31}$$

we prove that the endemic equilibrium point I^* for the continuous model which satisfies (4.4.29) also satisfies $a_0 < 1$. A sufficient condition for this is

$$a_0 \leq \frac{\phi[\lambda(I^*) + I^*\lambda^2(I^*)\phi - \lambda'(I^*)]}{(1 + I^*\lambda(I^*)\phi)^2} < 1. \tag{4.4.32}$$

Further simplification of (4.4.32) gives a quadratic inequality in ϕ:

$$[I^*\lambda^2(I^*)(1 - I^*)]\phi^2 + [\lambda(I^*) - \lambda'(I^*) - 2I^*\lambda(I^*)]\phi - 1 < 0,$$

which is factorized into

$$(\phi + \frac{b + \sqrt{b^2 + 4b}}{2a})(\phi + \frac{b - \sqrt{b^2 + 4b}}{2a}) < 0, \tag{4.4.33}$$

where

$$\begin{aligned} a &= I^*\lambda^2(I^*)(1 - I^*) > 0, \qquad\qquad (4.4.34)\\ b &= \lambda(I^*) - 2I^*\lambda(I^*) - \lambda'(I^*) > 0, \quad \text{by (4.4.29)}. \end{aligned}$$

Note that b is greater than 1 under the assumption (4.4.31). Thus, from (4.4.33), we have

$$\phi < \frac{-b + \sqrt{b^2 + 4b}}{2a}. \tag{4.4.35}$$

Furthermore,

$$\begin{aligned} \frac{-b + \sqrt{b^2 + 4b}}{2a} &> \frac{-b + \sqrt{b^2 + 2b + 1}}{2a}, \\ &= \frac{1}{2a}, \\ &> \frac{1}{2\max_{I\in[0,1]} \lambda^2(I)}, \quad \text{by} \quad (4.4.34). \end{aligned}$$

Combining (4.4.29) and (4.4.31), we impose the condition

$$\lambda'(I^*) < \min\left\{-1 - \lambda(I^*)\ ,\ -\frac{\lambda(I^*)}{I^*}\right\}, \tag{4.4.36}$$

which leads to the following final requirement on ϕ for (4.4.32), and therefore $a_0 < 1$, to be true:

$$\phi < \frac{1}{2\max_{I\in[0,1]} \lambda^2(I)}, \tag{4.4.37}$$

Thus, the denominator function ϕ is chosen as in (4.4.18) in order for the conditions given in (4.4.30) and (4.4.37) to be true and for the properties of the NSFD scheme resulting from the denominator function in (4.4.2) to be captured. The lengthy reasoning carried out above leads, in view of Theorem 4.4.6, to the desired local asymptotic stability of the endemic fixed point $I^* = I_e$.

Regarding the local asymptotic stability of the disease-free fixed point $I^* = 0$, we proceed as follows. The linearized equation (4.4.23) reads

$$I^{k+1} = \mathcal{R}_0\phi \sum_{i=0}^{k} a_{k-i} I^i, \tag{4.4.38}$$

where

$$a_{k-i} = \tilde{P}[(k-i)\Delta t], \quad i = 0,\ 1,\ 2,\ \cdots,\ k.$$

Since $0 \leq a_{k-i} = \tilde{P}[(k-i)\Delta t], \quad i = 0,\ 1,\ 2,\ \cdots,\ k$, with $\tilde{P}$ with in (4.3.10) satisfying the assumption (4.2.10) and $\int_0^\infty \tilde{P}(t)dt = 1$, it follows from the integral test that the series $\sum_{j=0}^\infty a_j$ is convergent such that

$$\sum_{j=0}^{\infty} a_j \leq 1.$$

It follows then from Theorem 4.4.7 that the disease-free fixed point is locally asymptotically stable for $\mathcal{R}_0 < 1$ and unstable for $\mathcal{R}_0 > 1$.

In summary, we have proved the following result.

Theorem 4.4.11. *If the denominator function ϕ is chosen as in (4.4.18), then we have the following properties for the NSFD scheme (4.4.3) or (4.4.6).*

1. *The disease-free fixed point is locally asymptotically stable for $\mathcal{R}_0 < 1$ and unstable for $\mathcal{R}_0 > 1$,*

2. *An endemic fixed point I_e is locally asymptotically stable if it satisfies Eq. (4.4.36).*

Remark 4.4.12. *Note that under the condition (4.4.36), the NSFD scheme (4.4.3) or (4.4.6) preserves Proposition 4.3.9 and Theorem 4.3.11. This implies that the bifurcation analysis of the SIS-VDE model (4.4.6) is the same as that given in Remark 4.3.16 for the SIS-VIE model (4.3.11). In particular, the NSFD scheme can exhibit the backward bifurcation.*

Theorem 4.4.11 is improved in the next result in terms of global asymptotic stability of fixed points.

Theorem 4.4.13. *For the NSFD scheme (4.4.3) or (4.4.6) with the denominator function ϕ in (4.4.18):*

1. *The disease-free fixed point is globally asymptotically stable if $\mathcal{R}_0 < \mathcal{R}_0^c$.*
2. *The endemic fixed point I_e guaranteed by Theorem 4.4.10 is globally asymptotically stable if $\mathcal{R}_0 > \mathcal{R}_0^m$.*

Proof: Let $(I^k)_{k\geq 0}$ be the dynamical system in $\mathbb{R}$ generated as in (4.4.20). We know that $(I^k)_{k\geq 0}$ is a bounded sequence and its limit set or the ω-limit set of an initial condition I^0 consists of the singleton I^* which is the locally asymptotically stable fixed point of the dynamical system (see Theorem 4.4.11). We want to show that I^* is globally asymptotically stable.

Since I^* is locally asymptotically stable, there exists $\delta > 0$ such that

$$|I^0 - I^*| < \delta \Rightarrow \lim_{k\to\infty} |I^k - I^*| = 0.$$

Let us initiate the sequence (I^k) from an arbitrary I^0. As the sequence $(I^k)_{k\geq 0}$ is bounded, Bolzano-Weierstrass theorem implies that there exists a subsequence $(I^{k_j})_{j\geq 0}$ of $(I^k)_{k\geq 0}$ which is convergent. Clearly we have

$$\lim_{j\to\infty} I^{k_j} = I^*.$$

Thus, there exists $j_0 \in \mathbb{N}$ such that

$$j \geq j_0 \Rightarrow |I^{k_j} - I^*| < \delta.$$

In particular

$$|I^{k_{j_0}} - I^*| < \delta$$

and

$$\lim_{\substack{k\to\infty \\ k\geq k_{j_0}}} |I^k - I^*| = \lim_{k\to\infty} |I^k - I^*| = 0.$$

For the case under consideration, we have

$$\omega(I^0) = \{I^*\} = \begin{cases} \{0\}, & \text{if } \mathcal{R}_0 < \mathcal{R}_0^c \\ \{I_e\}, & \text{if } \mathcal{R}_0 > \mathcal{R}_0^m. \end{cases}$$

This proves the theorem. □

Remark 4.4.14. *In line with Remark 4.4.12 and Theorem 4.4.13, the possibility of backward bifurcation for the NSFD scheme arises when* $\mathcal{R}_0^c < \mathcal{R}_0 < 1$.

4.4.3 Numerical simulations

In this subsection, we give numerical experiments for our nonstandard finite difference scheme (4.4.3) or (4.4.6). It is known that standard finite difference methods do not always preserve the dynamics of the corresponding differential equation. This was in particular illustrated for the Runge-Kutta method in Section 2.5. Further illustrations can be seen in [8]. Thus, it is not necessary to generate here simulations for the classical numerical methods for Volterra integral equations.

The theoretical analysis of this section is illustrated by considering a number of functions $\lambda(I)$.

Firstly, we take

$$\lambda(I) = -I^2 + I + 1/3. \tag{4.4.39}$$

For this function, we used the definitions in (4.3.21) and (4.3.22) to obtain $\mathcal{R}_0^c = 0.83$ and $\mathcal{R}_0^m = 1$. Indeed, by using Eq. (4.3.9),

$$f(I) = 3\lambda(I)(1-I) \quad \text{for } I \in [0,1]$$

and we arrive at

$$\max_{I\in[0,1]} f(I) = \frac{100}{83} = \frac{1}{\mathcal{R}_0^c} \quad \text{and} \quad \min_{I\in[0,1)} f(I) = 1 = \frac{1}{\mathcal{R}_0^m}.$$

Furthermore, this function satisfies conditions in Eq. (4.4.36). We take $q = 2$ in (4.4.18) and plot the NS-VDE (4.4.6). In accordance with Theorem 4.4.13, it is seen that this

NS-VDE displays the backward bifurcation (Fig. 4.3), as well as the GAS of both the disease-free fixed point (Fig. 4.4) and the endemic fixed point (Fig. 4.5).

Secondly, we consider the function

$$\lambda(I) = 1 + 5I. \tag{4.4.40}$$

By applying Eq. (4.3.9), we get

$$f(I) = (1 + 5I)(1 - I)$$

and from (4.3.21) and (4.3.22), we obtain $\mathcal{R}_0^c = \frac{5}{9}$ and $\mathcal{R}_0^m = 1$, respectively. In this case, Theorem 4.4.13 and Remark 4.4.14 are illustrated for $q = 102$. The backward bifurcation diagram is given in Fig 4.6. The excellent performance of the NS-VDE (4.4.6) to preserve the GAS of the disease-free and the endemic equilibria are illustrated in Fig 4.7 and Fig 4.9, respectively. The coexistence of LAS disease-free fixed point with LAS endemic fixed point is shown in Fig 4.8.

Thirdly, we take the function

$$\lambda(I) = 1 + 17I^4. \tag{4.4.41}$$

Then, similarly, $\mathcal{R}_0^c = 0.625$ and $\mathcal{R}_0^m = 1.26$. In this case, the numerical simulation of the nonstandard scheme (4.4.6) using the function in (4.4.41) is similar to the simulation using the function in (4.4.40) and the results obtained are illustrated in Fig. 4.10-4.13.

The facts that (4.4.40) and (4.4.41) satisfy rather the realistic condition $\lambda'(I^*) < \frac{\lambda(I^*)}{1-I^*}$ that holds for a LAS endemic equilibrium I^* of the continuous model than the stronger condition (4.4.36), suggests that Theorem 4.4.11 is valid under this realistic condition.

In all the figures, $\Delta t = 2$, a large value that is not acceptable for classical numerical methods.

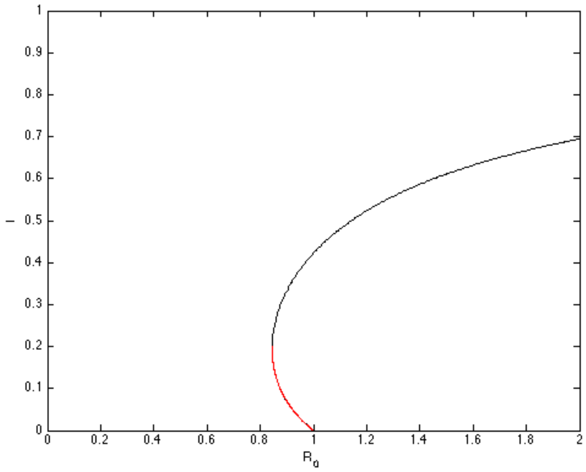

Figure 4.3: Backward bifurcation diagram for $\lambda(I) = -I^2 + I + 1/3$ and $\mathcal{R}_0 > 0.83 = \mathcal{R}_0^c$

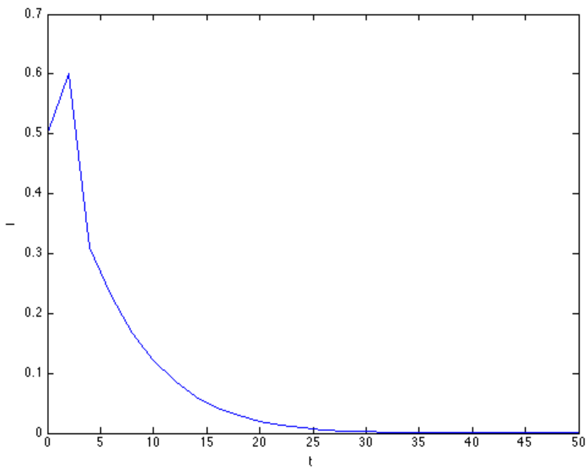

Figure 4.4: GAS of the disease-free fixed point with $\mu = 0.4,\ I_0(t) = \frac{1}{2}e^{-0.5t}$ and $P(t) = e^{-0.1t}$ for $\mathcal{R}_0 = \frac{2}{3} < 0.83$

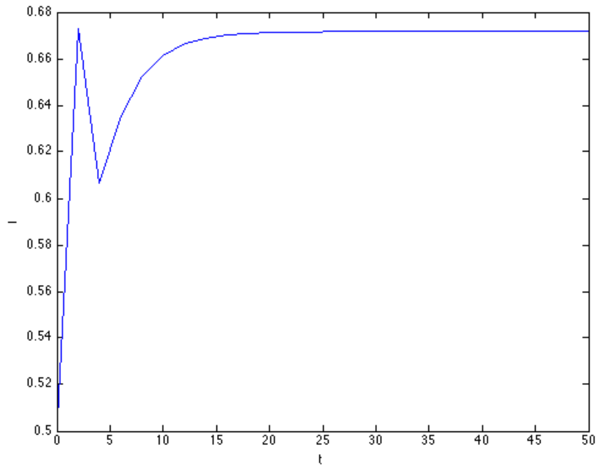

Figure 4.5: GAS of the endemic fixed point for $\mathcal{R}_0 = 5/3 > \mathcal{R}_0^m = 1$ with $\mu = 0.1,\ I_0(t) = \frac{1}{2}e^{-0.2t}$ and $P(t) = e^{-0.1t}$

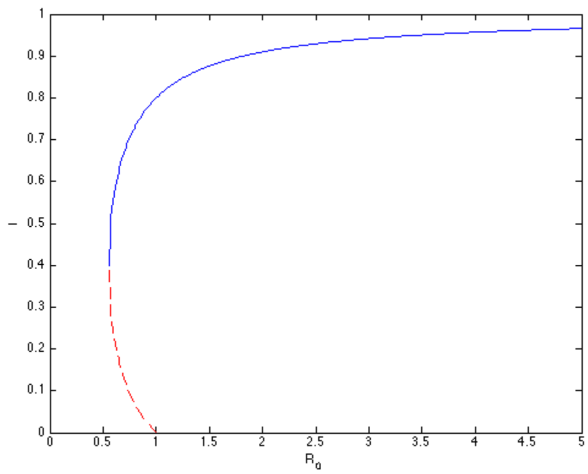

Figure 4.6: Backward bifurcation diagram for $\lambda(I) = 1 + 5I,\ \mathcal{R}_0 > \mathcal{R}_0^c = \frac{5}{9}$

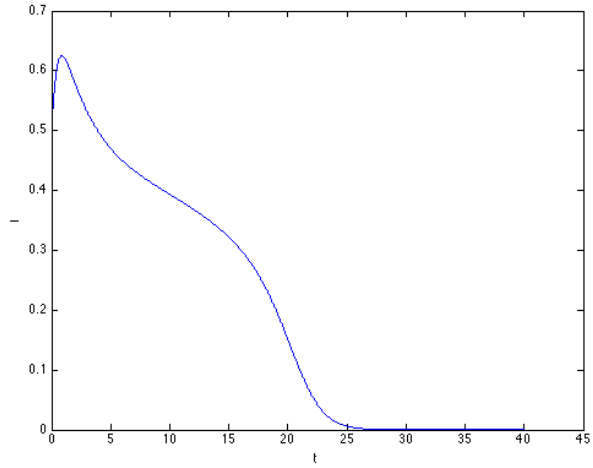

Figure 4.7: GAS of the disease-free fixed point for $\mathcal{R}_0 = 0.54 < \frac{5}{9} = \mathcal{R}_0^c$ with $\mu = 0.85,\ I_0(t) = \frac{1}{2}e^{-t}$ and $P(t) = e^{-t}$

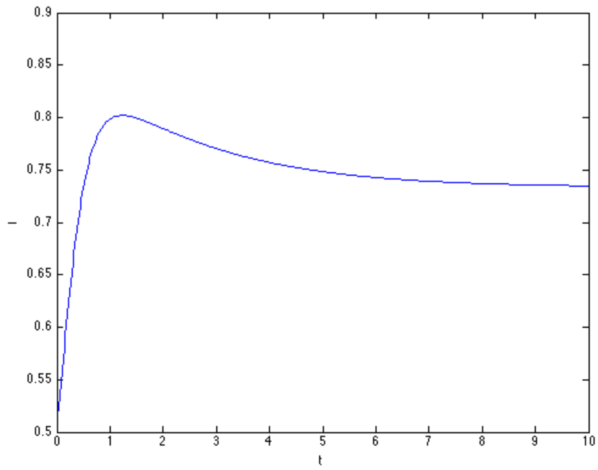

Figure 4.8: LAS of the endemic fixed point, $I_e = 0.74$ for $\frac{5}{9} = \mathcal{R}_0^c < \mathcal{R}_0 = 0.8 < 1$ with $\mu = 0.75,\ I_0(t) = \frac{1}{2}e^{-t}$ and $P(t) = e^{-t}$

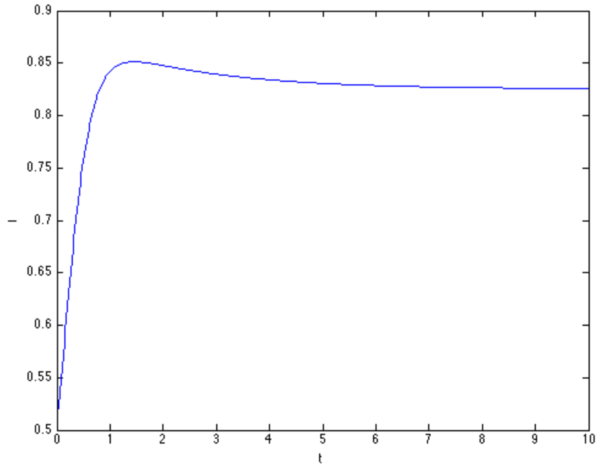

Figure 4.9: GAS of the endemic fixed point, $I_e = 0.83$ for $1 = \mathcal{R}_0^m < \mathcal{R}_0 = \frac{10}{9}$ with $\mu = 0.4,\ I_0(t) = \frac{1}{2}e^{-\frac{1}{2}t}$ and $P(t) = e^{\frac{-1}{2}t}$

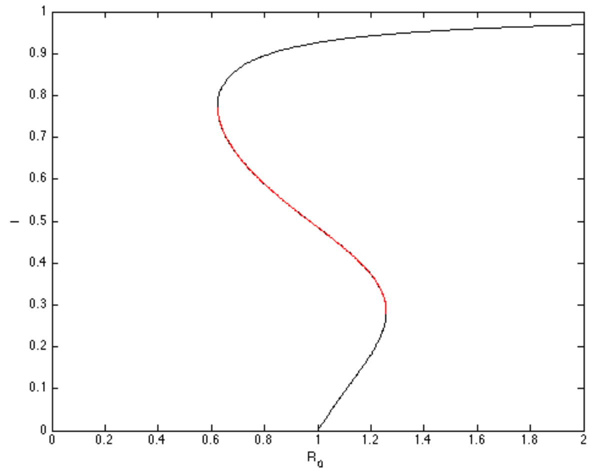

Figure 4.10: Backward bifurcation diagram for $\lambda(I) = 1 + 17I^4$ and $\mathcal{R}_0 \geq 0.625 = \mathcal{R}_0^c$

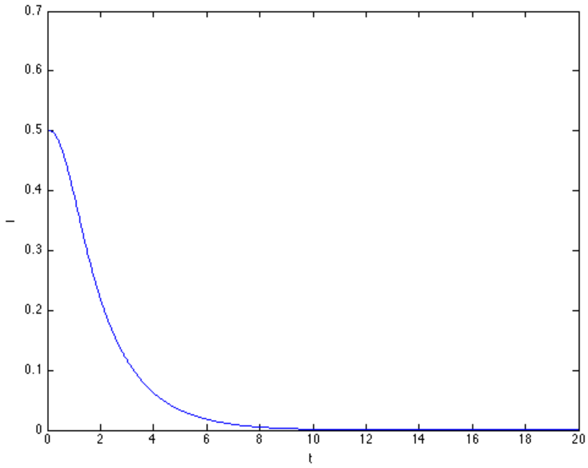

Figure 4.11: GAS of the disease-free fixed point for $0 < \mathcal{R}_0 = 0.6 < \mathcal{R}_0^c = 0.625$ with $\mu = 2/3,\ I_0(t) = \frac{1}{2}e^{-t}$ and $P(t) = e^{-t}$

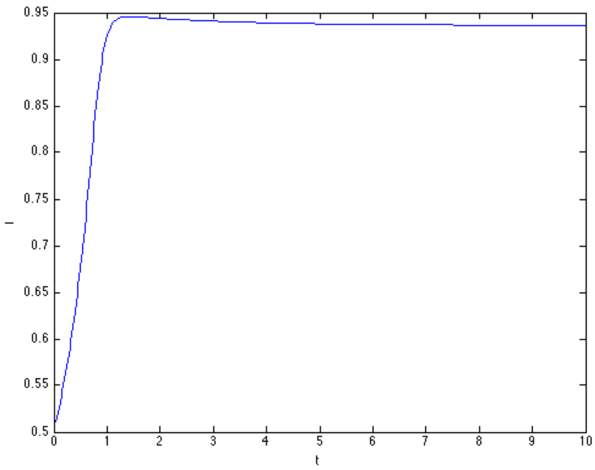

Figure 4.12: LAS of the endemic fixed point, $I_e = 0.936$ for $1 < \mathcal{R}_0 = \frac{10}{9} < \mathcal{R}_0^m = 1.26$ with $\mu = 0.4,\ I_0(t) = \frac{1}{2}e^{-\frac{1}{2}t}$ and $P(t) = e^{\frac{-1}{2}t}$

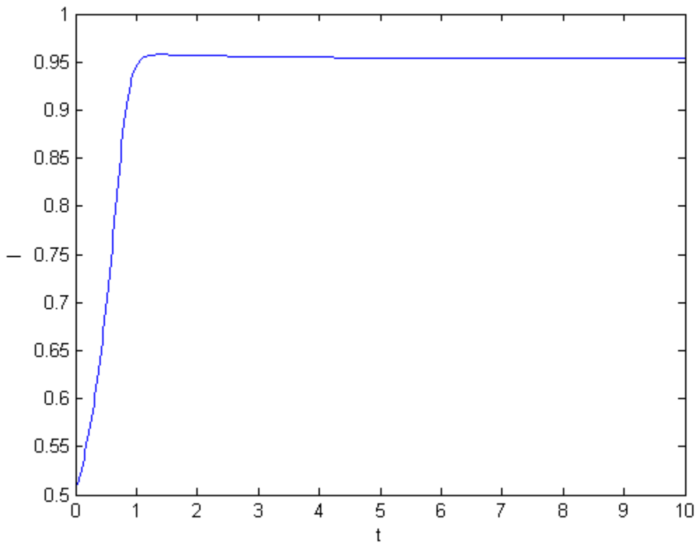

Figure 4.13: GAS of the endemic fixed point, $I_e = 0.968$ for $1 < \mathcal{R}_0^m = 1.26 < \mathcal{R}_0 = 2$ with $\mu = 0.3$, $I_0(t) = \frac{1}{2}e^{-\frac{1}{2}t}$ and $P(t) = e^{-0.2t}$

Chapter 5

Conclusion

In this thesis, we have investigated the SIS epidemiological model in three directions. In each direction, we did the quantitative and qualitative analysis as well as the computational analysis, through innovative NSFD scheme(s), with a strong focus on global stability results for both the continuous and the discrete models.

The first setting is that of the classical SIS model with constant contact rate, which is known to be a dynamical system on a suitable biologically feasible region having the value 1 of the basic reproduction number $\mathcal{R}_0$ as a transcritical bifurcation. That is the disease-free equilibrium is globally asymptotically stable when $\mathcal{R}_0 \leq 1$ whereas it is unstable and an additional globally asymptotically stable endemic equilibrium is born when $\mathcal{R}_0 > 1$ [14]. For this case, we designed two NSFD schemes and proved theoretically and computationally that they preserve the above-mentioned properties of the continuous model. We also designed a nonstandard Runge-Kutta method and proved that it has order 4. This is the first time that a higher order NSFD scheme is constructed.

In the second direction, the classical SIS model is extended into the SIS-diffusion model for the spread of diseases in space [67]. NSFD schemes used here are extensions of NSFD schemes for the classical SIS model. The stability results we obtained reads as for the classical continuous and discrete models. More precisely, the disease-free equilibrium is globally asymptotically stable for $\mathcal{R}_0 \leq 1$ and unstable for $\mathcal{R}_0 > 1$. In the latter case, there exists a globally asymptotically stable endemic equilibrium point. However, the proofs are more involved and based on the energy method [5].

The third setting is the SIS-Volterra integral equation model with the contact rate as a function of fraction of the total number of infective individuals as proposed by [79]. Following this reference, we specify sufficient conditions under which this model undergoes the backward bifurcation phenomenon, whereby there are two threshold parameters $\mathcal{R}_0^c$ and $\mathcal{R}_0^m$ such that the disease-free equilibrium is

- globally asymptotically stable for $\mathcal{R}_0 < \mathcal{R}_0^c$,
- locally asymptotically stable and coexists with a locally asymptotically stable endemic equilibrium for $\mathcal{R}_0^c < \mathcal{R}_0 < 1$, In this case, a backward bifurcation occurs.
- unstable and there exists a globally asymptotically stable endemic equilibrium for $\mathcal{R}_0 > \mathcal{R}_0^m$.

Furthermore, we designed a nonstandard-Volterra difference equation and proved theoretically and computationally that it replicates these properties. This is the first time to construct a dynamically consistent NSFD scheme for the SIS-Volterra integral equation model apart from a restrictive situation in [42, 63]. This is achieved by using Mickens's rules [61].

Possible extensions of this thesis that we will consider in future include:

- For the SIS-diffusion model:
    - ⋆ SIS advection-diffusion model [22].
- For the SIS-Volterra integral equation model:
    - ⋆ Fitting contact rates that depend on infective individuals with real data.
    - ⋆ Construction of dynamically consistent NSFD scheme when a general distribution function $P(t)$ of infective individuals and delay differential equations are involved.
    - ⋆ Volterra integral equation model based on other compartmental models such as SIR, SEIR, etc..
- Investigation of the SIR model which undergoes the backward bifurcation phenomenon (see [40]).

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