# Contents

# Notations

Throughout this thesis, we will assume the following notations:

- $\mathcal{B}(E)$ the Borel $\sigma$-field of any set $E \subset \mathbb{R}$ and $\mathcal{P}$ the predictable $\sigma$-field on $\Omega \times [0,T]$, where $\mathbb{R}$ denotes the set of real numbers;

- $\mathcal{C}$ the space of continuous functions;

- $\mathbb{L}^2_{-\varrho}(\mathbb{R})$- the space of measurable functions $k : [-\varrho, 0] \mapsto \mathbb{R}$, such that $\int_{-\varrho}^{0} |k(t)|^2 dt < \infty$, where $\varrho > 0$,.

- $\mathbb{S}^2_{-\varrho}(\mathbb{R})$- the space of bounded measurable functions $y : [-\varrho, 0] \mapsto \mathbb{R}$ such that
$$\sup_{t \in [-\varrho, 0]} |y(t)|^2 < \infty\,;$$

- $\mathbb{H}^2_{-\varrho,\nu}$- the space of product measurable functions $\upsilon : [-\varrho, 0] \times \mathbb{R} \mapsto \mathbb{R}$, such that
$$\int_{-\varrho}^{0} \int_{\mathbb{R}} |\upsilon(t,z)|^2 \nu(dz) dt < \infty\,;$$

- $\mathbb{L}^2(\mathbb{R})$- the space of random variables $\xi : \Omega \mapsto \mathbb{R}$, such that $\mathbb{E}[\,|\xi|^2] < \infty$;

- $L^2_\nu(\Omega)$- the space of measurable functions $\upsilon : \Omega \mapsto \mathbb{R}$ such that
$$\int_{\mathbb{R}} |\upsilon(z)|^2 \nu(dz) < \infty\,,$$
where $\nu$ is a $\sigma$-finite measure;

- $\mathbb{S}^2([0,T])$- the space of adapted càdlàg processes $Y : \Omega \times [0,T] \mapsto \mathbb{R}$ such that
$$\mathbb{E}[\sup_{t \in [0,T]} |Y(t)|^2] < \infty\,;$$

- $\mathbb{H}^2([0,T])$- denote the space of predictable processes $Z : \Omega \times [0,T] \to \mathbb{R}$ satisfying
$$\mathbb{E}\Big[\int_0^T |Z^2(t)| dt\Big] < \infty\,;$$

- $\mathbb{H}_N^2(\mathbb{R})$- the space of predictable processes $\Upsilon : \Omega \times [0,T] \times \mathbb{R} \mapsto \mathbb{R}$, such that
$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}} |\Upsilon(t,z)|^2 \nu(dz)dt\right] < \infty\,;$$
- $x \wedge y := \min\{x, y\}$;
- $A^T$ denote the transpose of the matrix $A$;
- $U^-$ is the negative part of $U$ defined by $U^- := \max\{-U, 0\}$ and $U^+$ is the positive part of $U$ given by $U^+ := \max\{U, 0\}$;
- $\langle \cdot, \cdot \rangle$ is the inner product defined as follows:
$$\langle a, b\rangle := \sum_{k=1}^{n} a_k b_k, \quad a, b \in \mathbb{R}^n\,;$$
- $\chi_A$ is a characteristic function defined by
$$\chi_A(x) := \begin{cases} 1, & \text{if } \ x \in A; \\ 0, & \text{otherwise.} \end{cases}$$

# Chapter 1

# Introduction

## 1.1 Background

A fundamental objective of a decision making is to come up with an optimal strategy in order to achieve the best expected outcome. The solution to such a problem is the main concern of an investor who needs to allocate his wealth over a certain or uncertain horizon. Mathematically, this problem can be formulated as a stochastic optimal control problem, which is the main task of this thesis. Essentially, the optimization problems are composed by three elements: decision variables, the objective functional and the constraints. When there is no constraint, it is called unconstrained optimization problem.

The two most common approaches that can be found in the literature, when investigating stochastic optimal control problems are: the Dynamic Programming Principle (DPP) and the Maximum Principle (MP). The DPP was developed in the 50's by R. Bellman [8]. The basic idea of its implementation is based on the following:

*An optimal policy has the property that whatever the previous decision was, the remaining decisions must constitute an optimal policy with regard to the state resulting from the previous decisions.* Bellman [8].

This approach leads to the so called Hamilton-Jacobi-Bellman (HJB) equation, which in general is a non-linear partial differential equation (PDE), with its solution not generally provided. In order to overcome this drawback, the notion of viscosity solutions was introduced by Crandall and Lions [19] in the early 80's. This technique made the DPP a powerful tool to solve stochastic optimal control problems.

Otherwise, the MP approach introduced by L. Pontryagin and his team in 1956 states that an optimal control problem can be divided into solving a forward backward differential equation system and a maximum condition on the Hamiltonian function. For stochastic control problems with jumps, this approach has been widely reported in the literature, see, e.g., Framstad *et. al.* [39], Øksendal and Sulem [77], An and Øksendal [1], Pamen [79], Pamen and Momeya [80], among others.

The first results for optimal investment-consumption problem in continuous time were obtained by R. Merton [65, 66] via DPP. Later, an alternative Martingale approach was developed by Karatzas *et al.* [50], Karatzas *et al.* [51], Karatzas and Shreve [52], among others. This method is based on the change of measure techniques, where an equivalent probability measure is derived using the well known Girsanov's Theorem.

Concerning investments, an interesting question that may arise during the investor's planning is related to the protection of the investor's dependents if a premature death occur. This suggests the inclusion of a life insurance in an optimal portfolio-consumption problem. Life insurance appears as an important tool to solve the question of life uncertainty. Since the optimal portfolio, consumption and life insurance problem by Richard [85] in 1975, many works in this direction have been reported in the literature. For example, Pliska and Ye [83] considered an optimal consumption and life insurance contract for a problem described by a risk-free asset. Duarte *et al.* [29] considered a problem of a wage earner who invests and buys a life insurance in a financial market with $n$ diffusion risky shares. Similar works include (Guambe and Kufakunesu [41], Huang *et al.* [47], Liang and Guo [60], Shen and Wei [88], among others).

In this thesis, we solve our stochastic optimal portfolios and life insurance problems in a jump-diffusion setup. This direction is motivated by many reasons. First, the existence of high frequency data on the empirical studies carried out by Cont [17], Tankov [94] and references therein, have shown that the analysis of price evolution reveals some sudden changes that cannot be explained by models driven by diffusion processes. Another reason is related to the presence of volatility clustering and regime switchings in the distribution of the risky share process, i.e., large changes in prices are often followed by large changes and small changes tend to be followed by small changes.

## 1.2 Outline of the thesis

This thesis treats various optimal portfolios and life insurance problems under jump-diffusion setup.

In the first part (Chapter 3), we consider a jump-diffusion problem with stochastic volatility. This problem has been solved in Mnif [68] via dynamic programming approach. The application of this approach in a jump-diffusion setting, results in a nonlinear parabolic partial differential equation (PPDE) which in general the solution is not obtained. In his paper, Mnif proves the existence of a smooth solution by reducing a nonlinear PPDE to a semi-linear one under certain conditions. To overcome these limitations, we propose a martingale approach developed by Karatzas *et al.* [50] and Karatzas and Shreve [52] in a diffusion process to solve the unrestricted problem. Then we solve a constrained optimization problem, where the constraint is of American put type. Considering a jump-diffusion model, a market is incomplete and consequently we have many martingale measures. We obtain the optimal investment, consumption and life insurance strategy by the convex optimization method. This method allow us to characterize the optimal martingale measure for the utility functions of the power type. In the literature, this method has also been applied by Castaneda-Leyva and Hernández-Hernández [13] in a optimal investment-consumption problem. They considered a stochastic volatility model described by diffusion processes. Similar works include (Liang and Guo [60], Michelbrink and Le [67] and references therein).

The optimal solution to the restricted problem is derived from the unrestricted optimal solution, applying the option based portfolio insurance (OBPI) method developed by El Karoui *et al.* [30]. The OBPI method consists in taking a certain part of capital and invest in the optimal portfolio of the unconstrained problem and the remaining part insures the position with American put. We prove the admissibility and the optimality of the strategy.

The main contribution of this chapter is the extension and combination of the results by Kronborg and Steffensen [55], Castaneda-Leyva and Hernández-Hernández [13], among others to a jump-diffusion setting with life insurance considerations.

In the second part (Chapter 4), we consider a similar problem as in Chapter 3. We consider a wage earner buying life insurance contract from various life insurance companies. We suppose that each company offers distinct pairwise contracts. This allows the wage earner to compare the premiums insurance ratio of the companies and buy the amount of the life insurance from the one offering the smallest premium payout ratio each time. We propose a maximum principle approach to solve this stochastic volatility jump-diffusion problem. This approach allows us to solve this problem in a more general setting. We prove a sufficient and necessary maximum principle in a general stochastic volatility problem. Then we apply these results to solve the wage earner investment, consumption and life insurance problem.

The third part (Chapter 5) of the thesis discusses an optimal investment , consumption and life insurance problem using the backward stochastic differential equations (BSDE) with jumps approach. Unlike the dynamic programming approach, this approach allows us to solve the problem in a more general non-Markovian case. For more details on the theory of BSDE with jumps, see e.g., Delong [26], Cohen and Elliott [16], and references therein. Our results extend, for instance, the paper by Cheridito and Hu [14] to a jump-diffusion setup and we allow the presence of life insurance and inflation risks. The inflation-linked products may be used to protect the future cash flow of the wage earner against inflation, which occurs from time to time in some developing economies. For more details on the inflation-linked derivatives, see e.g., Tiong [96], Mataramvura [61] and references therein. We consider a model described by a risk-free asset, a real zero coupon bond, an inflation-linked real money account and a risky asset under jump-diffusion processes. This type of processes are more appropriate for modeling the response to some important extreme events that may occur since they allow capturing some sudden changes in the price evolution, as well as, the consumer price index which cannot be explained by models driven by Brownian information. Such events happen due to many reasons, for instance, natural disasters, political situations, etc.

In Chapter 6, we consider an insurer's risk-based optimal investment problem with noisy memory. The financial market model setup is composed by one risk-free asset and one risky asset described by a hidden Markov regime-switching jump-diffusion process. The jump-diffusion models represent a valuable extension of the diffusion models for modeling the asset prices. They capture some sudden changes in the market such as the existence of high-frequency data, volatility clusters and regime switching. In this chapter, we consider a jump diffusion model, which incorporates jumps in the asset price as well as in the model coefficients, i.e., a Markov regime-switching jump-diffusion model. Furthermore, we consider the Markov chain to represent different modes of the economic environment such as, political situations, natural catastrophes or change of law. Such kind of models have been considered for option pricing of the contingent claim, see for example, Elliott *et. al* [36], Siu [92] and references therein. For stochastic optimal control problems, we mention the works by Bäuerle and Rieder [7], Meng and Siu [64]. In these works a portfolio asset allocation and a risk-based asset allocation of a Markov-modulated jump process model has been considered and solved via the dynamic programming approach. We also mention a recent work by Pamen and Momeya [80], where a maximum principle approach has been applied to an optimization problem described by a Markov-modulated regime switching jump-diffusion model.

We assume that the company receives premiums at the constant rate and pays the aggregate claims modeled by a hidden Markov-modulated pure jump process. We assume the existence of capital inflow or outflow from the insurer's current wealth, where the amount of the capital is proportional to the past performance of the insurer's wealth. Then, the surplus process is governed by a stochastic delay differential equation with the delay, which may be random. Therefore we find it reasonable to consider also a delay modeled by Brownian motion. In literature, a mean-variance problem of an insurer was considered, but the wealth process is given by a diffusion model with distributed delay, solved via the maximum principle approach (Shen and Zeng [89]). Chunxiang and Li [15] extended this mean-variance problem of an insurer to the Heston stochastic volatility case and solved using dynamic programming approach. For thorough discussion on different types of delay, we refer to Baños *et. al.* [6], Section 2.2.

We adopt a convex risk measure first introduced by Frittelli and Gianin [40] and Föllmer and Schied [38]. This generalizes the concept of coherent risk measure first introduced by Artzner *et. al.* [3], since it includes the nonlinear dependence of the risk of the portfolio due to the liquidity risks. Moreover, it relaxes a sub-additive and positive homogeneous properties of the coherent risk measures and substitute these by a convex property.

To solve our optimization problem, we first transform the unobservable Markov regime-switching problem into one with complete observation by using the so-called filtering theory, where the optimal Markov chain is also derived. For interested readers, we refer to Elliott *et. al.* [32], Elliott and Siu [35], Cohen and Elliott [16] and Kallianpur [49]. Then we formulate a convex risk measure described by a terminal surplus process as well as the dynamics of the noisy memory surplus over a period $[T-\varrho, T]$ of the insurer to measure the risks. The main objective of the insurer is to select the optimal investment strategy so as to minimize the risk. This is a two-player zero-sum stochastic delayed differential game problem. Using delayed backward stochastic differential equations (BSDE) with a jump approach, we solve this game problem by an application of a comparison principle for BSDE with jumps. Our modeling framework follows that in Elliott and Siu [34], later extended to the regime switching case by Peng and Hu [81].

Finally, we conclude the thesis and propose some possible directions for future research in Chapter 7.

## 1.3 Published papers and preprints

This thesis resulted in four papers on optimal portfolios and life insurance problems listed as follows:

1. C. Guambe and R. Kufakunesu, Optimal investment-consumption and life insurance with capital constraints, *Submitted*.

2. C. Guambe and R. Kufakunesu, On the optimal investment-consumption and life insurance selection problem with stochastic volatility, *Submitted*.

3. C. Guambe and R. Kufakunesu, Optimal investment-consumption and life insurance selection problem under inflation. A BSDE approach, *Optimization*, 2018, 67(4), 457-473.

4. R. Kufakunesu, C. Guambe and L. Mabitsela, Risk-based optimal portfolio of an insurer with regime switching and noisy memory, *Submitted*.

# Chapter 2

# Stochastic calculus and portfolio dynamics

Throughout this thesis, we consider a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$, where $\{\mathcal{F}_t\}_{t\in[0,T]}$ denotes an increasing family of $\sigma$-algebras which forms an information flow or filtration. We assume that this filtration satisfies the usual conditions[1].

A stochastic process $X(t) = X(t, \omega)$, $t \in [0, T]$, $\omega \in \Omega$ is a collection of random variables on $\Omega \times [0, T]$. The time parameter may be discrete or continuous. In this thesis, we only consider the continuous case. A stochastic process $X(t)$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is adapted to the filtration i.e., $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted, if each $X(t)$ is revealed at time $t$, that is, $X(t)$ is $\mathcal{F}_t$-measurable. $X(t)$ is always adapted to its history or natural filtration, which is the $\sigma$-algebra generated by $X(t)$. Moreover, $X(t)$ is progressively measurable with respect to the filtration if $X(t, \omega) : [0, T] \times \Omega \to \mathbb{R}$ is $\mathcal{B}[0, T] \times \{\mathcal{F}_t\}_{t\in[0,T]}$-measurable. Unless otherwise stated, we consider only processes that are càdlàg, i.e., right continuous with left limit.

[1] $\mathcal{F}_0$ contains all sets of $\mathbb{P}$-measure zero and $\mathcal{F}_t$ is right continuous, i.e., $\mathcal{F}_t = \mathcal{F}_{t+}$, where $\mathcal{F}_{t+} = \cap_{\epsilon>0}\mathcal{F}_{t+\epsilon}$.

## 2.1 Brownian motion and Lévy processes

**Definition 2.1.1.** *An $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted process $W := (W(t)\,,\, 0 \le t \le T)$ is called a Brownian motion if*

*(i) $W(0) = 0$ a.s.;*

*(ii) for $0 \le s < t \le T$, $W(t) - W(s)$ is independent of $\mathcal{F}_s$;*

*(iii) for $0 \le s < t \le T$, $W(t) - W(s)$ is a Gaussian random variable with mean zero and variance $t - s$, i.e., $W(t) - W(s) \sim \mathcal{N}(0\,,\, t - s)$;*

*(iv) for any $\omega \in \Omega$, the sample paths $W(t)$ are continuous functions.*

Note that there exists a modification of a Brownian motion to the discontinuous case, which counts the number of occurrence of some events in a certain interval. If the inter-arrival time between two events is exponentially distributed, such processes are called Poisson processes. This process counts the number of jump times in the interval. We introduce bellow the concept of random measure

**Definition 2.1.2.** *A function $N$ defined on $\Omega \times [0, T] \times \mathbb{R} \mapsto \mathbb{R}$ is called a random measure if*

*(i) for any $\omega \in \Omega$, $N(\omega, \cdot)$ is a $\sigma$-finite measure on $\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R})$;*

*(ii) for any $A \in \mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R})$, $N(\cdot, A)$ is a random variable on $(\Omega, \mathcal{F}, P)$.*

A random measure or jumps of discontinuous process $N$ is $\{\mathcal{F}_t\}_{0,T}$-predictable if for any $\{\mathcal{F}_t\}_{t\in[0,T]}$-predictable[2] process $X$, such that $\int_0^T \int_{\mathbb{R}} |X(t, z)| N(dt, dz)$ exists, the process $(\int_0^t \int_{\mathbb{R}} X(s, z) N(ds, dz)\,,\, 0 \le t \le T)$ is $\{\mathcal{F}_t\}_{t\in[0,T]}$-predictable. For any random measure $N$, we define a process

$$E_N(A) = \mathbb{E}\Big[\int_{[0,T]\times\mathbb{R}} \mathbf{1}_A(\omega, t, z) N(\omega, dt, dz)\Big]\,,\ A \in \mathcal{F} \otimes \mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R})\,.$$

[2]A predictable process is a real-valued stochastic process whose values are known, in a sense just in advance of time. Predictable processes are also called *previsible*.

Then we say that the random measure $N$ has a compensator $\nu$ if there exists an $\{\mathcal{F}_t\}_{t\in[0,T]}$-predictable random measure $\nu$ such that $E_\nu$ is $\sigma$-finite measure on $\mathcal{P} \times \mathcal{B}(\mathbb{R})$ and the measures $E_N$ and $E_\nu$ are identical on $\mathcal{P} \times \mathcal{B}(\mathbb{R})$. The compensated random measure is given by

$$\widetilde{N}(\omega, dt, dz) := N(\omega, dt, dz) - \nu(\omega, dt, dz)\,. \tag{2.1}$$

As was shown in (He *et. al.* [45], pp 295–297), the compensator is uniquely determined.

**Definition 2.1.3.** *A Lévy process is an* $\{\mathcal{F}_t\}_{t\in[0,T]}$*-adapted process* $X := (X(t)\,, 0 \le t \le T)$ *such that*

*(i)* $X(0) = 0$ *a.s.;*

*(ii) for* $0 \le s < t \le T$*,* $X(t) - X(s)$ *is independent of* $\mathcal{F}_s$*;*

*(iii) for* $0 \le s < t \le T$*,* $X(t) - X(s)$ *has the same distribution as* $X(t-s)$*;*

*(iv) the process* $X$ *is continuous in probability, i.e., for any* $t \in [0, T]$ *and* $\epsilon > 0$*,*

$$\lim_{s\to t} P(|X(t) - X(s)| > \epsilon) = 0.$$

This class of stochastic processes has been widely studied in the literature. For interested readers we refer to Applebaum [2], Kyprianou [58]. Some important examples of Lévy processes used in many applications are Brownian motions and Poisson processes. For each Lévy process, we have a Lévy measure $\nu$ which counts the expected number of jumps between the time interval $[0, T]$. This measure is defined by

$$\nu(A) := \mathbb{E}[\#\{t \in [0, T] \mid X(t) - X(t-) \neq 0,\ X(t) \in A\}],\ \ A \in \mathcal{B}(\mathbb{R})\,.$$

## 2.2 Jump-diffusion processes

In this thesis, we work with processes that are driven by Brownian motions and Poisson random measures, the so-called jump-diffusion processes. Thus, the Itô's formula for these processes play an important role in solving many different stochastic optimization problems.

Let $W$ be an $n$-dimensional Brownian motion independent to $m$-dimensional Poisson random measures $N$. We consider a jump-diffusion process $X(t)$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ of the form

$$dX(t) = \alpha(t)dt + \beta(t)dW(t) + \int_{\mathbb{R}} \gamma(t,z)\tilde{N}(dt,dz)\,, \tag{2.2}$$

where $\alpha$, $\beta$ and $\gamma$ are progressively measurable functions such that (2.2) is well defined. For the existence and uniqueness solution of the SDE of the form (2.2), we refer to Øksendal and Sulem [77], Theorem 1.19 or Applebaum [2], Theorem 6.2.3.

The following Theorem, gives the Itô's formula for multidimensional processes.

**Theorem 2.2.1.** *Let $X_i(t) \in \mathbb{R}\,, i = 1, ..., D$ be an Itô-Lévy process of the form*

$$dX_i(t) = \alpha_i(t,\omega)dt + \sum_{j=1}^{M} \beta_{ij}(t,\omega)dW_j(t) + \sum_{j=1}^{\ell} \int_{\mathbb{R}} \gamma_{ij}(t,z_j,\omega)\widetilde{N}_j(dt,dz_j), \tag{2.3}$$

*where $\alpha_i : [0,T] \times \Omega \to \mathbb{R}$, $\beta_i : [0,T] \times \Omega \to \mathbb{R}^M$ and $\gamma_i : [0,T] \times \mathbb{R}^\ell \times \Omega \to \mathbb{R}^\ell$ are adapted processes such that the integrals exist. Here $W_j(t)\,, j = 1, ..., M$ is 1-dimensional Brownian motion and*

$$\widetilde{N}_j(dt,dz_j) = N_j(dt,dz_j) - \mathbf{1}_{|z_j|<a_j}\nu_j(dz_j)dt,$$

*where $N_j$ are independent Poisson random measures with Lévy measures $\nu_j$ coming from $\ell$ independent (1-dimensional) Lévy processes $\eta_1, ..., \eta_\ell$ and $\mathbf{1}_{|z_j|<a_j}$ is a characteristic function, for some $a_j \in [0,\infty]$. Let $f \in \mathcal{C}^{1,2}([0,T] \times$*

$\mathbb{R}^N$). *Then* $Y(t) = f(t, X_1(t), ..., X_N(t))$ *is also an Itô-Lévy process and*

$$\begin{aligned} dY(t) &= \frac{\partial f}{\partial t}dt + \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(\alpha_i dt + \beta_i dW(t)) + \frac{1}{2}\sum_{i,j=1}^{N}(\beta\beta^T)_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}dt \\ &+ \sum_{k=1}^{\ell}\int_{|z_k|<a_k}\Big[f(t, X(t^-) + \gamma^{(k)}(t, z_k)) - f(t, X(t^-)) \\ &\qquad - \sum_{i=1}^{N}\gamma_i^{(k)}(t, z_k)\frac{\partial f}{\partial x_i}(X(t^-))\Big]\nu_k(dz_k)dt \\ &+ \sum_{k=1}^{\ell}\int_{|z_k|<a_k}\Big[f(t, X(t^-) + \gamma^{(k)}(t, z_k)) - f(t, X(t^-))\Big]\widetilde{N}_k(dt, dz_k), \end{aligned}$$

*where* $X(t) = (X_1(t), ..., X_N(t))$, $\beta \in \mathbb{R}^{N\times M}$, $W(t) = (W_1(t), ..., W_M(t))$ *and* $\gamma^{(k)} \in \mathbb{R}^\ell$ *is the column number* $k$ *of the* $N \times \ell$ *matrix* $\gamma$.

*Proof.* See Applebaum [2], Theorem 4.4.7. □

An immediate consequence of the Itô's formula is the Itô-isometry property. It is stated as follows

**Lemma 2.2.2.** (Itô-Lévy isometry) *Let* $X(t) \in \mathbb{R}$, $X(0) = 0$ *be an SDE* (2.2), *for* $\alpha = 0$. *Then*

$$\mathbb{E}[X^2(t)] = \mathbb{E}\Big[\int_0^t \beta^2(s)ds + \int_0^t\int_{\mathbb{R}}\gamma^2(s, z)\nu(dz)ds\Big]$$

*provided that the right hand side is finite.*

Another consequence mostly used in many applications is the so called Itô's product rule.

**Lemma 2.2.3.** *Let* $X(t)$ *and* $Y(t)$ *be two jump-diffusion processes of the form* (2.2). *The product of these processes is given by*

$$\begin{aligned} d(X(t)Y(t)) &= X(t-)dY(t) + Y(t-)dX(t) + \beta_1(t)\beta_2(t)dt \\ &+ \int_{\mathbb{R}}\gamma_1(t, z)\gamma_2(t, z)N(dt, dz)\,. \end{aligned}$$

Besides the SDE defined above, in this thesis, we will use another important type of SDEs driven by jump-diffusion processes that also depend on the past values of the solution. This type of SDEs are called stochastic delayed differential equations (SDDE). We consider the processes of the form

$$\begin{aligned} dX(t) &= \mu(t, X(t), X(t-\delta))dt + \sigma(t, X(t), X(t-\delta))dW(t) \\ &\quad + \int_{\mathbb{R}} \gamma(t, X(t), X(t-\delta), z)\tilde{N}(dt, dz)\,, \end{aligned} \tag{2.4}$$

where $X(t-\delta)$ means that the coefficients may depend also on the past values of the solution on the interval $[t-\delta, t]$, for some $\delta > 0$. For the existence and uniqueness solution of the SDDE of the form (2.4), we refer to Baños *et. al.* [6]. The Itô's formula is obtained in the similar way as in Theorem 2.2.1, however one needs to consider to notion of the directional derivative. The corresponding formula for jump-diffusion processes is given in Baños *et. al.* [6], Theorem 3.6.

## 2.3 Martingales for jump-diffusion processes and the Girsanov Theorem

The concept of martingales plays an important rule in proving some of the main results of this thesis. In fact, the results in Chapter 3 are based in duality martingale techniques and those in Chapter 5 on martingale optimality principle. Given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a martingale is defined as an $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted stochastic process $X(t)$ such that $\mathbb{E}[|X(t)|]$ is finite, for any $t \in [0, T]$ and

$$\mathbb{E}[X(t) \mid \mathcal{F}_s] = X(s), \quad \text{for all} \quad \mathrm{s} \leq \mathrm{t}\,, \quad \text{a.s.},$$

that is, $X(s)$ is the best guess of the future value, given all the information up to and including the present time $s$. If $\mathbb{E}[X(t) \mid \mathcal{F}_s] \geq X(s)$, *a.s.*, $X(s)$ is a submartingale and a supermartingale if $-X(s)$ is a submartingale.

The fundamental examples of martingales are Brownian motions and a compensated Poisson process.

Another important concept is a local martingale. We say that an adapted process $X(t)$ is a local martingale if there exists a sequence of stopping times[3] $\tau_1 \leq \tau_2 \leq \ldots \leq \tau_n \mapsto T$ (a.s.) such that each of the processes $(X(t \wedge \tau_n),\ t \in [0,T])$ is a martingale. The sequence $(\tau_i)_{i=1,2,\ldots,n}$ is called a fundamental sequence.

Very often in many applications, a stochastic process is not a martingale. However, one can transform it into a martingale applying the changing of measure. The Girsanov Theorem is a key result in this procedure.

**Theorem 2.3.1.** (Girsanov's Theorem for Itô-Lévy processes). *Let $W$ and $N$ be $(\mathbb{P}, \{\mathcal{F}_t\}_{t\in[0,T]})$-Brownian motion and $(\mathbb{P}, \{\mathcal{F}_t\}_{t\in[0,T]})$-random measure with compensator $\nu(dz)$. Moreover, consider $X(t)$ be a 1-dimensional Itô-Lévy process of the form*

$$dX(t) = \alpha(t,\omega)dt + \beta(t,\omega)dW(t) + \int_{\mathbb{R}} \gamma(t,z,\omega)\widetilde{N}(dt,dz),\ 0 \leq t \leq T\,.$$

*Assume there exist predictable processes $\theta(t) = \theta(t,\omega) \in \mathbb{R}$ and $\psi(t,z) = \psi(t,z,\omega) \in \mathbb{R}$ such that*

$$\beta(t)\theta(t) + \int_{\mathbb{R}} \gamma(t,z)\psi(t,z)\nu(dz) = \alpha(t),$$

*for a.s. $(t,\omega) \in [0,T] \times \Omega$ and such that the process*

$$\begin{aligned} Z(t) \quad := \quad & \exp\Big[-\int_0^t \theta(s)dW(s) - \frac{1}{2}\int_0^t \theta^2(s)ds \\ & + \int_0^t \int_{\mathbb{R}} \ln(1-\psi(s,z))\widetilde{N}(ds,dz) \\ & + \int_0^t \int_{\mathbb{R}} \{\ln(1-\psi(s,z)) + \psi(s,z)\}\nu(dz)ds\Big]\,,\ 0 \leq t \leq T \end{aligned}$$

*is well defined and satisfies $\mathbb{E}[Z(T)] = 1$. Furthermore, define the probability measure $\mathbb{Q}$ on $\mathcal{F}_T$ by $d\mathbb{Q}(\omega) = Z(T)d\mathbb{P}(\omega)$. Then $X(t)$ is a local martingale*

[3] We say that a random time $T : \Omega \to [0,\infty]$ is a *stopping time* of the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ if the event $(T \leq t) \in \{\mathcal{F}_t\}_{t\in[0,T]}$ for each $t \geq 0$.

*with respect to* $\mathbb{Q}$ *and*

$$\begin{aligned} W^Q(t) &= W(t) + \int_0^t \theta(s)ds\,,\ 0 \le t \le T, \\ \widetilde{N}^Q(t,A) &= N(t,A) - \int_0^t \int_{\mathbb{R}} (1+\psi(s,z))\nu(dz)ds\,,\ 0 \le t \le T,\ A \in \mathcal{B}(\mathbb{R}) \end{aligned}$$

*are* $(\mathbb{Q}, \{\mathcal{F}_t\}_{t\in[0,T]})$*-Brownian motion and* $(\mathbb{Q}, \{\mathcal{F}_t\}_{t\in[0,T]})$*-compensated random measure respectively.*

*Proof.* See Øksendal and Sulem [77], Theorem 1.31 and Delong [26], Theorem 2.5.1. □

Furthermore, we consider the martingale representation theorem for jump-diffusion processes. It states that any martingale $M(t) \in \mathcal{F}_t$ can be represented in terms of the sum of a Brownian motion and a compensated Poisson random measure.

**Theorem 2.3.2.** (Martingale representation theorem). *Any* $(\mathbb{P}, \{\mathcal{F}_t\}_{t\in[0,T]})$*-martingale* $M(t)$ *admits a representation*

$$M(t) = M(0) + \int_0^t \beta(s)dW(s) + \int_0^t \int_{\mathbb{R}} \gamma(s,z)\tilde{N}(ds,dz)\,, \qquad (2.5)$$

*where* $\beta$ *is predictable and square integrable and* $\gamma$ *is predictable marked process and square integrable with respect to* $\nu(dz)$*.*

Finally, we introduce the notion of martingales of bounded mean oscillation ($BMO$-martingales) for jump-diffusion processes as in Morlais [70]. This is an extension of the concept of continuous $BMO$-martingales introduced by Kazamaki [53]. We say that a martingale $M$ of the form (2.5) is in the class of $BMO$-martingales if there exists a constant $K > 0$, such that, for all $\mathcal{F}$-stopping times $\mathcal{T}$,

$$ess\sup_{\Omega} \mathbb{E}[[M]_T - [M]_{\mathcal{T}} \mid \mathcal{F}_{\mathcal{T}}] \le K^2 \qquad \text{and} \qquad |\Delta \mathrm{M}_{\mathcal{T}}| \le \mathrm{K}^2\,,$$

where $[M]$ denotes a quadratic variation of a process $M$. For the diffusion case, the $BMO$-martingale property follows from the first condition, whilst in a jump-diffusion case, we need to ensure the boundedness of the jumps of the local martingale $M$.

Let $M$ be a martingale of the form (2.5). We define a stochastic exponential $\mathcal{E}(M)$ by

$$\begin{aligned}\mathcal{E}(M) &= \exp\Big\{-\frac{1}{2}\int_0^t \beta^2(s)ds + \int_0^t\int_{\mathbb{R}}[\ln(1+\gamma(s,z)) - \gamma(s,z)]\nu(dz)ds \\ &\quad + \int_0^t \beta(s)dW(s) + \int_0^t\int_{\mathbb{R}} \ln(1+\gamma(s,z))\tilde{N}(ds,dz)\Big\}.\end{aligned}$$

Then, the following lemma, which follows from the application of the Itô's formula and the martingale representation theorem, relates the martingale property of the stochastic exponential to a *BMO*-martingale.

**Lemma 2.3.3.** (Kazamaki criterion). *Let $M$ be a BMO-martingale satisfying $\Delta M_t > -1$ $\mathbb{P}$-a.s. for all $t \in [0,T]$. Then $\mathcal{E}(M)$ is a true martingale.*

## 2.4 Backward stochastic differential equations

The theory of backward stochastic differential equation (BSDE) has become an important tool for solving stochastic optimal control problems. The main part of this thesis solves various stochastic optimization problems based on the theory of BSDEs. In this section, we introduce the concept of BSDEs and state the mein results.

Given the data $(\xi, f)$, where $\xi : \Omega \to \mathbb{R}$ is an $\mathcal{F}_T$-measurable random variable and $f$ is a $\mathcal{P}\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R})$-measurable function. We consider the following BSDE

$$\begin{aligned} dY(t) &= -f(t,Y(t),Z(t),\Upsilon(t,z))dt + Z(t)dW(t) \\ &\quad + \int_{\mathbb{R}}\Upsilon(t,z)\widetilde{N}(dt,dz); \\ Y(T) &= \xi, \end{aligned} \tag{2.6}$$

where the processes $Z$ and $\Upsilon$ are called *control processes* as they control an adapted process $Y$ so that it satisfies the terminal condition $\xi$.

**Definition 2.4.1.** *A triple* $(Y, Z, \Upsilon) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_N(\mathbb{R})$ *is said to be a solution to a BSDE* (2.6) *if*

$$\begin{aligned} Y(t) &= \xi + \int_t^T f(s, Y(s-), Z(s-), \Upsilon(s-, \cdot))ds - \int_t^T Z(s)dW(s) \\ &\quad - \int_t^T \int_{\mathbb{R}} \Upsilon(s, z)\widetilde{N}(ds, dz)\,,\; 0 \le t \le T\,. \end{aligned}$$

A pair $(\xi, f)$ is said to be a standard data for BSDE (2.6), if the following assumptions hold:

**(C1)** the terminal value $\xi \in \mathbb{L}^2(\mathbb{R})$;

**(C2)** the generator $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times L^2_\nu(\mathbb{R}) \mapsto \mathbb{R}$ is predictable, i.e., $f \in \mathcal{P} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(L^2_\nu(\mathbb{R}))$ and Lipschitz continuous in the sense that,

$$\begin{aligned} |f(\omega, t, y, z, v) - f(\omega, t, y', z', v')|^2 &\le K(|y - y'|^2 + |z - z'|^2 \\ &\quad + \int_{\mathbb{R}} |v(z) - v'(z)|^2 \nu(dz))\,, \end{aligned}$$

a.s., $(\omega, t) \in \Omega \times [0, T]$ a.e. for all $(y, z, v)$, $(y', z', v') \in \mathbb{R} \times \mathbb{R} \times L^2_\nu(\mathbb{R})$;

**(C3)**

$$\mathbb{E}[\int_0^T |f(t, 0, 0)|^2 dt] < \infty\,.$$

The following theorem is a classical result for the existence and uniqueness of the solution to the Lipschitz BSDE (2.6). For the proof we refer to Delong [26] or Cohen and Elliott [16]. However a quadratic-exponential BSDE with jumps will also play an important role in solving our optimization problem in Chapter 4. The existence and uniqueness result for this type of BSDE's is established in Morlais [70, 71].

**Theorem 2.4.1.** *Let* $(\xi, f)$ *be a standard data. Then the BSDE* (2.6) *has a unique solution* $(Y, Z, \Upsilon) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_N(\mathbb{R})$.

The key result of the BSDE approach for stochastic optimal control problems is the so called comparison principle for BSDEs. As the name suggest, it compare the solution of two BSDEs It is stated as follows:

**Theorem 2.4.2.** (The Comparison Principle). *Let $(\xi, \xi')$ and $(f, f')$ be two standard data for two BSDEs of the form (2.6), with solutions $(Y, Z, \Upsilon)$, $(Y', Z', \Upsilon') \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_N(\mathbb{R})$ respectively. Moreover, suppose that*

- $\xi \geq \xi'$, $\mathbb{P}$*-a.s.*
- $f(\omega, t, y, z, \upsilon) \geq f(\omega, t, y', z', \upsilon')$, $dt \times d\mathbb{P}$*-a.s.*
- $f(\omega, t, y, z, \upsilon) - f(\omega, t, y, z, \upsilon') \leq \int_{\mathbb{R}} \varphi(t, z)(\upsilon(z) - \upsilon'(z))\nu(dz)$*, a.s.* $(\omega, t) \in \Omega \times [0, T]$ *a.e. for all* $(y, z, \upsilon)$, $(y, z, \upsilon') \in \mathbb{R} \times \mathbb{R} \times L^2_\nu(\mathbb{R})$*, where* $\varphi : \Omega \times [0, T] \times \mathbb{R} \mapsto (-1, \infty)$ *is a predictable process such that* $\int_{\mathbb{R}} |\varphi(t, z)|^2 \nu(dz)$ *ia uniformly bounded.*

*Then $Y(t) \geq Y'(t)$, $t \in [0, T]$. In addition, if for some $A \in \{\mathcal{F}_t\}_{t \in [0,T]}$ we also have $(Y(t) - Y'(t))\chi_A = 0$, then $Y = Y'$ on $A \times [t, T]$, i.e., if $Y$ and $Y'$ meet, they remain the same from then onwards.*

*Proof.* For the proof, see Delong [26], Theorem 3.2.1 or Cohen and Elliott [16], Theorem 19.3.4. □

## 2.5 Portfolio dynamics under jump-diffusion processes

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which is given an $M$-dimensional Brownian motion $W(t) = (W_1(t), \ldots, W_M(t))$ and an $\ell$-dimensional Poisson random measure $N(t, A) = (N_1(t, A), \ldots, N_\ell(t, A))$ with a Lévy measure $\nu(A) = (\nu_1(A), \ldots, \nu_\ell(A))$, such that $W$ and $N$ are independent. Here, $W(0) = 0$ and $N(0, \cdot) = 0$ almost surely. This section is adopted from (Karatzas and Sreve [52], Section 1.1).

We suppose the existence of a risk-free share (money market) with price $S_0(t)$, $0 \leq t \leq T$ strictly positive, $\{\mathcal{F}_t\}_{t \in [0,T]}$-adapted and continuous defined by

$$dS_0(t) = r(t)S_0(t)dt\,,\;\; S_0(0) = 1, \quad \forall t \in [0, T]\,, \tag{2.7}$$

where a random and time-dependent, $\{\mathcal{F}_t\}_{t\in[0,T]}$-measurable $r(t) \geq 0$ is called the risk-free interest rate at time $t \in [0,T]$.

We introduce $D$ stocks with price per share $S_1(t); \ldots ; S_D(t)$ which are continuous, strictly positive and satisfy the following jump-diffusion SDE

$$\begin{aligned} dS_n(t) &= S_n(t)\Big[\alpha_n(t)dt + \sum_{m=1}^{M} \beta_{nm}(t)dW_m(t) \\ &\quad + \sum_{k=1}^{\ell} \int_{\mathbb{R}} \gamma_{nk}(t, z_k)\widetilde{N}_k(dt, dz_k)\Big], \quad \forall t \in [0,T], \\ S_n(0) &= s_n > 0, \end{aligned} \tag{2.8}$$

where $\alpha_n : [0,T]\times\Omega \to \mathbb{R}$, $\beta_n : [0,T]\times\Omega \to \mathbb{R}^M$ and $\gamma_n : [0,T]\times\mathbb{R}^\ell\times\Omega \to \mathbb{R}^\ell$ are adapted processes, for $n = 1, \ldots, D$, such that (2.8) is well defined.

**Definition 2.5.1.** *A financial market, hereafter denoted by $\mathcal{M}$, consists of*

*(i) a probability space $(\Omega, \mathcal{F}, \mathbb{P})$;*

*(ii) a positive constant $T$ called the terminal time;*

*(iii) an $M$-dimensional Brownian motion $\{W(t), \{\mathcal{F}_t\}; 0 \leq t \leq T\}$ and an $\ell$-dimensional Poisson random measure $\{N(t,\cdot), \{\mathcal{F}_t\}; 0 \leq t \leq T\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$, where $\{\mathcal{F}_t\}_{t\in[0,T]}$ is a filtration, with $W$ independent of $N$;*

*(iv) a progressively measurable risk-free rate process $r(\cdot)$ satisfying*

$$\int_0^T |r(t)|dt < \infty, \quad a.s.;$$

*(v) a progressively measurable $D$-dimensional mean-rate of return process $\alpha(t)$ satisfying*

$$\int_0^T \|\alpha(t)\|dt < \infty, \quad a.s.;$$

*(vi) a progressively measurable, $D \times M$-matrix-valued volatility process $\beta(t)$ satisfying*

$$\sum_{n=1}^{D}\sum_{m=1}^{M}\int_0^T \beta_{nm}^2(t)dt < \infty, \quad a.s.;$$

*(vii) a progressively measurable, $D \times \ell$-matrix-valued jump-coefficients process $\gamma(t, \cdot)$ satisfying*

$$\sum_{n=1}^{D} \sum_{k=1}^{\ell} \int_0^T \gamma_{nk}^2(t, z_k)\nu_k(dz_k)dt < \infty\,, \quad a.s.\,;$$

*(viii) a vector of positive constant initial stock prices $S(0) = (s_1, \ldots, s_D)^T$.*

We consider a financial market $\mathcal{M}$ consisting of a risk-free asset given by (2.7) and $D$ risky shares given by (2.8). The main objective of this section is to derive the dynamics of the value of a so-called *self-financing* portfolio in continuous time. For more details see e.g., (Björk [10], Chapter 6 and Karatzas and Sreve [52], Section 1.2), where a diffusion framework has been considered.

Let $0 = t_0 < t_1 < \cdots < t_k = T$ be a partition of the interval $[0, T]$.

**Assumption 2.1.**

$h_n(t_m)$ = the number of shares of stock $n$ held during the period $[t_m, t_{m+1})$, for $n = 1, \ldots, D$ and $m = 0, \ldots, k-1$;

$h_0(t_m)$ = the number of shares held in the risk-free asset;

$c(t_m)$ = the amount spent on consumption during the period $[t_m, t_{m+1})$.

We also assume that for $n = 0, 1, \ldots, D$, the random variable $h_n(t_m)$ is $\{\mathcal{F}_{t_m}\}_{t\in[0,T]}$-measurable, i.e., anticipation of the future is not permitted.

Let us define the value of the portfolios $V$ by the stochastic difference equation

$$\begin{aligned} V(0) &= 0\,; \\ V(t_{m+1}) - V(t_m) &= \sum_{n=0}^{D} h_n(t_m)\left[S_n(t_{m+1}) - S_n(t_m)\right]; \quad m = 0, \ldots, k-1\,. \end{aligned}$$

Then $V(t_m)$ is the amount of the portfolios during the period $[0, t_m]$. On the other hand, the value of the portfolios at today's price is given by

$$V(t_m) = \sum_{n=0}^{D} h_n(t_m)S_n(t_m)\,; \quad m = 0, \ldots, M\,,$$

if and only if there is no exogenous infusion or withdrawal of funds on the interval $[0, T]$. In this case, the trading is called *self-financing*.

Suppose that $h(\cdot) = (h_0(\cdot), \dots, h_D(\cdot))^T$ is an $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted process defined on the interval $[0, T]$, not just on the partition points $t_0, \dots, t_k$. The associated value process is now defined by the initial condition $V(0) = 0$ and the SDE

$$dV(t) = \sum_{n=0}^{D} h_n(t) dS_n(t); \quad \forall t \in [0, T] \, . \tag{2.9}$$

If we consider that the cost for the consumption rate $c(t_m)$ is given by $c(t_m)(t_{m+1} - t_m)$, the value process in continuous time becomes

$$dV(t) = \sum_{n=0}^{D} h_n(t) dS_n(t) - c(t)dt; \quad \forall t \in [0, T] \, . \tag{2.10}$$

We then give a mathematical definition of the main concepts.

**Definition 2.5.2.** *Let $S_0(t)$ be a risk-free price process given by (2.7) and $(S_n(t),\, t \in [0, T])$ be the risky price process given by (2.8), $n = 1, \dots, D$.*

(1) *A portfolio strategy $(h_0(\cdot), h(\cdot))$ for the financial market $\mathcal{M}$ consists of an $\{\mathcal{F}_t\}_{t\in[0,T]}$-progressively measurable real valued process $h_0(\cdot)$ and an $\{\mathcal{F}_t\}_{t\in[0,T]}$-progressively measurable, $\mathbb{R}^D$-valued process $h(\cdot) = (h_1(\cdot), \dots, h_N(\cdot))^T$;*

(2) *the portfolio $h(\cdot)$ is said to be Markovian if it is of the form $h(t, S(t))$, for some function $h : [0, T] \times \mathbb{R}^{D+1} \to \mathbb{R}^{D+1}$, that is, the value of the portfolio depends on the current value of the share $S(t)$;*

(3) *the value process $V$ corresponding to the portfolio $h$ is given by*

$$V(t) = \sum_{n=0}^{D} h_n(t) S_n(t) \, ;$$

(4) *a consumption process is an $\mathcal{F}_t$-adapted one-dimensional process $\{c(t);\, t \in [0, T]\}$;*

(5) *a portfolio-consumption pair* $(h, c)$ *is called self-financing if the value process* $V$ *satisfies the condition*

$$dV(t) = \sum_{n=0}^{D} h_n(t) dS_n(t) - c(t)dt; \quad \forall t \in [0, T] \,.$$

For computational purposes, it is often convenient to describe a portfolio in relative terms, i.e., we specify the relative proportion of the total portfolio value which is invested in the stock.

Define

$$\pi_n(t) := \frac{h_n(t)S_n(t)}{V(t)}\,; \quad n = 1, \ldots, D$$

and $\pi(\cdot) = (\pi_1(\cdot), \ldots, \pi_D(\cdot))^T$, where

$$\pi_0(t) = 1 - \sum_{n=1}^{D} \pi_n(t)\,.$$

From (2.7) and (2.8), the value process (2.10) becomes

$$\begin{aligned} dV(t) \;=\; & \Big[V(t)\Big(r(t) + \sum_{n=1}^{D} \pi_n(t)(\alpha_n(t) - r(t))\Big) - c(t)\Big]dt \\ & + V(t) \sum_{n=1}^{D} \sum_{m=1}^{M} \pi_n(t) \beta_{nm}(t) dW_m(t) \\ & + V(t) \sum_{n=1}^{D} \sum_{k=1}^{\ell} \pi_n(t) \int_{\mathbb{R}} \gamma_{nk}(t, z_k) \widetilde{N}_k(dt, dz_k); \; 0 \leq t \leq T\,, \end{aligned} \tag{2.11}$$

where

$$\int_0^T |\pi^T(t)(\alpha(t) - r(t)\mathbf{1})|dt < \infty; \quad \int_0^T \|\pi(t)\beta(t)\|^2 dt < \infty \tag{2.12}$$

and

$$\int_0^T \int_{\mathbb{R}} \|\pi(t)\gamma(t, z)\|^2 \nu(dz) dt < \infty \tag{2.13}$$

hold almost surely, where $\mathbf{1}$ represents a $D$-dimensional vector of units $\mathbf{1} = (1, \ldots, 1)$ and $\alpha \in \mathbb{R}^D$, $\beta \in \mathbb{R}^{D \times M}$ and $\gamma \in \mathbb{R}^{D \times \ell}$.

**Remark.** The definition of the value process in (2.11) does not take into account any cost for trading. A market in which there are no transaction costs is called *frictionless*.
The Conditions (2.12)-(2.13) are imposed in order to ensure the existence of the integrals in (2.11).
$\pi_0(\cdot) < 0$, means that the investor is borrowing money from the money market. The position $\pi_n(\cdot)$; $n = 1, \ldots, D$ in stock $n$ may be negative, which corresponds to the short-selling of the stocks.

## 2.6 Life insurance contract

Our main focus is to incorporate a life insurance contract on our stochastic optimization problems in order to protect the investors dependent in a case of a premature death. This section introduces the concept of life insurance contract and the hazard function. For more details see e.g. Pliska and Ye [83], Rotar [86], Chapter 7 and Melnikov [63], Chapter 3.

**Definition 2.6.1.** *A general life insurance contract is a vector* $((\xi(t), \delta(t))_{t\in[0,T]})$ *of* $t$*-portfolios, where for any* $t \in [0, T]$*, the portfolio* $\xi(t)$ *is interpreted as a payment of the insurer to the insurant (benefit) and* $\delta(t)$ *as a payment of the insurant to the insurer (premium), respectively taking place at time* $t$*.*

Let $\tau$ be the random *lifetime* or *age-at-death* of an individual. Set $F(t) = \mathbb{P}(\tau \leq t)$, the distribution function of $\tau$. We assume that an individual is alive at time $t = 0$, that is, once has been born, his/her lifetime is not equal to zero ($F(0) = 0$). We define the *survival function* $\bar{F}(t)$, by

$$\bar{F}(t) = \mathbb{P}(\tau > t \mid \mathcal{F}_t) = 1 - F(t)\,,$$

where $\{\mathcal{F}_t\}_{t\in[0,T]}$ is the filtration at time $t$. Clearly, $\bar{F}(0) = 1$ because $F(0) = 0$.

Hereafter, we assume that the distribution function $F(t)$ is continuous, thus the distribution has density $f(t) = F'(t)$. For an infinitesimal $\epsilon > 0$, we have that

$$\mathbb{P}(t < \tau \leq t + \epsilon) = f(t) \cdot \epsilon \,. \tag{2.14}$$

Consider $\mathbb{P}(t < \tau \leq t + \epsilon \mid \tau \geq t)$, the probability that an individual under consideration will die within the interval $[t, t + \epsilon]$, given that he/she has survived $t$ years, i.e., $\tau \geq t$. From (2.14), the *force of mortality* or *a hazard function* of $\tau$ is defined by

$$\begin{aligned} \mu(t) &:= \lim_{\epsilon \to 0} \frac{\mathbb{P}(t < \tau \leq t + \epsilon \mid \tau \geq t)}{\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{\mathbb{P}(t < \tau \leq t + \epsilon)}{\epsilon \mathbb{P}(\tau \geq t)} \\ &= \frac{1}{\bar{F}(t)} \lim_{\epsilon \to 0} \frac{F(t + \epsilon) - F(t)}{\epsilon} \\ &= \frac{f(t)}{\bar{F}(t)} \\ &= -\frac{d}{dt}(\ln(\bar{F}(t))) \,, \end{aligned} \tag{2.15}$$

provided that $\bar{F}(t) \neq 0$, $\forall t$. If $\bar{F}(t) = 0$, the force of mortality $\mu(t) = \infty$ by definition. The larger $\mu(t)$ is equivalent to the larger the probability that an individual of age $t$ will die soon, i.e., within a small time interval $[t, t + \epsilon]$.

From (2.15), the survival function of an individual is given by

$$\bar{F}(t) = \exp\left( - \int_0^t \mu(s) ds \right) \tag{2.16}$$

and consequently, the conditional probability density of death of the individual under consideration at time $t$, by

$$f(t) = F'(t) = \mu(t) \exp\left( - \int_0^t \mu(s) ds \right) . \tag{2.17}$$

**Remark.** The filtration $\{\mathcal{F}_t\}_{t \in [0,T]}$ is defined in such a way that it includes the information from the market as well as the information of lifetime of an individual.

## 2.7 Utility functions

As this thesis is devoted to solving stochastic optimal control problems, utility functions are of crucial importance in these problems as they measure a relative satisfaction of an investor. we introduce and give the properties of the utility functions to be considered. For more details see e.g. Karatzas *et. al.* [51] or Karatzas and Shreve [52], Chapter 3.

**Definition 2.7.1.** *A utility function is a concave, non-decreasing, upper semi-continuous function $U : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:*

$(i)$ *the half-line dom$(U) := \{x \in (0, \infty) : U(x) > -\infty\}$ is a nonempty subset of $[0, \infty)$;*

$(ii)$ *the derivative $U'$ is continuous, positive and strictly decreasing on the interior of dom$(U)$ and*

$$U'(0) := \lim_{x \to 0} U'(x) = \infty\,, \quad U'(\infty) := \lim_{x \to \infty} U'(x) = 0\,. \tag{2.18}$$

Given a function $\lambda : \mathbb{R} \to \mathbb{R}$ defined by

$$\lambda(x) := -x\frac{U''(x)}{U'(x)}\,.$$

A utility function $U$ is said to be of Constant Relative Risk Aversion (CRRA) type if $\lambda$ is a constant.

**Example 2.7.1.**
We consider some common examples of utility functions.

$$U^{(\delta)}(x) := \begin{cases} x^\delta/\delta, & \text{if } x > 0\,, \\ \lim_{\epsilon \to 0} \epsilon^\delta/\delta, & \text{if } x = 0\,, \\ -\infty, & \text{if } x < \infty\,, \end{cases} \tag{2.19}$$

for $\delta \in (-\infty, 1) \setminus \{0\}$.

For $\delta = 0$, set

$$U^{(0)}(x) := \begin{cases} \ln x, & \text{if } x > 0\,, \\ -\infty, & \text{if } x \leq \infty\,. \end{cases} \tag{2.20}$$

We also define a utility function of the exponential type as follows:

$$U(x) = e^{-\kappa x}, \quad \kappa > 0\,.$$

**Definition 2.7.2.** *Let $U$ be a utility function. We define a strictly decreasing, continuous inverse function $I : (0, \infty) \to (0, \infty)$ by $I(y) := (U'(y))^{-1}$. By analogous with* (2.18)*, $I$ satisfies*

$$I(0) := \lim_{y \to 0} I(y) = \infty\,, \qquad I(\infty) := \lim_{y \to \infty} I(y) = 0\,. \tag{2.21}$$

Define a function

$$\widetilde{U}(y) := \max_{x>0}[U(x) - xy] = U(I(y)) - yI(y)\,, \quad 0 < y < \infty\,, \tag{2.22}$$

which is the convex dual of $-U(-x)$, with $U$ extended to be $-\infty$ on the negative real axis. The function $\widetilde{U}$ is strictly decreasing, strictly convex and satisfies

$$\begin{aligned} \widetilde{U}'(y) &= -I(y)\,, \quad 0 < y < \infty \\ U(x) &= \min_{y>0}[\widetilde{U}(y) + xy] = \widetilde{U}(U'(x)) + xU'(x), \quad 0 < x < \infty\,. \end{aligned} \tag{2.23}$$

Then from (2.22) and (2.23), we have the following useful inequalities:

$$U(I(y)) \geq U(x) + y[I(y) - x]\,, \quad \forall x > 0, y > 0\,, \tag{2.24}$$

$$\widetilde{U}(U'(x)) \leq \widetilde{U}(y) - x[U'(x) - y]\,, \quad \forall x > 0, y > 0\,. \tag{2.25}$$

# Chapter 3

# On optimal investment-consumption and life insurance with capital constraints.

## 3.1 Introduction

Optimal consumption-investment problem by Merton [66] ushered a lot of extensions. In 1975, Richard [85] extended for the first time this problem to include life insurance decisions. Other references include Huang *et al.* [47], Pliska and Ye [83], Liang and Guo [60]. Recently, Kronborg and Steffensen [55] extended this problem to include capital constraints, previously introduced by Teplá [95] and El Karoui *et. al.* [30]. Most of the references mentioned above solved the problem under a diffusion framework.

As was pointed out by Merton and many empirical data, the analysis of the price evolution reveals some sudden and rare breaks (jumps) caused by external information flow. These behaviours constitute a very real concern of most investors. They can be modeled by a Poisson process, which has jumps occurring at rare and unpredictable time. For detailed information see e.g., Jeanblanc-Picque and Pontier [48], Runggaldier [87], Daglish [21], Øksendal and Sulem [77], Hanson [43] and references therein.

In this chapter, we consider a jump-diffusion problem with stochastic volatility as in Mnif [68]. In his paper, Mnif [68] solved the portfolio optimization problem using the dynamic programming approach. Applying this technique in a jump-diffusion model, the Hamilton-Jacobi-Bellman (HJB) equation associated to the problem is nonlinear, which in general the explicit solution is not provided. To prove the existence of a smooth solution, he reduced the nonlinearity of the HJB equation to a semi-linear equation under certain conditions. Here, we use a martingale approach developed by Karatzas *et al.* [50] and Karatzas and Shreve [52] in a diffusion process to solve the unrestricted problem. Then we solve a constrained optimization problem, where the constraint is of American put type. Considering a jump-diffusion model, a market is incomplete and consequently we have many martingale measures. We obtain the optimal investment, consumption and life insurance strategy by the convex optimization method. This method allow us to characterize the optimal martingale measure for the utility functions of the power type. In the literature, this method has also been applied by Castaneda-Leyva and Hernández-Hernández [13] in a optimal investment-consumption problem. They considered a stochastic volatility model described by diffusion processes. Similar works include (Liang and Guo [60], Michelbrink and Le [67] and references therein).

The optimal solution to the restricted problem is derived from the unrestricted optimal solution, applying the option based portfolio insurance (OBPI) method developed by El Karoui *et al.* [30]. The OBPI method consists in taking a certain part of capital and invest in the optimal portfolio of the unconstrained problem and the remaining part insures the position with American put. We prove the admissibility and the optimality of the strategy.

The structure of this chapter is organized as follows. In Section 3.2, we introduce the model and problem formulation of the Financial and the Insurance markets. In Section 3.3, we solve the unconstrained problem and a power utility function is considered. In Section 3.4, we solve the constrained problem and prove the admissibility of our strategy.

## 3.2 The Financial Model

We consider two dimensional Brownian motion $W = \{W_1(t); W_2(t),\ 0 \le t \le T\}$ associated to the complete filtered probability space $(\Omega^W, \mathcal{F}^W, \{\mathcal{F}_t^W\}, \mathbb{P}^W)$ such that $\{W_1(t), W_2(t)\}$ are correlated with the correlation coefficient $|\varrho| < 1$, that is, $dW_1(t) \cdot dW_2(t) = \varrho dt$. Moreover, we consider a Poisson process $N = \{N(t), \{\mathcal{F}_t^N\}, 0 \le t \le T\}$ associated to the complete filtered probability space $(\Omega^N, \mathcal{F}^N, \{\mathcal{F}_t^N\}, \mathbb{P}^N)$ with intensity $\lambda(t)$ and a $\mathbb{P}^N$-martingale compensated poisson process

$$\tilde{N}(t) := N(t) - \int_0^t \lambda(t)dt\,.$$

We assume that the intensity $\lambda(t)$ is Lebesgue integrable on $[0, T]$.

We consider the product space:

$$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P}) := (\Omega^W \times \Omega^N, \mathcal{F}^W \otimes \mathcal{F}^N, \{\mathcal{F}_t^W \otimes \mathcal{F}_t^N\}, \mathbb{P}^W \otimes \mathbb{P}^N)$$

where $\{\mathcal{F}_t\}_{t\in[0,T]}$ is a filtration satisfying the usual conditions. On this space, we assume that $W$ and $N$ are independent processes.

We consider a financial market consisting of a risk-free asset $B := (B(t)_{t\in[0,T]})$, a non-tradable index $Z := (Z(t)_{t\in[0,T]})$ which can be thought as an external economic factor, such as a temperature, a loss index or a volatility driving factor and a risky asset $S := (S(t))_{t\in[0,T]}$ correlated with $Z(t)$. This market is defined by the following jump-diffusion model:

$$\begin{aligned} dB(t) &= r(t)B(t)dt, \quad B(0) = 1, && (3.1)\\ dZ(t) &= \eta(Z(t))dt + dW_1(t), && (3.2)\\ dS(t) &= S(t)\Big[\alpha(t, Z(t))dt + \beta(t, Z(t))dW_1(t) + \sigma(t, Z(t))dW_2 && (3.3)\\ &\qquad + \gamma(t, Z(t))dN(t)\Big], \quad S(0) = s > 0\,, \end{aligned}$$

where $r : [0, T] \to \mathbb{R}$; $\alpha, \beta, \sigma, \gamma : [0, T] \times \mathbb{R} \to \mathbb{R}$ are measurable $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted bounded processes and $\gamma(t, R(t)) > -1$. With the latter condition and the continuity of $Z$, we guarantee that (4.12) is well defined.

To ensure the existence and uniqueness of the solution to (3.2), we assume a Lipschitz condition on the $\mathbb{R}$-valued function $\eta$:

(**A1**) There exists a positive constant $C$ such that

$$|\eta(y) - \eta(w)| \leq C|y - w|\,, \qquad y, w \in \mathbb{R}\,.$$

Under the above assumption, the solution to the SDE (3.2) is given by

$$Z(t) = z_0 + \int_0^t \eta(Z(s))ds + \int_0^t dW_1(s)\,. \tag{3.4}$$

Let us consider a policyholder whose lifetime is a nonnegative random variable $\tau$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and independent of the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. Moreover, suppose that $c(t)$ is the consumption rate of the policyholder, $\pi(t)$ the amount of the policyholder's wealth invested in the risky asset $S$ and $p(t)$ the sum insured to be paid out at time $t \in [0, T]$ for the life insurance upon the wage earner's death before time $T$. We assume that the strategy $(c(t), \pi(t), p(t))$ satisfies the following definition:

**Definition 3.2.1.** *The consumption rate $c$ is measurable, $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted process, nonnegative and*

$$\int_0^T c(t)dt < \infty, \quad a.s.$$

*The allocation process $\pi$ is an $\mathcal{F}_t$-predictable process with*

$$\int_0^t \pi^2(t)dt < \infty, \quad a.s.$$

*The insurance process $p$ is measurable, $\mathcal{F}_t$-adapted process, nonnegative and*

$$\int_0^T p(t)dt < \infty, \quad a.s.$$

Suppose that the policyholder receives a deterministic labor income of rate $\ell(t) \geq 0, \forall t \in [0, \tau \wedge T]$ and that the shares are divisible and can be traded continuously. Furthermore, we assume that there are no transaction costs, taxes or short-selling constraints in the trading, then after some calculations, the wealth process $X(t)\,,\ t \in [0, \tau \wedge T]$ is defined by the following SDE:

$$
\begin{aligned}
dX(t) &= [(r(t) + \mu(t))X(t) + \pi(t)(\alpha(t, Z(t)) - r(t)) + \ell(t) \\
&\quad -c(t) - \mu(t)p(t)]dt + \pi(t)\beta(t, Z(t))dW_1(t) \\
&\quad +\pi(t)\sigma(t, Z(t))dW_2(t) + \pi(t)\gamma(t, Z(t))dN(t)\,, \\
X(0) &= x > 0,
\end{aligned}
\tag{3.5}
$$

where $Z(t)$ is given by (3.4) and $\tau \wedge T := \min\{\tau, T\}$.

The expression $\mu(t)(p(t) - X(t))dt$ from the wealth process (3.5), corresponds to the risk premium rate to pay for the life insurance $p$ at time $t$. Notice that choosing $p > X$ corresponds to buying a life insurance and $p < X$ corresponds to selling a life insurance, that is buying an annuity (Kronborg and Steffensen [55]).

From Definition 3.2.1 and the conditions of $r, \alpha, \beta, \sigma, \gamma$, we see that the wealth process (3.5) is well defined and has a unique solution given by

$$
\begin{aligned}
X(t) &= x_0 e^{\int_0^t (r(s)+\mu(s))ds} + \int_0^t e^{\int_s^t (r(u)+\mu(u))du}\Big[\pi(s)(\alpha(s, Z(s)) - r(s)) \\
&\quad +\ell(s) - c(s) - \mu(s)p(s)\Big]ds + \int_0^t \pi(s)\beta(s, Z(s))e^{\int_s^t (r(u)+\mu(u))du}dW_1(s) \\
&\quad + \int_0^t \pi(s)\sigma(s, Z(s))e^{\int_s^t (r(u)+\mu(u))du}dW_2(s) \\
&\quad + \int_0^t \pi(s)\gamma(s, Z(s))e^{\int_s^t (r(u)+\mu(u))du}dN(s)\,.
\end{aligned}
\tag{3.6}
$$

We define a new probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ in which $S$ is a local martingale. The Radom-Nikodym derivative is given by:

$$\begin{aligned}\Lambda(t) \;:=\; & \exp\Big\{\int_0^t [(1-\psi(s))\lambda(s) - \frac{1}{2}\theta^2(s,Z(s),\psi(s)) - \frac{1}{2}\nu^2(s,\psi(s))]ds \\ & + \int_0^t \nu(s,Z(s),\psi(s))dW_1(s) + \int_0^t \theta(s,Z(s),\psi(s))dW_2(s) \\ & + \int_0^t \ln(\psi(s))dN(s)\Big\}. \end{aligned} \tag{3.7}$$

By Girsanov's Theorem, under $\mathbb{Q}$, we have that:

$$\begin{cases} dW_1^{\mathbb{Q},\psi}(t) = dW_1(t) - \nu(t,Z(t),\psi(t))dt\,, \\ dW_2^{\mathbb{Q},\psi}(t) = dW_2(t) - \theta(t,Z(t),\psi(t))dt\,, \\ d\tilde{N}^{\mathbb{Q}}(t) = dN(t) - \psi(t)\lambda(t)dt \end{cases}$$

are Brownian motions and compensated poisson random measure respectively, where (See Runggaldier [87], Section 4.)

$$\nu(t,Z(s),\psi(t)) = \frac{\beta(t,Z(t))}{\beta^2(t,Z(t))+\sigma^2(t,Z(t))}(r(t)-\alpha(t,Z(t))-\gamma(t,Z(t))\psi(t)\lambda(t))\,, \tag{3.8}$$

$$\theta(t,Z(t),\psi(t)) = \frac{\sigma(t,Z(t))}{\beta^2(t,Z(t))+\sigma^2(t,Z(t))}(r(t)-\alpha(t,Z(t))-\gamma(t,Z(t))\psi(t)\lambda(t))\,, \tag{3.9}$$

for any $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted bounded $\psi > 0$. We assume that $\beta^2(t,Z(t)) + \sigma^2(t,Z(t)) \neq 0$. Thus we have infinitely many martingale measures and consequently incomplete market.

Note that, from the boundedness of the associated parameters, the predictable processes $\nu, \theta$, are bounded. Then, one can prove that the stochastic exponential (3.7) is a positive martingale (see Delong [26], Proposition 2.5.1.).

From (3.8) and (3.9), we have that:

$$\begin{aligned} & [\pi(t)(\alpha(t,Z(t)) - r(t)) + \pi(t)\beta(t,Z(t))\nu(t,Z(t),\psi(t)) \\ & +\pi(t)\sigma(t,Z(t))\theta(t,Z(t),\psi(t)) + \pi(t)\gamma(t,Z(t))\psi(t)\lambda(t)] \;=\; 0\,, \end{aligned}$$

then under $\mathbb{Q}$, the dynamics of the wealth process is given by

$$\begin{aligned} dX(t) &= [(r(t)+\mu(t))X(t)+\ell(t)-c(t)-\mu(t)p(t)]\,dt \\ &\quad +\pi(t)\beta(t,Z(t))dW_1^{\mathbb{Q},\psi}(t)+\pi(t)\sigma(t,Z(t))dW_2^{\mathbb{Q},\psi}(t) \qquad (3.10) \\ &\quad +\pi(t)\gamma(t,Z(t))d\tilde{N}^{\mathbb{Q},\psi}(t)\,, \end{aligned}$$

which gives the following representation:

$$\begin{aligned} X(t) &= x_0 e^{\int_0^t (r(s)+\mu(s))ds} + \int_0^t e^{\int_s^t (r(u)+\mu(u))du}\Big[\ell(s)-c(s)-\mu(s)p(s)\Big]ds \\ &\quad + \int_0^t \pi(s)\beta(s,Z(s))e^{\int_s^t (r(u)+\mu(u))du}dW_1^{\mathbb{Q},\psi}(s) \\ &\quad + \int_0^t \pi(s)\sigma(s,Z(s))e^{\int_s^t (r(u)+\mu(u))du}dW_2^{\mathbb{Q},\psi}(s) \\ &\quad + \int_0^t \pi(s)\gamma(s,Z(s))e^{\int_s^t (r(u)+\mu(u))du}d\tilde{N}^{\mathbb{Q},\psi}(s)\,. \qquad (3.11) \end{aligned}$$

The following definition introduces the concept of admissible strategy.

**Definition 3.2.2.** *Define the set of admissible strategies $\{\mathcal{A}\}$ as the consumption, investment and life insurance strategies for which the corresponding wealth process given by (3.11) is well defined and*

$$X(t)+g(t)\geq 0, \quad \forall t\in[0,T], \tag{3.12}$$

*where $g$ is the time $t$ actuarial value of the future labor income defined by*

$$g(t):=\mathbb{E}\left[\int_t^T e^{-\int_t^s (r(u)+\mu(u))du}\ell(s)ds \mid \mathcal{F}_t\right]. \tag{3.13}$$

Since

$$\begin{aligned} \mathbb{E}^{\mathbb{Q},\psi}\left[\int_0^t \pi(s)\beta(s,Z(s))e^{\int_s^t (r(u)+\mu(u))du}dW_1^{\mathbb{Q},\psi}(s)\right] &= 0\,, \qquad (3.14) \\ \mathbb{E}^{\mathbb{Q},\psi}\left[\int_0^t \pi(s)\sigma(s,Z(s))e^{\int_s^t (r(u)+\mu(u))du}dW_2^{\mathbb{Q},\psi}(s)\right] &= 0\,, \qquad (3.15) \\ \mathbb{E}^{\mathbb{Q},\psi}\left[\int_0^t \pi(s)\gamma(s,Z(s))e^{\int_s^t (r(u)+\mu(u))du}d\tilde{N}^{\mathbb{Q},\psi}(s)\right] &= 0\,, \qquad (3.16) \end{aligned}$$

we see that the last three terms in (3.11) are $\mathbb{Q}$ local martingales and from (3.12), a supermartingale (see e.g., Karatzas *et al.* [51]). Then, the strategy $(c, \pi, p)$ is admissible if and only if $X(T) \geq 0$ and $\forall t \in [0, T]$,

$$\begin{aligned} X(t) + g(t) &= \mathbb{E}^{\mathbb{Q},\psi}\Big[\int_t^T e^{-\int_t^s (r(u)+\mu(u))du}[c(s) + \mu(s)p(s)]ds \quad (3.17) \\ &\quad + e^{-\int_t^T (r(u)+\mu(u))du} X(T) \mid \mathcal{F}_t\Big]. \end{aligned}$$

At time zero ($t = 0$), this means that the strategies have to fulfill the following budget constraint:

$$\begin{aligned} X(0) + g(0) &= \mathbb{E}^{\mathbb{Q},\psi}\Big[\int_0^T e^{-\int_0^t (r(u)+\mu(u))du}[c(t) + \mu(t)p(t)]dt \quad (3.18) \\ &\quad + e^{-\int_0^T (r(u)+\mu(u))du} X(T)\Big]. \end{aligned}$$

Note that the condition (3.12) allows the wealth to become negative, as long as it does not exceed in absolute value the actuarial value of future labor income $g(t)$ in (3.13) so that it prevent the family from borrowing against the future labor income.

As in Kronborg and Steffensen [55], the following remark is useful for the rest of the chapter.

**Remark.** Define for any $t \in [0, T]$

$$\begin{aligned} &Y(t) \quad (3.19) \\ &:= \int_0^t e^{-\int_0^s (r(u)+\mu(u))du}[c(s) + \mu(s)p(s) - \ell(s)]ds + X(t)e^{-\int_0^t (r(u)+\mu(u))du}. \end{aligned}$$

By (3.11) we have that the Conditions (3.14), (3.15) and (3.16) are fulfilled if and only if $Y$ is a martingale under $\mathbb{Q}$. The natural interpretation is that, under $\mathbb{Q}$, the discounted wealth plus discounted pension contributions should be martingales. It is useful to note that if $Y$ is a martingale under $\mathbb{Q}$, the dynamics of $X$ can be represented in the following form:

$$\begin{aligned} dX(t) &= [(r(t) + \mu(t))X(t) + \ell(t) - c(t) - \mu(t)p(t)]dt + \phi_1(t)dW_2^{\mathbb{Q},\psi}(t) \\ &\quad + \phi_2(t)dW_2^{\mathbb{Q},\psi}(t) + \varphi(t)d\tilde{N}^{\mathbb{Q},\psi}(t), \quad (3.20) \end{aligned}$$

for some $\{\mathcal{F}_t^W\}_{t\in[0,T]}$-adapted processes $\phi_1, \phi_2$ and $\{\mathcal{F}_t^N\}_{t\in[0,T]}$-adapted process $\varphi$, satisfying $\phi_1(t), \phi_2(t), \varphi(t) \in L^2$, for any $t \in [0,T]$, then under $\mathbb{Q}$, $Y$ is a martingale.

## 3.3 The Unrestricted problem

In this section, we solve our main optimization problem using the martingale duality method. Consider the concave, non-decreasing, upper semi-continuous and differentiable w.r.t. the second variable utility functions $U_k : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+\,, \quad k = 1,2,3\,.$

Let $\rho(t)$ be a deterministic function representing the policyholder's time preferences. The policyholder chooses his strategy $(c(t), \pi(t), p(t))$ in order to optimize the expected utility from consumption, legacy upon death and terminal pension. His strategy, therefore, fulfils the following:

$$\begin{aligned} & J(x,c,\pi,p) \\ := \; & \sup_{(\pi,c,p)\in\mathcal{A}'} \mathbb{E}\Big[\int_0^{\tau\wedge T} e^{-\int_0^s \rho(u)du} U_1(s,c(s))ds + e^{-\int_0^\tau \rho(u)du} U_2(\tau,p(\tau))\chi_{\{\tau\le T\}} \\ & \qquad + e^{-\int_0^T \rho(u)du} U_3(X(T))\chi_{\{\tau>T\}}\Big]. \end{aligned} \tag{3.21}$$

Here $\chi_A$ is an indicator function of set $A$. $\mathcal{A}'$ is the subset of the admissible strategies (feasible strategies) given by:

$$\begin{aligned} \mathcal{A}' \; := \; & \Big\{(c,\pi,p) \in \mathcal{A} \mid \mathbb{E}\Big[\int_0^{\tau\wedge T} e^{-\int_0^s \rho(u)du} \min(0, U_1(s,c(s)))ds \\ & + e^{-\int_0^\tau \rho(u)du} \min(0, U_2(\tau,p(\tau)))\chi_{\{\tau\le T\}} \\ & + e^{-\int_0^T \rho(u)du} \min(0, U_3(X(T)))\chi_{\{\tau>T\}}\Big] > -\infty\Big\}. \end{aligned} \tag{3.22}$$

The feasible strategy (3.22) means that it is allowed to draw an infinite utility from the strategy $(\pi,c,p) \in \mathcal{A}'$, but only if the expectation over the negative parts of the utility function is finite. It is clear that for a positive utility function, the sets $\mathcal{A}$ and $\mathcal{A}'$ are equal ( see eg., Kronborg and

Steffensen [55]). In order to solve the unrestricted control problem (3.21) one can use the Hamilton-Jacobi-Bellman (HJB) equation (e.g., Mnif [68]) or the combination of HJB equation with backward stochastic differential equation (BSDE) with jumps (Guambe and Kufakunesu [41]). In this Chapter, we use the duality martingale approach applied in (Karatzas *et al.* [51], Castaneda-Leyva and Hernández-Hernández [13], Kronborg and Steffensen [55]). This is due to the incompleteness of the market and the restricted problem in the next section, where its terms are derived from the martingale method in the unrestricted problem.

From (2.16) and (2.17), we can rewrite the policyholder's optimization problem (3.21) as (See, for example, Kronborg and Steffensen [55]):

$$\begin{aligned}
& J(x, c^*, \pi^*, p^*) \\
= & \sup_{(c,\pi,p)\in\mathcal{A}'} \mathbb{E}\Big[\int_0^T e^{-\int_0^s \rho(u)du}[\bar{F}(s)U_1(s,c(s)) + f(s)U_2(s,p(s))]ds \\
& \qquad + e^{-\int_0^T \rho(u)du}\bar{F}(T)U_3(X(T))\Big].
\end{aligned}$$

Hence,

$$\begin{aligned}
& J(x, c^*, \pi^*, p^*) \\
= & \sup_{(c,\pi,p)\in\mathcal{A}'} \mathbb{E}\Big[\int_0^T e^{-\int_0^s (\rho(u)+\mu(u))du}[U_1(s,c(s)) + \mu(s)U_2(s,p(s))]ds \\
& \qquad + e^{-\int_0^T (\rho(u)+\mu(u))du}U_3(X(T))\Big]. \qquad (3.23)
\end{aligned}$$

We now solve the main problem using the duality method. This approach allow us construct an auxiliary market $\mathcal{M}_{\hat{\psi}}$ related to the original one, by searching over a family of martingale measures, the *inf-sup* martingale measure $\hat{\psi}$ and so the hedging portfolio process in the auxiliary market, satisfies the portfolio constraints in the original market $\mathcal{M}_{\psi}$ and replicates exactly the contingent claim almost surely. This approach has been applied under diffusion in a number of papers, see, for instance, He and Pearson [44], Karatzas and Shreve [52], Section 5.8, Castañeda-Leyva and Hernandez-Hernandez [13], Liang and Guo [60]. Otherwise, one can complete the market by adding factitious risky assets in order to obtain a complete market, then apply the martingale approach to solve the optimal portfolio problem. For the market completion, we refer to Karatzas *et. al.* [51], Runggaldier [87], Section 4., Corcuera *et. al.* [18].

Thus, we define the associated dual functional $\Psi(\zeta, \psi)$ to the primal problem (3.23), where $\zeta$ is the Lagrangian multiplier, by:

$$\begin{aligned}
&\Psi(\hat{\zeta}, \hat{\psi}) \\
:= &\sup_{\zeta>0;\psi>0} \Big\{ \mathbb{E}\Big[ \int_0^T e^{-\int_0^s (\rho(u)+\mu(u))du} [U_1(s, c(s)) + \mu(s) U_2(s, p(s))] ds \\
&\qquad + e^{-\int_0^T (\rho(u)+\mu(u))du} U_3(X(T)) \Big] + \zeta(x + g(0)) \\
&- \zeta \Big\{ \mathbb{E}^{\mathbb{Q},\psi} \Big[ \int_0^T e^{-\int_0^t (r(u)+\mu(u))du} [c(t) + \mu(t) p(t)] dt \\
&\qquad + e^{-\int_0^T (r(u)+\mu(u))du} X(T) \Big] \Big\} \Big\}.
\end{aligned}$$

The dual problem that corresponds to the primal problem (3.23), consists of

$$\inf_{\zeta>0,\psi>0} \Psi(\zeta, \psi). \tag{3.24}$$

Note that (see Cuoco [20] or Karatzas *et al* [52], for more details)

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q},\psi} \Big[ \int_0^T e^{-\int_0^t (r(u)+\mu(u))du} [c(t) + \mu(t) p(t)] dt + e^{-\int_0^T (r(u)+\mu(u))du} X(T) \Big] \\
= &\mathbb{E}\Big[ \int_0^T e^{-\int_0^t (r(u)+\mu(u))du} \Gamma^\psi(t) [c(t) + \mu(t) p(t)] dt \\
&\qquad + e^{-\int_0^T (r(u)+\mu(u))du} \Gamma^\psi(T) X(T) \Big],
\end{aligned}$$

where we have defined the adjusted state price deflator $\Gamma$ by:

$$\begin{aligned}\Gamma^{\psi}(t) &:= \Lambda(t)e^{\int_0^t(\rho(s)-r(s))ds}\\ &= \exp\Big\{\int_0^t[\rho(s)-r(s)-\frac{1}{2}\theta^2(s,Z(s),\psi(s))-\frac{1}{2}\nu^2(s,Z(s),\psi(s))\\ &\quad+(1-\psi(s))\lambda(s)]ds+\int_0^t\nu(s,Z(s),\psi(s))dW_1(s)\\ &\quad+\int_0^t\theta(s,Z(s),\psi(s))dW_2(s)+\int_0^t\ln(\psi(s))dN(s)\Big\}, \qquad (3.25)\end{aligned}$$

which can be written in the SDE form by:

$$\begin{aligned}d\Gamma^{\psi}(t) &= \Gamma^{\psi}(t)\Big[(\rho(t)-r(t))dt+\nu(t,Z(s),\psi(t))dW_1(t) \qquad (3.26)\\ &\quad+\theta(t,R(t),\psi(t))dW_2(t)+(\psi(t)-1)d\tilde{N}(t)\Big].\end{aligned}$$

Then, from the definition of the Legendre-Transform (2.22), the dual functional $\Psi$ in (3.24) can be written as

$$\begin{aligned}\Psi(\zeta,\psi) &:= \mathbb{E}\Big[\int_0^T e^{-\int_0^s(\rho(u)+\mu(u))du}[\widetilde{U}_1(s,c(s))+\mu(s)\widetilde{U}_2(s,p(s))]ds\\ &\quad+e^{-\int_0^T(\rho(u)+\mu(u))du}\widetilde{U}_3(X(T))\Big]+\zeta(x+g(0)) . \qquad (3.27)\end{aligned}$$

The following theorem shows, under suitable conditions the relationship between the primal problem (3.23) and the dual problem (3.24).

**Theorem 3.3.1.** *Suppose that $\hat{\psi}>0$ and $\hat{\zeta}>0$. The strategy $(c^*(t),p^*(t))\in\mathcal{A}'$ and $X^*(T)>0$ defined by*

$$c^*(t)=I_1(t,\hat{\zeta}\Gamma^{\hat{\psi}}(t));\quad p^*(t)=I_2(t,\hat{\zeta}\Gamma^{\hat{\psi}}(t));\quad X^*(T)=I_3(\hat{\zeta}\Gamma^{\hat{\psi}}(T)),$$

*such that (3.18) is fulfilled, where $X^*(T)\in\{\mathcal{F}_T\}$ is measurable, is the optimal solution to the primal problem (3.23), while $(\hat{\psi},\hat{\zeta})$ is the optimal solution to the dual problem (3.24).*

*Proof.* By the concavity of the utility functions $U_k$, $k=1,2,3$, (see Karatzas *et al* [52]), we know that

$$U_k(t,x)\le U(t,I_k(t,x))-y(I_k(t,y)-x) .$$

Then it can be easily shown that

$$J(t,c(t),p(t),X(t)) \leq \inf_{\zeta>0,\psi>0} \Psi(\zeta,\psi)\,. \tag{3.28}$$

Hence, to finish the proof, we need to show that

$$\inf_{\zeta>0,\psi>0} \Psi(\zeta,\psi) \geq J(t,c(t),p(t),X(t))\,.$$

From (3.27), we know that

$$\begin{aligned}
&\inf_{\zeta>0,\psi>0} \Psi(\zeta,\psi)\\
=\;& \inf_{\zeta>0,\psi>0}\Big\{\mathbb{E}\Big[\int_0^T e^{-\int_0^s(\rho(u)+\mu(u))du}[\widetilde{U}_1(s,c(s))+\mu(s)\widetilde{U}_2(s,p(s))]ds\\
&\qquad\qquad +e^{-\int_0^T(\rho(u)+\mu(u))du}\widetilde{U}_3(X(T))\Big]+\zeta(x+g(0))\Big\}\\
\leq\;& \mathbb{E}\Big[\int_0^T e^{-\int_0^s(\rho(u)+\mu(u))du}[\widetilde{U}_1(s,c(s))+\mu(s)\widetilde{U}_2(s,p(s))]ds\\
&+e^{-\int_0^T(\rho(u)+\mu(u))du}\widetilde{U}_3(X(T))\Big]+\hat{\zeta}(x+g(0))\\
=\;& \mathbb{E}\Big[\int_0^T e^{-\int_0^s(\rho(u)+\mu(u))du}[U_1(s,c^*(s))+\mu(s)U_2(s,p^*(s))]ds\\
&+e^{-\int_0^T(\rho(u)+\mu(u))du}U_3(X^*(T))\Big]\\
&-\hat{\zeta}\Big\{\mathbb{E}\Big[\int_0^T e^{-\int_0^t(r(u)+\mu(u))du}\Gamma^{\hat{\psi}}(t)[c^*(t)+\mu(t)p^*(t)]dt\\
&+e^{-\int_0^T(r(u)+\mu(u))du}\Gamma^{\hat{\psi}}(T)X^*(T)\Big]\Big\}+\hat{\zeta}(x+g(0))\\
=\;& \mathbb{E}\Big[\int_0^T e^{-\int_0^s(\rho(u)+\mu(u))du}[U_1(s,c^*(s))+\mu(s)U_2(s,p^*(s))]ds\\
&\qquad\qquad +e^{-\int_0^T(\rho(u)+\mu(u))du}U_3(X^*(T))\Big]\\
\leq\;& \sup_{(c,p,\pi)\in\mathcal{A}'}\Big\{\mathbb{E}\Big[\int_0^T e^{-\int_0^s(\rho(u)+\mu(u))du}[U_1(s,c(s))+\mu(s)U_2(s,p(s))]ds\\
&\qquad\qquad +e^{-\int_0^T(\rho(u)+\mu(u))du}U_3(X(T))\Big]\Big\}\\
=\;& J(t,c(t),p(t),X(t))\,.
\end{aligned}$$

Then, using (3.28), we conclude the proof, i.e., $(c^*(t),p^*(t),X^*(T))$ is the optimal solution to the primal problem (3.23) and $(\hat{\psi},\hat{\zeta})$ is the optimal solution to the dual problem (3.24). □

**Remark.** Note that the optimal $(\hat{\psi}, \hat{\zeta})$ is not necessarily unique, thus for different choice of the initial wealth, one might obtain different $\hat{\psi}$ and $\hat{\zeta}$.

### 3.3.1 Results on the power utility case

In this section, we intend to derive the explicit solutions for the utility functions of the CRRA type given by:

$$U_1(t,x) = U_2(t,x) = U_3(t,x) = \begin{cases} \frac{e^{-\kappa t}}{\delta} x^\delta, & \text{if } \ x > 0, \\ \lim_{x\to 0} \frac{e^{-\kappa t}}{\delta} x^\delta, & \text{if } \ x = 0, \\ -\infty, & \text{if } \ x < 0, \end{cases} \tag{3.29}$$

for some $\delta \in (-\infty, 1) \setminus \{0\}$ and $t \in [0, T]$. Thus the inverse function $I_k$ is given by

$$I_k(t,x) = e^{-\frac{\kappa}{1-\delta} t} x^{-\frac{1}{1-\delta}}, \quad k = 1,2,3 \tag{3.30}$$

and the corresponding Legendre-Transform $\widetilde{U}_k$ by

$$\widetilde{U}_k(t,x) = U_k(t, I_k(t,x)) - x I_k(t,x) = \frac{1-\delta}{\delta} e^{-\frac{\kappa}{1-\delta} t} x^{-\frac{\delta}{1-\delta}}, \quad k = 1,2,3\,.$$

We define a function $\mathcal{N}(\psi)$ by

$$\begin{aligned} \mathcal{N}(\psi) \quad := \quad & \mathbb{E}\Big[\int_0^T e^{-\int_0^t (\rho(u)+\mu(u)+\frac{\kappa}{1-\delta} u) du} [\Gamma^\psi(t)]^{-\frac{\delta}{1-\delta}} [1+\mu(t)] dt \\ & + e^{-\int_0^T (\rho(u)+\mu(u)+\frac{\kappa}{1-\delta} u) du} [\Gamma^\psi(T)]^{-\frac{\delta}{1-\delta}} \Big]\,. \end{aligned} \tag{3.31}$$

Then the dual functional (3.27) is given by

$$\Psi(\zeta,\psi) = \frac{1-\delta}{\delta} \zeta^{-\frac{\delta}{1-\delta}} \mathcal{N}(\psi) + \zeta(x + g(0))\,. \tag{3.32}$$

Fixing $\psi > 0$ and taking the minimum on (3.32), we obtain the optimal $\hat{\zeta}$, given by

$$\hat{\zeta} = \left[\frac{x + g(0)}{\mathcal{N}(\psi)}\right]^{\delta - 1} .$$

Inserting this optimal $\hat{\zeta}$ to the above equation, we obtain

$$\Psi(\hat{\zeta}, \psi) = \frac{1}{\delta}(x + g(0))^{\delta} \mathcal{N}^{1-\delta}(\psi) .$$

Now, solving the dual problem (3.24) is equivalent to solving the following value function problem

$$V(t, Z(t)) = \inf_{\psi > 0} \mathcal{N}(\psi) , \quad \delta > 0 \tag{3.33}$$

or

$$V(t, Z(t)) = \sup_{\psi > 0} \mathcal{N}(\psi) , \quad \delta < 0 . \tag{3.34}$$

Note that from (3.26) and the Itô's formula (Theorem 2.2.1), yields

$$\begin{aligned}
& d[\Gamma^{\psi}(t)]^{-\frac{\delta}{1-\delta}} \\
= \; & [\Gamma^{\psi}(t)]^{-\frac{\delta}{1-\delta}} \Big\{ \Big[ -\frac{\delta}{1-\delta}(\rho(t) - r(t)) \\
& + \frac{\delta}{2(1-\delta)^2} (\nu^2(t, Z(t), \psi(t)) + \theta^2(t, Z(t), \psi(t))) \\
& + \big(\psi^{-\frac{\delta}{1-\delta}}(t) - 1 + \frac{\delta}{1-\delta}(\psi(t) - 1)\big)\lambda(t) \Big] dt - \frac{\delta}{1-\delta} \theta(t, Z(t), \psi(t)) dW_2(t) \\
& - \frac{\delta}{1-\delta} \nu(t, Z(t), \psi(t)) dW_1(t) + \Big(\psi^{-\frac{\delta}{1-\delta}}(t) - 1\Big) d\tilde{N}(t) \Big\} ,
\end{aligned}$$

which gives the following representation

$$\begin{aligned}
& \mathbb{E}\Big\{ [\Gamma^{\psi}(t)]^{-\frac{\delta}{1-\delta}} \Big\} \\
= \; & \mathbb{E}\Big[ \exp\Big\{ \int_0^t \Big[ \frac{\delta}{1-\delta}(r(u) - \rho(u)) \\
& + \frac{\delta}{2(1-\delta)^2} (\nu^2(u, Z(u), \psi(u)) + \theta^2(u, Z(u), \psi(u))) \\
& \quad + \big(\psi^{-\frac{\delta}{1-\delta}}(u) - 1 + \frac{\delta}{1-\delta}(\psi(u) - 1)\big)\lambda(u) \Big] du \Big\} \Big] ; \quad t \in [0, T] .
\end{aligned}$$

Then the function $\mathcal{N}(\psi)$ can be written as

$$\begin{aligned}\mathcal{N}(\psi) \;&=\; \mathbb{E}\Big[\int_0^T e^{-\int_0^t (\tilde{r}(u,Z(u),\psi(u))+\mu(u)+\frac{\kappa}{1-\delta}u)du}[1+\mu(t)]dt \\ &\quad +e^{-\int_0^T (\tilde{r}(u,Z(u),\psi(u))+\mu(u)+\frac{\kappa}{1-\delta}u)du}\Big]\,,\end{aligned} \tag{3.35}$$

where

$$\begin{aligned}&\tilde{r}(t,Z(t),\psi(t)) \\ =\;& -\frac{\delta}{1-\delta}r(t)+\frac{1}{1-\delta}\rho+\frac{\delta}{2(1-\delta)^2}(\nu^2(t,Z(t),\psi(t)) \\ &+\theta^2(t,Z(t),\psi(t)))+\Big(\psi^{-\frac{\delta}{1-\delta}}(t)-1+\frac{\delta}{1-\delta}(\psi(t)-1)\Big)\lambda(t)\,.\end{aligned} \tag{3.36}$$

Proceeding as in (3.7), we define a new probability measure $\widetilde{\mathbb{Q}}$ equivalent to $\mathbb{P}$, by

$$d\widetilde{\mathbb{Q}} = \Lambda^{-\frac{\delta}{1-\delta}}d\mathbb{P}\,.$$

By this change of measure, the external economic factor (3.2) can be written as

$$dZ(t) = \Big[\eta(Z(t)) - \frac{\delta}{1-\delta}\nu(t,Z(t),\psi(t))\Big]dt + dW_1^{\widetilde{\mathbb{Q}},\psi}(t)\,. \tag{3.37}$$

Now, the problem (3.33) with $\mathcal{N}(\psi)$ given by (3.35) can be solved using the dynamic programming approach. It is easy to see that the associated Hamilton-Jacobi-Bellman (HJB) equation satisfying $V(t,Z(t))$ is given by (see, Øksendal and Sulem [77], Theorem 3.1. for more details)

$$\begin{aligned}1+\mu(t)+V_t(t,z)+\frac{1}{2}V_{zz}(t,z)+\Big[\frac{1}{1-\delta}(\delta r(t)-\rho-\delta\lambda(t)+\kappa t)\\ -\lambda(t)+\mu(t)-\frac{\delta(r(t)-\alpha(t,z))^2}{2(1-\delta)^2[\beta^2(t,z)+\sigma^2(t,z)]}\Big]V(t,z)\\ +\Big[\eta(z)-\frac{\delta\beta(t,z)(r(t)-\alpha(t,z))}{(1-\delta)[\beta^2(t,z)+\sigma^2(t,z)]}\Big]V_z(t,z)-\inf_{\psi>0}\Big\{\Big(\psi^{-\frac{\delta}{1-\delta}}+\frac{\delta}{1-\delta}\psi\Big)\lambda(t)\\ +\frac{\delta(\gamma^2(t,z)\lambda^2(t)\psi^2-2(r(t)-\alpha(t,z))\gamma(t,z)\lambda(t)\psi)}{2(1-\delta)^2[\beta^2(t,z)+\sigma^2(t,z)]}V(t,z)\\ -\frac{\delta\beta(t,z)\gamma(t,z)\lambda(t)\psi}{(1-\delta)[\beta^2(t,z)+\sigma^2(t,z)]}V_z(t,z)\Big\} \;=\; 0\,.\end{aligned}$$

Hence, by the first order conditions of optimality, the optimal $\hat{\psi}$ is the solution of the following equation

$$\begin{aligned}\hat{\psi}^{-\frac{1}{1-\delta}} - \frac{\gamma^2(t,z)\lambda(t)V(t,z)}{(1-\delta)[\beta^2(t,z)+\sigma^2(t,z)]}\hat{\psi} + \Big[\frac{(r(t)-\alpha(t,z))\gamma(t,z)\lambda(t)V(t,z)}{(1-\delta)[\beta^2(t,z)+\sigma^2(t,z)]} \\ + \frac{\beta(t,z)\gamma(t,z)}{\beta^2(t,z)+\sigma^2(t,z)}V_z(t,z)\Big] - 1 &= 0\,,\end{aligned}$$

where $V(t,z)$ is the solution of the above second order partial differential equation. The problem (3.34) can be solved similarly.

Since we obtained the optimal $\hat{\zeta}$ and $\hat{\psi}$, from Theorem 3.3.1 and (3.30), we obtain the following expressions

$$c^*(t) = p^*(t) = \frac{X(t)+g(t)}{\mathcal{N}(t)} e^{-\frac{\kappa}{1-\delta}t}\,, \tag{3.38}$$

$$X^*(T) = \frac{X(t)+g(t)}{\mathcal{N}(t)} e^{-\frac{\kappa}{1-\delta}t}\left(\frac{\Gamma(T)}{\Gamma(t)}\right)^{-\frac{1}{1-\delta}}\,, \tag{3.39}$$

where

$$\begin{aligned}\mathcal{N}(t) &= \mathbb{E}\Big[\int_t^T e^{-\int_t^s (\tilde{r}(u,Z(u),\hat{\psi})+\mu(u)+\frac{\kappa}{1-\delta}(u-t)du}[1+\mu(s)]ds \\ &\quad + e^{-\int_t^T (\tilde{r}(u,Z(u),\hat{\psi})+\mu(u)+\frac{\kappa}{1-\delta}(u-t))du}\Big]\,.\end{aligned}$$

From (3.26), by Itô's formula we know that

$$\begin{aligned}&\left(\frac{\Gamma(T)}{\Gamma(t)}\right)^{-\frac{1}{1-\delta}} \\ &= \exp\Big\{\frac{1}{1-\delta}\int_t^T \Big[r(s)+\frac{1}{2}\nu^2(s,Z(s),\hat{\psi})+\frac{1}{2}\theta^2(s,Z(s),\hat{\psi})-\rho(s) \\ &\quad + [\hat{\psi}-1-\ln\hat{\psi}]\lambda(s)\Big]ds - \frac{1}{1-\delta}\Big[\int_t^T \nu(s,Z(s),\hat{\psi})dW_1(s) \\ &\quad + \int_t^T \theta(s,Z(s),\hat{\psi})dW_2(s) + \int_t^T \ln\hat{\psi}d\tilde{N}(s)\Big]\Big\}\,.\end{aligned}$$

Then we have:

$$\begin{aligned} dX^*(t) &= \mathcal{O}dt - \frac{1}{1-\delta}\nu(t)(X^*(t)+g(t))dW_1(t) \\ & -\frac{1}{1-\delta}\theta(t)(X^*(t)+g(t))dW_2(t) \\ & +\left(\hat{\psi}^{-\frac{1}{1-\delta}}(t)-1\right)(X^*(t)+g(t))dN(t), \end{aligned} \tag{3.40}$$

where $\mathcal{O} := \mathcal{O}(t, X^*(t), g(t))$. Comparing (3.40) with (3.5), we obtain the optimal allocation:

$$\begin{cases} \pi^*(t)\beta(t,Z(t)) = -\frac{1}{1-\delta}\nu(t,Z(t),\hat{\psi})(X^*(t)+g(t))\,, \\ \pi^*(t)\sigma(t,Z(t)) = -\frac{1}{1-\delta}\theta(t,Z(t),\hat{\psi})(X^*(t)+g(t))\,, \\ \pi^*(t)\gamma(t,Z(t)) = \left(\hat{\psi}^{-\frac{1}{1-\delta}}-1\right)(X^*(t)+g(t))\,. \end{cases} \tag{3.41}$$

Hence,

$$\pi^*(t) = \frac{\left(\hat{\psi}^{-\frac{1}{1-\delta}}-1\right) - \frac{1}{1-\delta}\nu(t,Z(t),\hat{\psi}) - \frac{1}{1-\delta}\theta(t,Z(t),\hat{\psi})}{\beta(t,Z(t))+\sigma(t,Z(t))+\gamma(t,Z(t))}(X^*(t)+g(t))\,. \tag{3.42}$$

Inserting (3.38) and (3.41) into (3.10) we obtain the following geometric SDE which can be solved applying the Itô formula.

$$\begin{aligned} & \frac{d(X^*(t)+g(t))}{X^*(t)+g(t)} \\ = \; & \left[r(t)+\mu(t)-\frac{1+\mu(t)}{\mathcal{N}(t)}\right]dt - \frac{1}{1-\delta}\nu(t,Z(t),\hat{\psi})dW_1^{\mathbb{Q},\hat{\psi}}(t) \\ & -\frac{1}{1-\delta}\theta(t,Z(t),\hat{\psi})dW_2^{\mathbb{Q},\hat{\psi}}(t) + \left(\hat{\psi}^{-\frac{1}{1-\delta}}(t)-1\right)d\widetilde{N}^{\mathbb{Q},\hat{\psi}}(t)\,. \end{aligned} \tag{3.43}$$

We conclude this section, summarizing our results in the following Lemma:

**Lemma 3.3.2.** *For the power utility functions* (3.29)*, the optimal investment-consumption-insurance strategy* $(c^*(t), \pi_1^*(t), \pi_2^*(t), p^*(t))$*,* $\forall t \in [0,T]$ *is given by*

$$c^*(t) = p^*(t) = \frac{X^*(t)+g(t)}{\mathcal{N}(t)}e^{-\frac{\kappa}{1-\delta}t}$$

*and*

$$\pi^*(t) = \frac{\left(\hat{\psi}^{-\frac{1}{1-\delta}} - 1\right) - \frac{1}{1-\delta}\nu(t, Z(t), \hat{\psi}) - \frac{1}{1-\delta}\theta(t, Z(t), \hat{\psi})}{\beta(t, Z(t)) + \sigma(t, Z(t)) + \gamma(t, Z(t))}(X^*(t) + g(t))\,.$$

## 3.4 The restricted control problem

In this section, we solve the optimal investment, consumption and life insurance problem for the constrained case. We obtain an optimal strategy for the case of continuous constraints (American put options) by using a so-called *option based portfolio insurance (OBPI) strategy*. The OBPI method consists in taking a certain part of capital and invest it in the optimal portfolio of the unconstrained problem and the remaining part insures the position with an American put. We prove the admissibility and the optimality of the strategy. For more details see e.g., El Karoui *et. al.* [30], Kronborg and Steffensen [55].

Consider the following problem

$$\sup_{(c,\pi,p)\in\mathcal{A}'} \mathbb{E}\Big[\int_0^T e^{-\int_0^s(\rho(u)+\mu(u))du}[U(c(s)) + \mu(s)U(p(s))]ds$$
$$+e^{-\int_0^T(\rho(u)+\mu(u))du}U(X(T))\Big]\,, \tag{3.44}$$

under the capital guarantee restriction

$$X(t) \geq k(t, D(t)),\ \forall t \in [0, T], \tag{3.45}$$

where

$$D(t) := \int_0^t h(s, X(s))ds\,,$$

for $k$ and $h$ deterministic functions of time. The guarantee (3.45) is covered by

$$k(t, \zeta) = 0 \tag{3.46}$$

and

$$k(t, \zeta) = x_0 e^{\int_0^t(r^{(g)}(s)+\mu(s))ds} + \zeta e^{\int_0^t(r^{(g)}(s)+\mu(s))ds}, \tag{3.47}$$

with

$$h(s, x) = e^{-\int_0^s(r^{(g)}(u)+\mu(u))du}[\ell(s) - c(s, x) - \mu(s)p(s, x)],$$

where $r^{(g)} \leq r$ is the minimum rate of return guarantee excess of the objective mortality $\mu$. Then

$$k(t,z) = x_0 e^{\int_0^t (r^{(g)}(s)-\mu(s))ds} + \int_0^t e^{\int_s^t (r^{(g)}(u)+\mu(u))ds}[\ell(s) - c(s) - \mu(s)p(s)]ds. \tag{3.48}$$

We still denote by $X^*, c^*, \pi^*$ and $p^*$ the optimal wealth, optimal consumption, investment and life insurance for the unrestricted problem (3.21), respectively. The optimal wealth for the unrestricted problem $Y^*(t) := X^*(t) + g(t)$ has the dynamics

$$\begin{aligned} dY^*(t) &= Y^*(t)\Big\{\Big[r(t)+\mu(t) - \frac{1+\mu(t)}{\mathcal{N}(t)}\Big]dt - \frac{1}{1-\delta}\nu(t,Z(t),\hat{\psi})dW_1^{\mathbb{Q},\hat{\psi}}(t) \\ &\quad - \frac{1}{1-\delta}\theta(t,Z(t),\hat{\psi})dW_2^{\mathbb{Q},\hat{\psi}}(t) + \Big(\hat{\psi}^{-\frac{1}{1-\delta}}(t) - 1\Big)\, d\widetilde{N}^{\mathbb{Q},\hat{\psi}}(t)\Big\}, \end{aligned} \tag{3.49}$$

$\forall t \in [0,T]$, $Y^*(0) = y_0$, where $y_0 := x_0 + g(0)$. Under $\mathbb{Q}$, the economic factor $Z$ is given by

$$dZ(t) = \Big[\eta(Z(t)) + \nu(t,Z(t),\hat{\psi}(t))\Big]\,dt + dW_1^{\mathbb{Q},\hat{\psi}}(t)\,. \tag{3.50}$$

Let $P^a_{y,\zeta}(t,T,k+g)$ denote the time-$t$ value of an American put option with strike price $k(s,D(s)) + g(s)$, $\forall s \in [t,T]$, where $D(t) = \zeta$ and maturity $T$ written on a portfolio $Y$, where $Y(s)$, $s \in [t,T]$ is the solution to (3.49), with $Y(t) = y$. By definition, the price of such put option is given by

$$\begin{aligned} P^a_{y,\zeta}(t,T,k+g) &:= \sup_{\tau_s \in \mathcal{T}_{t,T}} \mathbb{E}^{\mathbb{Q}}\Big[e^{-\int_t^{\tau_s}(r(u)+\mu(u))du}[k(\tau_s, D(\tau_s)) + g(\tau_s) \\ &\quad - Y(\tau_s)]^+ \Big| Y(t) = y, D(t) = \zeta\Big], \end{aligned}$$

where $\mathcal{T}_{t,T}$ is the set of stopping times $\tau_s \in [t,T]$.

As in Kronborg and Steffensen [55], we introduce the American put option-based portfolio insurance

$$\hat{X}^{(\varrho)}(t) := \varrho(t,D(t))Y^*(t) + P^a_{\varrho Y^*,D}(t,T,k+g) - g(t), \quad t \in [0,T], \tag{3.51}$$

for $\varrho : [0,T] \times \mathbb{R} \mapsto (0,1)$ defined by

$$\varrho(t, D(t)) = \varrho_0 \vee \sup_{s \le t} \left( \frac{b(s, D(s))}{Y^*(s)} \right), \tag{3.52}$$

where $b(t, D(t))$ is the exercise boundary of the American put option given by

$$b(t, \zeta) := \sup \left\{ y : P^a_{y,\zeta}(t, T, k+g) = (k(t,\zeta) + g(t) - y)^+ \right\} \tag{3.53}$$

and $\varrho_0 := \varrho(0, D(0))$ is determined by the budget constraint

$$\varrho(0, D(0))Y^*(0) + P^a_{\varrho Y^*, D}(0, T, k+g) - g(0) = x_0. \tag{3.54}$$

By definition of an American put option, $P^a_{\varrho Y^*,D}(t, T, k+g) \ge (k(t,d) + g(t) - \varrho(t, D(t))Y^*(t))^+$, $\forall t \in [0,T]$. Hence

$$\begin{aligned} \widehat{X}^{(\varrho)}(t) &:= \varrho(t, D(t))Y^*(t) + P^a_{\varrho Y^*,D}(t, T, k+g) - g(t) \\ &\ge \varrho(t, D(t))Y^*(t) + (k(t,\zeta) + g(t) - \varrho(t, D(t))Y^*(t))^+ - g(t) \\ &\ge k(t,\zeta), \forall t \in [0,T], \end{aligned}$$

i.e., $\widehat{X}^{(\varrho)}$ fulfils the American capital guarantee (3.45).

Under the optimal martingale measure $\hat{\psi}$, we recall some basic properties of American put options in a Black-Scholes market (Karatzas and Shreve, [52], Section 2.8. or Musiela and Rutkowski [73], pp. 219–221)

$$\begin{array}{ll} P^a_{y,\zeta}(t, T, k+g) = k(t,\zeta) + g(t) - y, & \forall (t,y,\zeta) \in \mathcal{C}^c \\ \frac{\partial}{\partial y} P^a_{y,\zeta}(t, T, k+g) = -1, & \forall (t,y,\zeta) \in \mathcal{C}^c \\ A P^a_{y,\zeta}(t, T, k+g) = (r(t) + \mu(t)) P^a_{y,\zeta}(t, T, k+g), & \forall (t,y,\zeta) \in \mathcal{C}, \end{array}$$

where from (3.49), the generator operator $A$ is given by (see e.g. Oksendal and Sulem [77], Theorem 1.22. Li *et. al.* [59])

$$
\begin{aligned}
& (A\phi)(y,z) \\
= \; & \frac{\partial \phi}{\partial t} + \Big(r(t)+\mu(t)-\frac{1+\mu(t)}{\mathcal{H}(t)}\Big) y \frac{\partial \phi}{\partial y} + \Big(\eta(z)+\nu(t,z,\hat{\psi})\Big)\frac{\partial \phi}{\partial z} + \frac{1}{2}\frac{\partial^2 \phi}{\partial z^2} \\
& + \frac{1}{2(1-\delta)^2}\Big[\nu^2(t,Z(t),\hat{\psi})+\theta^2(t,Z(t),\hat{\psi})\Big] y^2 \frac{\partial^2 \phi}{\partial y^2} - \frac{1}{1-\delta}\nu(t,z,\hat{\psi})\frac{\partial^2 \phi}{\partial y \partial z} \\
& + \Big[\phi(t, y\hat{\psi}^{-\frac{1}{1-\delta}}, z) - \phi(t,y,z) - y\Big(\hat{\psi}^{-\frac{1}{1-\delta}} - 1\Big)\frac{\partial \phi}{\partial y}\Big]\lambda(t)
\end{aligned}
$$

and

$$\mathcal{C} := \{(t,y,\zeta) : P^a_{y,\zeta}(t,T,k+g) > (k(t,\zeta)+g(t)-y)^+\}$$

defines the continuation region. $\mathcal{C}^c$ is the stopping region, that is, the complementary of the continuation region $\mathcal{C}$. From the exercise boundary given in (3.53), we can write the continuation region by

$$\mathcal{C} = \{(t,y,\zeta) : y > b(t,\zeta)\}.$$

Define a function $H$ by

$$H(t,y,\zeta) := y + P^a_{y,\zeta}(t,T,k+g) - g(t),$$

then we have

$$\widehat{X}^{(\varrho)}(t) = H(t, \varrho(t,D(t))Y^*(t), D(t)).$$

From the properties of $P^a_{y,\zeta}(t,T,k+g)$, we deduce that

$$
\begin{aligned}
H(t,y,\zeta) &= k(t,\zeta), \quad \forall (t,y,\zeta) \in \mathcal{C}^c, \\
\frac{\partial}{\partial y} H(t,y,\zeta) &= 0, \quad \forall (t,y,\zeta) \in \mathcal{C}^c && (3.55) \\
AH(t,y,\zeta) &= \frac{\partial}{\partial t} k(t,\zeta) + h(t,\zeta)\frac{\partial}{\partial \zeta} k(t,\zeta) \quad \forall (t,y,\zeta) \in \mathcal{C}^c, && (3.56) \\
AH(t,y,\zeta) &= (r(t)+\mu(t))P^a_{y,\zeta}(t,T,k+g) + \ell(t) - (r(t)+\mu(t))g(t) \\
& \quad + \Big(r(t)+\mu(t)-\frac{1+\mu(t)}{\mathcal{H}(t)}\Big) y \\
& \quad + \Big(P^a_{y\hat{\psi}^{-\frac{1}{1-\delta}},\zeta}(t,T,k+g) - P^a_{y,\zeta}(t,T,k+g)\Big)\lambda(t) \\
&= (r(t)+\mu(t))H(t,y,\zeta) + \ell(t) - \frac{1+\mu(t)}{\mathcal{H}(t)} y + \Big[H(t, y\hat{\psi}^{-\frac{1}{1-\delta}}, \zeta) \\
& \quad - H(t,y,\zeta) - y\Big(\hat{\psi}^{-\frac{1}{1-\delta}}(t) - 1\Big)\Big]\lambda(t), \quad \forall (t,y,\zeta) \in \mathcal{C}. && (3.57)
\end{aligned}
$$

**Proposition 3.4.1.** *Consider the strategy* $(\varrho c^*, \varrho\pi^*, \varrho p^*)$*, where* $\varrho$ *is defined by (3.52). Then, the strategy* $(\varrho c^*, \varrho\pi^*, \varrho p^*)$ *is admissible.*

*Proof.* For $\varrho$ constant and linearity of $Y^*(t), \forall t \in [0,T]$, we have that $\varrho(t, D(t))Y^*(t)$ and $Y^*(t)$ have the same dynamics. Then, using Itô's formula, (3.56)-(3.57), $(c^*(t), p^*(t))$ in Theorem 3.3.1 and the fact that $\varrho$ increases only at the boundary, we obtain (here $\frac{\partial}{\partial y}$ means differentiating with respect to the second variable)

$$
\begin{aligned}
& dH(t, \varrho(t, D(t))Y^*(t), D(t)) \\
= \; & [dH(t, \varrho Y^*(t), D(t))] + Y^*(t)\frac{\partial}{\partial y}H(t, \varrho(t, D(t))Y^*(t), D(t))d\varrho(t, D(t)) \\
= \; & AH(t, \varrho Y^*(t))dt - \frac{1}{1-\delta}\nu(t, Z(t), \hat{\psi})\varrho Y^*(t)\frac{\partial}{\partial y}H(t, \varrho Y^*(t), D(t))dW_1^{\mathbb{Q},\hat{\psi}}(t) \\
& -\frac{1}{1-\delta}\theta(t, Z(t), \hat{\psi})\varrho Y^*(t)\frac{\partial}{\partial y}H(t, \varrho Y^*(t), D(t))dW_2^{\mathbb{Q},\hat{\psi}}(t) \\
& + \left[H(t, \varrho Y^*(t)\hat{\psi}^{-\frac{1}{1-\delta}}(t), D(t)) - H(t, \varrho Y^*(t), D(t))\right] d\tilde{N}^{\mathbb{Q},\hat{\psi}}(t) \\
& + Y^*(t)\frac{\partial}{\partial y}H(t, \varrho(t, D(t))Y^*(t), D(t))d\varrho(t, D(t)) \\
= \; & \Big\{(r(t) + \mu(t))H(t, \varrho Y^*(t), D(t)) + \ell(t) - \varrho c^*(t) - \varrho\mu(t)p^*(t) \\
& + \Big[H(t, \varrho Y^*(t)\hat{\psi}^{-\frac{1}{1-\delta}}(t), D(t)) - H(t, \varrho Y^*(t), D(t)) \\
& - \varrho Y^*(t)\left(\hat{\psi}^{-\frac{1}{1-\delta}}(t) - 1\right)\Big]\lambda(t)\Big\}\mathbf{1}_{(\varrho(t,D(t))Y^*(t) > b(t,D(t)))}dt \\
& \left[\frac{\partial}{\partial t}k(t, D(t)) + h(t, D(t))\frac{\partial}{\partial d}k(t, D(t))\right]\mathbf{1}_{(\varrho(t,D(t))Y^*(t) \le b(t,D(t)))}dt \\
& + Y^*(t)\frac{\partial}{\partial y}H(t, \varrho(t, D(t))Y^*(t), D(t))\mathbf{1}_{(\varrho(t,D(t))Y^*(t) = b(t,D(t)))}d\varrho(t, D(t)) \\
& -\frac{1}{1-\delta}\nu(t, Z(t), \hat{\psi})\varrho Y^*(t)\frac{\partial}{\partial y}H(t, \varrho Y^*(t), D(t))dW_1^{\mathbb{Q},\hat{\psi}}(t) \\
& -\frac{1}{1-\delta}\theta(t, Z(t), \hat{\psi})\varrho Y^*(t)\frac{\partial}{\partial y}H(t, \varrho Y^*(t), D(t))dW_2^{\mathbb{Q},\hat{\psi}}(t) \\
& + \left[H(t, \varrho Y^*(t)\hat{\psi}^{-\frac{1}{1-\delta}}(t), D(t)) - H(t, \varrho Y^*(t), D(t))\right] d\tilde{N}^{\mathbb{Q},\hat{\psi}}(t).
\end{aligned}
$$

From (3.55), we know that $\frac{\partial}{\partial y}H(t, \varrho(t, D(t))Y^*(t), D(t)) = 0$ on the set

$\{(t,\omega) : \varrho(t,D(t))Y^*(t) = b(t,D(t))\}$, then

$$\begin{aligned}
& dH(t,\varrho(t,D(t))Y^*(t),D(t)) \\
= \; & \Big\{(r(t)+\mu(t))H(t,\varrho Y^*(t),D(t)) + \ell(t) - \varrho c^*(t) - \varrho\mu(t)p^*(t) \\
& + \Big[H(t,\varrho Y^*(t)\hat{\psi}^{-\frac{1}{1-\delta}}(t),D(t)) - H(t,\varrho Y^*(t),D(t)) \\
& - \varrho Y^*(t)\left(\hat{\psi}^{-\frac{1}{1-\delta}}(t) - 1\right)\Big]\lambda(t)\Big\}dt \\
& + \Big[\frac{\partial}{\partial t}k(t,D(t)) + h(t,D(t))\frac{\partial}{\partial d}k(t,D(t)) - [(r(t)+\mu(t))k(t,D(t)) + \ell(t) \\
& - \varrho(t,D(t))c^*(t) - \varrho(t,D(t))\mu(t)p^*(t)]\Big]\mathbf{1}_{(\varrho(t,D(t))Y^*(t)\le b(t,D(t)))}dt \\
& - \frac{1}{1-\delta}\nu(t,Z(t),\hat{\psi})\varrho Y^*(t)\frac{\partial}{\partial y}H(t,\varrho Y^*(t),D(t))dW_1^{\mathbb{Q},\hat{\psi}}(t) \\
& - \frac{1}{1-\delta}\theta(t,Z(t),\hat{\psi})\varrho Y^*(t)\frac{\partial}{\partial y}H(t,\varrho Y^*(t),D(t))dW_2^{\mathbb{Q},\hat{\psi}}(t) \\
& + \left[H(t,\varrho Y^*(t)\hat{\psi}^{-\frac{1}{1-\delta}}(t),D(t)) - H(t,\varrho Y^*(t),D(t))\right]d\tilde{N}^{\mathbb{Q},\hat{\psi}}(t).
\end{aligned}$$

Hence, since
$\{(t,\omega) : \varrho(t,D(t))Y^*(t) \le b(t,D(t))\} = \left\{(t,\omega) : \varrho(t,D(t)) = \frac{b(t,D(t))}{Y^*(t)}\right\}$ has a zero $dt \otimes d\mathbb{P}$-measure, we conclude that

$$\begin{aligned}
& dH(t,\varrho(t,D(t))Y^*(t),D(t)) \\
= \; & \Big\{(r(t)+\mu(t))H(t,\varrho Y^*(t),D(t)) + \ell(t) - \varrho c^*(t) - \varrho\mu(t)p^*(t) \\
& + \Big[H(t,\varrho Y^*(t)\psi^{-\frac{1}{1-\delta}}(t),D(t)) - H(t,\varrho Y^*(t),D(t)) \\
& - \varrho Y^*(t)\left(\psi^{-\frac{1}{1-\delta}}(t) - 1\right)\Big]\lambda(t)\Big\}dt \\
& - \frac{1}{1-\delta}\nu(t,Z(t),\hat{\psi})\varrho Y^*(t)\frac{\partial}{\partial y}H(t,\varrho Y^*(t),D(t))dW_1^{\mathbb{Q},\hat{\psi}}(t) \\
& - \frac{1}{1-\delta}\theta(t,Z(t),\hat{\psi})\varrho Y^*(t)\frac{\partial}{\partial y}H(t,\varrho Y^*(t),D(t))dW_2^{\mathbb{Q},\hat{\psi}}(t) \\
& + \left[H(t,\varrho Y^*(t)\hat{\psi}^{-\frac{1}{1-\delta}}(t),D(t)) - H(t,\varrho Y^*(t),D(t))\right]d\tilde{N}^{\mathbb{Q},\hat{\psi}}(t),
\end{aligned}$$

i.e. by (3.20), the strategy $(\varrho c^*, \varrho\pi^*, \varrho p^*)$ is admissible. □

We then state the main result of this section.

**Theorem 3.4.2.** *Consider the strategy* $(\widehat{c}, \widehat{\pi}, \widehat{p})$, $\forall t \in [0, T]$ *given by*

$$\widehat{c} = \frac{\varrho(t, D(t))Y^*(t)}{\mathcal{H}(t)} = \varrho(t, D(t))c^*(t), \tag{3.58}$$

$$\widehat{\pi} = \varrho(t, D(t))\pi^*(t), \tag{3.59}$$

$$\widehat{p} = \frac{\varrho(t, D(t))Y^*(t)}{\mathcal{H}(t)} = \varrho(t, D(t))p^*(t), \tag{3.60}$$

*where the strategy* $(c^*, \pi^*, p^*)$ *is defined in Lemma 3.3.2 combined with a position in an American put option written on the portfolio* $(\varrho(s, D(s))Y^*(s))$ *with strike price* $k(s, D(s)) + g(s)$, $\forall s \in [t, T]$ *and maturity* $T$*, where* $\varrho(s, D(s)), s \in [t, T]$ *is a function defined by (3.52). Then, the strategy is optimal for the American capital guarantee control problem given by (3.44)-(3.45).*

*Proof.* The proof follows similarly as Kronborg and Steffensen [55], Theorem 4.1. We omit the details.

□

# Chapter 4

# On the optimal investment-consumption and life insurance selection problem with stochastic volatility

## 4.1 Introduction

The problem of a wage earner who wants to invest and protect his dependent for a possible premature death has gained much concern in recent times. Since the research paper on portfolio optimization and life insurance purchase by Richard [85] appeared, a number of works in this direction have been reported in the literature. For instance, Pliska and Ye [83] studied an optimal consumption and life insurance contract for a problem described by a risk-free asset. Duarte *et al.* [29] considered a problem of a wage earner who invests and buys a life insurance in a financial market with $n$ diffusion risky shares. Similar works include (Guambe and Kufakunesu [41], Huang *et al.* [47], Liang and Guo [60], Shen and Wei [88], among others). In all the above-mentioned papers, a single life insurance contract was considered.

Recently, Mousa *et al.* [72], extended Duarte *et al.* [29] to consider a wage earner who buys life insurance contracts from $M > 1$ life insurance companies. Each insurance company offers pairwise distinct contract. This allows the wage earner to compare the premiums insurance ratio of the companies and buy the amount of life insurance from the one offering the smallest premium-payout ratio at each time. Using a dynamic programming approach, they solved the optimal investment, consumption and life insurance contracts in a financial market comprised by one risk-free asset and $n$ risky shares driven by diffusion processes. In this chapter, we extend their work to a jump-diffusion setup with stochastic volatility. This extension is motivated by the following reasons: First, the existence of high frequency data on the empirical studies carried out by Cont [17], Tankov [94] and references therein, have shown that the analysis of price evolution reveals some sudden changes that cannot be explained by models driven by diffusion processes. Another reason is related to the presence of volatility clustering in the distribution of the risky share process, i.e., large changes in prices are often followed by large changes and small changes tend to be followed by small changes.

To enable a full capture of these and other aspects, we consider a jump diffusion model with stochastic volatility similar to that in Mnif [68]. Using Dynamic programming approach, Mnif [68] proved the existence of a smooth solution of a semi-linear integro-Hamilton-Jacobi-Bellman (HJB) for the exponential utility function. Zeghal and Mnif [100] considered the same problem for power utility case. Under some particular assumptions, they also derived the backward stochastic differential equation (BSDE) associated with the semi-linear HJB. The drawback of the dynamic programming approach is that it requires the system to be Markovian. To overcome this limitation, we propose a maximum principle approach to solve this stochastic volatility jump-diffusion problem. This approach allows us to solve this problem in a more general setting. We prove a sufficient and necessary maximum principle in a general stochastic volatility problem. Then we apply these results to solve the wage earner investment, consumption and life insurance problem described earlier. In the literature, the maximum principle approach has been widely reported, see, for instance, Framstad *et. al.* [39], Øksendal and Sulem [77], An and Øksendal [1], Pamen [79], Pamen and Momeya [80], among others.

The rest of the chapter is organized as follows: in Section 4.2, we introduce our control problem and proof the sufficient and necessary maximum principle for a stochastic control problem with stochastic volatility. In Section 4.3, we give the characterization of the optimal strategies for an investment, consumption and life insurance problem applying the results of Theorem 4.2.1. Finally, we consider an example of a linear pure jump stochastic volatility model of Ornstein-Uhlenbeck type and an explicit optimal portfolio is derived.

## 4.2 Maximum principle for stochastic optimal control problem with stochastic volatility

As in the previous Chapter, let $T < \infty$ be a finite time horizon investment period, which can be viewed as a retirement time of an investor. Consider two independent Brownian motions $\{W_1(t); W_2(t),\, 0 \leq t \leq T\}$ associated to the complete filtered probability space $(\Omega^W, \mathcal{F}^W, \{\mathcal{F}_t^W\}, \mathbb{P}^W)$. Furthermore, we consider a Poisson process $N$ independent of $W_1$ and $W_2$, associated with the complete filtered probability space $(\Omega^N, \mathcal{F}^N, \{\mathcal{F}_t^N\}, \mathbb{P}^N)$ with the intensity measure $dt \times d\nu(z)$, where $\nu$ is the $\sigma$-finite Borel measure on $\mathbb{R} \setminus \{0\}$. A $\mathbb{P}^N$-martingale compensated Poisson random measure is given by:

$$\tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt\,.$$

We define the product space:

$$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P}) := (\Omega^W \times \Omega^N, \mathcal{F}^W \otimes \mathcal{F}^N, \{\mathcal{F}^W \otimes \mathcal{F}^N\}, \mathbb{P}^W \otimes \mathbb{P}^N)$$

where $\{\mathcal{F}_t\}_{t\in[0,T]}$ is a filtration satisfying the usual conditions.

Suppose that the dynamics of the state process is given by the following stochastic differential equation (SDE)

$$\begin{aligned} dX(t) &= b(t,X(t),Y(t),\pi(t))dt + \sigma(t,X(t),Y(t),\pi(t))dW_1(t) \qquad (4.1)\\ &\quad +\beta(t,X(t),Y(t),\pi(t))dW_2(t) \\ &\quad +\int_{\mathbb{R}} \gamma(t,X(t),Y(t),\pi(t),z)\tilde{N}(dt,dz)\,; \\ X(0) &= x \in \mathbb{R}\,, \end{aligned}$$

where the external economic factor $Y$ is given by

$$dY(t) = \varphi(Y(t))dt + \phi(Y(t))dW_2(t)\,. \qquad (4.2)$$

We assume that the functions $b, \sigma, \beta : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{A} \to \mathbb{R}$; $\gamma : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{A} \times \mathbb{R} \to \mathbb{R}$; $\varphi, \phi : \mathbb{R} \to \mathbb{R}$ are given predictable processes, such that (4.1) and (4.2) are well defined and (4.1) has a unique solution for each $\pi \in \mathcal{A}$. Here, $\mathcal{A}$ is a given closed set in $\mathbb{R}$.

Let $f : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{A} \to \mathbb{R}$ be a continuous function and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a concave function. We define the performance criterion by

$$\mathcal{J}(\pi) = \mathbb{E}\Big[\int_0^T f(t,X(t),Y(t),\pi(t))dt + g(X(T),Y(T))\Big]\,. \qquad (4.3)$$

We say that $\pi \in \mathcal{A}$ is an admissible strategy if (4.1) has a unique strong solution and

$$\mathbb{E}\Big[\int_0^T |f(t,X(t),Y(t),\pi(t))|dt + |g(X(T),Y(T))|\Big] < \infty\,.$$

The main problem is to find $\pi^* \in \mathcal{A}$ such that

$$\mathcal{J}(\pi^*) = \sup_{\pi \in \mathcal{A}} \mathcal{J}(\pi)\,.$$

The control $\pi^*$ is called an optimal control if it exists.

In order to solve this stochastic optimal control problem with stochastic volatility, we use the so called maximum principle approach. The beauty of this method is that it solves a stochastic control problem in a more general situation, that is, for both Markovian and non-Markovian cases. For the Markovian case, this problem has been solved using dynamic programming approach by Mnif [68]. Our approach may be considered as an extension of the maximum approach in Framstad *et. al.* [39] to the stochastic volatility case.

We define the Hamiltonian $\mathcal{H} : [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{A}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by:

$$\begin{aligned} & \mathcal{H}(t,X(t),Y(t),\pi(t),A_1(t),A_2(t),B_1(t),B_2(t),D_1(t,\cdot)) \\ = \; & f(t,X(t),Y(t),\pi(t)) + b(t,X(t),Y(t),\pi(t))A_1(t) + \varphi(Y(t))A_2(t) \\ & +\sigma(t,X(t),Y(t),\pi(t))B_1(t) + \beta(t,X(t),Y(t),\pi(t))B_2(t) + \phi(Y(t))B_3(t) \\ & + \int_{\mathbb{R}} \gamma(t,X(t),Y(t),\pi(t),z)D_1(t,z)\nu(dz)\,, \end{aligned} \tag{4.4}$$

provided that the integral in (4.4) converges. From now on, we assume that the Hamiltonian $\mathcal{H}$ is continuously differentiable w.r.t. $x$ and $y$. Then, the adjoint equations corresponding to the admissible strategy $\pi\in\mathcal{A}$ are given by the following backward stochastic differential equations (BSDEs)

$$\begin{aligned} dA_1(t) &= -\frac{\partial\mathcal{H}}{\partial x}(t,X(t),Y(t),\pi(t),A_1(t),A_2(t),B_1(t),B_2(t),D_1(t,\cdot))dt \\ &\quad +B_1(t)dW_1(t) + B_2(t)dW_2(t) + \int_{\mathbb{R}} D_1(t,z)\tilde{N}(dt,dz)\,, \qquad (4.5) \\ A_1(T) &= \frac{\partial g}{\partial x}(X(T),Y(T)) \qquad (4.6) \end{aligned}$$

and

$$\begin{aligned} dA_2(t) &= -\frac{\partial\mathcal{H}}{\partial y}(t,X(t),Y(t),\pi(t),A_1(t),A_2(t),B_1(t),B_2(t),D_1(t,\cdot))dt \\ &\quad +B_3(t)dW_1(t) + B_4(t)dW_2(t) + \int_{\mathbb{R}} D_2(t,z)\tilde{N}(dt,dz)\,, \qquad (4.7) \\ A_2(T) &= \frac{\partial g}{\partial y}(X(T),Y(T))\,. \qquad (4.8) \end{aligned}$$

The verification theorem associated to our problem is stated as follows:

**Theorem 4.2.1.** (Sufficient maximum principle) *Let $\pi^* \in \mathcal{A}$ with the corresponding wealth process $X^*$. Suppose that the pairs $(A_1^*(t), B_1^*(t), B_2^*(t), D_1^*(t,z))$ and $(A_2^*(t), B_3^*(t), B_4^*(t), D_2^*(t,z))$ are the solutions of the adjoint equations (4.5) and (4.7), respectively. Moreover, suppose that the following assumptions hold:*

*(i) The function $(x,y) \to g(x,y)$ is concave;*

*(ii) The function*

$$\mathcal{H}(t) = \sup_{\pi \in \mathcal{A}} \mathcal{H}(t, X(t), Y(t), \pi, A_1^*(t), A_2^*(t), B_1^*(t), B_2^*(t), D_1^*(t,z))$$

*is concave and*

$$\begin{aligned} & \mathcal{H}^*(t, X, Y, \pi^*, A_1^*, A_2^*, B_1^*, B_2^*, D_1^*) \\ = & \sup_{(\pi,c,p)\in\mathcal{A}} \mathcal{H}(t, X, Y, \pi, A_1^*, A_2^*, B_1^*, B_2^*, D_1^*)\,. \end{aligned}$$

*Furthermore, we assume the following:*

$$\mathbb{E}\Big[\int_0^T (X^*(t))^2\Big((B_1^*(t))^2 + (B_2^*(t))^2 + \int_{\mathbb{R}} (D_1^*(t,z))^2 \nu(dz)\Big)dt\Big] < \infty\,;$$

$$\mathbb{E}\Big[\int_0^T (Y(t))^2\Big((B_3^*(t))^2 + (B_4^*(t))^2 + \int_{\mathbb{R}} (D_2^*(t,z))^2 \nu(dz)\Big)dt\Big] < \infty\,;$$

$$\begin{aligned} \mathbb{E}\Big[\int_0^T \Big\{ & (A_1^*(t))^2\Big((\sigma(t, X(t), Y(t), \pi^*(t)))^2 + (\beta(t, X(t), Y(t), \pi^*(t)))^2 \\ & + \int_{\mathbb{R}} (\gamma(t, X(t), Y(t), \pi^*(t), z))^2 \nu(dz)\Big) + (A_2^*(t))^2 (\phi(Y(t)))^2\Big] dt\Big] \quad < \quad \infty\,. \end{aligned}$$

*Then, $\pi^* \in \mathcal{A}$ is an optimal strategy with the corresponding optimal state process $X^*$.*

*Proof.* Let $\pi \in \mathcal{A}$ be an admissible strategy and $X(t)$ the corresponding wealth process. Then, following Framstad *et. al.* [39], Theorem 2.1., we have:

$$
\begin{aligned}
\mathcal{J}(\pi^*) - \mathcal{J}(\pi) &= \mathbb{E}\Big[\int_0^T (f(t, X^*(t), Y^*(t), \pi^*(t)) - f(t, X(t), Y(t), \pi(t)))dt \\
&\quad +(g(X^*(T), Y^*(T)) - g(X(T), Y(T)))\Big] \\
&= \mathcal{J}_1 + \mathcal{J}_2 \,.
\end{aligned}
$$

By condition $(i)$ and the integration by parts rule (Øksendal and Sulem [77], Lemma 3.6.), we have

$$
\begin{aligned}
& \mathcal{J}_2 \\
= \;& \mathbb{E}\Big[g(X^*(T),Y^*(T)) - g(X(T),Y(T))\Big] \\
\geq \;& \mathbb{E}\Big[(X^*(T)-X(T))A_1^*(T) + (Y^*(T)-Y(T))A_2^*(T)\Big] \\
= \;& \mathbb{E}\Big[\int_0^T (X^*(t)-X(t))dA_1^*(t) + \int_0^T A_1^*(t)(dX^*(t)-dX(t)) \\
& + \int_0^T (Y^*(t)-Y(t))dA_2^*(t) + \int_0^T A_2^*(t)(dY^*(t)-dY(t)) \\
& + \int_0^T [(\sigma(t,X^*(t),Y^*(t),\pi^*(t)) - \sigma(t,X(t),Y(t),\pi(t)))B_1^*(t) \\
& +(\beta(t,X^*(t),Y^*(t),\pi^*(t)) - \sigma(t,X(t),Y(t),\pi(t)))B_2^*(t)]dt \\
& + \int_0^T \int_{\mathbb{R}} (\gamma(t,X^*(t),Y^*(t),\pi^*(t),z) \\
& -\gamma(t,X^*(t),Y^*(t),\pi^*(t),z))D_1^*(t,z)\nu(dz)dt \\
& + \int_0^T (\phi(Y^*(t)) - \phi(Y(t)))B_3^*(t)dt\Big] \\
= \;& \mathbb{E}\Big[-\int_0^T (X^*(t)-X(t))\frac{\partial \mathcal{H}^*}{\partial x}(t)dt - \int_0^T (Y^*(t)-Y(t))\frac{\partial \mathcal{H}^*}{\partial y}(t)dt \\
& + \int_0^T (A_1^*(t)b(t,X^*(t),Y^*(t),\pi^*(t)) - b(t,X(t),Y(t),\pi(t)))dt \\
& + \int_0^T (\varphi(Y^*(t)) - \varphi(Y(t)))A_2^*(t)dt + \int_0^T (\phi(Y^*(t)) - \phi(Y(t)))B_3^*(t)dt \\
& + \int_0^T [(\sigma(t,X^*(t),Y^*(t),\pi^*(t)) - \sigma(t,X(t),Y(t),\pi(t)))B_1^*(t) \\
& +(\beta(t,X^*(t),Y^*(t),\pi^*(t)) - \sigma(t,X(t),Y(t),\pi(t)))B_2^*(t)]dt \\
& + \int_0^T \int_{\mathbb{R}} (\gamma(t,X^*(t),Y^*(t),\pi^*(t),z) \\
& -\gamma(t,X^*(t),Y^*(t),\pi^*(t),z))D_1^*(t,z)\nu(dz)dt\Big] \,,
\end{aligned}
$$

where we have used the notation

$$
\mathcal{H}^*(t) = \mathcal{H}(t,X^*(t),Y^*(t),\pi^*(t),A_1^*(t),A_2^*(t),B_1^*(t),B_2^*(t),B_3^*(t),D_1^*(t,\cdot))\,.
$$

On the other hand, by definition of $\mathcal{H}$ in (4.4), we see that

$$
\begin{aligned}
& \mathcal{J}_1 \\
= \;& \mathbb{E}\Big[\int_0^T (f(t,X^*(t),Y^*(t),\pi^*(t)) - f(t,X(t),Y(t),\pi(t)))dt\Big] \\
= \;& \mathbb{E}\Big[\int_0^T [\mathcal{H}(t,X^*(t),Y^*(t),\pi^*(t),A_1^*(t),A_2^*(t),B_1^*(t),B_2^*(t),B_3^*(t),D_1^*(t,\cdot)) \\
& -\mathcal{H}(t,X^*(t),Y^*(t),\pi^*(t),A_1^*(t),A_2^*(t),B_1^*(t),B_2^*(t),B_3^*(t),D_1^*(t,\cdot))]dt \\
& -\int_0^T A_1^*(t)(A_1^*(t)b(t,X^*(t),Y^*(t),\pi^*(t)) - b(t,X(t),Y(t),\pi(t)))dt \\
& -\int_0^T (\varphi(Y^*(t)) - \varphi(Y(t)))A_2^*(t)dt + \int_0^T (\phi(Y^*(t)) - \phi(Y(t)))B_3^*(t)dt \\
& -\int_0^T [(\sigma(t,X^*(t),Y^*(t),\pi^*(t)) - \sigma(t,X(t),Y(t),\pi(t)))B_1^*(t) \\
& -(\beta(t,X^*(t),Y^*(t),\pi^*(t)) - \sigma(t,X(t),Y(t),\pi(t)))B_2^*(t)]dt \\
& -\int_0^T \int_{\mathbb{R}} (\gamma(t,X^*(t),Y^*(t),\pi^*(t),z) \\
& -\gamma(t,X^*(t),Y^*(t),\pi^*(t),z))D_1^*(t,z)\nu(dz)dt\Big]\,.
\end{aligned}
$$

Then, summing the above two expressions, we obtain

$$
\begin{aligned}
& \mathcal{J}_1 + \mathcal{J}_2 \\
= \;& \mathbb{E}\Big[\int_0^T [\mathcal{H}(t,X^*(t),Y^*(t),\pi^*(t),A_1^*(t),A_2^*(t),B_1^*(t),B_2^*(t),B_3^*(t),D_1^*(t,\cdot)) \\
& -\mathcal{H}(t,X(t),Y(t),\pi(t),A_1^*(t),A_2^*(t),B_1^*(t),B_2^*(t),B_3^*(t),D_1^*(t,\cdot))]dt \\
& -\int_0^T (X^*(t) - X(t))\frac{\partial \mathcal{H}^*}{\partial x}(t)dt - \int_0^T (Y^*(t) - Y(t))\frac{\partial \mathcal{H}^*}{\partial y}(t)dt\,.
\end{aligned}
$$

By the concavity of $\mathcal{H}$, i.e., conditions $(i)$ and $(ii)$, we have

$$
\begin{aligned}
& \mathbb{E}\Big[\int_0^T [\mathcal{H}(t,X^*(t),Y^*(t),\pi^*(t),A_1^*(t),A_2^*(t),B_1^*(t),B_2^*(t),B_3^*(t),D_1^*(t,\cdot)) \\
& -\mathcal{H}(t,X(t),Y(t),\pi(t),A_1^*(t),A_2^*(t),B_1^*(t),B_2^*(t),B_3^*(t),D_1^*(t,\cdot))]dt\Big] \\
\geq \;& \mathbb{E}\Big[\int_0^T (X^*(t) - X(t))\frac{\partial \mathcal{H}^*}{\partial x}(t)dt + \int_0^T (Y^*(t) - Y(t))\frac{\partial \mathcal{H}^*}{\partial y}(t)dt \\
& +\int_0^T (\pi^*(t) - \pi(t))\frac{\partial \mathcal{H}^*}{\partial \pi}(t)dt\Big]\,.
\end{aligned}
$$

Then, by the maximality of the strategy $\pi^* \in \mathcal{A}$ and the concavity of the Hamiltonian $\mathcal{H}$,

$$\begin{aligned} & \mathbb{E}\Big[\int_0^T [\mathcal{H}(t, X^*(t), Y^*(t), \pi^*(t), A_1^*(t), A_2^*(t), B_1^*(t), B_2^*(t), B_3^*(t), D_1^*(t,\cdot)) \\ & -\mathcal{H}(t, X(t), Y(t), \pi(t), A_1^*(t), A_2^*(t), B_1^*(t), B_2^*(t), B_3^*(t), D_1^*(t,\cdot))]dt\Big] \\ \geq \ & \mathbb{E}\Big[\int_0^T (X^*(t) - X(t))\frac{\partial \mathcal{H}^*}{\partial x}(t)dt + \int_0^T (Y^*(t) - Y(t))\frac{\partial \mathcal{H}^*}{\partial y}(t)dt\Big]. \end{aligned}$$

Hence $\mathcal{J}(\pi^*) - \mathcal{J}(\pi) = \mathcal{J}_1 + \mathcal{J}_2 \geq 0$. Therefore, $\mathcal{J}(\pi^*) \geq \mathcal{J}(\pi)$, that is, the strategy $\pi^* \in \mathcal{A}$ is optimal. □

Note that the sufficient maximum principle presented in Theorem 4.2.1 is based on the concavity of the Hamiltonian, however, this condition does not hold in many concrete situations. Below, we relax this condition and state the necessary maximum principle (also called equivalent maximum principle) for our control problem. Thus, we further consider the following assumptions.

- For all $s \in [0,T]$ and all bounded $\{\mathcal{F}_s\}_{s\in[0,T]}$-measurable random variable $\alpha(\omega)$, the control $\xi(t) := \chi_{[s,T]}(t)\alpha(\omega)$ belongs to the admissible strategy $\mathcal{A}$.

- For all $\pi, \zeta \in \mathcal{A}$, with $\zeta$ bounded, there exists $\epsilon > 0$ such that the control $\pi(t) + \ell\zeta(t) \in \mathcal{A}$, for all $\ell \in (-\epsilon; \epsilon)$.

- We define the derivative processes

$$x_1(t) := \frac{d}{d\ell}X^{\pi+\ell\zeta}(t)\Big|_{\ell=0} \quad \text{and} \quad \mathrm{y}_1(\mathrm{t}) := \frac{\mathrm{d}}{\mathrm{d}\ell}\mathrm{Y}^{\pi+\ell\zeta}(\mathrm{t})\Big|_{\ell=0}.$$

  Then, for all $\pi, \zeta \in \mathcal{A}$, with $\zeta$ bounded, the above derivatives exist and belong to $L^2([0,T]\times\Omega)$, and (4.1) and (4.2),

$$\begin{aligned} dx_1(t) \ = \ & \\ x_1(t)\Big[&\frac{\partial b}{\partial x}(t)dt + \frac{\partial \sigma}{\partial x}(t)dW_1(t) + \frac{\partial \beta}{\partial x}(t)dW_2(t) + \int_{\mathbb{R}}\frac{\partial \gamma}{\partial x}(t,z)\tilde{N}(dt,dz)\Big] \\ +y_1(t)\Big[&\frac{\partial b}{\partial y}(t)dt + \frac{\partial \sigma}{\partial y}(t)dW_1(t) + \frac{\partial \beta}{\partial y}(t)dW_2(t) + \int_{\mathbb{R}}\frac{\partial \gamma}{\partial y}(t,z)\tilde{N}(dt,dz)\Big] \\ +\zeta(t)\Big[&\frac{\partial b}{\partial \pi}(t)dt + \frac{\partial \sigma}{\partial \pi}(t)dW_1(t) + \frac{\partial \beta}{\partial \pi}(t)dW_2(t) + \int_{\mathbb{R}}\frac{\partial \gamma}{\partial \pi}(t,z)\tilde{N}(dt,dz)\Big], \end{aligned}$$

where we have used the notation $\frac{\partial b}{\partial x}(t) = \frac{\partial b}{\partial x}(t, X(t), Y(t), \pi(t))$, $\frac{\partial \sigma}{\partial x}(t) = \frac{\partial \sigma}{\partial x}(t, X(t), Y(t), \pi(t))$, etc. Moreover,

$$dy_1(t) = y_1(t)[\varphi'(Y(t))dt + \phi'(Y(t))dW_2(t)]\,.$$

**Theorem 4.2.2.** (Necessary maximum principle) *Let $\pi^* \in \mathcal{A}$ with corresponding solutions $X^*(t)$, $(A_1^*(t), B_1^*(t), B_2^*(t), D_1^*(t,\cdot))$, $(A_2^*(t), B_3^*(t), B_4^*(t), D_2^*(t.\cdot))$ of (4.1), (4.5) and (4.7) respectively, and the derivative processes $x_1(t)$ and $y_1(t)$ given above. Moreover, assume the following integrability conditions:*

$$\begin{aligned}\mathbb{E}\Big\{\int_0^T (A_1^*)^2(t)\Big[x_1^2(t)\Big(\Big(\frac{\partial\sigma}{\partial x}(t)\Big)^2+\Big(\frac{\partial\beta}{\partial x}(t)\Big)^2+\int_{\mathbb{R}}\Big(\frac{\partial\gamma}{\partial x}(t,z)\Big)^2\nu(dz)\Big)\\ y_1^2(t)\Big(\Big(\frac{\partial\sigma}{\partial y}(t)\Big)^2+\Big(\frac{\partial\beta}{\partial y}(t)\Big)^2+\int_{\mathbb{R}}\Big(\frac{\partial\gamma}{\partial y}(t,z)\Big)^2\nu(dz)\Big)\\ \zeta^2(t)\Big(\Big(\frac{\partial\sigma}{\partial \pi}(t)\Big)^2+\Big(\frac{\partial\beta}{\partial \pi}(t)\Big)^2+\int_{\mathbb{R}}\Big(\frac{\partial\gamma}{\partial \pi}(t,z)\Big)^2\nu(dz)\Big)\Big]dt\\ +\int_0^T (A_2^*)^2(t)y_1^2(t)(\phi'(Y(t)))^2dt\Big\}\\ < \infty\end{aligned} \tag{4.9}$$

*and*

$$\begin{aligned}\mathbb{E}\Big\{\int_0^T x_1^2(t)\Big[(B_1^*)^2(t)+(B_2^*)^2(t)+\int_{\mathbb{R}}(D_1^*)^2(t,z)\nu(dz)\Big]dt\\ \int_0^T y_1^2(t)\Big[(B_3^*)^2(t)+(B_4^*)^2(t)+\int_{\mathbb{R}}(D_2^*)^2(t,z)\nu(dz)\Big]dt\Big\} \;<\; \infty\,.\end{aligned}$$

*Then the following are equivalent*

1. $\frac{d}{d\ell}\mathcal{J}(\pi+\ell\zeta)\Big|_{\ell=0} = 0$ *for all bounded* $\zeta \in \mathcal{A}$;
2. $\frac{d\mathcal{H}}{d\pi}(t, X^*(t), Y(t), \pi^*(t), A_1^*(t), A_2^*(t), B_1^*(t), B_2^*(t), D_1^*(t,z)) = 0$ *for all* $t \in [0,T]$.

*Proof.* From (4.3), we have that

$$\begin{aligned}\frac{d}{d\ell}\mathcal{J}(\pi+\ell\zeta)\Big|_{\ell=0} &= \mathbb{E}\Big[\int_0^T\Big(\frac{\partial f}{\partial x}(t)x_1(t)+\frac{\partial f}{\partial y}(t)y_1(t)+\frac{\partial f}{\partial \pi}(t)\zeta(t)\Big)dt\\ &\quad+\frac{\partial g}{\partial x}(X(T),Y(T))x_1(T)+\frac{\partial g}{\partial y}(X(T),Y(T))y_1(T)\Big]_{\ell=0}.\end{aligned}$$

Let

$$I(t) := \mathbb{E}\Big[\frac{\partial g}{\partial x}(X(T),Y(T))x_1(T) + \frac{\partial g}{\partial y}(X(T),Y(T))y_1(T)\Big].$$

By Itôs formula, the dynamics of $x_1$ and $y_1$ and (4.9), we get

$$\begin{aligned}
I(t) &= \mathbb{E}\Big[\frac{\partial g}{\partial x}(X(T),Y(T))x_1(T) + \frac{\partial g}{\partial y}(X(T),Y(T))y_1(T)\Big] \\
&= \mathbb{E}\Big[A_1(T)x_1(T) + A_2(T)y_1(T)\Big] \qquad (4.10)\\
&= \mathbb{E}\Big[\int_0^T x_1(t)\Big(A_1(t)\frac{\partial b}{\partial x}(t) + B_1(t)\frac{\partial \sigma}{\partial x}(t) + B_2(t)\frac{\partial \beta}{\partial x}(t) \\
&\quad + \int_{\mathbb{R}} \frac{\partial \gamma}{\partial x}(t,z)D_1(t,z)\nu(dz) - \frac{\partial \mathcal{H}}{\partial x}(t)\Big)dt \\
&\quad + \int_0^T y_1(t)\Big(A_1(t)\frac{\partial b}{\partial y}(t) + A_2(t)\varphi'(Y(t)) + B_1(t)\frac{\partial \sigma}{\partial y}(t) + B_2(t)\frac{\partial \beta}{\partial y}(t) \\
&\quad + B_4(t)\phi'(Y(t)) + \int_{\mathbb{R}} \frac{\partial \gamma}{\partial y}(t,z)D_1(t,z)\nu(dz) - \frac{\partial \mathcal{H}}{\partial y}(t)\Big)dt \\
&\quad + \int_0^T \zeta(t)\Big(A_1(t)\frac{\partial b}{\partial \pi}(t) + B_1(t)\frac{\partial \sigma}{\partial \pi}(t) + B_2(t)\frac{\partial \beta}{\partial \pi}(t) \\
&\quad + \int_{\mathbb{R}} \frac{\partial \gamma}{\partial \pi}(t,z)D_1(t,z)\nu(dz)(t)\Big)dt\Big].
\end{aligned}$$

On the other hand, by definition of the Hamiltonian (4.4), we have

$$\begin{aligned}
\nabla_{x,y,\pi}\mathcal{H}(t) &= \frac{\partial \mathcal{H}}{\partial x}(t)x_1(t) + \frac{\partial \mathcal{H}}{\partial y}(t)y_1(t) + \frac{\partial \mathcal{H}}{\partial \pi}(t)\zeta(t) \\
&= x_1(t)\Big[\frac{\partial f}{\partial x}(t) + A_1(t)\frac{\partial b}{\partial x}(t) + B_1(t)\frac{\partial \sigma}{\partial x}(t) + B_2(t)\frac{\partial \beta}{\partial x}(t) \\
&\quad + \int_{\mathbb{R}} \frac{\partial \gamma}{\partial x}(t,z)D_1(t,z)\nu(dz)\Big] \\
&\quad + y_1(t)\Big[\frac{\partial f}{\partial y}(t) + A_1(t)\frac{\partial b}{\partial y}(t) + A_2(t)\varphi'(Y(t)) + B_1(t)\frac{\partial \sigma}{\partial y}(t) \\
&\quad + B_2(t)\frac{\partial \beta}{\partial y}(t) + B_3(t)\phi'(Y(t)) + \int_{\mathbb{R}} \frac{\partial \gamma}{\partial x}(t,z)D_1(t,z)\nu(dz)\Big] \\
&\quad + \zeta(t)\Big[\frac{\partial f}{\partial \pi}(t) + A_1(t)\frac{\partial b}{\partial \pi}(t) + B_1(t)\frac{\partial \sigma}{\partial \pi}(t) + B_2(t)\frac{\partial \beta}{\partial \pi}(t) \\
&\quad + \int_{\mathbb{R}} \frac{\partial \gamma}{\partial \pi}(t,z)D_1(t,z)\nu(dz)\Big].
\end{aligned}$$

Combining this and (4.10), we get

$$\frac{d}{d\ell}\mathcal{J}(\pi+\ell\zeta)\Big|_{\ell=0} = \mathbb{E}\Big[\int_{\mathbb{R}} \frac{\partial \mathcal{H}}{\partial \pi}(t)\zeta(t)dt\Big]\,,$$

Then, we conclude that

$$\frac{d}{d\ell}\mathcal{J}(\pi+\ell\zeta)\Big|_{\ell=0} = 0\,,$$

for all bounded $\zeta \in \mathcal{A}$ implies that the same holds in particular for $\zeta \in \mathcal{A}$ of the form

$$\zeta(t) := \chi_{[s,T]}(t)\alpha(\omega),\ t \in [0,T]$$

for a fixed $s \in [0,T)$, where $\alpha(\omega)$ is a bounded $\{\mathcal{F}_{t_0}\}_{t_0\in[0,T)}$-measurable random variable. Therefore,

$$\mathbb{E}\Big[\int_s^T \mathbb{E}\Big[\frac{\partial \mathcal{H}}{\partial \pi}(t) \mid \mathcal{F}_t\Big]\alpha(t)dt\Big] = 0$$

Differentiating with respect to $s$, we have

$$\mathbb{E}\Big[\frac{\partial \mathcal{H}}{\partial \pi}(s) \mid \mathcal{F}_t\Big] = 0\,, \quad a.a.\ \ s \in [0,T)\,.$$

This proves that $1 \Rightarrow 2$.

Conversely, using the fact that every bounded $\zeta \in \mathcal{A}$ can be approximated by a linear combination of the form $\pi(t) + \ell\zeta(t) \in \mathcal{A}$, The above arguments can be reversed to show that $2 \Rightarrow 1$ as in Pamen and Momeya [80], Theorem 3.5.

□

## 4.3 Application to optimal investment- consumption and life insurance selection problem

We consider a financial market consisting of one risk-free asset $(B(t))_{0\le t\le T}$ and one risky asset $(S(t))_{0\le t\le T}$. Their respective prices are given by the following SDE:

$$\begin{aligned} dB(t) &= r(t)B(t)dt\,, \quad B(0)=1\,, && (4.11)\\ dS(t) &= S(t)\Big[\alpha(t,Y(t))dt+\beta(t,Y(t))dW_1(t)+\sigma(t,Y(t))dW_2(t) \\ &\qquad + \int_{\mathbb{R}} \gamma(t,Y(t),z)\widetilde{N}(dt,dz)\Big]\,, \quad S(0)=s>0\,, && (4.12) \end{aligned}$$

where $Y$ is a continuous time economic external factor governed by

$$dY(t) = g(Y(t))dt + dW_1(t)\,. \tag{4.13}$$

Here, the associated parameters in the model satisfy the following assumptions:

(**A1**) The interest rate $r(t)$ is positive, deterministic and integrable for all $t \in [0,T]$. The mean rate of return $\alpha$, the volatilities $\beta\,,\sigma$ and the dispersion rate $\gamma > -1$, are $\mathbb{R}$-valued functions are assumed to be continuously differentiable functions ($\in \mathcal{C}^1$) and bounded. Note that, by the continuity of $Y$, the process $S$ in (4.12) is well defined on $[0,T]$. We also assume the following integrability condition:

$$\mathbb{E}\left[\int_0^T (\beta^2(t,y)+\sigma^2(t,y)+\int_{\mathbb{R}\setminus\{0\}} |\gamma(t,y,z)|^2\nu(dz))dt\right] < \infty\,.$$

Suppose that $g \in \mathcal{C}^1(\mathbb{R})$ with the first derivative bounded, i.e., $|g'(y)| \le K$ and satisfy a Lipschitz condition on the $\mathbb{R}$-valued function $g$:

(**A2**) There exists a positive constant $C$ such that:

$$|g(y)-g(w)| \le C|y-w|\,, \qquad y,w\in\mathbb{R}\,.$$

Consider a wage earner whose life time is a nonnegative random variable $\tau$ defined on the probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in[0,T]},\mathbb{P})$. As in Mousa *et al.* [72], we suppose the existence of an insurance market composed of $M$ insurance companies, with each insurance company continuously offering life insurance contracts. We assume that the wage earner is paying premium insurance rate

$p_n(t)$, at time $t$ for each company $n = 1, 2, \ldots, M$. If the wage earner dies, the insurance companies will pay $p_n(\tau)/\eta_n(\tau)$ to his/her beneficiary. Here, $\eta_n > 0$ is the $n$th insurance company premium-payout ratio. Additionally, we assume that the $M$ insurance companies under consideration offer pairwise distinct contracts in the sense that $\eta_{n_1}(t) \neq \eta_{n_2}(t)$, for every $n_1 \neq n_2$, a.e. When he/she dies, the total legacy is given by:

$$\mathcal{J}_n(\tau) := X(\tau) + \sum_{n=1}^{M} \frac{p_n(\tau)}{\eta_n(\tau)}\,, \tag{4.14}$$

where $X(\tau)$ is the wealth process of the wage earner at time $\tau \in [0, T]$.

Let $c(t)$ denote the consumption rate of the wage earner and $\pi(t)$ the fraction of the wage earner's wealth invested in the risky share at time $t$, satisfying the following integrability condition.

$$\int_0^T [c(t) + \pi^2(t)]dt < \infty, \quad \text{a.s.} \tag{4.15}$$

Moreover, we assume that the shares are divisible, continuously traded and there are no transaction costs, taxes or short-selling constraints in the trading. Then the wealth process $X(t)$ is defined by the following SDE:

$$\begin{aligned}
dX(t) &= \left[X(t)(r(t) + \pi(t)\mu(t, Y(t))) - c(t) - \sum_{n=1}^{M} p_n(t)\right] dt \qquad (4.16)\\
&\quad +\pi(t)\beta(t, Y(t))X(t)dW_1(t) + \pi(t)X(t)\sigma(t, Y(t))dW_2(t)\\
&\quad +\pi(t)X(t)\int_{\mathbb{R}} \gamma(t, Y(t), z)\widetilde{N}(dt, dz)\,, \quad t \in (0, \tau \wedge T]\,,\\
X(0) &= x > 0\,,
\end{aligned}$$

where $\mu(t, Y(t)) := \alpha(t, Y(t)) - r(t)$ is the appreciation rate and $\tau \wedge T := \min\{\tau, T\}$. We assume that $\mu(t, Y(t)) > 0$, i.e., the expected return of the risk share is higher than the interest rate.

Let $\rho(t) > 0$ be deterministic process denoting the discount rate process. We define the utility functions $U_i : [0, T] \times \mathbb{R}_+ \to \mathbb{R}_+\,,\ \ i = 1, 2, 3$ as the concave, non-decreasing, continuous and differentiable functions with respect to the second variable, and the strictly decreasing continuous inverse functions $I_i : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+ \,, \quad i = 1,2,3\,,$ by

$$I_i(t,x) = \left(\frac{\partial U_i(t,x)}{\partial x}\right)^{-1} , \quad i = 1,2,3 \text{ and}. \tag{4.17}$$

Let $p(t) := (p_1(t), \ldots, p_M(t))$ be the vector of the insurance rates paid at the insurance companies. The wage earner faces the problem of choosing the optimal strategy $\mathcal{A} := \{(\pi, c, p) := (\pi(t), c(t), p(t))_{t\in[0,T]}\}$ which maximizes the discounted expected utilities from the consumption during his/her lifetime $[0, \tau \wedge T]$, from the wealth if he/she is alive until the terminal time $T$ and from the legacy if he/she dies before time $T$. This problem can be defined by the following performance functional (for more details see, e.g., Pliska and Ye [83], Øksendal and Sulem [77], Azevedo et. al. [4], Guambe and Kufakunesu [41]).

$$\begin{aligned} & J(0,x,\pi,c,p) \\ := \ & \mathbb{E}\Big[\int_0^{\tau\wedge T} e^{-\int_0^s \rho(u)du} U_1(s,c(s))ds \\ & + e^{-\int_0^\tau \rho(u)du} U_2(\tau, \mathcal{J}(\tau))\chi_{\{\tau\leq T\}} + e^{-\int_0^T \rho(u)du} U_3(X(T))\chi_{\{\tau>T\}}\Big], \end{aligned} \tag{4.18}$$

where $\chi_A$ is a characteristic function of the set $A$.

The set of strategies $\mathcal{A} := \{(\pi, c, p) := (\pi(t), c(t), p(t))_{t\in[0,T]}\}$ is said to be admissible if, in addition to the integrability condition (4.15), the SDE (4.16) has a unique strong solution such that $X(t) \geq 0$, $\mathbb{P}$-a.s. and

$$\begin{aligned} \mathbb{E}\Big[\int_0^{\tau\wedge T} & e^{-\int_0^s \rho(u)du} U_1(s,c(s))ds + e^{-\int_0^\tau \rho(u)du} U_2(\tau, \mathcal{J}(\tau))\chi_{\{\tau\leq T\}} \\ & + e^{-\int_0^T \rho(u)du} U_3(X(T))\chi_{\{\tau>T\}}\Big] < \infty\,. \end{aligned}$$

Note that from the conditional survival probability of the wage earner (2.16) and the conditional survival probability density of death of the wage earner (2.17), we can write the dynamic version of the functional (4.18) by:

$$\begin{aligned} J(t,x,\pi,c,p) &= \mathbb{E}_{t,x}\Big[\int_t^T e^{-\int_t^s(\rho(u)+\lambda(u))du}[U_1(s,c(s))+\lambda(s)U_2(s,\mathcal{J}(s))]ds \\ &\qquad +e^{-\int_t^T(\rho(u)+\lambda(u))du}U_3(X(T))\Big]. \end{aligned} \tag{4.19}$$

Thus, the problem of the wage earner is to maximize the above dynamic performance functional under the admissible strategy $\mathcal{A}$. Therefore, the value function $V(t,x,y)$ can be restated in the following form:

$$V(t,x,y) = \sup_{(\pi,c,p)\in\mathcal{A}} J(t,x,\pi,c,p)\,. \tag{4.20}$$

Applying the results in the previous section to solve the above problem, we define the Hamiltonian $\mathcal{H} : [0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times(0,1)\times\mathbb{R}^M\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by:

$$\begin{aligned} &\mathcal{H}(t,X(t),Y(t),c(t),\pi(t),p(t),A_1(t),A_2(t),B_1(t),B_2(t),B_3(t),D_1(t)) \\ =\;& e^{-\int_0^t(\rho(s)+\lambda(s))ds}[U_1(t,c(t))+\lambda(t)U_2(t,\mathcal{J}(t))] \\ &+\Big[X(t)(r(t)+\pi(t)\mu(t,Y(t)))-c(t)-\sum_{n=1}^M p_n(t)\Big]A_1(t)+g(Y(t))A_2(t) \\ &+\pi(t)X(t)(\beta(t,Y(t))B_1(t)+\sigma(t,Y(t))B_2(t))+B_3(t) \\ &+\pi(t)X(t)\int_{\mathbb{R}}\gamma(t,Y(t),z)D_1(t,z)\nu(dz)\,. \end{aligned} \tag{4.21}$$

The adjoint equations corresponding to the admissible strategy $(\pi,c,p)\in\mathcal{A}$ are given by the following BSDEs

$$\begin{aligned} &dA_1(t) = \\ &-\frac{\partial\mathcal{H}}{\partial x}(t,X(t),Y(t),c(t),\pi(t),p(t),A_1(t),A_2(t),B_1(t),B_2(t),B_3(t),D_1(t))dt \\ &+B_1(t)dW_1(t)+B_2(t)dW_2(t)+\int_{\mathbb{R}}D_1(t,z)\tilde{N}(dt,dz)\,; \end{aligned} \tag{4.22}$$

$$A_1(T) = e^{-\int_0^T(\rho(s)+\lambda(s))ds}U_3'(X(T))$$

where $U' := U_x$ and

$$
\begin{aligned}
dA_2(t) &= \\
&-\frac{\partial \mathcal{H}}{\partial y}(t, X(t), Y(t), c(t), \pi(t), p(t), A_1(t), A_2(t), B_1(t), B_2(t), B_3(t), D_1(t))dt \\
&+B_1(t)dW_3(t) + B_4(t)dW_2(t) + \int_{\mathbb{R}} D_2(t,z)\tilde{N}(dt,dz)\,; \qquad (4.23) \\
&A_2(T) = 0\,.
\end{aligned}
$$

To solve our optimization problem, we consider the power utility functions of the CRRA type defined as follows $U_i(t,x) = U_i(x) = \kappa_i \frac{x^\delta}{\delta}\,, \quad i = 1,2,3,$ where $\delta \in (-\infty, 1) \setminus \{0\}$ and $\kappa_i > 0$ are constants. Thus, the inverse function (4.17) is given by $I_i(t,x) = I_i(x) = \left(\frac{x}{\kappa_i}\right)^{-\frac{1}{1-\delta}}$.

The following theorem gives the characterization of the optimal strategy.

**Theorem 4.3.1.** *Suppose that the assumptions* (**A1**) − (**A2**) *and the integrability condition* (4.15) *hold. Then the optimal strategy* $(c^*, p^*, \pi^*) \in \mathcal{A}$ *for the problem* (4.20) *is given by:*

(*i*) *the optimal consumption process is given by*

$$
\begin{aligned}
c^*(t,x,y) &= I_1\left(t, \frac{A_1^*(t)}{\kappa_1}(t) e^{\int_0^t (\rho(s)+\lambda(s))ds}\right) \\
&= \left(\frac{A_1^*(t)}{\kappa_1}\right)^{\frac{1}{\delta-1}} e^{\frac{1}{\delta-1}\int_0^t (\rho(s)+\lambda(s))ds}\,; \qquad (4.24)
\end{aligned}
$$

(*ii*) *for each* $n \in \{1, 2, \ldots, M\}$, *the optimal premium insurance* $p_n(t,x,y)$ *is given by*

$$
\begin{aligned}
p_n^*(t,x,y) &= \begin{cases} \max\left\{0, \left[I_2\left(t, \frac{\eta_n(t)}{\kappa_2 \lambda(t)} A_1^*(t) e^{\int_0^t (\rho(s)+\lambda(s))ds}\right) - x\right]\right\}, & \textit{if } n = n^*(t) \\ 0, & \textit{otherwise}\,, \end{cases} \\
&= \begin{cases} \max\left\{0, \eta_n(t)\left[\left(\frac{\eta_n(t) A_1^*(t)}{\kappa_2 \lambda(t)}\right)^{\frac{1}{\delta-1}} e^{\frac{1}{\delta-1}\int_0^t (\rho(s)+\lambda(s))ds} - x\right]\right\}, & \textit{if } n = n^*(t) \\ 0, & \textit{otherwise}\,, \end{cases} \qquad (4.25)
\end{aligned}
$$

*where* $n^*(t) = \arg\min_{n\in\{1,2,\ldots,M\}}\{\eta_n(t)\}$

(*iii*) *and, the optimal allocation* $\pi^*(t,x,y) \in (0,1)$ *is the solution of the following equation*

$$\begin{aligned}\beta(t,y)h_y(t,y) - \Big\{\mu(t,y) - (1-\delta)\left(\beta^2(t,y)+\sigma^2(t,y)\right)\pi \\ - \int_{\mathbb{R}}\left[1-(1+\pi\gamma(t,y,z))^{\delta-1}\right]\gamma(t,y,z)\nu(dz)\Big\}h(t,y) &= 0\,,\end{aligned}$$

*where* $h \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R})$ *to be specified later in the proof.*

*Proof.* From the Hamiltonian function (4.21) and the definition of the utility functions $U_1$, $U_2$, we can deduce the following conditions:

$$\begin{aligned}\mathcal{H}_{cc} &= e^{-\int_0^t(\rho(s)+\lambda(s))ds}\frac{\partial^2 U_1}{\partial c^2}(t,c) < 0 \\ \mathcal{H}_{p_{n_1}p_{n_2}} &= e^{-\int_0^t(\rho(s)+\lambda(s))ds}\frac{\lambda(t)}{\eta_{n_1}\eta_{n_2}}\frac{\partial^2 U_2}{\partial x^2}\left(t, x+\sum_{n=1}^{M}\frac{p_n}{\eta_n(t)}\right) < 0\,.\end{aligned}$$

Thus, it is sufficient to obtain the optimal consumption and insurance $(c^*, p^*)$ by applying the first order conditions of optimality. Then from (4.21) we have the following:

(*i*) The optimal consumption $c^*(t,x,y)$ is obtained from the following

$$-A_1(t) + e^{-\int_0^t(\rho(s)+\lambda(s))ds}\frac{\partial U_1}{\partial c}(t,c) = 0\,.$$

From (4.17), the optimal consumption can explicitly be obtained by

$$\begin{aligned}c^*(t,x,y) &= I_1\left(t, \frac{A_1^*(t)}{\kappa_1}e^{\int_0^t(\rho(s)+\lambda(s))ds}\right) \\ &= \left(\frac{A_1^*(t)}{\kappa_1}\right)^{\frac{1}{\delta-1}} e^{\frac{1}{\delta-1}\int_0^t(\rho(s)+\lambda(s))ds}\,;\end{aligned}$$

(*ii*) the optimal premium insurance $p_n^*(t,x,y)$ is obtained using the Kuhn-Tucker conditions of optimality. As in Mousa *et al.* [72], we are looking for the solutions $(p_1(t,x,y);\ldots;p_M(t,x,y);\xi_1(t,x,y);\ldots;\xi_M(t,x,y))$ in the following system

$$\begin{cases} -A_1(t) + \frac{\lambda(t)}{\eta_n(t)}e^{-\int_t^0(\rho(s)+\lambda(s))ds}\frac{\partial U_2}{\partial x}\left(t, x+\sum_{n=1}^{M}\frac{p_n}{\eta_n(t)}\right) = -\xi_n(t,x,y) \\ p_n(t,x,y) \geq 0\,; \quad \xi_n(t,x,y) \geq 0\,; \\ p_n(t,x,y)\xi_n(t,x,y) = 0, \quad \forall n = 1,2,\ldots M\,. \end{cases} \tag{4.26}$$

First, suppose that $n_1 \neq n_2$. If we have $\xi_{n_1}(t,x,y) = \xi_{n_2}(t,x,y)$, for some $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}$, one must have $\eta_{n_1}(t) = \eta_{n_2}(t)$. Thus, from the assumption that all the insurance companies offer distinct contracts, we obtain that for every $n_1, n_2 \in \{1,2,\ldots,M\}$, such that $n_1 \neq n_2$, then $\xi_{n_1}(t,x,y) \neq \xi_{n_2}(t,x,y)$; $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}$, a.e. Therefore, there is at most one $n \in \{1,2,\ldots,M\}$ such that $p_n(t,x,y) \neq 0$.

Then from the first equation in the system (4.26),

$$\eta_{n_1}(A_1(t) - \xi_{n_1}(t,x,y)) = \eta_{n_2}(A_1(t) - \xi_{n_2}(t,x,y)) \,.$$

Hence, we can conclude that if $\xi_{n_1}(t,x,y) > \xi_{n_2}(t,x,y)$, then $\eta_{n_1}(t) > \eta_{n_2}(t)$. Moreover, if $\xi_{n_1}(t,x,y) = 0$ for some $t \in [0,T]$, $\eta_{n_1}(t) < \eta_{n_2}(t)$, $\forall n_2 \in \{1,2,\ldots,M\}$ such that $n_1 \neq n_2$. From this point, let $n^*(t) = \arg\min_{n \in \{1,2,\ldots,M\}}\{\eta_n(t)\}$, then either $p_n(t,x,y) = 0$ or $p_{n^*}(t,x,y) > 0$ is the solution to the equation

$$-A_1(t) + \frac{\lambda(t)}{\eta_{n^*}(t)} e^{-\int_t^0 (\rho(s)+\lambda(s))ds} \frac{\partial U_2}{\partial x}\left(t, x + \frac{p_{n^*}}{\eta_{n^*}(t)}\right) = 0,$$

which gives the required solution

$$\begin{aligned} p_n^*(t,x,y) &= \begin{cases} \max\left\{0,\ \left[I_2\left(t,\ \frac{\eta_n(t)}{\kappa_2\lambda(t)} A_1^*(t) e^{\int_0^t (\rho(s)+\lambda(s))ds}\right) - x\right]\right\}, & \text{if } \ n = n^*(t) \\ 0, & \text{otherwise}\,, \end{cases} \\ &= \begin{cases} \max\left\{0,\ \eta_n(t)\left[\left(\frac{\eta_n(t)A_1^*(t)}{\kappa_2\lambda(t)}\right)^{\frac{1}{\delta-1}} e^{\frac{1}{\delta-1}\int_0^t (\rho(s)+\lambda(s))ds} - x\right]\right\}, & \text{if } \ n = n^*(t) \\ 0, & \text{otherwise}\,; \end{cases} \end{aligned}$$

(*iii*) Since the expression involving $\pi$ in the Hamiltonian $\mathcal{H}$ (4.21) is linear, for the maximum investment $\pi^*$, we have the following relation

$$\mu(t,y)A_1^*(t) + \beta(t,y)B_1^*(t) + \sigma(t,y)B_2^*(t) + \int_{\mathbb{R}} \gamma(t,y,z)D_1^*(t,z)\nu(dz) = 0\,. \tag{4.27}$$

To obtain the optimal portfolio, we first solve the adjoint BSDE equations (4.22) and (4.23). From the terminal condition of the adjoint equation (4.22), we try the solution of the first adjoint equation $A_1^*(t)$ of the form

$$A_1^*(t) = X(t)^{\delta-1} e^{-h(t,Y(t))}, \quad h(T,Y(T)) = \int_0^T (\rho(u) + \lambda(u)) du\,. \tag{4.28}$$

On the other hand, for the optimal strategy $(c^*, p_n^*, \pi^*)$, we have

$$dA_1^*(t) = -\eta_{n^*} A_1^*(t)dt + B_1^*(t)dW_1(t) + B_2^*(t)dW_2(t) + \int_{\mathbb{R}} D_1^*(t,z)\tilde{N}(dt,dz)\,. \tag{4.29}$$

Applying the Itô's product rule in (4.28) and from (4.16), (4.24) and (4.25), we obtain

$$\begin{aligned}
& dA_1^*(t) \\
= \;& -x(t)^{\delta-1} e^{-h(t,y)} \Big\{ h_t(t,y) + g(y)h_y(t,y) + \frac{1}{2} h_{yy}(t,y) - \frac{1}{2}(h_y(t,y))^2 \\
& + \frac{1}{2}(\delta-1)\pi^*(t)\beta(t,y)h_y(t,y) - \Big[ (\delta-1)[r(t) + \mu(t,y)\pi^*(t) + \eta_{n^*}(t)] \\
& + \frac{1}{2}(\delta-1)(\delta-2)(\pi^*(t))^2(\beta^2(t,y) + \sigma^2(t,y)) \\
& + \int_{\mathbb{R}} \big[ (1+\pi^*(t)\gamma(t,y,z))^{\delta-1} - 1 - (\delta-1)\pi^*(t)\gamma(t,y,z) \big] \nu(dz) \Big] \\
& + (1-\delta) e^{\frac{1}{1-\delta}h(t,y)} e^{\int_0^t (\rho(s)+\lambda(s))ds} \left[ 1 + \eta_{n^*}(t) \left( \frac{\eta_{n^*}(t)}{\kappa_2 \lambda(t)} \right)^{\frac{1}{\delta-1}} \right] \Big\} dt \\
& + ((\delta-1)\pi^*(t)\beta(t,y) - h_y(t,y)) x(t)^{\delta-1} e^{-h(t,y)} dW_1(t) \\
& + (\delta-1)\pi^*(t) x(t)^{\delta-1} \sigma(t,y) e^{-h(t,y)} dW_2(t) \\
& + x(t)^{\delta-1} e^{-h(t,y)} \int_{\mathbb{R}} \big[ (1+\pi^*(t)\gamma(t,y,z))^{\delta-1} - 1 \big] \tilde{N}(dt,dz)\,.
\end{aligned}$$

Comparing with the adjoint equation (4.29), we get:

$$B_1^*(t) = ((\delta-1)\pi^*(t)\beta(t,y) - h_y(t,y)) x(t)^{\delta-1} e^{-h(t,y)}\,; \tag{4.30}$$

$$B_2^*(t) = (\delta-1)\pi^*(t)\sigma(t,y) x(t)^{\delta-1} e^{-h(t,y)}\,; \tag{4.31}$$

$$D_1^*(t) = x(t)^{\delta-1} e^{-h(t,y)} \big[ (1+\pi^*(t)\gamma(t,y,z))^{\delta-1} - 1 \big]\,. \tag{4.32}$$

Furthermore, $h$ is a solution of the following backward partial differential equation (PDE)

$$\begin{aligned} & h_t(t,y) + (g(y) + \frac{1}{2}(\delta-1)\pi^*(t)\beta(t,y))h_y(t,y) + \frac{1}{2}h_{yy}(t,y) - \frac{1}{2}(h_y(t,y))^2 \\ & \quad + K(t) + (1-\delta)e^{\frac{1}{1-\delta}h(t,y)}e^{\int_0^t(\rho(s)+\lambda(s))ds}\left[1 + \eta_{n^*}(t)\left(\frac{\eta_{n^*}(t)}{\kappa_2\lambda(t)}\right)^{\frac{1}{\delta-1}}\right] \quad = \quad 0\,, \end{aligned} \tag{4.33}$$

with a terminal condition $h(T, Y(T)) = e^{-\int_0^T(\rho(u)+\lambda(u))du}$, where

$$\begin{aligned} K(t) &= -(\delta-1)[r(t) + \mu(t,y)\pi^*(t) + \delta\eta_{n^*}(t) \\ &\quad + \frac{1}{2}(\delta-1)(\delta-2)(\pi^*(t))^2(\beta^2(t,y) + \sigma^2(t,y)) \\ &\quad + \int_{\mathbb{R}}[(1 + \pi^*(t)\gamma(t,y,z))^{\delta-1} - 1 - (\delta-1)\pi^*(t)\gamma(t,y,z)\big]\nu(dz)\,. \end{aligned}$$

Under the assumptions (**A1**) and (**A2**), there exists a unique solution $h \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R})$ of the above PDE. (Pham [82], Theorem 4.1).

Substituting (4.28), (4.30), (4.31), (4.32) into (4.27), we obtain

$$\begin{aligned} & \beta(t,y)h_y(t,y) - \Big\{\mu(t,y) + (\delta-1)\pi^*(t)(\beta^2(t,y) + \sigma^2(t,y)) \\ & + \int_{\mathbb{R}} \gamma(t,y,z)\big[(1 + \pi^*(t)\gamma(t,y,z))^{\delta-1} - 1\big]\nu(dz)\Big\} = 0\,. \end{aligned}$$

Following the similar idea in Benth *et. al.* [9], define a function $f$, by

$$\begin{aligned} f(\pi) &= \beta(t,y)h_y(t,y) - \Big\{\mu(t,y) + (\delta-1)\pi^*(t)(\beta^2(t,y) + \sigma^2(t,y)) \\ &\quad + \int_{\mathbb{R}} \gamma(t,y,z)\big[(1 + \pi^*(t)\gamma(t,y,z))^{\delta-1} - 1\big]\nu(dz)\Big\}\,. \end{aligned}$$

For $\pi = 0$, $f(\pi) = \beta(t,y)h_y(t,y) - \mu(t,y) > 0$ implies $\mu(t,y) < \beta(t,y)h_y(t,y)$. Since

$$f'(\pi) = -(1-\delta)\Big[\beta^2(t,y) + \sigma^2(t,y) + \int_{\mathbb{R}}(1+\pi(t)\gamma(t,y,z))^{\delta-2}\gamma^2(t,y,z)\nu(dz)\Big] < 0.$$

Then, there exists a unique solution $\pi^*(t) \in (0,1)$ if $f(1) > 0$, i.e.,

$$\begin{aligned} & \mu(t,y) + (\delta-1)(\beta^2(t,y) + \sigma^2(t,y)) \\ + & \int_{\mathbb{R}} \gamma(t,y,z)\big[(1 + \gamma(t,y,z))^{\delta-1} - 1\big]\nu(dz) \quad < \quad \beta(t,y)h_y(t,y), \end{aligned}$$

$\square$

For the second adjoint equation, note that from (4.27), we obtain the following relation

$$\frac{\partial\mu}{\partial y}(t,y)A_1^*(t)+\frac{\partial\beta}{\partial y}(t,y)B_1^*(t)+\frac{\partial\sigma(t,y)}{\partial y}B_2^*(t)+\int_{\mathbb{R}}\frac{\partial\gamma}{\partial y}(t,y,z)D_1^*(t,z)\nu(dz)=0\,. \tag{4.34}$$

Then, for optimal strategy, the second adjoint equation (4.23), can be written as

$$dA_2^*(t)=-g'(y)A_2^*(t)dt+B_3^*(t)dW_1(t)+B_4^*(t)dW_1(t)+\int_{\mathbb{R}}D_2^*(t,z)\tilde{N}(dt,dz)\,. \tag{4.35}$$

Which is a linear BSDE with jumps. Since the terminal condition is $A^*(T)=0$, by applying the techniques for solving linear BSDE with jumps (Delong [26], Propositions 3.3.1 and 3.4.1), we obtain $A_2^*(t)=B_3^*(t)=B_4^*(t)=D_2^*(t,z)=0$.

The corresponding wealth process equation (4.16) for the optimal solutions becomes

$$\begin{aligned} dX^*(t) &= X(t)\Big[G(t)dt+\pi^*(t)[\beta(t,y)dW_1(t)+\sigma(t,y)dW_2(t)] \\ &\quad +\pi^*(t)\int_{\mathbb{R}}\gamma(t,y,z)\tilde{N}(dt,dz)\Big]\,, \end{aligned}$$

where

$$\begin{aligned} G(t) &= r(t)+\pi^*(t)\mu(t,y)+\eta_{n^*}(t) \\ &\quad -e^{\frac{1}{1-\delta}h(t,y)}\Big[\kappa_1^{-\frac{1}{\delta-1}}+\Big(\frac{\eta_{n^*}(t)}{\kappa_2\lambda(t)}\Big)^{\frac{1}{\delta-1}}e^{\frac{1}{\delta-1}\int_0^t(\rho(s)+\lambda(s))ds}\Big]\,, \end{aligned}$$

which gives the following solution

$$\begin{aligned} X(t) &= x\exp\Big\{\int_0^t[G(s)-\frac{1}{2}(\pi^*(s))^2(\beta^2(s,y)+\sigma^2(s,y))]ds \\ &\quad +\int_0^t\int_{\mathbb{R}}[\ln(1+\pi^*(s)\gamma(s,y,z))-\pi^*(s)\gamma(s,y,z)]\nu(dz)ds \\ &\quad +\int_0^t\pi^*(s)[\beta(s,y)dW_1(s)+\sigma(s,y)dW_2(s)] \\ &\quad +\int_0^t\int_{\mathbb{R}}\ln(1+\pi^*(s)\gamma(s,y,z))\tilde{N}(ds,dz)\Big\}\,. \end{aligned}$$

Finally, the value function of the problem (4.20) can be characterized as the solution of the following BSDE

$$\begin{aligned} dV(t,x,y) &= -\mathcal{H}(t,x^*,y,c^*,\pi^*,p^*,A_1^*,A_2^*,B_1^*,B_2^*,D_1^*)dt + B_1^*(t)dW_1(t) \\ &\quad + B_2^*(t)dW_2(t) + \int_{\mathbb{R}} D_1^*(t,z)\tilde{N}(dt,dz)\,; \\ V(T,x,y) &= \kappa_3 e^{\int_0^T [\rho(t)+\lambda(t)]dt} \frac{X(T)^\delta}{\delta}\,. \end{aligned}$$

**Example 4.3.1.** The following example specifies the results in Theorem 4.3.1 to a well known stochastic volatility model of Ornstein-Uhlenbeck type and an explicit portfolio strategy is derived. Let $N$ be the Poisson process, with intensity $\nu > 0$. We consider the following model dynamics

$$\begin{aligned} B(t) &= 1\,; \\ dS(t) &= S(t)[(\alpha_0 + \alpha_1 Y(t))dt + \gamma d\tilde{N}(t)]\,; \\ dY(t) &= -bY(t)dt + dW(t)\,, \end{aligned}$$

where $\alpha_0, \alpha_1, \gamma \in \mathbb{R}$ and $b > 0$. Suppose that we have a constant mortality rate $\lambda > 0$, constant insurance premium rates $\eta_n > 0$, $n = 1, 2, \ldots, M$, discount rate $\rho > 0$ and $\kappa_1 = \kappa_1 = \kappa_3 = 1$. Then the Hamiltonian is given by

$$\begin{aligned} &\mathcal{H}(t, X(t), Y(t), A_1(t), A_2(t), B(t), D_1(t)) \\ = \; &\frac{1}{\delta} e^{-(\rho+\lambda)t} \Big[ (c(t))^\delta + \lambda (X + \sum_{n=1}^M \frac{p_n(t)}{\eta_n})^\delta \Big] \\ &+ [X(t)\pi(t)(\alpha_0 + \alpha_1 Y(t)) - c(t) - \sum_{n=1}^M \frac{p_n(t)}{\eta_n}] A_1(t) \\ &- bY(t)A_2(t) + B(t) + \pi(t)X(t)\gamma D(t)\nu\,. \end{aligned}$$

Then, following Theorem 4.3.1, we can easily see that the optimal portfolio is given by

$$\pi^*(t) = \frac{1}{\gamma}\Big[ \Big( \frac{\gamma\nu - \alpha_0 - \alpha_1 y}{\gamma\nu} \Big)^{\frac{1}{\delta - 1}} - 1 \Big]\,,$$

where $y$ is given by $Y(t) = e^{-bt}y_0 + \int_0^t e^{-b(t-s)}dW(s)$.
Moreover, the optimal consumption and insurance are given by

$$c^*(t) = e^{\frac{1}{\delta-1}(\rho+\lambda)t}(A_1^*(t))^{\frac{1}{\delta-1}}\,, \quad p_{n^*}^*(t) = \left[\left(\frac{\eta_{n^*}}{\lambda}A_1^*(t)\right)^{\frac{1}{\delta-1}} e^{\frac{1}{\delta-1}(\rho+\lambda)t} - x\right],$$

Where $A_1^*(t)$ is part of a solution of the following linear BSDE

$$dA_1^*(t) = -\eta_{n^*}A_1^*(t)dt + B^*(t)dW(t) + D^*(t)d\tilde{N}(t).$$

Hence, $A_1^*(t) = e^{-\rho T}\mathbb{E}\Big[e^{\eta_{n^*}(T-t)}(X(T))^{\delta-1} \mid \mathcal{F}_t\Big]$. $B^*$ and $D^*$ can be derived by the martingale representation theorem. See Delong [26], Propositions 3.3.1 and 3.4.1. Thus, for this pure jump Poisson process of Ornstein-Uhlenbeck type, we have derived an explicit optimal portfolio strategy.

# Chapter 5

# Optimal investment-consumption and life insurance selection problem under inflation

## 5.1 Introduction

The problem of asset allocation with life insurance consideration is of great interest to the investor because it protects their dependents if a premature death occurs. Since the optimal portfolio, consumption and life insurance problem by Richard [85] in 1975, many works in this direction have been reported in the literature. (See, e.g., Pliska and Ye [83], Guambe and Kufakunesu [41], Han and Hu [42], among others).

In this chapter, we discuss an optimal investment, consumption and life insurance problem using the backward stochastic differential equations (BSDE) with jumps approach. Unlike the dynamic programming approach applied, for instance, in Han and Hu [42], this approach allows us to solve the problem in a more general non-Markovian case. For more details on the theory of BSDE with jumps, see e.g., Delong [26], Cohen and Elliott [16], Morlais [71], and references therein. Our results extend, for instance, the paper by Cheridito and Hu [14] to a jump diffusion setup and we allow the presence of life insurance and inflation risks. Inflation is described as a percentage change of a particular reference index. The inflation-linked products may be used to protect the future cash flow of the wage earner against inflation, due to its rapid escalation in some developing economies. Therefore, it make sense to model the inflation-linked products using jump-diffusion processes. For more details on the inflation-linked derivatives, see e.g., Tiong [96], Mataramvura [61] and references therein. We consider a model described by a risk-free asset, a real zero coupon bond, an inflation-linked real money account and a risky asset under jump-diffusion processes. These type of processes are more appropriate for modeling the response to some important extreme events that may occur since they allow capturing some sudden changes in the price evolution, as well as, the consumer price index that cannot be explained by models driven by Brownian information. Such events happen due to many reasons, for instance, natural disasters, political situations, etc.

The corresponding quadratic-exponential BSDE with jumps relies on the results by Morlais [71], Morlais [70], where the existence and uniqueness properties of the quadratic-exponential BSDE with jumps have been proved. Thus, we are also extending the utility maximization problem in Morlais [70] by including consumption and life insurance. Similar works include Hu *et. al.* [46], Xing [98], Siu [91], Øksendal and Sulem [78], among others.

This chapter is organized as follows: in Section 5.2, we introduce the inflation risks and the related assets: the real zero coupon bond, the inflation-linked real money account, and the risky asset. We also introduce the insurance market and we state the main problem under study. Section 5.3 is the main section of this paper, we present the general techniques of the BSDE approach and we prove the main results in the exponential and power utility function. Finally, in Section 5.4, we give some concluding remarks.

## 5.2 Model formulation

Suppose we have a wage earner investing in a finite investment period $T < \infty$, which can be interpreted as a retirement time. Consider a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$, where $\{\mathcal{F}_t\}_{t\in[0,T]}$ is a filtration satisfying the usual conditions. Denote by $W_r$ and $W_n$ the Brownian motions underlying the risks driven by the real and nominal term structures. We also define the Brownian motions $W_I$ and $W_S$, the drivers in the inflation rate and the risky asset. We assume that $W_r$, $W_n$, $W_I$ and $W_S$ are independent processes. Note that if we allow the correlations among $W_r$, $W_n$, $W_I$ and $W_S$, i.e., $dW_k(t)dW_I(t) = \rho_{kI}dt$; $dW_k(t)dW_S(t) = \rho_{kS}dt$ for $k \in \{r, n\}$ and $dW_I(t)dW_S(t) = \rho_{IS}dt$, where $\rho_{ij}$ are the correlation coefficients, the optimization problem may result in a highly nonlinear BSDE with jumps which the existence and uniqueness of its solution has not yet been established and it is out of the scope of this Chapter. Moreover, we consider a Poisson process $N$ independent of $W_r$, $W_n$, $W_I$ and $W_S$, associated with the complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ with the intensity measure $dt \times d\nu(z)$, where $\nu$ is the $\sigma$-finite Borel measure on $\mathbb{R} \setminus \{0\}$. A $\mathbb{P}$-martingale compensated Poisson random measure is given by:

$$\tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt\,.$$

Let $r$ denote the real and $n$ the nominal forward rates, defined for $k \in \{r, n\}$, by:

$$f_k(t,T) := f_k(0,T) + \int_0^t \alpha_k(s,T)ds + \int_0^t \sigma_k(s,T)dW_k(s) + \int_0^t \gamma_k(s,T,z)\tilde{N}(ds,dz)\,,$$

where the coefficients $\alpha_k(t,T)$, $\sigma_k(t,T)$ and $\gamma_k(t,T,z)$ are $\{\mathcal{F}_t\}_{t\in[0,T]}$-predictable bounded processes, satisfying the following condition:

$$\int_0^T \left[|\alpha_k(t,T)| + \sigma_k^2(t,T) + \int_{\mathbb{R}} \gamma_k^2(t,T,z)\nu(dz)\right] dt < \infty, \quad \text{a.s.}$$

We denote by $r_k(t) = f_k(t,t)$ the corresponding spot rate at time $t$.

It is well known that the price of the real (nominal) bond is given by

$$P_k(t,T) := \exp\left\{-\int_t^T f_k(t,s)ds\right\}.$$

See Björk [10], Chapter 22, for more details in the diffusion case. An application of Itô's formula yields:

$$dP_k(t,T) = P_k(t,T)\left\{a_k(t,T)dt + b_k(t,T)dW_k(t) + \int_{\mathbb{R}} c_k(t,T,z)\tilde{N}(dt,dz)\right\},$$

where

$$b_k(t,T) := -\int_t^T \sigma_k(t,s)ds\,; \quad c_k(t,T,z) := -\int_t^T \gamma_k(t,s,z)ds$$

and

$$a_k(t,T) := r_k(t) - \int_t^T \alpha_k(t,s)ds + \frac{1}{2}\|b_k(t,T)\|^2 - \int_{\mathbb{R}} c_k(t,T,z)\nu(dz)\,.$$

We suppose the existence of an inflation index $I(t)$, i.e., the consumer price index (CPI) governed by the following SDE

$$dI(t) = I(t)\Big[\mu_I(t)dt + \sigma_I(t)dW_I(t) + \int_{\mathbb{R}} \gamma_I(t,z)\tilde{N}(dt,dz)\Big],$$

where the expected inflation rate $\mu_I(t)$, the volatility $\sigma_I(t)$ and the dispersion rate $\gamma_I(t,z) > -1$ are $\mathcal{F}_t$-predictable bounded processes, satisfying the following integrability condition

$$\int_0^T \Big[|\mu_I(t)| + \sigma_I^2(t) + \int_{\mathbb{R}} \gamma_I^2(t,z)\nu(dz)\Big]dt < \infty, \quad \text{a.s.}$$

The financial market consists of four assets, namely a real (nominal) money account $B_k(t)$ defined by

$$B_k(t) = \exp\left\{\int_0^t r_k(s)ds\right\}.$$

A real zero coupon bond price $P_r^*(t,T)$ defined as

$$P_r^*(t,T) = I(t)P_r(t,T).$$

Applying the Itô's product rule, we have the following dynamics

$$\begin{aligned} dP_r^*(t,T) &= I(t)dP_r(t,T) + P_r(t,T)dI(t) + d[I(t),P_r(t,T)] \\ &= P_r^*(t,T)\Big[\tilde{A}(t,T)dt + b_r(t,T)dW_r(t) + \sigma_I(t)dW_I(t) \\ &\quad + \int_{\mathbb{R}} \tilde{C}(t,T,z)\tilde{N}(dt,dz)\Big], \end{aligned}$$

where

$$\tilde{A}(t,T) := a_r(t,T) + \mu_I(t) + \int_{\mathbb{R}} c_r(t,T,z)\gamma_I(t,z)\nu(dz)$$

and

$$\tilde{C}(t,T,z) := c_r(t,T,z) + \gamma_I(t,z) + c_r(t,T,z)\gamma_I(t,z)\,.$$

We also define the inflation-linked real money account $B_r^*(t)$ by

$$B_r^*(t) := I(t)B_r(t).$$

Then, by Itô's formula, we can easily see that $B_r^*$ is governed by the following SDE:

$$dB_r^*(t) = B_r^*(t)\Big[(r_r(t)+\mu_I(t))dt + \sigma_I(t)dW_I(t) + \int_{\mathbb{R}} \gamma_I(t,z)\tilde{N}(dt,dz)\Big]\,.$$

Finally, we define the risky asset price by the following geometric jump-diffusion process

$$dS(t) = S(t)\Big[\mu_S(t)dt + \sigma_S(t)dW_S(t) + \int_{\mathbb{R}} \gamma_S(t,z)\tilde{N}(dt,dz)\Big]\,,$$

where the mean rate of return $\mu_S(t)$, the volatility $\sigma_S(t)$ and the dispersion rate $\gamma_S(t,z) > -1$ are $\{\mathcal{F}_t\}_{t\in[0,T]}$-predictable bounded processes, satisfying the following integrability condition

$$\int_0^T \Big[|\mu_S(t)| + \sigma_S^2(t) + \int_{\mathbb{R}} \gamma_S^2(t,z)\nu(dz)\Big]dt < \infty, \quad \text{a.s.}$$

For later use, we define the following processes (also called market price of risks) $\varphi_1 := \frac{\tilde{A}-r_r}{b_r}$, $\varphi_2 := \frac{\mu_I}{\sigma_I}$ and $\varphi_3 := \frac{\mu_S - r_r}{\sigma_S}$, provided that $b_r, \sigma_I, \sigma_S \neq 0$.

As in the previous two Chapters, we consider a wage earner whose lifetime is a nonnegative random variable $\tau$ defined on the probability space

$(\Omega, \mathcal{F}, \mathbb{P})$. We suppose the existence of an insurance market, where the term life insurance is continuously traded. We assume that the wage earner is paying premiums at the rate $p(t)$, at time $t$ for the life insurance contract and the insurance company will pay $p(t)/\eta(t)$ to the beneficiary for his death, where the $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted process $\eta(t) > 0$ is the premium insurance ratio. When the wage earner dies, the total legacy to his beneficiary is given by

$$\ell(t) := X(t) + \frac{p(t)}{\eta(t)},$$

where $X(t)$ is the wealth process of the wage earner at time $t$ and $p(t)/\eta(t)$ the insurance benefit paid by the insurance company to the beneficiary if death occurs at time $t$.

Let $c(t)$ be the consumption rate of the wage earner and $\theta(t) := (\theta_1(t), \theta_2(t), \theta_3(t))$ be the vector of the amounts of the wage earner's wealth invested in the real zero coupon bond $P_r^*$, the inflation-linked real money account $B_r^*$ and the risky asset $S$ respectively, satisfying the following integrability condition.

$$\int_0^T \Big[c(t) + p(t) + \sum_{i=1}^{3} \theta_i^2(t)\Big] dt < \infty, \quad \text{a.s.} \tag{5.1}$$

Furthermore, we assume that the shares are divisible, continuously traded and there are no transaction costs, taxes or short-selling constraints in the trading. Then the wealth process $X(t)$ is defined by the following (SDE):

$$\begin{aligned} dX(t) &= [r_r(t)X(t) + \langle\theta(t), \hat{\mu}(t)\rangle - c(t) - p(t)]dt + \theta_1(t)b_r(t,T)dW_r(t) \\ &\quad +(\theta_1(t) + \theta_2(t))\sigma_I(t)dW_I(t) + \theta_3(t)\sigma_S(t)dW_S(t) \\ &\quad + \int_{\mathbb{R}} \langle\theta(t), \hat{\gamma}(t,T,z)\rangle \tilde{N}(dt,dz)\,, \quad t \in [0, \tau \wedge T]\,, \\ X(0) &= x > 0\,, \end{aligned} \tag{5.2}$$

where $\hat{\mu}(t) := (\tilde{A}(t,T) - r_r(t), \mu_I(t), \mu_S(t) - r_r(t))$ and $\hat{\gamma}(t,T,z) := (\tilde{C}(t,T,z), \gamma_I(t,z), \gamma_S(t,z))$.

The wage earner faces the problem of choosing the optimal strategy $\mathcal{A} := \{(\theta, c, p) := (\theta(t), c(t), p(t))_{t\in[0,T]}\}$ which maximizes the discounted expected utilities from the consumption during his/her lifetime $[0, \tau \wedge T]$, from the wealth if he/she is alive until the terminal time $T$ and from the legacy if he/she dies before time $T$. Suppose that the discount process rate $\varrho(t)$ is positive and $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted process. This problem can be defined by the following performance functional (for more details see, e.g., Pliska and Ye [83], Øksendal and Sulem [77], Guambe and Kufakunesu [41]).

$$\begin{aligned} J(0, x_0; \theta, c, p) \quad &:= \quad \mathbb{E}\Big[\int_0^{\tau\wedge T} e^{-\int_0^s \varrho(u)du} U(c(s))ds \qquad (5.3)\\ &+ e^{-\int_0^\tau \varrho(u)du} U(\ell(\tau))\chi_{\{\tau\leq T\}} + e^{-\int_0^T \varrho(u)du} U(X(T))\chi_{\{\tau>T\}}\Big], \end{aligned}$$

where $\chi_A$ is a characteristic function defined on a set $A$ and $U$ is the utility function for the consumption, legacy and terminal wealth.

Note that from the conditional survival probability of the wage earner (2.16) and the conditional survival probability density of death of the wage earner (2.17), we can write a dynamic version of the functional (5.3) by:

$$\begin{aligned} J(t, x, \theta, c, p) \quad &= \quad \mathbb{E}_{t,x}\Big[\int_t^T e^{-\int_t^s (\varrho(u)+\lambda(u))du}[U(c(s)) + \lambda(s)U(\ell(s))]ds \\ &+ e^{-\int_t^T (\varrho(u)+\lambda(u))du} U(X(T)) \mid \mathcal{F}_t\Big]. \qquad (5.4) \end{aligned}$$

Thus, the problem of the wage earner is to maximize the above dynamic performance functional under the admissible strategy $\mathcal{A}$. Therefore, the value function $V(t, x)$ can be restated in the following form:

$$V(t, x, r) = ess \sup_{(\theta,c,p)\in\mathcal{A}} J(t, x, \theta, c, p)\,. \qquad (5.5)$$

The set of strategies $\mathcal{A} := \{(\theta, c, p) := (\theta(t), c(t), p(t))_{t\in[0,T]}\}$ is said to be admissible if the SDE (5.2) has a unique strong solution such that $X(t) \geq 0$, $\mathbb{P}$-a.s. and

$$\mathbb{E}_{t,x}\Big[\int_t^T e^{-\int_t^s(\varrho(u)+\lambda(u))du}[U(c(s))+\lambda(s)U(\ell(s))]ds \\ +e^{-\int_t^T(\varrho(u)+\lambda(u))du}U(X(T)) \mid \mathcal{F}_t\Big] < \infty\,.$$

The nonnegative condition of the wealth process contains a non-borrowing constraints that prevents the family from borrowing for consumption and life insurance at any time $t \in [0,T]$.

In order to solve our optimization problem using the quadratic-exponential BSDE's with jumps and to make the proofs easier, we introduce, in addition to the integrability condition (5.1), the constraints in the admissible strategy $\mathcal{A}$ as follows: let $\mathcal{C} \subset \mathcal{P}$ and $\mathcal{D} \subset \mathcal{P}$, where $\mathcal{P}$ denotes the set of real valued predictable processes $(c(t))_{0\le t\le T}$, $(p(t))_{0\le t\le T}$, and $\mathcal{Q} \subset \mathcal{P}^{1\times 3}$, where $\mathcal{P}^{1\times 3}$ represents the set of all predictable processes $(\theta_1(t),\theta_2(t),\theta_3(t))_{0\le t\le T}$. In the exponential case, we assume that the admissible strategy $(c(t),p(t),\theta(t)) \in \mathcal{C}\times\mathcal{D}\times\mathcal{Q}$. For the power utility case, the consumption, investment and life insurance strategies will be denoted by their fractions of the total wealth, that is, $c=\xi X$, $\theta=\pi X$, and $p=\zeta X$. We assume that $(\xi(t),\zeta(t),\pi(t)) \in \mathcal{C}\times\mathcal{D}\times\mathcal{Q}$.

We assume that $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{Q}$ are closed and compact sets.

## 5.3 The BSDE approach to optimal investment, consumption and insurance

In this section, we solve the optimal investment, consumption and life insurance problem under inflation using the BSDE with jumps approach. For more details on the theory of BSDEs with jumps see, e.g., Delong [26]. We then consider two utility functions, namely, the exponential utility and the power utility. The techniques we use are similar to Morlais [70], Cheridito and Hu [14], Xing [98].

Define the following BSDE with jumps:

$$\begin{aligned} dY(t) &= -h(t, r(t), Y(t), Z_1(t), Z_2(t), Z_3(t), \Upsilon(t,\cdot), \theta(t), c(t), p(t))dt \qquad (5.6) \\ &\quad + Z_1(t)dW_r(t) + Z_2(t)dW_I(t) + Z_3(t)dW_S(t) + \int_{\mathbb{R}} \Upsilon(t,z)\tilde{N}(dt,dz); \\ Y(T) &= 0\,. \end{aligned}$$

The aforementioned approach is based on the martingale optimality principle as follows: Consider the process

$$\begin{aligned} \mathcal{R}(t) &= \int_0^t e^{-\int_0^s (\varrho(u)+\lambda(u))du}[U(c(s)) + \lambda(s)U(\ell(s))]ds \\ &\quad + e^{-\int_t^T (\varrho(u)+\lambda(u))du} U(X(t) - Y^r(t))\,, \end{aligned}$$

with the initial condition $\mathcal{R}(0) = U(x - Y^r(0))$. Here, $X(t)$ represents the wealth process (5.2) and $Y^r(t)$ part of the solution $(Y^r, Z_1, Z_2, Z_3, \Upsilon)$ of the BSDE with jumps (5.6). Applying the generalized Itô's formula, we have

$$\begin{aligned} & d\mathcal{R}(t) \\ = \; & e^{-\int_0^t (\varrho(s)+\lambda(s))ds}\Big\{[\Lambda(t, y, z_1, z_2, z_3, \upsilon, \theta, c, p) + h(t, r, y, z_1, z_2, z_3, \upsilon)]dt \\ & U'(X(t) - Y^r(t))[(\theta_1(t)b_r(t,T) + z_1)dW_r(t) \\ & +((\theta_1(t) + \theta_2(t))\sigma_I(t) + z_2)dW_I(t) + (\theta_3(t)\sigma_S(t) + z_3)dW_S(t)] \\ & + \int_{\mathbb{R}} [U(X(t) - Y^r(t) + \langle \theta(t), \hat{\gamma}(t,T,z)\rangle + \upsilon(t,z)) \\ & \qquad -U(X(t) - Y^r(t))]\tilde{N}(dt,dz)\Big\}\,, \end{aligned}$$

where

$$\begin{aligned} & \Lambda(t, y, z_1, z_2, z_3, \upsilon, \theta, c, p) \\ = \; & -\Big\{[U(c(t)) + \lambda(t)U(\ell(t)) - (\varrho(t) + \lambda(t))U(X(t) - Y^r(t))] \\ & +U'(X(t) - Y^r(t))[r_r(t)X(t) + \langle \theta(t), \hat{\mu}(t)\rangle - c(t) - p(t)] \\ & +\frac{1}{2}U''(X(t) - Y^r(t))[(\theta_1(t)b_r(t,T) + z_1)^2 + ((\theta_1(t) + \theta_2(t))\sigma_I(t) + z_2)^2 \\ & +(\theta_3(t)\sigma_S(t) + z_3)^2] + \int_{\mathbb{R}} [U(X(t) - Y^r(t) + \langle \theta(t), \hat{\gamma}(t,T,z)\rangle + \upsilon(t,z)) \\ & -U(X(t) - Y^r(t)) - U'(X(t) - Y^r(t))(\langle \theta(t), \hat{\gamma}(t,T,z)\rangle + \upsilon(t,z))]\nu(dz)\Big\}\,. \end{aligned}$$

Note that we can write $\mathcal{R}$ as

$$\begin{aligned}\mathcal{R}(t) &= \mathcal{R}(0) + \int_0^t e^{-\int_0^s (\varrho(u)+\lambda(u))du}[\Lambda(s, y, z_1, z_2, z_3, \upsilon, \theta, c, p) \\ &\quad + h(s, r, y, z_1, z_2, z_3, \upsilon)]ds + \{a \ \ local \ \ martingale\}. \quad (5.7)\end{aligned}$$

We define the generator $h$ by

$$h(s, r, y, z_1, z_2, z_3, \upsilon) = \inf_{(\theta,c,p)\in\mathcal{A}} \Lambda(s, y, z_1, z_2, z_3, \upsilon, \theta, c, p)\,.$$

Then we can see that (5.7) is a decreasing process, hence $\mathcal{R}$ is a local supermartingale and we can choose a strategy $(\theta^*, c^*, p^*)$ such that the drift process in (5.7) is equal to zero, therefore, $\mathcal{R}$ is a local martingale and prove that $(\theta^*, c^*, p^*)$ is the optimal strategy.

We will establish the existence and uniqueness properties of the solution $(Y^r, Z_1, Z_2, Z_3, \Upsilon) \in \mathbb{S}(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_\nu(\mathbb{R})$ of the BSDE with jumps (5.6), as well as the characterization of the optimal strategy $(\theta^*, c^*, p^*)$, for the specific utilities in the following subsections.

### 5.3.1 The exponential utility

We consider the exponential utility function of the form

$$U(x) = -e^{-\delta x}, \quad \delta > 0\,. \tag{5.8}$$

The functional (5.4), is then given by

$$\begin{aligned}J(t) &= -\mathbb{E}_{t,x}\Big[\int_t^T e^{-\int_t^s (\varrho(u)+\lambda(u))du}[e^{-\delta c(s)} + \lambda(s)e^{-\delta \ell(s)}]ds \\ &\quad + e^{-\int_t^T (\varrho(u)+\lambda(u))du} \cdot e^{-\delta X_t(T)} \mid \mathcal{F}_t\Big]\,. \end{aligned} \tag{5.9}$$

We then state the main result of this subsection.

**Theorem 5.3.1.** *Suppose that the utility function is given by (5.8). Then the optimal value function of the optimization problem (5.5) is given by*

$$V(t, x, r) = -\exp(-\delta(x - Y^r(t))), \tag{5.10}$$

*where $Y^r(t)$ is part of the solution $(Y^r, Z_1, Z_2, Z_3, \Upsilon) \in \mathbb{S}(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_\nu(\mathbb{R})$ of the BSDE with jumps (5.6), with terminal condition $Y^r(T) = 0$ and the generator $h$ given by*

$$\begin{aligned}
& h(t, r, y, z_1, z_2, z_3, \upsilon(\cdot)) \qquad (5.11)\\
= \;& (1 - r_r(t))X(t) + \frac{1}{\delta}\Big(1 + \eta(t) - \varrho(t) - \lambda(t) + \ln\delta + \eta(t)\ln\Big(\frac{\delta\lambda(t)}{\eta(t)}\Big)\Big)\\
& -(1 + \frac{\eta(t)}{\delta})y + \inf_\theta\Big\{\frac{\delta}{2}\Big[\Big|\theta_1(t)b_r(t,T) - \Big(z_1 + \frac{\varphi_1(t)}{\delta}\Big)\Big|^2\\
& +\Big|(\theta_1(t) + \theta_2(t))\sigma_I(t) - \Big(z_2 + \frac{\varphi_2(t)}{\delta}\Big)\Big|^2 + \Big|\theta_3(t)\sigma_S(t) - \Big(z_3 + \frac{\varphi_3(t)}{\delta}\Big)\Big|^2\Big]\\
& +\frac{1}{\delta}\int_{\mathbb{R}}[\exp(\delta(\upsilon(t,z) - \langle\theta(t), \hat{\gamma}(t,T,z)\rangle)) - 1 - \delta(\upsilon(t,z)\\
& \qquad -\langle\theta(t), \hat{\gamma}(t,T,z)\rangle)]\nu(dz)\Big\} - (\varphi_1(t)z_1 + \varphi_2(t)z_2 + \varphi_3(t)z_3)\\
& -\frac{1}{2\delta}\Big(\varphi_1^2(t) + \varphi_2^2(t) + \varphi_3^2(t)\Big).
\end{aligned}$$

*Furthermore, the optimal admissible strategy $(\theta^*(t), c^*(t), p^*(t))$ is given by*

$$c^*(t) = X^{(\theta^*, c^*, p^*)}(t) - Y^r(t) + \frac{1}{\delta}\ln\delta; \qquad p^*(t) = \eta(t)\Big[\frac{1}{\delta}\ln\Big(\frac{\delta\lambda(t)}{\eta(t)}\Big) - Y^r(t)\Big]$$

*and*

$$\begin{aligned}
& \theta^*(t) \qquad (5.12)\\
= \;& \inf_\theta\Big\{\frac{\delta}{2}\Big[\Big|\theta_1(t)b_r(t,T) - \Big(z_1 + \frac{\varphi_1(t)}{\delta}\Big)\Big|^2\\
& +\Big|(\theta_1(t) + \theta_2)\sigma_I(t) - \Big(z_2 + \frac{\varphi_2(t)}{\delta}\Big)\Big|^2 + \Big|\theta_3(t)\sigma_S(t) - \Big(z_3 + \frac{\varphi_3(t)}{\delta}\Big)\Big|^2\Big]\\
& +\frac{1}{\delta}\int_{\mathbb{R}}[\exp(\delta(\upsilon(t,z) - \langle\theta(t), \hat{\gamma}(t,T,z)\rangle)) - 1 - \delta(\upsilon(t,z)\\
& \qquad -\langle\theta(t), \hat{\gamma}(t,T,z)\rangle)]\nu(dz)\Big\}.
\end{aligned}$$

Note that for the optimal investment strategy $\theta^*(t) = (\theta_1^*(t), \theta_2^*(t), \theta_3^*(t))$, the solution (5.12) is not explicit. We then obtain an explicit solution for a special case where there is no jumps, that is $\nu = 0$. Applying the first order condition of optimality in (5.12), we prove that the optimal strategy $\theta^*(t) = (\theta_1^*(t), \theta_2^*(t), \theta_3^*(t))$ is given by the following corollary.

**Corollary 5.3.2.** *Assume that* $\nu = 0$*, then the optimal portfolio strategy* $(\theta_1^*(t), \theta_2^*(t), \theta_3^*(t))$*, for all* $t \in [0, T]$ *is given by*

$$\begin{aligned}
\theta_1^*(t) &= \frac{\tilde{A}(t,T) - r_r(t) - \mu_I(t)}{\delta b_r^2(t,T)} + \frac{Z_1(t)}{b_r(t,T)} \\
\theta_2^*(t) &= \frac{1}{\delta}\Big[\Big(\frac{1}{\sigma_I^2(t)} + \frac{1}{b_r^2(t,T)}\Big)\mu_I(t) - \frac{\tilde{A}(t,T) - r_r(t)}{b_r^2(t,T)} + \frac{Z_2(t)}{\sigma_I(t)} - \frac{Z_1(t)}{b_r(t,T)}\Big] \\
\theta_3^*(t) &= \frac{\mu_S(t) - r_r(t)}{\delta \sigma_S^2(t)} + \frac{Z_3(t)}{\sigma_S(t)}\,,
\end{aligned}$$

*where* $(Z_1(t), Z_2(t), Z_3(t))$ *is part of the solution* $(Y^r, Z_1, Z_2, Z_3)$ *of the following BSDE.*

$$\begin{aligned}
dY^r(t) &= -h(t, r, Y^r(t), Z_1(t), Z_2(t), Z_3(t), \theta^*(t), c^*(t), p^*(t))dt + Z_1(t)dW_r(t) \\
&\quad + Z_2(t)dW_I(t) + Z_3(t)dW_S(t); \\
Y^r(T) &= 0\,.
\end{aligned}$$

Before we prove the main theorem of this subsection, we establish the assumptions for the existence and uniqueness solution of a BSDE with quadratic growth. Suppose we are given a BSDE (5.6), with terminal condition $Y^r(T) = 0$ and a generator $h$ given by (5.11). From the boundedness of the associated parameters, there exists a constant $K > 0$ such that

$$\begin{aligned}
& |h(t, r, y, z_1, z_2, z_3, \upsilon)| \\
\leq\; & K\Big(1 + |y| + |z_1|^2 + |z_2|^2 + |z_3|^2 \\
& + \frac{1}{\delta}\int_{\mathbb{R}} [\exp(\delta(\upsilon(t,z) - \langle\theta(t), \hat{\gamma}\rangle)) - 1 - \delta(\upsilon(t,z) - \langle\theta(t), \hat{\gamma}\rangle)]\nu(dz)\Big)\,.
\end{aligned} \tag{5.13}$$

Moreover,

$$|h(t, r, y, z_1, z_2, z_3, \upsilon) - h(t, r, y', z_1', z_2', z_3', \upsilon)| \leq K\Big(|y - y'| + \sum_{i=1}^{3}(1 + |z_i| + |z_i'|)|z_i - z_i'|\Big) \tag{5.14}$$

and

$$|h(t, r, y, z_1, z_2, z_3, \upsilon) - h(t, r, y, z_1, z_2, z_3, \upsilon')| \leq \int_{\mathbb{R}} \Phi(\upsilon, \upsilon')(\upsilon - \upsilon')\nu(dz)\,,$$

where

$$\begin{aligned}\Phi(\upsilon,\upsilon') &= \sup_{\theta}\Big(\int_0^1 m'(s(\upsilon-\langle\theta,\hat{\gamma}\rangle)+(1-s)(\upsilon'-\langle\theta,\hat{\gamma}\rangle)(z))ds\Big)\chi_{\upsilon\geq\upsilon'}\\ &\quad+\inf_{\theta}\Big(\int_0^1 m'(s(\upsilon-\langle\theta,\hat{\gamma}\rangle)+(1-s)(\upsilon'-\langle\theta,\hat{\gamma}\rangle)(z))ds\Big)\chi_{\upsilon<\upsilon'},\end{aligned}$$

for the function $m$ defined by $m(x)=\frac{\exp(\delta x)-1-\delta x}{\delta}$.

Then, it follows from Morlais [70], [71], Theorems 1-2, that the BSDE with jumps (5.6), with terminal condition $Y^r(T)=0$ and a generator (5.11) has a unique solution $(Y^r,Z_1,Z_2,Z_3,\Upsilon)\in\mathbb{S}(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})\times\mathbb{H}^2_\nu(\mathbb{R})$.

**Remark.** Note that the terminal condition of the BSDE (5.6) need not to be zero, for instance, if one consider an investor receiving a lump sum payment $E$ at the terminal time $T$, then $Y^r(T)=E(r(T))$.

*Proof of Theorem 5.3.1.*

Define a family of processes

$$\begin{aligned}\mathcal{R}_1^{(\theta,c,p)}(t) &= -\int_0^t e^{-\int_0^s(\varrho(u)+\lambda(u))du}[e^{-\delta c(s)}+\lambda(s)e^{-\delta\ell(s)}]ds \\ &\quad -e^{-\int_0^t(\varrho(u)+\lambda(u))du}\cdot e^{-\delta(X^{(\theta,c,p)}(t)-Y^r(t))}.\end{aligned}\tag{5.15}$$

We aim to construct the process $\mathcal{R}_1^{(\theta,c,p)}$ such that for each strategy $(\theta,c,p)\in\mathcal{A}$, it is a super-martingale and there exists a strategy $(\theta^*,c^*,p^*)\in\mathcal{A}$ such that $\mathcal{R}_1^{(\theta^*,c^*,p^*)}$ is a martingale.

Applying the Itô's formula for the process (5.15), we have

$$
\begin{aligned}
& d\mathcal{R}_1(t) \qquad (5.16)\\
= \; & \delta e^{-\int_0^t(\varrho(u)+\lambda(u))du} \cdot e^{-\delta(X(t)-Y^r(t))}\Big\{\Big[-e^{\delta(X(t)-Y^r(t))}\left[e^{-\delta c(t)}+\lambda(t)e^{-\delta\ell(t)}\right]\\
& +\frac{1}{\delta}(\varrho(t)+\lambda(t))+r_r(t)X(t)+\langle\theta(t),\hat{\mu}(t)\rangle-c(t)-p(t)+h(t,z_1,z_2,z_3,\upsilon(\cdot))\\
& -\frac{\delta^2}{2}\Big[(z_1-\theta_1 b_r(t,T))^2+(z_2-(\theta_1(t)+\theta_2(t))\sigma_I(t))^2+(z_3-\theta_3(t)\sigma_S(t))^2\Big]\\
& +\frac{1}{\delta}\int_{\mathbb{R}}[1-\delta(\langle\theta(t),\hat{\gamma}(t,T,z)\rangle-\upsilon(t,z))\\
& -\exp(-\delta(\langle\theta(t),\hat{\gamma}(t,T,z)\rangle-\upsilon(t,z)))]\nu(dz)\Big]dt-(z_1-\theta_1(t)b_r(t,T))dW_r(t)\\
& -(z_2-(\theta_1(t)+\theta_2(t)))\sigma_I(t)dW_I(t)-(z_3-\theta_3(t)\sigma_S(t))dW_S(t)\\
& +\int_{\mathbb{R}}[1-\exp(-\delta(\langle\theta(t),\hat{\gamma}(t,T,z)\rangle-\upsilon(t,z)))]\tilde{N}(dt,dz)\Big\}
\end{aligned}
$$

with the initial condition $\mathcal{R}_1(0)=-\exp(-\delta(x-Y^r(0)))$.

Note that the drift process of the family $\mathcal{R}_1$ is given by

$$
\begin{aligned}
& A(t) \qquad (5.17)\\
= \; & \delta e^{-\int_0^t(\varrho(u)+\lambda(u))du} \cdot e^{-\delta(X(t)-Y^r(t))}\Big\{-e^{\delta(X(t)-Y^r(t))}\left[e^{-\delta c(t)}+\lambda(t)e^{-\delta\ell(t)}\right]\\
& +\frac{1}{\delta}(\varrho(t)+\lambda(t))+r_r(t)X(t)+\langle\theta(t),\hat{\mu}(t)\rangle-c(t)-p(t)+h(t,r,z_1,z_2,z_3,\upsilon(\cdot))\\
& -\frac{\delta^2}{2}[(z_1-\theta_1 b_r(t,T))^2+(z_2-(\theta_1(t)+\theta_2(t))\sigma_I(t))^2+(z_3-\theta_3(t)\sigma_S(t))^2]\\
& +\frac{1}{\delta}\int_{\mathbb{R}}[1-\delta(\langle\theta(t),\hat{\gamma}(t,T,z)\rangle-\upsilon(t,z))\\
& -\exp(-\delta(\langle\theta(t),\hat{\gamma}(t,T,z)\rangle-\upsilon(t,z)))]\nu(dz)\Big\}.
\end{aligned}
$$

Therefore, the process $\mathcal{R}_1$ is a local super-martingale if the drift process $A(t)$ is non-positive. This holds true if the generator $h$ is defined as follows

$$
\begin{aligned}
& h(t,r,y,z_1,z_2,z_3,\upsilon) && (5.18)\\
= \; & \inf_c \left\{ e^{\delta(X(t)-Y^r(t))} \cdot e^{-\delta c(t)} + c(t) \right\} + \inf_p \left\{ \lambda(t) e^{\delta(X(t)-Y^r(t))} \cdot e^{-\delta \ell(t)} + p(t) \right\} \\
& -\frac{1}{\delta}(\varrho(t)+\lambda(t)) - r_r(t)X(t) + \inf_\theta \Big\{ \frac{\delta}{2} \Big[ \big| \theta_1(t) b_r(t,T) - \big(z_1 + \frac{\varphi_1(t)}{\delta}\big)\big|^2 \\
& + \big| (\theta_1(t)+\theta_2(t))\sigma_I(t) - \big(z_2 + \frac{\varphi_2(t)}{\delta}\big)\big|^2 + \big|\theta_3(t)\sigma_S(t) - \big(z_3 + \frac{\varphi_3(t)}{\delta}\big)\big|^2 \Big] \\
& + \frac{1}{\delta}\int_{\mathbb{R}} [\exp(\delta(\upsilon(t,z) - \langle \theta(t), \hat{\gamma}(t,T,z)\rangle)) - 1 - \delta(\upsilon(t,z) \\
& - \langle \theta(t), \hat{\gamma}(t,T,z)\rangle)]\nu(dz) \Big\} - (\varphi_1(t)z_1 + \varphi_2(t)z_2 + \varphi_3(t)z_3) \\
& - \frac{1}{2\delta}\left(\varphi_1^2(t) + \varphi_2^2(t) + \varphi_3^2(t)\right) ,
\end{aligned}
$$

provided that $\frac{1}{\delta}\int_{\mathbb{R}}[\exp(\delta(\upsilon(t,z) - \langle\theta(t),\hat{\gamma}\rangle)) - 1 - \delta(\upsilon(t,z) - \langle\theta(t),\hat{\gamma}\rangle)]\nu(dz)$ is finite, for any $\theta \in \mathcal{A}$. Due to the boundedness of the associated parameters, for any $z_1, z_2, z_3 \in \mathbb{R}$, the generator $h(t, z_1, z_2, z_3, \upsilon)$ is almost surely finite. Solving the three minimization problems in (5.18), leads to

$$
c^*(t) = X^{(\theta^*,c^*,p^*)}(t) - Y^r(t) + \frac{1}{\delta}\ln\delta; \qquad p^*(t) = \eta(t)\Big[\frac{1}{\delta}\ln\Big(\frac{\delta\lambda(t)}{\eta(t)}\Big) - Y^r(t)\Big] ,
$$

and $\theta^*(t)$ in (5.12), where $X^{(\theta^*,c^*,p^*)}$ is the wealth process associated to $(\theta^*, c^*, p^*)$ and $Y^r$ is part of the solution $(Y^r, Z_1, Z_2, Z_3, \Upsilon)$ of the BSDE with jumps (5.6), with terminal condition $Y^r(T) = 0$ and the generator $h$ given by (5.11).

To prove the super-martingale property of $\mathcal{R}_1^{(\theta,c,p)}$, we consider a function $\Psi(t) = e^{-\delta X(t)}$. Applying the generalized Itô's formula and the dynamics of $X(t)$ in (5.2), we have

$$
\begin{aligned}
& d\Psi(t) \\
= \ & \Psi(t)\Big\{\big[-\delta[r_r(t)X(t)+\langle\theta(t),\hat{\mu}(t)\rangle-c(t)-p(t)] \\
& +\frac{\delta}{2}\left[\theta_1^2(t)b_r^2(t,T)+(\theta_1(t)+\theta_2(t))^2\sigma_I^2(t)+\theta_3^2(t)\sigma_S^2(t)\right] \\
& +\int_{\mathbb{R}}[\exp(-\delta\langle\theta(t),\hat{\gamma}(t,T,z)\rangle)-1+\delta\langle\theta(t),\hat{\gamma}(t,T,z)\rangle]\nu(dz)\big]dt \\
& -\delta\theta_1(t)b_r(t,T)dW_r(t)-\delta(\theta_1(t)+\theta_2(t))\sigma_I(t)dW_I(t)-\delta\theta_3(t)\sigma_S(t)dW_s(t) \\
& +\int_{\mathbb{R}}[\exp(-\delta\langle\theta(t),\hat{\gamma}(t,T,z)\rangle)-1]\tilde{N}(dt,dz)\Big\}\,.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \Psi(t) \\
= \ & \Psi(0)\mathcal{E}\Big(-\int_0^t\delta\theta_1(t)b_r(t,T)dW_r(t)-\int_0^t\delta(\theta_1(t)+\theta_2(t))\sigma_I(t)dW_I(t) \qquad (5.19)\\
& -\int_0^t\delta\theta_3(t)\sigma_S(t)dW_s(t)+\int_0^t\int_{\mathbb{R}}[\exp(-\delta\langle\theta(t),\hat{\gamma}(t,T,z)\rangle)-1]\tilde{N}(dt,dz)\Big)e^{K(t)}\,,
\end{aligned}
$$

where $\mathcal{E}(M)$ denotes the stochastic exponential of $M$ and

$$
\begin{aligned}
K(t) \ = \ & \int_0^t\big[-\delta[r_r(s)X(s)+\langle\theta(s),\hat{\mu}(s)\rangle-c(s)-p(s)] \\
& +\frac{\delta}{2}\left[\theta_1^2(s)b_r^2(s,t)+(\theta_1(s)+\theta_2(s))^2\sigma_I^2(s)+\theta_3^2(s)\sigma_S^2(s)\right] \\
& +\int_{\mathbb{R}}[\exp(-\delta\langle\theta(s),\hat{\gamma}(s,t,z)\rangle)-1+\delta\langle\theta(s),\hat{\gamma}(s,t,z)\rangle]\nu(dz)\big]ds\,.
\end{aligned}
$$

Hence, $K(t)$ is a bounded process due to the boundedness of the associated parameters and that the strategy $(c(t),p(t),\theta(t))\in\mathcal{C}\times\mathcal{D}\times\mathcal{Q}$. Furthermore, thanks to the boundedness of the associated parameters and $\exp(-\delta\langle\theta(t),\hat{\gamma}(t,T,z)\rangle)-1>-1$, the local martingale process

$$
\begin{aligned}
& M(t) \\
:= \ & -\int_0^t\delta\theta_1(t)b_r(t,T)dW_r(t)-\int_0^t\delta(\theta_1(t)+\theta_2(t))\sigma_I(t)dW_I(t) \\
& -\int_0^t\delta\theta_3(t)\sigma_S(t)dW_s(t)+\int_0^t\int_{\mathbb{R}}[\exp(-\delta\langle\theta(t),\hat{\gamma}(t,T,z)\rangle)-1]\tilde{N}(dt,dz)
\end{aligned}
$$

satisfies the $BMO$-martingale property. Then, by *Kazamaki's criterion* (Lemma 2.3.3), the stochastic exponential $\mathcal{E}$ in (5.19), is a true martingale. Hence $\Psi$ is uniformly integrable. Then we can conclude that $\mathcal{R}_1^{(\theta,c,p)}$ is a supermartingale, i.e.,

$$\mathcal{R}_1^{(\theta,c,p)}(0) \geq \mathbb{E}[\mathcal{R}_1^{(\theta,c,p)}(T)]\,.$$

On the other hand, $A(t) \equiv 0$, for the strategy $(\theta^*, c^*, p^*)$, hence $\mathcal{R}_1^{(\theta^*,c^*,p^*)}$ is a true martingale. Therefore, (5.10) hold, which completes the proof.

□

### 5.3.2 The power utility case

Let $\pi(t) := (\pi_1(t), \pi_2(t), \pi_3(t))$, $t \in [0,T]$ be the vector of the portfolio weights invested in $P_r^*(t,T)$, $B_r^*(t)$ and $S(t)$ respectively. Define the relative consumption rate $\xi(t)$ and the relative premium insurance rate $\zeta(t)$ by their fraction of the total wealth, i.e., $\xi(t) := \frac{c(t)}{X(t)}$ and $\zeta(t) := \frac{p(t)}{X(t)}$. We suppose that the strategy $(\pi(t), \xi(t), \zeta(t))$ satisfies the integrability condition similar to (5.1) for $(\theta(t), c(t, p(t)))$. Define $\mathcal{A}^{\S}$ as the admissible strategy for $(\pi(t), \xi(t), \zeta(t))$. Then, the wealth process $X(t)$ becomes

$$\begin{aligned} dX(t) \;\; = \;\; & X(t)\Big\{[r_r(t) + \langle \pi(t), \hat{\mu}(t)\rangle - \xi(t) - \zeta(t)]dt + \pi_1(t)b_r(t,T)dW_r(t) \\ & +(\pi_1(t)+\pi_2(t))\sigma_I(t)dW_I(t) + \pi_3(t)\sigma_S(t)dW_S(t) \\ & + \int_{\mathbb{R}} \langle \pi(t), \hat{\gamma}(t,T,z)\rangle \tilde{N}(dt,dz)\Big\}\,, \quad X(0) = x > 0\,, \end{aligned} \tag{5.20}$$

which, by Itô's formula, gives the following solution

$$\begin{aligned}
& X(T) \\
= \;& x \exp\Big\{ \int_0^T \Big[ r_r(t) + \langle \pi(t), \hat{\mu}(t) \rangle - \xi(t) - \zeta(t) - \frac{1}{2} [\pi_1^2(t) b_r^2(t,T) \\
& + (\pi_1(t) + \pi_2(t))^2 \sigma_I^2(t) + \pi_3^2(t) \sigma_S^2(t)] \\
& + \int_{\mathbb{R}} [\ln(1 + \langle \pi(t), \hat{\gamma}(t,T,z) \rangle) - \langle \pi(t), \hat{\gamma}(t,T,z) \rangle] \nu(dz) \Big] dt \\
& + \int_0^T \pi_1(t) b_r(t,T) dW_r(t) + \int_0^T (\pi_1(t) + \pi_2(t)) \sigma_I(t) dW_I(t) \\
& + \int_0^T \pi_3(t) \sigma_S(t) dW_S(t) + \int_0^T \int_{\mathbb{R}} \ln(1 + \langle \pi(t), \hat{\gamma}(t,T,z) \rangle) \tilde{N}(dt,dz) \Big\} .
\end{aligned}$$

Consider the following utility function

$$U(x) = \frac{x^\kappa}{\kappa}, \qquad \kappa \in (-\infty, 1) \setminus \{0\} . \tag{5.21}$$

The functional (5.4) can be written as

$$\begin{aligned}
\mathcal{J}(t) \;=\; & \mathbb{E}_{t,x} \Big[ \frac{1}{\kappa} \int_t^T e^{-\int_t^s (\varrho(u) + \lambda(u)) du} \left[ (\xi(s))^\kappa + \lambda(s) \left( 1 + \frac{\zeta(s)}{\eta(s)} \right)^\kappa \right] (X(s))^\kappa ds \\
& + e^{-\int_t^T (\varrho(u) + \lambda(u)) du} \frac{(X(T))^\kappa}{\kappa} \mid \mathcal{F}_t \Big] .
\end{aligned} \tag{5.22}$$

Define a function $B(t)$ as

$$\begin{aligned}
B(t) \;=\; & \int_0^t \Big[ r_r(s) + \langle \pi(s), \hat{\mu}(s) \rangle - \xi(s) - \zeta(s) - \frac{1}{2} [\pi_1^2(s) b_r^2(s,t) \\
& + (\pi_1(s) + \pi_2(s))^2 \sigma_I^2(s) + \pi_3^2(s) \sigma_S^2(s)] \\
& + \int_{\mathbb{R}} [\ln(1 + \langle \pi(s), \hat{\gamma}(s,t,z) \rangle) - \langle \pi(s), \hat{\gamma}(s,t,z) \rangle] \nu(dz) \Big] ds \\
& + \int_0^t \pi_1(s) b_r(s,t) dW_r(s) + \int_0^t (\pi_1(s) + \pi_2(s)) \sigma_I(s) dW_I(s) \\
& + \int_0^t \pi_3(s) \sigma_S(s) dW_S(s) + \int_0^t \int_{\mathbb{R}} \ln(1 + \langle \pi(s), \hat{\gamma}(s,t,z) \rangle) \tilde{N}(ds,dz) .
\end{aligned}$$

Then the wealth process can be written as $X(t) = x e^{B(t)}$.

The main result of this subsection is given by the following theorem.

**Theorem 5.3.3.** *Suppose that the utility function is given by (5.21). Then, the optimal value function is given by*

$$V(t,x,r) = \frac{x^\kappa}{\kappa} e^{Y^r(t)}, \tag{5.23}$$

*where $Y^r$ is part of the solution $(Y^r, Z_1, Z_2, Z_3, \Upsilon) \in \mathbb{S}(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_\nu(\mathbb{R})$ of the following BSDE with jumps*

$$\begin{aligned} dY(t) &= -h_1(t,r,Y^r(t),Z_1(t),Z_2(t),Z_3(t),\Upsilon(t,\cdot))dt + Z_1(t)dW_r(t) \\ &\quad + Z_2(t)dW_I(t) + Z_3(t)dW_S(t) + \int_{\mathbb{R}} \Upsilon(t,z)\tilde{N}(dt,dz); \qquad (5.24) \\ Y^r(T) &= 0 \end{aligned}$$

*with the generator*

$$\begin{aligned} h_1(t,r,y,z_1,z_2,z_3,\upsilon(\cdot)) &= \Big\{\frac{1}{\kappa}\Big(1+\lambda(t)\Big(\frac{\eta(t)}{\lambda(t)}\Big)^{-\frac{\kappa}{1-\kappa}}\Big) - \Big(1+\eta(t)\Big(\frac{\eta(t)}{\lambda(t)}\Big)^{-\frac{1}{1-\kappa}}\Big)\Big\} e^{-\frac{1}{1-\kappa}y} \qquad (5.25) \\ &\quad -\frac{1}{\kappa}(\varrho(t)+\lambda(t)) + r_r(t) + \inf_\pi\Big\{\frac{\kappa-1}{2}\Big[\Big|\pi_1(t)b_r(t,T) + \frac{z_1+\varphi_1(t)}{\kappa-1}\Big|^2 \\ &\quad + \Big|(\pi_1(t)+\pi_2(t))\sigma_I(t) + \frac{z_2+\varphi_2(t)}{\kappa-1}\Big|^2 + \Big|\pi_3(t)\sigma_S(t) + \frac{z_3+\varphi_3(t)}{\kappa-1}\Big|^2\Big] \\ &\quad + \int_{\mathbb{R}} [(1+\langle\pi(t),\hat{\gamma}(t,T,z)\rangle)^\kappa e^{\kappa\upsilon(t,z)} - 1 - \kappa\langle\pi(t),\hat{\gamma}(t,T,z)\rangle - \upsilon(t,z)]\nu(dz)\Big\} \\ &\quad - \frac{1}{2(\kappa-1)}[(z_1+\varphi_1(t))^2 + (z_2+\varphi_2(t))^2 + (z_3+\varphi_3(t))^2] + \frac{1}{\kappa}(z_1^2+z_2^2+z_3^3). \end{aligned}$$

*Moreover, the optimal strategy $(\pi^*(t), \xi^*(t), \zeta^*(t))$ is given by*

$$\xi^*(t) = e^{-\frac{1}{1-\kappa}Y^r(t)}, \qquad \zeta^*(t) = \eta(t)\Big[\Big(\frac{\eta(t)}{\lambda(t)}\Big)^{-\frac{1}{1-\kappa}} e^{-\frac{1}{1-\kappa}Y^r(t)} - 1\Big]$$

*and*

$$\begin{aligned} \pi^*(t) &= \inf_\pi\Big\{\frac{\kappa-1}{2}\Big[\Big|\pi_1(t)b_r(t,T) + \frac{z_1+\varphi_1(t)}{\kappa-1}\Big|^2 \\ &\quad + \Big|(\pi_1(t)+\pi_2(t))\sigma_I(t) + \frac{z_2+\varphi_2(t)}{\kappa-1}\Big|^2 + \Big|\pi_3(t)\sigma_S(t) + \frac{z_3+\varphi_3(t)}{\kappa-1}\Big|^2\Big] \qquad (5.26) \\ &\quad + \int_{\mathbb{R}} [(1+\langle\pi(t),\hat{\gamma}(t,T,z)\rangle)^\kappa e^{\kappa\upsilon(t,z)} - 1 - \kappa\langle\pi(t),\hat{\gamma}(t,T,z)\rangle - \upsilon(t,z)]\nu(dz)\Big\}. \end{aligned}$$

Note that the generator $h_1$ in (5.25) has an exponential growth in $Y$. However, due to the boundedness of the associated parameters, it satisfies the monotonicity condition, i.e., there exists a constant $K \geq 0$ such that $y(h_1(t,r,y,z_1,z_2,z_3,\upsilon(\cdot) - h_1(t,r,0,z_1,z_2,z_3,\upsilon(\cdot)) \leq K|y|^2$. Moreover, it can be seen that the conditions (5.13)-(5.14) are satisfied. Then, by (Briand and Hu [11] and Morlais [70]), the BSDE with jumps (5.24) has a unique solution $(Y^r, Z_1, Z_2, Z_3, \Upsilon) \in \mathbb{S}(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_\nu(\mathbb{R})$.

Similar to Corollary 5.3.2, the optimal investment strategy $\pi^*(t) = (\pi_1^*(t), \pi_2^*(t), \pi_3^*(t))$ for the special case of not having jumps ($\nu = 0$), is given by the following corollary.

**Corollary 5.3.4.** *Assume that $\nu = 0$, then the optimal portfolio strategy $(\pi_1^*(t), \pi_2^*(t), \pi_3^*(t))$, for all $t \in [0, T]$ is given by*

$$\begin{aligned}
\pi_1^*(t) &= \frac{1}{1-\kappa}\Big[\frac{\tilde{A}(t,T) - r_r(t) - \mu_I(t)}{b_r^2(t,T)} + \frac{Z_1(t)}{b_r(t,T)}\Big] \\
\pi_2^*(t) &= \frac{1}{1-\kappa}\Big[\Big(\frac{1}{\sigma_I^2(t)} + \frac{1}{b_r^2(t,T)}\Big)\mu_I(t) - \frac{\tilde{A}(t,T) - r_r(t)}{b_r^2(t,T)} + \frac{Z_2(t)}{\sigma_I(t)} - \frac{Z_1(t)}{b_r(t,T)}\Big] \\
\pi_3^*(t) &= \frac{1}{1-\kappa}\Big[\frac{\mu_S(t) - r_r(t)}{\sigma_S^2(t)} + \frac{Z_3(t)}{\sigma_S(t)}\Big],
\end{aligned}$$

*where $(Z_1(t), Z_2(t), Z_3(t))$ is part of the solution $(Y^r, Z_1, Z_2, Z_3)$ of the following BSDE.*

$$\begin{aligned}
dY^r(t) &= -h_1(t, Y^r(t), Z_1(t), Z_2(t), Z_3(t), \pi^*(t), \xi^*(t), \zeta^*(t))dt + Z_1(t)dW_r(t) \\
&\quad + Z_2(t)dW_I(t) + Z_3(t)dW_S(t); \\
Y^r(T) &= 0\,.
\end{aligned}$$

*Proof of Theorem 5.3.3.*

Consider the process

$$\begin{aligned}
\mathcal{R}_2(t) &= \frac{1}{\kappa}\int_0^t e^{-\int_0^s(\varrho(u)+\lambda(u))du}\Big[(\xi(s))^\kappa + \lambda(s)\Big(1 + \frac{\zeta(s)}{\eta(s)}\Big)^\kappa\Big](X(s))^\kappa ds \\
&\quad + e^{-\int_0^t(\varrho(u)+\lambda(u))du}\frac{(X(t))^\kappa}{\kappa}e^{Y^r(t)},
\end{aligned}$$

with initial condition $\mathcal{R}_2(0) = \frac{x^\kappa}{\kappa}e^{Y^r(0)}$. Applying the generalized Itô's formula, we obtain

$$
\begin{aligned}
d\mathcal{R}_2(t) \quad = \quad & e^{-\int_0^t(\varrho(u)+\lambda(u))du}(X(t))^\kappa e^{Y(t)}\Big\{\Big[\frac{1}{\kappa}e^{-Y^r(t)}\Big((\xi(t))^\kappa+\lambda(t)\Big(1+\frac{\zeta(t)}{\eta(t)}\Big)^\kappa\Big) \\
& -\frac{1}{\kappa}(\varrho(t)+\lambda(t))+r_r(t)+\langle\pi(t),\hat{\mu}(t)\rangle-\xi(t)-\zeta(t)-h_1(t,r,y,z_1,z_2,z_3,\upsilon) \\
& +\frac{1}{2}(\kappa-1)[\pi_1^2(t)b_r^2(t,T)+(\pi_1(t)+\pi_2(t))^2\sigma_I^2(t)+\pi_3^2(t)\sigma_S^2(t)] \\
& +\frac{1}{2\kappa}(z_1^2+z_2^2+z_3^2)+\pi_1(t)b_r(t,T)z_1+(\pi_1(t)+\pi_2(t))\sigma_I(t)z_2+\pi_3(t)\sigma_S(t)z_3 \\
& +\frac{1}{\kappa}\int_{\mathbb{R}}[(1+\langle\pi(t),\hat{\gamma}(t,T,z)\rangle)^\kappa e^{\kappa\upsilon(t,z)}-1-\kappa\langle\pi(t),\hat{\gamma}(t,T,z)\rangle-\upsilon(t,z)]\nu(dz)\Big]dt \\
& +\frac{1}{\kappa}[(z_1+\kappa\pi_1(t)b_r(t,T))dW_r(t)+(z_2+\kappa(\pi_1(t)+\pi_2(t))\sigma_I(t))dW_I(t) \\
& +(z_3+\kappa\pi_3(t)\sigma_S(t))dW_S(t)]+\frac{1}{\kappa}\int_{\mathbb{R}}[(1+\langle\pi(t),\hat{\gamma}(t,T,z)\rangle)^\kappa e^{\kappa\upsilon(t,z)}-1 \\
& +\kappa\ln(1+\langle\pi(t),\hat{\gamma}(t,T,z)\rangle)-\upsilon(t,z)]\tilde{N}(dz,dt)\Big\}\,. \qquad (5.27)
\end{aligned}
$$

Note that similar to the exponential case, we can easily see that the process $\mathcal{R}_2$ is a local super-martingale if the generator $h_1$ is given by

$$
\begin{aligned}
h_1(t,r,y,z_1,z_2,z_3,\upsilon) \quad = \quad & \inf_{\xi}\Big\{\frac{1}{\kappa}e^{-Y^r(t)}(\xi(t))^\kappa-\xi(t)\Big\}+\inf_{\zeta}\Big\{\frac{1}{\kappa}\lambda(t)e^{-Y^r(t)}\Big(1+\frac{\zeta(t)}{\eta(t)}\Big)^\kappa-\zeta(t)\Big\} \\
& +\inf_{\pi}\Big\{\frac{\kappa-1}{2}\Big[\big|\pi_1(t)b_r(t,T)+\frac{z_1+\varphi_1(t)}{\kappa-1}\big|^2 \\
& +\big|(\pi_1(t)+\pi_2(t))\sigma_I(t)+\frac{z_2+\varphi_2(t)}{\kappa-1}\big|^2+\big|\pi_3(t)\sigma_S(t)+\frac{z_3+\varphi_3(t)}{\kappa-1}\big|^2\Big] \\
& +\int_{\mathbb{R}}[(1+\langle\pi(t),\hat{\gamma}(t,T,z)\rangle)^\kappa e^{\kappa\upsilon(t,z)}-1-\kappa\langle\pi(t),\hat{\gamma}(t,T,z)\rangle-\upsilon(t,z)]\nu(dz)\Big\} \\
& -\frac{1}{2(\kappa-1)}\big[(z_1+\varphi_1(t))^2+(z_2+\varphi_2(t))^2+(z_3+\varphi_3(t))^2\big] \\
& +\frac{1}{\kappa}[z_1^2+z_2^2+z_3^3]-\frac{1}{\kappa}(\varrho(t)+\lambda(t))+r_r(t)\,.
\end{aligned}
$$

Solving the three minimization problems, provided that the associated parameters are bounded $\{\mathcal{F}_t\}_{t\in[0,T]}$-predictable, we obtain the candidate optimal strategy

$$
\xi^*(t)=e^{-\frac{1}{1-\kappa}Y^r(t)}, \qquad \zeta^*(t)=\eta(t)\left[\left(\frac{\eta(t)}{\lambda(t)}\right)^{-\frac{1}{1-\kappa}}e^{-\frac{1}{1-\kappa}Y^r(t)}-1\right]
$$

and $\pi^*(t)$ in (5.26). Where $Y^r$ is part of the solution $(Y^r,Z_1,Z_2,Z_3,\Upsilon)\in\mathbb{S}(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})\times\mathbb{H}^2_\nu(\mathbb{R})$ of the BSDE with jumps (5.24), with terminal condition $Y^r(T)=0$ and the generator $h_1$ given by (5.25).

To prove the super-martingale property, we consider the following function $(X(t))^\kappa$. Applying the generalized Itô's formula, we have

$$
\begin{aligned}
& d(X(t))^\kappa \\
= \; & (X(t))^\kappa \Big\{ \Big[ r_r(t) + \langle \pi(t), \hat{\mu}(t) \rangle - \xi(t) - \zeta(t) + \frac{\kappa}{2}(\kappa-1)[\pi_1^2(t) b_r^2(t,T) \\
& + (\pi_1(t)+\pi_2(t))^2 \sigma_I^2(t) + \pi_3^2(t)\sigma_S^2(t)] \\
& + \int_{\mathbb{R}} [1 - \kappa \ln(1 + \langle \pi(t), \hat{\gamma}(t,T,z) \rangle) - (1 + \langle \pi(t), \hat{\gamma}(t,T,z) \rangle)^{-\kappa}] \nu(dz) \Big] dt \\
& + \kappa \pi_1(t) b_r(t,T) dW_r(t) + \kappa(\pi_1(t)+\pi_2(t))\sigma_I(t) dW_I(t) \\
& + \kappa \pi_3(t) \sigma_S(t) dW_S(t) + \kappa \int_{\mathbb{R}} \ln(1 + \langle \pi(t), \hat{\gamma}(t,T,z) \rangle) \tilde{N}(dt,dz) \Big\} .
\end{aligned}
$$

Hence

$$
\begin{aligned}
& (X(t))^\kappa \\
= \; & (X(0))^\kappa \mathcal{E}\Big( \kappa \pi_1(t) b_r(t,T) dW_r(t) + \kappa(\pi_1(t)+\pi_2(t))\sigma_I(t) dW_I(t) \\
& + \kappa \pi_3(t)\sigma_S(t) dW_S(t) + \kappa \int_{\mathbb{R}} \ln(1 + \langle \pi(t), \hat{\gamma}(t,T,z) \rangle) \tilde{N}(dt,dz) \Big) e^{Q(t)} ,
\end{aligned}
$$

where

$$
\begin{aligned}
Q(t) \; = \; & \int_0^t \Big[ r_r(s) + \langle \pi(s), \hat{\mu}(s) \rangle - \xi(s) - \zeta(s) + \frac{\kappa}{2}(\kappa-1)[\pi_1^2(s) b_r^2(s,t) \\
& + (\pi_1(s)+\pi_2(s))^2 \sigma_I^2(s) + \pi_3^2(s)\sigma_S^2(s)] \\
& + \int_{\mathbb{R}} [1 - \kappa \ln(1 + \langle \pi(s), \hat{\gamma}(s,t,z) \rangle) - (1 + \langle \pi(s), \hat{\gamma}(s,t,z) \rangle)^{-\kappa}] \nu(dz) \Big] ds .
\end{aligned}
$$

Hence, $Q(t)$ is a bounded process due to the boundedness of the associated parameters and the fact that the strategy $(\xi(t), \zeta(t), \pi(t)) \in \mathcal{C} \times \mathcal{D} \times \mathcal{Q}$. Moreover, using similar arguments of a $BMO$-martingale property as in the proof of Theorem 5.3.1, we can easily see that $(X(t))^\kappa$ is uniformly integrable. Then $\mathcal{R}_2^{(\theta,c,p)}$ is a super-martingale, i.e.,

$$
\mathcal{R}_2^{(\theta,c,p)}(0) \geq \mathbb{E}[\mathcal{R}_2^{(\theta,c,p)}(T)] .
$$

On the other hand, the drift process in (5.27) is equal to zero for the strategy $(\theta^*, c^*, p^*)$, hence $\mathcal{R}_1^{(\theta^*,c^*,p^*)}$ is a true martingale. Therefore, (5.23) hold.

$\square$

**Remark.** We point out that, when there is no inflation and jumps in the model, the results obtained in Theorems 5.3.1 and 5.3.3 relate on the results in Cheridito and Hu [14]. Similar results have been obtained by Xing [98] for the Espein-Zin utility type. Moreover, when there is no consumption and life insurance rates, these results are similar to those in Hu *et. al.* [46] for the diffusion case and Morlais [70] in the jump-diffusion case.

## 5.4 Conclusion

In this Chapter, we solved an optimal investment, consumption and life insurance problem using the BSDE techniques. We considered the presence of inflation-linked asset, which normally helps the investors to manage the inflation risks that in general are not completely observable. Under jump diffusion market, we derived the optimal strategy for the exponential and power utility functions. This work extends, for instance, the paper by Cheridito and Hu [14], by allowing the presence of inflation risks, life insurance and jumps in the related assets. Furthermore, it appears as an alternative approach to the dynamic programming approach applied in Han and Hung [42], were a similar problem was considered under a stochastic differential utility. We noted that the generator of the associated BSDE is of quadratic growth in the controls $z_1$, $z_2$, $z_3$ and exponential in $\upsilon(\cdot)$, the similar BSDEs with jumps that the existence and uniqueness results have been proved by Morlais [70], [71]. Furthermore, we derived the explicit solutions for the optimal portfolio for a special case without jumps.

# Chapter 6

# Risk-based optimal portfolio of an insurer with regime switching and noisy memory

## 6.1 Introduction

Stochastic delay equations are equations whose coefficients depend also on past history of the solution. They appear naturally in economics, life science, finance, engineering, biology, etc. In Mathematics of Finance, the basic assumption of the evolution price process is that they are Markovian. In reality, these processes possess some memory which cannot be neglected. Stochastic delay control problems have received much interest in recent times and these are solved by different methods. For instance, when the state process depends on the discrete and average delay, Elsanosi *et. al.* [37] studied an optimal harvesting problem using the dynamic programming approach. On the other hand, a maximum principle approach was used to solve optimal stochastic control systems with delay. See e.g., Øksendal and Sulem [76], Pamen [79]. When the problem allows a noisy memory, i.e., a delay modeled by a Brownian motion, Dahl *et. al* [22] proposed a maximum principle approach with Malliavin derivatives to solve their problem. For detailed information on the theory of stochastic delay differential equations (SDDE) and their applications to stochastic control problems, see, e.g., Baños *et. al.* [6], Kuang [56], Mohammed [69] and references therein.

In this chapter, we consider an insurer's risk-based optimal investment problem with noisy memory. The financial market model setup is composed by one risk-free asset and one risky asset described by a hidden Markov regime-switching jump-diffusion process. The jump-diffusion models represent a valuable extension of the diffusion models for modeling the asset prices. They capture some sudden changes in the market such as the existence of high-frequency data, volatility clusters and regime switching. It is important to note that in the Markov regime-switching diffusion models, we can have random coefficients possibly with jumps, even if the return process is a diffusion one. In this chapter, we consider a jump diffusion model, which incorporates jumps in the asset price as well as in the model coefficients, i.e., a Markov regime-switching jump-diffusion model. Furthermore, we consider the Markov chain to represent different modes of the economic environment such as, political situations, natural catastrophes or change of law. Such kind of models have been considered for option pricing of the contingent claim, see for example, Elliott *et. al* [36], Siu [92] and references therein. For stochastic optimal control problems, we mention the works by Bäuerle and Rieder [7], Meng and Siu [64]. In these works a portfolio asset allocation and a risk-based asset allocation of a Markov-modulated jump process model has been considered and solved via the dynamic programming approach. We also mention a recent work by Pamen and Momeya [80], where a maximum principle approach has been applied to an optimization problem described by a Markov-modulated regime switching jump-diffusion model.

In this chapter, we assume that the company receives premiums at the constant rate and pays the aggregate claims modeled by a hidden Markov-modulated pure jump process. We assume the existence of capital inflow or outflow from the insurer's current wealth, where the amount of the capital is proportional to the past performance of the insurer's wealth. Then, the surplus process is governed by a stochastic delay differential equation with the delay, which may be random. Therefore we find it reasonable to consider also a delay modeled by Brownian motion. In literature, a mean-variance problem of an insurer was considered, but the wealth process is given by a diffusion model with distributed delay, solved via the maximum principle approach (Shen and Zeng [89]). Chunxiang and Li [15] extended this mean-variance problem of an insurer to the Heston stochastic volatility case and solved using dynamic programming approach. For thorough discussion on different types of delay, we refer to Baños *et. al.* [6], Section 2.2.

We adopt a convex risk measure first introduced by Frittelli and Gianin [40] and Föllmer and Schied [38]. This generalizes the concept of coherent risk measure first introduced by Artzner *et. al.* [3], since it includes the nonlinear dependence of the risk of the portfolio due to the liquidity risks. Moreover, it relaxes a sub-additive and positive homogeneous properties of the coherent risk measures and substitute these by a convex property.

When the risky share price is described by a diffusion process and without delay, such kind of risk-based optimization problems of an insurer have been widely studied and reported in literature, see e.g., Elliott and Siu [34, 35], Siu [90–92], Peng and Hu [81]. For a jump-diffusion case, we refer to Mataramvura and Øksendal [62].

To solve our optimization problem, we first transform the unobservable Markov regime-switching problem into one with complete observation by using the so-called filtering theory, where the optimal Markov chain is also derived. For interested readers, we refer to Elliott *et. al.* [32], Elliott and Siu [35], Cohen and Elliott [16] and Kallianpur [49]. Then we formulate a convex risk measure described by a terminal surplus process as well as the dynamics of the noisy memory surplus over a period $[T-\varrho, T]$ of the insurer to measure the risks. The main objective of the insurer is to select the optimal investment strategy so as to minimize the risk. This is a two-player zero-sum stochastic delayed differential game problem. Using delayed backward stochastic differential equations (BSDE) with a jump approach, we solve this game problem by an application of a comparison principle for BSDE with jumps. Our modeling framework follows that in Elliott and Siu [34], later extended to the regime switching case by Peng and Hu [81].

The rest of the chapter is organized as follows: In Section 6.2, we introduce the dynamic of state process described by SDDE in the Hidden Markov regime switching jump-diffusion market. In Section 6.3, we use the filtering theory to turn the model into one with complete observation. We also derive the optimal Markov chain. Section 6.4, is devoted to the formulation of our risk-base optimization problem as a zero-sum stochastic delayed differential game problem, which is then solved in Section 6.5. Finally, in Section 6.6, we derive the explicit solutions for a particular case of a quadratic penalty function and we give an example to show how one can apply these results in a concrete situation.

## 6.2 Model formulation

Suppose we have an insurer investing in a finite investment period $T < \infty$. Consider a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$, where $\{\mathcal{F}_t\}_{t\in[0,T]}$ is a filtration satisfying the usual conditions. Let $\Lambda(t)$ be a continuous time finite state hidden Markov chain defined on $(\Omega, \mathcal{F}, \mathbb{P})$, with a finite state space $\mathcal{S} = \{e_1, e_2, \ldots, e_D\} \subset \mathbb{R}_D$, $e_j = (0, \ldots, 1, 0, \ldots, 0) \in \mathbb{R}^D$, where $D \in \mathbb{N}$ is the number of states of the chain, and the $j$th component of $e_n$ is the Kronecker delta $\delta_{nj}$, for each $n, j = 1, 2, \ldots, D$. $\Lambda(t)$ describes the evolution of the unobserved state of the model parameters in the financial market over time, i.e., a process which collects factors that are relevant for the model, such as, political situations, laws or natural catastrophes (see, e.g. Bauerle and Rieder [7], Elliott and Siu [35]). The main property of the Markov chain $\Lambda$ with the canonical state space $\mathcal{S}$ is that, any nonlinear function of $\Lambda$, is linear in $\Lambda$, i.e., $\varphi(\Lambda) = \langle\varphi, \Lambda\rangle$. For detailed information, see, for instance, Elliott *et. al.* [32]. When $D = 2$, the state space $\mathcal{S} = \{e_1, e_2\}$, where $e_1$ can be considered as a state with the economy in expansion and $e_2$

the state with the economy in recession.

To describe the probability law of the chain $\Lambda$, we define a family of intensity matrix $A(t) := \{a_{ji}(t);\ t \in [0,T]\}$, where $a_{ji}(t)$ is the instantaneous transition intensity of the chain $\Lambda$ from state $e_i$ to state $e_j$ at time $t \in [0,T]$. Then it was proved in Elliott *et. al.* [32], that $\Lambda$ admits the following semi-martingale dynamics:

$$\Lambda(t) = \Lambda(0) + \int_0^t A(s)\Lambda(s)ds + \Phi(t)\,,$$

where $\Phi$ is an $\mathbb{R}^D$-valued martingale with respect to the natural filtration generated by $\Lambda$.

To describe the dynamics of the financial market, we consider a Brownian motion $W(t)$ and a compensated Markov regime-switching Poisson random measure $\tilde{N}_\Lambda(dt,dz) := N(dt,dz) - \nu_\Lambda(dz)dt$, with the dual predictable projection $\nu_\Lambda$ defined by

$$\nu_\Lambda(dt,dz) = \sum_{j=1}^{D} \langle \Lambda(t-), e_j\rangle \varepsilon_j(t)\nu_j(dz)dt\,,$$

where $\nu_j$ is the conditional Levy measure of the random jump size and $\varepsilon_j$ is the intensity rate when the Markov chain $\Lambda$ is in state $e_j$ . We suppose that the processes $W$ and $N$ are independent.

We consider a financial market consisting of one risk-free asset $(B(t))_{0\le t\le T}$ and one risky asset $(S(t))_{0\le t\le T}$. Their respective prices are given by the following regime-switching SDE:

$$\begin{aligned} dB(t) &= r(t)B(t)dt\,, \quad B(0) = 1\,, \\ dS(t) &= S(t)\Big[\alpha^\Lambda(t)dt + \beta(t)dW(t) + \int_{\mathbb{R}} \Big(e^z - 1\Big)N(dt,dz)\Big] \\ &= S(t)\Big[\Big(\alpha^\Lambda(t) + \sum_{j=1}^{D}\int_{\mathbb{R}}\Big(e^z - 1\Big)\langle \Lambda(t-), e_j\rangle\varepsilon_j(t)\nu_j(dz)\Big)dt \\ &\qquad + \beta(t)dW(t) + \int_{\mathbb{R}}\Big(e^z - 1\Big)\tilde{N}_\Lambda(dt,dz)\Big]\,, \end{aligned} \tag{6.1}$$

with initial value $S(0) = s > 0$. We suppose that the instantaneous interest rate $r(t)$ is a deterministic function, the appreciation rate $\alpha(t)$ is modulated by the Markov chain $\Lambda$, as follows:

$$\alpha^{\Lambda}(t) \ := \ \langle \alpha(t), \Lambda(t) \rangle = \sum_{j=1}^{D} \alpha_j(t) \langle \Lambda(t), e_j \rangle \,,$$

$\alpha_j$ represents the appreciation rate, when the Markov chain is in state $e_j$ of the economy. We suppose that $\alpha(t)$ is $\mathbb{R}^D$-valued $\{\mathcal{F}_t\}_{t\in[0,T]}$-predictable and uniformly bounded processes on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Otherwise, the volatility rate $\beta(t)$ is an $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted uniformly bounded process. Note that we may consider a Markov modulated volatility process, however it would lead in a complicated, if not possible filtering issue in the following section. As was pointed out by Siu [92] and references therein, the other reason is that, the volatility can be determined from a price path of the risky share, i.e., the volatility is observable. For Markov modulated volatility models, see Elliott *et. al.* [33].

We now model the insurance risk by a Markov regime-switching pure jump process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We follow the modeling framework of Elliott and Siu [35], Siu [91], Pamen and Momeya [80].

Consider a real valued pure jump process $Z := \{Z(t);\ t \in [0, T]\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $Z$ denotes the aggregate amount of the claims up to time $t$. Then, we can write $Z$ as

$$Z(t) = \sum_{0<s\leq t} \Delta Z(s)\,; \quad Z(0) = 0, \quad \mathbb{P} - \text{a.s.}, \quad \mathrm{t} \in [0, \mathrm{T}] \,,$$

where $\Delta Z(s) := Z(s) - Z(s-)$, for each $s \in [0, T]$, represents the jump size of $Z$ at time $s$.

Suppose that the state space of the claim size $\mathcal{Z}$ is $(0, \infty)$. Consider a random measure $N^0(\cdot, \cdot)$ defined on a product space $[0, T] \times \mathcal{Z}$, which selects the random claim arrivals and size $z := Z(s) - Z(s-)$, at time $s$. The aggregate insurance claim process $Z$ can be written as

$$Z = \int_0^t \int_0^\infty z N^0(ds, dz); \quad t \in [0, T] \,.$$

Define, for each $t \in [0,T]$,

$$M(t) := \int_0^t \int_0^\infty N^0(ds,dz); \quad t \in [0,T].$$

$M(t)$ counts the number of claim arrivals up to time $t$. Suppose that under $\mathbb{P}$, $M := \{M(t),\, t \in [0,T]\}$ is a conditional Poisson process on $(\Omega, \mathcal{F}, \mathbb{P})$, given the information about the realized path of the chain, with intensity $\lambda^\Lambda(t)$ modulated by the Markov chain given by

$$\lambda^\Lambda(t) := \langle \lambda(t), \Lambda(t) \rangle = \sum_{j=1}^D \lambda_j \langle \Lambda(t), e_j \rangle,$$

where $\lambda_j$ is the $j$th entry of the vector $\lambda$ and represents the intensity rate of $M$ when the Markov chain is in the state space $e_j$.

Let $f_j(z)$, $j = 1, \dots, D$ be the probability density function of the chain size $z = Z(s) - Z(s-)$, when $\Lambda(t-) = e_j$. Then the Markov regime-switching compensator of the random measure $N^0(\cdot,\cdot)$ under $\mathbb{P}$, is given by

$$\nu^0_\Lambda(ds,dz) := \sum_{j=1}^D \langle \Lambda(s-), e_j \rangle \lambda_j(s) f_j(dz) ds.$$

Therefore, a compensated version of the random measure is given by

$$\tilde{N}^0_\Lambda(ds,dz) = N^0(ds,dz) - \nu^0_\Lambda(ds,dz).$$

We suppose that $\tilde{N}^0_\Lambda$ is independent of $W$ and $\tilde{N}_\Lambda$.

Let $p(t)$ be the premium rate at time $t$. We suppose that the premium rate process $\{p(t),\, t \in [0,T]\}$ is $\{\mathcal{F}_t\}_{t \in [0,T]}$-progressively measurable and uniformly bounded process on $(\Omega, \mathcal{F}, \mathbb{P})$, taking values on $(0,\infty)$. Let $R := \{R(t),\, t \in [0,T]\}$ be the insurance risk process of the insurance company without investment. Then, $R(t)$ is given by

$$\begin{aligned} R(t) &:= r_0 + \int_0^t p(s)ds - Z(t) \\ &= r_0 + \int_0^t p(s)ds - \int_0^t \int_0^\infty z N^0(ds,dz). \end{aligned}$$

Let $\pi(t)$ be the amount of the money invested in the risky asset at time $t$. We denote the surplus process by $X(t)$, then we formulate the surplus process with delay, which is caused by the capital inflow/outflow function from the insurer's current wealth. We suppose that the capital inflow/outflow function is given by

$$\varphi(t, X(t), \bar{Y}(t), U(t)) = (\vartheta(t) + \xi)X(t) - \vartheta(t)\bar{Y}(t) - \xi U(t)\,,$$

where $\vartheta(t) \geq 0$ is uniformly bounded function of $t$, $\xi \geq 0$ is a constant and

$$Y(t) = \int_{t-\varrho}^{t} e^{\zeta(s-t)} X(s) dW_1(s)\,; \qquad \bar{Y}(t) = \frac{Y(t)}{\int_{t-\varrho}^{t} e^{\zeta(s-t)} ds}\,; \quad U(t) = X(t-\varrho)\,.$$

Here, $Y$, $\bar{Y}$, $U$ represent respectively the integrated, average and pointwise delayed information of the wealth process in the interval $[t-\varrho, t]$. $\zeta \geq 0$ is the average parameter and $\varrho \geq 0$ the delay parameter. $W_1$ is an independent Brownian motion.

The parameters $\vartheta$ and $\xi$ represent the weights proportional to the past performance of $X - \bar{Y}$ and $X - U$, respectively. A good performance ($\varphi > 0$), may bring to the insurer more wealth, so that he can pay part of the wealth to the policyholders. Otherwise, a bad performance ($\varphi < 0$) may force the insurer to use the reserve or look for further capital in the market to cover the losses in order to achieve the final performance.

**Remark.** According to the definition of our capital inflow/outflow function, we take a noisy memory into account, thus generalizing the inflow/outflow function considered in Shen and Zeng [89]. To the best of our knowledge, this kind of noisy delay has just been applied in a stochastic control problem recently by Dahl *et. al.* [22] using a maximum principle techniques with Malliavin derivatives. Unlike in Dahl *et. al.* [22], we suppose that the noisy delay is derived by an independent Brownian motion. We believe that this assumption is more realistic since the delay of the information may not be caused by the same source of randomness as the one driving the stock price.

Furthermore, when the delay is driven by the same noisy with the asset price, the filtering theory we apply in the next section, fails to turn the model into one with complete observations, as the dynamics of $Y(t)$ would still be dependent on some hidden parameters. Under derivative pricing, such kind of delays have been applied to consider some stochastic volatility models, but with the delay driven by independent Poisson process, see, e.g., Swishchuk [93].

Note that we can write the noisy memory information $Y$ in a differential form by

$$\begin{aligned} dY(t) &= -\zeta Y(t)dt + X(t)dW_1(t) - e^{-\zeta\varrho}X(t-\varrho)dW_1(t-\varrho) \qquad (6.2) \\ &= -\zeta Y(t)dt + X(t)(1 - e^{-\zeta\varrho}\chi_{[0,T-\varrho]})dW_1(t) \quad t \in [0,T]\,, \end{aligned}$$

where $\chi_A$ denotes the characteristic function defined in a set $A$.

Then, the surplus process of the insurer is given by the following stochastic delay differential equation (SDDE) with regime-switching

$$\begin{aligned} & dX(t) \qquad (6.3) \\ =\; & [p(t) + r(t)X(t) + \pi(t)(\alpha^\Lambda(t) - r(t)) - \varphi(t, X(t), \bar{Y}(t), U(t))]dt \\ & + \pi(t)\beta(t)dW(t) + \pi(t)\int_{\mathbb{R}}\Big(e^z - 1\Big)N(dt,dz) - \int_0^\infty zN^0(dt,dz) \\ =\; & \Big[p(t) + (r(t) - \vartheta(t) - \xi)X(t) + \pi(t)(\alpha^\Lambda(t) - r(t)) + \bar{\vartheta}(t)Y(t) \\ & -\xi U(t) + \sum_{j=1}^D \langle \Lambda(t-), e_j\rangle \Big(\pi(t)\int_{\mathbb{R}}\Big(e^z - 1\Big)\varepsilon_j(t)\nu_j(dz) \\ & - \int_0^\infty \lambda_j(t) z f_j(dz)\Big)\Big]dt + \pi(t)\beta(t)dW(t) \\ & + \pi(t)\int_{\mathbb{R}}\Big(e^z - 1\Big)\tilde{N}_\Lambda(dt,dz) - \int_0^\infty z\tilde{N}^0_\Lambda(dt,dz)\,, \quad t \in [0,T]\,, \\ X(t) =\; & x_0 > 0, \quad t \in [-\varrho, 0]\,. \end{aligned}$$

The portfolio process $\pi(t)$ is said to be admissible if it satisfies the following:

1. $\pi(t)$ is $\{\mathcal{F}_t\}_{t\in[0,T]}$-progressively measurable and $\int_0^T |\pi(t)|^2 dt < \infty$, $\mathbb{P}$-a.s.

2. The SDDE (6.3) admits a unique strong solution;

3.
$$\sum_{j=1}^{D}\Big\{\int_0^T |p(t) + (r_j(t) - \vartheta(t) - \xi)X(t) + \pi(t)(\alpha_j(t) - r_j(t))$$
$$+\bar{\vartheta}(t)Y(t) - \xi U(t)|dt + \int_0^T \Big[\int_{\mathbb{R}} (\pi(t))^2(t)\Big(e^z - 1\Big)^2 \varepsilon_j(t)\nu_j(dz)$$
$$+ \int_0^\infty z^2 \lambda_j(t) f_j(dz) + \pi^2(t)\beta^2(t)\Big] dt\Big\} < \infty\,.$$

We denote the space of admissible investment strategy by $\mathcal{A}$.

We end this section, clarifying the information structure of our main problem. We define $\mathbb{F} := \{\mathcal{F}_t \mid t \in [0,T]\}$ be the $\mathbb{P}$-augmentation of the natural filtration generated by the risky asset $S(t)$ and the insurance risk process $R(t)$. This denotes the observable filtration in the market model. Let $\mathcal{G}_t := \mathcal{F}_t^\Lambda \vee \mathcal{F}_t$. $\mathbb{G} := \{\mathcal{G}_t \mid t \in [0,T]\}$ represents the full information structure of the model, where $\mathcal{F}^\Lambda$ is the filtration generated by the market chain $\Lambda$.

## 6.3 Reduction by the filtering theory

As we are working with an unobservable Markov regime-switching model, one needs to reduce the model into one with complete observations. We adopt the filtering theory for this reduction. This is a classical approach and it has been widely applied in stochastic control problems. See, for example, Bäuerle and Rieder [7], Elliott *et. al.* [32], Elliott and Siu [35], Siu [90], and references therein. We proceed as in Siu [92].

Consider the following $\{\mathcal{G}_t\}_{t\in[0,T]}$-adapted process $\widehat{W} := \{\widehat{W}(t),\, t \in [0,T]\}$ defined by

$$\widehat{W}(t) := W(t) + \int_0^t \frac{\alpha^\Lambda(s) - \hat{\alpha}^\Lambda(s)}{\beta(s)} ds\,, \quad t \in [0,T]\,,$$

where $\hat{\alpha}$ is the optional projection[1] of $\alpha$ under $\mathbb{P}$, with respect to the filtration $\{\mathcal{G}_t\}$, i.e., $\hat{\alpha}^{\Lambda}(t) = \mathbb{E}[\alpha^{\Lambda}(t) \mid \mathcal{G}_t]$, $\mathbb{P}$-a.s. Then it was shown that $\widehat{W}$ is a Brownian motion. See e.g., Elliott and Siu [35] or Kallianpur [49], Lemma 11.3.1.

Let $\hat{\Lambda}$ be the optional projection of the Markov chain $\Lambda$. For the jump part of the risk share $N$ and the insurance risk $N^0$, we consider the following:

$$\hat{\nu}(dt, dz) := \sum_{j=1}^{D} \langle \hat{\Lambda}(t-), e_j \rangle \varepsilon_j(t) \nu_j(dz) dt \quad \text{and}$$

$$\hat{\nu}^0(dt, dz) := \sum_{j=1}^{D} \langle \hat{\Lambda}(t-), e_j \rangle \lambda_j(t) \nu_j^0(dz) dt \, .$$

Define the compensated random measures $\widehat{N}(dt, dz)$ and $\widehat{N}^0(dt, dz)$ by

$$\begin{aligned} \widehat{N}(dt, dz) &:= N(dt, dz) - \hat{\nu}(dt, dz) \\ \widehat{N}^0(dt, dz) &:= N^0(dt, dz) - \hat{\nu}^0(dt, dz) \, . \end{aligned}$$

Then, it can be shown that the following processes are martingales with respect to the filtration $\mathcal{G}$. (See Elliott [31]):

$$\begin{aligned} \widehat{M} &:= \int_0^t \int_{\mathbb{R}} \Big( e^z - 1 \Big) \widehat{N}(dt, dz) \\ \widehat{M}^0 &:= \int_0^t \int_0^{\infty} z \widehat{N}^0(dt, dz) \, . \end{aligned}$$

Therefore, the surplus process $X(t)$ for any $t \in [0, T]$, can be written, under $\mathbb{P}$, as:

$$\begin{aligned} dX(t) &= \Big[ p(t) + (r(t) - \vartheta(t) - \xi) X(t) + \pi(t)(\hat{\alpha}^{\Lambda}(t) - r(t)) + \bar{\vartheta}(t) Y(t) - \xi U(t) \\ &\quad + \sum_{j=1}^{D} \langle \hat{\Lambda}(t-), e_j \rangle \Big( \pi(t) \int_{\mathbb{R}} \Big( e^z - 1 \Big) \varepsilon_j(t) \nu_j(dz) - \int_0^{\infty} \lambda_j(t) z f_j(dz) \Big) \Big] dt \\ &\quad + \pi(t) \beta(t) d\widehat{W}(t) + \pi(t) \int_{\mathbb{R}} \Big( e^z - 1 \Big) \widehat{N}_{\Lambda}(dt, dz) - \int_0^{\infty} z \widehat{N}_{\Lambda}^0(dt, dz) \, , \\ X(t) &= x_0 > 0, \quad t \in [-\varrho, 0] \, . \end{aligned} \tag{6.4}$$

[1]Let $X$ be a measurable bounded process. An optional projection is defined as a process $Y$ such that $\mathbb{E}[X(\mathcal{T}) \mathbf{1}_{\mathcal{T} < \infty} \mid \mathcal{F}_{\mathcal{T}}] = Y(\mathcal{T}) \mathbf{1}_{\mathcal{T} < \infty}$ a.s., for every stopping time $\mathcal{T}$. See Cohen and Elliott [16], Chapter 7 or Nikeghbali [74], Chapter 4.

Note that the dynamics (6.2) remains, since it is driven by an independent Brownian motion.

We then use the reference probability approach to derive a filtered estimate $\hat{\Lambda}$ of the Markov chain $\Lambda$ following the discussions in Siu [92], Section 8.3.2.

Let $\varphi(t) \in \mathbb{R}^D$, such that $\varphi_j(t) = \alpha_j(t) - \frac{1}{2}\beta^2(t)$, $j = 1, 2, \ldots, D$. Define, for any $t \in [0, T]$, the following functions

$$\begin{aligned}
\Psi_1(t) &:= \int_0^t \langle \varphi(s), \Lambda(s) \rangle ds + \int_0^t \beta(s) dW(s)\,; \\
\Psi_2(t) &:= \int_0^t \int_{\mathbb{R}} \Big( e^z - 1 \Big) N(ds, dz)\,; \\
\Psi_3(t) &:= \int_0^t \int_0^\infty z N^0(ds, dz)\,.
\end{aligned}$$

Write $\mathbb{P}^*$, for a probability measure on $(\Omega, \mathcal{G})$, on which the observation process does not depend on the Markov chain $\Lambda$. Define, for each $j = 1, 2, \ldots, D$,

$$F_j(t, z) := \frac{\lambda_j(t) f_j(dz)}{f(dz)} \quad \text{and} \quad \mathcal{E}_{\mathrm{j}}(\mathrm{t}, \mathrm{z}) := \frac{\varepsilon_{\mathrm{j}}(\mathrm{t}) \nu_{\mathrm{j}}(\mathrm{dz})}{\nu(\mathrm{dz})}\,.$$

Consider the following $\mathcal{G}_t$-adapted processes $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ defined by putting

$$\begin{aligned}
\Gamma_1(t) &:= \exp\Big( \int_0^t \beta^{-2}(s) \langle \varphi(s), \Lambda(s) \rangle d\Psi_1(s) - \frac{1}{2} \int_0^t \beta^{-4}(s) \langle \varphi(s), \Lambda(s) \rangle^2 ds \Big)\,; \\
\Gamma_2(t) &:= \exp\Big[ - \int_0^t \sum_{j=1}^D \langle \Lambda(s-), e_j \rangle \Big( \int_{\mathbb{R}} (\mathcal{E}_j(s, z) - 1) \nu(dz) \Big) ds \\
&\quad + \int_0^t \int_{\mathbb{R}} \Big( \sum_{j=1}^D \langle \Lambda(s-), e_j \rangle \ln(\mathcal{E}_j(s, z)) \Big) N(ds, dz) \Big]\,; \\
\Gamma_3(t) &:= \exp\Big[ - \int_0^t \sum_{j=1}^D \langle \Lambda(s-), e_j \rangle \Big( \int_0^\infty (F_j(s, z) - 1) f(dz) \Big) ds \\
&\quad + \int_0^t \int_0^\infty \Big( \sum_{j=1}^D \langle \Lambda(s-), e_j \rangle \ln(F_j(s, z)) \Big) N^0(ds, dz) \Big]\,.
\end{aligned}$$

Consider the $\{\mathcal{G}_t\}_{t \in [0,T]}$-adapted process $\Gamma := \{\Gamma(t),\, t \in [0, T]\}$ defined by

$$\Gamma(t) := \Gamma_1(t) \cdot \Gamma_2(t) \cdot \Gamma_3(t).$$

Note that the process $\Gamma$ is a local martingale and $\mathbb{E}[\Gamma(T)] = 1$. Under some strong assumptions (predictability and boundedness), it can be shown that $\Gamma$ is a true martingale. See, for instance, Proposition 2.5.1 in Delong [26] or Elliott *et. al.* [36], Section 3.

The main goal of the filtering process is to evaluate the $\{\mathcal{G}_t\}_{t\in[0,T]}$-optional projection of the Markov chain $\Lambda$ under $\mathbb{P}$. To that end, let, for each $t \in [0, T]$,

$$\mathbf{q}(t) := \mathbb{E}^*[\Gamma(t)\Lambda(t) \mid \mathcal{G}_t]\,,$$

where $\mathbb{E}^*$ is an expectation under the reference probability measure $\mathbb{P}^*$. The process $\mathbf{q}(t)$ is called an unnormalized filter of $\Lambda(t)$.

Define, for each $j = 1, 2, \ldots, D$ the scalar valued process $\gamma_j := \{\gamma_j(t),\, t \in [0, T]\}$ by

$$\begin{aligned}
&\gamma_j(t)\\
:= \;& \exp\Big(\int_0^t \varphi_j(s)\beta^{-2}(s)d\Psi_1(s) - \frac{1}{2}\int_0^t \varphi_j^2(s)\beta^{-4}(s)ds + \int_0^t (1-\varepsilon_j(s))ds\\
&+ \int_0^t (1-f_j(s))ds + \int_0^t \ln(\mathcal{E}_j(s))dN(s) + \int_0^t \ln(F_j(s))dN^0(s)\Big)\,.
\end{aligned}$$

Consider a diagonal matrix $\mathbf{L}(t) := \mathbf{diag}(\gamma_1(\mathrm{t}), \gamma_2(\mathrm{t}), \ldots, \gamma_{\mathrm{D}}(\mathrm{t}))$, for each $t \in [0, T]$. Define the transformed unnormalized filter $\{\bar{\mathbf{q}}(t),\, t \in [0, T]\}$ by

$$\bar{\mathbf{q}}(t) := \mathbf{L}^{-1}(t)\mathbf{q}(t)\,.$$

Note that the existence of the inverse $\mathbf{L}^{-1}(t)$ is guaranteed by the definition of $\mathbf{L}(t)$ and the positivity of $\gamma_j(t)$, $j = 1, 2, \ldots, D$.

Then, it has been shown (see Elliott and Siu [35], Theorem 4.2.), that the transformed unnormalized filter $\bar{\mathbf{q}}$ satisfies the following linear order differential equation

$$\frac{d\bar{\mathbf{q}}(t)}{dt} := \mathbf{L}^{-1}(t)A(t)\mathbf{L}(t)\bar{\mathbf{q}}(t)\,, \qquad \bar{\mathbf{q}}(0) = \mathbf{q}(0) = \mathbb{E}[\Lambda(0)]\,.$$

Hence, by a version of the Bayes rule, the optimal estimate $\hat{\Lambda}(t)$ of the Markov chain $\Lambda(t)$ is given by

$$\hat{\Lambda} := \mathbb{E}[\Lambda(t) \mid \mathcal{G}_t] = \frac{\mathbb{E}^*[\Gamma(t)\Lambda(t) \mid \mathcal{G}_t]}{\mathbb{E}^*[\Gamma(t) \mid \mathcal{G}_t]} = \frac{\mathbf{q}(t)}{\langle \mathbf{q}(t), \mathbf{1}\rangle}\,.$$

## 6.4 Risk-based optimal investment problem

In this section, we introduce the optimal investment problem of an insurer with regime-switching and delay. We consider a problem where the objective is to minimize the risk described by the convex risk measure, with the insurer not only concerned with the terminal wealth, but also with the integrated noisy memory surplus over the period $[T-\varrho, T]$. This **problem** is then described as follows: Find the investment strategy $\pi(t) \in \mathcal{A}$ which minimizes the risks of the terminal surplus and the integrated surplus, i.e., $X(T) + \kappa Y(T)$, where $\kappa \geq 0$ denotes the weight between $X(T)$ and $Y(T)$. This allow us to incorporate the terminal wealth as well as the delayed wealth at the terminal time $T$ in the performance functional.

Since we are dealing with a measure of risk, we will use the concept of convex risk measures introduced in Föllmer and Schied [38] and Frittelli and Rosazza [40], which is the generalization of the concept of coherent risk measures proposed by Artzner *et. al.* [3].

**Definition 6.4.1.** *Let $\mathcal{K}$ be a space of all lower bounded $\{\mathcal{G}_t\}_{t\in[0,T]}$-measurable random variables. A convex risk measure on $\mathcal{K}$ is a map $\rho : \mathcal{K} \to \mathbb{R}$ such that:*

1. *(translation) If $\epsilon \in \mathbb{R}$ and $X \in \mathcal{K}$, then $\rho(X+\epsilon) = \rho(X) - \epsilon$;*
2. *(monotonicity) For any $X_1, X_2 \in \mathcal{K}$, if $X_1(\omega) \leq X_2(\omega)$; $\omega \in \Omega$, then $\rho(X_1) \geq \rho(X_2)$;*
3. *(convexity) For any $X_1, X_2 \in \mathcal{K}$ and $\varsigma \in (0,1)$,*
$$\rho(\varsigma X_1 + (1-\varsigma)X_2) \leq \varsigma\rho(X_1) + (1-\varsigma)\rho(X_2)\,.$$

Following the general representation of the convex risk measures (see e.g., Theorem 3, Frittelli and Rasozza [40]), also applied by Mataramvura and Øksendal [62], Elliott and Siu [34], Meng and Siu [64], among others, we assume that the risk measure $\rho$ under consideration is as follows:

$$\rho(X) = \sup_{Q\in\mathcal{M}_a} \{\mathbb{E}^{\mathbb{Q}}[-X] - \eta(Q)\},$$

where $\mathbb{E}^{\mathbb{Q}}$ is the expectation under $\mathbb{Q}$, for the family $\mathcal{M}_a$ of probability measures and for some penalty function $\eta : \mathcal{M}_a \to \mathbb{R}$.

In order to specify the penalty function, we first describe a family $\mathcal{M}_a$ of all measures $Q$ of Girsanov type. We consider a robust modeling setup, given by a probability measure $\mathbb{Q} := Q^{\theta_0,\theta_1,\theta_2}$, with the Radon-Nikodym derivative given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = G^{\theta_0,\theta_1,\theta_2}(t)\,, \quad 0 \le t \le T\,.$$

The Radon-Nikodym $G^{\theta_0,\theta_1,\theta_2}(t)\,, \quad t \in [0, T+\varrho]$, is given by

$$\begin{aligned} dG^{\theta_0,\theta_1,\theta_2}(t) &= G^{\theta_0,\theta_1,\theta_2}(t^-)\Big[\theta_0(t)d\widehat{W}(t) + \theta_1(t)dW_1(t) \\ &\quad + \int_0^\infty \theta_0(t)\widehat{N}^0_\Lambda(dt,dz) + \int_{\mathbb{R}} \theta_2(t,z)\widehat{N}_\Lambda(dt,dz)\Big]\,, \qquad (6.5) \\ G^{\theta_0,\theta_1,\theta_2}(0) &= 1, \\ G^{\theta_0,\theta_1,\theta_2}(t) &= 0\,, \quad t \in [-\varrho, 0)\,. \end{aligned}$$

The set $\Theta := \{\theta_0, \theta_1, \theta_2\}$ is considered as a set of scenario control. We say that $\Theta$ is admissible if $\theta_2(t,z) > -1$ and

$$\mathbb{E}\Big[\int_0^T \Big\{\theta_0^2(t) + \theta_1^2(t) + \int_{\mathbb{R}} \theta_2^2(t,z)\nu_\Lambda(dz)\Big\}dt\Big] < \infty\,.$$

Then, the family $\mathcal{M}_a$ of probability measures is given by

$$\mathcal{M}_a := \mathcal{M}(\Theta) = \{\mathbb{Q}^{\theta_0,\theta_1,\theta_2} \,:\, (\theta_0,\theta_1,\theta_2) \in \Theta\}\,.$$

Let us now specify the penalty function $\eta$. Suppose that for each $(\pi, \theta_0, \theta_1, \theta_2) \in \mathcal{A}\times\Theta$ and $t \in [0,T]$, $\pi(t) \in \mathbf{U}_1$ and $\theta(t) = (\theta_0(t), \theta_1(t), \theta_2(t,\cdot)) \in \mathbf{U_2}$, where $\mathbf{U}_1$ and $\mathbf{U}_2$ are compact metric spaces in $\mathbb{R}$ and $\mathbb{R}^3$.

Let $\ell : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbf{U}_1 \times \mathbf{U}_2 \to \mathbb{R}$ and $h : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be two bounded measurable convex functions in $\theta(t) \in \mathbf{U}_2$ and $(X(T), Y(T)) \in \mathbb{R}\times\mathbb{R}$, respectively. Then, for each $(\pi,\theta) \in \mathcal{A}\times\Theta$,

$$\mathbb{E}\Big[\int_0^T |\ell(t, X(t), Y(t), Z(t), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t,\cdot))|dt + |h(X(T), Y(T))|\Big] < \infty\,.$$

As in Mataramvura and Øksendal [62], we consider, for each $(\pi, \theta) \in \mathcal{A} \times \Theta$, a penalty function $\eta$ of the form

$$\begin{aligned}
& \eta(\pi, \theta_0, \theta_1, \theta_2) \\
:= \; & \mathbb{E}\Big[\int_0^T \ell(t, X(t), Y(t), Z(t), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t, \cdot))dt + h(X(T), Y(T))\Big] .
\end{aligned}$$

Then, we define a convex risk measure for the terminal wealth and the integrated wealth of an insurer, i.e., $X(T) + \kappa Y(T)$, for $\kappa \geq 0$, given the information $\{\mathcal{G}_t\}_{t \in [0,T]}$ associated with the family of probability measures $\mathcal{M}_a$ and the penalty function $\eta$, as follows:

$$\rho(X(T), Y(T)) := \sup_{(\theta_0, \theta_1, \theta_2) \in \Theta} \Big\{ \mathbb{E}^{\mathbb{Q}}[-(X^\pi(T) + \kappa Y^\pi(T))] - \eta(\pi, \theta_0, \theta_1, \theta_2) \Big\} .$$

As in Elliott and Siu [34], the main objective of the insurer is to select the optimal investment process $\pi(t) \in \mathcal{A}$ so as to minimizes the risks described by $\rho(X(T), Y(T))$. That is, the optimal problem of an insurer is:

$$\mathcal{J}(x) := \inf_{\pi \in \mathcal{A}} \Big\{ \sup_{(\theta_0, \theta_1, \theta_2) \in \Theta} \Big\{ \mathbb{E}^{\mathbb{Q}}[-(X^\pi(T) + \kappa Y^\pi(T))] - \eta(\pi, \theta_0, \theta_1, \theta_2) \Big\} \Big\} . \quad (6.6)$$

Note that $\mathbb{E}^{\mathbb{Q}}[-(X^\pi(T) + \kappa Y^\pi(T))] = \mathbb{E}[-(X^\pi(T) + \kappa Y^\pi(T)) G^{\theta_0, \theta_1}(T)]$ (See Cuoco [20] or Karatzas *et. al.* [51], Lemma 3.5.3 for more details). Then from the form of the penalty function,

$$\begin{aligned}
& \bar{\mathcal{J}}(x) \\
= \; & \inf_{\pi \in \mathcal{A}} \sup_{(\theta_0, \theta_1, \theta_2) \in \Theta} \mathbb{E}\Big[ -(X^\pi(T) + \kappa Y^\pi(T)) G^{\theta_0, \theta_1, \theta_2}(T) \\
& - \int_0^T \ell(t, X(t), Y(t), Z(t), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t, \cdot))dt - h(X(T), Y(T)) \Big] \\
= \; & \mathcal{J}(x), \quad \text{say.}
\end{aligned}$$

For each $(\pi, \theta) \in \mathcal{A} \times \Theta$, suppose that

$$\begin{aligned}
&\mathcal{V}^{\pi,\theta}(x) \\
:= \quad &\mathbb{E}\Big[-(X^{\pi}(T)+\kappa Y^{\pi}(T))G^{\theta_0,\theta_1,\theta_2}(T) \\
&-\int_0^T \ell(t,X(t),Y(t),Z(t),\pi(t),\theta_0(t),\theta_1(t),\theta_2(t,\cdot))dt - h(X(T),Y(T))\Big] .
\end{aligned}$$

Then,

$$\mathcal{J}(x) = \inf_{\pi\in\mathcal{A}} \sup_{(\theta_0,\theta_1,\theta_2)\in\Theta} \mathcal{V}^{\pi,\theta}(x) = \mathcal{V}^{\pi^*,\theta^*}(x)\,,$$

that is, the insurer selects an optimal investment strategy $\pi$ so as to minimize the maximal risks, whilst the market reacts by selecting a probability measure indexed by $((\theta_0,\theta_1,\theta_2))\in\Theta$ corresponding to the worst-case scenario, where the risk is maximized. To solve this game problem, one must select the optimal strategy $(\pi^*,\theta_0^*,\theta_1^*,\theta_2^*)$ from the insurer and the market, respectively, as well as the optimal value function $\mathcal{J}(x)$.

## 6.5 The BSDE approach to a game problem

In this section, we solve the risk-based optimal investment problem of an insurer using delayed BSDE with jumps. Delayed BSDEs may arise in insurance and finance, when one wants to find an investment strategy which should replicate a liability or meet a purpose depending on the past values of the portfolio. For instance, under participating contracts in life insurance endowment contracts, we have a so called *performance-linked payoff*, that is, the payoff from the policy is related to the performance of the portfolio held by the insurer. Thus, the current portfolio and the past values of the portfolio have an impact on the final value of the liability. For more discussions on this and more applications of delayed BSDEs see Delong [25].

Define the following delayed BSDE with jumps:

$$\begin{aligned}
d\mathcal{Y}(t) &= -\mathcal{W}(t,\pi(t),\theta(t))dt + K_1(t)d\widehat{W}(t) + K_2(t)dW_1(t) \\
&\quad + \int_{\mathbb{R}} \Upsilon_1(t,z)\widehat{N}_\Lambda(dt,dz) + \int_0^\infty \Upsilon_2(t,z)\widehat{N}_\Lambda^0(dt,dz); \\
\mathcal{Y}(T) &= h(X(T),Y(T))\,,
\end{aligned} \tag{6.7}$$

where

$$\begin{aligned}\mathcal{W}(t,\pi(t),\theta(t)) \quad := \quad & \mathcal{W}(t,\mathcal{Y}(t),\mathcal{Y}(t-\varrho),K_1(t),K_1(t-\varrho),K_2(t),K_2(t-\varrho), \\ & \Upsilon_1(t,\cdot),\Upsilon_1(t-\varrho,\cdot),\Upsilon_2(t,\cdot),\Upsilon_2(t-\varrho,\cdot),\pi(t),\theta(t))\,.\end{aligned}$$

We assume that the generator $\mathcal{W}:\Omega\times[0,T]\times\mathbb{S}^2(\mathbb{R})\times\mathbb{S}^2_{-\varrho}(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})\times\mathbb{H}^2_{-\varrho}(\mathbb{R})\times\mathbb{H}^2_{\nu}(\mathbb{R})\times\mathbb{H}^2_{-\varrho,\nu}(\mathbb{R})\mapsto\mathbb{R}$ satisfy the following Lipschtz continuous condition, i.e., there exists a constant $C>0$ and a probability measure $\eta$ on $([-\varrho,0],\mathcal{B}([-\varrho,0]))$ such that

$$\begin{aligned}\mathcal{W}(t,\pi(t),\theta(t))-\tilde{\mathcal{W}}(t,\pi(t),\theta(t)) \quad \leq \quad & C\Big(\int_{-\varrho}^{0}|y(t+\zeta)-\tilde{y}(t+\zeta)|^2\eta(d\zeta)+\int_{-\varrho}^{0}|k_1(t+\zeta)-\tilde{k}_1(t+\zeta)|^2\eta(d\zeta) \\ & +\int_{-\varrho}^{0}|k_2(t+\zeta)-\tilde{k}_2(t+\zeta)|^2\eta(d\zeta)+\int_0^T|y(t)-\tilde{y}(t)|^2dt \\ & +\int_{-\varrho}^{0}\int_{\mathbb{R}}|\upsilon_1(t+\zeta,z)-\tilde{\upsilon}_1(t+\zeta,z)|^2\nu(dz)\eta(d\zeta)+\int_0^T|k_2(t)-\tilde{k}_2(t)|^2dt \\ & +\int_{-\varrho}^{0}\int_{\mathbb{R}}|\upsilon_2(t+\zeta,z)-\tilde{\upsilon}_2(t+\zeta,z)|^2\nu(dz)\eta(d\zeta) \\ & +\int_0^T\int_{\mathbb{R}}|\upsilon_1(t,z)-\tilde{\upsilon}_1(t,z)|^2\nu(dz)dt+\int_0^T\int_{\mathbb{R}}|\upsilon_2(t,z)-\tilde{\upsilon}_2(t,z)|^2\nu(dz)dt\Big)\,.\end{aligned}$$

Then, if $h\in\mathbb{L}^2$ and the above Lipschitz condition is satisfied, one can prove the existence and uniqueness solution $(\mathcal{Y},K_1,K_2,\Upsilon_1,\Upsilon_2)\in\mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})\times\mathbb{H}^2_{\nu}(\mathbb{R})\times\mathbb{H}^2_{\nu}(\mathbb{R})$ of a delayed BSDE with jumps (6.7). See Delong and Imkeller [27] and Delong [26] for more details. In practice, $\mathcal{Y}$ denotes a replicating portfolio, $K_1,K_2,\Upsilon_1,\Upsilon_2$ represent the replicating strategy, $h(X(T),Y(T))$ is a terminal liability and $\mathcal{W}$ models the stream liability during the contract life-time.

The key result for solving our delayed stochastic differential game problem is based on the following theorem.

**Theorem 6.5.1.** *Suppose that there exists a strategy $(\hat{\pi}(t),\hat{\theta}(t))\in\mathbf{U}_1\times\mathbf{U}_2$ such that*

$$\begin{aligned}\mathcal{W}(t,y,k_1,k_2,\upsilon(\cdot),\hat{\pi}(t),\hat{\theta}(t)) \quad &= \quad \inf_{\pi\in\mathcal{A}}\sup_{(\theta_0,\theta_1,\theta_2)\in\Theta}\mathcal{W}(t,y,k_1,k_2,\upsilon(\cdot),\pi,\theta) \qquad (6.8)\\ &= \quad \sup_{(\theta_0,\theta_1,\theta_2)\in\Theta}\inf_{\pi\in\mathcal{A}}\mathcal{W}(t,y,k_1,k_2,\upsilon(\cdot),\pi,\theta)\,,\end{aligned}$$

*that is, $\mathcal{W}$ satisfy the Isaac's condition. Furthermore, suppose that there exists a unique solution $(\mathcal{Y}^{\pi,\theta}(t),K_1^{\pi,\theta}(t),K_2^{\pi,\theta}(t),\Upsilon_1^{\pi,\theta}(t,\cdot),\Upsilon_2^{\pi,\theta}(t,\cdot))\in\mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})\times\mathbb{H}^2_{\nu}(\mathbb{R})\times\mathbb{H}^2_{\nu}(\mathbb{R})$ of the BSDE (6.7), for all $(\pi,\theta)\in\mathcal{A}\times\Theta$.*

*Then, the value function $\mathcal{J}(x)$ is given by $\mathcal{Y}^{\hat{\pi},\hat{\theta}}(t)$. Moreover, the optimal strategy of the problem* (6.6) *is given by*

$$\begin{cases} \pi^*(t) = \hat{\pi}(t, Y(t), K_1(t), K_2(t), \Upsilon_1(t,\cdot), \Upsilon_2(t,\cdot))\,, \\ \theta^*(t) = \hat{\theta}(t, Y(t), K_1(t), K_2(t), \Upsilon_1(t,\cdot), \Upsilon_2(t,\cdot))\,. \end{cases} \tag{6.9}$$

*Proof.* The proof is based on the comparison principle for BSDEs with jumps as follows, (see Theorem 2.4.2). Define three generators $\phi_1$, $\phi_2$ and $\phi_3$ by

$$\begin{aligned} \phi_1(t,y,k_1,k_2,\upsilon(\cdot)) &= \mathcal{W}(t,y,k_1,k_2,\upsilon_1(\cdot),\upsilon_2(\cdot),\hat{\pi}(t),\theta(t)) \\ \phi_2(t,y,k_1,k_2,\upsilon(\cdot)) &= \mathcal{W}(t,y,k_1,k_2,\upsilon_1(\cdot),\upsilon_2(\cdot),\hat{\pi}(t),\hat{\theta}(t)) \\ \phi_3(t,y,k_1,k_2,\upsilon(\cdot)) &= \mathcal{W}(t,y,k_1,k_2,\upsilon_1(\cdot),\upsilon_2(\cdot),\pi(t),\hat{\theta}(t)) \end{aligned}$$

and the corresponding BSDEs

$$\begin{aligned} d\mathcal{Y}_1(t) &= -\phi_1(t,y,k_1,k_2,\upsilon(\cdot))dt + K_1(t)d\widehat{W}(t) + K_2(t)dW_1(t) \\ &\quad + \int_{\mathbb{R}} \Upsilon_1(t,z)\widehat{N}_\Lambda(dt,dz) + \int_0^\infty \Upsilon_2(t,z)\widehat{N}^0_\Lambda(dt,dz)\,, \\ \mathcal{Y}_1(T) &= h(X(T),Y(T))\,. \end{aligned}$$

$$\begin{aligned} d\mathcal{Y}_2(t) &= -\phi_2(t,y,k_1,k_2,\upsilon(\cdot))dt + K(t)d\widehat{W}(t) + K_2(t)dW_1(t) \\ &\quad + \int_{\mathbb{R}} \Upsilon_1(t,z)\widehat{N}_\Lambda(dt,dz) + \int_0^\infty \Upsilon_2(t,z)\widehat{N}^0_\Lambda(dt,dz) \\ \mathcal{Y}_2(T) &= h(X(T),Y(T))\,. \end{aligned}$$

and

$$\begin{aligned} d\mathcal{Y}_3(t) &= -\phi_3(t,y,k_1,k_2,\upsilon(\cdot))dt + K(t)d\widehat{W}(t) + K_2(t)dW_1(t) \\ &\quad + \int_{\mathbb{R}} \Upsilon_1(t,z)\widehat{N}_\Lambda(dt,dz) + \int_0^\infty \Upsilon_2(t,z)\widehat{N}^0_\Lambda(dt,dz)\,, \\ \mathcal{Y}_3(T) &= h(X(T),Y(T))\,. \end{aligned}$$

From (6.8), we have

$$\phi_1(t,y,k_1,k_2,\upsilon(\cdot)) \le \phi_2(t,y,k_1,k_2,\upsilon(\cdot)) \le \phi_3(t,y,k_1,k_2,\upsilon(\cdot)).$$

Then, by comparison principle, $\mathcal{Y}_1(t) \leq \mathcal{Y}_2(t) = \mathcal{J}(x) \leq \mathcal{Y}_3(t)$, for all $t \in [0,T]$. By uniqueness, we get $\mathcal{Y}_2(t) = \mathcal{V}^{\pi^*,\theta^*}$. Hence, the optimal strategy is given by (6.9). □

In order to solve our main problem, note that from the dynamics of the processes $X(t)$, $Y(t)$ and $G^{\theta_0,\theta_1,\theta_2}$ in (6.3), (6.2) and (6.5), respectively and applying the Itô's differentiation rule for delayed SDEs with jumps (See Baños *et. al.* [6], Theorem 3.6), we have:

$$
\begin{aligned}
& d[(X(t)+\kappa Y(t))G^{\theta_0,\theta_1,\theta_2}(t)] \\
= \; & G^{\theta_0,\theta_1,\theta_2}(t)\Big[p(t)+(r(t)-\vartheta(t)-\xi)X(t)+\pi(t)(\hat{\alpha}^{\Lambda}(t)-r(t)) \\
& +(\bar{\vartheta}(t)-\kappa\zeta)Y(t)-\xi U(t)+\pi(t)\beta(t)\theta_0(t)+\theta_1(t)\kappa X(t)(1-e^{-\zeta\varrho}\chi_{[0,T-\varrho]}) \\
& +\sum_{j=1}^{D}\langle\hat{\Lambda}(t-),e_j\rangle\Big(\pi(t)\int_{\mathbb{R}}\Big(e^z-1\Big)\theta_2(t,z)\varepsilon_j(t)\nu_j(dz) \\
& -\int_0^\infty \lambda_j(t)z(1+\theta_0(t))f_j(dz)\Big)\Big]dt+G^{\theta_0,\theta_1,\theta_2}(t)[(\pi(t)\beta(t)+X(t)\theta_0(t))d\widehat{W}(t) \\
& +(\theta_1(t)+X(t)(1-e^{-\zeta\varrho}\chi_{[0,T-\varrho]}))dW_1(t)] \\
& +G^{\theta_0,\theta_1,\theta_2}(t)\int_{\mathbb{R}}[(1+\theta_2(t,z))\pi(t)\Big(e^z-1\Big)+X(t)\theta_2(t,z)]\widehat{N}_{\Lambda}(dt,dz) \\
& -G^{\theta_0,\theta_1,\theta_2}(t)\int_0^\infty[(1+\theta_0(t))z-X(t)\theta_0(t)]\widehat{N}^0_{\Lambda}(dt,dz)\ ,.
\end{aligned}
$$

Thus, for each $(\pi,\theta_0,\theta_1,\theta_2)$,

$$
\begin{aligned}
& \mathcal{J}(x) \\
= \; & \mathbb{E}\Big\{-\int_0^T\Big[G^{\theta_0,\theta_1,\theta_2}(t)\Big[p(t)+(r(t)-\vartheta(t)-\xi)X(t)+\pi(t)(\hat{\alpha}^{\Lambda}(t)-r(t)) \\
& +(\bar{\vartheta}(t)-\kappa\zeta)Y(t)-\xi U(t)+\pi(t)\beta(t)\theta_0(t)+\theta_1(t)\kappa X(t)(1-e^{-\zeta\varrho}\chi_{[0,T-\varrho]}) \\
& +\sum_{j=1}^{D}\langle\hat{\Lambda}(t-),e_j\rangle\Big(\pi(t)\int_{\mathbb{R}}\Big(e^z-1\Big)\theta_2(t,z)\varepsilon_j(t)\nu_j(dz) \\
& -\int_0^\infty\lambda_j(t)z(1+\theta_0(t))f_j(dz)\Big)\Big] \\
& +\ell(t,X(t),Y(t),U(t),\pi(t),\theta_0(t),\theta_1(t),\theta_2(t,\cdot))\Big]dt-h(X(T),Y(T))\Big\}\,.
\end{aligned}
$$

We now define, for each $(t,X,Y,U,\pi,\theta_0,\theta_1,\theta_2)\in[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbf{U}_1\times\mathbf{U}_2$, a function

$$\begin{aligned}
& \tilde{\ell}(t, X(t), Y(t), U(t), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t,\cdot)) \\
= \; & G^{\theta_0,\theta_1,\theta_2}(t)\Big[p(t) + (r(t) - \vartheta(t) - \xi)X(t) + \pi(t)(\hat{\alpha}^{\Lambda}(t) - r(t)) \\
& +(\bar{\vartheta}(t) - \kappa\zeta)Y(t) - \xi U(t) + \pi(t)\beta(t)\theta_0(t) + \theta_1(t)\kappa X(t)(1 - e^{-\zeta\varrho}\chi_{[0,T-\varrho]}) \\
& + \sum_{j=1}^{D}\langle\hat{\Lambda}(t-), e_j\rangle\Big(\pi(t)\int_{\mathbb{R}}\Big(e^z - 1\Big)\theta_2(t,z)\varepsilon_j(t)\nu_j(dz) \\
& - \int_0^{\infty}\lambda_j(t)z(1+\theta_0(t))f_j(dz)\Big)\Big] \\
& +\ell(t, X(t), Y(t), U(t), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t,\cdot))\,.
\end{aligned}$$

Then,

$$\mathcal{J}(x) \;=\; -x_0 + \mathbb{E}\Big[-\int_0^T \tilde{\ell}(t, X(t), Y(t), U(t), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t,\cdot))dt - h(X(T), Y(T))\Big]\,.$$

Define, for each $(\pi, \theta) \in \mathcal{A} \times \Theta$, a functional

$$\tilde{\mathcal{J}}(x) = \mathbb{E}\Big[-\int_0^T \tilde{\ell}(t, X(t), Y(t), U(t), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t,\cdot))dt - h(X(T), Y(T))\Big]\,.$$

Then, the stochastic differential delay game problem discussed in the previous section is equivalent to the following problem:

$$\tilde{\mathcal{V}}(t,x) = \inf_{\pi\in\mathcal{A}} \sup_{(\theta_0,\theta_1,\theta_2)\in\Theta} \tilde{\mathcal{J}}(x)\,.$$

We now define the Hamiltonian of the aforementioned game problem $\mathcal{H}$ : $[0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbf{U}_1 \times \mathbf{U}_2 \to \mathbb{R}$ as follows:

$$\begin{aligned}
& \mathcal{H}(t, X(t), Y(t), U(t), K_1(t), K_2(t), \Upsilon_1(t,\cdot), \Upsilon_2(t,\cdot), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t,\cdot)) \\
:= \; & -\tilde{\ell}(t, X(t), Y(t), U(t), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t,\cdot))\,.
\end{aligned}$$

In order for the Hamiltonian $\mathcal{H}$ to satisfy the Issac's condition, we require that $\mathcal{H}$ is convex in $\pi$ and concave in $(\theta_0, \theta_1, \theta_2)$. Moreover, for the existence and uniqueness solution of the corresponding delayed BSDE with jumps, the Hamiltonian should satisfy the Lipschitz condition. From the boundedness of the associate parameters, we prove that $\mathcal{H}$ is indeed Lipschitz.

**Lemma 6.5.2.** *The Hamiltonian $\mathcal{H}$ is Lipschitz continuous in $X$, $Y$ and $U$.*

*Proof.* Since $(\pi, (\theta_0, \theta_1, \theta_2)) \in \mathbf{U}_1 \times \mathbf{U}_2$ and $\ell$ is bounded, $\tilde{\ell}$ is bounded. Then $\mathcal{H}$ is uniformly bounded with respect to $(t, X(t), Y(t), U(t), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t, \cdot))$. To prove the Lipschitz condition, we suppose that $\mathcal{H}$ is not Lipschitz continuous in $(K_1(t), K_2(t), \Upsilon_1(t, \cdot), \Upsilon_2(t, \cdot))$, uniformly in $(t, X(t), Y(t), U(t))$. Then, there exist two points $(K_1(t), K_2(t), \Upsilon_1(t, \cdot), \Upsilon_2(t, \cdot))$, $(\tilde{K}_1(t), \tilde{K}_2(t), \tilde{\Upsilon}_1(t, \cdot), \tilde{\Upsilon}_2(t, \cdot))$ such that

$$
\begin{aligned}
&|\mathcal{H}(t, X(t), Y(t), U(t), K_1(t), K_2(t), \Upsilon_1(t, \cdot), \Upsilon_2(t, \cdot), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t, \cdot)) \\
&-\mathcal{H}(t, X(t), Y(t), U(t), \tilde{K}_1(t), \tilde{K}_2(t), \tilde{\Upsilon}_1(t, \cdot), \tilde{\Upsilon}_2(t, \cdot), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t, \cdot))|
\end{aligned}
$$

is unbounded. However, since $\mathcal{H}$ is uniformly bounded with respect to $(t, X(t), Y(t), U(t), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t, \cdot))$, we have

$$
\begin{aligned}
&|\mathcal{H}(t, X(t), Y(t), U(t), K_1(t), K_2(t), \Upsilon_1(t, \cdot), \Upsilon_2(t, \cdot), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t, \cdot)) \\
&-\mathcal{H}(t, X(t), Y(t), U(t), \tilde{K}_1(t), \tilde{K}_2(t), \tilde{\Upsilon}_1(t, \cdot), \tilde{\Upsilon}_2(t, \cdot), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t, \cdot))| \quad \leq \\
&|\mathcal{H}(t, X(t), Y(t), U(t), K_1(t), K_2(t), \Upsilon_1(t, \cdot), \Upsilon_2(t, \cdot), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t, \cdot))| \\
&+|\mathcal{H}(t, X(t), Y(t), U(t), \tilde{K}_1(t), \tilde{K}_2(t), \tilde{\Upsilon}_1(t, \cdot), \tilde{\Upsilon}_2(t, \cdot), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t, \cdot))| \quad < \quad \infty \,,
\end{aligned}
$$

which contradicts the assumption. Then $\mathcal{H}$ is Lipschitz continuous. □

Then, following Theorem 6.5.1, we establish the relationship between the value function of the game problem and the solution of a delayed BSDE with jumps. Thus, the value function $\tilde{\mathcal{J}}(t, x)$ is given by the following noisy memory BSDE:

$$
\begin{aligned}
d\tilde{\mathcal{J}}(t) \quad = \quad & -\mathcal{H}(t, X(t), Y(t), U(t), K_1(t), K_2(t), \Upsilon_1(t, \cdot), \Upsilon_2(t, \cdot), \pi^*(t), \theta_0^*(t), \theta_1^*(t), \theta_2^*(t, \cdot))dt \\
& +K_1(t)d\widehat{W}(t) + K_2(t)dW_1(t) + \int_{\mathbb{R}} \Upsilon_1(t, z)\widehat{N}_\Lambda(dt, dz) + \int_{\mathbb{R}} \Upsilon_2(t, z)\widehat{N}^0_\Lambda(dt, dz) \,,
\end{aligned}
$$

with the terminal condition $\tilde{\mathcal{J}}(T) = h(X(T), Y(T))$.

In fact, the existence and uniqueness of the solution to the above delayed BSDE with jumps is guaranteed from the Lipschitz condition proved in Lemma 6.5.2. Then, The solution of the delayed BSDE is given by

$$
\begin{aligned}
\tilde{\mathcal{J}}(t) \quad &= \quad \mathbb{E}\Big[h(X(T), Y(T)) \\
&\qquad - \int_t^T \tilde{\ell}(s, X(s), Y(s), U(s), \pi^*(s), \theta_0^*(s), \theta_1^*(s), \theta_2^*(s, \cdot))ds \mid \mathcal{G}_t\Big] \\
&= \quad \mathcal{V}(\pi^*, \theta_1^*, \theta_2^*, \theta_3^*) \,,
\end{aligned}
$$

which is the optimal value function from Theorem 6.5.1.

## 6.6 A quadratic penalty function case

In this section, we consider a convex risk measure with quadratic penalty. We derive explicit solutions when $\ell$ is quadratic in $\theta_0$, $\theta_1$, $\theta_2$ and identical zero in $h$. The penalty function under consideration here, may be related to the entropic penalty function considered, for instance, by Delbaen *et. al.* [24]. It has also been adopted by Elliott and Siu [34], Siu [90] and Meng and Siu [64]. We obtain the explicit optimal investment strategy and the optimal risks for this case of a risk-based optimization problem with jumps, regime switching and noisy delay. Finally, we consider some particular cases and we see using some numerical parameters, how an insurer can allocate his portfolio.

Suppose that the penalty function is given by

$$\begin{aligned}
& \ell(t, X(t), Y(t), Z(t), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t, \cdot)) \\
:= \; & \frac{1}{2(1-\delta)}\Big(\theta_0^2(t) + \theta_1^2(t) + \int_{\mathbb{R}} \theta_2^2(t,z)\nu_{\hat{\Lambda}}(dz)\Big) G^{\theta_0,\theta_1,\theta_2}(t)\,,
\end{aligned}$$

where $1-\delta$ is a measure of an insurer's relative risk aversion and $\delta < 1$. Then, the Hamiltonian $\mathcal{H}$ becomes:

$$\begin{aligned}
& \mathcal{H}(t, X(t), Y(t), U(t), K_1(t), K_2(t), \Upsilon_1(t,\cdot), \Upsilon_2(t,\cdot), \pi(t), \theta_0(t), \theta_1(t), \theta_2(t,\cdot)) \\
= \; & -G^{\theta_0,\theta_1,\theta_2}(t)\Big[p(t) + (r(t) - \vartheta(t) - \xi)X(t) + \pi(t)(\hat{\alpha}^{\Lambda}(t) - r(t)) \\
& +(\bar{\vartheta}(t) - \kappa\zeta)Y(t) - \xi U(t) + \pi(t)\beta(t)\theta_0(t) + \theta_1(t)\kappa X(t)(1 - e^{-\zeta\varrho}\chi_{[0,T-\varrho]}) \\
& + \sum_{j=1}^{D} \langle \hat{\Lambda}(t-), e_j\rangle \Big(\pi(t)\int_{\mathbb{R}} \Big(e^z - 1\Big)\theta_2(t,z)\varepsilon_j(t)\nu_j(dz) \\
& - \int_0^{\infty} \lambda_j(t) z(1+\theta_0(t)) f_j(dz)\Big)\Big] \\
& -\frac{1}{2(1-\delta)}\Big(\theta_0^2(t) + \theta_1^2(t) + \int_{\mathbb{R}} \theta_2^2(t,z)\nu_{\hat{\Lambda}}(dz)\Big) G^{\theta_0,\theta_1,\theta_2}(t)\,.
\end{aligned}$$

Applying the first order condition for maximizing the Hamiltonian with respect to $\theta_0$, $\theta_1$ and $\theta_2$, and minimizing with respect to $\pi$, we obtain the following

$$
\begin{aligned}
\pi^*(t) &= \frac{\hat{\alpha}^{\Lambda}(t) - r(t) + (1-\delta)\beta(t)\Big(\sum_{j=1}^{D}\langle\hat{\Lambda}(t-), e_j\rangle \int_0^\infty z\lambda_j(t) f_j(dz)\Big)}{(1-\delta)\Big(\beta^2(t) + \sum_{j=1}^{D}\langle\hat{\Lambda}(t-), e_j\rangle \int_{\mathbb{R}}\Big(e^z - 1\Big)^2 \varepsilon_j(t)\nu_j(dz)\Big)}, \\
\theta_0^*(t) &= (1-\delta)\Big[\sum_{j=1}^{D}\langle\hat{\Lambda}(t-), e_j\rangle \int_0^\infty z\lambda_j(t) f_j(dz) \\
&\quad - \frac{\hat{\alpha}^{\Lambda}(t) - r(t) + (1-\delta)\beta(t)\Big(\sum_{j=1}^{D}\langle\hat{\Lambda}(t-), e_j\rangle \int_0^\infty z\lambda_j(t) f_j(dz)\Big)}{(1-\delta)\Big(\beta^2(t) + \sum_{j=1}^{D}\langle\hat{\Lambda}(t-), e_j\rangle \int_{\mathbb{R}}\Big(e^z - 1\Big)^2 \varepsilon_j(t)\nu_j(dz)\Big)}\beta(t)\Big], \\
\theta_1^*(t) &= (\delta - 1)\kappa X(t)(1 - e^{-\zeta\varrho}\chi_{[0,T-\varrho]})
\end{aligned}
$$

and

$$
\theta_2^*(t,z) = (\delta-1)z\frac{\hat{\alpha}^{\Lambda}(t) - r(t) + (1-\delta)\beta(t)\Big(\sum_{j=1}^{D}\langle\hat{\Lambda}(t-), e_j\rangle \int_0^\infty z\lambda_j(t) f_j(dz)\Big)}{(1-\delta)\Big(\beta^2(t) + \sum_{j=1}^{D}\langle\hat{\Lambda}(t-), e_j\rangle \int_{\mathbb{R}}\Big(e^z - 1\Big)^2 \varepsilon_j(t)\nu_j(dz)\Big)}.
$$

Then, the value function of the game problem is given by the following BSDE:

$$
\begin{aligned}
& d\mathcal{J}(t) \\
= \;& G^{\theta_0^*,\theta_1^*,\theta_2^*}(t)\Big[p(t) + (r(t) - \vartheta(t) - \xi)X(t) + \pi^*(t)(\hat{\alpha}^{\Lambda}(t) - r(t)) \\
& + (\bar{\vartheta}(t) - \kappa\zeta)Y(t) - \xi U(t) + \pi^*(t)\beta(t)\theta_0^*(t) + \theta_1^*(t)\kappa X(t)(1 - e^{-\zeta\varrho}\chi_{[0,T-\varrho]}) \\
& + \sum_{j=1}^{D}\langle\hat{\Lambda}(t-), e_j\rangle\Big(\pi^*(t)\int_{\mathbb{R}}\Big(e^z - 1\Big)\theta_2^*(t,z)\varepsilon_j(t)\nu_j(dz) \\
& - \int_0^\infty \lambda_j(t) z(1 + \theta_0^*(t)) f_j(dz)\Big)\Big] \\
& + \frac{1}{2(1-\delta)}\Big((\theta_0^*)^2(t) + (\theta_1^*)^2(t) + \int_{\mathbb{R}}(\theta_2^*)^2(t,z)\nu_{\hat{\Lambda}}(dz)\Big)\Big]dt + K_1(t)d\widehat{W}(t) \\
& + K_2(t)dW_1(t) + \int_{\mathbb{R}}\Upsilon_1(t,z)\widehat{N}_{\Lambda}(dt,dz) + \int_{\mathbb{R}}\Upsilon_2(t,z)\widehat{N}_{\Lambda}^0(dt,dz).
\end{aligned}
$$

**Example 6.6.1.** Suppose that the the driving processes $\tilde{N}$ and $\tilde{N}^0$ are Poisson processes $N$ and $N^0$, with the jump intensities $\lambda$ and $\lambda^0$. Under noisy delay modeling, we consider the following cases:

Case 1. We suppose that there is no regime switching in the model, then the optimal investment strategy is given by

$$\pi^*(t) = \frac{\alpha(t) - r(t)}{(1-\delta)(\beta^2(t) + \lambda)} \,.$$

To be concrete, we assume that the interest rate $r = 4.5\%$, the appreciation rate $\alpha = 11\%$, the volatility $\beta = 20\%$, the insurer's relative risk aversion $\delta = 0.5$ and the jump intensity given by $\lambda = 0.5$. Then the optimal portfolio invested in the risky asset is given by $\pi^* = 0.24074$, i.e., 24.074% of the wealth should be invested in the risky share.

Case 2. We suppose existence of two state Markov chain $\mathcal{S} = \{e_1, e_2\}$, where the states $e_1$ and $e_2$ represent the expansion and recession of the economy respectively. By definition, $\langle \hat{\Lambda}(t), e_1 \rangle = \mathbb{P}(X(t) = e_1 \mid \mathcal{F}_t)$ and $\langle \hat{\Lambda}(t), e_2 \rangle = 1 - \mathbb{P}(X(t) = e_1 \mid \mathcal{F}_t)$. Let $\alpha_i, r_i, \lambda_i, \lambda_i^0$ be the associate parameters when the economy is in state $e_i$, $i = 1, 2$. Then the optimal portfolio is given by

$$\begin{aligned}\pi^*(t) \quad &= \quad \frac{[\alpha_1(t) - r_1(t) - (\alpha_2(t) - r_2(t)) + (1-\delta)\beta(t)(\lambda_1^0(t) - \lambda_2^0(t))]\mathbb{P}(X(t) = e_1 \mid \mathcal{F}_t)}{(1-\delta)[\beta^2(t) + \lambda_2(t) + (\lambda_1(t) - \lambda_2(t))\mathbb{P}(X(t) = e_1 \mid \mathcal{F}_t)]} \\ &\quad + \frac{\alpha_2(t) - r_2(t) + (1-\delta)\beta(t)\lambda_2^0}{(1-\delta)[\beta^2(t) + \lambda_2(t) + (\lambda_1(t) - \lambda_2(t))\mathbb{P}(X(t) = e_1 \mid \mathcal{F}_t)]} \,.\end{aligned}$$

In this case, we consider the following parameters: $\alpha_1 = 13\%$, $\alpha_2 = 9\%$, $r_1 = t\%$, $r_2 = 9\%$, $\beta = 20\%$, $\lambda_1^0 = \lambda_1 = 0.5$, $\lambda_2 = \lambda_2^0 = 0.7$, $\delta = 0.5$ and $\mathbb{P}(X = e_1) = 70\%$. Then $\pi^* = 0.28$, i.e., 28% of the wealth should be invested in the risky share.

# Chapter 7

# Conclusion and future research

In this thesis, we focused in the theoretical aspects of the stochastic optimal portfolio theory and its applications in a jump-diffusion portfolio optimization problems.

In the first part, we solved a stochastic volatility optimization problem with American capital constraints. We first solved the unconstrained problem via the martingale duality method, where explicit solutions were derived for the power utility case. Similar results can be obtained for the exponential and logarithmic utility functions. Then the constrained problem was solved from the unconstrained optimal solution by the application of the so-called option based portfolio insurance approach. The results in this chapter generalize the existing ones in the literature to the jump-diffusion case.

In the second part, a maximum principle method was applied to solve the stochastic volatility optimization problem. We proved the necessary and sufficient maximum principle theorems. These results allow us to generalize the maximum principle theorems to the stochastic volatility case. Then we applied the results to solve an optimal investment, consumption and life insurance problem, generalizing the results in Mousa *et. al.* [72].

In the third part, we considered the BSDE techniques to solve the stochastic optimization problem. We assumed the presence of inflation linked assets, namely, a consume price index and a zero coupon bond price. The inclusion of inflation linked products it is important since it helps the investors to manage the inflation risks that are in general not completely observable. We assumed independence of the noises driving the associated processes. Using the theory of quadratic-exponential BSDEs, we derived the optimal strategies for the exponential and power utility functions. In the literature, these results extend, for instance, the paper by Cheridito and Hu [14] by allowing the presence of inflation risks, life insurance and jumps. Note that if there are correlations between the driving processes, the optimization problem results in a highly non-linear BSDE with jumps which the existence and uniqueness of its solution has not yet been established. Therefore, the existence and uniqueness of such BSDEs with jumps represents an interesting research problem.

Finally, we turned our attention to a risk-based optimization problem of an insurer in a regime-switching model with noisy memory.Using a robust optimization modelling, we formulated the problem as a zero-sum stochastic differential delay game problem between the insurer and the market with a convex risk measure of the terminal surplus and delay. This type of risk measure allows that a diversification of investments does not increase the risks. To turn the model from partial observation to complete observation setup, we used the filtering theory techniques, then, by the BSDE approach, we solved the game problem. The model in this chapter combined and generalized several components:

- an asset market where prices follow a regime-switching jump-diffusions with unobservable states;
- capital inflows/outflows which are subject to delays of different forms;
- premium and claim processes which are close to standard actuarial settings.

Then, we considered an example to show the applicability of the model for the quadratic penalty case.

Overall, this thesis presents results on the linkage of the classical stochastic optimal portfolio problems with life insurance considerations. These results are generally based on the theory of BSDEs with jumps, which are in general hard to solve. Therefore, the need to apply Malliavin Calculus as to enhance solvability of some stochastic optimal control and risk allocation problems is of potential interest. It would be interesting to look as well at what happens in an maximization problem using this FBSDE approach when the investment affects prices as well as in a more general utility functions (law invariant)

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