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SUMMARY

Two well-known methods of improving the reliability of a system are

(i) provision of redundants units, and
(ii) repair maintenance.

In a redundant system more units are made available for performing the system function when fewer are required exactly. There are two major types of redundancy- parallel and standby. In this thesis we confine to both these redundant systems. A series system is also studied.

Some of the typical assumptions made in the analysis of redundant systems are

(i) the repair times are assumed to be exponential
(ii) the system measures are modeled but not estimated
(iii) the system is available continuously
(iv) environmental factors not affecting the system
(v) the failures take place only in one stage
(vi) the switching device is perfect
(vii) system reliability for given chance constraints
(viii) the time required to transfer a unit from the standby state to the operating stage is negligible (instantaneous switchover)
(ix) the failures and repairs are independent

However, we frequently come across systems where one or more of these assumptions have to be dropped. This is the motivation for the detailed study of the models presented in this thesis.

In this thesis we present several models of redundant systems relaxing one or more of these assumptions simultaneously. More specifically it is a study of stochastic models of redundant repairable systems with ‘rest period’ for the operator, non-instantaneous switchover, imperfect switch, intermittent use, and series system optimization.

The thesis contains seven chapters. Chapter 1 is introductory in nature and contains a brief description of the mathematical techniques used in the analysis of redundant systems.

In chapter 2, a two unit system with Erlangian repair time is studied by relaxing the assumptions (i) and (ii). The difference- differential equations are formulated for the state probabilities, and the system measures like reliability and the availability are obtained over a long run. The asymptotic interval estimation is studied for these system measures. The model has been illustrated numerically.

In chapter 3, an n unit system operating intermittently, and in a random environment is studied, by relaxing the assumptions (iii) and (iv). In an intermittently used system, the mean number of disappointments is one of the important measures, which has been obtained for this system in the steady state.

In chapter 4, the assumption (v) and (vi) are relaxed. In most of the models studied earlier in reliability analysis is the study of system measures like reliability and availability. In this chapter, profit analysis of a single unit system with three possible modes of the failure of the unit is studied .This chapter consists of two models: in model 1, the unit goes under repair (if a repairman is available) the moment it fails partially, whereas in model 2 the unit goes under repair at complete failure. The repairman appears in, and disappears from, the system randomly. A comparison between these two models has been studied, after calculating numerically the profit and the MTSF.

Contrary to the previous chapters, stochastic optimization is studied using the Branch and Bound technique in chapter 5 (relaxing the assumption (vii)). In this chapter, an n unit system operating in a random environment is considered. The environment determines the number of units required for the satisfactory performance of the system. Assuming that a unit in standby can fail and that the environment is described by a Markov process, we obtained expressions for the distribution and the moments of the time to the first disappointment, and the expected number of disappointments over an arbitrary interval (0, t].

In chapter 6, the assumption (viii) is relaxed. The reliability, availability and the busy period analysis is studied with the assumption of the non-instantaneous switchover (the time taken from standby state to the operating state is non-negligible random variable). It is also assumed that the unit has three possible failure modes (normal, partial and total failure). Numerical example illustrated the results obtained.

The assumption (ix) is relaxed in chapter 7, and a two-unit cold standby system with the provision of rest for the operating unit is studied. Also, the failure and repair times of each unit assumed to be correlated by taking their joint density as bivariate exponential. The system is observed at suitable regenerative epochs to obtain various reliability characteristics of interest, such as the distribution of time to system failure and its mean, and the steady-state probabilities of the system being in up or down states or under repair. Earlier results are verified as particular cases. Numerical example illustrated the results obtained.

CHAPTER 1

INTRODUCTION

1.1 Introduction

Reliability theory is one of the most important branches of Operations Research and Systems Engineering. Any systems analysis in order to be complete, must give due consideration to system reliability. With remarkable advances made in electronics engineering, military and communication systems have become more sophisticated and when such systems fail, very serious situations arise. Thus in the present day context, high system reliability has become very important from the view point of both makers and the users.

A system designer is often faced with problems of determining the various system measures like reliability, availability and interval reliability etc. He also has to suggest ways by which the efficiency of a given system can be improved. Due to the nature of the subject, the methods of Probability Theory and Mathematical Statistics are necessary to study and solve the problems that arise in reliability theory.

Many mathematical models have been proposed to evaluate various measures of system performance and methods of improving them. These models, which describe the various operational characteristics of the system taking into account its essential features, can be studied only with the help of probability theory. The present work is a study of some mathematical models representing the behaviour of a few complex systems. Introduction of redundancy and repair maintenance are two important methods of improving system reliability.

The manufacturing of tools and special equipment is part of human nature. At first experience, faults and accidents were the only schools for learning to make safer and more reliable equipment. Before structural design became an engineering science, the reliability of a bridge was tested with a team of elephants. It is collapsed, a stronger bridge was built and tested again! Obviously, these methods could not continue and is human skills developed a wide variety of very reliable items and structures were designed and

manufactured. One example is the undersea telephone cables built by Bell Telephone Laboratories.

Man's earliest preoccupation with reliability was undoubtedly related to weaponry. Interest as a result of the terrible non-reliability of electronic weapons systems used during World War II. Increasingly complex systems, such as the first missiles, also emphasized the importance of successful operation of equipment in a specific environment during a certain time period. The V-1 missile, developed in Germany with high quality parts and careful attention, was catastrophic: the first 10 missiles either exploded on the launching pad, or landed short of their targets.

Technological developments lead to an increase in the number of complicated systems as well as an increase in the complexity of the systems themselves. With remarkable advancements made in electronics and communications, systems became more and more sophisticated. Because of their varied nature, these problems have attracted the attention of scientists from various disciplines especially the systems engineers, software engineers and the applied probabilists. An overall scientific discipline, called *reliability theory,* that deals with the methods and techniques to ensure the maximum effectiveness of systems (from known qualities of their component parts) has developed. '*Reliability theory introduces quantitative indices of the quality of production' (*Gnedenko et al. (1969)) and these are carried through from the design and subsequent manufacturing process to the use and storage of technological devices.

Engineers, Scientists and Government leaders are all concerned with increasing the reliability of manufactured goods and operating systems. As '*Unreliability has consequences in cost, time wasted, the psychological effect of inconvenience, and in certain instances personal and national security'* (Lloyd & Lipow (1962)). In 1963 the first journal on reliability, IEEE-Transactions on Reliability saw the light.

Due to the very nature of the subject, the methods of Probability theory and Mathematical statistics (information theory, queuing theory, linear and nonlinear programming, mathematical logic, the methods of statistical simulation on electronic computers, demography, manufacturing, etc.), play an important role in the problem solving of reliability theory. Other areas include contemporary medicine, reliable software systems, geoastronomy, irregularities in neuronal activity, interactions of physiological growth, fluctuations in business investments, and many more. In human behaviour mathematical models based on probability theory and stochastic process are helpful in rendering realistic modelling for social mobility of individuals, industrial mobility of labour, educational advancements, diffusion of information and social networks. In the biological sciences stochastic models were first used by Watson and Galton (1874) in a study of extinction of families. Research on population genetics, branching process, birth and death process, recovery, relapse, cell survival after irradiation, the flow of particles through organs, etc, then followed. In business management, analytical models evolved for the purchasing behaviour of the individual consumer, credit risk and term structure. Income determination under uncertainty and more related subjects. Traffic flow theory is a well known field for stochastic models and studies have been developed for traffic of pedestrians, freeways, parking lots, intersections, etc. (Erasmus,2005)

Problems encountered in the design of highly reliable technical systems have led to the development of high accuracy methods of reliability analysis. Two major problems can be identified, namely:

- Creating classes of probability-statistical models that can be used in the description of the reliability behaviour of the systems, and
- Developing mathematical models for the examination of the reliability characteristic of a class of systems.

Considering only redundant systems the classical examples are the models of Markov processes with a finite set of states (in particular birth and death processes) (Gnedenko et al. (1969)), Barlow (1984), Gertbakh (1989) and
Kovalenko et al. (1997)), the renewal process method (Cox (1962)), the semi-Markov process method and its generalisations (Cinlar (1975a, b)), generalized semi-Markov process (GSMP) method (Rubenstein (1981)), special models for coherent systems (Aven (1966)) and systems in random and variable environment (Ozekici (1996)) and Finkelstein (1999a, b, c)), van Schoor (2005), Muller (2005).

Depending on the nature of the research, the applicable form of reliability theory can be introduced to each. A stochastic analysis is made based on some good probability characteristics. It is, however, not simply a case of changing terminology in standard probability theory (say, "random variable" changes to "lifetime"), but reliability distinguishes itself by providing answers and solutions to a series of new problems not solved in the "standard" probability theory framework. Gertbakh (1989) points out that reliability,

- of a system is based on the information regarding the reliability of the system's components

- gives a mathematical description of the ageing process with the introduction of several formal notations of ageing (failure rate, etc.)
- introduces well-developed techniques of *renewal theory*
- introduces *redundancy* to achieve optimal use of standby components (an excellent introduction to redundant systems is given in Gnedenko et al. (1969))
- includes the theory of optimal preventative maintenance (Beichelt and Fischer (1980))
- is a study of statistical inference (often from censored data)

Generally, the mathematical problems of lifetime studies of technical objects (reliability theory) and of biological entities (survival analysis) are similar, differing only in the notation. The term "lifetime" therefore does not apply to lifetimes in the strictest literal sense, but can be used in the figurative sense. The idea is that the statistical analysis done in this thesis should be true in any of the applicable disciplines, although the notation is mostly as for engineering
(systems, components, units, etc). With minor modifications the discipline can be changed to biological, or financial, or any other disciplines.

1.2 FAILURE

'*A failure is the result of a joint action of many unpredictable, random processes going on inside the operating system as well as in the environment in which the system is operating.'* (Gertbakh (1989)). Functioning is therefore seriously impeded or completely stopped at a certain moment in time and all failures have a stochastic nature. In some cases the time of failure is easily observed. But if units deteriorate continuously, determination of the moment of failure is not an easy task. In this study we assume that failure of a unit can be obtained exactly.

Failure of a system is called a *disappointment* or a *death* and failure results in the system being in the down state. This can also be referred to as a breakdown (Finkelstein (1999a)).

Zacks (1992) points out that there are two types of data to consider, namely:

- data from continuous monitoring of a unit until failure is observed
- data from observations made at discrete time points, therefore failure counts

Villermeur (1992) gives an extensive list of possible failures and inter-dependent failures. There are catastrophic failures, determined by a sharp change in the parameters and drift failures (the result of wear or fatique), arising as a result of gradual change in the values of the parameters. (Muller, 2005).

1.3 REDUNDANCY AND DIFFERENT TYPES OF REDUNDANT SYSTEMS

In a *redundant system* more units are built into it than is actually necessary for proper system performance. Redundancy can be applied in more than one way

and a definite distinction can be made between *parallel* and *standby* (sequential) redundancy. In parallel redundancy the redundant units form part of the system from the start, whereas in a standby system, the redundant units do not form part of the system from the start (until they are needed).

1.3.1 Parallel systems

A parallel redundant system with n units is one in which all units operate simultaneously, although system operation requires at least one unit to be in operation. Hence a system failure only occurs when all the components have failed.

Let k be a non-negative integer, such that k < n, counting the number of units in an n-unit system. It is customary to refer to such a system as k-out-of-n system.

1.3.2 k-out-of-n: F system

If k-out-of-n system fails, when k units fail, it is called an F-system. The functioning of a minimum number of units ensures that the system is up (Sfakianakis and Papastavridis (1993)).

1.3.3 k-out-of-n: G-system

A G-system is operational if and only if at least k units of the system are operational. Recent work related to this topic can be seen in Zhang and Lorn (1998) and Liu (1998). Suppose a radar network has n radar control stations covering a certain area: the system can be operable if and only if at least k of these stations are operable. In other words, to ensure functioning of the system it is essential that a minimum number of units, k, are functioning.

Lately attention moved to load-sharing k-out-of-n: G systems, where

- the serving units share the load
- the failure rate of a component is affected by the magnitude of the load it shares.

1.3.4 n-out-of-n: G system

A series that consists of *n* units and when the failure of any one unit causes the system to fail. Although this type of system is not redundant system, as all the units are in series and have to be operational, it can still be considered as a special case of a k-out-of-n system. There are many papers on the reliability of these systems. Scheuer (1988) studied reliability for *shared-load k-out-of-n:* G systems, where there is an increasing failure rate in survivors, assuming identically distributed components with constant failure rates. Shao and Lamberson (1991) considered the same scenario, but with imperfect switching. Then Huamin (1998) published a paper on the influence of work-load sharing in non-identical, non-repairable components, each having an arbitrary failure time distribution. He assumed that the failure time distribution of the components can be represented by the accelerated failure time model, which is also a proportional hazards model when base-line reliability is Weibull. (Muller, 2005)

1.4 REPAIRABLE SYSTEM

In order to increase the system reliability, failed units may be replaced by new ones. However when this proves to be very expensive, resort is made to repair the failed units. On failure, a unit is sent to a repair facility. If the repair facility is not free, failed units queue up for repair. The life time of a unit while online, while in standby and the repair time are all independent random variables. It is assumed that the distribution functions of these random variables are known and that they have probability density functions.

Barlow (1962) had considered some repairman problems and they have much in common with queuing problems. Rau (1964) had discussed the problem of finding the optimum value of m in an m out of n: G system for maximizing reliability.

Ascher (1968) has pointed out some inconsistencies in the modelling of repairable systems by renewal theory. Several authors, notably Barlow and Proschan (1965), Sandler (1963), Shooman (1968), Buzacott (1970) and Doyon and Berssenbrugge (1968) have used continuous time discrete state Markov renewal process model for describing the behaviour of a repairable system.

These conceptionally simple methods are not practically feasible for systems with large number of states. Gaver (1963), Gnedenko et al (1969), Osaki (1969, 70 a, b) and Srinivasan (1966) have employed the techniques of Semi-Markov processes for finding the reliability of a system with exceptional failures. By the use of Semi-Markov processes, Kumagi (1971) studied the effect of different failure distributions on the availability through numerical calculations. Branson and Shah (1971) studied repairable systems with arbitrary failure distributions using Semi-Markov Processes. Srinivasan and Subramanian (1977), Venkatakrishnan

(1975), Ravichandran (1979), Natarajan (1980), Sarma (1982), Botha (2001), Muller (2005) have used

regeneration point technique to analyse repairable systems with many, though not all, arbitrary distributions. More references in related topics can be found in the review papers by Subba Rao and Natarajan (1970), Osaki and Nakagawa (1976), Pierskalla and Voelker (1976) and Lie, Wang and Tillman (1977) and Kumar and Agarwal (1980), Gopalan (2004).

1.5 SYSTEMS WITH NON-INSTANTANEOUS SWITCHOVER

In the study of redundant systems it is generally assumed that when the unit operating online fails, the unit in standby is automatically switched online and the switchover from the standby state to online state is instantaneous. Srinivasan (1968), Osaki (1972), Khalil (1977), Subramanian and Ravichandran (1978 a), Gopalan and Marathe (1978, 80), Singh et al (1979) and Kalpakam and Shahul Hameed (1980), Subramanian and Sarma (1982) have studied redundant systems incorporating non-negligible switchover times.

1.6 SYSTEMS WITH IMPEREFCT SWITCH

To transfer a unit from the standby state to the online state, a device known as ‘switching device’ is required. Generally we assume that the switching device is perfect in the sense that it does not fail. However; there are practical situations where the switching device can also fail. This has been pointed out by Gnedenko et al (1969). Such systems in which the switching device can fail are called systems with imperfect switch.

Chow (1971), Osaki (1972), Nakagawa and Osaki (1975 a), Nakagawa (1977), Venkatakrishnan (1975), Prakash and Kumar (1970), Srinivasan and Subramanian (1980) and Subramanian and Natarajan (1980), Subramanian & Sarma (1984) have considered models where the switching device can also fail.

1.7 INTERMITTENTLY USED SYSTEMS

In almost all the models of redundant systems studied so far, it is assumed that the system under consideration is needed all the time. But in some practical
situations continuous failure free performance may not be necessary. In such cases we have to take into consideration the fact that the system can be in down state during certain intervals without any real consequence. In this case the probability that the system is in the up state is not an important measure; what is really important is the probability that the system is available when it is needed. Gaver (1964) pointed out that is pessimistic to evaluate the performance of an intermittently used system solely on the basis of the distribution of the time to failure. Srinivasan (1966), Nakagawa et al (1976), Srinivasan and Bhaskar (1979 a, b, c), Kapur and Kapoor (1978, 80) extended Gaver's results for two-unit systems. Detailed study of an n-unit intermittently used system is made. The statistical inference of some of these models has been studied recently by Yadavalli et al (2000, 2001).

1.8 MEASURES OF SYSTEM PERFORMANCE

The previous sections briefly describe the various types of redundant systems discussed in the literature. In this section some of the important measures of system performance useful in different contexts are discussed (Barlow and Proschan (1965), Gnedenko et al (1969)).

(a) Reliability:

Reliability is the probability that the system will perform satisfactorily for a given period of time in its intended application. Let $\{\xi(t), t \geq 0\}$ be the performance process of the system; then for a fixed t, $\xi(t)$ is a binary random variable which takes the value 1 if the system operates satisfactorily at a given time t, and takes the value 0 otherwise.

Then the reliability R (t) is given by

$$\begin{aligned} R(t) &= \Pr[\text{system is up in } (0, t]] \\ &= \Pr[\xi(u) = 1; \text{ for all u such that } 0 \leq u \leq t] \end{aligned}$$

The expectation of the random variable representing the duration of time measured from the point the system starts operating till the instant it fails for the first time is called Mean time to System Failure (MTSF).It can be obtained from R (t) from the relation

$$\text{MTSF} = \int_0^\infty R(u)du$$

(b) Pointwise Availability:

This is defined as the 'probability that the system is able to operate within the tolerances at a given instant of time'. In symbols:

$$\text{Pointwise availability } A(t) = \Pr[\xi(t) = 1]$$

(c) Asymptotic or Steady-State Availability:

Steady-state availability $A_\infty = \lim_{t \to \infty} A(t)$.

It can be shown (Barlow and Proschan (1975)) that this is equal to the expected fraction per unit time in the long run that the system operates satisfactorily.

(d) Interval Reliability:

The interval reliability R (t, x) is the probability that the system is up in the interval $[t, t + x]$.

Hence:

R (t, x) = Pr [ξ (u) = 1, for all u such that t ≤ u ≤ t + x]

We observe that the reliability R(x) and the pointwise availability *A (t)* can be obtained from the interval reliability R (t, x) by putting t = 0 and x = 0 respectively.

(e) Limiting interval reliability:

This is defined as the limit of R (t, x) as t→∞, and hence is denoted by $R_\infty(x)$, which is the ordinary reliability function.

(f) Mean number of events in (0, t):

Let N(x, t) denote the number of particular type of event (like break down etc.) in $\left(x, x+t\right]$.Then the mean number of events in (0, t) is given by

$$E\,[N\,(0, t)] = \int_0^t h_1(u)du$$

where $h_1(t)$ is the first order product density of the events (product densities are defined in a subsequent section in this chapter) .The stationary rate of occurrence of those events is given by:

$$E\,[N] = \lim_{t\to\infty} \frac{E[N(0,t)]}{t}$$

1.9 TECHNIQUES USED IN THE ANALYSIS OF REDUNDANT SYSTEMS.

This section is a compilation of the techniques used in the analysis of redundant repairable systems.

1.9.1 Renewal Theory

Renewal theory forms an important in the study of stochastic processes and applied probability models, and is extensively used by many to study specific reliability problems. Feller (1968) made significant contributions to renewal theory giving the proper lead.

Smith (1958) gave an extensive review and highlighted the applications of renewal theory to a variety of problems. A lucid account of renewal theory is given by Cox (1962).

Definition 1.1

A renewal process is a sequence of independent, non-negative and identically distributed random variables $\{Y_i$, i = 1, 2,...} which are not all zero with probability one.

We assume that these random variables are defined on the same probability space and have finite mean μ.A renewal process is completely determined by means of f (.), the pdf of X_i. Associated with a renewal process is a r.v *N (t)* which represents the number of renewals in the time interval (0, t]; *N (t)* is also known as the renewal counting process (Parzen, 1962, Beichelt and Fatti (2002).

If policy 0 is the practical background of a renewal process, then Y_i denotes the time between the $(i-1)-th$ and the $i-th$ renewal. If at time $t=0$ policy 0 has already been in effect for a while, then Y_1 is a residual lifetime in the sense of section 1.2.3. However, the age of the

system working at time $t=0$ need not to be known. But if at time $t=0$ a new system started working, then all the random variables $Y_1, Y_2,...$ are identically distributed.

Let the random variables $Y_2, Y_3,...$ be identically distributed as Y with distribution function $F(t)=P(Y\le t)$, whereas Y_1 has distribution function $F_1(t)=P(Y_1\le t)$.

Definition 1.2 (see Beichelt and Fatti, 2002)

A renewal process is called delayed if $F_1(t) \not\equiv F(t)$ and ordinary if $F_1(t) \equiv F(t)$.

Since, by assumption, the renewal occur in negligible time, T_n defined by

$$T_n = \sum_{i=1}^{n} Y_i \text{ ; n = 1,2,…;}$$

is the time point at which the nth failure (renewal) takes place. Hence, T_n is called a renewal time. The time intervals between two neighbouring renewals are called renewal cycles.

Let the renewal counting process $\{N(t), t \geq 0\}$ be defined by

$$N(t) = \begin{cases} \max(n;\ T_n \leq t) \\ 0 \quad for \quad t < T_1 \end{cases}$$

$N(t)$ is the random number of renewals occurring in $(0, t]$.Since $N(t) \geq n$ if and only if $T_n \leq t$,

$$F_{T_n}(t) = P(T_n \leq t) = P(N(t) \geq n),$$

where, because of the independence of the $Y_i, F_{T_n}(t)$ is the convolution of F_1 with the $(n-1)-th$ convolution power of F.

$$F_{T_n}(t) = F_1 © F^{(n-1)}(t),\ F^{(0)}(t) \equiv 1,\ t \geq 0;\ n = 1,\ 2,...$$

If the densities $f_1(t)=F_1'(t)$ and $f(t)=F'(t)$ exist, then the density of T_n is

$$f_{T_n}(t)=f_1 © f^{(n-1)}(t)\ ,\ f^{(0)}(t)\equiv 1,\ t\geq 0; n=1,2,...$$

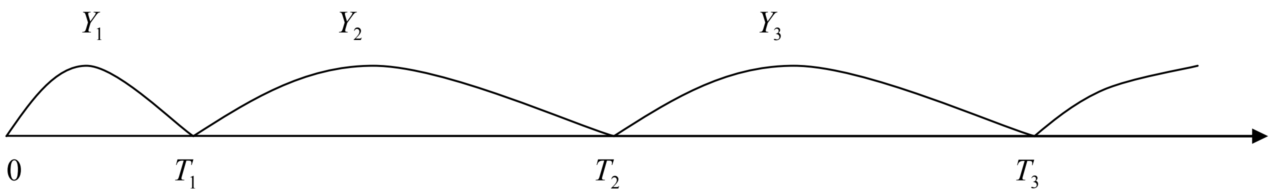

Figure: illustration of a renewal process

Definition 1.3

The expected value of *N (t)* is called the renewal function and is denoted by $H(t)$. The derivative of $H(t)$, if it exists, is denoted by $h(t)$ and is called the renewal density. The quantity $h(t)dt$ is the probability that a renewal occurs in $(t,t+dt)$.

The renewal density satisfies the following famous integral equation, known as the functional equation of renewal theory.

$$h(t)=f(t)+\int_0^t f(u)h(t-u)du$$

The solution of the above equation is:

$$h(t) = \sum_{n=1}^{\infty} f^{(n)}(t)$$

where $f^{(n)}(t)$ is the n-fold convolution of f(t).

We now briefly indicate how renewal theory has been used in the solution of reliability problems. Srinivasan et al (1971) used renewal theory to obtain some operating characteristics of a one unit system. The integral equation of renewal theory was used by Gnedenko et al (1969) to obtain MTSF of a two-unit standby system. Osaki (1970b) applied the integral equation to study several redundant systems. Buzacott (1971) used renewal theoretic arguments to study some priority redundant systems.

1.9.2 SEMI-MARKOV AND MARKOV RENEWAL PROCESS

Now we consider a stochastic process which makes transitions from one state to another in accordance with a Markov chain but the amount of time spent in each state before a transition is probabilistic. Denoting the state space by the set of non-negative integers {0,1, 2….}. Let the transition probabilities be given by p_{ij}, i,j = 0,1,2,…..Let $F_{ij}(t)$, t> 0 be

the conditional distribution function of the sojourn time in state i, given that the next transition will be into state j.

Let

$$Q_{ij}(t) = p_{ij} F_{ij}(t), \text{ i, j} = 0,1,2,\ldots..$$

Then $Q_{ij}(t)$ denotes the probability that the process makes a transition into state j in an amount of time less than or equal to t given that it just entered state i at t = 0.The functions $Q_{ij}(t)$ satisfy the following conditions:

$$Q_{ij}(0) = 0, \qquad Q_{ij}(\infty) = p_{ij};$$

$$Q_{ij}(t) \geq 0, \qquad i,j = 0,1,2,\ldots$$

$$\sum_{j=0}^{\infty} Q_{ij}(t) = 1$$

Let J_0 denote the initial state of the process and J_n (n = 1, 2…) the state of the process after the n-th transition has occurred. Then the process {J_n, n = 0, 1, 2…} is a Markov Chain with transition probabilities P_{ij}.This is called the embedded Markov Chain. Let $N_i(t)$ denote the number of transitions into state i in (0, t] and define

$$N(t) = \sum_{i=0}^{\infty} N_i(t)$$

Now define a stochastic process {Z (t), t≥0} where Z (t) = i, denotes that the process is in state i at time t. Then it is clear that Z (t) = $J_n(t)$

Definition 1.4

The stochastic process {Z (t), t ≥ 0} is called a Semi-Markov process (SMP).

Definition 1.5

The vector stochastic process $\{N_1(t), N_2(t), \ldots, t \geq 0\}$ is called a Markov Renewal Process (MRP).

Thus the SMP records the state of the process at each time point, while the MRP is a counting process which keeps track of the number of visits to each state. Denote by X_i the random variable denoting the time interval between two successive visits to state i of the process $\{Z(t), t \geq 0\}$.Then we observe that $\{x_i\}$ is a renewal process for i = 0,1,2,…. Detailed treatments of SMP and MRP can be found in Pyke (1961 a, b), Cinlar (1975 a) and Ross (1970).

The survey by of Cinlar (1975 b) demonstrates the usefulness of the theory of MRP and SMP in applications. Barlow et al (1965) used these processes to determine the MTSF of a two unit system. Srinivasan (1968), Cinlar (1975 b), Osaki (1970 a, 1972). Arora (1976 a, b), Nakagawa and Osaki (1974, 1976), and Nakagawa (1974) have used the theory of SMP to discuss some reliability problems.

1.9.3 STOCHASTIC POINT PROCESSES

Stochastic point processes are more general than those considered in the earlier sections. Since point processes have been studied by many with varying backgrounds, there have been several definitions of the point processes each appearing quite natural from the view point of the particular problem under study. [See for example Bartlett (1966), Bhaba (1950), Harris (1963) and Khinchine (1955)]. A comprehensive definition of point process is due to Moyal (1962) who deals with such processes in a general space which is not necessarily Euclidean.

Roughly speaking a stochastic point process can be defined as continuous time parameter discrete state space stochastic process.

1.9.4 PRODUCT DENSITIES

One of the ways of characterizing a general stochastic point process is through product densities (Ramakrishnan (1950, 1958), Srinivasan (1974)).These densities are analogous of the renewal density in the case of non-renewal processes.

Let N (x, t) denote the random variable representing the number of events in the interval (t, t + x), d_x N(x , t) the events in the interval (t + x, t + x +dx) and p(n , x , t) = Pr[N (x , t) = n].

The product density of order n is defined as:

$$h_n(x_1, x_2, \ldots, x_n) = \lim_{\Delta_1, \Delta_2, \ldots, \Delta_n \to 0} \frac{E\left[\prod_{i=1}^{n} N(\Delta_1, x_i)\right]}{\Delta_1 \Delta_{2 \ldots} \Delta_n}$$

$$x_1 \neq x_2 \neq \ldots x_n .$$

A process is said to be regular if the probability of occurrence of more than one event in an interval of length is o (Δ). For such process we have:

$$h_n(x_1, x_2, \ldots, x_n) = \lim \frac{\Pr\left[N(\Delta_i, x_i) \geq 1, i = 1,2,\ldots,n\right]}{\Delta_1 \Delta_2 \ldots \Delta_n}, x_1 \neq x_2 \neq \ldots \neq x_n$$

These densities represent the probability of an event in each of the intervals $(x_{1,} x_1 + \Delta_1)$, $(x_2, x_2 + \Delta_2), \ldots, (x_n, x_n + \Delta_n)$.

Even though the functions $h_n(.,.,.,..)$ are called densities, it is important to note that their integration will not give probabilities but will yield the factorial
moments. The ordinary moments can be obtained by relaxing the condition that all x_i are distinct.

1.9.5 REGENERATIVE STOCHASTIC PROCESSES

The idea of regeneration point was first introduced by Bellman and Harris (1948) while studying population point processes. Feller (1949), in the theory of recurrent events, dealt with a special case of regeneration points. Later on, Smith (1955) generalized Feller's results and dealt with more general stochastic point processes possessing such regeneration points, familiarity known as regenerative processes. A formal theory of such processes has been developed by Kingman (1964).

A regenerative event R of a stochastic process {X (t)} is an event that is characterized by the property that if it is known that R happens at $t = t_1$, then the knowledge of the history of the process prior to t_1 loses its predictive value. In some special cases, the event R is the only characteristic, so that the process regenerates itself with each occurrence of R.

In more general cases, in addition to the occurrence of R, knowledge of X (t) is necessary for the prediction of the process. The renewal process can be thought of as a general point process in which each point at which the event R occurs is a regeneration point. The occurrence of an event at

$t = t_1$ uniquely determines the distribution of events from any collection of segments of points $t \geq t_1$. If we further specialize to the case when the intervals between successive events are exponentially distributed, we notice that any point (not necessarily a point where an event occurs) on the time axis is a regeneration point. Gnedenko (1964), Srinivasan and Gopalan (1973 a, b), Birolini (1974, 75), Srinivasan and Subramanian (1977), Hines (1987), Hargreaves (2002),

Botha (2001), Muller (2005) have used such regenerative events to study some reliability problems.

1.9.6 CONCLUDING REMARKS AND SCOPE OF WORK

Reliability theory is a very important branch of systems engineering and operations and deals with general method of evaluating the various measures of performance of a system that may be subject to gradual deterioration. Several models of redundant systems have been studied in the literature and the following are some of the typical assumptions made in analyzing such systems:

(i) the repair times are assumed to be exponential

(ii) the estimated study of the system measures has not been made.

(iii) the system is available continuously

(iv) Environmental factors not affecting the system

(v) The failures take place in one mode

(vi) The switching device is perfect

(vii) System reliability evaluated for given chance constraints

(viii) The switchover time required to transfer a unit from the standby state to online stage is negligible.

(ix) the failures and repairs are assumed to be independent.

However, we frequently come across systems in which one or more of these assumptions have to be dropped and hence there is an increasing need for studying models in which at least some of these assumptions could be relaxed. That is the motivation for the detailed study of the models presented in this thesis. This thesis is a study of some redundant repairable systems with 'rest period' for the operator, non-instantaneous switchover, imperfect switch, intermittent use and optimization study.

CHAPTER 2

A STUDY OF A TWO UNIT PARALLEL SYSTEM WITH ERLANGIAN REPAIR TIME

A modified version of this chapter has been published in Bul.Al.Math.soc, vol.19, 2005

2.1 INTRODUCTION

In order to improve the reliability, availability, quality and safety operational systems the well known tools to be used are redundancy, repair and preventive maintenance, etc. (Birollini et al, (1994)). Most of the past studies of reliability systems are confined to obtaining expressions for various measures of system performance and do not consider the associated inference problems. Chandrasekhar and Natarajan (1994), Yadavalli et al (2001), (2002) have considered a two unit parallel system and obtained exact confidence limits for the steady state availability of the system, when the failure rate of an operative unit is constant and the repair time of the failed unit is a two stage Erlang distribution. The Bayesian methods for these problems were subsequently studied by Yadavalli et al (2003).

In general, the failure- free time and repair time are independent random variables. Thus there is need to study a model by relaxing this imposed condition. An attempt is made in this paper to study a two-unit parallel system, wherein the failure rate of a unit is constant and the repair time distribution is a two Erlangian distribution under the assumption that an operative unit has a zero failure rate if a failed unit is in the second stage of repair. Apart from expressions for the system reliability, MTBF, availability and steady state availability, we obtain a CAN estimator and an asymptotic confidence interval for the steady state availability of the system and the MLE of the system reliability.

2.2 MODEL AND ASSUMPTIONS

The system under consideration is a two unit parallel system with a single repair facility, subjected to the following assumptions:

(i) The units are similar and statistically independent. Each unit has a constant failure rate λ.

(ii) There is only one repair facility and the repair time distribution is a two stage Erlangian distribution with probability density function (p.d.f) given by

$$g(y) = (2\mu)^2 e^{-2\mu y} y, \; 0 < y < \infty, \; \mu > 0 \qquad (2.1)$$

(iii) Each unit is new after repair.

(iv) Switch is perfect and the switchover is instantaneous.

(v) An operative unit has a zero failure rate if a failed unit is in the second stage of repair.

2.3 ANALYSIS OF THE SYSTEM

To analyze the behaviour of the system, we note that at any time t, the system may be in any one of the mutually exclusive and exhaustive states

So:	Both units are operating
S1:	One unit is operating and the other is in the first stage of repair.

S2:	One unit is operating and the other is in the second stage of repair.
S3:	One unit is in the first stage of repair and the other is waiting for repair
S4:	One unit is in the second stage of repair and the other is waiting for repair.

.

Since an Erlang distribution is the distribution of the sum of two independent and identically distributed exponential random variables, the stochastic

process describing the behaviour of the system is a Markov process. Let $p_i(t)$, i = 0, 1, 2, 3, 4 be the probability that the system is in state at S_i time t. Clearly, the infinitesimal generator of the Markov process is given by:

$$Q = \begin{matrix} S_0 \\ S_1 \\ S_2 \\ S_3 \\ S_4 \end{matrix} \begin{bmatrix} -2\lambda & 2\lambda & 0 & 0 & 0 \\ 0 & -(\lambda+2\mu) & 2\mu & \lambda & 0 \\ 2\mu & 0 & -2\mu & 0 & 0 \\ 0 & 0 & 0 & -2\mu & 2\mu \\ 0 & 2\mu & 0 & 0 & -2\mu \end{bmatrix} \quad (2.2)$$

It should be noted that states S_0, S_1 and S_2 are up-states, whereas S_3 and S_4 are down states. We assume that initially, both the units are operative.

2.3.1. Reliability

The system reliability R (t) is the probability of failure free operation of the system in (0, t]. To derive an expression for the reliability of the system, we restrict the transitions of the Markov process to the system, we restrict the transitions of the Markov process to the system up-states namely S_0, S_1 and

S_2.Using the infinitesimal generator given in (2.2) pertaining to these up-states and standard probabilities arguments, we obtain the following system of differential-difference equations

$$p_0'(t) = -2\lambda p_o(t) + 2\mu p_2(t)$$

$$p_1'(t) = 2\lambda p_0(t) - (\lambda + 2\mu) p_1(t)$$

$$p_2'(t) = 2\mu p_1(t) - 2\mu p_2(t).$$

With the condition $p_0(0) = 1$ and $p_i(0) = 0$ for $i = 1, 2$. Thus,

$$\sum_{i=0}^{2} p_i(t) = 1.$$

Let L_i(s) be the Laplace transform of p_i (t), i = 0, 1, 2. Taking Laplace transforms for $p_i(t)$, we get

$$(s + 2\lambda) p_0^*(s) - 2\mu p_2^*(s) = 1$$

$$(s + \lambda + 2\mu) p_1^*(s) - 2\lambda p_0^*(s) = 0$$

$$(s + 2\mu) p_2^*(s) - 2\mu p_1^*(s) = 0$$

$$R(t) = \sum_{i=1}^{3} \frac{[(\alpha_i + 2\mu)(\alpha_i + 3\lambda + 2\mu) + 4\lambda\mu]}{\prod_{j=1}^{3} (\alpha_i - \alpha_j)} e^{\alpha_i t} \tag{2.3}$$

where α_1, α_2, and α_3 are the roots of the cubic equation:

$$s^3 + (3\lambda + 4\mu)s^2 + \left(2\lambda^2 + 10\lambda\mu + 4\mu^2\right)s + 4\lambda^2\mu = 0$$

2.3.2 Mean Time Before Failure (MTBF)

The system mean time before failure is given by

$$MTBF = L_0(0) + L_1(0) + L_2(0) = \frac{5\lambda + 2\mu}{2\lambda^2}$$

2.3.3 System Availability

The system availability A (t) is the probability that the system operates (within the tolerances) at a given instant of time t.

Using the infinitesimal generator given in (2.2), we obtain the following system of differential-difference equations:

$$p_0'(t) = -2\lambda p_0(t) + 2\mu p_2(t) \tag{2.4}$$

$$p_1'(t) = 2\lambda p_0(t) - (\lambda + 2\mu)p_1(t) + 2\mu p_4(t) \tag{2.5}$$

$$p_2'(t) = 2\mu p_1(t) - 2\mu p_2(t) \tag{2.6}$$

$$p_3'(t) = \lambda p_1(t) - 2\mu p_3(t) \tag{2.7}$$

$$p_4'(t) = 2\mu p_3(t) - 2\mu p_4(t) \tag{2.8}$$

with the condition $p_o(0) = 1$ and

$$\sum_{i=0}^{4} p_i(t) = 1 \tag{2.9}$$

Taking the Laplace transforms for the equations (2.4) – (2.8), we get

$$(s+2\lambda)p_0^*(s) - 2\mu p_2^*(s) = 1 \tag{2.10}$$

$$(s+\lambda+2\mu)p_1^*(s) - 2\lambda p_0^*(s) - 2\mu p_4^*(s) = 0 \tag{2.11}$$

$$(s+2\mu)p_2^*(s) - 2\mu p_1^*(s) = 0 \tag{2.12}$$

$$(s+2\mu)p_3^*(s) - \lambda p_1^*(s) = 0 \tag{2.13}$$

$$(s+2\mu)p_4^*(s) - 2\mu p_3^*(s) = 0 \tag{2.14}$$

solving the equations (2.10) – (2.14) using the relation (2.9), we get

$p_i^*(s)$, $i = 0, 1, 2, \ldots, 4$.

Inverting $p_i^*(s)$, we get

$$p_0(t) = \frac{\mu^2}{(\lambda+\mu)^2} + 8\lambda\mu^2 \sum_{i=1}^{3} \frac{(\alpha_i + 2\mu)}{\alpha_i(\alpha_i + 2\lambda)\prod_{\substack{i=1 \\ j\approx 1}}^{3}(\alpha_i - \alpha_j)} e^{-\alpha_i t} \tag{2.15}$$

$$p_1(t) = \frac{\lambda\mu}{(\lambda+\mu)^2} + 2\lambda \sum_{i=1}^{3} \frac{(\alpha_i + 2\mu)^2}{\alpha_i \prod_{\substack{i=1 \\ j\approx 1}}^{3}(\alpha_i - \alpha_j)} e^{-\alpha_i t} \tag{2.16}$$

$$p_2(t) = \frac{\lambda\mu}{(\lambda+\mu)^2} + 4\lambda\mu \sum_{i=1}^{3} \frac{1}{\alpha_i} \frac{(\alpha_i + 2\mu)}{\prod_{\substack{i=1 \\ j\approx 1}}^{3} (\alpha_i - \alpha j)} e^{-\alpha_i t} \tag{2.17}$$

$$p_3(t) = \frac{\lambda^2}{2(\lambda+\mu)^2} + 2\lambda^2 \sum_{i=1}^{3} \frac{1}{\alpha_i} \frac{(\alpha_i + 2\mu)}{\prod_{\substack{i=1 \\ j\approx 1}}^{3} (\alpha_i - \alpha_j)} e^{-\alpha_i t} \tag{2.18}$$

$$p_4(t) = \frac{\lambda^2}{2(\lambda+\mu)^2} + 4\lambda^2\mu \sum_{i=1}^{3} \frac{1}{\alpha_i \prod_{\substack{i=1 \\ j\approx 1}}^{3} (\alpha_i - \alpha_j)} e^{-\alpha t t} \tag{2.19}$$

where α_1, α_2 and α_3 are the roots of the cubic equation.

$$s^3 + 3(\lambda + 2\mu)s^2 + 2(\lambda^2 + 8\lambda\mu + 6\mu^2)s + 8\mu(\lambda+\mu)^2 = 0 \tag{2.20}$$

Since S_0, S_1 and S_2 are the up-states, the availability of the system is given by:

$$A(t) = p_0(t) + p_1(t) + p_2(t) \tag{2.21}$$

$$A(t) = \frac{\mu(2\lambda+\mu)}{(\lambda+\mu)^2} + 8\lambda\mu \sum_{i=1}^{3} \frac{(\alpha_i + 2\mu)e^{-\alpha t\, t_i}}{\alpha_i(\alpha_i + 2\lambda)\prod_{j=1}^{3}(\alpha_i - \alpha_j)} + 2\lambda \sum_{i=1}^{3} \frac{(\alpha_i + 2\mu)^2}{\alpha_i \prod_{j=1}^{3}(\alpha_i - \alpha_j)} e^{-\alpha_i t}$$
$$+ 4\mu \sum_{i=1}^{3} \frac{(\alpha_i + 2\mu)}{\alpha_i \prod_{j=1}^{3}(\alpha_i - \alpha_j)}$$

2.3.4 Steady State Availability

The system steady state availability is given by:

$$A_\infty = \lim_{t \to \infty} A(t) = \frac{\mu(2\lambda + \mu)}{(\lambda + \mu)^2} \qquad (2.22)$$

which is in agreement with Mohammed Abu-Salih *et al.* (1990).

In the following sections, we obtain a CAN estimator, a 100(1- α) % asymptotic confidence interval for steady state availability of the system and the MLE of the system reliability.

2.4 CONFIDENCE INTERVAL FOR STEADY-STATE AVAILABILITY OF THE SYSTEM

Let $X_1, X_2, \ldots, X_n$ be a random sample of failure free-times of a unit with probability density function (p.d.f) given by

$$f(x) = \lambda e^{-\lambda x}; 0 < x < \infty, \lambda > 0 \qquad (2.23)$$

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample of the repair times with the p.d.f given by $g(y) = \mu e^{-\mu y}$. It is clear that $E(\overline{X}) = \frac{1}{\lambda}$ and $E\left(\frac{\overline{Y}}{2}\right) = \frac{1}{\mu}$, where $\overline{X}$ and $\overline{Y}$ are respectively the sample means of the failure-free times and the repair times of s unit. It can be shown that $\overline{X}$ and $\frac{\overline{Y}}{2}$ are respectively the maximum likehood estimators (MLE's) of $\frac{1}{\lambda}$ and $\frac{1}{\mu}$.

Let $\theta_1 = \dfrac{1}{\lambda}$ and $\theta_2 = \dfrac{1}{\mu}$. Clearly, the steady state availability, given by (2.16), reduces to

$$A_\infty = \frac{\theta_1(\theta_1 + 2\theta_2)}{(\theta_1 + \theta_2)^2}$$

Hence, the MLE of A_∞ is given by

$$A_\infty = \frac{4\overline{X}(\overline{X} + \overline{Y})}{(2\overline{X} + \overline{Y})^2} \qquad (2.24)$$

It should be noted that A_∞ is real valued differential function in X and Y. Now consider the following application of the multiplicative central limit theorem (Rao, 1974).

Suppose that $T_1^{'}$, $T_2^{'}$, $T_3^{'}$, . . . are independent and identically distributed k-dimensional random variables such that:

$$T_n' = (T_{1n},\ T_{2n}, . . ., \ T_{kn})$$

has first and second order moments

$$E(T_n) = \mu \ \text{ and } \ D(T_n) = \Sigma .$$

Define the sequence of random variables $\overline{T}_n = (\overline{T}_{1n},\ \overline{T}_{2n}, . . ., \overline{T}_{kn})$, $n = 1,\ 2$ where:

$$\overline{T}_{in} = \frac{1}{n}\sum_{j=1}^{n} T_{ij}, \qquad i = 1, 2, \ldots, k \quad and \quad j = 1, 2, \ldots, n$$

Then, $\sqrt{n}\left(\overline{T}_n - \mu\right) \xrightarrow{d} N(0, \Sigma)$ as n $\rightarrow$ ∞. Hence, the applying the Multivariate Central Limit theorem, it follows that:

$$\sqrt{n}\left[\left(\overline{X}, \frac{\overline{Y}}{2}\right) - (\theta_1, \theta_2)\right] \xrightarrow{d} N(0, \Sigma) \quad as \quad n \rightarrow \infty$$

where the dispersion matrix $\sum = \left(\left(\sigma_{ij}\right)\right)_{2x2}$ is given by $\sum = diag\left(\theta_1^2, \frac{\theta_2^2}{2}\right)_{2x2}$

Again from Rao (1974), we have:

$$\sqrt{n}\left(\hat{A}_\infty - A_\infty\right) \xrightarrow{d} N\left(0, \sigma^2(\theta)\right) \quad as \quad n \rightarrow \infty,$$

where $\theta = (\theta_1, \theta_2)$ and

$$\sigma^2(\theta) = \sum_{i=1}^{2}\left(\frac{\partial A_\infty}{\partial \theta_i}\right)^2 \sigma_{ii} = \frac{6\theta_1^2\theta_2^4}{(\theta_1 + \theta_2)^6}$$

Consequently $\hat{A}_\infty$ is a CAN estimator of A_∞:

Let $\sigma^2(\hat{\theta})$ be the estimator of $\sigma^2(\theta)$ obtained by replacing θ by a consistent estimator $\hat{\theta}$ namely:

$$\hat{\theta} = \left(\overline{X}, \ \frac{\overline{Y}}{2} \right).$$

Moreover, let $\hat{\theta}^2 = \sigma^2(\hat{\theta})$. Since $\sigma^2(\theta)$ is a continuous function of θ, $\hat{\sigma}_2^2$ is a consistent estimator of $\sigma^2(\theta)$, i.e. $\hat{\sigma}_2^2 \xrightarrow{p} \sigma^2(\theta)$ as n→∞.

By Slutskey's theorem

$$\frac{\sqrt{n}\left(\hat{A}_\infty - A_\infty\right)}{\hat{\sigma}} \xrightarrow{d} N(0, 1)$$

That is,

$$\mathrm{P}\left[-k_{\frac{\alpha}{2}} < \frac{\sqrt{n\left(\hat{A}_\infty - A_\infty\right)}}{\hat{\sigma}} < k_{\frac{\alpha}{2}} \right] = (1-\alpha),$$

where $k_{\frac{\alpha}{2}}$ is obtainable from normal tables. Hence, a 100 (1 – α) % asymptotic confidence interval for A_∞ is given by $\hat{A}_\infty \pm k_{\frac{\alpha}{2}} . \frac{\sigma}{\sqrt{n}}$

2.5 MLE OF SYSTEM RELIABILITY

Since $\overline{X}$ and $\frac{\overline{Y}}{2}$ are the MLE's of $\frac{1}{\lambda}$ and $\frac{1}{\mu}$ respectively, we obtain by

applying a method given in Zacks (1972), the MLE of system reliability as

$$\hat{R}(t) = \sum_{i=1}^{3} \frac{\left[\left(\hat{\alpha}_i \overline{Y} + 4\overline{X} + 3\overline{Y}\right) + 8\overline{Y}\right]}{\overline{X}\,\overline{Y}^2 \prod_{j=1}^{3} \left(\hat{\alpha}_{i-} \hat{\alpha} j\right)} e^{\alpha \hat{i}} t,$$

where $\hat{\alpha}_{1,} \hat{\alpha}_2 and \hat{\alpha}_3$ are the roots of the cubic equation

$$\left(\overline{X}\,\overline{Y}\right)^2 s^3 + \overline{X}\,\overline{Y}\left(8\overline{X} + 3\overline{Y}\right)s^2 + \left(16\overline{X}^2 + 20\overline{X}\,\overline{Y} + 2\overline{Y}^2\right)s + 8\overline{Y} = 0.$$

2.6 NUMERICAL ILLUSTRATION

For $A_\infty = \dfrac{\mu(2\lambda + \mu)}{(\lambda + \mu)^2}$

When λ = 0.01, 0.015, 0.02, 0.025 and μ = 0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.35.

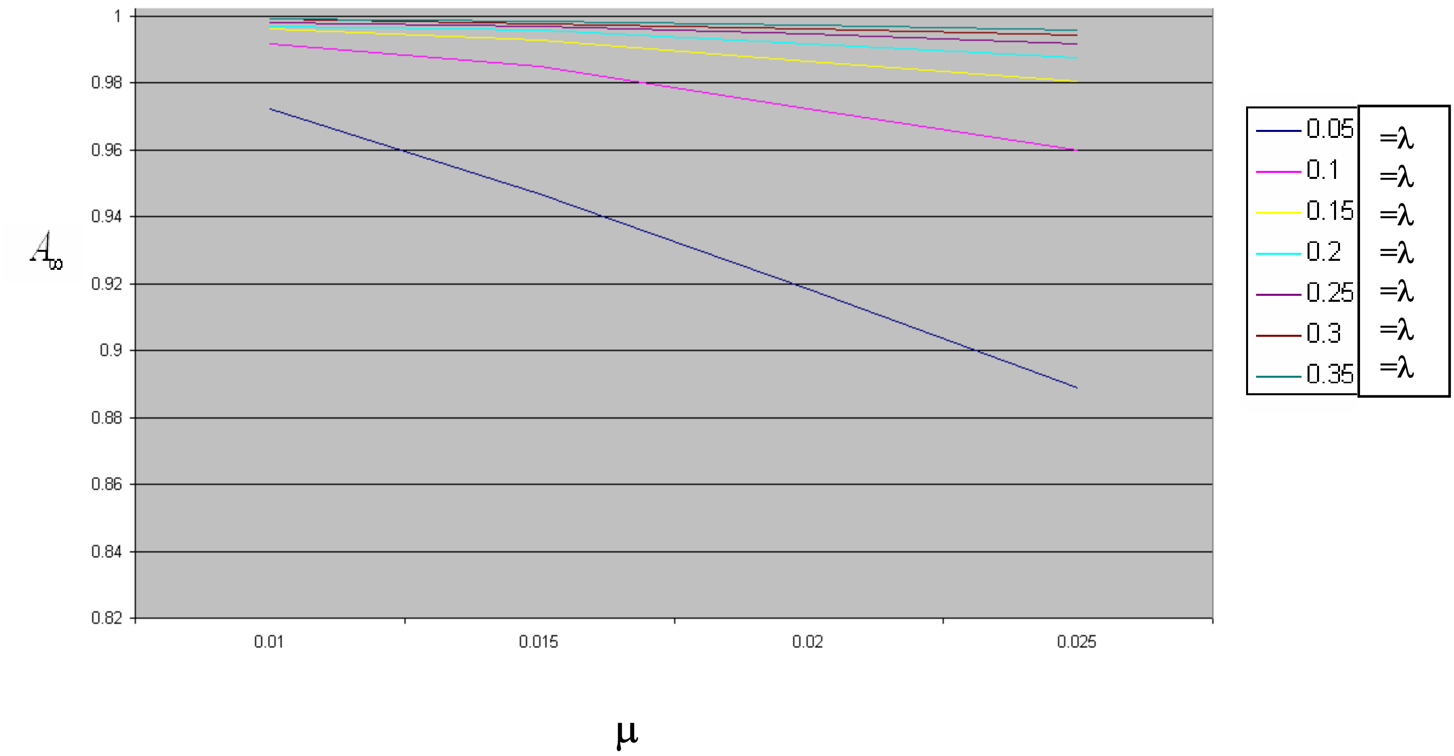

Figure 2.1

The λ and μ values are chosen from an exponential data available (Yadavalli et al, 2005)

From Figure 2.1, it is observed that as repair time increases, the steady state availability decreases.

Table 2.1 : CONFIDENCE INTERVALS FOR THE MODEL

For

λ=0.10

n	μ	95% CI	99% CI
100	0.01	(0.8101,0.9991)	(0.7811,0.9999)
	0.015	(0.7006,0.8192)	(0.6933,0.8399)
	0.02	(0.5218,0.6368)	(0.6066,0.6566)
	0.025	(0.5006,0.6019)	(0.4888,0.5771)
200	0.01	(0.8332,0.9673)	(0.8206,0,9709)
	0.015	(0.7161,0.8006)	(0.7988,0.8113)
	0.02	(0.6314,0.7091)	(0.6111,0.7108)
	0.025	(0.5822,0.6641)	(0.5316,0.5669)
2000	0.01	(0.8608,0.8992)	(0.8541,0.9053)
	0.015	0.6879,0.7227)	(0.6790,0.7306)
	0.02	0.6041,0.6330)	(0.5991,0.6376)
	0.025	(0.5911,0.6130)	(0.5444,0.5619)

Table 2.1 presents the 95% and 99% confidence intervals for different simulated samples. It can be observed that, as n increases, the steady state availability decreases.

CONCLUSION:

A two-unit system with Erlangian repair time is studied in this chapter. The system of simultaneously differential equations is developed to obtain the availabilities analytically. The asymptotic confidence limits for steady state availability are studied at the end of this chapter. A numerical example illustrated the results. The results show that, as n increases A_∞ decreases.

CHAPTER 3

AN n UNIT SYSTEM OPERATING IN A RANDOM ENVIRONMENT

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3.1 INTRODUCTION

In the probabilistic analysis of multi-unit redundant systems it is usually assumed that a constant number of units perform the system operation at all times. However, we have situations in which this assumption is not true. For example, to increase the thermal power plant availability, an additional induced draft fan (ID fan) may be installed in 200 MW sets, though two ID fans are normally used to handle flue gas and fly ash during full load operation of the plant, i.e., the load on a system may change randomly (see Das and Acharya, 1988). Again, in a telecommunication network, the success of sending a message from an origin to a destination depends upon the existence of at least one path connecting the origin to a destination depends upon the existence of at least one path connecting the origin with the destination with all units determining the path in the operable state. Therefore, the number of units required for sending the message successfully at any time is determined by the availability of units in the intermediate stations and the locations of the origin and destination. Hence the number of units required for the satisfactory performance of the system may depend on the environment in which the system is functioning and the environment is also changing with time.

Sharafali et al (1988) have considered a two-unit n system with similar assumption and obtained expressions for the mean time to the first disappointment and expected number of disappointments in an interval. (see Limnios and Cocozza (1992)).

An attempt is made in this chapter to study a system consisting of n units with the assumption that the number of units required for the satisfactory performance of the system at any time t is prescribed by

the state of a randomly changing environment described by a Markov process $\{Y(t): t \geq 0\}$. The model is discussed in detail in the following section.

3.2 THE MODEL AND ASSUMPTIONS

The system under consideration is an n unit system with a single repair facility. Precisely; the assumptions of the model are as follows:

(i) There are n identical units in the system, which are statistically independent. The failure rate of an operable unit during the need period is a constant.

(ii) The environment determining the number of units required for the satisfactory performance of the system at any time t is a Markov process $\{Y(t): t \geq 0\}$ with the state space $\{0,1,2,\ldots,n\}$.It may be noted that the environment process is independent of the system behaviour.

(iii) The infinitesimal generator of the environment process $\{ Y(t) : t \geq 0 \}$ is given by:

$$A = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ . \\ . \\ . \\ \\ n \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & \ldots & n \end{array} \\ \begin{bmatrix} -\lambda_0 & \lambda_{01} & \lambda_{02} & \ldots & \lambda_{0n} \\ \lambda_{10} & -\lambda_1 & \lambda_{12} & \ldots & \lambda_{1n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ & & & & \\ & & & & \\ & & & & \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \lambda_{n0} & \lambda_{n1} & \lambda_{n2} & \ldots & -\lambda_n \end{bmatrix} \end{array}$$

(iv) If any time t, Y (t) = i, then i (i = 0,1,….,n) of the n identical units are online (if operable) and the remaining operable units will be kept as warm standbys. These i units which online behave like a series system.

(v) Whenever an online unit fails a standby unit if operable is switched online instantaneously.

(vi) A unit in standby can also fail and its failure rate is a constant.

(vii) The failed units are taken up for repair in FIFO order. However, a repair for a failed unit cannot commence, when the environment process is in state zero. Repair is perfect and the repair rate is a constant ‘μ’.

(viii) Whenever the number of units in the operable state is less than the number of units required at that instant of time for the satisfactory performance of the system, the system enters the down state.

(ix) When the system is in the down state, an operable unit cannot fail.

3.3 THE NOTATION

n	number of units in the system
a	constant failure rate of an operable unit during the need period
b	constant failure rate of a standby unit (b < a)
μ	constant repair rate of a failed unit
$\{Y(t): t \geq 0\}$	state of the environment process that determines the number of units required for the satisfactory performance of the system
Matrix A	the infinitesimal generator of $\{Y(t): t \geq 0\}$
$X(t)$	number of failed units in the system at time t
E and Q	respectively the state space and infinitesimal generator of the Markov Process $\{(X(t), Y(t)): t \geq 0\}$

3.4 STATE OF THE SYSTEM

Let X (t) represent the number of failed units at time t and Y(t), the number of units required for the satisfactory performance of the system at time t. Clearly {(X (t), Y (t)): t≥0} is a Markov Process on the state space:

$$E = E_0 \cup E_1 \cup E_2 \cup \ldots \cup E_n$$

where

E_i = {(i , 0),(i , 1),…,(i , n)}, i = 0, 1, 2,. . ., n.

Let $\Delta(\underline{e}_j)$ be diagonal matrix of order (n + 1) with the first leading (j + 1) diagonal elements being the integers 0, 1, 2,…, j and the remaining elements zero. That is:

$$\Delta(\underline{e}_j) = \text{diag}\,(0, 1, 2, j-1, j, \ldots, 0, 0, 0),\ j = 1,2,\ldots,n$$

Also, let $\Delta(\underline{f}_i)$ be a diagonal matrix of order (n + 1) with the first leading (n-i+1) diagonal elements being the integers (n – i) , (n – i – 1), (n – i – 2), . . ., 2, 1, 0 and the remaining elements
(n-i). that is:

$$\Delta(\underline{f}_i) = \text{diag}\,(n-i, n-i-1, n-i-2, \ldots, 2, 1, 0)$$

Then, the infinitesimal generator of this process is easily seen to be:

$$E_o \quad E_1 \quad E_2 \quad \ldots \quad E_j \quad \ldots \quad E_n$$

$$
\mathbf{Q} = \begin{array}{c} E_0 \\ E_1 \\ E_2 \\ . \\ . \\ . \\ E_i \\ . \\ . \\ . \\ E_n \end{array}
\begin{bmatrix}
\tilde{Q}_{00} & \tilde{Q}_{01} & \tilde{Q}_{02} & \cdots & \tilde{Q}_{oj} & \cdots & \tilde{Q}_{on} \\
\tilde{Q}_{10} & \tilde{Q}_{11} & \tilde{Q}_{12} & \cdots & \tilde{Q}_{1j} & \cdots & \tilde{Q}_{1n} \\
\tilde{Q}_{20} & \tilde{Q}_{21} & \tilde{Q}_{22} & \cdots & \tilde{Q}_{2j} & \cdots & \tilde{Q}_{2n} \\
\cdots & & & & & & \\
 & & & & & & \\
 & & & & & & \\
\tilde{Q}_{i0} & \tilde{Q}_{i1} & \tilde{Q}_{i2} & \cdots & \tilde{Q}_{ij} & \cdots & \tilde{Q}_{in} \\
\cdots & & & & & & \\
 & & & & & & \\
 & & & & & & \\
\tilde{Q}_{n0} & \tilde{Q}_{n1} & \tilde{Q}_{n2} & \cdots & \tilde{Q}_{nj} & \cdots & Q_{nn}
\end{bmatrix} \tag{3.1}
$$

where the partitioned matrices $\tilde{Q}_{ij}$ are given by:

$\tilde{Q}_{00} = \mathrm{A} - \mathrm{a}\,\Delta(\underline{e}_n) - \mathrm{b}\,\Delta(\underline{f}_0)$,

$\tilde{Q}_{nn} = \mathrm{A} - \mu\mathrm{I}$

$\tilde{Q}_{i,i} = \mu\mathrm{I}$ for $i = 1, 2, \ldots, n$,

$\tilde{Q}_{i,i} = \mathrm{A} - \mu\mathrm{I} - \mathrm{a}\,\Delta(\underline{e}_{n-i}) - \mathrm{b}\,\Delta(\underline{f}_i)$,

$\tilde{Q}_{i,i+1} = a\Delta(\underline{e}_{n-i}) + b\Delta(\underline{f}_i)$

and

$\tilde{Q}_{i,j} = 0$, for other values of i and j.

It may be noted that Q is a square matrix of order $(n + 1)^2 \times (n + 1)^2$. Let

$$p_{ij}(t) = \Pr\{X(t) = i;\ Y(t) = j\};\ \ i, j = 0,\ 1,\ 2, \ldots, n$$

represent the probability that the system is in state (i, j) at time t. Also let:

$$\underline{\mathrm{p}}(\mathrm{t}) = (p_{00}(t),\ \ p_{01}(t),\ \ p_{nm}(t))$$

To derive an expression for $\underline{\mathrm{p}}(\mathrm{t})$, we note that $\underline{\mathrm{p}}(\mathrm{t})$ satisfies the Kolmogorov equation which leads to:

$$\frac{\mathrm{d}\underline{\mathrm{p}}(\mathrm{t})}{\mathrm{dt}} = \underline{\mathrm{p}}(\mathrm{t}).\mathrm{Q}$$

Solving this differential equation, we obtain:

$$\underline{\mathrm{p}}(\mathrm{t}) = \underline{\mathrm{p}}(0).\mathrm{e}^{\mathrm{Qt}} \tag{3.2}$$

where

$\underline{\mathrm{p}}(0)$ is the vector of initial state probabilities.

3.5 STATIONARY DISTRIBUTION

Let $\underline{\pi} = (\underline{\pi}_0, \underline{\pi}_1, \underline{\pi}_2, \ldots, \underline{\pi}_n)$ where $\underline{\pi}_k = (\pi_{k0,}\ \pi_{k1,}\ \ldots,\ \pi_{kn})$ for k = 0, 1, 2, n is the stationary distribution corresponding to the Markov process $\{(X\ (t),\ Y\ (t)):\ t \geq 0\}$. This is the solution of the equation:

$$\underline{\pi}\,\mathbf{Q} = \underline{0} \tag{3.3}$$

with

$$\sum_{k=0}^{n} \underline{\pi}_k\, \underline{e} = 1 \tag{3.4}$$

where

$\underline{\mathbf{e}} = (1,1,1)^T$.

Equation (3.3) gives:

$$\underline{\pi}_0\left(A - a\Delta(\underline{e}_n) - b\Delta(\underline{f}_0)\right) + \mu\underline{\pi}_1 = \underline{0} \tag{3.5}$$

$$\underline{\pi}_0\left(a\Delta(\underline{e}_n) + b\Delta(\underline{f}_0)\right) + \underline{\pi}_1\left(A - \mu I - a\Delta(\underline{e}_{n-1}) - b\Delta(\underline{f}_1) + \mu\underline{\pi}_2\right) = \underline{0} \tag{3.6}$$

$$\underline{\pi}_1\left(a\Delta(\underline{e}_{n-1}) + b\Delta(\underline{f}_1)\right) + \underline{\pi}_2\left(A - \mu I - a\Delta(\underline{e}_{n-2}) - b\Delta(\underline{f}_2)\right) + \mu\underline{\pi}_3 = 0 \tag{3.7}$$

$$\cdots$$

$$\underline{\pi}_{n-2}\left(a\Delta(\underline{e}_2) + b\Delta(\underline{f}_{n-2})\right) + \underline{\pi}_{n-1}\left(A - \mu I - a\Delta(\underline{e}_1) - b\Delta(\underline{f}_{n-1})\right) + \mu\underline{\pi}_n = \underline{0} \tag{3.8}$$

$$\underline{\pi}_{n-1}\left(a\Delta(\underline{e}_1) + b\Delta(\underline{f}_{n-1})\right) + \underline{\pi}_n\left(A - \mu I\right) = \underline{0} \tag{3.9}$$

Addition of all these equations in (3.5) – (3.9) yields $(\underline{\pi}_0 + \underline{\pi}_1 + ... + \underline{\pi}_n)$ A = 0. These, together with equation (3.4) implies that $(\underline{\pi}_0 + \underline{\pi}_1 + ... + \underline{\pi}_n)$ must be the invariant measure of the Markov process {(X(t),Y(t)):t ≥ 0} with the generator A. Assume that a possess the invariant measure and let it be η. Hence:

$$\underline{\pi}_0 + \underline{\pi}_1 + ... + \underline{\pi}_n = \underline{\eta} \tag{3.10}$$

We can express $\underline{\pi}_0, \underline{\pi}_1, . . ., \underline{\pi}_n$ in terms of $\underline{\pi}_0$ by solving (3.5) – (3.9). Using equation (3.10), we can get explicit expression for ($\underline{\pi}_0, \underline{\pi}_1, . . ., \underline{\pi}_n$).

3.6 TIME TO THE FIRST DISAPPOINTMENT

The system is said to be in a state of disappointment if the number of operable units at any time is less than the number of units required for the satisfactory performance of the system at that instant of time. i.e.,
n – X (t) < Y (t). In other words, X (t) + Y (t) > n.

Clearly, the set of states of disappointment is:

D ={(1,n), (2,n-1), (2,n), (3,n-2), (3,n-1), (3,n), (n,1), (n,2), (n,n-1), (n,n)}

Let U be the set of upstates, which is the complement of D. By suitably altering the rows and coloumns, we can partition the matrix Q as:

$$\mathbf{Q} = \begin{array}{c} \\ U \\ D \end{array}\!\!\begin{array}{c} \begin{array}{cc} U & D \end{array} \\ \begin{bmatrix} Q_U & B_D \\ B_U & Q_D \end{bmatrix} \end{array} \tag{3.11}$$

Let T_D represent the time to the first disappointment. To obtain the distribution of the random variable T_D, we lump together the states of disappointment of the Markov Process $\{(X(t), Y(t)): t \geq 0\}$ into a single absorbing state D. We obtain the absorbing Markov Process with generator:

$$\mathbf{Q}' = \begin{bmatrix} Q_U & B_D \underline{e} \\ \underline{0} & 0 \end{bmatrix} \tag{3.12}$$

Let us assume that the process starts in a state in U and so let $\tilde{P}_U(0)$ be the row vector of the initial state probabilities corresponding to this situation. Now the time to the first disappointment is the same as the time to absorption in the Markov process with the generator Q′ given in (3.12). If $G_D(t)$ is the distribution function of the random variables T_D, then:

$$G_D(t) = 1 - \tilde{P}_U(0)\, e^{Q_U T}\, \underline{e}, t>0 \tag{3.13}$$

It may be noted that the distribution function $G_D(t)$ given in (3.13) corresponds to the distribution function of a continuous PH-distribution with representation ($\tilde{P}_U(0), Q_U$) (See Neuts, 1981).

The raw moments are given by:

$$E\left(T_D^K\right)=\left(-1\right)\, k!\,\widetilde{P}_U\left(0\right)\, Q_U^{-k}\, \underline{e}, k=0,1,2 \qquad (3.14)$$

3.7 MEAN NUMBER OF DISAPPOINTMENTS

To derive an expression for the mean number of disappointments in an arbitrary interval (0, t], we consider the point process generated by the events corresponding to the occurrence of a disappointment. Let $h_D(.)$ be the first order product density of a disappointment (See Srinivasan, 1974). Then $h_D(t)dt$ is the probability that a disappointment occurs in
(t, t + dt). By considering the various possibilities of entering into the states of disappointment, we have:

$$h_D\left(t\right)=a\sum_{i=1}^{n} ip_{(n-i),i}\left(t\right)+\sum_{j=1}^{n}\sum_{i=0}^{n-j}\sum_{k=1}^{j}\lambda_{i,n-j+k}P_{ji}\left(t\right) \qquad (3.15)$$

where $P_{ij}\left(t\right)$ can be obtained from (3.2).

The above result is in agreement with Chandrasekhar and Natarajan (1999). It may be noted that $h_D(t)$ given in (3.15) is independent of the constant failure rate b of the standby unit.

The expected value of N (D, t), the number of disappointments in (0, t] is given by :

$$E\left[N\left(D,t\right)\right]=\int_0^t h_D\left(u\right)du \qquad (3.16)$$

3.8 MEAN STATIONARY RATE OF DISAPPOINTMENTS

The mean stationary rate of disappointments is given by:

$$E[N(D,t)] = \lim_{t \to \infty} h_D(t)$$

and can be easily obtained from (15) by replacing $p_{ki}(t)$ by π_{ki}.

3.9 CONCLUSIONS

This chapter is a study of a more general system in the sense that the results corresponding to several systems can be deduced as special cases as shown below.

3.9.1 Two unit system

For n = 2, we have:

$$h_D(t) = 2ap_{02}(t) + \lambda p_{10}(t) + (\lambda_{12} + a)p_{11}(t) + \lambda_0 p_{20}(t)$$

This result is in agreement with Sharafali *et al*. (1988).

3.9.2 Intermittently used n unit standby redundant system (Yadavalli,1982)

We observe that the results corresponding to an intermittently used n unit standby redundant system can be obtained as a particular case of the model discussed in this paper by taking the state space of the stochastic process $\{Y(t): t \geq 0\}$ to be consisting of only two states 0 and 1 representing the ‘need’ and ‘no need’ states respectively.

CHAPTER 4

BUSY PERIOD ANALYSIS OF A TWO UNIT SYSTEM WITH PREVENTITIVE MAINTENANCE AND IMPERFECT SWITCH

4.1 INTRODUCTION

To increase the effectiveness of a system, a unit that has failed is renewed. The renewal can assume various forms. Several authors carried analyses of systems with two or three modes under the assumption that whenever the operative unit fails, it goes to repair immediately and after the completion of repair the server goes off. Srinivasan and Gopalan (1973) studied a two-unit system with warm standby and a single repair facility. Murari and Goyal (1984) made a comparison of two-unit cold standby reliability models; in

“Model 1” the repairman always remains with the system after the failure of the unit. Goel and Sharma (1989) analysed a two-unit standby system with two failure modes and slow switching. Makaddis (1999) considered a system with three modes and an administrative delay in repair. However, in practice, there may be occasions when the repairman appears in and disappears from the system randomly with some probabilities. In this Chapter, two models have been studied. In both the models, a single-unit repairable system with three possible modes of the unit- normal (N), partial failure (P) and total failure (F) – is examined.

In “Model 1”, if the repairman finds the unit in P-mode, then he takes the unit under repair while the unit is operating, whereas in “Model 2”, the partially failed unit does not go under repair but repair is started only when the unit fails completely. Using the regeneration point technique, the various measures of system performance such as MTSF, steady state availability(A_∞), busy period analysis of the repair facility, expected number of visits by a repairman, and the profit analysis, are studied, for each model. Numerical example illustrated some of the results obtained. Two models have been compared on the basis of

numerical results by carrying out MTSF and profit analysis for a particular case when repair time distributions are exponential.

4.2 SYSTEM DESCRIPTION

1. The system consists of a single unit. Initially, we assume that the unit is operating. The unit fails through a partial failure.

2. There is a single repair facility which appears in and disappears from the system randomly.
3. The life time of a unit and time of appearance and disappearance of the repairman are negative exponential, whereas the repair times are arbitrarily distributed.
4. The repairman cannot leave the system while repairing the unit.
5. Switch is perfect and the switchover is instantaneous.

4.3 NOTATION

E_0	state of the system at $t=0$
E	set of regeneration states $(S_0 - S_5)$ for each model
$\overline{E}$	set of non-regenerative states for each model
a, b	The rates of appearance and disappearance of repairman
$g_i(t), G_i(t)$	pdf and cdf of the repair time in phase $i = 1, 2$.
$q_{ij}(t), Q_{ij}(t)$	pdf and cdf of time for one-step transition from regenerative states S_i to S_j.

$\Phi_i(t)$	cdf of time to system failure where starting state $E_0 = S_i \in E$
$A_i(t)$	P [system is up at t $\mid$ S_i at t = 0]
$B_i(t)$	P [repairman is busy in repair at t $\mid$ S_i at t = 0]
$N_i(t)$	Expected number of visits by the repairman to state i in $(0,t]$
A, NA	repair facility is available/not available.
λ_1	failure rate from N-mode to P-mode
$M_i(t)$	P [system is up at t without passing through any regenerative state or returning to itself $\mid$ S_i at $t = 0$]
μ_i	mean sojourn time in state S_i
L.S.T	Laplace-Stieltjes transform

L.T Laplace transforms

$$\widetilde{Q}_{ij}(s)=\int_0^\infty e^{-st}\,dQ_{ij}(t)$$

$$q*_{ij}(s)=\int_0^\infty e^{-st}q_{ij}(t)dt$$

$$\mu_i=\sum_i t\,d\,Q_{ij}(t)=-\sum_j q_{ij}^{*'}(0)=-\sum_j Q_{ij}'(0)$$

$$=\sum_j m_{ij}\quad i=1,2,\ldots$$

m_{ij} = *contribution to mean sojourn time in state* S_i *when the transition to* $S_j=-\widetilde{Q}_{ij}(0)=-q_{ij}^{*'}(0)$

Ⓢ Laplace-Stieltjes convolution

© Laplace convolution

4.4 SYMBOLS FOR STATES OF THE SYSTEM

N_0 : Unit in N-mode, and operative

P : Unit in partial failure mode

P_r : Unit in partial failure mode and under repair

F_r : Unit in complete failure and under repair

F : Unit in failure mode

A, NA : repairman is available / not available

W_r : Unit waiting for repair

Using the above symbols the system may be in one of the following states:

$S_0 = (N_{0,} NA)$, $S_1 = (N_{0,} A)$, $S_2 = (P, NA)$.

$S_3 = (P_r, A)$, $S_4 = (W_r, NA)$.

$S_5 = (F_r, A)$ for "Model 1".

$S_0 = (N_0, NA)$,

$S_1 = (N_0, A)$,

$S_2 = (P, NA)$,

$S_3 = (P, A)$,

$S_4 = (W_r, NA)$,

$S_5 = (F_r, A)$ (for "Model 2")

4.5 RELIABILITY ANALYSIS (MODEL 1)

Here the repairman repairs the unit only when it is P-mode (Figure 4.1).Meantime to system failure analysis gives:

$$\Phi_0(t) = Q_{01}(t) \circledS \Phi_1(t) + Q_{02}(t) \circledS \Phi_2(t)$$

$$\Phi_1(t) = Q_{10}(t) \circledS \Phi_0(t) + Q_{13}(t) \circledS \Phi_3(t)$$

$$\Phi_2(t) = Q_{23}(t) \circledS \Phi_3(t) + Q_{24}(t)$$

$$\Phi_3(t) = Q_{31}(t) \circledS \Phi_1(t) + Q_{35}(t) \tag{4.1}$$

$$d\ Q_{01}(t) = ae^{-at} e^{-\lambda_1 t} dt\ ;$$

$$d\ Q_{02}(t) = \lambda_1 e^{-\lambda_1 t} dt\ .$$

$$dQ_{10}(t) = be^{-bt} e^{-\lambda_1 t} dt;$$

$$dQ_{23}(t) = \lambda_2 e^{-\lambda_2 t} dt\,.$$

$$dQ_{24}(t) = \lambda_2 e^{-\lambda_2 t} e^{-at} dt;$$

$$dQ_{35}(t) = \lambda_2 e^{-\lambda_2 t} \overline{G}(t) dt.$$

$$dQ_{31}(t) = g_1(t) e^{-\lambda_2 t} dt;$$

$$dQ_{45}(t) = a e^{-at} dt;$$

$$dQ_{51}(t) = g_2(t) dt \qquad (4.2)$$

Letting t $\to\infty$, using $Q_{ij}(\infty) = P_{ij}$,

$$p_{01} = \frac{a}{a+\lambda_1}; \qquad p_{02} = \frac{\lambda_1}{a+\lambda_1}.$$

$$p_{10} = \frac{b}{b+\lambda_1}; \qquad p_{13} = \frac{\lambda_1}{b+\lambda_1}.$$

$$p_{23} = \frac{a}{a+\lambda_2}; \qquad p_{24} = \frac{\lambda_2}{a+\lambda_2}$$

$$p_{35} = 1 - g_1^*(\lambda_2); \qquad p_{31} = g_1^*(\lambda_2);$$

$$p_{45} = 1 = P_{51} \tag{4.3}$$

$$\mu_0 = \int_0^{\infty} P(T < t)dt = \frac{1}{a + \lambda_1}$$

$$\mu_1 = \frac{1}{b + \lambda_1}$$

Figure 4.1 (Model 1)

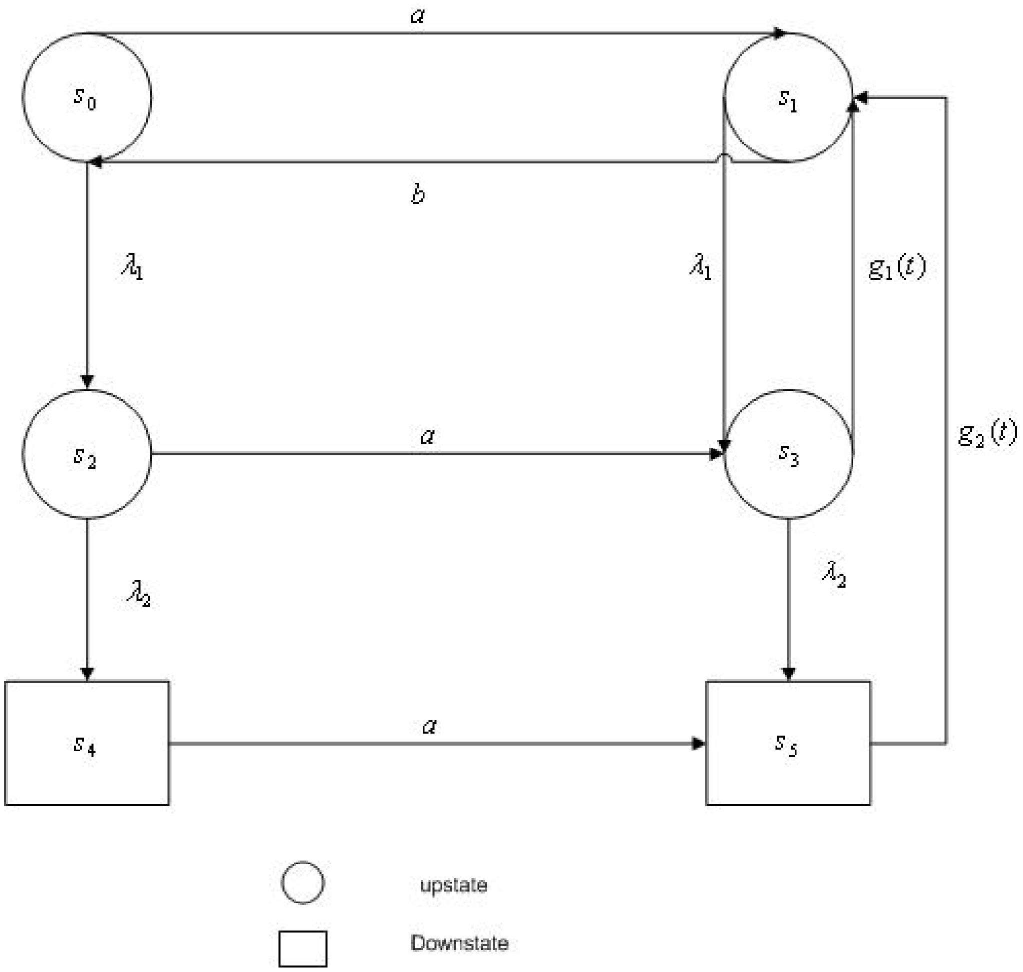
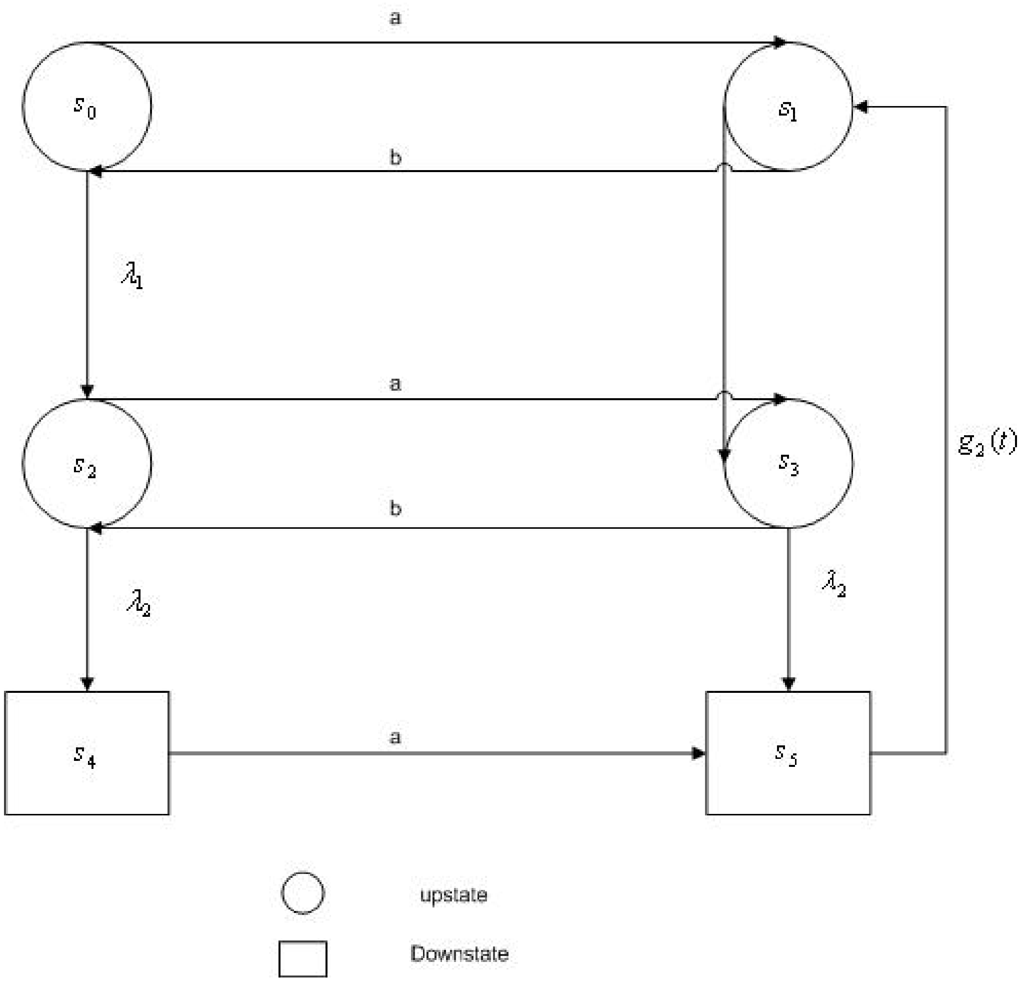


Figure 4.2 (Model 2)



$$\mu_2 = \frac{1}{a+\lambda_2} \quad , \qquad \mu_3 = \frac{1-g^*(\lambda_2)}{\lambda_2}$$

$$\mu_4 = \frac{1}{a} \quad , \qquad \mu_5 = \int_0^\infty g_2(t)dt \tag{4.4}$$

It can easily be verified that

$$p_{01} + p_{02} = p_{13} + p_{10} = p_{23} + p_{24} = p_{31} + p_{35} = p_{45} = p_{51}.$$

$$\mu_0 = m_{01} + m_{02}; \mu_1 = m_{10} + m_{13}; \mu_2 = m_{24} + m_{23.}$$

$$\mu_3 = m_{31} + m_{35}; \mu_4 = m_{45}; \mu_5 = m_{51}.$$

Taking Laplace-Stieltjies transform of these relations and solving for

$\tilde{\Phi}_0(s)$, we have

$$\int \tilde{\Phi}_0(s) = \frac{N_1(s)}{D_1(s)} \qquad (4.5)$$

where

$$N_1(s) = \tilde{Q}_{01}(s)\tilde{Q}_{13}(s)\tilde{Q}_{35}(s) + \tilde{Q}_{02}(s)\{\tilde{Q}_{24}(s)\left(1 - \tilde{Q}_{13}(s)\tilde{Q}_{31}(s) + \tilde{Q}_{23}(s)\tilde{Q}_{35}(s)\right)\}.$$

$$D_1(s) = 1 - \tilde{Q}_{13}(s)\tilde{Q}_{31}(s) - \tilde{Q}_{01}(s)\tilde{Q}_{10}(s) - \tilde{Q}_{23}(s)\tilde{Q}_{31}(s)\tilde{Q}_{10}(s)\tilde{Q}_{02}(s)$$

The mean time to system failure is found to be (MTSF)

$$T_1 = \lim_{s \to 0} \frac{1-\tilde{\Phi}_0(s)}{s} \tag{4.6}$$

$$T_1 = \frac{N_1}{D_1}$$

$$N_1 = \mu_0(1 - p_{13}p_{31}) + \mu_1(p_{01} + p_{23}p_{31}p_{02}) + \mu_2\{p_{02}(1 - p_{13}p_{23})\} + \mu_3(p_{23}p_{02} + p_{01}p_{13})$$

$$D_1 = 1 - p_{13}p_{31} - p_{01}p_{10} - p_{23}p_{31}p_{10}p_{02}$$

4.6 AVAILABILITY ANALYSIS

$A_i(t) =$ P [system is up at t | S_i at t = 0],

then

$$A_0(t) = M_0(t) + q_{01}(t) © A_1(t) + q_{02}(t) © A_2(t)$$

$$A_1(t) = M_1(t) + q_{10}(t) © A_0(t) + q_{13}(t) © A_3(t)$$

$$A_2(t) = M_2(t) + q_{23}(t) © A_3(t) + q_{24}(t) © A_4(t)$$

$$A_3(t) = M_3(t) + q_{31}(t) © A_1(t) + q_{35}(t) © A_5(t)$$

$$A_4(t) = q_{45}(t) © A_5(t)$$

$$A_5(t) = q_{51}(t) © A_1(t) \tag{4.7}$$

where

$$M_0(t) = e^{-(a+\lambda_1)t}$$

$$M_1(t) = e^{-(b+\lambda_1)t}$$

$$M_2(t) = e^{-(a+\lambda_2)t}$$

$$M_3(t) = e^{-\lambda_2 t}\,\overline{G}(t)$$

Using the Laplace-transforms, we obtain

$$A_0^*(s) = \frac{N_2(s)}{D_2(s)}$$

$$N_2(s) = M_0^*(s)\left[1 - q_{13}^*(s)q_{31}^*(s) - q_{13}^*(s)q_{35}^*(s)q_{51}^*(s)\right]$$

$$+ M_1^*(s)\begin{bmatrix} q_{02}^*(s)q_{23}^*(s)q_{35}^*(s)q_{51}^*(s) + q_{02}^*(s)q_{24}^*(s)q_{45}^*(s)q_{51}^*(s) + q_{01}^*(s) + q_{01}^*(s) \\ + q_{02}^*(s)q_{23}^*(s)q_{31}^*(s) \end{bmatrix}$$

$$+ M_2^*(s)\left[q_{02}^*(s)\left\{1 - q_{13}^*(s)q_{31}^*(s)q_{13}^*(s)q_{35}^*(s)q_{51}^*(s)\right\}\right]$$

$$+ M_3^*(s)\left[q_{02}^*(s)q_{24}^*(s)q_{45}^*(s)q_{51}^*(s)q_{13}^*(s) + q_{01}^*(s)q_{13}^*(s) + q_{03}^*(s)q_{23}^*(s)\right]$$

and

$$D_2(s) = 1 - q_{13}^*(s)q_{31}^*(s) - q_{35}^*(s)q_{51}^*(s)q_{13}^*(s) - q_{10}^*(s)q_{01}^*(s) -$$
$$q_{10}^*(s)q_{02}^*(s)q_{23}^*(s)\left\{q_{31}^*(s) + q_{35}^*(s)q_{51}^*(s)\right\}$$
$$- q_{45}^*(s)q_{51}^*(s)q_{10}^*(s)q_{01}^*(s)q_{24}^*(s)$$

For $A_0^*(s)$, we can obtain the steady state availability , A_∞

$$A_\infty = \lim_{s\to 0} s\, A_0^*(s) = \frac{N_2}{D_2}$$

where

$$N_2 = (\mu_0 + \mu_2 p_{02})p_{10} + \mu_1 + \mu_3 [p_{02}\{p_{24}p_{13} + p_{23}\} + p_{01}p_{13}]$$

$$D_2 = (\mu_0 + \mu_2 p_{02})p_{10} + \mu_1 + \mu_2(p_{13} + p_{10}p_{02}p_{23})$$
$$+ \mu_4 p_{10}p_{02}p_{24} + \mu_5[p_{13}p_{35} + p_{10}p_{02}(1 - p_{23}p_{31})]$$

4.7 BUSY PERIOD ANALYSIS

$B_i(t) =$ P [the repairman is busy at t | S_i at t = 0]

$$B_0(t) = q_{01}(t) © B_1(t) + q_{02}(t) © B_2(t)$$

$$B_1(t) = q_{10}(t) © B_0(t) + q_{13}(t) © B_3(t)$$

$$B_2(t) = q_{23}(t) © B_3(t) + q_{24}(t) © B_4(t)$$

$$B_3(t) = M_3(t) + q_{31}(t) © A_1(t) + q_{35}(t) © B_5(t)$$

$$B_4(t) = q_{45}(t) © B_5(t)$$

$$B_5(t) = M_5(t) + q_{51}(t) © B_1(t)$$

where

$$M_3(t) = e^{-\lambda_2 t}\,\overline{G_1}(t)$$

$$M_5(t) = \overline{G}_2(t)$$

Using the Laplace transforms, we can find

$$B_0^*(s) = \frac{N_3(s)}{D_2(s)}$$

where

$$N_3(s) = M_3^*(s)\left[q_{01}^*(s)q_{01}^*(s)q_{13}^*(s) + q_{02}^*(s)q_{45}^*(s)q_{13}^*(s)\right]$$

$$+ M_5^*(s)\left[q_{35}^*(s)\left(q_{13}^*(s)q_{01}^*(s) + q_{02}^*(s)q_{23}^*(s)\right) + q_{02}^*(s)q_{24}^*(s)q_{45}^*(s)\{1 - q_{13}^*(s)q_{31}^*(s)\}\right]$$

and $D_2(s)$ is already given earlier. In the long run, the function of time for which the system is under repair is

$$B_\infty = \lim_{t\to 0} B_0(t) = \lim_{s\to 0} s\,B_0^*(s)$$
$$= \frac{N_3}{D_2}$$

$$N_3 = \mu_3\left[p_{01}p_{13} + p_{02}\left(p_{23} + p_{24}p_{13}\right)\right] + \mu_5\left[p_{35}\left(p_{01}p_{13+}\, p_{02}p_{23}\right) + p_{02}p_{24}\left(1 - p_{13}p_{31}\right)\right]$$

4.8 EXPECTED NUMBER OF VISITS BY REPAIR FACILITY

The equations for $N_0(t), N_1(t), N_2(t), N_3(t)$ *and* $N_4(t)$ *are given by*

$$N_0(t)=Q_{01}(t)Ⓢ[1+N_1(t)]+Q_{02}(t)Ⓢ N_2(t)$$

$$N_1(t)=Q_{10}(t)Ⓢ N_0(t)+Q_{13}(t)Ⓢ N_3(t)$$

$$N_2(t)=Q_{23}(t)Ⓢ[1+N_3(t)]+Q_{24}(t)Ⓢ N_4(t)$$

$$N_3(t)=Q_{31}(t)Ⓢ N_1(t)+Q_{35}(t)Ⓢ N_5(t)$$

$$N_4(t)=Q_{45}(t)Ⓢ[1+N_5(t)]$$

$$N_5(t)=Q_{51}(t)Ⓢ N_1(t)$$

Taking L.S.T of these equations, and solving for $\tilde{N}_0(s)$, we get

$$\tilde{N}_0(s)=\frac{N_4(s)}{D_2(s)}.$$

$$N_4(s)=\left[\widetilde{Q}_{01}(s)+\widetilde{Q}_{02}(s)\left\{\widetilde{Q}_{23}(s)+\widetilde{Q}_{24}(s)\widetilde{Q}_{45}(s)\right\}\right]\left[1-\widetilde{Q}_{13}(s)\widetilde{Q}_{31}(s)-\widetilde{Q}_{51}(s)\widetilde{Q}_{13}(s)\widetilde{Q}_{35}(s\right.$$

In steady state the number of visits per unit time is

$$N_\infty=\lim_{t\to\infty}\frac{N_0(t)}{t}=\frac{\widetilde{N}_4}{D_2}$$

where $\widetilde{N}_4 = p_{10.}$

4.9 MODEL 2

Here the repairman repairs the unit only when it is in the F-mode (figure 4.2). The equations for $\Phi_0(t), \Phi_1(t)$ *and* $\Phi_2(t)$ are the same as in model 1.

The additional equation is

$$\Phi_3(t)=\Phi_{32}(t)\,Ⓢ\,\Phi_1(t)+Q_{35}(t)$$

Transition probabilities $Q_{01}(t), Q_{02}(t), Q_{13}(t), Q_{10}(t), Q_{23}(t), Q_{24}(t)\, and\, Q_{45}(t)$ are the same as in model 1. The additional probabilities are

$$dQ_{35}(t)=\lambda_2\, e^{-(b+\lambda_2)t}\,dt$$

$$dQ_{32}(t)=b\,e^{-(b+\lambda_2)t}\,dt$$

$$dQ_{51}(t)=g(t)dt$$

Letting $t\to\infty$ and using

$$Q_{ij}(\infty)=p_{ij} \quad ,we\ get$$

$$p_{35}=\frac{\lambda_2}{b+\lambda_2}\,,\ p_{32}=\frac{b}{b+\lambda_2}$$

It can be easily verified that

$$p_{01}+p_{02}=1=p_{10}+p_{13}=p_{23}+p_{24}=p_{35}+p_{32}=p_{45}=p_{51}$$

The mean sojourn times (μ_0,μ_1,μ_2) are the same as in model 1. The additional times are

$$\mu_3=\frac{1}{b+\lambda_2},\quad \mu_5=\int_0^\infty \overline{G}(t)dt.$$

Now , proceeding in a similar manner as in model 1, we have the MTSF as:

$$MTSF=\frac{N}{D}$$

where

$$N=(\mu_0+\mu_1 p_{01})(1-p_{32}p_{23})+\mu_2(p_{02}+p_{01}p_{13}p_{32})+\mu_1(p_{01}p_{13}+p_{02}p_{23})$$

$$D = (1 - p_{01} p_{10})(1 - p_{32} p_{23})$$

4.10 AVAILABILITY ANALYSIS (MODEL 2)

The equations for $A_0(t), A_1(t) and\ A_2(t)$ are the same as in model 1. The additional equation is

$$A_3(t) = M_3(t) + q_{32}(t) © A_2(t) + q_{35}(t) © A_5(t)$$

The steady state availability A_∞ for "Model 2" is

$$A_\infty = \frac{N_2}{D_2} \quad \text{where}$$

$$N_2 = \mu_2[1 - p_{13}p_{35} - p_{35}(p_{13}p_{24} + p_{23})] + \mu_1[p_{02}(p_{23}p_{35} + p_{24}) + p_{01}(1 - p_{23}p_{32})]$$

$$+ \mu_2[p_{01}p_{13}p_{32} + p_{02}(1 - p_{13}p_{35})] + \mu_3 p_{02}p_{24}p_{13}$$

and

$$D_2 = [\mu_0 p_{10} + \mu_1 + \mu_5\{p_{13} + p_{02}p_{10}\}][1 - p_{23}p_{32}] +$$
$$[\mu_2 + \mu_4 p_{24}][p_{13}p_{32} + p_{10}p_{02}] + \mu_3[p_{13} + p_{10}p_{02}p_{23}]$$

4.11 BUSY PERIOD ANALYSIS (MODEL 2)

The equations for $B_0(t), B_1(t), B_2(t)$ and $B_4(t)$ are the same as in model 1. The additional equations are

$$B_3(t) = q_{32}(t) © B_2(t) + q_{35}(t) © B_5(t),$$

$$B_5(t) = M_5(t) + q_{51}(t) © B_1(t)$$

where $M_5(t) = \overline{G}(t)$

In the long run, the function of time of which the system is under repair is given by

$$B_\infty = \lim_{t \to \infty} B_0(t) = \frac{N_3}{D_2}$$

where

$$N_3 = \mu_5 p_{01} p_{13}(p_{35} + p_{32} p_{24}) - p_{02}(p_{23} p_{35} + p_{24})$$

4.12 EXPECTED NUMBER OF VISITS BY REPAIR FACILITY

(Model 2)

The equations for $N_0(t), N_1(t), N_2(t)$ *and* $N_4(t)$ are the same as in "Model 1". The additional equation is

$$N_3(t) = Q_{32}(t) Ⓢ N_2(t) + Q_{35}(t) Ⓢ N_5(t).$$

In the steady state the number of visits per unit time is given by

$$N_\infty = \lim_{t \to \infty} \left| \frac{N_0(t)}{t} \right| = \frac{N_4}{D_2}$$

where

$N_4 = p_{01}\{p_{13}p_{32} - p_{10}(1 - p_{23}p_{32})\} + p_{02}(1 - p_{13}p_{35})$ and D_2 is already specified.

4.13 PROFIT ANALYSIS

The expected up time and down time of the system and the busy period of the repairman in $(0, t]$ are

$$\mu_{up}(t) = \int_0^t A_0(\mu)d\mu$$

$$\mu_{dn}(t) = t - \mu_{up}(t) = \int_0^t B_0(u)du$$

so that $\mu_{up}^*(s) = \frac{A_0^*(s)}{s} = \frac{B_0^*(s)}{s}$

$$\mu_{dn}^*(s) = \frac{1}{s^2} - \mu_{up}^*(s)$$

Now expected profit incurred in $(0, t]$

= Expected total revenue in $(0, t]$

- Expected total repair in $(0, t]$

- Expected cost of visits by repairman in $(0, t]$.

For "Model 1" and "Model 2", we have the profit functions as follows:

$$p_1 = k_1 A_\infty - k_2 B_\infty - k_3 N_\infty$$

$$p_2 = k_1 A_\infty - k_2 B_\infty - k_3 N_\infty$$

k_1 = revenue per unit up time of the system

k_2 = cost per unit time for which the repairman is busy

k_3 = cost per unit visits by the repair facility

Table 4.1

$\lambda_2 = 1.2,\ \mu_1 = 1.1,\ \mu_2 = 1.2, a = 0.3, b = 0.7,\ \mu = 1.1, k_1 = 400,\ k_2 = 30, k_3 = 100$

MEAN TIME TO SYSTEM FAILURE

Failure rate	**Model I**	**Model 2**
0.16	9.6123	8.6617
0.17	9.4888	8.2501
0.18	8.7677	7.9101
0.19	8.4711	7.5223

0.20	8.2506	7.2551
0.21	7.7569	6.9915
0.22	7.5116	6.7962

Table 4.2

$a = 0.3, b = 0.7, \mu_1 = 1.1, \ \mu_2 = 1.2, \mu = 1.1, k_1 = 400, k_2 = 20, \lambda_1 = 1.2, \lambda_2 = 1$

PROFIT

Failure rate	**Model 1**	**Model 2**
0.16	245.0015	272.5101
0.17	244.1121	269.3112
0.18	239.8162	262.8716

0.19	236.6617	259.9867
0.20	233.3351	256.6226
0.21	230.6318	254.1512
0.22	226.8813	251.8664

4.14 NUMERICAL ILLUSTRATION

In these models, λ_i , μ_i are taken from (Yadavalli et al, 2005).

From Table 4.1, we conclude that as failure rate increases the mean time to system failure decreases. For both models as the failure rate increases the MTSF of the system decreases.

From Table 4.2 we conclude that for both models as the failure rate increases the profit of the system decreases. It is clear that “Model 2” is more beneficial than “Model 1”.

Chapter 5

RELIABILITY STOCHASTIC OPTIMIZATION WITH BRANCH AND BOUND TECHNIQUE

An Application of Stochastic Programming with Branch and Bound technique - n stage series system with m chance constraints.

A modified version of this chapter is submitted to South African Journal of Science.

5.1 INTRODUCTION

In the past three decades, numerous reliability optimization techniques have been proposed (Tillman et al 1977, 1980, Kuo et al. 1987, Chen 1992). Stochastic programming models for general redundancy-optimization problems have been studied by (Zhao et al 2003). Stochastic programming models arise as reformulations or extensions of reliability optimization problems with random parameters. Moreover, the resource elements vary and it is reasonable to regard them as stochastic variables. Problems in this area are not easy to solve. Most researchers in this area concentrated on developing approximate solution methods as optimal solutions. However, efficiency in the complex theoretical aspect is usually not considered. Quality statements mostly remain restricted to convergence to an optimal solution without accompanying implications on the running time of the algorithms for attaining most accurate solutions. Very recently the complexity of stochastic programming problems has been addressed, confirming these problems are harder than most combinatorial optimization problems.

This chapter addresses chance constrained reliability stochastic optimization (CCRSO) problem. Chance constraint programming technique has been first proposed by (Charnes and Cooper, 1954). The objective is to maximize system reliability for the given chance constraints. A methodology is illustrated to determine optimal solutions to n stage series system with m chance constraints of the redundancy allocation problem. Various cases of randomness have been discussed with known distributions like Uniform, Normal, and Log-normal distributions, when the resource variables are random. Once the real number solution is obtained using the technique of chance constraint, the B&B technique is used to obtain the integer solution. In this chapter, a 4-stage series system with two chance constraints is numerically illustrated for the redundancy allocation problem.

This chapter has been organized as follows, stochastic integer programming problem for n stage series system with m chance constraint discussed and then the required algorithm to get integer solution is provided along with numerical example, which illustrate the model effectively.

1.1 Stochastic Integer Programming (SIP): *n* Stage Series System with *m* Chance Constraints

The chance constraint optimization problem for n stage series system with m chance constraint can be formulated as

$$\text{Max } R_s(X) = \prod_{j=1}^{n}\left[1-(1-r_j)^{x_j}\right] \tag{5.1}$$

Subject to, $P\left[g_i(x) \le b_i\right] \ge 1-\alpha_i$, $i = 1, 2, \ldots, m$; $x_j \ge 1, j = 1, 2, \ldots, n$, where resource vector b is random in nature,

R_s - reliability of the system

r_j, q_j - reliability, unreliability of components j; $r_j + q_j \equiv 1$

x_j - number of components used at stage j

$g_i(x)$ -chance constraint i

b_i - amount of resource i available (random

α_i - level of significance.

5.1.1 Case 1: b follows uniform distribution

Let $b_i \sim U(l_i, u_i)$, the constraint in system (1) is equivalent to $g_i(x) \le \tau_i$, where $\beta_i = 1 - \alpha_i$,

$$\int_{\tau_i}^{u_i} \left(\frac{dx}{u_i - l_i} \right) = \beta_i$$

$\tau_i = \alpha_i u_i + \beta_i l_i$.

Hence, the deterministic equivalent of system (5.1) is:

$$\text{MaxR}_s(X) = \prod_{j=1}^{n} \left[1 - (1 - r_j)^{x_j} \right]$$

(5.2)

subject to

$g_i(x) \le \alpha_i u_i + \beta_i l_i$, $i = 1, 2, \ldots, m$; $x_j \ge 1, j = 1, 2, \ldots, n$.

5.1.2 Case 2: b follows normal distribution

Let $b_i \sim N\left(b_i, \sigma_{bi}{}^2\right)$, where $\mu_{b_i}, \sigma^2_{b_i}$ are mean and variance of the normal random variable b_i. Using the i^{th} chance constraint of the system (5.1), restate the chance constraint as $P\left[b_i \geq g_i(x)\right] \geq 1 - \alpha_i$, i = 1, 2,…, m, this expression can be further stated as $P\left[(b_i - \mu_{b_i})/\sigma_{b_i} \geq (g_i(x) - \mu_{b_i})/\sigma_{bi}\right] \geq 1 - \alpha_i$, $i = 1, 2, \ldots, m$.

Using the cumulative density function of the standard normal random variable, it can be simplified as:

$$1 - \Phi[(g_i(x) - \mu_{b_i})/\sigma_{bi}] \geq 1 - \alpha_i, \quad i=1,2,\ldots, m,$$

where

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left(\frac{t^2}{2}\right) dt.$$

This can be further simplified as

$$\Phi[(g_i(x) - \mu_{b_i})/\sigma_{bi}] \leq \Phi(-K_{\alpha_i}), \; i = 1, 2, \ldots, m.$$

The chance constraint can be transformed into deterministic constraint as

$$g_i(x) \leq \mu_{b_i} - \sigma_{b_i} K_{\alpha_i}, \; i = 1,2, \ldots, m.$$

Hence, the deterministic equivalent of system (1) is:

$$\text{MaxR}_s(\text{X}) = \prod_{j=1}^{n}\left[1 - (1 - r_j)^{x_j}\right] \qquad (5.3)$$

subject to

$$g_i(x) \le \mu_{b_i} - \sigma_{b_i} K_{\alpha_i},\ i = 1, 2, \ldots, m; \quad x_j \ge 1,\ j = 1, 2, \ldots, n.$$

5.1.3 Case 3: b follows log-normal distribution

Let $b_i \sim LN(\mu_I, \sigma_I{}^2)$, where μ_i, σ_i^2 are the mean and variance of the log normal random variable b_i. Using the i^{th} chance constraint of the system (5.1), we restate the chance constraint as

$$P[\ln b_i \ge \ln g_i(x)] \ge 1 - \alpha_i, \quad i = 1, 2, \ldots, m.$$

This expression can be further stated as $P[(\ln b_i - \mu_i)/\sigma_i \ge (\ln g_i(x) - \mu_i)/\sigma_i] \ge 1 - \alpha_i$, $i = 1, 2, \ldots, m$. The following deterministic i^{th} constraint is obtained by similar arguments made in case 2.

$$g_i(x) \le \exp(\mu_i - \sigma_i K_{\alpha_i}),\ i = 1, 2, \ldots, m.$$

Hence, the deterministic equivalent of system (1) is:

$$\text{MaxR}_s(\text{X}) = \prod_{j=1}^{n} \left[1 - (1 - r_j)^{x_j}\right] \tag{5.4}$$

subject to

$g_i(x) \le \exp(\mu_i - \sigma_i K_{\alpha i})$, $i = 1, 2, \ldots, m$; $x_j \ge 1$, $j = 1, 2, \ldots, n$.

5.2 General Algorithm

1. Convert the deterministic form of chance constraint into a linear constraint, adopting the technique of sequential linear programming (Rao 2000, Jeeva et al 2002,2004 , Charles and Dutta, 2003).

2. Code any one of the system (5.2) – (5.4) along with respective linearized constraint in MATLAB or LINGO and generate optimal solutions by inputting initial values using random function (in later stages one can use the existing real solution to generate integer solution using the step below given).

3. Apply the branch and bound algorithm given below to get integer solutions.

5.3 Branch-and-bound (B&B) technique

The B&B technique for CCRSO for stochastic optimization is given below:

1. Solve the problem as if all the variables were real numbers i.e. not integers, using the general algorithm given above. This solution is the upper bound (for maximization problem) of the CCRSO problem.

2. Choose one variable at a time that has a non-integer value, say x_j and branch that variable to the next higher integer value for one problem and to the next lower integer value for the other. The real valued solution of the variable j can be expressed as $x_j = [x_j] + x_j^*$, where $[x_j]$ is the integer part of x_j and x_j^* is the fractional part of x_j, $0 < x_j^* < 1$. The lower bound and upper bound constraints of the two mutually exclusive problems are $x_j = [x_j]$ and $x_j = [x_j] + 1$, respectively. Add these two constraints to both branched problems.

3. Now the variable x_j is an integer in either branch. Fix the integer of x_j for the following steps of branch-and-bound. Select the branch that yields the maximum objective function with all constraints satisfied. Then repeat step 2 on another variable $x_k \neq x_j$ for each of the new sub problems until all variables become integers.

4. Stop the particular branch if the solution is not satisfying the constraints of the original problem else stop the branch when all the desired integer values are obtained.

5.4 NUMERICAL EXAMPLE

Example 1

A four-stage system with chance constraints is formulated as a pure stochastic integer programming problem using the data given in table 5.1. The decision variables, X = (x_1,…,x_4), are the number of redundancies at each stage. The problem is formulated as in Case 1.

Table 5.1: Data for Example 1

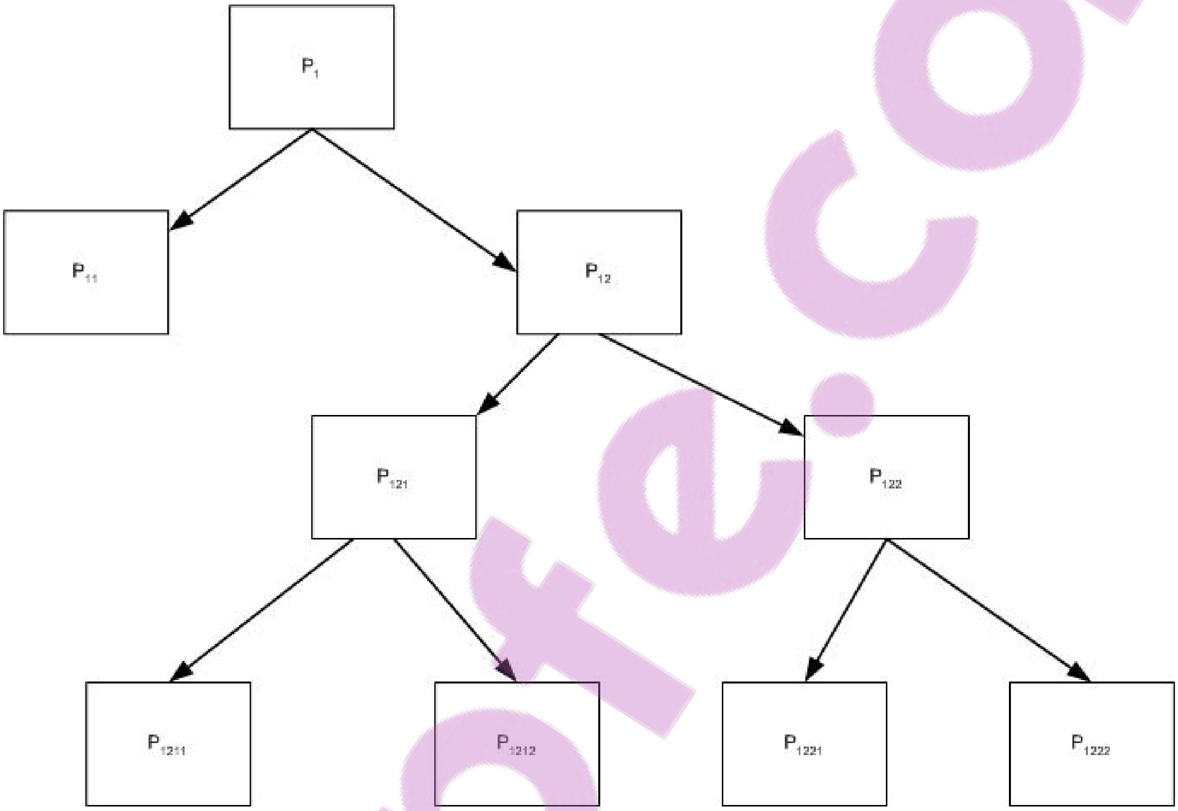
Stage, *j*	**1**	**2**	**3**	**4**	**Available Resource**			
$\mathbf{r}_j$	0.75	0.80	0.75	0.85		l_i	u_i	α_i
$\mathbf{c}_{1j}$	1.5	3.3	3.2	4.4	$\mathbf{b}_1$	50	60	0.10
$\mathbf{c}_{2j}$	4.0	5.0	7.0	9.0	$\mathbf{b}_2$	110	140	0.15

Table 5.2: Solutions for Example 1

S. No.	Initial guess (obtained using rand())				x_1	x_2	x_3	x_4	$R_s(X)$
	x_1	x_1	x_2	x_3					
1	1.9501	1.2311	1.6068	1.4860	7.7656	9.5884	1.0344	1	1
2	1.8913	1.7621	1.4565	1.0185	10.857	8.2167	1	1	1
3	1.8214	1.4447	1.6154	1.7919	8.4843	8.6375	1	1.4931	1
4	1.9218	1.7382	1.1763	1.4057	7.7650	6.2088	1	1	0.9999
5	1.9355	1.9169	1.4103	1.8936	10.226	7.5664	1	1.7028	1
6	1.0579	1.3529	1.8132	1.0099	11.370	7.6831	1.3097	1	1
7	1.1389	1.2028	1.1987	1.6038	10.706	8.0460	1	1.1794	1
8	1.2722	1.1988	1.0153	1.7468	10.125	7.9687	1	1.4356	1
9	1.4451	1.9318	1.4660	1.4186	12.011	6.5778	1	1	1
10	1.8462	1.5252	1.2026	1.6721	9.3136	8.5091	1.0046	1.3034	1

With the data given in table 1, the real solutions are obtained using the general algorithm, which is exhibited in table 2. This paper suggests that the real solution be further elaborated by the B&B technique. Let us take one solution X = (11.3697, 7.6831, 1.3097, 1.0000) from Table 5. 2. Now the integer solution is obtained using B&B technique. The following figure 1 gives clear picture about B&B network.

Figure 5. 1: A B&B Network Representation for Example 1



P1 : x_1 =11.3697; x_2 =7.6831; x_3 =1.3097; x_4 =1.0000; R=1.0000

P11 : **Fathomed**

P12 : x_1 =11.1175; x_2 =7.1284; x_3 =2.0000; x_4 =1.0000; R=1.0000

P121 : x_1 =11.1175; x_2 =7.0000; x_3 =2.0000; x_4 =1.0000; R=1.0000

P122 : x_1 =9.2000; x_2 =8.0000; x_3 =2.0000; x_4 =1.0000; R=1.0000

P1211 : **x_1 =11.0000; x_2 =7.0000; x_3 =2.0000; x_4 =1.0000; R=1.0000**

P1212 : **Fathomed**

P1221 : **x_1 =9.0000; x_2 =8.0000; x_3 =2.0000; x_4 =1.0000; R=1.0000**

P1222 : **Fathomed**

Alternative optimal integer is obtained from the B&B process, X = (11, 7, 2, 1) and X = (9, 8, 2, 1).

5.5 CONCLUSION

The combination of the chance constraint technique and the B&B technique takes advantage of an exact method and enumerative method. In this paper the chance constraint technique, using MATLAB program, quickly reaches real solutions that is close to optimum. In addition, the B&B technique generates many sets of integer solutions. The competitive alternatives provide the management with several options and flexibility. Since a good approximation is obtained by the chance constraint technique, it does not take many branches for the B&B technique to reach the integer solution. The B&B algorithm given in this paper can be directly applied to the mixed integer stochastic programming problem (MISPP). For MISPP, only the integer variables need to be enumerated by the B&B procedure. The real variables are free of restriction after each step of the B&B technique.

CHAPTER 6

A two unit cold standby system with non-instantaneous switchover

6.1 INTRODUCTION

Gopalan et al (1984) have analysed a single-server two-unit cold standby system subject to a slow switch and have obtained expressions for the expected switchover time of unit from standby to operative state in (0,t] and the expected repair time of a unit in (0,t]. Sharma et al (1986) modified that model by taking a two-unit warm standby system and obtained several reliability characteristics. They did not take into account the partial failure mode. The purpose of the present chapter is to study a two-unit cold standby system with three modes of the system subject to slow switch. The system fails totally only through the partial failure mode. When a unit fails partially, its repair starts immediately and the installation of a new unit in place of a partially failed unit remains operative. Regenerative point technique is used for the analysis.

6.2 MODEL ASSUMPTION

The system compromises two identical units. Initially one is operative and the other is a cold standby.

(1) Each unit is has three possible modes: normal (N), partial failure

(P) and total failure (F).

(2) The system fails totally only through the partial failure mode.

(3) The failure and switchover times are distributed negative exponentially whereas the repair times of units are distributed arbitrarily.

(4) When a unit fails partially, repair of the partially failed unit starts

instantaneously and installation of the standby for operation is not

permitted.

(5) When a unit fails completely from the partially failed state, repair

of the failed unit and installation of the standby for operation start

simultaneously and independently.

(6) The repaired system is as good as new.

6.3 NOTATION

α, β	Constant failure rates from N to P and P to F modes
η	Constant rate of switchover time of a unit from standby state to operative state
$f(t), F(t)$	pdf and cdf of repair time of a unit from P state
$g(t), G(t)$	pdf and cdf of repair time of a unit from F state

Symbols for states of the system

N_0, N_5	system operative in N mode
P_{or}	unit operative in P mode and under repair mode
F_r	unit in F mode and under repair
F_R	unit in F mode and its repair continued from earlier state
F_w	system in F mode and waiting for repair
bso	standby being switched over

Thus the following states are possible:

$$S_0 = (N_0, N_5); S_1 = (P_{or}, N_5); S_2 = (F_r, bso);$$
$$S_3 = (N_s, bso); S_4 = (F_R, N_0); S_5 = (F_w, P_{or});$$
$$S_6 = (F_r, N_0); S_7 = (F_r, F_w).$$

Up states-$S_{0,} S_1, S_4 - S_6$; down states-$S_{2,} S_3, S_7$.

The underlined states are non-regenerative. Possible states and transitions are shown
in Figure 6.1.

6.4. TRANSITION PROBABILITIES AND SOJOURN TIMES

Let $T_0(=0), T_1, \ldots$ denote the epochs at which the system enters any state $S_i \in E$ and X_n be the state visited at time T_{n+}, i.e. just after the transition at T_n. Then $\{X_n, T_n\}$ is a Markov renewal process with state space. Let

$$Q_{ij}(t) = P\left[X_{n+1} = j, T_{n+T} T_{n \le t} X_n = i\right];$$

then the transition probability matrix of embedded Markov Chain is

$$P = \left(P_{ij}\right) = \left(\left(Q_{ij}(\infty)\right)\right) = Q(\infty),$$

with non-zero elements

$$P_{01} = P_{30} = P_{72} = 1, P_{10} = 1 - P_{12} = \widetilde{F}(\beta),$$

$$P_{23} = 1 - P_{24} = \widetilde{G}(\eta), \quad P_{20}{}^{(4)} = \eta\left[\frac{\widetilde{G}(\alpha) - \widetilde{G}(\eta)}{\eta - \alpha}\right],$$

$$P_{25}^{(4)} = 1 - \frac{\left[\eta \tilde{G}(\alpha) - \alpha \tilde{G}(\eta)\right]}{(\eta - \alpha)},$$

$$P_{56} = 1 - P_{57} = \tilde{F}(\beta), \qquad P_{60} = 1 - P_{65} = \tilde{G}(\alpha).$$

Evidently,

$$P_{10} + P_{12} = 1, P_{23} + P_{24} = 1, P_{23} + P^{(4)}{}_{20} + P^{(4)}{}_{25} = 1,$$

$$P_{56} + P_{57} = 1, P_{60} + P_{65} = 1.$$

Mean sojourn times μ_i in state S_i are

$$\mu_0 = \frac{1}{\alpha},\ \mu_1 = \mu_5 = \frac{\left[1 - \tilde{F}(\beta)\right]}{\beta},$$

$$\mu_2 = \frac{\left[1 - \tilde{G}(\eta)\right]}{\eta}, \mu_3 = \frac{1}{\eta},$$

$$\mu_6 = \frac{\left[1 - \tilde{G}(\alpha)\right]}{\alpha},$$

$$\mu_7 = \int_0^\infty \overline{G}(t)\,dt$$

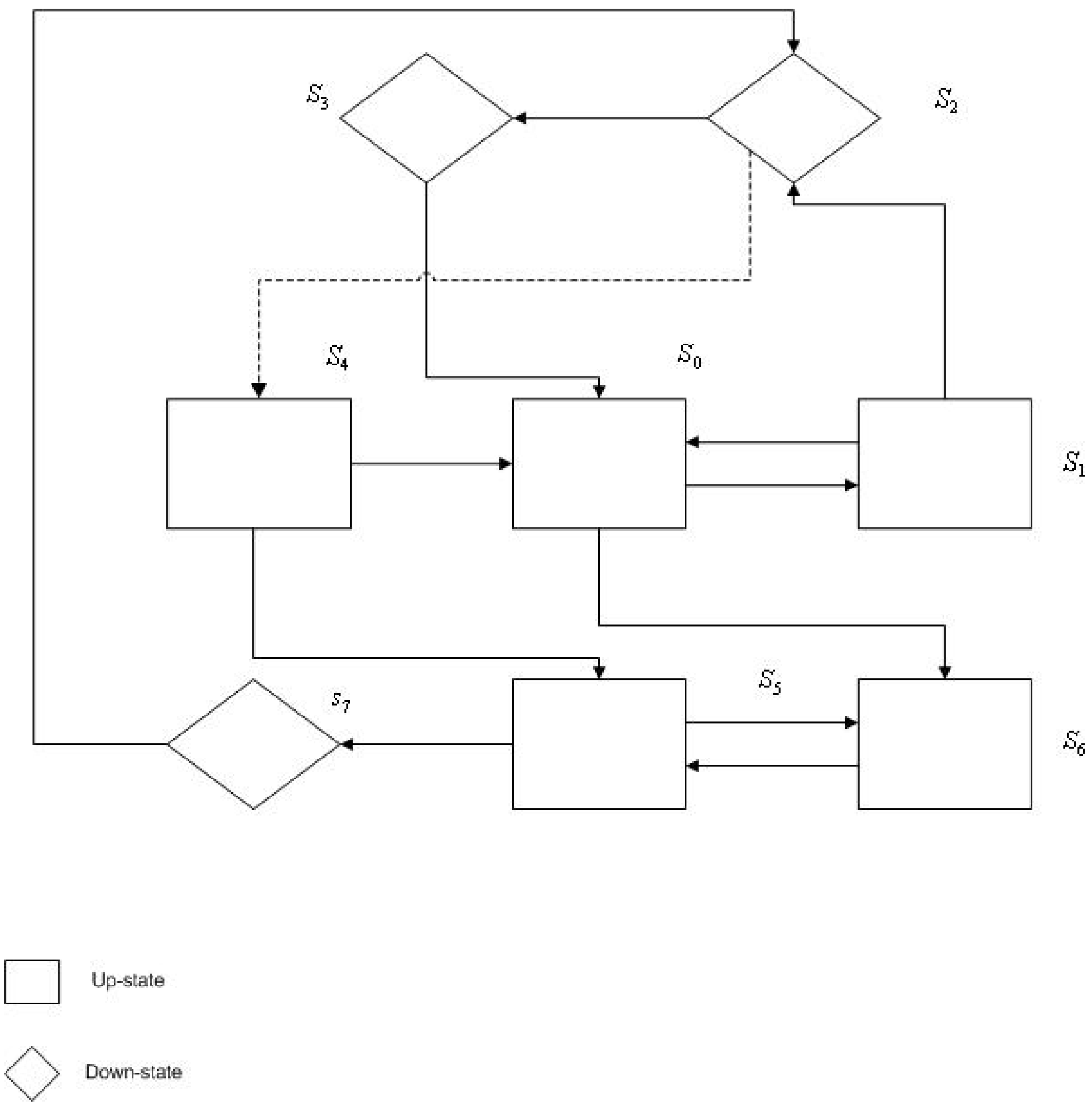


Figure 6.1

6.5 TIME TO SYSTEM FAILURE

Time to system failure can be regarded as the first passage to the failed state. To obtain it we consider down states as absorbing. We obtain the following recursive relations for $\pi_i(t)$, the cdf of time to system failure when the system starts from state S_i

$$\pi_0(t) = Q_{01}(t) \circledS \pi_1(t) \tag{6.1}$$

$$\pi_1(t) = Q_{10}(t) \circledS \pi_0(t) + Q_{12}(t) \tag{6.2}$$

$$\pi_5(t) = Q_{56}(t) \circledS \pi_6(t) + Q_{57}(t) \tag{6.3}$$

$$\pi_6(t) = Q_{60}(t) \circledS \pi_0(t) + Q_{65}(t) \copyright \pi_5(t) \tag{6.4}$$

Taking Laplace-Stieltjies transforms of equations (1)-(4) and solving for $\tilde{\pi}_0(s)$, we have

$$\tilde{\pi}_0(s) = \frac{\tilde{Q}_{01}\tilde{Q}_{12}\left(1-\tilde{Q}_{56}\tilde{Q}_{65}\right)}{\left(1-\tilde{Q}_{01}\tilde{Q}_{10}\right)\left(1-\tilde{Q}_{56}\tilde{Q}_{65}\right)}$$

$$= \frac{\tilde{Q}_{01}\tilde{Q}_{12}}{\left(1-\tilde{Q}_{01}\tilde{Q}_{10}\right)}, \tag{6.5}$$

where, for brevity, the argument ‘s’ is omitted.

The mean time to system failure (MTSF), when the system starts from S_0, is

$$MTSF = E(T) = \frac{[D_r'(0) - N_r'(0)]}{D_r(0)} = \frac{(\mu_0 + \mu_1)}{P_{12}}. \tag{6.6}$$

6.6 AVAILABILITY ANALYSIS

Let $M_i(t)$ be the probability that the system which started from state S_i has reached time t without making any transition into any other regenerative state belonging to E. By probabilistic arguments, we have

$$M_0(t) = e^{-\alpha t},$$

$$M_1(t) = M_5(t) = e^{-\beta t}\overline{F}(t),$$

$$M_6(t) = e^{-\alpha t}\overline{G}(t).$$

From then theory of regenerative process, the pointwise availabilities $A_i(t)$ of a system which has started from a given regenerative point are seen to satisfy the following recursive relations:

$$A_0(t) = M_0(t) + q_{01}(t) © A_1(t) \tag{6.7}$$

$$A_1(t) = M_1(t) + q_{10}(t) © A_0(t) + q_{12}(t) © A_2(t) \tag{6.8}$$

$$A_2(t) = q_{20}^{(4)}(t) © A_0(t) + q_{23}(t) © A_3(t) + q_{25}^{(4)}(t) © A_5(t) \tag{6.9}$$

$$A_3(t) = q_{30}(t) © A_0(t) \tag{6.10}$$

$$A_5(t) = M_5(t) + q_{56}(t) © A_6(t) + q_{57}(t) © A_7(t) \tag{6.11}$$

$$A_6(t) = M_6(t) + q_{60}(t) © A_0(t) + q_{65}(t) © A_5(t) \tag{6.12}$$

$$A_7(t) = q_{72}(t) © A_2(t). \tag{6.13}$$

Taking Laplace-transforms of equations (6.7)-(6.12) and solving for $A_0^*(s)$ we have

$$A^*{}_0 = \frac{N_1(s)}{D_1(s)} \tag{6.14}$$

$$M_0(t) = e^{-\alpha t},$$

$$M_1(t) = M_5(t) = e^{-\beta t}\overline{F}(t),$$

$$M_6(t) = e^{-\alpha t}\overline{G}(t).$$

Where

$$N_1(s) = \left(1 - q_{56}^* q_{65}^* - q_{25}^{*(4)} q_{57}^* q_{72}^*\right) \times \left(M_0^* + q_{01}^* M_1^*\right)$$
$$+ q_{01}^* q_{12}^* q_{25}^{*(4)} \times \left(M_5^* + q_{56}^* M_6^*\right)$$

$$D_1(s) = \left(1 - q_{56}^* q_{65}^* - q_{25}^{*(4)} q_{57}^* q_{72}^*\right) \times \left(1 - q_{01}^* q_{10}^*\right)$$
$$- q_{01}^* q_{12}^* \left[\left(1 - q_{56}^* q_{65}^*\right) \times \left(q_{20}^{*(4)} + q_{23}^* q_{30}^*\right) + q_{25}^* q_{56}^* q_{60}^*\right]$$

The steady-state availability of the system is

$$A_0 = \lim_{s \to 0} s \; A_0^*(s) = \frac{N_1}{D_1'} . \qquad (6.15)$$

$$N_1 = \left(1 - P_{56} P_{65} - P_{25}^{(4)} P_{57}\right)\left(\mu_0 + P_{01}\mu_1\right) + P_{01} P_{12} P_{25}^{(4)}\left(\mu_1 + P_{56}\mu_6\right)$$

$$D_1' = \left(\mu_0 + \mu_1\right)\left(1 - P_{56}P_{65} - P_{25}^{4} P_{57}\right) + P_{12}\left(1 - P_{56}P_{65}\right)\left[\frac{\eta\mu_6 - \alpha\mu_2}{(\eta - \alpha)} + P_{23}\mu_3\right] +$$
$$P_{12} P_{25}^{(4)}\left(\mu_1 + P_{56}\mu_6 + P_{57}\mu 7\right)$$

6.7 BUSY PERIOD ANALYSIS

As defined earlier, $B_i(t)$ is the probability that the system is under repair at time t given that the system entered regenerative state s_i at $t = 0$. By probabilistic arguments we have

$$B_0(t) = q_{01}(t) © B_1(t) \qquad (6.16)$$

$$B_1(t) = W_1(t) + q_{10}(t) \copyright B_0(t) + q_{12}(t) \copyright B_2(t) \tag{6.17}$$

$$B_2(t) = W_2(t) + q_{20}{}^{(4)} \copyright B_0(t) + q_{23}(t) \copyright B_3(t) + q_{25}^{(4)} \copyright B_5(t) \tag{6.18}$$

$$B_3(t) = q_{30}(t) \copyright B_0(t) \tag{6.19}$$

$$B_5(t) = W_5(t) + q_{56}(t) \copyright B_6(t) + q_{57}(t) \copyright B_7(t) \tag{6.20}$$

$$B_6(t) = W_6(t) + q_{60}(t) \copyright B_0(t) + q_{65}(t) \copyright B_5(t) \tag{6.21}$$

$$B_7(t) = W_7(t) + q_{72}(t) \copyright B_2(t) \tag{6.22}$$

where

$$W_1(t) = W_5(t) = e^{-\beta t}\overline{F}(t),$$

$$W_6(t) = e^{-\alpha t}\overline{G}(t),$$

$$W_7(t) = \overline{G}(t),$$

$$W_2(t) = \left(\eta e^{-\alpha t} - \alpha e^{-\eta t}\right)\frac{\overline{G}(t)}{(\eta - \alpha)}.$$

Taking Laplace-transforms of relations (6.16)-(6.22) we have

$$B_0(s) = \frac{N_2(s)}{D_1(s)} \tag{6.23}$$

where

$$N_2(s) = q_{01}^*\left(1 - q_{56}^* q_{65}^* - q_{25}^* q_{57}^* q_{72}^*\right) \times W_1^* + q_{01}^* q_{12}^*$$
$$\left[\left(1 - q_{56}^* q_{65}^*\right) W_2^* + q_{25}^{*(4)}\left(W_5^* + q_{56}^* W_6^* + q_{57}^* W_7^*\right)\right]$$

In the long run, the fraction of time for which the system is under repair is given by

$$B_0 = \lim_{t \to \infty} B_0(t) = \lim_{s \to 0} s\, B_0^*(s) = \frac{N_2}{D_1'} \tag{6.24}$$

where, in terms of

$$W_1^*(0) = W_5^*(0) = \mu_1$$

$$W_6^*(0) = \mu_6\,,$$

$$W_7^*(0) = \mu_{7,}$$

$$W_2^*(0) = \frac{(\eta\mu_6 - \alpha\mu_2)}{(\eta - \alpha)},$$

we have

$$N_2 = \left(1 - P_{56}P_{65} - P_{25}^{(4)}P_{57}\right)\mu_1 + \left[\left(P_{25}^{(4)}\left(\mu_1 + P_{56}\mu_6 + P_{57}\mu_7\right) + \left(1 - P_{56}P_{65}\right)\left(\eta\mu_6 - \alpha\mu_2\right)\right)\right]P_{12.}$$

6.8 EXPECTED NUMBER OF VISITS BY THE REPAIRFACILITY

We define $V_i(t)$ as the expected number of visits by the repairman in $(0,t]$ given that the system initially starts from regenerative states S_i. By probabilistic arguments, we have the following recursive relations:

$$V_0(t) = Q_{01}(t) © \left[1 + V_1(t)\right] \tag{6.25}$$

$$V_1(t) = Q_{10}(t) © V_0(t) + Q_{12}(t) © V_2(t) \tag{6.26}$$

$$V_2(t) = Q_{20}^{(4)} © V_0(t) + Q_{23}(t) © V_3(t) + Q^{(4)}{}_{25}(t) © V_5(t) \tag{6.27}$$

$$V_3(t) = Q_{30}(t) © V_0(t) \tag{6.28}$$

$$V_5(t) = Q_{56}(t) © V_6(t) + Q_{57}(t) © V_5(t) \tag{6.29}$$

$$V_6(t) = Q_{60}(t) © V_0(t) + Q_{65}(t) © V_5(t) \tag{6.30}$$

$$V_7(t) = Q_{72}(t) © V_2(t) \tag{6.31}$$

Taking the Laplace-Stieltjes transforms of the above equations and solving for $\widetilde{V}_0(s)$, we have

$$\widetilde{V}_0(s)=\frac{N_3(s)}{D_2(s)} \tag{6.32}$$

where

$$N_3(s)=\widetilde{Q}_{01}\left(1-\widetilde{Q}_{56}\widetilde{Q}_{65}-\widetilde{Q}_{25}{}^{(4)}\widetilde{Q}_{57}\widetilde{Q}_{72}\right)$$

$$D_2(s)=\left(1-\widetilde{Q}_{56}Q_{65}-\widetilde{Q}_{25}{}^{(4)}\widetilde{Q}_{57}\widetilde{Q}_{72}\right)\left(1-\widetilde{Q}_{01}\widetilde{Q}_{10}\right)$$
$$-\widetilde{Q}_{01}\widetilde{Q}_{12}\left[\left(1-\widetilde{Q}_{56}\widetilde{Q}_{65}\right)\left(\widetilde{Q}_{20}{}^{(4)}+\widetilde{Q}_{23}\widetilde{Q}_{30}\right)+\widetilde{Q}^{(4)}{}_{25}Q_{56}\widetilde{Q}_{60}\right]$$

In the steady state, the number of visits per unit time is given by

$$V_0=\lim_{t\to\infty}\frac{V_0(t)}{t}=\lim_{s\to 0}s^2\,\widetilde{V}_0(s)=\frac{N_3}{D_1'}, \tag{6.33}$$

where

$$N_3=\left(1-P_{56}P_{65}-P_{25}{}^{(4)}P_{57}\right).$$

6.9 SWITCHOVER ANALYSIS

We define $I_i(t)$ as the probability that the standby unit being switched is under switching device in $(0,t]$, given that the system entered regenerative state S_i at $t=0$. By probabilistic arguments, we have

$$I_0(t)=q_{01}(t)\copyright I_1(t) \tag{6.34}$$

$$I_1(t) = q_{10}(t) © I_0(t) + q_{12}(t) © I_2(t) \tag{6.35}$$

$$I_2(t) = H_2(t) + q_{20}{}^{(4)}(t) © I_0(t) + q_{23}(t) © I_3(t) + q_{25}{}^{(4)}(t) © I_5(t) \tag{6.36}$$

$$I_3(t) = H_3(t) + q_{30}(t) © I_0(t) \tag{6.37}$$

$$I_5(t) = q_{56}(t) © I_6(t) + q_{57}(t) © I_7(t) \tag{6.38}$$

$$I_6(t) = q_{60}(t) © I_0(t) + q_{65}(t) © I_5(t) \tag{6.39}$$

$$I_7(t) = q_{72}(t) © I_2(t) \tag{6.40}$$

where

$$H_2(t) = e^{-\eta t}\overline{G}(t),$$

$$H_3(t) = e^{-\eta t}.$$

Taking the Laplace-transforms of relations (6.34) – (6.40), we have

$$I^*{}_0(s) = \frac{N_4(s)}{D_1(s)}, \tag{6.41}$$

$$N_4(s) = q^*{}_{01}q^*{}_{12}\left(1 - q_{56}{}^*q_{65}{}^*\right)\left(H^*{}_2 + q^*{}_{23}H_3{}^*\right)$$

In the steady-state, the fraction of time for which the system is under switch activation is given by

$$I_0 = \lim_{t\to\infty} I_0(t) = \lim_{s\to 0} sI^*{}_0(s) = \frac{N_4}{D_1'} \tag{6.42}$$

where, in terms of

$$H_2^*(0) = \mu_2,$$

$$H^*{}_3(0) = \mu_3,$$

we have

$$N_4 = P_{12}(1 - P_{56}P_{65})(\mu_2 + P_{23}\mu_3).$$

6.10 COST ANALYSIS

(1) The expected uptime of the system in $(0,\ t]$ is

$$\mu^*{}_{up} = \int_0^t A_0(u)du$$

so that

$$\mu^*{}_{up}(s) = \frac{A^*{}_0(s)}{s} \tag{6.43}$$

(2) The expected duration of the repairman's busy time in $(0,t]$ is

$$\mu_b(t) = \int_0^t B_0(u)du$$

so that

$$\mu^*{}_b(s) = \frac{B^*{}_0(s)}{s} \tag{6.44}$$

(3) The expected switchover time of the standby unit in $(0,t]$ is

$$\mu_1(t) = \int_0^t I_0(\mu)du\,,$$

so that

$$\mu^*{}_I(s) = \frac{I^*{}_0(s)}{s} \tag{6.45}$$

The expected total cost (gain) incurred in $(0,t]$ is

$$G(t) = C_1\mu_{up}(t) - C_2\mu_b(t) - C_3V_0(t) - C_4\mu_1(t) \tag{6.46}$$

where C_1 is the revenue per unit up time, C_2 is the cost per unit for which the system is under

repair, C_3 is the cost per visit by the repairman and C_4 is the cost per unit time for which the

system is under switch activation device.

The expected profit per unit time in the steady state is

$$G = \lim_{t \to \infty} \frac{G(t)}{t} = \lim_{s \to 0} s^2 G^*(s)$$

$$= C_1 A_0 - C_2 B_0 - C_3 V_0 - C_4 I_0 \qquad (6.47)$$

CHAPTER 7

A COMPLEX SYSTEM WITH CORRELATED FAILURES

7.1 INTRODUCTION

Most reliability models assume the continuous operation of the unit (or system) until a failure occurs. However, situations may arise where the unit (or system) needs rest after its operation for some time [Muller, 2005]. Very few attempts have been made in this direction. Murari and Muruthachalan (1981), Sarma (1982), Botha (2002), Hargreaves (2003) considered a two-unit system with a provision for rest for the system. The working and the rest are assumed to be random variables with negative exponential distributions. However, the idea of preparation time for the system may prove expensive as no output is obtained from the system during rest. This situation can be avoided in a two-unit cold standby system by providing rest to each unit alternately and operating the other unit when one requires rest. Further, in repairable systems, the dependence of repair time on the failure time of unit is a common experience of systems engineers, but this fact has also been ignored so far by reliability researchers. Keeping these factors in view, we analyse in this chapter a two-unit cold standby system with independent failure and repair times, with provision for the rest of the operative unit.

7.2 SYSTEM DESCRIPTION

1. The system consists of two identical units; initially, one is operative and the other is kept as a cold standby.

2. After operating for a random amount of time, the operating unit may require rest and again become fit for operation. The operating time and rest periods are independent random variables which are distributed exponentially.
3. As soon as the operative unit goes to rest, the standby unit starts operation.

4. There is a single repair facility
5. The repair facility is available instantaneously to repair the failed unit. The failure and repair are distributed according to bivariate exponential law.
6. Both units cannot go for rest simultaneously.
7. If the operative unit fails (after operating for time $X = x$) while the other unit is under repair. The unit failed later is repaired first and its repair time Y follows the bivariate exponential density jointly with X. The repair time already spent in the repair of the earlier failed unit is wasted and the further repair time Y' of this unit need not depend on x. It is assumed to have an independent negative exponential distribution with parameter θ.

7.3 NOTATION

Let O, S, R, F_r, F_{rc} and F_{wr}, denote respectively the operative, standby, under rest, under repair, under repair from previous state, and waiting for repair states of the unit. With these notation, the possible states of the system are:

Up states		Down states	
S_0	(O, S)	S_3	(R, F_r)
S_1	(O, R)	S_4	(F_{rc}, R)
S_2	(F_r, O)		
		S_6	(F_{wr}, F_r)
S_5	$(\mathrm{F_{rc}}, O)$		
S_7	(O, F_r)		

The possible transitions together with the corresponding transition probability density functions are shown in Figure 7.1

X, Y	:random variables representing respectively the failure and repair times of a unit.
$f(x, y)$	:joint pdf of (X, Y)

$$f(x,y) = \lambda\,\mu\,(1-r)e^{-\lambda x - \mu y}\, I_0\left(2\sqrt{\lambda\,\mu\,x\,y}\right); \qquad x, y, \lambda, \mu > 0; |r| < 1 \qquad (7.1)$$

with

$$I_0\left(2\sqrt{\lambda\,\mu\,r\,x\,y}\right) = \sum_{k=0}^{\infty} \frac{(\lambda\,\mu\,r\,x\,y)^k}{(k!)^2}$$

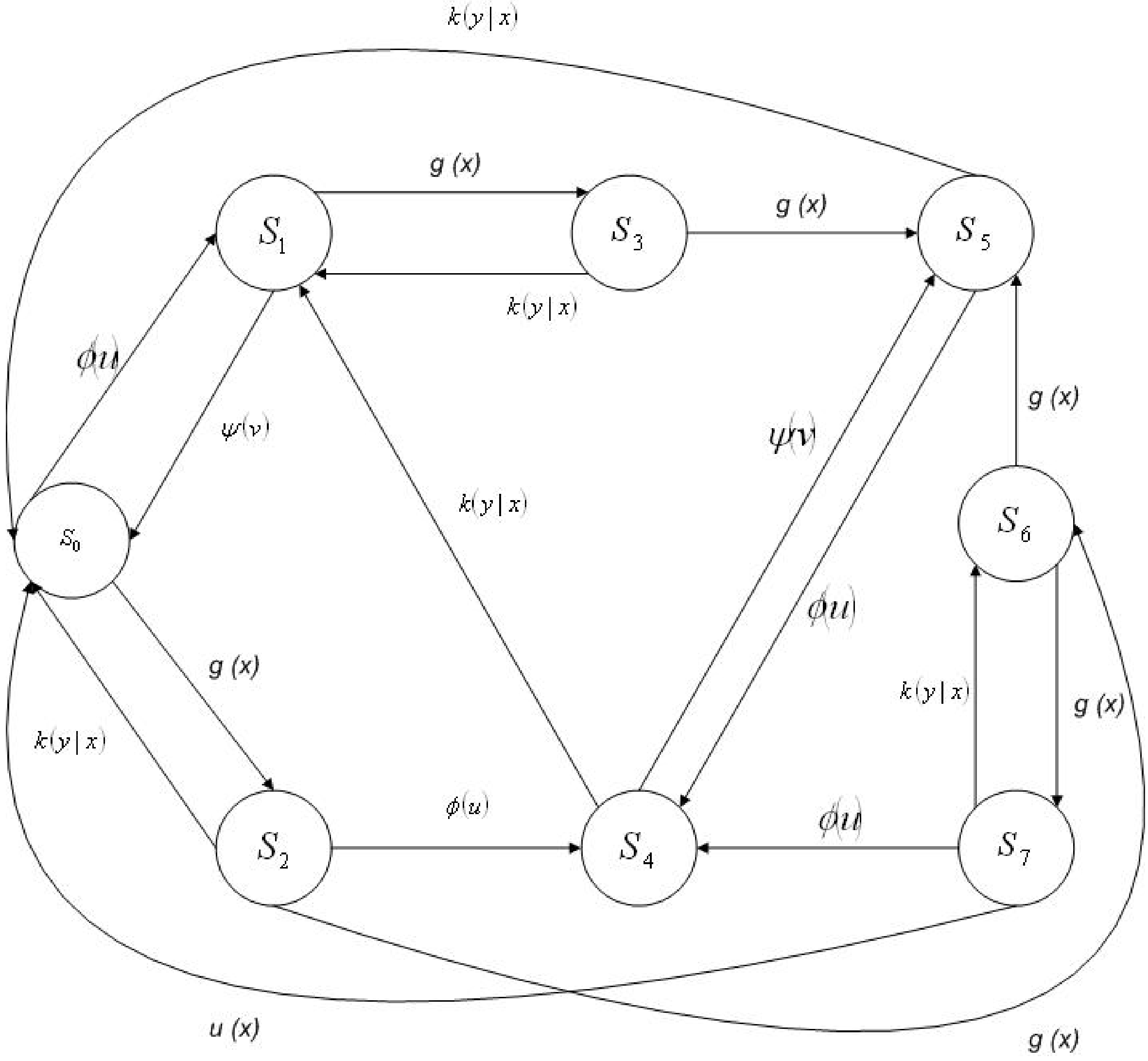

k(y|x)
g (x)
g (x)
S1
S3
S5
k(y|x)
φ(u)
ψ(v)
ψ(v)
g (x)
k(y|x)
S0
S6
φ(u)
g (x)
k(y|x)
k(y|x)
g (x)
φ(u)
φ(u)
S2
S4
S7
u (x)
g (x)

FIGURE 7.1

$g(x), G(x)$:pdf and cdf of X

$g(x) = \lambda(1-r)e^{-\lambda(1-r)x\lambda}$; x > 0

$G(x) = 1 - e^{-\lambda(1-r)x}$; $|r| < 1.$

$h(y) = \mu(1-r)e^{-\mu(1-r)y}$; y > 0

$H(y) = 1 - e^{-\mu(1-r)y}$; $|r| < 1.$

$k(y/x), k(y/x)$: :conditional pdf and cdf of y given x

$$k(y/x) = \mu\, e^{-\mu y - \lambda r x}\, I_0\left(2\sqrt{\lambda\,\mu\, r\, x\, y}\right)$$

$$k(y/x) = \int_0^y k(t/x)\,dt \qquad ; x, y, \lambda, \mu > 0, |r| < 1.$$

$u(y'), U(y')$: pdf and cdf of Y', the random variable representing the repair time of a unit whose repair was interrupted.

$u(y') = \theta\, e^{-\theta y'}$; $\theta, y' > 0$

$U(y') = 1 - e^{-\theta y'}$

$\phi(u), \Phi(u)$:pdf and cdf of the working period of a unit

$\phi(u) = \alpha\, e^{-\alpha\,\mu}\ ;\ \alpha\,, u > 0$

$\Phi(v) = 1 - e^{-\alpha\, u}$

$\varphi(v), \psi(v)$: pdf and cdf of the rest period of a unit.

$\varphi(v) = \beta\, e^{-\beta\, v};\ \ \beta, r > 0$

$\psi(v) = 1 - e^{-\beta\, v}$

$q_{ij}(k, l, \ldots.)$:pdf of transition time from state S_i *to* S_j (both regenerative) passing through $S_k, S_l, \ldots..$

$Q_{ij}^{(k,l,\ldots.)}$:cdf of transition time from state S_i *to* S_j passing through $S_k, S_l, \ldots.$

$p_{ij/x}^{(k,l,\ldots)}$:steady state probability of transformation from state S_i *to* S_j (or first return to state S_i *if* $j = i$) through states $S_k, S_l, \ldots.$ given that the system entered S_i after a sojourn for time x in the preceeding state.

$p_{ij}^{(k,l,...)}$:steady state probability of transition from state S_i to S_j (both regenerative) passing through $S_k, S_l,...$.

p_{ij} : steady-state probability of direct transition from state S_i to S_j given that the system entered states S_i after a sojourn time x in the preceeding state.

$v_i(t)$: cdf of sojourn time in state S_i

7.4 TRANSITION PROBABILITIES AND SOJOURN TIMES

We know that $p_{67} = 1$.

We first obtain the steady-state conditional probabilities as follows:

$$p_{20/x} = \int \mu\, e^{-\mu y - \lambda r x} I_0\left(2\sqrt{\lambda \mu r x y}\right) e^{-[\lambda(1-r)+\alpha]y} dy$$

$$= \frac{\mu}{\mu'} \exp\left[-\lambda r x\left(1 - \frac{\mu}{\mu'}\right)\right]$$

where

$$\mu' = \mu + \alpha + \lambda(1-r)$$

$$= p_{50/x}$$

$$p_{24/x} = \alpha \int e^{-[\lambda(1-r)+\alpha]y} dy \int_y^\infty \mu\, e^{-\mu \Im - \lambda r x} I\left(2\sqrt{\lambda\, \mu\, r\, x\, y}\right) dz$$

$$= \frac{\alpha}{\lambda(1-r)+\alpha} \left\{ 1 - \frac{\mu}{\mu'} \exp\left\{ -\lambda\, r\, x \left(1 - \frac{\mu}{\mu'} \right) \right\} \right\}$$

$$= p_{54/x}$$

$$p_{26/x} = \int \lambda(1-r) e^{-[\lambda(1-r)+\alpha]x} dy \left(\int_y^\infty \mu\, e^{-\mu r - \lambda r x} I_0\left(2\sqrt{\lambda\, \mu\, r\, x\, \Im}\right) dr \right)$$

$$= \frac{\lambda(1-r)}{\lambda(1-r)+\alpha} \left\{ 1 - \frac{\mu}{\mu'} \exp\left\{ -\lambda\, r\, x \left(1 - \frac{\mu}{\mu'} \right) \right\} \right\}$$

$$= p_{56/x}$$

$$p_{34/x} = \int e^{-\beta y} \mu\, e^{-\mu y - \lambda r x}\ I_0\left(2\sqrt{\lambda\, \mu\, r\, x\, y}\right) dy$$

$$= \frac{\mu}{\mu+\beta} e^{-\lambda r x \left(1 - \frac{\mu}{\mu'} + \beta \right)}$$

$$= p_{41/x}$$

$$p_{35/x} = \int \beta\, e^{-\beta y} dy \int_y^\infty \mu\, e^{-\mu z - \lambda r x} I_0\left(2\sqrt{\lambda\, \mu\, r\, x\, z}\right) dz$$

$$= \int \beta\, e^{-\beta y}\, \overline{K}\, (\mathrm{y} \mid \mathrm{x})$$

$$= 1 - \frac{\mu}{\mu+\beta} e^{-\lambda r x \left(1 - \frac{\mu}{\mu'} + \beta \right)}$$

$$= p_{45/x}$$

$$p_{67/x} = \int \mu\, e^{-\mu y - \lambda r x} I_0\left(2\sqrt{\lambda\, \mu\, r\, x\, y}\right) dy$$

Using these conditional probabilities we obtain the following unconditional probabilities:

$$p_{20} = \int p_{20/x}\, g(x)\, dx$$

$$= \int \left(\frac{\mu}{\mu'}\right) \exp\left\{-\lambda\, r\, x\left(1 - \frac{\mu'}{\mu}\right)\right\}\left[(1-r)e^{-(1-r)x}\, dx\right]$$

$$= \frac{\mu(1-r)}{(\lambda+\mu)(1-r)+\alpha}$$

$$= p_{50} \quad = \quad (A, \ \ say)$$

$$p_{24} = \int \frac{\alpha}{\lambda(1-r)+\alpha}\left[1 - \mu\mu'^{-1} \exp\left\{-\lambda\, r\, x\left(1 - \mu\mu'^{-1}\right)\right\}\right]\lambda(1-r)\exp\left[-\lambda(1-r)x\right]dx$$

$$= \frac{\alpha}{(\lambda+\mu)(1-r)+\alpha}$$

$$= p_{54} \quad = \quad (A_4, \ \ say)$$

Similarly,

$$p_{26} = p_{56} = \frac{(\lambda+\mu)(1-r)}{(\lambda+\mu)(1-r)+\alpha} \quad (= A_3,\ say)$$

$$p_{31} = p_{41} = \frac{\mu(1-r)}{\mu(1-r)+\beta} \quad (= \mathbf{B},\ say)$$

$$p_{35} = p_{45} = \frac{\beta}{\mu(1-r)+\beta} \quad (\overline{B} = 1 - B)$$

$$p_{67} = 1$$

The other unconditional transition probabilities are

$$p_{70} = \frac{\theta}{\theta+\alpha+\lambda(1-r)}; (= c_1; say)$$

$$p_{74} = \frac{\alpha}{\theta+\alpha+\lambda(1-r)}; (= c_2, say)$$

$$p_{76} = \frac{\lambda(1-r)}{\theta+\alpha+\lambda(1-r)}; (= c_3, say)$$

$$p_{01} = \frac{\alpha}{\lambda(1-r)+\alpha}, (= D, say)$$

$$p_{02} = \frac{\lambda(1-r)}{\lambda(1-r)+\alpha}, (= \overline{D} = 1 - D, \quad say)$$

$$p_{10} = \frac{\beta}{\lambda(1-r)+\beta}, (= E, \quad say)$$

$$p_{13} = \frac{\lambda(1-r)}{\lambda(1-r)+\beta}, (= \overline{E} = 1 - E, \quad say)$$

Hence the non-zero elements of the transition probability matrix

$$p = \left[p_{ij}^{(k,l,\ldots)}\right] = \left[Q_{ij}^{(k,l,\ldots)}(\infty)\right]$$

are

$$p_{01} = D, \quad p_{01}^{(2,4)} = \overline{D}\, A_2\, B$$

$$p_{01}^{(2,4,5,4)} = \overline{D}\, A_2\, \overline{B}\, A_2$$

$$p_{00}^{(2)} = \overline{D}\, A_1$$

$$p_{00}^{(2,4,5)} = \overline{D}\, A_2\, \overline{B}\, A_1$$

$$p_{07}^{(2,6)} = p_{04}^{(2)} = \overline{D}\, A_3$$

$$p_{07}^{(2,4,5,6)} = p_{04}^{(2,4,5)} = \overline{D}\, A_2\, \overline{B}\, A_3$$

$$p_{10} = E \;\; ; \; p_{10}^{(3,5)} = \overline{E}\,\overline{B}\, A_1$$

$$p_{11}^{(3)} = \overline{E}.B; \quad p_{11}^{(3,5,4)} = \overline{E}\,\overline{B}\, A_2\, \overline{B}\, A_3$$

$$p_{10}^{(3,5,4,5,6)} = p_{16}^{(3,5,4,5)} = \overline{E}\,\overline{B}\;\, A_2\, \overline{B}\, A_1$$

$$p_{11}^{(3,5,4,5,4)} = \overline{E}\,\overline{B}\, A_2\, \overline{B}\, A_2$$

$$p_{70} = C_1, \quad p_{70}^{(4,5)} = C_2\, \overline{B}\, A_1$$

$$p_{71}^{(4)} = C_2 B, \quad p_{71}^{(4,5,4)} = C_2\, \overline{B}\, A_2$$

$$p_{77}^{(6)} = C_3, \quad p_{77}^{(4,5,6)} = C_2\, \overline{B}\, A_3$$

These transition probabilities are seen to satisfy the following relations.

$$p_{01} + p_{01}^{(2,4)} + p_{01}^{(3,4,5,4)} + p_{00}^{(2)} + p_{00}^{(2,4,5)} + p_{07}^{(2,6)} + p_{07}^{(2,4,5,6)} = 1 \quad (7.2)$$

$$p_{10} + p_{10}^{(3,5)} + p_{10}^{(3,5,4,5)} + p_{11}^{(3)} + p_{11}^{(3,5,4)} + p_{11}^{(3,5,4,5,4)} + p_{17}^{(3,5,6)} + p_{17}^{(3,5,4,5,6)} = 1 \quad (7.3)$$

$$p_{70} + p_{70}^{(4,5)} + p_{71}^{(4)} + p_{71}^{(4,5,4)} + p_{77}^{(6)} + p_{77}^{(4,5,6)} = 1 \quad (7.4)$$

The sojourn times in various regenerative states are

$$\mu_0 = [\alpha + \lambda(1-r)]^{-1} \quad (7.5)$$

$$\mu_1 = [\beta + \lambda(1-r)]^{-1} \quad (7.6)$$

$$\mu_2 = [\alpha + \theta + \lambda(1-r)]^{-1} \quad (7.7)$$

7.5 MEANTIME TO SYSTEM FAILURE

Time to system failure can be regarded as the first passage time to the failed states S_i $(i = 3,4,6)$. Considering the states as absorbing we have, by simple probabilistic reasoning

$$\pi\, dt = Q_{00}^{(2)}(t) \circledS \pi_0(t) + Q_{01}(t) \circledS \pi_1(t) + Q_{04}^{(2)}(t) + Q_{06}^{(2)}(t) \quad (7.8)$$

$$\pi_1(t) = Q_{01}(t) \circledS \pi_0(t) + Q_{13}(t) \tag{7.9}$$

Taking Laplace-Stieltjes transform and solve for $\tilde{\pi}_0(s)$, we get

$$\tilde{\pi}_0 = \frac{\left[\tilde{Q}_{04}^{(2)} + \tilde{Q}_{06}^{(2)} + \tilde{Q}_{01}\tilde{Q}_{13}\right]}{1 - \tilde{Q}_{00}^{(2)} - \tilde{Q}_{01}\tilde{Q}_{10}} \tag{7.10}$$

which gives

$$MTSF = \frac{m_0 + p_{01}m_1}{1 - p_{00}^{(2)} - p_{01}p_{10}}$$

where

$$m_0 = m_{01} + m_{00}^{(2)} + m_{04}^{(2)} + m_{06}^{(2)}$$

$$m_1 = m_{10} + m_{13}$$

and

$m_{ij}^{(k,j,\ldots)}$ have their usual meaning.

7.6 AVAILABILITY ANALYSIS

Let $A_i(t)$= P [the system is up at any time t | S_i at t = 0]

From the arguments used in the theory of regenerative processes,

$$
\begin{aligned}
A_0(t) &= \left\{q_{00}^{(2)}(t) + q_{00}^{(2,4,5)}(t)\right\} © A_0(t) \\
&+ \left\{q_{01}(t) + q_{01}^{(2,4)}(t) + q_{01}^{(2,4,5,4)}(t)\right\} © A_1(t) \\
&+ \left\{q_{07}^{(2,6)}(t) + q_{07}^{(2,4,5,6)}(t)\right\} © A_7(t) \\
&+ e^{-\{\lambda(1-r)+\alpha\}t} + Q_{02}(t) © e^{-\{(\lambda+\mu)(1-r)+\alpha\}t}
\end{aligned}
\tag{7.11}
$$

$$
\begin{aligned}
A_1(t) &= \left\{q_{10}(t) + q_{10}^{(3,5)}(t) + q_{10}^{(3,5,4,5)}(t)\right\} © A_0(t) \\
&+ \left\{q_{11}^{(3)}(t) + q_{11}^{(3,5,4)}(t) + q_{11}^{(3,5,4,5,4)}(t)\right\} © A_1(t) \\
&+ \left\{q_{17}^{(3,5,6)}(t) + q_{17}^{(3,5,4,5,6)}(t)\right\} © A_7(t) + e^{-\{\lambda(1-r)+\beta\}t}
\end{aligned}
\tag{7.12}
$$

$$
\begin{aligned}
A_7(t) &= \left\{q_{70}(t) + q_{70}^{(4,5)}(t)\right\} © A_0(t) \\
&+ \left\{q_{71}^{(4)}(t) + q_{71}^{(4,5,4)}(t)\right\} © A_1(t) \\
&+ \left\{q_{77}^{(6)}(t) + q_{77}^{(4,5,6)}\right\} © A_7(t) + e^{-\{\theta+\alpha+\lambda(1-r)\}t}
\end{aligned}
\tag{7.13}
$$

Taking Laplace transforms for (7.11) – (7.13) and solving for $A_0^*(s)$, we get

$$A_0^*(s) = \frac{N_1(s)}{D_1(s)}$$

where

$N_1(s)$

$$
\begin{aligned}
&= \frac{1}{[\lambda(1-r)+\alpha+s]}\left[\begin{matrix}\left(1-q^{(3)*}{}_{11} - q^{(3,5,4)*}{}_{11} - q^{(3,5,4,5,4)*}{}_{11}\right) \\ \times\left(1-q^{(6)*}{}_{77} - q^{(4,5,4)*}{}_{77}\right) - \left(q_{71}^{(4)*} + q_{71}^{(4,5,4)*}\right)\times\left(q_{17}^{(3,5,4,5,6)*} + q_{71}^{(3,5,6)*}\right.\end{matrix}\right. \\
&+ \frac{1}{\lambda(1-r)+\beta+s}\left[\left(q_{01}^{*} + q_{01}^{(2,4)*}\right)\times\left(1-q_{77}^{(6)*} - q_{77}^{(4,5,6)*}\right)\right] + q_{01}^{(2,4,5,4)*} \\
&+ \left(q_{07}^{(2,6)*} + q_{07}^{(2,4,5,6)*}\right)\left(q_{71}^{(4)*} + q_{71}^{(4,5,4)*}\right)
\end{aligned}
$$

$$+\frac{1}{\theta+\alpha+\lambda(1-r)+s}\begin{bmatrix}\left(q_{01}^{*}+q_{01}^{(2,4)}+q_{01}^{(2,4,5,4)*}\right)\times\left(q_{17}^{(3,5,6)*}+q_{17}^{(3,5,4,5,6)*}\right)+\left(q_{07}^{(2,6)*}+q_{07}^{(2,} \right. \\ \times\left(1-q_{11}^{(3)*}-q_{11}^{(3,5,4)*}-q_{11}^{(3,5,4,5,4)*}\right)\end{bmatrix}$$

and

$$D_1(s)=\left(1-q_{00}^{(2)*}-q_{00}^{(2,4,5)}\right)\begin{bmatrix}\left(1-q_{11}^{(3)*}-q_{11}^{(3,5,4)*}-q_{11}^{(3,5,4,5,4)*}\right)\left(1-q_{77}^{(6)*}-q_{77}^{(4,5,6)*}\right) \\ -\left(q_{71}^{(4)*}+q_{71}^{(4,5,4)*}\right)\left(q_{17}^{(3,5,4,5,6)*}+q_{17}^{(3,5,6)*}\right)\end{bmatrix}-$$

$$\left(q_{01}^{*}+q_{01}^{(2,4)*}+q_{01}^{(2,4,5,4)*}\right)\begin{bmatrix}\left(q_{10}^{*}+q_{10}^{(3,5)*}+q_{10}^{(3,5,4,5)*}\right)\left(1-q_{77}^{(6)*}-q_{77}^{(4,5,6)*}\right)+ \\ \left(q_{70}^{*}+q_{70}^{(4,5)*}\right)\left(q_{17}^{(3,5,6)*}+q_{17}^{(3,5,4,5,6)*}\right)\end{bmatrix}$$

$$-\left(q_{07}^{(2,6)*}+q_{07}^{(2,4,5,6)*}\right)\begin{bmatrix}\left(q_{10}^{*}+q_{10}^{(3,5)*}+q_{10}^{(3,5,4,6)*}\right)\left(q_{71}^{(4)}+q_{71}^{(4,5,4)*}\right)+\left(q_{70}^{*}+q_{70}^{(4,5)*}\right) \\ \left(1-q_{11}^{(3)*}-q_{11}^{(3,5,4)*}-q_{11}^{(3,5,4,5,4)*}\right)\end{bmatrix}$$

The steady state availability of the system is

$$A_{\infty}=\frac{\frac{1}{\lambda(1-r)+\alpha}.U_1+\frac{1}{\lambda(1-r)+\beta}.U_2+\frac{1}{\theta+\alpha+\lambda(1-r)}.U_3}{n_0U_1+n_1U_2+n_7U_3}$$

where

$$U_1=\begin{bmatrix}\left(1-p_{11}^{(3)}-p_{11}^{(3,5,4)}-p_{11}^{(3,5,4,5,4)}\right)\left(1-p_{77}^{(6)}-p_{77}^{(4,5,6)}\right)-\left(p_{17}^{(3,5,6)}+p_{17}^{(3,5,4,5,6)}\right) \\ \left(p_{71}^{(4)}+p_{71}^{(4,5,4)}\right)\end{bmatrix}$$

$$U_2=\left[\left(1-p_{00}^{(2)}-p_{00}^{(2,4,5)}\right)\left(1-p_{77}^{(6)}-p_{77}^{(4,5,6)}\right)-\left(p_{07}^{(2,6)}+p_{07}^{(2,4,5,6)}\right)\left(p_{70}+p_{70}^{(4,5)}\right)\right]$$

$$U_3 = \begin{bmatrix} \left(1 - p_{00}^{(2)} - p_{00}^{(2,4,5)}\right)\left(1 - p_{11}^{(3)} - p_{11}^{(3,5,4)} - p_{11}^{(3,5,4,5,4)}\right) \\ - \left(p_{01} + p_{01}^{(2,4)} + p_{01}^{(2,4,5,4)}\right)\left(p_{10} + p_{10}^{(3,5)} + p_{10}^{(3,5,4,5)}\right) \end{bmatrix}$$

$$n_0 = \sum_j m_{0j}^{(k,l,....)}$$

$$n_1 = \sum_j m_{0j}^{(k,l,....)}$$

$$n_7 = \sum_j m_{7j}^{(k,l,....)}; \qquad k,\ l = 2,\ 3,\ 4,\ 5,\ 6.$$

Therefore the interval availability (Sarma, 1982), for the interval $(0,\ t)$ is

$$A_0(t) = \frac{1}{t}\int_0^t A_0(u)\,du \tag{7.14}$$

so that

$$A_0^*(s) = \int_0^\infty \frac{A_0^*(u)}{u}\,du \tag{7.15}$$

The inherent (limiting interval) availability of the system is

$$A_0(\infty) = \lim_{t\to\infty} A_0(t) = \lim_{s\to 0} s^2\, L\left[\int_0^t A_0(u)\,du\right]$$

$$= \lim_{s\to 0} s\, A_0^*(s) = A_\infty$$

7.6 BUSY PERIOD ANALYSIS

By probabilistic arguments, we obtain the following equations for $\beta_i(t)$.

$\beta_i(t) = P$[the repairman is busy at t | S_i at t = 0]

$$\beta_0(t) = \left\{q_{00}^{(2)}(t) + q_{00}^{(2,4,5)}(t)\right\} © \beta_0(t)$$
$$+ \left\{q_{01}(t) + q_{01}^{(2,4)}(t) + q_{00}^{(2,4,5,4)}(t)\right\} © \beta_1(t)$$
$$+ \left\{q_{07}^{(2,6)}(t) + q_{07}^{(2,4,5,4)}(t)\right\} © \beta_7(t) \quad (7.16)$$

$$\beta_1(t) = \left\{q_{10}(t) + q_{10}^{(3,5)}(t) + q_{10}^{(3,5,4,5)}(t)\right\} © \beta_0(t)$$
$$+ \left\{q_{11}^{(3)}(t) + q_{11}^{(3,5,4)}(t) + q_{11}^{(3,5,4,5,4)}(t)\right\} © \beta_1(t)$$
$$+ \left\{q_{17}^{(3,5,6)}(t) + q_{17}^{(3,5,4,56)}(t)\right\} © \beta_7(t) \quad (7.17)$$

$$\beta_7(t) = \left\{q_{70}(t) + q_{70}^{(4,5)}\right\} © \beta_0(t)$$
$$+ \left\{q_{71}^{(4)}(t) + q_{71}^{(4,5,4)}(t)\right\} © \beta_1(t)$$
$$+ \left\{q_{77}^{(6)}(t) + q_{77}^{(4,5,6)}(t)\right\} © \beta_7(t) + e^{-\theta t} \quad (7.18)$$

Taking the Laplace transforms for (4.16) – (4.18) and solve for $\beta_0^*(s)$, *we get*

$$\beta_0(s) = \frac{N_3(s)}{D_2(s)} \quad (7.19$$

where

$$N_3(s) = \frac{1}{\theta + s}\left[\left(q_{01}^{*} + q_{01}^{(3,5,4)*} + q_{01}^{(3,5,4,5,4)*}\right)\left(q_{17}^{(3,5,6)*} + q_{17}^{(3,5,4,5,6)*}\right) + \left(1 - q_{11}^{(3)*} - q_{11}^{(3,5,4)*} - \right.\right.$$

$$\times\left(q_{07}^{(2,4)*} + q_{07}^{(2,4,5,6)*}\right)$$

and $D_2(s)$ is same as $D_1(s)$.

Then the steady state probability that the repairman will be busy is

$$\beta_\infty = \lim_{t \to \infty} \beta_o(t) = \lim_{s \to 0} s\beta_0^*(s) = \frac{N_2^*(s)}{D_2^*(s)}$$

$$N_3^*(0) = \frac{U_3}{\theta}$$

The expected busy period of the repairman in $(0,\ t]$ is

$$\mu_b(t) = \int_0^t \beta_0(u)\,du \tag{7.20}$$

so that

$$U_b^*(s) = \frac{\beta_0^*(s)}{s} \tag{7.21}$$

and the expected idle period of the repairman in $(0, t]$ is

$$\mu_I(t) = t - \mu_b(t) \tag{7.22}$$

so that

$$\mu_I^*(s) = \frac{1}{s^2} - \mu_b^*(s) \tag{7.23}$$

As $\beta_0^*(s)$ is known explicitly, these quantities can easily be calculated.

7.7 PROFIT ANALYSIS

The expected up-time of the system in $(0,t]$ can be calculated from the pointwise availability as

$$\mu_u(t) = \int_0^t A_0(v)\,dv$$

so that

$$\mu^*{}_u(s) = \frac{A_0^*(s)}{s} \tag{7.24}$$

Let k_0 represent the expected revenue per unit up-time and k_1, the expected repair cost per unit time, then the expected profit in $(0,t]$ is

$$G(t) = k_0\,\mu_u(t) - k_1\,\mu_b(t) \tag{7.25}$$

The expected net profit per unit time in the long run is

$$G = \lim_{t\to\infty}\frac{G(t)}{t} = k_0\ A_\infty - k_1\beta_\infty$$

7.8 SPECIAL CASES

1. When the failure and repair times are independent; i.e. $r = 0$

$$MTSF = \frac{\mu_0 + p_{01}\mu_1 + p_{02}\mu_2}{1-(p_{02}p_{20} + p_{01}p_{10})} \quad (7.26)$$

$$A_\infty = \frac{N_1}{D_1} \quad ; \beta_\infty = \frac{N_2}{D_2}$$

$$; D_1 = D_2$$

$$N_1 = \mu_0[(1-p_{26})-(p_{23}p_{32})]$$
$$+\mu_1[(1-p_{26})(p_{01}+p_{31})+p_{23}(p_{02}p_{31}-p_{01}p_{32})]$$
$$+\mu_2[p_{02}+p_{13}(p_{01}p_{32}-p_{02}p_{31})]$$

$$D_1 = \mu_0[p_{20}(1-p_{13}p_{31})-p_{10}p_{23}p_{32}]+\mu_1[p_{01}(1-p_{26})-p_{23}p_{32}]$$
$$+\mu_2[p_{02}(1-p_{13}p_{31})+p_{01}p_{13}p_{32}]$$
$$+\mu_3[p_{13}(1-p_{26})-p_{13}p_{02}p_{20}]+\mu_6[p_{26}(1-p_{13}p_{31}-p_{01}p_{10})]$$

$$N_2 = (\mu_2 + p_{26}\mu_6)[p_{01}p_{13}p_{32}+p_{02}(1-p_{13}p_{31})]+\mu_3[p_{01}p_{13}(1-p_{26})-p_{02}p_{23}]$$

2. When $\phi(u) = \psi(v) = 0$ and then the states $S_1, S_3, S_4\ and\ S_5$ do not exist. Then

$$MTSF = \frac{\mu + 2\lambda}{\lambda^2(1-r)} \tag{7.27}$$

$$A_\infty = \left[\frac{1 + 2\lambda^2(1-r)}{\theta(\mu + 2\lambda)}\right]^{-1} \tag{7.28}$$

$$\beta_\infty = \frac{\lambda^2}{\theta(\mu + 2\lambda) + 2\lambda^2(1-r)} \tag{7.29}$$

3. When there is no provision for rest and failure and repair times are independent

$$MTSF = \frac{\mu + 2\lambda}{\lambda^2} \tag{7.30}$$

$$A_\infty = \frac{\mu(2\lambda + \mu)}{\mu(2\lambda + \mu) + 2\lambda^2} \tag{7.31}$$

$$\beta_\infty = \frac{\lambda^2}{\mu(2\lambda + \mu) + \lambda^2} \tag{7.32}$$

7.10 NUMERICAL ILLUSTRATION

When $\mu = \theta = 10$; $\alpha = \beta = 0$.

Table 7.1

Profit

λ	r =-0.5	r=0	r=0.5
0	100.0010	100.0911	100.1101
2	92.9110	93.1525	94.6616
4	79.1502	81.5612	85.0315
6	66.8816	73.6116	80.1506
8	54.1606	61.4441	69.7012
10	43.2806	53.3315	62.1111
12	40.0015	47.6106	56.0152
14	36.1585	42.6150	49.1566
16	32.6617	39.9915	45.8106

Table 7.2

Profit

λ	$\alpha = 4, \beta = 3$	$\alpha = \beta = 0$
0	100.0911	100.0911
2	90.9106	93.1525
4	77.5505	81.5612
6	67.8819	73.6116
8	62.5531	61.4441
10	53.3316	53.3315
12	49.1606	47.6106
14	44.1629	42.6150
16	42.8718	39.4915

Table 7.2

CONCLUSIONS:

From **Table 7.1** we conclude that as failure rate increases the mean time to system failure (MTSF) decreases. For both models as the failure rate increases the MTSF of the system decreases but as the failure rate continues increasing MTSF goes on decreasing.

From **Table 7.2** we conclude that for both models as the failure rate increases the profit of the system decreases but comparatively less when the failure rate increases less; model 2 is more beneficial than model 1 and as the failure rate continues increase the profit difference goes on decreasing. As cost per visit of the repairman increases, the profit of the system decreases.

To observe the effect of correlation and rest on the profit (in the steady state), we plot the profit function against λ,
setting $\mu = \theta = 10, k_0 = 100 \;\; and \;\; k_1 = 20$. The curves so obtained are shown in Table 7.1 and 7.2 respectively. In Table 7.1, in addition, we set $\alpha = \beta = 0$ and obtain three different curves for profit function vs λ .Taking r=-0.5, 0.0 and 0.5 respectively. In table 7.2, we put r=0 in addition to the values of μ, θ, k_0, k_1 and obtain two different values of profit function against λ ,one
with $\alpha = 4, \; \beta = 3$ (i.e. when there is provision for rest) and the other with $\alpha = \beta = 0$ (i.e. when there is no provision for rest).
These values reveal two important facts:

1. The profit/unit time (in steady state) decreases with respect to the increase in λ. However, for the same λ the profit increases with increases in r. Thus a high positive correlation between failure and repair times tends to increase the profit earned by the system in steady state.

The effect of providing rest for the operative unit depends on the proportion of values of λ and μ. Although in both cases (i.e. when $\alpha = \beta = 0$ or when $\alpha = 4, \beta = 3$) the profit decreases with increase in failure rate, a favourable effect of providing rest is observed when only when $\lambda > \mu$, i.e. when the failure rate is higher than the repair rate. As long as $\lambda < \mu$, the provision of rest is nothing but a costly burden on the systems manager, and when $\lambda = \mu$, the profit with or without rest is the same, so there is no advantage in providing rest. Thus, one must avoid providing rest as long as $\lambda \leq \mu$. But since, in practice, most of the time, the failure rate is much higher than the repair rate, a considerable increase in profit can be obtained by providing rest to the operative unit and taking output from the standby unit during the rest time of the operating unit.

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