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Chapter 1

INTRODUCTION

This dissertation describes an investigation into an apparent paradox that sometimes occurs when additional capacity is added to a road network. This phenomenon was presented by Braess [9] in 1968 and occurs when adding extra capacity to a network results in an increase in the total travel time on the network. This is counter-intuitive, and is known as Braess' paradox.

Braess used a very small network to illustrate the paradox and it could be considered a contrived example. However, cases of it occurring on actual city street networks have been reported.

Knödel [38] cited in Murchland [43] states that the effect could occur in reality as was shown by a case in Stuttgart. Major road investments in the city centre, in the vicinity of the Schlossplatz, failed to yield the benefits that had been expected. The benefits were only obtained when a cross street, the lower part of Königstrasse, was withdrawn from use by traffic.

The New York Times [57] reports on a second case that could be a real-world example of Braess' Paradox. In 1990, Lucius J Riccio, New York City's Transportation Commissioner decided to close 42nd Street on Earth Day (22 April). According to the article, 42nd Street is "as every New Yorker knows is always congested" To everyone's surprise, no traffic jam occurred in the vicinity of 42nd Street on Earth Day. Traffic flow actually improved when 42nd Street was closed.

This introduction continues with a brief description of the traffic assignment problem. Descriptions of different assignment techniques that are used are also given. More detail is provided on the user equilibrium assignment technique, which is the most commonly used assignment technique. Braess' paradox is then presented with the network that he used. This is followed by a literature survey of references to Braess' and other paradoxes that occur in transportation networks. An analysis of real world occurrences of Braess' paradox is provided. In this analysis an example where 186 new road sections were proposed for a regional road network were tested for the occurrence of Braess' paradox. When considered individually a number of these proposals result in the occurrence of Braess' paradox. Methods are derived to reduce the amount of computational effort required to eliminate all the proposed sections of road that will result in Braess' paradox. This will result in a more cost efficient road construction programme for the future.

1.1 The Traffic Assignment Problem

1.1.1 Representation of the Transportation Network

A transportation network can be represented by a set of nodes and a set of links (usually they are directed links) connecting the nodes. The links represent the roads in the network and the nodes represent intersections and interchanges. The term "network" is used to represent both the physical structure and its mathematical representation.

Links can also represent transit lines (bus and rail routes) and nodes can represent bus stops, transfer stations, etc.

The area that is covered by the network is usually divided into traffic zones. Each traffic zone is represented by a node known as the centroid. These centroids are the "source" and "sink" nodes where traffic originates and to which traffic is destined. Centroids are connected to the network by means of centroid connectors.

Once the set of centroids has been defined, the movement of traffic over a transportation network can be expressed in terms of an origin-destination matrix. This matrix specifies the flow between every origin centroid and every destination centroid in the network. Centroids can be both origins and destinations.

The links have associated with them a travel impedance, which can include many factors such as travel time, safety, cost of travel, etc. The major component of this impedance is travel time, which is often used as the only measure of link impedance. There are three reasons for using only travel time:

- Studies have indicated that it is the primary deterrent for flow,
- most other measures of travel impedance are highly correlated with travel time, and
- it is easier to measure than many of the other possible components of impedance.

The impedance experienced by many transportation systems is a function of the usage of these systems. Due to congestion, the travel time on streets is an increasing function of flow. Therefore, a performance function rather than a constant travel time measure should be associated with each of the links of the network. The performance function relates the travel time on each link to the flow on the link. A typical link performance function is shown in Figure 1.1. (This section is based on parts of chapter 1 of Sheffi [51].)

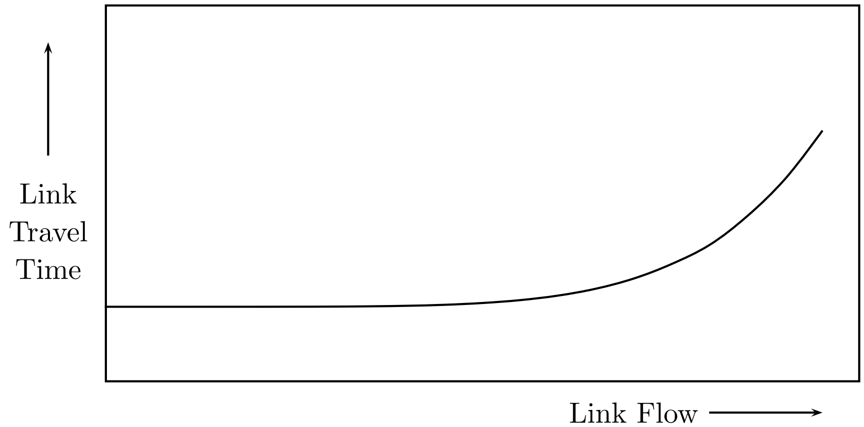


Figure 1.1: Typical Link Performance Function

1.2 Equilibrium in Transportation Networks

The idea of equilibrium in the analysis of transportation networks arises from the dependence of the link travel times on the link flows. Assuming that a number of motorists wish to travel between a given origin and a given destination, which are connected by a number of possible paths, how will the motorists be distributed among the possible paths? If all the motorists were to use the same path (initially the shortest path in terms of travel time), this path would become more congested. This would result in the travel time on this path increasing. A point might be reached where this is no longer the shortest path in terms of travel time. Some of the motorists would then divert to an alternative path that, however, might also be congested.

The system will be in a state of equilibrium when no motorist will want to change from his present path to an alternative path. This is known as the user equilibrium situation where every motorist seeks to minimize his travel time. There is second situation where some motorists travel for longer than the minimum time, but the total travel time of all motorists on the network is minimized. This second situation is known as system optimal situation.

In 1952 Wardrop [62] proposed two principles of route choice that would result in the above two states of equilibrium. They were:

Def 2.1. User Equilibrium: The journey times in all routes used are equal and less than those which would be experienced by a single vehicle on any unused route.

Def 2.2. System Optimal: The average journey time over all routes is a minimum.

These two principles of Wardrop correspond to the following two sets of circumstances:

- Drivers choose their routes independently in their own best interests in the light of the traffic conditions resulting from the choice of others (user equilibrium).

- Drivers cooperate in their choice of routes so as to produce a pattern of traffic flows giving the maximum benefit to the community (system optimal).

It is generally accepted that in practice Wardrop's first principle is the more likely basis for network equilibrium.

Although Wardrop is credited with the above two principles of equilibrium, similar ideas had been expressed by others. Correa et al. [17] cite Kohl [39] as saying in 1841 that travellers minimize their individual travel times.

Florian [28] cites Knight [37] as describing the following traffic pattern that he called equilibrium in 1924:

"Suppose that between two points there are two highways, one of which is broad enough to accommodate without crowding all the traffic which may care to use it, but it is poorly graded and surfaced; while the other is a much better road, but narrow and quite limited in capacity. If a large number of trucks operate between the two termini and are free to choose either of the two routes, they will tend to distribute themselves between the roads in such proportions that the cost per unit of transportation, or effective returns per unit of investment, will be the same for every truck on both routes. As more trucks use the narrower and better road, congestion develops, until at a certain point it becomes equally profitable to use the broader but poorer highway."

The problem of assigning or allocating all motorists to the various paths or routes is known as the traffic assignment problem. The simplest form of assignment is known as an all-or-nothing assignment where all trips are assigned to the shortest possible route without taking the effects of congestion into account.

Different traffic assignment techniques are described in the following section.

1.3 Some Traffic Assignment Techniques

1.3.1 All-or-nothing assignment

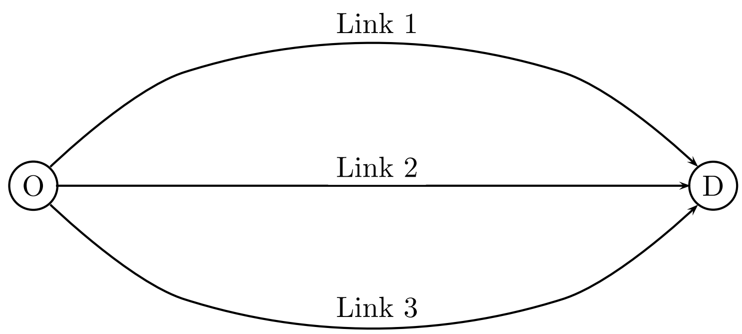
This is the simplest form of traffic assignment. In this procedure, every origin-destination (O-D) flow, between an origin node and a destination node, is assigned to all the links that are on the minimum travel time path connecting the two nodes. All other paths connecting the two nodes are not assigned any flow.

During this process, the link travel times are assumed to be fixed (not dependent on the flow on them).

Many early transportation studies used the all-or-nothing traffic assignment procedure based on empty network travel times. The travel times on the links after the flows have been assigned to them will, in most cases, be different from the times on which the assignment was based. Since this assignment method does not take into account the dependence between flows and travel time, it, in effect, ignores the equilibrium problem.

1.3.2 Incremental assignment

This is a heuristic technique where a portion of the origin-destination matrix is assigned to the network at each iteration using the all-or-nothing method. After each assignment, the travel times on all links are recalculated taking the assigned flows into consideration, before the next portion of the matrix is assigned. Sheffi [51] provides an example using the network shown in Figure 1.2 to illustrate the method.



$$t_1 = 10\left[1 + 0.15\left(\tfrac{x_1}{2}\right)^4\right]$$
$$t_2 = 20\left[1 + 0.15\left(\tfrac{x_2}{4}\right)^4\right]$$
$$t_3 = 25\left[1 + 0.15\left(\tfrac{x_3}{3}\right)^4\right]$$

Total flow from O to D = 10 flow units

Figure 1.2: Network Example with Three Links and one O-D Pair

The algorithm for the incremental assignment is as follows (Sheffi, [51]):

Step 0: *Preliminaries.* Divide each origin-destination entry into N equal portions (i.e. set $q_{rs}^n = q_{rs}/N$). Set $n = 1$ and $x_a^0 = 0, \quad \forall \ a$.

Step 1: *Update.* Set $t_a^n = t_a\left(x_a^{n-1}\right), \quad \forall \ a$.

Step 2: *Incremental loading.* Perform an all-or-nothing assignment based on $\{t_a^n\}$, but using only the trip rates q_{rs}^n for each O-D pair. This yields a flow pattern $\{w_a^n\}$.

Step 3: *Flow summation.* Set $x_a^n = x_a^{n-1} + w_a^n, \quad \forall \ a$.

Step 4: *Convergence test.* If $n = N$, stop (the current set of link flows is the solution), otherwise set $n = n + 1$ and go to step 1.

where:

t_a^n = time on link a at iteration n
w_a^n = incremental loading on link a at iteration n

x_a^n = loading on link a after iteration n

q_{rs} = total flow from origin r to destination s

q_{rs}^n = flow from r to s at iteration n

Table 1.1 shows the application of the incremental assignment algorithm to the network shown in Figure 1.2 when four equal increments of 2.5 flow units are applied to the network.

Table 1.1: Incremental Assignment Algorithm Applied to the Network in Figure 1.2

Iteration	Algorithm Step	Link 1	Link 2	Link 3
1	Update	$t_1^1 = 10$	$t_2^1 = 20$	$t_3^1 = 25$
	Incremental loading	$w_1^1 = 2.5$	$w_2^1 = 0$	$w_3^1 = 0$
	Summation	$x_1^1 = 2.5$	$x_2^1 = 0$	$x_3^1 = 0$
2	Update	$t_1^2 = 14$	$t_2^2 = 20$	$t_3^2 = 25$
	Incremental loading	$w_1^2 = 2.5$	$w_2^2 = 0$	$w_3^2 = 0$
	Summation	$x_1^2 = 5.0$	$x_2^2 = 0$	$x_3^2 = 0$
3	Update	$t_1^3 = 69$	$t_2^3 = 20$	$t_3^3 = 25$
	Incremental loading	$w_1^3 = 0$	$w_2^3 = 2.5$	$w_3^3 = 0$
	Summation	$x_1^3 = 5.0$	$x_2^3 = 2.5$	$x_3^3 = 0$
4	Update	$t_1^4 = 69$	$t_2^4 = 20.5$	$t_3^4 = 25$
	Incremental loading	$w_1^4 = 0$	$w_2^4 = 2.5$	$w_3^4 = 0$
	Summation	$x_1^4 = 5.0$	$x_2^4 = 5.0$	$x_3^4 = 0$
	Travel time at end	$t_1^* = 68.6$	$t_2^* = 27.3$	$t_3^* = 25.0$

As shown in Table 1.1, the incremental assignment method also does not necessarily converge or result in a set of flows that represent the user equilibrium flow pattern. The two used travel paths (links 1 and 2) do not have equal travel times and the travel times on these two links are higher than that on the unused path (link 3).

1.3.3 Capacity restraint assignment

This heuristic technique is also sometimes called the iterative assignment technique. This method involves a number of all-or-nothing assignments in which the travel times resulting from the previous assignment are used in the current iteration. This method does not necessarily converge. To overcome this problem the algorithm is terminated after a given number of iterations, N. The equilibrium flow pattern is then taken to be the average flow for each link over the last four iterations. This method does not necessarily result in a true equilibrium flow pattern.

The algorithm for the capacity restraint assignment method is as follows (Sheffi [51]):

Step 0: *Initialization.* Perform all-or-nothing assignment based on $t_a^0 = t_a(0), \ \forall \ a$. Obtain a set of link flows $\{x_a^0\}$. Set iteration counter $n = 1$.

Step 1: *Update.* Set $t_a^n = t_a\left(x_a^{n-1}\right), \quad \forall \ a$.

Step 2: *Network loading.* Assign all trips to the network using all-or-nothing assignment based on travel times $\{t_a^n\}$. This yields a set of link flows $\{x_a^n\}$.

Step 3: *Stopping rule.* If $n = N$, go to step 4. Otherwise set $n = n + 1$ and go to step 1.

Step 4: *Averaging.* Set $x_a^* = \frac{1}{4}\sum_{i=0}^{3} x_a^{N-i} \quad \forall a$ and stop. ($\{x_a^*\}$ are the final link flows.)

Table 1.2 shows the results of using this algorithm on the example shown in Figure 1.2.

Table 1.2: Capacity Restraint Algorithm Applied to the Network in Figure 1.2

Iteration	**Algorithm Step**	**Link 1**	**Link 2**	**Link 3**
0	Initialization	$t_1^0 = 10$ $x_1^0 = 10$	$t_2^0 = 20$ $x_2^0 = 0$	$t_3^0 = 25$ $x_3^0 = 0$
1	Update Loading	$t_1^1 = 947$ $x_1^1 = 0$	$t_2^1 = 20$ $x_2^1 = 10$	$t_3^1 = 25$ $x_3^1 = 0$
2	Update Loading	$t_1^2 = 10$ $x_1^2 = 10$	$t_2^2 = 137$ $x_2^2 = 0$	$t_3^2 = 25$ $x_3^2 = 0$
3	Update Loading	$t_1^3 = 947$ $x_1^3 = 0$	$t_2^3 = 20$ $x_2^3 = 10$	$t_3^3 = 25$ $x_3^3 = 0$
	Average	$x_1^* = 5.0$ $t_1^* = 68.6$	$x_2^* = 5.0$ $t_2^* = 27.3$	$x_3^* = 0$ $t_3^* = 25$

As shown in Table 1.2, the capacity restraint method does not necessarily converge. In fact in this case it "flip-flops" between links 1 and 2 and link 3 does not get loaded at all.

To remedy this situation, the algorithm can be modified as follows. Instead of using the travel time obtained in the last iteration for the new loading, a combination of the last two travel times obtained is used. This introduces a "smoothing" effect. The steps of the modified capacity restraint algorithm (using weights of 0.75 and 0.25 for the averaging process) are as follows (Sheffi, [51]):

Step 0: *Initialization.* Perform all-or-nothing assignment based on $t_a^0 = t_a(0), \forall\, a$. Obtain a set of link flows $\{x_a^0\}$. Set iteration counter $n = 1$.

Step 1: *Update.* Set $\tau_a^n = t_a\left(x_a^{n-1}\right), \ \forall\, a$.

Step 2: *Smoothing.* Set $t_a^n = 0.75 t_a^{n-1} + 0.25\tau_a^n, \ \forall\, a$.

Step 3: *Network loading.* Assign all trips to the network using all-or-nothing assignment based on travel times $\{t_a^n\}$. This yields a set of link flows $\{x_a^n\}$.

Step 4: *Stopping rule.* If $n = N$, go to step 5. Otherwise set $n = n + 1$ and go to step 1.

Step 5: *Averaging.* Set $x_a^* = \frac{1}{4}\sum_{i=0}^{3} x_a^{N-i} \quad \forall a$ and stop. ($\{x_a^*\}$ are the final link flows.)

The smoothing is accomplished by creating a temporary link-travel-time variable τ_a^n, which is not used as the travel time for the next iteration (see step 1). Instead it is averaged together with the travel time used in the last iteration, t_a^{n-1}, to obtain the link travel time for the current iteration, t_a^n.

The application of this modified capacity restraint algorithm using the network shown in Figure 1.2 is shown in Table 1.3.

Table 1.3: Modified Capacity Restraint Algorithm Applied to the Network in Figure 1.2

Iteration	**Algorithm Step**	**Link 1**	**Link 2**	**Link 3**
0	Initialization	$t_1^0 = 10$	$t_2^0 = 20$	$t_3^0 = 25$
		$x_1^0 = 10$	$x_2^0 = 0$	$x_3^0 = 0$
1	Update	$\tau_1^1 = 947.5$	$\tau_2^1 = 20$	$\tau_3^1 = 25$
	Smoothing	$t_1^1 = 244.4$	$t_2^1 = 20$	$t_3^1 = 25$
	Loading	$x_1^1 = 0$	$x_2^1 = 10$	$x_3^1 = 0$
2	Update	$\tau_1^2 = 10$	$\tau_2^2 = 137.2$	$\tau_3^2 = 25$
	Smoothing	$t_1^2 = 185.8$	$t_2^2 = 49.3$	$t_3^2 = 25$
	Loading	$x_1^2 = 0$	$x_2^2 = 0$	$x_3^2 = 10$
3	Update	$\tau_1^3 = 10$	$\tau_2^3 = 20$	$\tau_3^3 = 488$
	Smoothing	$t_1^3 = 141.8$	$t_2^3 = 42.0$	$t_3^3 = 140.7$
	Loading	$x_1^3 = 0$	$x_2^3 = 10$	$x_3^3 = 0$
4	Update	$\tau_1^4 = 10$	$\tau_2^4 = 137.2$	$\tau_3^4 = 25$
	Smoothing	$t_1^4 = 108.9$	$t_2^4 = 65.8$	$t_3^4 = 111.8$
	Loading	$x_1^4 = 0$	$x_2^4 = 10$	$x_3^4 = 0$
5	Update	$\tau_1^5 = 10$	$\tau_2^5 = 137.2$	$\tau_3^5 = 25$
	Smoothing	$t_1^5 = 84.2$	$t_2^5 = 83.6$	$t_3^5 = 90.1$
	Loading	$x_1^5 = 0$	$x_2^5 = 10$	$x_3^5 = 0$
6	Update	$\tau_1^6 = 10$	$\tau_2^6 = 137.2$	$\tau_3^6 = 25$
	Smoothing	$t_1^6 = 65.6$	$t_2^6 = 97.2$	$t_3^6 = 73.8$
	Loading	$x_1^6 = 10$	$x_2^6 = 0$	$x_3^6 = 0$
	Average	$x_1^* = 2.5$	$x_2^* = 5.0$	$x_3^* = 2.5$
		$t_1^* = 13.7$	$t_2^* = 27.3$	$t_3^* = 26.8$

1.3.4 Equilibrium assignment

As described in the following section it is possible to formulate the assignment problem as a mathematical program, the solution of which provides the user equilibrium flow pattern on the road network. This is the method that was used in the 1985 PWV Transportation Study (and subsequent studies).

1.4 The Assignment Problem as a Mathematical Program

Beckmann et al. [4], (cited by Sheffi [51]), developed the following transformation which enabled the user equilibrium problem to be solved as a nonlinear programming problem. The equilibrium assignment problem is to find the link flows, $\mathbf{x}$, that will satisfy the user equilibrium criterion when all the origin-destination entries, $\mathbf{q}$, have been appropriately assigned. This link-flow pattern can be obtained by solving the following mathematical program (Sheffi, [51]):

$$\min z(x) = \sum_a \int_0^{x_a} t_a(\omega) d\omega$$

subject to

$$\sum_k f_k^{rs} = q_{rs} \qquad \forall\, r, s$$

$$f_k^{rs} \geq 0 \qquad \forall\, k, r, s$$

and $$x_a = \sum_r \sum_s \sum_k f_k^{rs} \delta_{a,k}^{rs}$$

where

x_a = flow on arc a; $\mathbf{x} = (...,x_a,....)$
t_a = travel time on arc a; $\mathbf{t} = (...,t_a,....)$
f_k^{rs} = flow on path k connecting O-D pair $r-s$; $\mathbf{f}^{rs} = (...., f_k^{rs}, ...)$
q_{rs} = trip rate between origin r and destination s
$\delta_{a,k}^{rs}$ = indicator variable; $\delta_{a,k}^{rs} \begin{cases} = & 1 \text{ if link } a \text{ is on path } k \text{ between O-D pair } rs \\ = & 0 \text{ otherwise} \end{cases}$

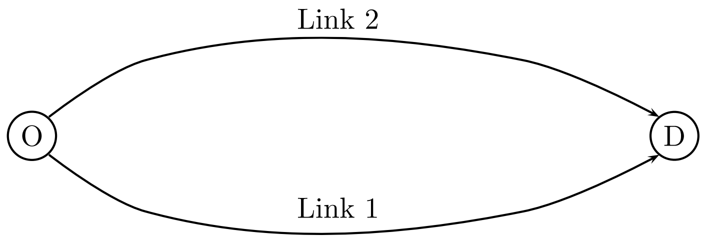
In this formulation, the objective function is the sum of the integrals of the link performance functions. This function does not have any intuitive economic or behavioural interpretation. It should be viewed strictly as a mathematical construct that is utilized to solve equilibrium problems.

The first set of constraints represents a set of flow conservation constraints. These constraints state that the flow on all paths connecting each O-D pair has to equal the O-D trip rate. This means that all O-D trips have to be assigned to the network. The second set of constraints are non-negativity conditions and are required to ensure that the solution of the problem will be physically meaningful.

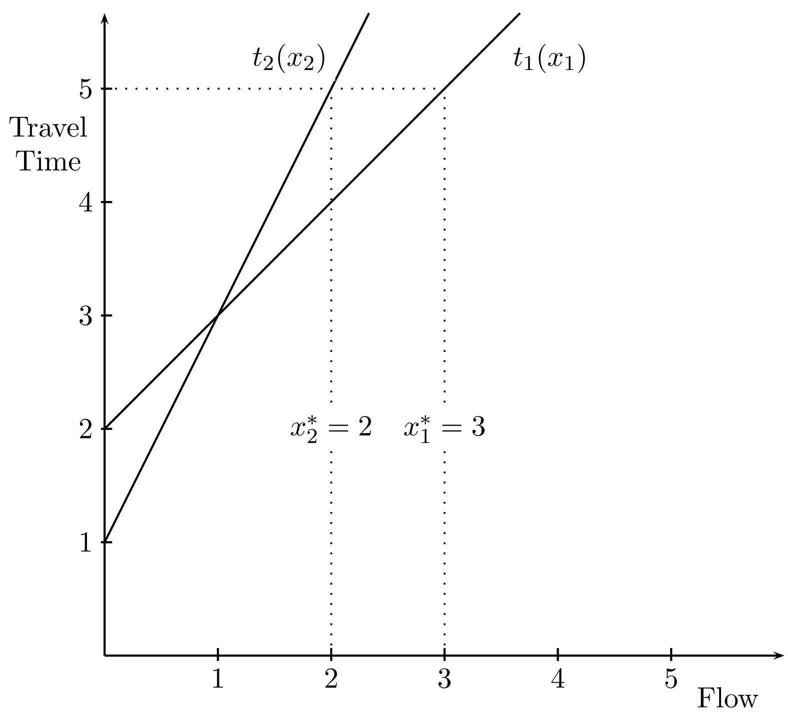
The objective function above is formulated in terms of link flows while the flow conservation constraints are expressed in terms of path flows. The third set of constraints are definitional incidence relationships which bring the network structure into the formulation.

It has been proved that this converges to a unique solution (see Sheffi, [51], pp. 63-69).

Sheffi [51] provides the following simple example showing the solution of an equilibrium assignment on a two-link network as shown in Figure 1.3.



(a) Two-link network



(b) The performance functions and the equilibrium solution

Figure 1.3: Equilibrium Example

The network shown in Figure 1.3 has two paths (which are also links), leading from the origin, O, to the destination, D. The volume-delay curves for the two links are given by:

$$t_1 = 2 + x_1$$

$$t_2 = 1 + 2x_2$$

The O-D flow, q, is 5 units of flow, that is,

$$x_1 + x_2 = 5$$

The equilibrium condition for this example can be expressed as

$$t_1 \leq t_2 \text{ if } x_1 > 0 \text{ and } t_1 \geq t_2 \text{ if } x_2 > 0$$

In this example it can be verified by inspection that both paths will be used at equilibrium and that the last equation can therefore be written more simply (given that $x_1 > 0$ and $x_2 > 0$) as

$$t_1 = t_2$$

The equilibrium problem then, is to solve four equations (the two volume-delay curves, the flow conservation condition ($x_1 + x_2 = 5$), and the user-equilibrium condition ($t_1 = t_2$), in four unknowns: x_1, x_2, t_1, and t_2. The solution to this set of equations is

$$x_1 = 3 \text{ flow units}$$

$$x_2 = 2 \text{ flow units}$$

$$t_1 = t_2 = 5 \text{ time units}$$

When the problem is formulated as a minimization program, the result is the following:

$$\min \mathrm{z}(\mathbf{x}) = \int_0^{x_1}(2+\omega)d\omega + \int_0^{x_2}(1+2\omega)d\omega$$

subject to

$$x_1 + x_2 = 5$$

$$x_1, x_2 \geq 0$$

To set the problem up as a simple one-dimensional unconstrained minimization, $x_2 = 5 - x_1$ can be substituted into the objective function and into the remaining (non-negativity) constraints to get the problem

$$\min \mathrm{z}(\mathbf{x}) = \int_0^{x_1}(2+\omega)d\omega + \int_0^{5-x_1}(1+2\omega)d\omega$$

subject to

$$x_1 = 0 \text{ and } 5 - x_1 = 0$$

To solve this program, the constraints can be relaxed and the objective function can be minimized as in an unconstrained program. If the solution satisfies the constraints, it is valid for the constrained program as well. Carrying out the integration and collecting similar terms, the objective function becomes

$$z(x_1) = 1.5x_1^2 - 9x_1 + 30$$

This function attains its minimum at $x_1^* = 3$, where $\frac{dz(x_1)}{dx_1} = 0$. This solution satisfies the two constraints ($x_1 = 0$ and $5 - x_1 = 0$) and is therefore a minimum of the constrained program as well. The original flow conservation constraint guarantees that $x_2^* = 2$ and indeed, the solution of the mathematical program is identical to the solution of the equilibrium equations. (Sheffi, [51])

As shown above, the flow pattern that minimizes the user-equilibrium equivalent program is, a user-equilibrium solution. This program includes a convex (non-linear) objective function and a linear constraint set. The convex combinations method (Frank-Wolfe) method is a suitable method to solve this problem. Using this method, a series of all-or-nothing assignments are combined with the results from the previous step until a stopping criterion is reached (see Sheffi, [51], pp. 117 -119). The choice of an appropriate stopping criterion is discussed in Chapter 5 of this document.

The following algorithm describes the process to find a user equilibrium flow pattern using the Frank-Wolfe method:

Step 0: *Initialization.* Perform an all-or-nothing assignment based on $t_a = t_a(0)$, a. This yields $\{x_a^1\}$. Set iteration counter $n = 1$.

Step 1: *Update.* Set $t_a^n = t_a\left(x_a^n\right), \ \forall\, a$.

Step 2: *Direction finding.* Perform an all-or-nothing assignment based on $\{t_a^n\}$. This yields a set of (auxiliary) flows $\{y_a^n\}$.

Step 3: *Line search.* Find a_n that solves

$$\min_{0 \leq a \leq 1} \sum_a \int_0^{x_a^n + a(y_a^n - x_a^n)} t_a(\omega) d\omega$$

Step 4: *Move.* Set $x_a^{n+1} = x_a^n + a_n\left(y_a^n - x_a^n\right), \ \forall\, a$.

Step 5: *Convergence test.* If a convergence criterion is met, stop (the current solution, x_a^{n+1}, is the set of equilibrium link flows); otherwise set $n = n + 1$ and go to step 1.

The issue of an appropriate convergence criterion will be discussed in Chapter 4.

The result of applying the Frank-Wolfe (convex combinations) algorithm to the three link network shown in Figure 1.2 is shown in Table 1.4.

The results in Table 1.4 show that that the convergence towards equilibrium is much better than those of the heuristic methods shown earlier. After five iterations the flows are close to equilibrium with the travel times being almost equal. As shown in Table 1.4, flow is taken away from the congested paths and assigned to less congested paths during each iteration. This process equalizes the travel times among all the paths and moves the system towards equilibrium. The marginal contribution of each successive iteration to the reduction in the value of the objective function decreases with each iteration. (Sheffi, [51])

1.5 Braess' Paradox

In 1968 Braess [9], [10] presented an example of an equilibrium assignment problem that produces an apparently paradoxical result. In this example, the addition of a new link to the network results in an increase in the total travel time on the network, instead of the decrease that would be intuitively expected. This phenomenon has become known as Braess' Paradox and is discussed below.

Figure 1.4 shows a simple network including one O-D pair that is connected by four links and

Table 1.4: Frank-Wolfe Algorithm Applied to the Network in Figure 1.2

Iteration	**Algorithm Step**	**Link 1**	**Link 2**	**Link 3**	**Objective Function**	**Step Size**
0	Initialization	$t_1^0 = 10.0$	$t_2^0 = 20.0$	$t_3^0 = 25.0$		
		$x_1^0 = 10.00$	$x_2^0 = 0.00$	$x_3^0 = 0.00$		
1	Update	$t_1^1 = 947.5$	$t_2^1 = 20.0$	$t_3^1 = 25.0$	1975.00	
	Direction	$y_1^1 = 0$	$y_2^1 = 10$	$y_3^1 = 0$		0.596
	Move	$x_1^2 = 4.04$	$x_2^2 = 5.96$	$x_3^2 = 0$		
2	Update	$t_1^2 = 35.0$	$t_2^2 = 35.0$	$t_3^2 = 25.0$	197.00	
	Direction	$y_1^2 = 0$	$y_2^2 = 0$	$y_3^2 = 10$		0.161
	Move	$x_1^3 = 3.39$	$x_2^3 = 5.00$	$x_3^3 = 1.61$		
3	Update	$t_1^3 = 22.3$	$t_2^3 = 27.3$	$t_3^3 = 25.3$	189.98	
	Direction	$y_1^3 = 10$	$y_2^3 = 0$	$y_3^3 = 0$		0.035
	Move	$x_1^4 = 3.62$	$x_2^4 = 4.83$	$x_3^4 = 1.55$		
4	Update	$t_1^4 = 26.1$	$t_2^4 = 26.3$	$t_3^4 = 25.3$	189.44	
	Direction	$y_1^4 = 0$	$y_2^4 = 0$	$y_3^4 = 10$		0.020
	Move	$x_1^5 = 3.54$	$x_2^5 = 4.73$	$x_3^5 = 1.72$		
5	Update	$t_1^5 = 24.8$	$t_2^5 = 25.8$	$t_3^5 = 25.4$	189.33	
	Direction	$y_1^5 = 10$	$y_2^5 = 0$	$y_3^5 = 0$		0.007
	Move	$x_1^6 = 3.59$	$x_2^6 = 4.70$	$x_3^6 = 1.71$		
	Update	$t_1^6 = 25.6$	$t_2^6 = 25.7$	$t_3^6 = 25.4$	189.33	

two paths. The figure shows the two paths (numbered 1 and 2) and the link performance functions for the four links. Assume that six units of flow travel between O and D (ie q=6). The user equilibrium flow pattern for this network can be solved by inspection (due to the travel time symmetry of the two paths). It is obvious that half of the flow would use each path and that the solution would be:

$$f_1 = f_2 = 3 \text{ flow units}$$

or, in terms of link flows,

$$x_1 = x_2 = x_3 = x_4 = 3 \text{ flow units.}$$

The associated link travel times are:

$$t_1 = 53, t_2 = 53, t_3 = 30, t_4 = 30 \text{ time units}$$

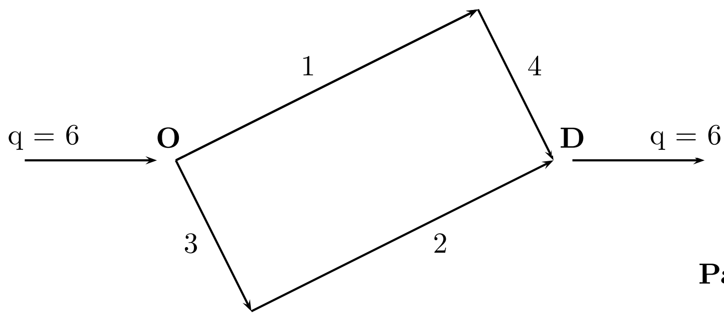
and the path times are

$$c_1 = c_2 = 83 \text{ time units, satisfying the user equilibrium criterion.}$$

The total travel time on the network is 498 (flow-time) units.

Figure 1.5 shows the network expanded to include a new link connecting the two intermediate nodes. The figure shows this added (fifth) link, the performance function for this link, and the new path (number 3) resulting from the addition of the link.

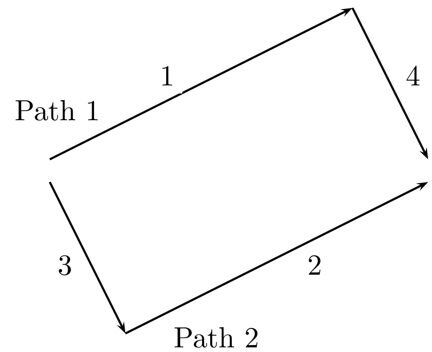
The old UE flow pattern is no longer an equilibrium solution since,



Link Performance Data

$$t_1(x_1) = 50 + x_1$$
$$t_2(x_2) = 50 + x_2$$
$$t_3(x_3) = 10x_3$$
$$t_4(x_4) = 10x_4$$

Path Definitions



User Equilibrium Solution

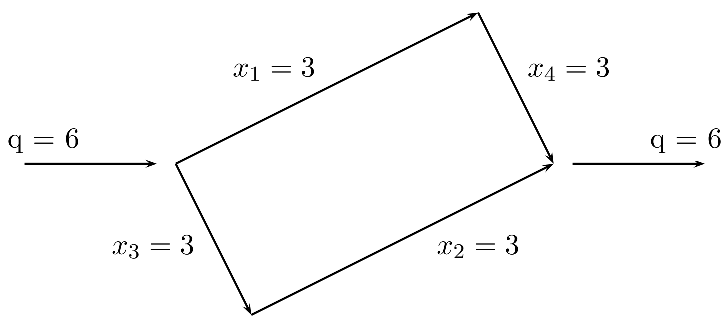


Figure 1.4: Intial Braess' Network with Equilibrium Flows

$x_1 = 3, x_2 = 3, x_3 = 3, x_4 = 3, x_5 = 0$ flow units

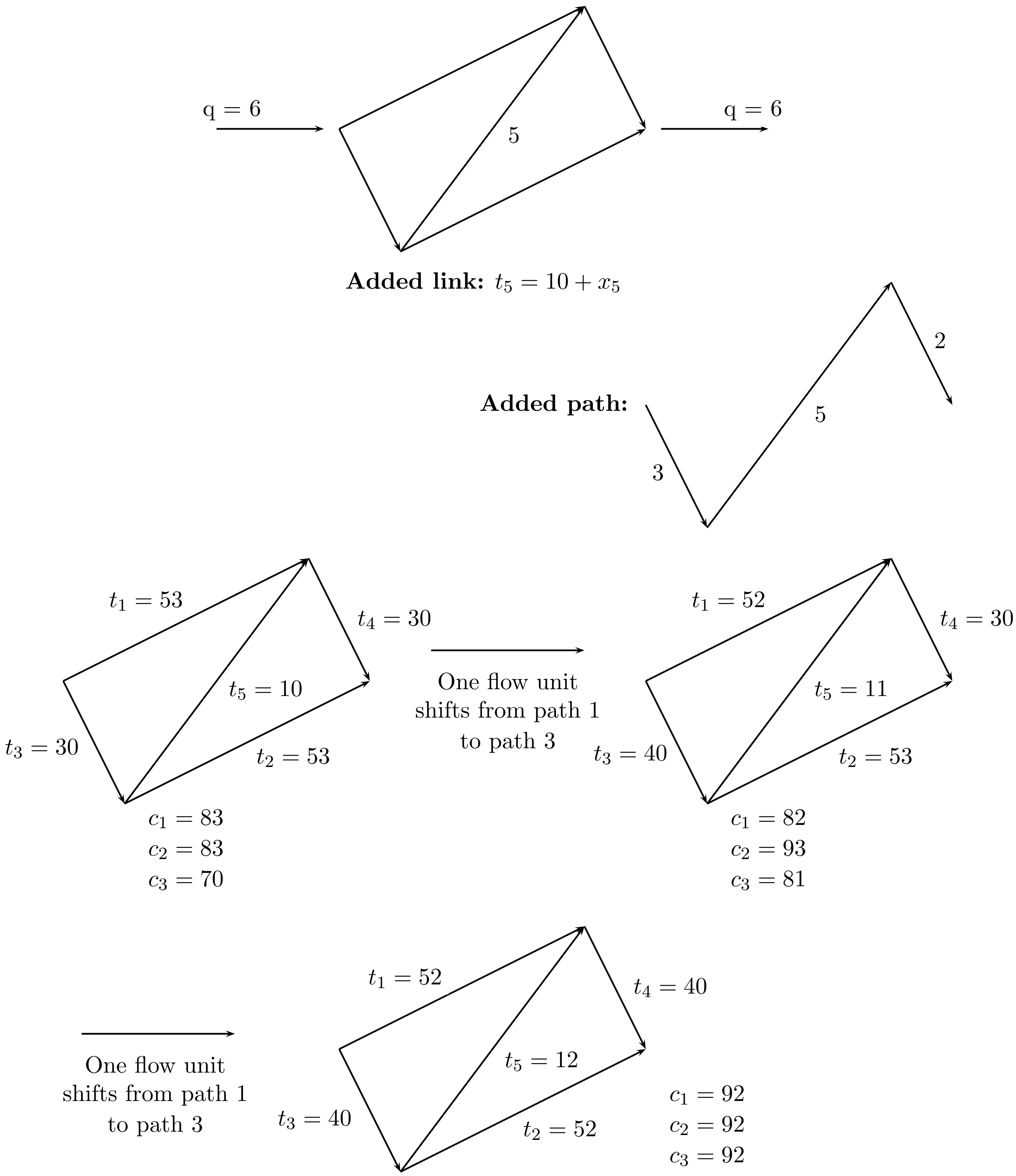


Figure 1.5: Braess' Network with Additional Link and New Solution

with path travel times being

$$c_1 = 83, c_2 = 83, c_3 = 70 \text{ time units.}$$

The travel time on the unused path (path 3) is lower than the travel times on the two used paths so this is not an equilibrium solution. Figure 1.5 shows a possible sequence of assignment of flow units that would result in an equilibrium solution. The equilibrium flow pattern for the new network is given by the solution

$$x_1 = 2, x_2 = 2, x_3 = 4, x_4 = 4, x_5 = 2 \text{ flow units}$$

with path flows

$$f_1 = f_2 = f_3 = 2 \text{ flow units}$$

and path travel times

$$c_1 = c_2 = c_3 = 92 \text{ time units.}$$

It is important to note that the total travel time on the network in now 552 (flow-time) units compared to the 498 (flow-time) units before the fifth link was added. Therefore, the addition of the link has resulted in the travel time of each traveller increasing from 83 to 92 time units and the total travel time increasing from 498 (flow-time) units to 552 (flow-time) units. The addition of the new link has therefore made the situation worse. This apparently counter-intuitive result is known as Braess' Paradox.

It should be noted that the user equilibrium objective function (see Chapter 2) did in fact decrease, from 399 before the link addition to 388 after the link addition.

Sheffi [51] explains the paradox by pointing out that in user equilibrium the individual choice of route is carried out with no consideration of the effect of this action on other network users.

Figures 1.4 and 1.5 show the three paths that traffic can follow in the augmented network.

If one considers the paths shown in Figures 1.4 and 1.5, then it is obvious that, due to symmetry, the flows on paths 1 and 2 will be equal at equilibrium.

In the case of the original network, if the total flow is Q, then the flows on all the links will be $0.5Q$.

In the case of the augmented network, if the flow on link 5 is P (i.e. on path 3), then the flows on the various links are as follows:

$$\text{Link1} = 0.5Q - 0.5P$$

$$\text{Link2} = 0.5Q - 0.5P$$

$$\text{Link3} = 0.5Q + 0.5P$$

$$\text{Link4} = 0.5Q + 0.5P$$

$$\text{Link5} = P$$

Then,

$$\begin{aligned}
\text{Time on link 1 } &= 50 + 0.5Q - 0.5P \\
\text{Time on link 4 } &= 10(0.5Q + 0.5P) \\
\text{Time on link 5 } &= 10 + P \\
\text{Time on path 1} &= \text{time on path 2} \\
&= 50 + 0.5Q - 0.5P + 5Q + 5P \\
\text{Time on path 3} &= 2(5Q + 5P) + 10 + P \\
&= 10 + 10Q + 11P
\end{aligned}$$

At equilibrium, the times on all the paths are equal, therefore

$$\begin{aligned}
50 + 5.5Q + 4.5P &= 10 + 10Q + 11P \\
40 - 4.5Q &= 6.5P \\
P &= \tfrac{80}{13} - \tfrac{9}{13}Q
\end{aligned}$$

If, as in the example shown in the figures, $Q = 6$, then $P = \frac{80}{13} - \frac{54}{13} = 2$.

The above numerical analysis did not form part of Braess' paper and has been added by the author.

Chapter 2

LITERATURE REVIEW

References in Chapter 1 to Braess' Paradox in the literature and will not be repeated here.

2.1 Game Theory and Equilibria

2.1.1 Nash equilibrium

A number of authors have pointed out that the equilibrium assignment is an example of a Nash equilibrium, e.g. Dafermos and Sparrow [21] as cited by Steinberg and Zangwill [54].

The Nash equilibrium (named after John Nash who proposed it) is part of game theory. Game theory is an approach to the study of human behaviour involving a number of disciplines such as mathematics and economics. The formal conception of game theory as part of economic theory was done by von Neumann and Morgenstern in their 1944 book, Theory of Games and Economic Behavior [61].

A strategic game can be defined as consisting of [44]:

- A set of players
- For each player, a set of actions (sometimes called strategies)
- For each player, a payoff function that gives the player's payoff to each list of the players' actions.

It is assumed that in a strategic game:

- Each player chooses the action that is best for him, given his beliefs about the other players' actions.
- Every player's belief about the other player's actions is correct.

Definition 4.1: A **Nash equilibrium** of a strategic game is an action profile (list of actions, one for each player) with the property that no player can increase his payoff by choosing a different action, given that the other players' actions remain fixed.

John Nash introduced his concept of equilibrium in 1950 and the "Nash Equilibrium" has probably become the most widely used "solution concept" in game theory [42]. A strategic game may have no Nash equilibrium, may have a single Nash equilibrium, or may have many Nash equilibria [44].

Although the concept is generally referred to as the "Nash Equilibrium", it in fact goes as far back as 1838 and Cournot [18] (cited by [29]). Therefore it is also sometimes referred to as a "Cournot-Nash equilibrium".

Cournot's equilibrium model is an economic model which deals with a noncooperative oligopoly where firms choose output levels independently. He assumed that each firm acts independently and tries to maximize its profit. According to Jaquier [35] it was the first, and is probably still the most widely used model of this type.

The following are some exampless of Cournot and Nash equilibria that illustrate the concepts:

The following example of a Cournot equilibrium is based on Jaquier [35] where there are two firms F1 and F2 in a noncooperative duopoly. The question is, what strategy should F1 use to decide on its output level? The answer depends on what firm F1 believes about firm F2's behaviour. If firm F1 believes that F2 will sell q_2 it can then determine the output q_1 that will maximize its profit. If q is the total output that would maximize the profit of a monopoly, then

$$q_1 = q - q_2$$

It is assumed that the following functions apply:

Cost function of each of the two firms: $C = 0.28q$

(in order to simply the example, it is assumed that there are no fixed costs)

Market demand function: $q = 1000 - 1000p$

Market price function: $p = 1 - 0.001q$

According to Cournot's equilibrium model, the decision on F1's output level then depends F1's estimate of what F2's output will be.

If F1 estimates that F2 will produce 200 units of the product. Then to calculate the optimum output for F1 one can use the following "best-response function" or "reaction function":

$$q1 = 360 - 0.5q2$$

where 360 is the optimum quantity in the case of a monopoly.

Therefore $q1 = 260$ will be the output which maximizes F1's profit.

Note: 360 is the optimal output in the case of a monopoly (only one firm in this market).

The best response function can be derived as follows:

The profit for F1 is the total income minus the total cost of producing q_1 items out of the total production of q items.

Total cost $= 0.28q_1$

Total income $= [1 - 0.001(q_1 + q_2)]q_1$

$$= q_1 - 0.001q_1^2 - 0.001q_1q_2$$

Therefore profit $= q_1 - 0.001q_1^2 - 0.001q_1q_2 - 0.28q_1$

$$= 0.72q_1 - 0.001q_1^2 - 0.001q_1q_2$$

Differentiating with respect to q_1 and setting equal to zero to obtain the maximum gives

$0.72 - 0.002q_1 - 0.001q_2 = 0$

or

$q1 = 360 - 0.5q2$

This situation is summarized in the following table.

Table 2.1: Summary of Output and Profit of F1 and F2

Firms	**Market Price**	**Output**	**Revenue**	**Cost**	**Profit**
F1	0.54	260	140.4	72.8	67.6
F2	0.54	200	108.0	56.0	52.0
Industry	0.54	460	248.4	128.8	119.6

The Cournet equilibrium occurs when $q1 = q2$.

$q_1 = 360 - 0.5q_2$, and

$q_2 = 360 - 0.5q_1$

$q_1 = 360 - 0.5(360 - 0.5q_1)$

$q_1 = 360 - 180 + 0.25q_1$

$q_1 - 0.25q_1 = 180$

$0.75q_1 = 180$

$q_1 = q_2 = 240$

One of the classic examples in game theory is the Prisoners' Dilemma. The description of this problem that is given below is based on McCain [42].

Al and Bob are two burglars who caught near the scene of a burglary. Each is given the "third degree" separately by the police. They each have to choose whether or not to confess and implicate the other. If neither of them confess, then both them will be sentenced to one year in prison on a charge of carrying a concealed weapon. If each of them confesses and implicates the other then both of them will go to prison for 10 years. However, if one should confess and implicate the other, and the other burglar does not confess, the one who has confessed will be allowed to go free and the other one will be sentenced to 20 years on the maximum charge.

The strategies in this game are "confess" or "do not confess". The payoffs (actually penalties) are the sentences to be served. It is possible to show all of this compactly in a "payoff table" of a type that is pretty standard in game theory. The payoff table for the Prisoners' Dilemma is shown in Figure 2.1.

The table shown in Figure 2.1 is read as follows: Each of the prisoners chooses one of two strategies. Al chooses a column and Bob chooses a row. The two numbers in each of the cells show the results for the two prisoners when the corresponding pair of strategies is chosen. The number to the left of the comma shows the payoff to the person who chooses the rows (Bob) and the number to the right of the comma gives the payoff to the person who chooses the columns (Al). Therefore should they both confess, they both get 10 years in prison, But if Al confesses while Bob does not then Bob gets 20 years while Al goes free.

		Al	
		Confess	Don't
Bob	Confess	10,10	0,20
	Don't	20,0	1,1

Figure 2.1: Payoff Table For Prisoners' Dilemma

The questions is what strategies are "rational" if they both want to minimize the time that they will spend in prison? Al's reasoning may be as follows: "There are two possibilities: Bob can confess or Bob can remain silent. Suppose Bob confesses. I will then get 20 years if I don't confess, or 10 years if I do confess, therefore in that case it would be best to confess. However, if Bob doesn't confess, and neither do I, I will get one year, but should

I confess I will go free. Either way, it is better if I confess. Therefore, I will confess."

Bob, however, will presumably reason in the same way. Therefore they both confess and both go to prison for 10 years. Yet, if they had both acted "irrationally" by keeping quiet, they could have both spent only one year in prison.

What has happened in this situation is that the two prisoners have fallen into something called a "dominant strategy equilibrium" [42].

Definition 4.2: **Dominant Strategy**: Let an individual player in a game evaluate separately each of the strategy combinations he may face, and for each combination choose from his own strategies the one that gives the best payoff. If the same strategy is chosen for each of the different combinations of strategies the player might face, then that strategy is called a "dominant strategy" for that player in that game.

Definition 4.3: **Dominant Strategy Equilibrium**: If, in a game, each player has a dominant strategy, and each player plays the dominant strategy, then that combination of (dominant) strategies and the corresponding payoffs are said to constitute the dominant strategy equilibrium for that game.

McCain [42] states that the Prisoners' Dilemma problem is powerful since it can be related to a number of interactions in the modern world where individually rational actions result in persons being worse off in terms of their own self-interested purposes. He provides the following examples: arms races, road congestion, pollution, the depletion of fisheries and the over-exploitation of some subsurface water resources. The Prisoners' Dilemma is a Nash Equilibrium where the prisoners' make less than optimum choices.

The following example has a single Nash equilibrium. Consider, the game shown in Figure 2.2 [44].

		Player 2	
		L	R
Player 1	T	2,2	0,3
	B	3,0	1,1

Figure 2.2: Example of Strategic Game

The table shown in Figure 2.2 shows the payoffs to the two players as follows:

The top row of the table shows the payoffs when Player 1 chooses T (depending on the choice of Player 2). If Player 2 chooses L, then both players have a payoff of 2 (shown in the top left corner of the table). If Player 2 chooses R, then Player 1 has a payoff of 0 and Player 2 has a payoff of 3 (top right corner of the table). Similar comments apply to the bottom half of the table where Player 1 chooses B.

There are four action profiles (T,L), (T,R), (B,L) and (B,R) and each can be examined in turn to check whether it is a Nash equilibrium.

(T,L): By choosing B rather T, player 1 obtains a payoff of 3 rather than 2, given player 2's action. Thus (T,L) is not a Nash equilibrium. [Player 2 can also increase his payoff (from 2 to 3) by choosing R rather than L.]

(T,R): By choosing B rather than T, player 1 obtains a payoff of 1 rather than 0, given player 2's action. Thus (T,R) is not a Nash equilibrium.

(B,L): By choosing R rather than L, player 2 obtains a payoff of 1 rather than 0, given player 1's action. Thus (B,L) is not a Nash equilibrium.

(B,R): Neither player can increase his payoff by choosing an action different from his current one. Thus this action profile is a Nash equilibrium.

This game therefore has a unique Nash equilibrium, (B,R). In this equilibrium both players are worse off than they are in action profile (T,L). They would like to achieve (T,L) but their individual incentives point them to (B,R)[44].

McCain [42] provides the following example where there are multiple Nash equilibria.

There are two radio stations (WIRD and KOOL) that have to choose the format for their broadcasts. They can choose between three possible formats: Country-Western (CW), Industrial Music (IM) or all-news (AN). The audiences for these three formats are 50, 30 and 20 per cent respectively. If both choose the same format then they will split the audience for that format equally. If they choose different formats then each will get the total audience for that format. In this case the payoffs are proportional to the audience shares. The audience shares are shown in Table 2.3

Scrutinizing Table 2.3 shows that there are two Nash equilibria, the upper cell in the middle column and the middle row in the left-hand column. In both of these cases one station chooses CW and gets a 50 per cent market share, while the other chooses IM and gets 30 per cent of the market share. It does not matter which station chooses which format. In this example there is no dominant strategy equilibrium.

It may appear as it makes little difference which station chooses which format, since

- the total payoff is the same in both cases, i.e. 80
- both are efficient as there is no larger payoff than 80.

However, the multiplication of equilibria creates a danger. There is the danger that both stations will choose the more profitable CW format. This will result in them splitting the

		KOOL		
		CW	IM	AN
WIRD	CW	25,25	50,30	50,20
	IM	30,50	15,15	30,20
	AN	20,50	20,30	10,10

Figure 2.3: Payoff Table with Multiple Nash Equilibria

market and each getting only 25 per cent of the market. In fact there is an even greater danger that both stations might assume that the other station will choose CW and then choose IM. This will once again split the market, leaving both stations with only 15 per cent of the market. A game of this type raises a "coordination problem". How do the two stations coordinate their choices so as to avoid the mutually inferior result of splitting the market? Games such as this are also sometimes called coordination games.

The examples given so far all refer to games in which there are only two players. McCain [42] provides an example with many participants. This example deals with the choice of transportation mode (car or bus) by a large number of identical commuters. The situation is that as more car commuters drive their cars to work, congestion increases and it takes longer to get to work and the payoffs are lower for both car and bus commuters.

Figure 2.4 shows this. In this figure the proportion of commuters who use cars is shown on the horizontal axis (from 0 to 1). The vertical axis shows the payoffs for the commuters. The upper (red) line shows the payoffs for the car commuters. As expected it decreases as the proportion of car commuters increases. The lower (blue) line shows the payoffs for the bus commuters.

In this case the payoff for car users is higher than the payoff for bus users regardless of the proportion of commuters travelling by car. Therefore commuting by car is the dominant strategy, and in a dominant strategy equilibrium all commuters will drive to work in their cars. The result is that all the commuters will have negative payoffs, whereas, if they all used buses, they would all have positive payoffs. Therefore if all the commuters choose their mode of transportation with self-interested rationality, they will all choose the strategy that

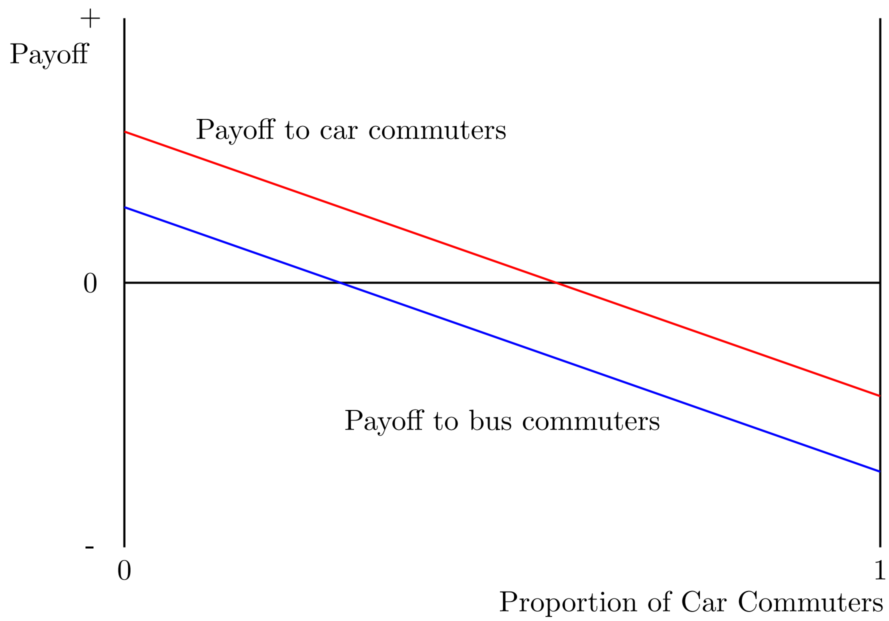


Figure 2.4: Payoff to Commuters (A)

makes them individually better off, but all are worse off as a result.

McCain [42] points out that this is an extension of the Prisoners' Dilemma. This is because there is a dominant strategy equilibrium which makes everyone worse off. McCain also states that it is probably not a very "realistic" model of the choice of transportation mode since some people do use the buses. He then proposes the model shown in Figure 2.5 as being more realistic.

In Figure 2.5 the buses are slowed somewhat by congestion but not as much as the cars. Therefore the payoff to the bus users decreases at a slower rate than the payoffs to those using cars. When the proportion of people using cars reaches q the payoff to the bus commuters overtakes the payoff to the car commuters. For proportions of commuters greater than q the payoff to those using their cars is worse than for those travelling by bus.

There is no longer a dominant strategy equilibrium. However, there is a Nash equilibrium when a fraction q of the commuters drive cars. That it is a Nash equilibrium ca be seen, since starting at q, if a single bus commuter shifts to using a car, he moves into the region to the right of q where car commuters are worse off. On the other hand, starting from q, should one car user change to using a bus that will move him into the region to the left of q where the bus users are worse off. No one can improve their situation by changing mode.

McCain refers to this example of being an instance of "the tragedy of the commons"[1].

[1] This term derives from a parable that was published by William Forster Lloyd in his 1833 book on population (as cited by [41]).

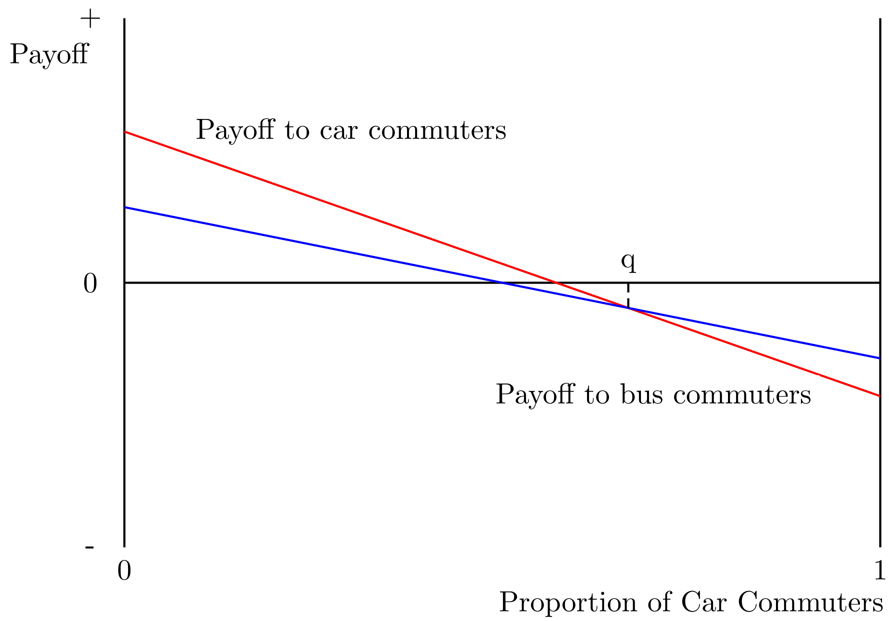


Figure 2.5: Payoff to Commuters (B)

The roads are a common resource that are available to all car and bus users. However, people using cars make more intensive use of this common resource causing the resource to be degraded (congested). The people using cars gain a private advantage by choosing to make more intensive use of the common resource, at least while the resource is relatively undegraded. The "tragedy" results from the fact this intensive use leads to the degradation of the resource until everybody is worse off.

McCain describes the classical example as presented by Lloyd [41] where reference is made to common pastures, on which theoretically, each of the farmers will increase the size of their herd until the pasture is overgrazed and everyone is worse off.

Hollander and Prashker [33] present a number of examples from transport literature where concepts from non-cooperative game theory have been incorporated into various models. These include examples such as choice of mode, the choice of using a road segment or not, choice of departure time, and choice of size of car.

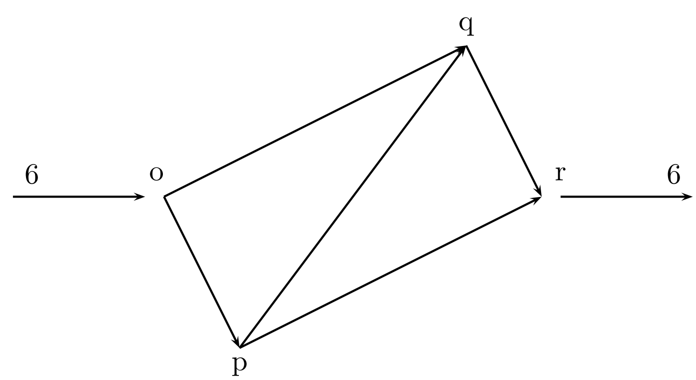
Fisk [26] as cited by Hollander and Prashker [33] mentioned that the user equilibrium principle that was introduced by Wardrop [62] is an example of a game. The reason for this is that it satisfies the conditions of a Nash equilibrium where no driver can reduce his/her travel time by changing their choice of path. This may result in Braess' paradox with the travel times being worse after the addition of a link to the network.

2.2 Other Braess' Networks

There are other examples of networks that exhibit Braess' paradox properties. Some examples of these are included below.

2.2.1 Murchland

Murchland [43] provides a modified example using a similar network to that of Braess but with different travel-time flow relationships (volume delay functions). His added link has a travel-time flow relationship of $t_{pq} = 0$, i.e. it is an uncongesting link (the travel time does not vary with traffic flow using it). This results in all the traffic using the new link. This example is shown in Figure 2.6.



Travel Time Function	Before		After	
	Flow	Time	Flow	Time
$t_{op} = x_{op} * 23/3$	3	23	6	46
$t_{qr} = x_{op} * 23/3$	3	23	6	46
$t_{oq} = 46$	3	23	0	46
$t_{pr} = 46$	3	23	0	46
$t_{pq} = 0$	-	-	6	46

Figure 2.6: Murchland's Modified Braess Example

In this case the travel times before the addition of link pq is 69 on both routes. After the addition of the link, the travel times are 92 on all three routes. Therefore in this case, the addition of a link with zero travel time increases all travel times by one third.

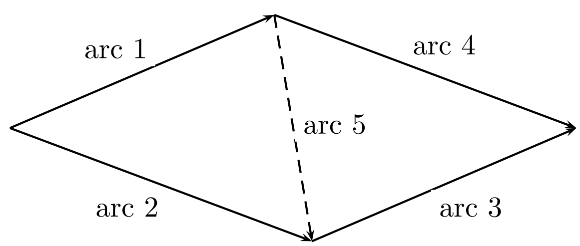
Although not providing an example, Murchland claims that it would be possible to construct a similar example with a sliding scale agricultural subsidy. In this case each producer is better off if he produces more, but the scale operates in such a way that minimum production by all would actually give each the greatest profit.

Murchland also states the following: “It seems to me that the importance of Braess’ paradox for practical networks will only become apparent when sufficiently accurate congested traffic assignment calculations become available and the phenomenon emerges, or fails to emerge, during systematic searches for optimal link additions.”

2.2.2 LeBlanc

LeBlanc [40] also produced a modified version of Braess’ network and this is shown in Figure 2.7.

Originally the total travel time on each of the two routes is 338.4 time units and the total travel time on the network is 2 030.4 time units. After the addition of arc 5, the travel time on the three possible routes is 367.4 time units with the total travel time on the network being 2 204.4 time units.



Travel Time Function	Before		After	
	Flow	**Time**	**Flow**	**Time**
$A_1(x_1) = 40 + 0.5(x_1)^4$	3	80.5	4	168.0
$A_2(x_2) = 185 + 0.9(x_2)^4$	3	257.9	2	199.4
$A_3(x_3) = 40 + 0.5(x_3)^4$	3	80.5	4	168.0
$A_4(x_4) = 185 + 0.9(x_4)^4$	3	257.9	2	199.4
$A_5(x_5) = 15.4 + (x_5)^4$	-	-	2	31.4

Figure 2.7: LeBlanc’s Modified Braess Example

Although [40] does not mention how the equilibrium problem was solved, it can be done using the same method that was used for the Braess’ example in the previous chapter.

If the total flow on the network is Q and P is the flow on the new link, then

$$\text{Flow on arc 1} = \text{Flow on arc 3}$$
$$= 0.5Q + 0.5P$$
$$\text{Flow on arc 2} = \text{Flow on arc 4}$$
$$= 0.5Q - 0.5P$$
$$\text{Flow on arc 5} = \text{P}$$
$$\text{Time on arc 1} = \text{Time on arc 3}$$
$$= 40 + 0.5(0.5Q + 0.5P)^4$$
$$= 40 + 0.03125Q^4 + 0.125Q^3P + 0.1875Q^2P^2 + 0.125QP^3 + 0.03125P^4$$
$$\text{Time on arc 2} = \text{Time on arc 4}$$
$$= 185 + 0.9(0.5Q - 0.5P)^4$$
$$= 185 + 0.05625Q^4 - 0.225Q^3P + 0.3375Q^2P^2 - 0.225QP^3 + 0.05625P^4$$
$$\text{Time on arc 5} = 15.4 + P^4$$

At equilibrium, the total time on arcs 1, 5 and 3 equals the total time on arcs 2 and 3, or

$$95.4 + 1.0625P^4 + 0.25P^3Q + 0.375P^2Q^2 + 0.25PQ^3 + 0.0625Q^4$$
$$= 225 + 0.0875P^4 - 0.1P^3Q + 0.525P^2Q^2 - 0.1PQ^3 + 0.0875Q^4$$

or

$$0 = -129.6 + 0.975P^4 + 0.35P^3Q - 0.155P^2Q^2 + 0.35PQ^3 + 0.025Q^4$$

If $Q = 6$ the this equation becomes

$$0 = -162 + 0.975P^4 + 2.15P^3 - 5.4P^2 + 75.6P$$

$P = 2$ is the only meaningful solution to this equation (the other solutions being -6 and two imaginary numbers).

2.2.3 Fisk

Fisk [24] presented two examples of traffic paradoxes. The first is based on the network shown in Figure 2.8.

For this network the cost functions are:

$$S_{ab} = f_{ab};\ S_{bc} = f_{bc};\ S_{ac} = f_{ac} + 90$$

where f is the flow on the link. The initial flows between the nodes are:

$$g_{ab} = 1;\ g_{bc} = 100;\ g_{ac} = 20.$$

Let x and y be respectively the flows on paths (abc) and (ac), then the travel times on the

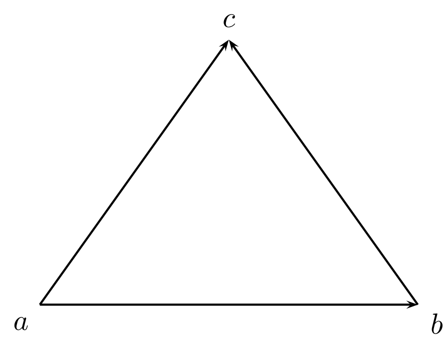


Figure 2.8: Fisk's Three-Link Network

links are as follows:

$$u_{ab} = 1 + x$$

$$u_{bc} = 100 + x$$

$$u_{ac} = 90 + y$$

At equilibrium

$$u_{ac} = u_{ab} + u_{bc}$$

or, $$90 + y = 1 + x + 100 + x$$

$$= 101 + 2x$$

and $$x + y = 20$$

$$x, y \geq 0$$

Solving for x and y, gives $x = 3$, $y = 17$,

and the total travel time on the network is 12 444.

If g_{ab} is now increased from 1 to 4 and the calculations are repeated,
then $x' = 2$ and $y' = 18$. The total travel time on the network is 12 384. This means that increasing the demand g_{ab} by 3 has resulted in a decrease in the total travel time on the network by 60 units.

Fisk provides a second example that has two modes (cars and transit) using the network. The network is shown in Figure 2.9.

In the case of a two-mode model, the transit cost for link a, t_a, is calculated as a function of the car cost. This is because the speed of the transit vehicles (unless they have a separate right-of-way) is dependent on the speed of the cars. Fisk uses the following relationship:

$$t_a = S_a(f_a) + p_a$$

where p_a is a flow independent penalty term for link a. The flow f_a is that resulting

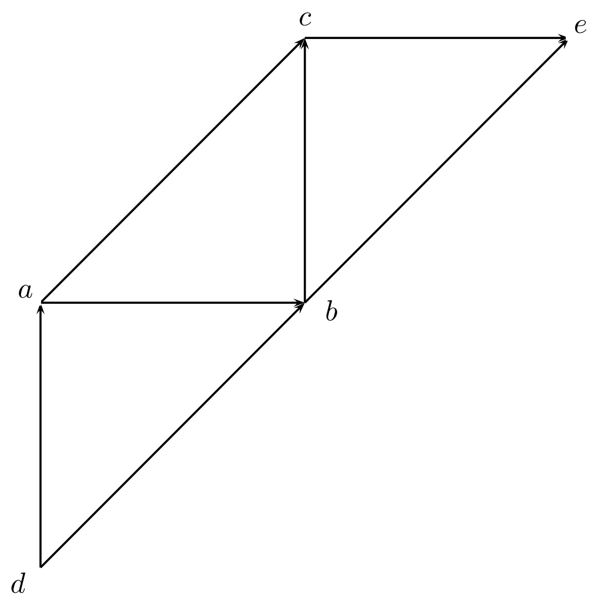


Figure 2.9: Network for Two-Mode Equilibrium Example

from a user equilibrium assignment of the set (g_i) on the network plus a constant term incorporating the contribution of transit traffic.

P_i designates the set of paths between origin-destination (O-D) pair i. In Fisk's example she set

$P_i = (dabe)$, i.e. there is one path for O-D pair i going from d to a to b to e

$P_j = (abc), (ac)$, i.e. there are two paths between O-D pair j from a to c

$P_k = (bc)$

and the transit path between O-D pair i is

$T_i = (dbce)$

The transit cost between O-D pair i is given by

$v_i = S_{db} + S_{bc} + S_{ce} + \text{constant}$

The car travel time between a pair of nodes is u_i. If g_i is increased to g_i', then to attain equilibrium some cars between a and c will divert from (abc) to (ac) since the travel time will have increased on (ab).

Then at the new equilibrium $S_{bc}' < S_{bc}$ while $S_{db}' = S_{db}$ and $S_{ce}' = S_{ce}$. Thus $v_i' < v_i$ and $u_i' > u_i$. In other words, an increase in cars trips and car travel time for O-D pair i has resulted in a reduced transit cost for O-D pair i.

Fisk states that the corollary of this is that a reduction in the number of cars may lead to an increase in the transit cost.

2.2.4 Fisk and Pallottino

Fisk and Pallottino [25] used a triangular network as shown in Figure 2.8 and provide examples where increasing the travel cost or reducing the capacity of a link in the network results in a decrease in the total travel time on the network. They also provide two examples that occurred on the modelled network for the city of Winnipeg showing that Braess' paradox can occur in real-world networks.

These examples differed from most of the others that are presented in the literature in that they were from a larger network and that BPR (Bureau of Public Roads) curves were used to represent the link delay functions. These are realistic forms of flow delay equations and are commonly used in assignment models. Unfortunately the paper does not report what stopping criterion was used to stop the assignment algorithm (see Chapter 5).

2.2.5 Steinberg and Stone

Steinberg and Stone [53] presented a slightly more complex network that exhibits Braess' paradox depending on the link performance function on one of the links. (The authors mention that it was brought to their attention after their paper had been accepted for publication that a similar paradox had been presented by Frank [32]). The authors show that increasing the congestion effect along a route can result in the abandonment of a different route.

Figure 2.10 shows the network and the four possible routes between the origin and destination.

Figure 2.11 shows the cost functions on the different arcs. Arc e_{32} has a cost of $\theta f + 2$ where θ is a parameter such that $\theta \geq 0$. If one increases the parameter θ, then the congestion is increased on route R_3 only.

When θ is less than 2 ($0 < \theta < 2$), all four routes have positive flows at equilibrium. These four flows and the associated user costs (or travel times) are shown in Table 2.2. Since the flows are all greater than zero, the user costs are all identical. By symmetry, R_1 and R_2 have the same flow, but the flows along R_3 and R_4 are different.

It can be seen from Table 2.2 that the flow on R_4 is decreasing in θ and vanishes at $\theta = 2$. For $\theta \geq 2$ the user cost along R_4 is greater than along the other routes and therefore R_4 attracts no trips. The flows and user costs for $\theta \geq 2$ are shown in Table 2.3.

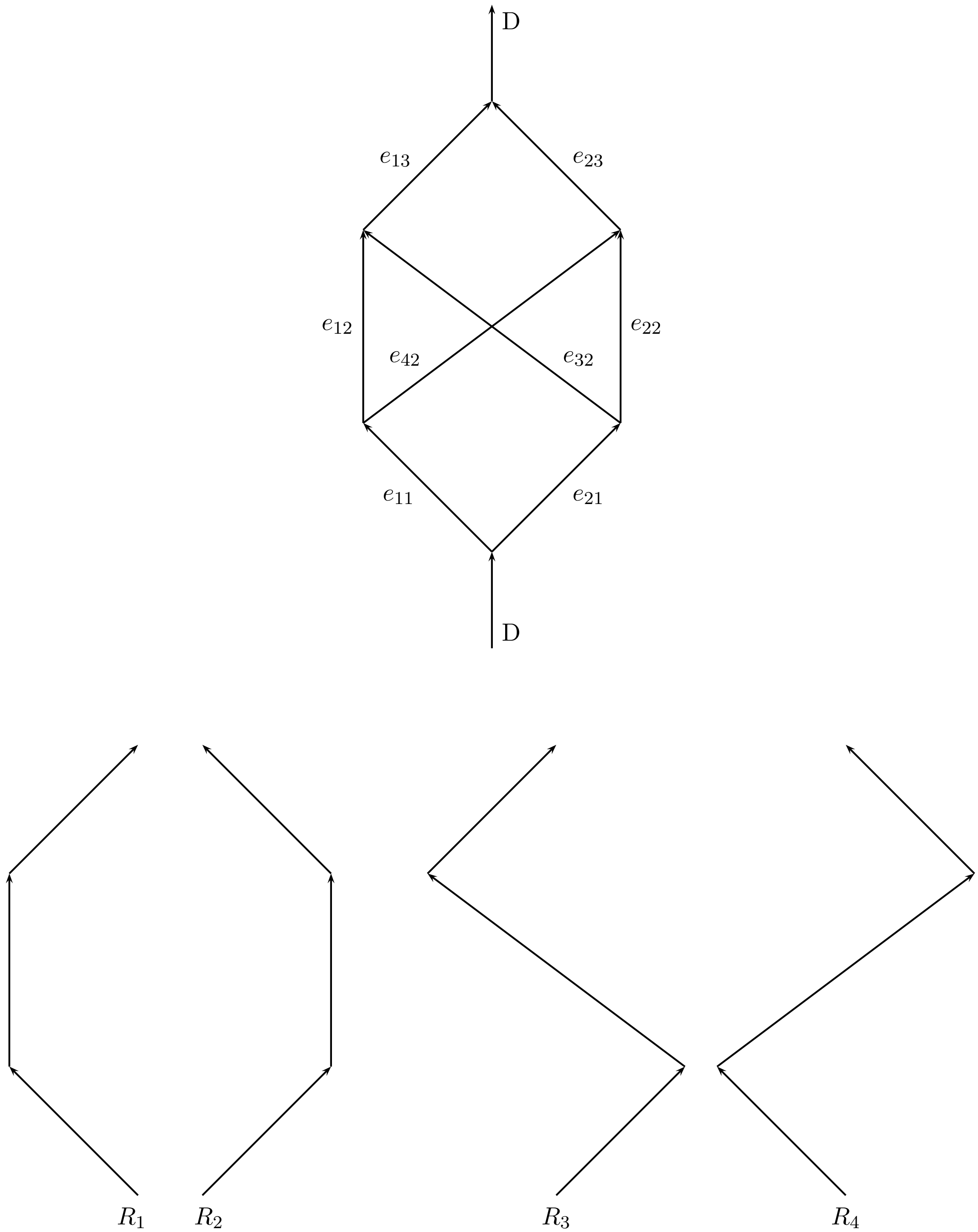


Figure 2.10: Network and routes

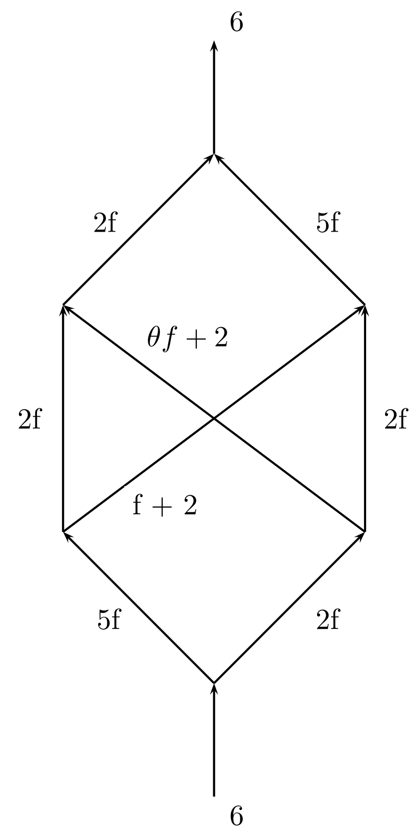


Figure 2.11: The network with cost functions and demand

Table 2.2: Flows and user costs for $0 < \theta < 2$

Route	R_1	R_2	R_3	R_4
Flow	$\frac{38\theta+42}{11\theta+37}$	$\frac{38\theta+42}{11\theta+37}$	$\frac{118}{11\theta+37}$	$\frac{20-10\theta}{11\theta+37}$
User cost	$\frac{292\theta+714}{11\theta+37}$	$\frac{292\theta+714}{11\theta+37}$	$\frac{292\theta+714}{11\theta+37}$	$\frac{292\theta+714}{11\theta+37}$

2.3 Other Traffic Paradoxes

2.3.1 Smith

Smith [52] showed that for certain network configurations, the total travel costs could be decreased by increasing travel cost locally at a congested intersection. This situation may be

Table 2.3: Flows and user costs for $\theta \geq 2$

Route	R_1	R_2	R_3	R_4
Flow	$\frac{6\theta+14}{2\theta+9}$	$\frac{6\theta+14}{2\theta+9}$	$\frac{26}{2\theta+9}$	0
User cost	$\frac{54\theta+178}{2\theta+9}$	$\frac{54\theta+178}{2\theta+9}$	$\frac{54\theta+178}{2\theta+9}$	$\frac{64\theta+158}{2\theta+9}$

explained by the increased local costs forcing the motorists towards the system equilibrium.

2.3.2 Cohen and Kelly

Cohen and Kelly [16] presented an example where a Braess type paradox occurs in a queuing network. The before and after networks are shown in Figure 2.12. The networks contain two kinds of servers, FCFS and IS. FCFS denotes a single-server queue with a first-come-first-served queue discipline. Service times are assumed to be independent exponential random variables with mean $1/\phi$ time units, where $\phi > 0$ is fixed throughout. If individuals arrive at a stationary FCFS queue in a Poisson stream with a mean of x individuals per unit of time, where $x < \phi$, the mean time in the queue of an individual is $1/(\phi - x)$.

IS denotes an infinite-server queue at which an individual is delayed by some random amount of time, the average of which is independent of the number of individuals awaiting service. In the initial network, there is an average delay of 2 units in the two IS queues.

The augmented queuing network shown in the figure differs from the initial network by having an IS queue with a mean delay of 1 time unit added to it.

Cohen and Kelly prove the following theorem showing that for certain parameter values, if individuals choose a route from entry to exit so as to minimize their average transit time, given the choice of other individuals, then at equilibrium the mean transit time in the augmented network is strictly larger than the mean transit time in the initial network.

Theorem 4.1. let 2λ denote the total traffic departing from node A and assume $2\lambda > \phi - 1 > \lambda > 0$. Then the mean transit time in the initial network is strictly larger than the mean transit time in the initial network is strictly less than 3 time units, while the mean transit time in the augmented network equals 3 time units.

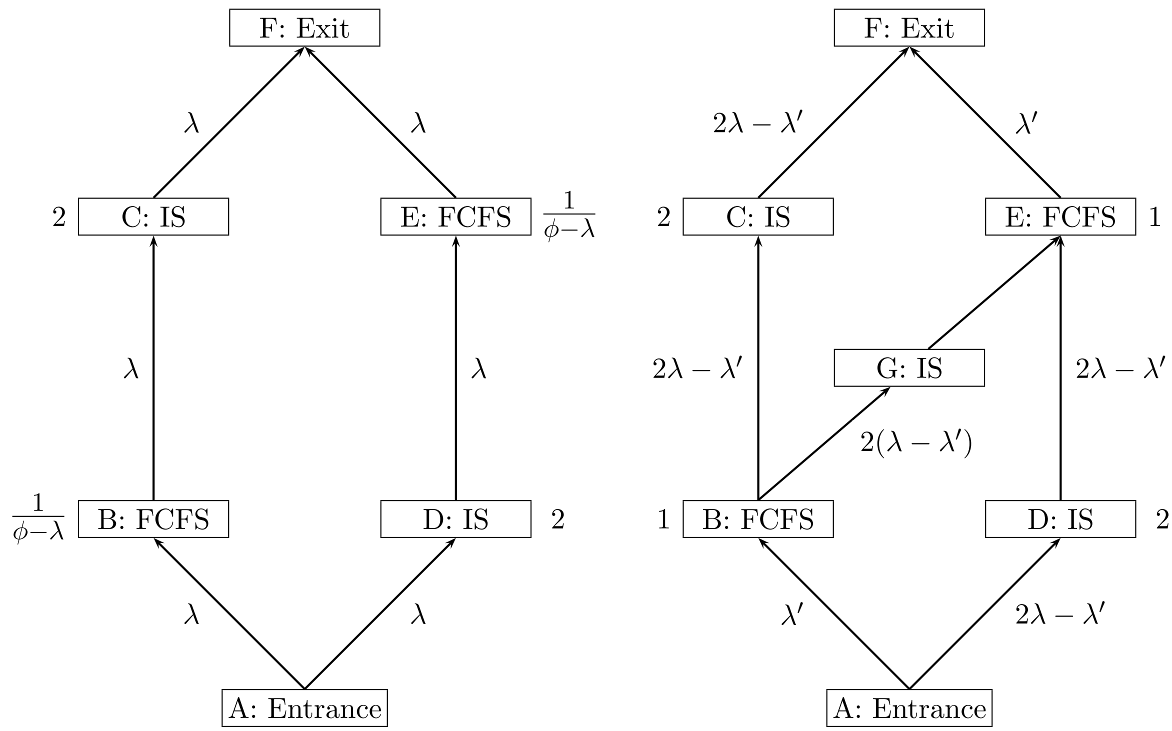


Figure 2.12: Initial and Augmented Queuing Networks

2.3.3 Sheffi

Sheffi [51] presents an example of a paradox in a stochastic user equilibrium (SUE) model. The other models described in this document are deterministic in that it assumed that there is perfect knowledge allowing travellers to always select the path with the shortest travel time or least cost. This assumption is relaxed in stochastic models where a random component is included in the traveller's perception of travel time.

The networks are shown in Figure 2.13 with(a) showing the original network and (b) the augmented network. Each link of the original network has an actual (measured) travel time of 1 time unit. There are two paths between the origin and the destination, each with a measured travel time of 2 time units. The addition of the centre link with measured travel time of 0.1 time units in (b) provides one additional (longer) path from the origin to the destination.

However, due to its stochastic nature the model will assign some trips to the longer path and the total system travel time will therefore increase. For example in Figure 2.13, the new average travel time will be between 2.0 and 2.1 units of time per traveller, compared with 2.0 units before the introduction of the new link.

Therefore, in this example, the total travel time increased after the network was seemingly improved. If the users of the system were interviewed, however, they would all have reported

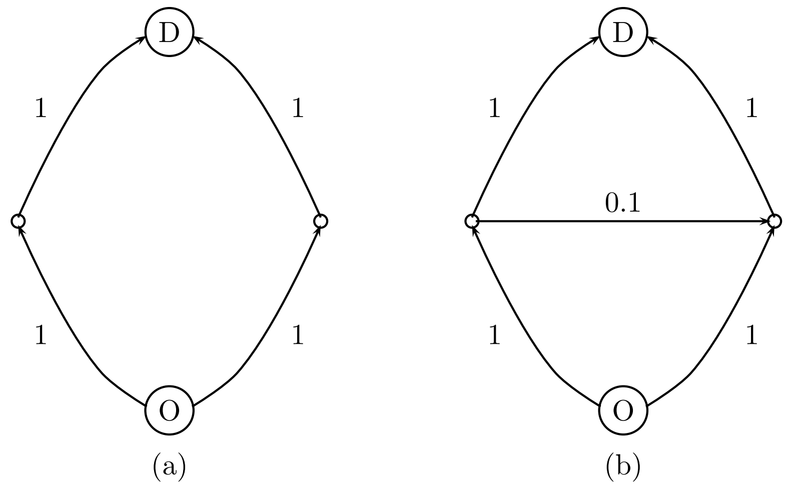


Figure 2.13: Networks for Paradox Example for a Stochastic Equilibrium model

an improvement in the travel time. Thus the travel time as travellers perceive it did decrease even though the total measured travel time increased as a result of the improvement.

2.3.4 Downs-Thomson

Arnott et al. [2] describe what has been called the Downs-Thomson paradox [22], [56]. This is as follows.

Suppose there is a roadway along a route which is served also by rail. If travel costs by road rise with the number of drivers, while rail costs are independent of ridership; then at equilibrium, just that number of travellers will drive to raise travel costs by road to equal costs by rail. If road capacity is now expanded, users will shift to the road until it as congested as before.

If the railway has to balance its budget, the loss of revenue will force it to increase fares and cut service, inducing more passengers to switch, so that travel costs on both modes end up higher than before the road expansion.

2.3.5 Arnott, De Palma and Lindsey

Arnott et al. [2] provide an example where they consider a Y-shaped highway corridor with one bottleneck on each arm and a third bottleneck downstream. Two groups of commuters use this corridor, each passing through one of the upstream bottlenecks and then the downstream bottleneck. They show that expanding the capacity of one of the upstream bottlenecks can raise travel costs because reduced congestion upstream is more than offset by increased congestion downstream.

This is the principle followed in ramp-metering where the traffic entering a freeway from an on-ramp is restricted so as to reduce the congestion on the freeway and also the total delay in the system.

2.3.6 Chen and Hsueh

Chen and Hsueh [13] describe a discrete-time dynamic user-optimal departure time/route choice model. In this model, travellers' departure times are not fixed, but chosen so as to minimize their travel times. They provide a couple of simple examples to demonstrate the model including one using a similar network to that of Braess where Braess' paradox is shown to occur.

2.3.7 Bean, Kelly and Taylor

Bean et al. [3] cite the following authors that describe paradoxes similar to the Braess' paradox:

It has been shown that paradoxes similar to Braess'paradox can occur in, for example, mechanical and electrical networks (Cohen and Horowitz [15]), water-supply networks (Calvert and Keady [11]) and queueing networks under fixed (Cohen and Kelly [16], see Figure 2.12 and the accompanying description) and dynamic (Calvert et al. [12]) routing schemes. Also Whitt [63] discusses an example where the efficiency of a waiting room decreases as the waiting capacity is increased.

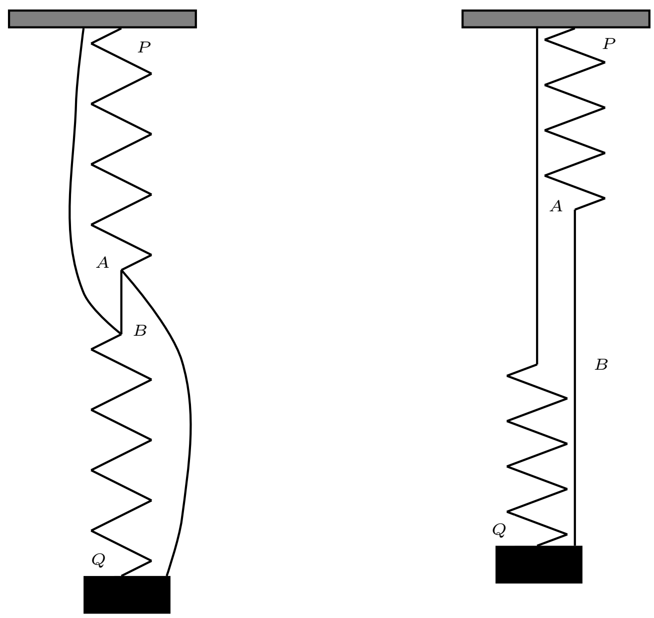
Bean et al. show that a Braess' paradox can occur in a loss network. Loss networks are used to model multi-resource access problems where requests for access that cannot be fully met are denied and lost. The classical example is the circuit-switched telephone network. The authors provide two examples. One for a network operating under fixed routing and the second with a network in which alternative routing is allowed.

2.3.8 Braess' paradox in computer networks

Although the majority of the work described in this dissertation refers to road networks, Braess' paradox can occur in other networks such as computer networks. A considerable amount of work has been published on the paradox in computer metworks, e.g. Kameda et al. [27]. Much of the work described by Roughgarden in his book on Selfish Routing [50] deals with computer networks.

2.3.9 Chen

Chen [14] investigated the mechanical analogue of Braess' paradox as proposed by Cohen and Horowitz [15]. This is shown in Figure 2.14. In part (a) of Figure 2.14 the system



(a)

(b)

Figure 2.14: Mechanical Analogue of Braess' Paradox

consists of two springs in series, $P - A$ and $B - Q$, that are connected by a string between A and B. There are also two "safety" strings between P and B and between A and Q. A weight with a mass of M is suspended from Q.

The question is, what happens to the position of the weight with mass M, if the string A-B is cut? Cohen and Horowitz showed that for certain combinations of strength of springs, length of strings and mass of weight, the weight will rise instead of dropping as could be expected. This can be explained by the fact that initially the springs are acting in series, but act in parallel after the string is cut.

Cohen and Horowitz carried our a number of experiments and claimed that the weight rises as soon as the string $A - B$ is cut. Chen, however, proved that the weight will initially drop before rising. This drop may be for a very short period that may be difficult to observe in practice.

2.4 Predicting the Occurrence of Braess' Paradox

In real world networks, it is generally not possible to predict whether adding a new link will results in Braess' paradox or not. LeBlanc [40] said the following in this regard: "When dealing with a network with many origins and destinations, it is not clear whether adding an arc will increase or decrease the congestion at equilibrium."

Steinberg and Zangwill [54] produced a theorem showing that, under what they term "reasonable assumptions", Braess' paradox is as likely to occur as not. Whether this is always the case in real world networks will be tested later in this dissertation. It is also unclear whether it will be applicable (or tractable) in a real world situation where a number of road improvements may be implemented in a single year and where an improvement will probably be used by routes connecting many origin-destination pairs and there are thousands of O-D pairs to be considered.

Frank [32], [31] analysed Braess' original networks and developed a series of theorems which give the necessary and sufficient conditions for the existence of all different types of Braess' flows.

Steinberg and Zangwill [54] analysed an arbitrary network and showed that adding a link will increase costs if the ratio of the determinants of two large matrices satisfy certain conditions. However, Dafermos and Nagurney [20] point out that these conditions cannot be checked a priori in a computationally efficient way.

Dafermos and Nagurney [19] also produced formulae that could be used to determine, under certain conditions, the change in traveller's costs on every O-D pair induced by the addition of a new path. These can be used to determine whether Braess' paradox occurs in the network. This requires what Dafermos and Nagurney term as "lengthy calculations". This can be seen from the fact that for a network not to exhibit Braess' paradox, the ratio of the determinants of two matrices must be non-positive. Parts of these matrices consist of the sub-matrices A and B, where:

A is a K by Q matrix

B is a Z by Q matrix

K is the number of links in the network

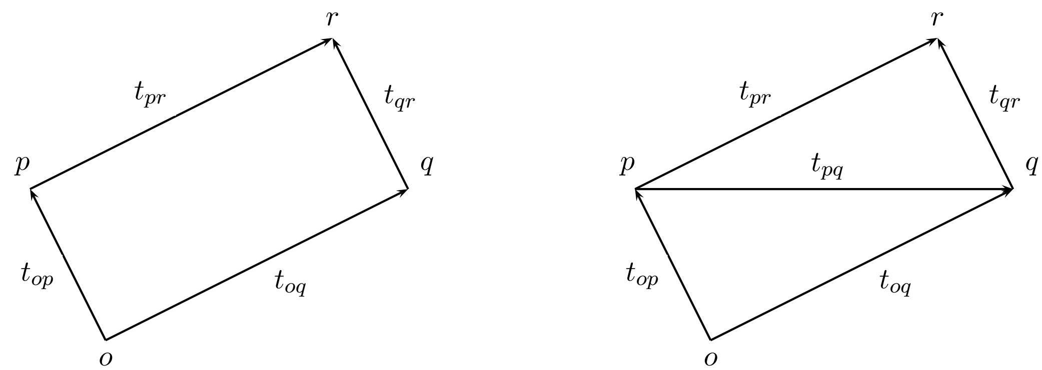
Q is the number of paths, and

Z is the number of O-D pairs

These parameters will lead to very large matrices in real-world networks where there are hundreds (possibly thousands) of zones and thousands of links. As an example, the latest regional model for the province of Gauteng has 899 zones or 808 201 O-D pairs, and there are bigger models in existence. Determining all the used paths in a real-world network will be a major task. It is therefore thought that this methodology is not practical for real-world examples.

2.4.1 Pas and Principio

Pas and Principio [45] analysed the classical Braess' Paradox network as shown in Figure 2.15 with a total demand for travel from origin o to destination r of Q and the travel time functions for the different links are as shown in Figure 2.15.



$$
\begin{aligned}
t_{op} &= \beta_1 f_{op} \\
t_{qr} &= \beta_1 f_{qr} \\
t_{oq} &= \alpha_1 + \beta_2 f_{oq} \\
t_{pr} &= \alpha_1 + \beta_2 f_{pr} \\
t_{pq} &= \alpha_2 + \beta_2 f_{pq}
\end{aligned}
$$

(a) Four-link network (b) Five-link network

Figure 2.15: Classical Braess' Networks

Their analysis resulted in the following inequality for the values of Q where Braess' paradox will occur:

$$\frac{2(\alpha_1-\alpha_2)}{3\beta_1+\beta_2} < Q < \frac{2(\alpha_1-\alpha_2)}{\beta_1-\beta_2}$$

The above inequality therefore implies that there are values of demand, Q, where the demand is too low for for Braess' paradox to occur and also where the demand is too high for Braess' paradox to occur. That is, Braess' paradox will not occur when congestion levels are too low or too high. Using the values of α and β given in Figures 1.4 and 1.5 results in the range $2.58 < Q < 8.89$ where Braess' paradox will occur.

Pas and Principio provide a figure which gives a graphical representation of how the flow on the different routes in the classical Braess' network vary as the demand Q varies. This Figure is reproduced as Figure 2.16. There is an apparent error in the figure in [45] with the start of the range where Braess' paradox occurs shown as 3.64 instead of 2.58. This has been corrected in Figure 2.16.

Pas and Principio also cite an example of a network presented by Arnott and Small [1] where the time functions on the links of the network shown in Figure 2.15 are the following:

$$t_{op} = 0.01 \times f_{op}$$

$$t_{qr} = 0.01 \times f_{qr}$$

$$t_{oq} = 15$$

$$t_{pr} = 15$$

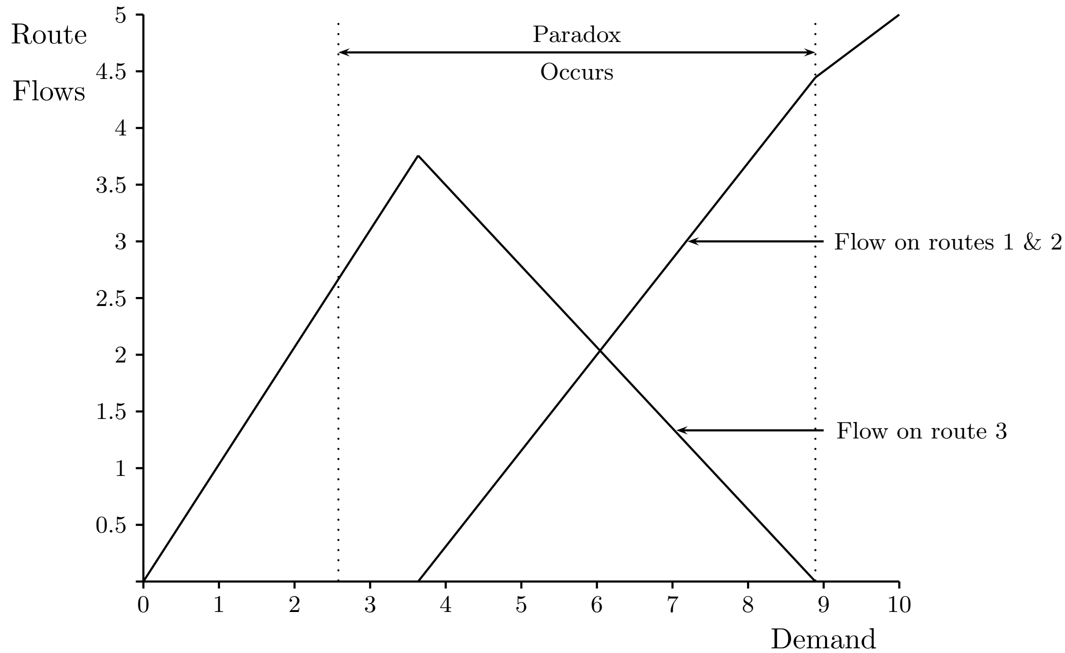


Figure 2.16: Route flows as a function of demand - Braess' example

$$t_{pq} = 7.5$$

In this case the range where Braess' paradox occurs is $500 < Q < 1500$ (Arnott and Small used a value of $Q = 1000$ in their example).

The figure showing the flows on the different routes in this case is reproduced as Figure 2.17.

It is interesting to note that in both the Braess' example and Arnott and Small's example, there is no flow on the new link once the demand reaches the upper limit of the range where Braess' paradox occurs.

Similar analysis to those shown in Figures 2.16 and 2.17 will be presented in Chapter 5 of this dissertation.

2.4.2 Valiant and Roughgarden

Valiant and Roughgarden [59] describe how a number of recent papers have studied strategies that will allocate additional capacity to a network without causing Braess' paradox to occur. They then raise the question about the prevalence of Braess' paradox. If it is a rare occurrence in what they term "selfish routing" networks, the the strategies would probably be superfluous for real-world networks. However, if it is a widespread phenomenon, then the problem of adding capacity to a network should be treated with care. They use the term "selfish routing" to describe the situation in the equilibrium assignment where each motorist is interested only in minimizing his own travel time.

This problem is the basis of their paper and they summarize the problem in the following

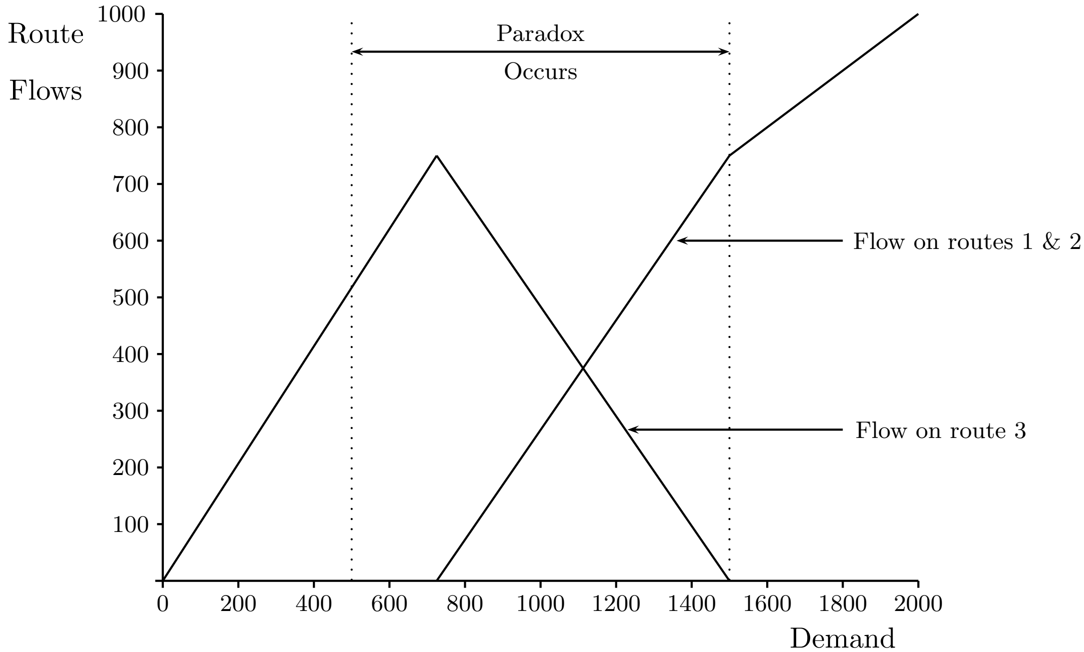


Figure 2.17: Route flows as a function of demand - Arnott and Small's example

question:

> *Is Braess' paradox a "pathological" example, or a pervasive phenomenon in selfish routing networks.*

Working with a model of random networks they prove the following:

- With high probability as $n \to \infty$ there is a choice of traffic rate such that the Braess ratio of a random network is strictly greater than 1.
- There is a constant $\rho > 1$ such that, with high probability as $n \to \infty$, there is a choice of traffic rate such that the Braess ratio of a random network is at least ρ.

Where:

> The Braess ratio of a network is the largest factor by which the removal of one or more links can improve the travel time at an equilibrium flow.
>
> "With high probability" means with probability tending to 1 as $n \to \infty$
>
> n is the number of nodes (or vertices as Valiant and Roughgarden term them) in the random network.

The large random networks that Valiant and Roughgarden worked with had links that had random linear delay functions, a single source node, and a single sink node.

By proving the statements listed above, Valiant and Roughgarden showed that with for the random networks that they defined, Braess' paradox is a widespread phenomenon.

They applicability of Valiant and Roughgarden's to real-world road networks is unclear since:

- For road networks the volume delay functions are not linear.
- Valiant and Roughgarden used only a single origin-destination pair for their trips. It is possible that in a real-world road network, while a certain link or set of links may exhibit Braess' paradox for trips between one origin-destination pair this may be outweighed by the benefits of the link or links for trips between other O-D pairs and also using the same links.

The question of the prevalence of Braess' paradox in real-world road networks will be addressed in Section 5.1.

Roughgarden also published a book entitled "Selfish Routing and the Price of Anarchy" [50]. This book provides an extensive bibliography on the subject of selfish routing including Braess' paradox.

2.5 Eliminating Braess' Paradox

Fisk [24] said that "sensitivity tests should be performed on a network before policies designed to reduce congestion are implemented, since such policies are usually aimed at reducing origin to destination and total travel costs."

LeBlanc [40] proposed a solution to the network design problem that takes Braess' paradox into account. The network design problem refers to selecting the optimal set of proposed improvements to the network taking budget constraints into account. In other words, the authorities often have a number of road projects that they would like to construct but budget constraints will not permit them all to be built. The problem is then to select that set of the projects that will provide the optimum benefit using the funds available.

LeBlanc's solution made use of a branch and bound algorithm. In order to avoid Braess' paradox, LeBlanc used user equilibrium (UE) and system optimal (SO) rules to define the lower and upper bounds. This is based on the fact that for the same network, the UE assignment has a total travel time greater than or equal to the total travel time from the SO traffic assignment. In addition, a network with new links results in a smaller SO total travel time. However, as pointed out by Zeng and Mouskos [64] the method rapidly becomes intractable as the number of variables increases and is therefore only applicable for very small networks.

Foulds [30] presented a heuristic algorithm to eliminate those arcs which show Braess' paradox from a network. This procedure is iterative in that it repeats the following sequence of steps:

First a traffic assignment process is used to find user equilibrium flows. These flows are then used to calculate the arc costs that will be experienced at equilibrium. That is the arc flows for each arc are substituted into the volume delay (link flow) expression. These arc costs are temporarily assumed constant. A branch and bound routine is then used to find the optimal set S, of arcs to eliminate so as to minimize the total user cost. It is assumed that the arc costs are likely to be realistic estimates of those that are likely to be actually experienced. The traffic assignment process is then repeated with the arcs in S permanently deleted. The cycle of traffic assignment and branch and bound is repeated until either no further arcs are eliminated or congestion fails to continue to decrease at which point the procedure is terminated.

The present author considers it incorrect to eliminate a set of arcs at one time. An arc that initially exhibits Braess' paradox may no longer do so after one of the other arcs has been eliminated. This is due to the change in flows that will occur after the removal of one arc and which will affect the flows on some of the other routes.

2.5.1 User Optimal vs System Optimal

As noted in Section 2.2, the following two types of equilibrium are possible in a traffic network:

- User equilibrium, where the journey times in all routes used are equal and less than those which would be experienced by a single vehicle on any unused route.
- System optimal, where the average journey time over all routes is a minimum.

In a normal road network, user equilibrium applies since travellers will attempt to minimize their travel times even though this may have a negative impact on other users of the system, whereas system optimal requires cooperation between travellers.

Kelly [36] said the following with regard to attempting to force drivers towards the system optimal situation:

Traffic dependent tolls are sufficient to force the system to an equilibrium which minimizes average network delay: the tolls charge drivers for the delays they cause to others. The study of tolls has long been a topic of central importance in economics, and it is interesting to note that Pigou [46] used a simple two-link traffic network to illustrate that taxation could 'create an "artificial" situation superior to the "natural" one'. (We note in passing that developments in electronics now make feasible the practical application of both route guidance and road pricing [60].

Chapter 3

DESCRIPTION OF THE MODELS USED IN ANALYSIS

Two different models were used to test large networks for the effects of Braess' Paradox. Both of these models were developed by the PWV Consortium for the then Transvaal Provincial Administration: Roads Branch. In Gauteng this organisation has now been replaced by the Gauteng Department of Public Transport, Roads and Works (Gautrans). The PWV Consortium was made up of five consulting engineering firms and one firm of town and regional planners. The name PWV refers to the Pretoria-Witwatersrand-Vereeniging area of the old Transvaal Province and included most of the present province of Gauteng. In addition, other adjacent areas such as Brits, Witbank and Sasolburg were included in the modelled area. Brief descriptions of the two models are provided below.

3.1 1985 PWV Update Model

The first PWV model was developed in the mid 1970s with 1975 as the base year. This model was updated in the 1980s with 1985 as the base year. This model had the years 2000, 2010 and 2025 as target years for which forecasts were made. This model consisted of 626 zones of which 36 were external zones. External zones represent the traffic entering and leaving the study area where roads cross the boundary of the study area. Further details of the model can be found in [47].

As part of the study a construction programme of the road improvements needed each year up until 2000 was drawn up. These recommended road improvements took the form of either new roads or the construction of additional lanes on existing roads. In total 186 road sections were recommended for construction between 1988 and 2000. The philosophy followed in recommending improvements to the network was that construction should take place to relieve congestion on roads when the operating conditions reached a level of service of D. The concept of level of service will be discussed at the end of this chapter.

Since a detailed construction programme had been drawn up it was possible to construct a modelled road network for each of the years between 1988 and 2000. Trip matrices for each of these years were obtained by interpolating between the matrices for 1985 and 2000. This enabled 13 different network scenarios to be tested for the effects of Braess' Paradox. In this testing all the roads that had been added from 1988 onwards were tested for Braess' Paradox for each year between 1988 and 2000.

3.2 The Vectura Study

The Vectura (or PWV Public Transport Study) had 1991 as its base year and as its name implies, greater emphasis was placed on public transport. This model had forecasts for the years 2000 and 2010. The model contained 668 zones of which 35 were external zones. There were the following two reasons for the increase in the number of zones:

- A number of zones to the north-west of Pretoria were subdivided due to the development that was taking place in the area.
- The study area was expanded to include the area around Carletonville which had been included in the province of Gauteng.

In addition to the updated land use information that was included in the model, a new set of volume-delay curves was used in the model. These curves were validated by means of travel time surveys and showed an improvement in the representation of increased travel time due to congestion on the roads. Further details of the model can be found in [48] and [49].

In an exercise carried out on behalf of Gautrans, an investigation was done into which were the critical roads that should be constructed by the years 2000 and 2010. A different philosophy was followed in that road improvements were only included in order to bring relief to roads experiencing level of service F. This provided two further sets of road improvements that could be tested for Braess' Paradox using a different model.

3.3 Level of Service

The concept of level of service uses qualitative measures that characterize the operational conditions within a traffic stream and their perception by motorists and passengers. The descriptions of individual levels of service characterize these conditions in terms of such factors as speed and travel time, freedom to manoeuvre, traffic interruptions, and comfort and convenience.

Six levels of service are defined for each type of facility making up the road network. They are given letter designations, from A to F, with level of service (LOS) A representing the best operating conditions and LOS F the worst. Each level of service represents a range of operating conditions.

The volume of traffic that can be served under the stop-and-go conditions of LOS F is generally accepted as being lower than possible at LOS E; consequently, the flow at level of service E is the value that corresponds to the maximum flow rate, or capacity, of the facility. For most design or planning purposes, however, levels of service D or C are usually used since they ensure a more acceptable quality of service to the road users.

As mentioned above, six levels of service are defined for each facility type. As an example, the following are descriptions of the different levels of service on freeways (from [58].

- LOS A describes primarily free-flow operations. Average operating speeds at the free-flow speed generally prevail. Vehicles are almost completely unimpeded in their ability to manoeuvre within the traffic stream. The average vehicle spacing affords the motorist with a high level of physical and psychological comfort.

- LOS B also represents reasonably free flow, and speeds at the free-flow speed are generally maintained. The ability to manoeuvre within the traffic stream is only slightly restricted, and the general level of physical and psychological comfort provided to drivers is still high.

- LOS C provides for flow with speeds still at or near the free-flow speed of the freeway. Freedom to manoeuvre within the traffic stream is notably restricted at LOS C, and lane changes require more vigilance on the part of the driver. The driver now experiences a noticeable increase in tension because of the additional vigilance required for safe operation.

- LOS D is the level at which speeds begin to decline slightly with increasing flows. Freedom to manoeuvre within the traffic stream is more noticeably limited, and the driver experiences reduced physical and psychological comfort levels.

- LOS E, at its lower boundary, describes operation at capacity. Operations in this level are volatile, because there are virtually no usable gaps in the traffic stream. Manoeuvrability within the traffic stream is extremely limited, and the level of physical and psychological comfort afforded the driver is extremely poor.

- LOS F describes breakdowns in vehicular flow. Such conditions generally exist within queues forming behind breakdown points.

Chapter 4

THE IMPORTANCE OF USING APPROPRIATE STOPPING CRITERIA IN THE EQUILIBRIUM ASSIGNMENT

4.1 The equilibrium assignment problem

As mentioned earlier, Beckmann et al. [4] showed that the equilibrium assignment problem could be transformed into an equivalent optimisation problem. This can be solved using the Franke-Wolfe algorithm to combine the results of successive all-or-nothing assignments in an iterative manner. Each all-or-nothing assignment uses the link travel times obtained using the link volumes resulting from the previous iteration of the process.

It has been proved that that this process converges to a unique solution (see Sheffi, [51], pp. 63-69). Since it is an iterative procedure, the question is how many iterations need to be performed? This question was addressed by Boyce et al. [8], Bloy [6], [7] and Blaschuk and Hunt [5].

4.2 Stopping Criteria for the Equilibrium Assignment Problem

Evans [23] proved that the addition of the all-or-nothing auxiliary assignment always leads to an improved solution. He also showed that provided enough iterations are performed, the Frank-Wolfe procedure converges to the equilibrium solution. However, the successive improvements become smaller with each iteration and it may take a very large number of iterations to reach convergence in a real-world network.

Thomas [55] lists the following three basic types of stopping rules:

- Those that look at the differences between estimates of quantities, usually link flows or costs, derived in successive iterations and, on the basis of the differences, decide whether or not continuation of the process is likely to bring about significant changes.
- Those which measure the agreement between the latest assumed link costs and assigned flows and the assumed cost/flow relationships.
- Those that consider the potential improvement that may result from continuing with more iterations.

Sheffi [51] has the following to say concerning convergence and the number of iterations required:

In solving the UE program over a large network, each iteration involves a significant computational cost, due primarily to the effort required to calculate the shortest paths. It is important then, that, a good answer is achieved after a relatively small number of iterations.

In practice, this is not a major problem for two reasons. First, the convergence pattern of the convex combinations algorithm is such that the first few iterations are the most "cost effective." In other words, the flow pattern after only a few iterations is not very far from equilibrium. Second, the convergence criteria used in practice are not very stringent and thus convergence can be achieved after only a small number of iterations. This is because the accuracy of the input data does not warrant the effort needed to obtain an extremely accurate equilibrium flow pattern.

Sheffi provides a figure similar to Figure 4.1 that shows the convergence rate for three levels of congestion on what he calls a "medium-sized" network.

He continues by saying: *In actual applications, only four to six iterations are usually sufficient to find the equilibrium flow pattern over large urban networks. This number reflects common practice in terms of trade-offs among analytical accuracy, data limitations, and budget, given typical congestion levels.*

Results obtained when plotting the results obtained using the Gautrans model appear to support Sheffi's statements. Figure 4.2 shows the value of the objective function of the optimization problem plotted against the number of iterations. (This is the objective function used in expressing the equilibrium assignment problem as a mathematical program - see Section 2.4).

Probably the most popular transportation modelling program in the world, EMME/2, (approximately 2000 licences in 73 countries worldwide) provides the following three stopping criteria with defaults [34]:

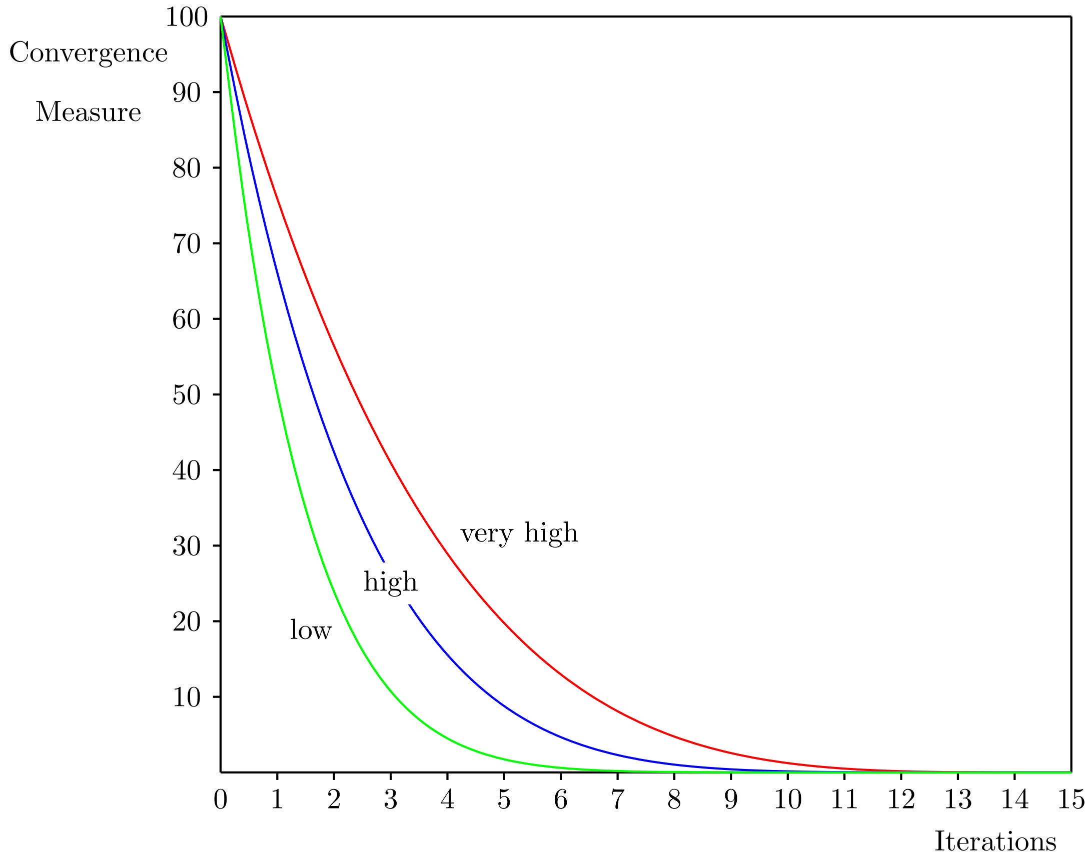


Figure 4.1: Rate of Convergence Depending on Level of Congestion - Sheffi [51]

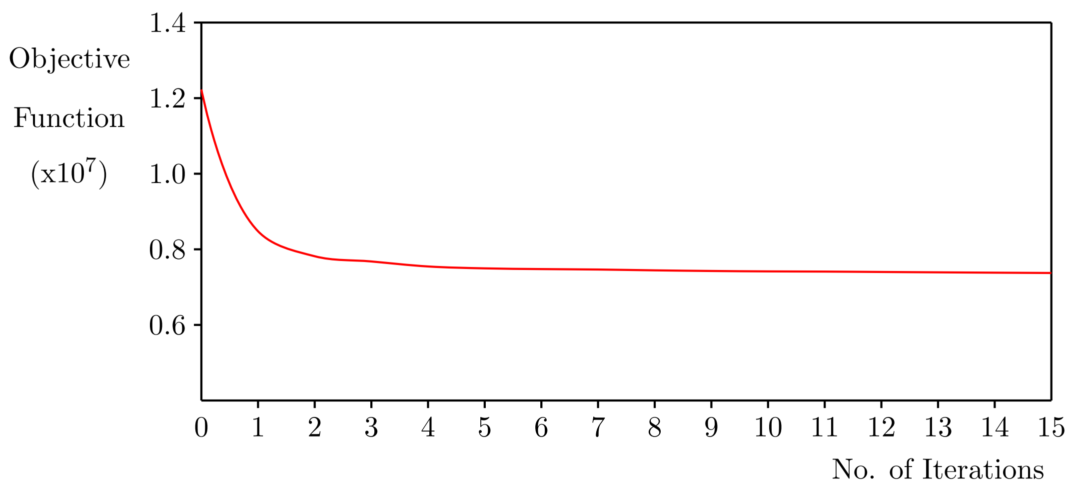


Figure 4.2: Objective Function vs Number of Iterations: Gautrans Model

- Maximum number of iterations (default = 15).

- The admissible value, in percentage, for the relative gap (default = 0.50). The relative gap is an estimate of the difference between the current assignment and a perfect equilibrium assignment, in which all paths used for a given O-D pair would have exactly the same time. It is computed using the best lower bound and the current value of the objective function.

- The admissible value, in minutes, for the normalized gap (default = 0.50). The normalized gap, or trip time difference, is the absolute difference between the mean trip time of the current assignment and the mean minimal trip time. The mean trip time is the average trip time used in the previous iteration while the mean minimal trip time is the average time computed using the shortest paths of the current iteration.

The relative gap decreases strictly from one iteration to the next, whereas the trip time difference does not necessarily have this property. In a perfect equilibrium assignment, both the relative gap and the normalized gap are zero [34].

Boyce et al. [8] investigated the relative gap required in order to produce stable link flows when comparing two scenarios to decide whether a new project should be built or not. They concluded that a relative gap of 0.01 was required to obtain the desired level of stability in the link flows. Their analysis was based on the examination of figures (such as Figure 4.3) showing the difference in link flows for different values of relative gap. As shown in Figure 4.3, only a limited number of assignments were done with the relative gap varying by a factor of 10. However, there is no indication of what happens at intermediate values of relative gap, e.g. 0.05.

It should also be remembered that with the increased computing power that is now available, the cost of extra iterations is much lower than it was in 1985 when Sheffi wrote his book.

This author [6], [7] proposed that there could be two possible stopping criteria, depending on for what purpose the results of the assignments are to be used. (Blaschuk and Hunt [5] made a similar proposal). They are the following:

- The link volumes are to be used in the geometric and/or pavement design of a road. In this case the degree of convergence does not have to be too stringent since a small difference in the assigned volumes will not affect the number of lanes required on the road or the pavement design.

- A decision has to be made concerning the financial benefit of a new road or a number of proposed new roads have to be ranked according to their economic benefits. These are usually calculated based on construction costs and the benefits brought about by the reduction in total travel times and distances resulting from the inclusion of the new roads in the network. It is more important to have a greater degree of convergence

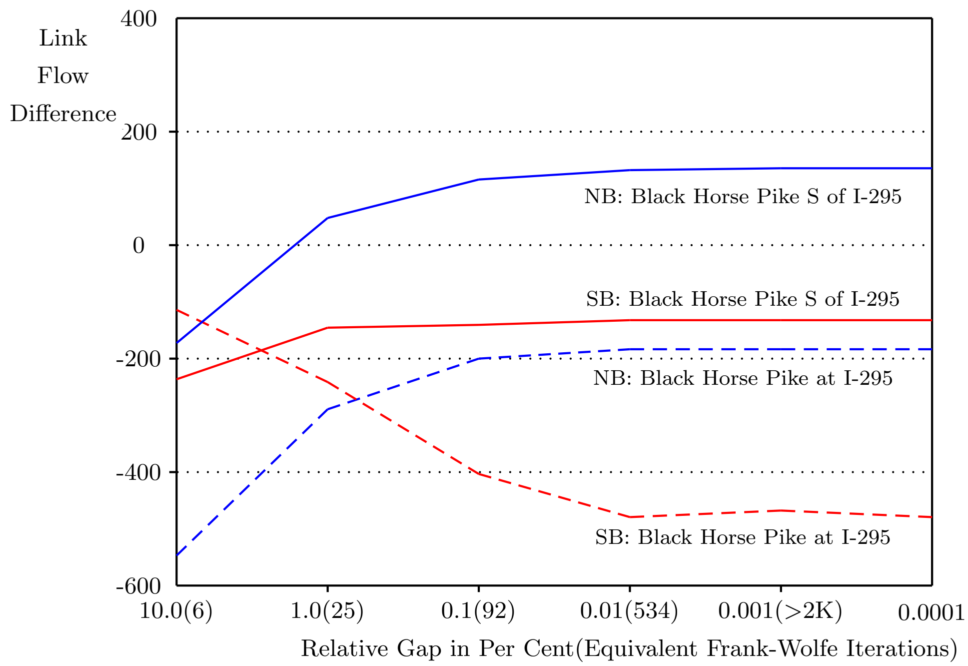


Figure 4.3: Link Flow Differences vs Relative Gap (Boyce et al [8])

in this case because the effect of a small difference in volumes can be quite large when the volumes are near capacity (see Figure 1.1).

4.3 Investigation into the Effect of Different Stopping Criteria (Bloy [6], [7])

In investigating the effect of using different stopping criteria on the results of user equilibrium assignments use was made of the Gautrans Model (also known as the Vectura Model). The investigation was carried out using the EMME/2 computer software. The macro writing capabilities of this program made it easy to write a program that could test a number of alternatives overnight and over weekends without having to set up each alternative manually.

In order to test the two possible cases mentioned in Section 6.2 above, two investigations were carried out. These were the following:

- A proposed 2000 network had been developed as part of a previous study. This network included those roads that should have been constructed already in order to relieve congestion on the network. These proposed roads were evaluated by removing them from the network, one at a time, and then doing a trip assignment. A benefit-cost ratio for each road was then obtained by calculating the cost of the difference in total travel times and distances that resulted from removing the roads from the network and the estimated costs of constructing the roads. The proposed roads were then ranked according to their benefit-cost ratios. This ranking was carried out using different stopping criteria to see when these rankings stabilized.

- The trip matrix for the year 2000 was assigned on the 2000 network using different stop criteria. The assigned volumes were compared to screen line counts[1] that were done in 2000. In addition an analysis was carried out in order to determine how the assigned link volumes obtained with different numbers of iterations and different values of relative gap differed from those for the ultimate equilibrium assignment. Two different models were used in this exercise, the 1985 PWV Model and the existing Gautrans Model. Although both models cover the same area, they are different in that they have different volume delay functions and, as a result different rates of convergence.

4.3.1 Effect of number of iterations on the financial evaluation of projects

Boyce et al. [8] in their analysis used relative gaps differing by a factor of 10 (10, 1, 0.1, etc) and then showed the differences in volumes that were obtained graphically in arriving at their conclusion that a relative gap of 0.01 was required for stability results. The purpose of this part of the study was to use benefit-cost ratios of different projects to have a more quantitative evaluation and to evaluate what happens for intermediate values of relative gap.

A total of 32 different projects were proposed to alleviate the congestion on the provincial road network in Gauteng. The economic value of these projects was evaluated by comparing the total vehicles hours and vehicle kilometres on the network with and without the projects in the network. Benefit-cost ratios for each project were calculated using average values for time and vehicle operating costs and the estimated costs of the projects. The projects were then ranked according to the resulting benefit-cost ratios. This was done for different numbers of iterations and relative gap so as to determine at what point the ranking become stable and/or when they were the same as obtained with the ultimate equilibrium solution. The "ultimate" equilibrium solution was obtained by using a very large number for the number of iterations and very small values of the relative and normalized gaps as stopping criteria.

The results obtained using between 15 and 30 iterations are shown in Table 4.1. Table 4.2 shows the rankings obtained with different values of the relative gap. For reasons of conciseness, the different projects have been assigned numbers from 1 to 32, and only the top 15 in the rankings are included in the tables.

[1] Screen lines are imaginary lines drawn across the modelled area (usually horizontally and vertically). Traffic counts are done where these lines cross the road network and these counts are are are used for calibration purposes in the modelling process.

Table 4.1: Ranking of Projects for Different Numbers of Iterations

Rank	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**
1	12	12	12	25	12	25	25	25	28	28	5	30	30	30	30	30
2	7	25	25	12	25	12	12	28	5	5	30	25	25	25	25	25
3	25	10	10	7	7	10	28	12	12	25	25	5	32	32	32	32
4	10	7	7	28	10	7	7	10	10	30	17	32	5	5	5	5
5	28	28	28	10	17	17	10	17	11	12	10	28	7	7	11	17
6	17	17	17	17	28	28	17	7	17	10	32	7	28	11	7	11
7	23	11	23	23	23	23	23	11	22	17	7	11	11	1	17	7
8	21	23	11	11	11	21	11	5	25	22	12	22	8	24	10	10
9	11	21	21	21	21	1	21	21	21	32	28	12	12	28	19	28
10	1	1	1	1	1	11	1	22	1	7	22	21	21	22	28	19
11	31	31	31	31	14	22	22	1	23	23	23	1	1	21	16	16
12	3	14	14	22	16	2	2	23	2	21	21	24	22	23	23	23
13	22	20	16	16	22	14	16	2	7	2	11	23	24	17	1	24
14	2	16	2	2	2	3	14	3	3	3	2	2	4	12	22	22
15	14	2	3	3	3	14	3	24	32	18	3	3	23	2	2	2

Table 4.2: Ranking of Projects for Different Values of Relative Gap

Rank	**1.0**	**0.5**	**0.4**	**0.3**	**0.2**	**0.10**	**0.08**	**0.07**	**0.06**	**0.05**	**0.04**	**0.03**	**0.02**	**0.01**	**Ult**
1	11	25	7	7	25	25	25	25	25	25	25	25	25	25	25
2	25	10	25	17	7	7	7	7	7	7	7	7	7	7	7
3	7	10	7	12	25	17	17	17	17	17	17	17	17	17	17
4	17	14	11	21	21	21	21	21	21	21	21	21	21	21	21
5	6	28	25	28	12	28	28	28	28	28	28	28	28	28	28
6	7	21	23	28	23	23	23	23	23	23	23	23	23	23	23
7	24	16	28	17	23	12	31	31	31	31	31	31	31	31	31
8	14	1	32	31	10	31	12	12	12	12	12	12	12	12	12
9	4	13	23	3	31	10	10	10	10	10	10	10	10	10	10
10	12	24	12	11	3	1	1	3	1	3	1	1	3	1	1
11	16	23	31	24	18	3	3	1	3	1	3	3	1	3	3
12	28	21	1	18	1	16	18	11	11	11	11	16	11	11	11
13	1	4	3	26	32	24	11	16	16	16	16	11	16	16	16
14	22	31	14	16	24	11	24	18	24	24	24	24	24	24	24
15	31	18	13	32	11	18	16	24	18	6	6	18	18	18	18

The results shown in Table 4.1 show that the ranking of the different projects varies widely depending on the number of iterations carried out. An example of this is project number 7, whose position in the rankings fluctuates considerably. Another example is project 30, it is ranked first when 26 to 30 iterations are performed, but only appears in the top fifteen after 24 iterations. In fact, when the number of iterations are from 15 to 21, project 30 is the worst ranked of all the projects, and ends up being ranked 29th when a relative gap of 0.01 is used as the stopping criterion.

Looking at Table 4.2, it can be seen that the rankings obtained using a relative gap of 0.01 are the same as the "ultimate" or final equilibrium assignments. The final equilibrium results were obtained by using a large number of iterations and very small values for the relative and normalized gaps as stopping criteria. For relative gaps of 0.02 and 0.03, the rankings of some projects (1 and 3, 11 and 16) alternate in the rankings. As can be seen from Table 4.3 the benefit-cost ratios of these projects are almost identical when small values are used for the relative gap and that therefore these differences are insignificant. With a relative gap of 0.04 the differences in the rankings become more pronounced (project 6 appears in the top 15 and there are more differences further down the rankings.

Table 4.3 shows the Benefit-Cost ratios for the top 15 projects that are obtained when different values are used for the relative gap as the stopping criterion. In Figure 4.4, the benefit-cost ratios for the top 12 projects are shown for when different values of relative gap are used as the stopping criterion. It can be seen that the benefit-cost ratios are fairly stable up to a relative gap of 0.03, after which they begin to fluctuate. This fluctuation becomes large for relative gap values of greater than 0.1.

Table 4.3: Benefit-Cost Ratios of Projects Using Different Values for Relative Gap as Stopping Criteria

Final Rank	**Proj No.**	**Ult**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.10**	**0.20**	**0.30**	**0.40**	**0.50**
1	25	5.83	5.90	5.98	6.15	6.09	5.93	6.30	6.10	6.09	6.36	5.69	5.44	7.48	8.40
2	7	5.39	5.42	5.44	5.50	5.60	5.47	5.70	5.55	5.64	5.60	6.43	5.91	5.13	4.84
3	17	4.65	4.66	4.65	4.78	4.65	4.46	4.59	4.40	5.30	5.06	5.69	3.22	5.41	8.30
4	21	3.94	3.91	3.90	3.95	3.97	3.93	4.09	4.16	4.04	3.86	3.94	3.87	3.06	3.96
5	28	3.90	3.77	3.77	3.77	3.65	3.73	3.72	3.48	3.60	3.82	3.62	3.58	4.49	6.58
6	23	3.52	3.50	3.53	3.56	3.59	3.48	3.60	3.50	3.55	3.65	3.51	3.38	2.95	5.19
7	31	3.22	3.20	3.18	3.22	3.21	3.23	3.26	3.27	3.24	3.28	3.11	2.55	3.61	4.39
8	12	3.10	3.11	3.17	3.15	3.16	3.09	3.25	3.12	3.20	3.29	3.87	4.90	4.10	3.52
9	10	2.88	2.83	2.76	2.85	3.15	2.81	2.75	2.69	2.75	3.09	3.25	0.37	4.35	6.21
10	1	2.53	2.55	2.43	2.48	2.49	2.27	2.50	2.37	2.64	2.74	1.78	0.73	2.98	5.88
11	3	2.46	2.46	2.47	2.47	2.48	2.46	2.48	2.48	2.47	2.49	2.47	2.44	2.63	2.59
12	11	1.95	1.96	1.98	1.87	2.07	1.82	2.12	2.13	1.89	1.72	1.32	2.31	2.64	3.04
13	16	1.94	1.90	1.84	1.88	1.97	1.78	1.94	2.04	1.75	2.14	0.95	0.94	0.65	4.81
14	24	1.77	1.76	1.72	1.80	1.85	1.69	1.70	1.61	1.85	1.73	1.39	2.20	0.81	3.05
15	18	1.38	1.37	1.33	1.39	1.37	1.26	1.55	1.82	2.22	1.66	1.93	1.10	1.77	2.48

The results shown in Tables 4.1 to 4.3 and Figure 4.4, and discussed above, therefore appear to confirm the conclusions of Boyce et al. [8]. However, there may be cases where time limitations justify the use of slightly less stringent stopping criteria (a relative gap of 0.02

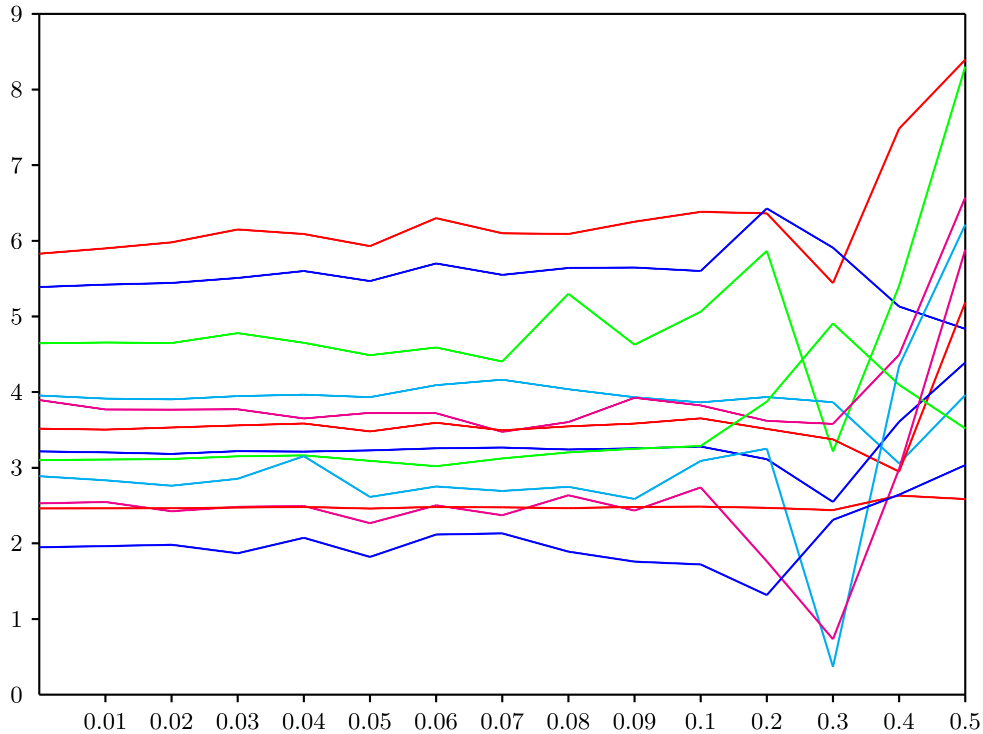


Figure 4.4: Benefit-Cost Ratios for the Top 12 Projects Using Different Stopping Criteria)

or 0.03). The question of time requirements will be addressed later in this chapter.

Table 4.4 shows the final rankings for some projects that were not in the final top 12 as well as the highest ranking that they achieved when using different stopping criteria.

The results shown in Table 4.4 show that there can be a large difference between the final ranking and the highest ranking of a project when different stopping criteria are used. For example, projects 5, 19 and 30 were all the top ranked project at some point. However, their final rankings were 31, 32 and 29 respectively. In other words three out of the four worst projects were ranked number one at some stage. This illustrates the importance of using small values of relative gap as the stopping criterion when using assignment results in the economic evaluation of projects.

As a matter of interest, Table 4.5 shows a comparison of the results obtained using LeBlanc's network [40] where the equilibrium coditions can be computed analytically. As in LeBlanc's example, a total demand of 6 units were assigned to the network. The results for the flow on the added link (computed to be 2) and the travel time on the route including the added link (computed to be 367.4) are compared.

Table 4.4: Projects not Ranked in Top 12

Project No.	Final Ranking	Final B/C Ratio	Highest Ranking	Stopping Criterion at Highest Ranking
2	21	0.46	5	R Gap = 1.40
4	20	0.59	5	R Gap = 1.80
5	31	-0.32	1	25 Iterations
6	16	1.37	5	R Gap = 1.20
14	24	0.37	4	R Gap = 1.55
19	32	-1.26	1	R Gap = 1.60
24	14	1.77	2	R Gap = 1.40
4	20	0.59	5	R Gap = 1.80
30	29	-0.01	1	26 to 30 Iterations
32	25	0.36	3	27 to 30 Iterations

Table 4.5: Effect of Using Different Values of Relative Gap on LeBlanc's Network

Relative Gap	No. of Iterations	Volume on new link	Time on new route
1.00	10	2.013861	369.62808
0.50	12	2.007315	368.57338
0.20	15	2.002823	367.85223
0.10	17	2.001499	367.64002
0.05	20	2.000581	367.49289
0.02	22	2.000309	367.44939
0.01	25	2.000120	367.41914
0.00	48	2.000007	367.40113

4.3.2 The effect of stopping criteria on the correlation between modelled volumes and traffic counts

As shown above, in order to obtain stable assignment results to be used in the financial evaluation of projects it is necessary to use very small values of the relative gap for the stopping criterion. However, many traffic assignments are used to provide traffic volumes that are used to assist engineers in the design of new facilities. The question therefore is what stopping criterion would be suitable to provide reliable information for this purpose.

One way of doing this would be to compare assigned volumes with traffic counts. In 2000 a number of screen line counts were carried out for the calibration of a new model for Gauteng. A total of 79 stations were counted providing 158 directional counts that could be compared with assignments results. Once again assignments were carried out using a wide range of number of iterations and relative gaps as stopping criteria. The results of this investigation are summarized in Table 4.6 (I = number of iterations, RG = relative gap).

The results shown in Table 4.6 indicate that the degree of convergence of the equilibrium assignment process has virtually no effect on the correlation between modelled and counted

Table 4.6: Regression Results: Assigned Volumes vs Traffic Counts for Different Stopping Criteria

	I = 15	**I = 30**	**RG = 1.0**	**RG = 0.5**	**RG = 0.1**	**RG = 0.01**
R^2	0.750	0.746	0.747	0.746	0.745	0.744
Intercept	21.237	18.517	20.960	18.475	17.386	17.959
Slope	0.867	0.867	0.864	0.866	0.866	0.865

volumes. This would seem to agree with Sheffi's statement that most of the convergence takes place in the first few iterations. It should, however, be pointed out that many of the screen line counts were done in rural and semi-rural areas where there is little congestion. Different results may have been obtained if more counts in congested city areas had been used.

The fact that the degree of convergence has little or no effect when comparing modelled with counted volumes and yet has a significant effect in the finacial evaluation of projects can be explained as follows. Consider a road operating at or near its capacity of 4000 vehicles per hour. A difference of plus or minus 100 vehicles would have a very small effect on the regression coefficients shown in Table 4.6 with 158 points being considered. However, the same differences would have a much larger effect on the total time travelled since the road would be operating at the steep section of its link performance function as shown in Figure 1.1.

The results shown in Table 4.6 appear to show that, when compared with counts, a relatively small number of iterations are sufficient to provide reliable estimates of traffic volumes. However, it should be remembered that only a small proportion of the total number of links are included in the comparison. The question that then arose was: how close to the final equilibrium volumes are the link volumes when different stopping criteria are used? It was thought that it would be useful to produce a table that would provide modellers with an idea of the level of confidence they could have in their results compared to the final equilibrium solution.

In order to obtain answers that might be generally applicable, it was decided to test more than a single scenario. Two different models were used, the 1985 PWV Study and the 1991 Vectura Study (Gautrans) models. Although both models modelled the same area, they are different models having different volume delay functions and matrices. As a result, their convergence characteristics are quite different. In addition, six different scenarios using each model were investigated. These different scenarios were obtained by using different networks and different matrices to give different congestion levels (e.g. matrices for 1985, 2000 and 2010 were assigned to the 2000 network). The results of this analysis are shown in Table 4.7.

An example of how Table 4.7 should be understood is that with a relative gap of 0.5, 94.9 per cent of the links will have assigned volumes within 10 per cent of those for the final equilibrium solution. It is believed that this table could provide modellers with an indication of the stopping criterion to use so as to obtain a required level of confidence in their results.

Table 4.7: Cumulative Percentage Within Given Percentage of "Equilibrium" Link Volumes

Cum %	RG=1.0	RG=0.5	RG=0.4	RG=0.3	RG=0.2	RG=0.1	RG=0.05	RG=0.01
0	19.2	23.4	25.0	27.4	30.8	37.7	45.8	69.6
1	45.0	55.5.4	59.0	64.0	70.5	80.5	88.3	97.3
2	61.5	71.9	74.9	79.1	83.7	90.2	94.5	98.9
5	82.1	88.4	90.0	92.0	94.1	96.7	98.3	99.7
10	91.8	94.9	95.7	96.7	97.6	98.7	99.4	99.9
25	97.5	98.5	98.7	99.1	99.4	99.7	99.8	100
50	99.0	99.4	99.6	99.7	99.8	99.9	99.9	100
100	99.6	99.8	99.8	99.9	99.9	100	100	100

Table 4.6 indicates that the stopping criterion used has very little effect on the correlation between counts and assigned volumes. However, Table 4.7 shows that if one wants the assigned volumes to be near the equilibrium volumes then more stringent stopping criteria should be used. Since modellers generally refer to the results as being from an equilibrium assignment they should be fairly close to the actual equilibrium solution. For design purposes, it is not necessary that they be very close, a relative gap of 0.1 or 0.2 would be probably be sufficient since that would mean that approximately 95 per cent of the link volumes are within five per cent of the equilibrium solution.

The results obtained using the number of iterations as the stopping criteria varied much more widely between the different scenarios tested than did those using relative gap. This suggests that relative gap should rather be used as a stopping criterion since it is more likely to result in similar degrees of convergence for different models.

4.3.3 Computational effort

The equilibrium assignment process is an iterative process and the improvement resulting from each successive iteration gets progressively smaller. Therefore, the relationship between the relative gap and the number of iterations is not a linear one. Table 4.8 shows the time taken to achieve different relative gaps. It should be noted that these are specific to the Gautrans Model as it existed at the time of the investigation, (668 zones, 7326 nodes including centroids and 18133 links), and the computer on which the model was run (1.20 Ghz processor). However, the table could be used to give a modeller an indication of the times required once they had done an assignment using one of the stopping criteria, for example relative gap = 0.5. Boyce et al. [8] stated that they thought that a reasonable solution time for a large scale network would be an overnight run, or up to 12 - 14 hours. Using the EMME/2 macro language it is easy to test a number of alternatives overnight or over a weekend.

Table 4.8: Computational Effort Required for Different Stopping Criteria Gautrans Model

Relative Gap	No. of Iterations	Time (min:sec)
1.90	15	0:56
0.50	40	2:24
0.40	45	2:42
0.30	55	3:16
0.20	72	4:16
0.10	113	6:44
0.05	184	10:58
0.04	218	13:26
0.03	275	16:28
0.02	376	22:12
0.01	692	40:10

4.4 Conclusions and Recommendations

Bloy reached the following conclusions from his [6], [7] investigation:

- Modellers should be very wary about using default values for stopping criteria when doing assignments.
- An objective measure such as the relative gap is preferable to the number of iterations as a stopping criterion.
- Different stopping criteria can be used depending on the purpose of the assignment (economic evaluation or inputs for design).
- The speed of modern computers means that doing extra iterations is no longer the drawback that it used to be.

Bloy [6], [7] made the following recommendations:

- When doing assignments where the results will be used in economic evaluation, a relative gap of 0.01 is the most appropriate stopping criterion.
- If the results of the assignment are to be used as input in the design of roads, the values contained in Table 4.8 should be used as an indication of the relative gap needed to provide the required confidence that the results are close to the equilibrium solution.

Blaschuk and Hunt [5] tested different values of relative gap in the range 0.01 to 0.10 and compared the total vehicle hours travelled on the network and the volumes on a few selected links on a model for Calgary in Canada. They recommended that a relative gap of 0.01

be used for both economic evaluation and link volumes for design purposes. They did, however, qualify the recommendation for link volumes for design purposes by saying that a relative gap of 0.05 could be used in order to save time.

Chapter 5

NEW INVESTIGATIONS INTO BRAESS' PARADOX

5.1 How Often Does Braess' Paradox Occur?

As explained in Chapter 3 tests were done using data from the 1985 PWV Update Study. In this study which had 1985 as the base year, a network was developed for the year 2000. This was done by adding road improvements for each year from 1988 to 2000. In the tests, the results of which follow, each year was tested separately by assigning a matrix which was derived by interpolating between the matrices which had been developed as part of the study for the years 1985 and 2000.

For each year the road improvements for that year as well as for the preceding years were tested to see whether they resulted in Braess' paradox or not. For example, there were 23 road projects for 1988 and another eight were added for 1989, all 31 projects were then tested for 1989. The reason for doing this was to check to see whether any of the projects for 1988 showed Braess' paradox in later years.

The results of this testing are shown in Tables 5.1 to 5.8 for different stopping criteria. The tables contain the following information: the number of road improvement projects (either new roads or additional lanes), the number of projects showing Braess' paradox, the largest Braess' paradox (the largest reduction in total vehicle-hours resulting from the removal of a project), and a list of the projects showing Braess' paradox together with the magnitude of the Braess' paradox in parenthesis. For conciseness, the projects have been numbered rather than giving a description.

The following statements can be made after examining the results given in Tables 5.1 to 5.8:

- The results obtained when using a stopping criterion of 15 iterations (used by some practitioners) are disturbing, with up to 58 per cent of the projects apparently showing

Braess' paradox (in 1996).

- In general, the number of projects apparently showing Braess' paradox decreases as the stopping criterion becomes more stringent (there are some exceptions to this).

- The magnitude of the decrease in total vehicle-hours (the size of the apparent Braess' paradox) tends to decrease as the stopping criterion becomes more stringent. Once again there are exceptions to this.

- The number of projects showing Braess' paradox does not appear to be monotonic or proportional to the number of projects being considered. As an illustration of this consider the following extract from Table 5.8:

Year	**No. of Projects**	**No. of Braess' Projects**
1992	48	2
1993	77	5
1994	91	2
1995	107	0

- A project which shows Braess' paradox in one or more years, may not show it in succeeding years, e.g. project number 76 in 1993 and 1994 with relative gap = 0.01 as the stopping criterion shows the paradox but does not in later years. This shows that the addition of further roads may result in a "better" network with less cases of Braess' paradox.

- As described in Chapter 4, it is better to use a more stringent stopping criterion such as the relative gap = 0.01 when doing financial type analyses such as checking for Braess' paradox. This implies that the results are nearer the true equilibrium condition than would otherwise be the case.

- Valiant and Roughgarden [59] proved that for large random networks with linear delay functions and a single origin-destination pair, Braess' paradox occurs with high probability (see Section 2.4.2).

The analysis shown here, particularly in Table 5.8, shows that under certain flow and network conditions examples of Braess' paradox are likely to occur in sets of road network improvements.

Table 5.1: Braess' Paradox in 1985 Update Study with 15 Iterations

Year	No. of Projects	No. of Braess' Projects	Largest Braess' Paradox	Braess' Projects (size of paradox)
1988	23	3	27	2(7); 3(3); 11(27)
1989	31	1	13	18(13)
1990	44	7	31	8(9); 10(9); 18(14); 29(31); 33(6) 37(2); 38(3)
1991	48	12	18	4(4); 8(10); 18(12); 19(10); 20(9); 28(2) 29(17); 34(4); 38(2); 44(7); 45(18); 49(5)
1992	64	28	62	1(29); 4(21); 6(9); 10(41); 11(44); 14(2); 16(12); 18(27); 19(53); 20(23) 21(24); 25(1); 29(20); 33(21); 42(11) 43(7); 45(24); 46(36); 47(6); 48(3) 49(4);51(17); 53(10); 57(38); 58(62) 60(7); 61(39); 64(14)
1993	77	19	58	3(5); 8(3); 12(7); 18(2); 20(1) 29(9); 38(7); 39(13); 43(5); 48(1) 51(3); 55(11); 58(29); 59(5); 64(11) 67(7); 71(2); 74(29); 76(58)
1994	91	6	21	54(6); 55(12); 73(9); 76(10); 86(4) 88(21)
1995	107	9	20	11(3); 18(1); 49(2); 54(3); 70(4) 79(5); 88(20); 97(4); 98(3)
1996	123	71	70	1(22); 3(23); 4(49); 6(2); 8(35) 10(30); 11(34); 12(21); 13(19); 18(20) 19(36); 20(40); 22(29); 23(21); 25(23) 28(14); 29(30); 33(21); 34(23); 37(8) 38(24); 43(29); 46(28); 48(8); 51(51) 53(32); 54(22); 55(10); 56(25); 58(47) 60(17); 61(3); 64(38); 66(5); 68(34) 70(25); 71(21); 72(10); 73(28); 74(46) 76(55); 79(10); 80(56); 81(20); 83(41) 84(39); 88(31); 89(70); 90(45); 91(32) 93(43); 94(17); 96(4); 98(26); 99(26) 100(25); 102(16); 105(21); 106(32); 109(12) 110(8); 111(10); 113(36); 114(20); 116(39) 117(38); 119(31); 120(20); 121(21); 124(39)

Table 5.2: Braess' Paradox in 1985 Update Study with 15 Iterations - continued

Year	No. of Projects	No. of Braess' Projects	Largest Braess' Paradox	Braess' Projects (size of paradox)
1997	148	36	30	4(17); 12(12); 20(4); 29(30); 33(20); 34(3); 38(4) 43(6) 53(18); 55(9); 58(20); 60(2); 69(2); 70(23) 71(19); 75(15); 80(7); 86(16); 88(8); 89(16) 90(5); 93(15); 97(6); 114(29); 127(14); 128(23) 129(15); 131(7); 137(21); 138(6); 140(6) 141(10); 143(23); 144(3); 145(16); 146(10)
1998	158	52	66	4(37); 10(16); 11(48); 13(13); 28(18) 38(14); 43(8); 51(17) 54(12); 55(27); 61(17) 67(5); 68(2); 69(22); 70(32); 71(17); 73(1) 76(35); 80(22); 83(35); 84(36); 86(29); 88(20) 97(16); 98(14); 102(66); 105(12); 111(4) 113(28); 114(28); 116(30); 119(9); 120(15) 124(19); 126(14); 127(52); 128(7); 129(3) 130(3); 132(5); 134(21); 139(19); 140(35) 143(23); 146(6); 147(14); 148(32); 149(5) 151(14); 153(13); 154(2); 158(46)
1999	165	42	52	4(16); 8(17); 10(3); 18(16); 20(52); 34(9) 38(28); 51(31) 55(25); 58(27); 61(22); 69(46) 74(25); 75(31); 76(20); 83(13); 84(4); 89(2) 93(5); 97(32); 98(17); 101(6); 109(10) 111(51); 114(4); 116(14); 120(1); 121(15) 125(18); 128(23); 129(10); 130(11); 131(26) 133(31); 140(9); 141(2); 144(29); 147(10) 151(8); 157(15); 161(16); 163(33)
2000	186	79	84	11(44); 19(30); 21(45); 22(23); 23(10); 25(72) 28(43); 29(25); 34(65) 35(25); 43(2); 49(35) 51(2); 53(71); 54(19); 57(2); 60(13); 61(46) 62(10); 68(1); 70(22); 71(83); 73(13); 74(26) 76(16); 79(64); 80(37); 84(40); 86(35); 88(66) 89(53); 90(64); 91(39); 93(16); 94(30); 96(1) 111(81); 113(32); 116(82); 117(79); 119(44) 122(66); 123(82); 125(21); 126(37); 131(62) 133(44); 134(36); 135(29); 137(17); 139(67) 140(66); 141(7); 142(26); 144(34); 145(38) 146(59); 149(20); 150(84); 151(58); 155(2) 157(17); 161(8); 163(80); 165(32); 168(43) 170(4); 172(10); 173(56); 174(63); 175(76) 177(69); 178(27); 180(15); 181(65); 184(54) 185(27); 186(3); 187(59)

Table 5.3: Braess' Paradox in 1985 Update Study with Relative Gap = 0.20

Year	No. of Projects	No. of Braess' Projects	Largest Braess' Paradox	Braess' Projects (size of paradox)
1988	23	10	27	1(10); 4(10); 8(3); 10(14); 11(14) 16(5); 18(11); 19(11); 20(27); 23(8);
1989	31	11	28	1(23); 3(13); 4(27); 10(8); 11(16); 13(3) 18(28); 19(1); 25(6); 27(11); 28(6)
1990	44	2	2	18(1); 28(2)
1991	48	11	12	3(6); 10(9); 16(3); 23(3); 27(8); 34(6) 37(2); 38(12); 44(1); 46(3); 48(87
1992	64	9	16	20(6); 24(11); 29(11); 38(3); 49(3) 51(3); 53(9); 55(16); 58(14)
1993	77	1	24	76(24)
1994	91	40	63	3(12); 4(27); 8(9); 10(1); 11(6) 13(9); 16(6); 18(11); 20(23); 23(9) 28(9); 34(23); 37(4); 38(25); 43(1) 46(3); 49(18); 51(4); 53(27); 55(6) 58(5); 60(1); 64(10); 67(7); 68(10) 69(15); 70(12); 71(13); 73(3); 74(4) 75(10); 76(63); 79(2); 80(13); 83(31) 84(19); 85(6); 86(9); 88(24); 91(22
1995	107	6	4	29(1); 58(2); 70(4); 73(2); 79(3); 89(4)
1996	123	29	16	11(8); 18(4); 20(14); 29(13); 34(11) 38(5); 51(4); 53(2); 55(8); 58(13) 69(4); 70(11); 74(14); 79(4); 80(8) 84(2); 88(11); 89(1); 90(4); 91(14) 93(4); 97(10); 98(10); 109(5); 111(16) 113(2); 114(8); 116(11); 123(9)
1997	148	14	8	38(4); 58(2); 69(2); 71(3); 79(6) 83(7); 88(2); 89(8); 100(2); 111(7) 119(6); 128(8); 129(1); 131(4)
1998	158	5	10	20(10); 71(3); 79(2); 88(2); 97(1)
1999	165	11	16	18(3); 38(16); 58(7); 79(10); 83(1); 89(12) 102(2); 144(16); 148(1); 153(1); 161(1)
2000	186	27	27	4(1); 20(27); 34(14); 38(7); 52(10) 58(3); 68(3); 69(11) 84(19); 89(14) 97(5); 102(8); 116(1); 128(18); 132(10) 136(2); 153(1); 154(4); 161(6); 163(4) 170(4); 174(10); 175(1); 176(4); 181(16) 187(15); 188(6)

Table 5.4: Braess' Paradox in 1985 Update Study with Relative Gap = 0.10

Year	No. of Projects	No. of Braess' Projects	Largest Braess' Paradox	Braess' Projects (size of paradox)
1988	23	8	19	1(2); 4(15); 8(18); 10(3); 16(9) 18(15); 20(19); 23(12)
1989	31	14	17	1(10); 3(8); 4(3); 5(1); 8(3) 10(4); 13(2); 16(10); 18(17); 20(4) 23(11); 25(3); 27(14); 28(10)
1990	44	9	10	8(10); 11(5); 16(6); 18(7); 20(8) 29(6); 34(7); 38(10); 40(3)
1991	48	1	7	38(7)
1992	64	1	3	20(3)
1993	77	3	40	20(1); 69(2); 76(40)
1994	91	3	24	69(3); 76(24); 83(4)
1995	107	1	1	20(1)
1996	123	0	-	
1997	148	8	6	20(6); 38(6); 52(6); 67(3); 69(6) 97(2); 144(5); 148(1)
1998	158	5	5	52(2); 58(5); 128(1); 144(1); 148(3)
1999	165	1	6	88(6)
2000	186	19	12	11(1); 34(1); 38(2); 52(3); 69(1) 84(4); 89(4); 93(1) 104(8); 113(6) 128(5); 130(9); 132(9); 146(1); 148(2) 151(2); 153(3); 170(8); 181(12)

Table 5.5: Braess' Paradox in 1985 Update Study with Relative Gap = 0.05

Year	No. of Projects	No. of Braess' Projects	Largest Braess' Paradox	Braess' Projects (size of paradox)
1988	23	4	11	16(1); 18(11); 20(8); 23(4)
1989	31	3	5	4(1); 18(5); 27(4)
1990	44	0	-	
1991	48	2	3	29(1); 49(3)
1992	64	0	-	
1993	77	4	44	34(1); 38(3); 71(3); 76(44)
1994	91	1	27	76(27)
1995	107	6	3	34(1); 38(2); 68(1); 69(3); 82(3); 97(2)
1996	123	4	5	34(5); 38(3); 73(5); 113(1)
1997	148	2	3	104(1); 148(3)
1998	158	0	-	
1999	165	3	3	20(2); 104(1); 148(3)
2000	186	4	3	88(3); 89(2); 170(1); 175(1)

Table 5.6: Braess' Paradox in 1985 Update Study with Relative Gap = 0.03

Year	**No. of Projects**	**No. of Braess' Projects**	**Largest Braess' Paradox**	**Braess' Projects (size of paradox)**
1988	23	4	7	4(1); 16(1); 18(7); 20(5)
1989	31	3	4	18(4); 20(1); 27(1)
1990	44	0	-	
1991	48	2	2	29(1); 49(2)
1992	64	4	4	20(1); 34(4); 38(1); 58(1)
1993	77	4	42	38(2); 69(1); 74(1); 76(42)
1994	91	1	33	76(33)
1995	107	9	4	34(2); 38(1); 52(2); 58(2); 69(2) 83(1); 88(1); 89(4) 103(1)
1996	123	2	4	38(1); 73(4)
1997	148	3	1	89(1); 130(1); 144(1)
1998	158	11	5	34(2); 38(4); 58(1); 69(1); 89(5); 104(4) 128(1); 130(3); 144(2); 148(2)
1999	165	1	2	58(2)
2000	186	7	3	20(1); 36(1); 52(2); 88(2); 89(1) 128(2); 153(3)

Table 5.7: Braess' Paradox in 1985 Update Study with Relative Gap = 0.02

Year	**No. of Projects**	**No. of Braess' Projects**	**Largest Braess' Paradox**	**Braess' Projects (size of paradox)**
1988	23	2	7	18(7); 20(3)
1989	31	2	4	18(4); 20(1)
1990	44	5	4	20(1); 28(4); 29(2); 34(2); 38(4)
1991	48	0	-	
1992	64	2	1	20(1); 38(1)
1993	77	3	38	20(1); 38(1); 76(38)
1994	91	2	34	76(34); 89(1)
1995	107	0	-	
1996	123	5	3	27(1); 36(2); 38(1); 52(3); 73(3)
1997	148	2	4	38(1); 73(4)
1998	158	2	1	58(1); 128(1)
1999	165	5	2	20(2); 52(2); 58(1); 88(1); 153(1)
2000	186	8	2	27(1); 36(1); 52(1); 128(2) 130(1); 148(1); 153(1); 175(2)

Table 5.8: Braess' Paradox in 1985 Update Study with Relative Gap = 0.01

Year	**No. of Projects**	**No. of Braess' Projects**	**Largest Braess' Paradox**	**Braess' Projects (size of paradox)**
1988	23	1	2	18(2)
1989	31	2	6	18(6); 20(1)
1990	44	4	1	20(1); 29(1); 34(1); 38(1)
1991	48	5	2	20(2); 28(1); 29(2); 34(1); 38(1)
1992	64	2	1	20(1); 29(1)
1993	77	5	40	27(2); 38(1); 52(1); 69(1); 76(40)
1994	91	2	35	20(1); 76(35)
1995	107	0	-	
1996	123	1	1	73(1)
1997	148	0	-	
1998	158	1	1	58(1)
1999	165	1	2	58(2)
2000	186	3	1	27(1); 36(1); 128(1)

5.2 Range Where Braess' Paradox Occurs

5.2.1 Some Theoretical Analysis

Pas and Principio [45] analysed the original Braess network and derived a range of values of the total demand on the network for which Braess' paradox would occur. They also produced Figure 2.16 showing how the flows on the different paths in the network varied when the total demand on the network increased.

The link performance functions in the Braess network were linear (see Figure 1.4) and all the lines in Figure 2.16 were also linear. In order to test what happens when the link performance functions are non-linear, a similar figure was produced for the network shown by LeBlanc in his paper [40], (see Figure 2.7). The results of this analysis are shown in Figure 5.1. It can be seen from Figure 5.1 that initially all of the flow in path 3 and the line representing this flow and the demand is linear. Once flow starts being loaded onto paths 1 and 2, the flow on the paths is no longer linear. Unlike the Pas and Principio example using Braess' network, there is flow on path 3 after Braess' paradox no longer occurs. Although Figure 5.1 shows the flow on path 3 decreasing, it does reach a point where it starts to increase again. Braess' paradox occurs for demand values from 2.86 to 7.45 in this example.

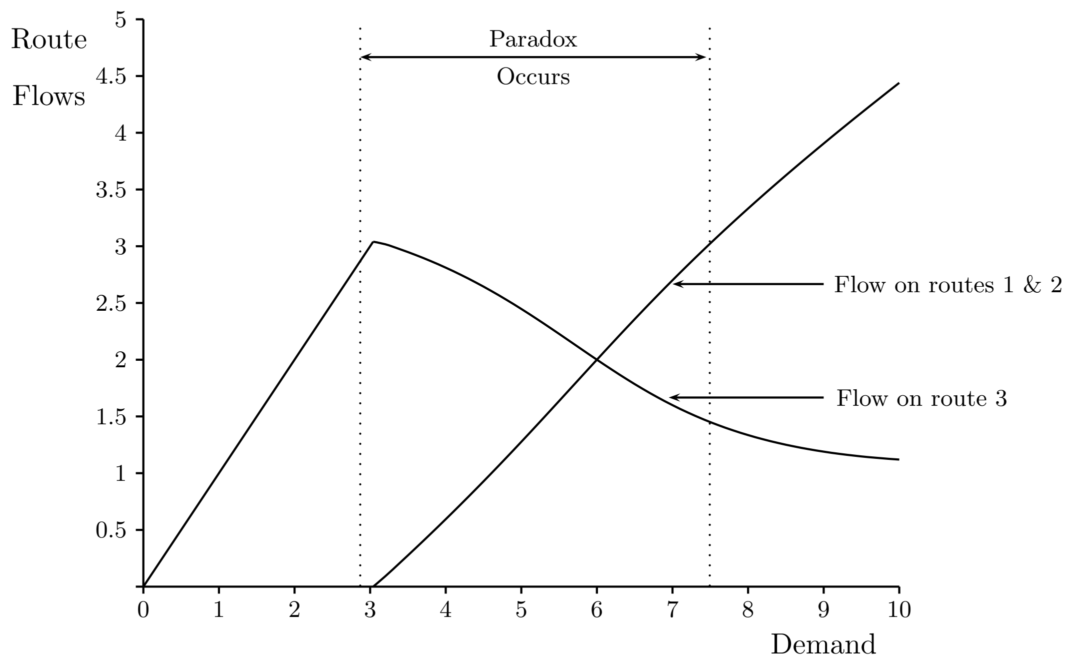


Figure 5.1: Route flows as a function of demand - LeBlanc's example

In both of the networks from the Braess and Leblanc examples, the link performance functions are functions of only the flow on the links and the capacity of the links is not taken into account. It was therefore not possible to determine the levels of congestion at which Braess' paradox occurred.

In order to obtain some idea as to whether it is necessary to have high levels of congestion, we have constructed a simple network that uses the link performance functions from a real world model. This network is shown in Figure 5.2 and the details of the links are provided in Table 5.9.

The link performance functions were according to the Bureau of Public Roads (BPR) formula which has the following form:

$$t = t_0 \left[1 + a\left(\frac{x}{c_p}\right)^b\right]$$

where:

$$\begin{aligned} t &= \text{travel time under congested conditions} \\ t_0 &= \text{free flow travel time} \\ x &= \text{assigned link volume} \\ c_p &= \text{practical capacity, = 0.75 of the nominal capacity} \\ a, b &= \text{constants, values of 0.15 and 4 were used} \end{aligned}$$

Table 5.9: Link characterstics

Link	Length (km)	Free flow speed (km/h)	Capacity (vph)
1	1.56	60	830
2	1.56	60	830
3	0.75	70	920
4	0.75	70	920
5	1.56	110	1110

A plot similar to that produced by Pas and Principio is shown in Figure 5.3. In this case Braess' paradox occurs for total demand, Q, between 508.25 and 873.99 vehicles per hour. Figure 5.3 has the same general form as Figure 5.1 for the LeBlanc network. Similarly to the discussion about Figure 5.1, the flow on path 3 decreases until it reaches a minimum and then starts to increase at higher volumes than shown in Figure 5.3.

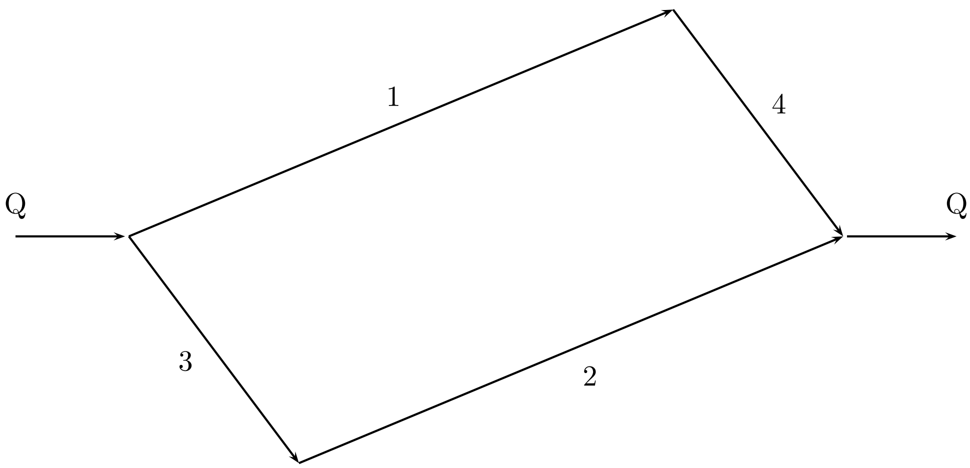
Figure 5.3 and the range of 508.25 and 873.99 were derived using the EMME/2 computer program and using different values for Q. This can also be done analytically by substituting the values from Table 5.9 in the BPR formula and computing the link travel times as follows:

For the augmented 5-link network

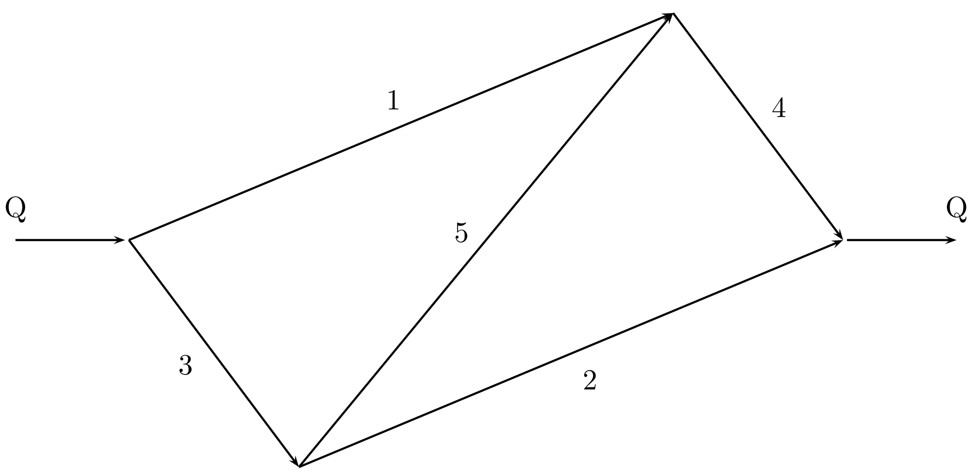
Assume that the total flow on the networks is Q and that the flow using link 5 in the augmented network is P on the network shown in Figure 5.2.

Then the times on the various links are as follows:

$$t_1 = t_{01}\left[1 + 0.15\left(\frac{0.5Q - 0.5P}{0.75 \times 830}\right)^4\right]$$



(a) Original network



(b) Augmented network

Figure 5.2: Braess type network with BPR link performance functions

$$t_2 = t_{02}\left[1 + 0.15\left(\frac{0.5Q - 0.5P}{0.75 \times 830}\right)^4\right]$$

$$t_3 = t_{03}\left[1 + 0.15\left(\frac{0.5Q + 0.5P}{0.75 \times 920}\right)^4\right]$$

$$t_4 = t_{04}\left[1 + 0.15\left(\frac{0.5Q + 0.5P}{0.75 \times 920}\right)^4\right]$$

$$t_5 = t_{05}\left[1 + 0.15\left(\frac{5P}{0.75 \times 1110}\right)^4\right]$$

where t_{0n} in the free flow travel time on link n

Then the time on route 1 is

$$t_1\left[1 + 0.15\left(\frac{0.5Q - 0.5P}{0.75 \times 830}\right)^4\right] + t_4\left[1 + 0.15\left(\frac{0.5Q + 0.5P}{0.75 \times 920}\right)^4\right]$$

$$= 1.56\left[1 + 0.15\left(\frac{0.5Q - 0.5P}{622.5}\right)^4\right] + \frac{9}{14}\left[1 + 0.15\left(\frac{0.5Q + 0.5P}{690}\right)^4\right]$$

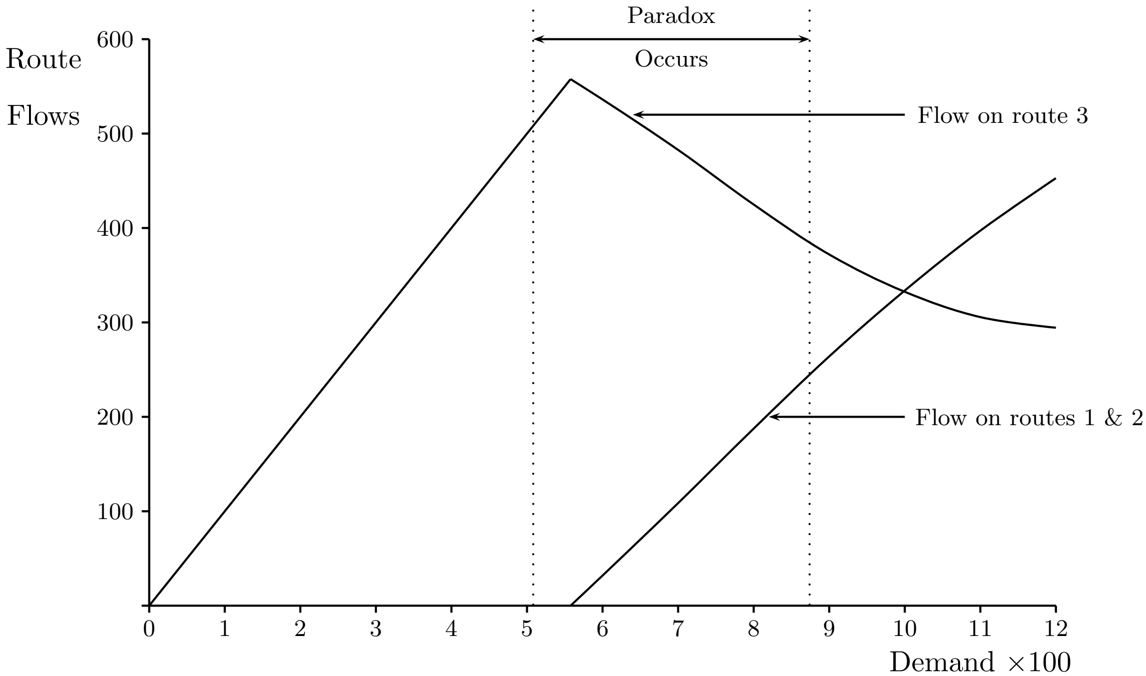


Figure 5.3: Route flows as a function of demand - BPR example

The time on route 3 is

$$t_3\left[1+0.15\left(\frac{0.5Q+0.5P}{0.75\times 920}\right)^4\right]+t_5\left[1+0.15\left(\frac{P}{0.75\times 1110}\right)^4\right]+t_4\left[1+0.15\left(\frac{0.5Q+0.5P}{0.75\times 920}\right)^4\right]$$

$$= 2\left\{\frac{9}{14}\left[1+0.15\left(\frac{0.5Q+0.5P}{690}\right)^4\right]\right\}+\frac{9.36}{11}\left[1+0.15\left(\frac{P}{832.5}\right)^4\right]$$ (links 3 and 4 are identical)

At equilibrium the time on route 1 is equal to the time on route 3 (also on route 2), therefore

$$1.56\left[1+0.15\left(\frac{0.5Q-0.5P}{622.5}\right)^4\right]+\frac{9}{14}\left[1+0.15\left(\frac{0.5Q+0.5P}{690}\right)^4\right]$$

$$= 2\left\{\frac{9}{14}\left[1+0.15\left(\frac{0.5Q+0.5P}{690}\right)^4\right]\right\}+\frac{9.36}{11}\left[1+0.15\left(\frac{P}{832.5}\right)^4\right]$$

or

$$\frac{9.36}{11}\left[1+0.15\left(\frac{P}{832.5}\right)^4\right]+\frac{9}{14}\left[1+0.15\left(\frac{0.5Q+0.5P}{690}\right)^4\right]-1.56\left[1+0.15\left(\frac{0.5Q-0.5P}{622.5}\right)^4\right]$$

$$=0$$

The above expression was simplified using Maxima to produce the following expression (approximately):

$$-7.0807173805064854E-14Q^4+4.9593451575338179E-13Q^3R$$

$$-4.24843042830389E-13Q^2R^2+4.9593451575338179E-13QP^3$$

$$+1.9492090885609397E-13P^4-0.066233766233766$$

$$=0$$

Different values of Q can be inserted in the above equation and the equation solved to provide the corresponding value of P at equilibrium. These values of Q and P can then be used to calculate the travel times on all five links and thus the whole network.

The total travel time on the network is obtained by summing the travel times on all the links, i.e.

$$
\begin{aligned}
& t_{01}\left[1+0.15\left(\tfrac{0.5Q-0.5P}{0.75\times 830}\right)^4\right]+t_{02}\left[1+0.15\left(\tfrac{0.5Q-0.5P}{0.75\times 830}\right)^4\right]+t_{03}\left[1+0.15\left(\tfrac{0.5Q+0.5P}{0.75\times 920}\right)^4\right] \\
& +t_{04}\left[1+0.15\left(\tfrac{0.5Q+0.5P}{0.75\times 920}\right)^4\right]+t_{05}\left[1+0.15\left(\tfrac{5P}{0.75\times 1110}\right)^4\right] \\
= {} & 1.56\left[1+0.15\left(\tfrac{0.5Q-0.5P}{622.5}\right)^4\right]+1.56\left[1+0.15\left(\tfrac{0.5Q-0.5P}{622.5}\right)^4\right]+\tfrac{9}{14}\left[1+0.15\left(\tfrac{0.5Q+0.5P}{690}\right)^4\right] \\
& +\tfrac{9}{14}\left[1+0.15\left(\tfrac{0.5Q+0.5P}{690}\right)^4\right]+\tfrac{9.36}{11}\left[1+0.15\left(\tfrac{P}{832.5}\right)^4\right] \\
= {} & 3.12\left[1+0.15\left(\tfrac{0.5Q-0.5P}{622.5}\right)^4\right]+\tfrac{9}{14}\left[1+0.15\left(\tfrac{0.5Q+0.5P}{690}\right)^4\right]+\tfrac{9.36}{11}\left[1+0.15\left(\tfrac{P}{832.5}\right)^4\right]
\end{aligned}
$$

The original 4-link network

For a total flow of Q on the network, the total travel time on the network is

$$
\begin{aligned}
& t_{01}\left[1+0.15\left(\tfrac{0.5Q}{0.75\times 830}\right)^4\right]+t_{02}\left[1+0.15\left(\tfrac{0.5Q}{0.75\times 830}\right)^4\right] \\
& +t_{03}\left[1+0.15\left(\tfrac{0.5Q}{0.75\times 920}\right)^4\right]+t_{04}\left[1+0.15\left(\tfrac{0.5Q}{0.75\times 920}\right)^4\right] \\
= {} & 3.12\left[1+0.15\left(\tfrac{0.5Q}{622.5}\right)^4\right]+\tfrac{9}{7}\left[1+0.15\left(\tfrac{0.5Q}{690}\right)^4\right]
\end{aligned}
$$

The existence of Braess' Paradox

Whether Braess' Paradox exists for a particular value of Q or not, can be tested by subtracting the total time for the 4-link network from the total time for the original augmented 5-link network. If the result is positive then Braess' Paradox occurs (the travel time on the augmented network is greater than the travel time on the original network).

Analytically this can be expressed as follows:

$$
\begin{aligned}
& 3.12\left[1+0.15\left(\tfrac{0.5Q-0.5P}{622.5}\right)^4\right]+\tfrac{9}{14}\left[1+0.15\left(\tfrac{0.5Q+0.5P}{690}\right)^4\right]+\tfrac{9.36}{11}\left[1+0.15\left(\tfrac{P}{832.5}\right)^4\right] \\
& -3.12\left[1+0.15\left(\tfrac{0.5Q}{622.5}\right)^4\right]-\tfrac{9}{7}\left[1+0.15\left(\tfrac{0.5Q}{690}\right)^4\right] \\
= {} & 0
\end{aligned}
$$

Once again this this can be simplified using Maxima to obtain (approximately):

$$-1.2837330021324927E-11Q^4-6.728103007070801E-13PQ^3$$
$$+1.3282741818603035E-12P^2Q^2-6.728103007070801E-13P^3Q$$
$$+2.2140560311831672E-13P^4+3.328051948051948$$
$$=0$$

The equation derived previously to obtain P for different values of Q for Figure 5.3 can be used and the values for P and Q inserted in the above expression to see whether Braess' paradox exists or not for different values of Q. If the above expression is positive then Braess' paradox exists for that particular value of Q.

A common measure of the level of congestion on a road is the volume/capacity (V/C) ratio. The V/C ratios on the different links at which Braess' paradox occurs are given in Table 5.10. The statistics in Table 5.10 show that in this particular case Braess' paradox occurs at relatively low levels of congestion. A V/C ratio of 0.68 equates to a level of service of C where a level of service of D, although somewhat congested, is considered to be acceptable.

Table 5.10: Link flows and V/C ratios at which Braess' paradox occurs

Link	**Capacity (vph)**	**Flow where paradox starts**	**V/C**	**Flow where paradox ends**	**V/C**
1	830	0	0	245.15	0.30
2	830	0	0	245.15	0.30
3	920	508.25	0.55	628.85	0.68
4	920	508.25	0.55	628.85	0.68
5	1110	508.25	0.46	383.71	0.35

Another way of representing the range of values for the total demand where Braess'paradox occurs is shown in Figure 5.4. In this figure the difference in costs on the original network and the augmented network is plotted against the total demand. If this difference is negative, i.e. the cost on the augmented network is higher than the cost on the original network, then Braess' paradox occurs.

Figure 5.5 shows the three paths traffic can follow in the augmented network (paths 1 and 2 are also the paths that are used in the original network).

If one considers the paths shown in Figure 5.5, then it is obvious that, due to symmetry, the flows on paths 1 and 2 will be equal at equilibrium (the lengths and capacities of these links are the same).

In the case of the original network, the flows on all the links will be $0.5Q$.

In the case of the augmented network, if the flow on link 5 is P (i.e. on path 3), then the flows on the various links are as follows:

$$\text{Link1} = 0.5Q - 0.5P$$

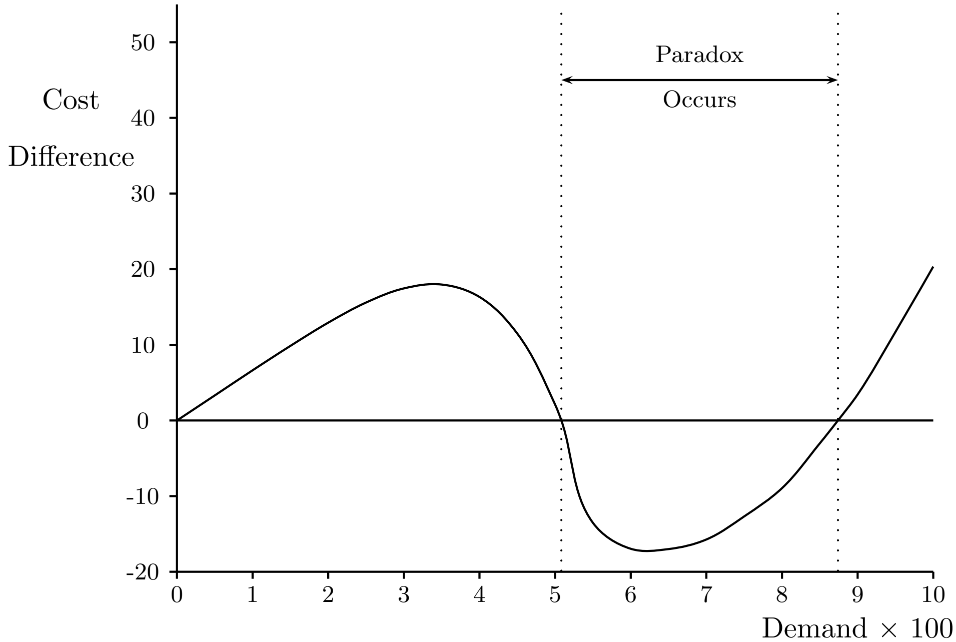


Figure 5.4: Cost on original network - cost on augmented network

$$\text{Link2} = 0.5Q - 0.5P$$

$$\text{Link3} = 0.5Q + 0.5P$$

$$\text{Link4} = 0.5Q + 0.5P$$

$$\text{Link5} = P$$

With the additional link in the augmented network, links 3 and 4 carry more flow than they would have in the original network (if there is any flow on path 3). Therefore, for Braess' paradox to occur the additional cost caused by the extra loading on links 3 and 4 exceeds the reduced costs by having lower loads on links 1 and 2 and the new link.

As a result of the above, increasing the capacity of links 3 and 4 should reduce the probability of Braess' paradox occurring. This was tested by increasing the capacity of links 3 and 4 in steps of ten per cent of their original capacities. This was found to be the case, with the range of values over which Braess' paradox decreasing until it no longer occurred. The results of this analysis are shown in Figure 5.6 and Table 5.11.

Figure 5.6 was derived using the EMME/2 computer software. However, it is possible to derive the family of curves shown in Figure 5.6 analytically. This can be done by for different values of Q and by substituting 0.75 times the different capacities in place of 690 in the formulae discussed in connection with Figure 5.3.

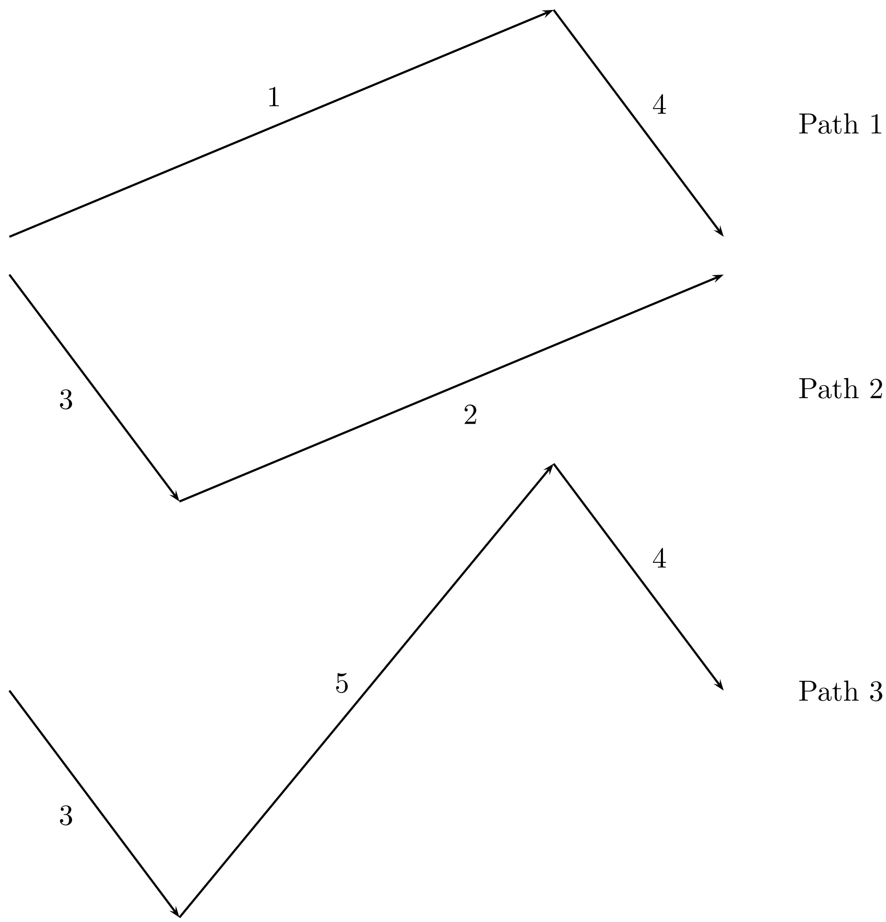


Figure 5.5: Paths through the augmented network shown in Figure 5.2

Table 5.11: Link flows at which Braess' paradox occurs for different capacities on links 3 and 4

Capacity (vph)	Flow where paradox starts	Flow where paradox ends
920	508.25	873.99
1012	548.59	831.02
1104	584.92	780.97
1196	617.04	726.01
1288	644.91	667.33

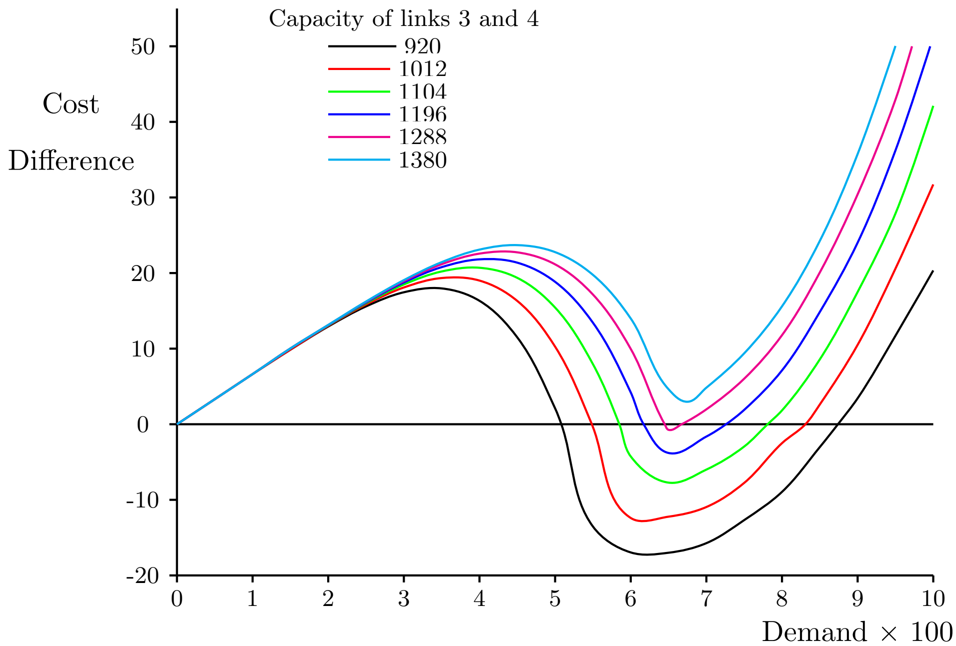


Figure 5.6: Cost on original network - cost on augmented network for different capacities on links 3 and 4

5.2.2 A Real-World Network

An attempt was made to determine whether Braess' paradox occurs over a range of values in a real world network. Once again data from the 1985 PWV Update Study [47] was used. In the first test, the entire trip matrix was multiplied by a range of factors and the effect of removing projects from the network was tested using the resulting matrices. This exercise was carried out for the 1989, 1993 and 1999 networks and matrices. For the 1989 and 1993 networks and matrices, the range of factors used was from 0.5 to 3.0. In the case of the 1999 networks and matrices, the factors ranged from 0.5 to 2.5 since it had been observed in the other two cases that any pattern that there was in the results had been established by then.

The complete set of results for the three years is shown in the Appendix of this dissertation. Tables 5.12 to 5.14 below show the results using selected projects from the three years. The projects in these tables were selected so as to give a good representation of the different types of results obtained. In these tables a negative value indicates the occurrence of Braess' paradox.

The results for selected projects in 1989 are shown in Table 5.12.

Looking at the results shown in Table 5.12 enables one to make the following observations:

- Project 4 starts by exhibiting Braess' paradox (difference = -1) when a factor of 0.5 is used. When the original matrix is multiplied by a factor of 0.6 there is no difference between the networks. For factors larger than 0.6 the differences are larger than zero and show a generally increasing trend (no Braess' paradox).

- Project 23 shows no Braess' paradox initially but does show the paradox when a factor of 0.7 is used. For larger factors there is no Braess' paradox with the differences showing an increasing trend. This suggests part of the "sinusoidal-shaped" curve shown in Figure 5.4.

- Some projects show a steady increase in the difference in travel times as the number of trips increases (projects 7, 13 and 14).

- Project 8 starts with a zero difference, the difference then increases to a local maximum (factor = 0.7) after which it decreases to a local minimum (factor = 0.8). After reaching a local minimum the difference then show a steady increase. Although there are no negative differences indicating Braess' paradox, a plot of the difference values would produce a "sinusoidal-shaped" curve similar to that shown in Figure 5.4. Projects 10 and 25 show similar results.

- Apart from with a factor of 0.6, project 18 shows Braess' paradox with the size of the paradox generally increasing as the size of the factor by which the original matrix was multiplied increases.

Table 5.12: 1989: Differences with and without projects when trip matrix is multiplied by a factor

Factor	Projects							
	4	**7**	**8**	**10**	**13**	**14**	**18**	**23**
0.5	-1	10	0	0	0	14	-1	0
0.6	0	19	3	3	1	23	0	0
0.7	2	27	5	1	4	29	-3	-1
0.8	1	34	1	0	6	33	-4	0
0.9	2	47	2	2	11	44	-3	0
1.0	4	57	7	9	15	48	-6	2
1.1	6	71	8	14	25	58	-6	3
1.2	6	90	16	19	33	79	-7	5
1.3	6	97	29	30	42	84	-9	8
1.4	6	111	37	43	54	96	-8	11
1.5	11	150	56	66	77	119	-13	16
1.6	16	172	81	82	95	136	-15	23
1.7	17	236	100	110	109	162	-12	29
1.8	22	257	111	153	148	173	-32	38
1.9	41	283	164	168	185	203	-36	53
2.0	72	346	202	203	218	233	-12	77
2.1	74	400	238	240	271	259	-21	100
2.2	85	457	282	295	333	307	-23	110
2.3	116	557	393	369	425	353	-29	167
2.4	143	667	466	450	512	438	-57	235
2.5	161	750	548	527	621	520	-75	285

- While the size of the differences show general trends there is some oscillation. It is thought that this could be due to the fact that the links in question are being used by trips between more than one origin-destination pair. In addition, as the number of trips in the matrix increases, additional O-D pairs may be served by the links as trips are diverted to new routes due to increasing congestion.

Table 5.13 shows the results for selected projects for the 1993 network and trip matrix.

Most of the examples shown in Table 5.13 show Braess' paradox for some of the smaller factors, but the paradox then disappears as the size of the factors used increases. In addition, some of the examples initially show no paradox for a factor of 0.5, but as the size of the factor then increases, Braess' paradox occurs over a short range before no longer occurring (projects 38, 69 and 70). This suggests the existence of the "sinusoidal-shaped" curve shown in Figure 5.4.

In the case of project 5, there is no Braess' paradox and the improvement brought about by including the project increases steadily as the size of the factor increases. Project 76 always shows Braess' paradox and the magnitude of the paradox shows a generally increasing trend.

Table 5.13: 1993: Differences with and without projects when trip matrix is multiplied by a factor

	Projects								
Factor	**5**	**18**	**34**	**38**	**58**	**67**	**69**	**70**	**76**
0.5	3	-2	0	0	-3	-2	0	1	-9
0.6	5	-1	0	-2	2	2	0	-1	-12
0.7	12	2	1	-1	0	4	-1	3	-16
0.8	17	4	0	-1	0	6	0	0	-21
0.9	23	4	0	-1	-1	5	0	1	-28
1.0	35	10	0	-1	0	11	-1	4	-40
1.1	47	11	1	-1	0	15	0	6	-41
1.2	67	11	2	0	-1	19	0	8	-54
1.3	92	22	5	1	3	24	2	11	-57
1.4	121	21	9	1	2	30	1	19	-80
1.5	151	20	11	1	-2	37	5	21	-93
1.6	203	28	16	2	1	47	5	30	-87
1.7	260	29	17	5	1	59	11	37	-91
1.8	333	44	20	6	8	86	15	51	-96
1.9	409	47	31	4	7	108	16	64	-114
2.0	542	81	36	10	17	124	29	79	-89
2.1	639	72	46	8	26	164	38	100	-86
2.2	775	88	64	11	35	196	43	125	-66
2.3	946	77	83	8	36	279	64	139	-125
2.4	1134	149	99	14	36	313	72	176	-81
2.5	1322	176	145	17	64	385	82	229	-107

Table 5.14 shows some of the results when the network and matrix for 1989 were used in this exercise.

Apart from project 142 which shows a steady increase in the difference values the examples show similar results. The other selected projects all show Braess' paradox for a range of small factors with the paradox disappearing as the size of the factor increases. Once again this suggests that, at least in some cases, there is evidence of the existence of the "sinusoidal-shaped" curve that was shown in Figure 5.4.

Although the results shown in Tables 5.12 to 5.14 suggest the existence of the "sinusoidal-shaped" curve, there was some concern that with the whole matrix being multiplied by the factors might possibly lead to some distortion. This is because when the whole matrix is multiplied by the factors, trips that would not normally use the routes in question might be diverted to the routes due to congestion. Also high congestion levels on routes other than the one being considered might mask the effects on the route in question.

A second test was therefor carried out where an attempt was made to isolate the trips using the links in question. This was done by doing select link assignments on the specific projects and then applying factors to only the matrices obtained from the assignment.

Table 5.14: 1999: Differences with and without projects when trip matrix is multiplied by a factor

	Projects								
Factor	**3**	**8**	**34**	**70**	**76**	**89**	**102**	**137**	**142**
0.5	2	5	0	3	-1	1	3	0	20
0.6	-3	-2	-3	-3	-3	-4	-1	-4	21
0.7	3	6	-1	0	8	-2	7	1	29
0.8	9	5	-1	1	14	0	9	2	35
0.9	11	17	1	4	22	1	10	5	44
1.0	21	31	2	7	21	1	8	10	51
1.1	25	45	3	11	47	0	17	16	74
1.2	26	74	5	11	40	3	21	25	92
1.3	61	93	10	18	77	9	13	41	94
1.4	77	123	13	18	102	10	15	59	108
1.5	80	171	15	37	152	10	22	92	118
1.6	96	220	33	50	178	7	33	131	132
1.7	129	268	44	72	206	7	41	142	159
1.8	171	351	69	89	311	7	60	177	205
1.9	235	498	89	118	368	12	109	220	306
2.0	336	618	111	137	410	17	144	331	384
2.1	420	770	141	190	407	26	152	390	474
2.2	480	972	167	231	390	27	192	223	582
2.3	617	1193	207	285	488	38	260	570	751
2.4	768	1449	265	372	609	49	304	700	909
2.5	900	1752	304	424	766	54	378	835	1200

These factored matrices were then combined with the remainder of the trip matrix and the projects were tested for Braess' paradox.

A select link assignment is a procedure where the whole matrix is assigned to the network and during the assignment process those trips that use selected links are identified. It is also possible to identify the matrix of trips that use the selected links.

The results of this process are shown in Tables 5.15 to 5.17 for selected projects for 1989, 1993 and 1999.

Table 5.15 shows the results for selected projects in the 1989 network. The results obtained were similar to those obtained when using multiples of the whole matrix.

Project 18 in Table 5.15 appears to show part of the "sinusoidal-shaped" curve first shown in Figure 5.4. With a factor of 0.5 Braess' paradox occurs and the magnitude of the paradox increases until at a factor of 1.1 it reaches a maximum. At higher values for the factor it decreases until there is no Braess' paradox when a factor 0f 1.8 is used. This is different from the results shown in Table 5.12 where the magnitude of the Braess' paradox kept on increasing for project 18.

Table 5.15: 1989: Differences with and without projects when select link matrix using the project is multiplied by a factor

Factor	Projects						
	4	**11**	**18**	**23**	**26**	**28**	**29**
0.5	0	4	-3	1	3	0	1
0.6	0	4	-4	1	4	2	2
0.7	0	6	-5	0	6	2	2
0.8	0	5	-6	-1	8	1	4
0.9	3	8	-6	1	14	1	4
1.0	4	8	-6	2	19	1	3
1.1	4	9	-8	3	25	1	2
1.2	7	9	-5	3	31	3	2
1.3	7	12	-4	4	36	1	3
1.4	10	14	-4	7	44	1	4
1.5	8	14	-3	10	57	2	2
1.6	8	15	-4	13	64	1	5
1.7	10	18	-2	15	75	1	1
1.8	12	17	0	20	86	2	5
1.9	14	18	0	23	96	1	2
2.0	15	19	1	29	111	0	3
2.1	14	18	2	33	120	1	3
2.2	16	19	4	38	128	0	3
2.3	19	17	3	42	144	1	2
2.4	24	18	3	46	158	2	4
2.5	22	20	5	51	175	2	4

Table 5.16 shows the results when factors were used to multiply the select link matrices for selected projects in 1993. All of the selected projects show Braess' paradox at some stage. However, the paradox no longer occurs when larger factors are used.

In the case of project 76, Braess' paradox no longer occurs when a factor of 2.4 is used. This differs from the case when the entire matrix is multiplied by factors where the size of the paradox keeps on increasing (in general terms) for project 76 (see Table 5.13).

The reason for the different results for project 76 can be explained with the assistance of Figure 5.7. Figure 5.7 shows the network in the vicinity of project 76. The roads ADE and FDCG are freeways and HIJ is a two-lane, two-way arterial road. ABC is a new freeway (project 76) in the new network. There is an existing interchange at D. Originally there was an interchange between the freeway and the arterial near C. With the introduction of the new road and a new interchange at C, the old interchange would no longer be possible due to the proximity to C. As a result there would no longer be an interchange between the freeway and the arterial.

When the whole matrix is multiplied by factors, the traffic not only increases on the new freeway but also on the arterial road. Since the traffic on the arterial can no longer access the freeway in this area it has to remain on the arterial for a longer distance than was the case originally. The arterial has a lower capacity than the freeway and as a result the additional delays on the arterial exceed any benefit that might result from the inclusion of project 76.

When only the select link matrix which uses project 76 is multiplied by factors the traffic on the arterial remains the same. As shown in Table 5.16, initially the delays caused on the arterial by not having access to the freeway are larger than the benefits to those using the new freeway. Eventually with increasing volumes on the new road (when it is multiplied by larger factors), the benefits to the freeway users outweigh the delays to the arterial users.

Table 5.16: 1993: Differences with and without projects when select link matrix using the project is multiplied by a factor

	Projects						
Factor	**34**	**38**	**58**	**67**	**69**	**70**	**76**
0.5	-1	0	-1	0	-1	-1	-49
0.6	0	0	1	-1	0	0	-46
0.7	0	0	0	-1	0	2	-42
0.8	1	0	1	3	2	2	-43
0.9	-2	-1	-1	8	0	2	-41
1.0	0	-1	0	14	-2	2	-40
1.1	0	0	1	14	0	5	-39
1.2	1	1	3	17	1	7	-27
1.3	2	2	1	22	2	10	-23
1.4	3	2	1	29	3	10	-20
1.5	4	0	0	36	0	8	-23
1.6	8	0	-1	44	2	7	-28
1.7	7	-1	-2	52	1	11	-30
1.8	8	0	0	64	-1	12	-31
1.9	6	4	1	74	3	14	-26
2.0	9	2	2	83	3	18	-21
2.1	10	4	3	91	3	20	-15
2.2	12	2	4	104	7	19	-11
2.3	15	4	6	117	5	21	-4
2.4	13	3	9	131	7	21	7
2.5	17	4	12	140	7	27	14

Table 5.17 shows the results for 1999 when the select link matrices for selected projects are multiplied by factors.

The following observations can be made by referring to Table 5.17 and comparing the results with those shown in Table 5.14:

- Projects 3 and 8 do not show Braess' paradox in Table 5.17 when only the select link matrices are multiplied by factors. However, both showed Braess' paradox when

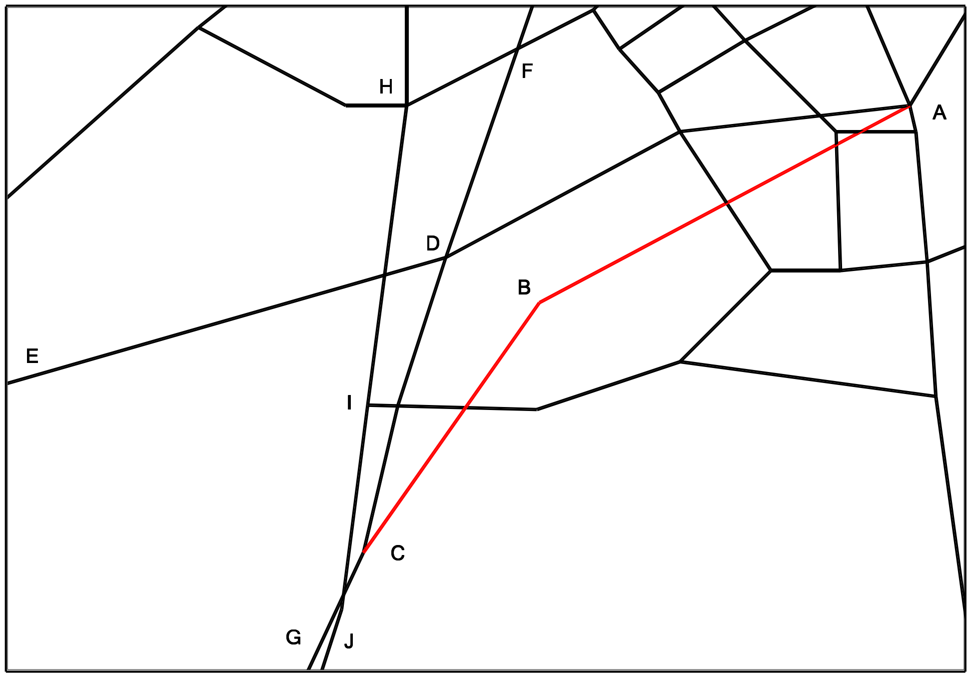


Figure 5.7: The network in the vicinity of project 76

the whole matrices were multiplied by factors. This can be explained by the fact, that with factors less than 1 in the select link case, traffic volumes on adjacent routes remains high and when the select link volumes are reduced some trips from these adjacent routes divert to the route in question. In the situation where the entire matrix is multiplied by a factor less than 1 traffic on adjacent routes is also reduced.

- Project 70 shows the "sinusoidal-shaped" curve with the difference being zero initially, then increasing before decreasing and then finally increasing again. It does not exhibit Braess' paradox however, since the differences remain positive throughout.
- Project 76 shows Braess' paradox initially but this disappears when using a much smaller factor than was the case in 1993 (Table 5.16). This is because between 1993 and 1999, the arterial route is widened providing additional capacity so that the delays on it are not as large as they were previously.

The results of the analysis shown in Tables 5.12 to 5.17 make it possible to draw the following conclusions:

- Not all new projects show Braess' paradox, even at relatively low volumes.
- Where Braess' paradox does occur it generally only occurs at lower volumes.
- There is evidence that the "sinusoidal-shaped" curve shown in Figure 5.4 and derived from a simple network also occurs in real world networks.

Table 5.17: 1999: Differences with and without projects when select link matrix using the project is multiplied by a factor

	Projects								
Factor	**3**	**8**	**10**	**18**	**34**	**70**	**76**	**102**	**137**
0.5	3	6	15	5	1	0	-14	4	0
0.6	9	14	17	6	1	3	-6	4	1
0.7	13	14	18	5	0	2	3	7	2
0.8	16	14	21	6	-1	1	12	8	3
0.9	18	19	21	7	2	4	19	9	5
1.0	21	31	24	10	2	7	21	8	10
1.1	25	41	27	9	2	9	31	7	14
1.2	26	47	28	9	3	10	43	8	19
1.3	27	55	29	12	5	8	51	8	26
1.4	28	56	28	12	6	13	60	9	32
1.5	34	58	31	14	7	15	65	13	39
1.6	42	66	35	13	8	16	72	15	43
1.7	47	72	36	15	7	17	82	18	50
1.8	52	77	39	17	9	19	88	20	59
1.9	57	84	41	18	9	19	95	20	66
2.0	58	92	42	20	12	23	98	19	75
2.1	60	99	45	22	13	27	102	21	85
2.2	63	105	44	24	16	29	106	21	94
2.3	69	111	46	26	17	29	109	22	104
2.4	76	126	47	28	18	32	113	22	114
2.5	81	131	50	28	21	31	113	22	127

5.3 Eliminating Braess' Paradox

In Section 5.2.1 it was shown that Braess' paradox could be eliminated from a small network by increasing the capacity on links upstream and downstream of the new link.

If it is possible to eliminate Braess' paradox from the small BPR network in the manner described in Section 5.2.1, it may well be possible to do so in a real-world situation as well. This possibility was tested using three examples where Braess' paradox was shown to occur. The examples chosen were from the 1989, 1993 and 1999 networks.

Figure 5.8 shows a portion of the PWV Update network in the 1989 network. In this case the road shown in blue was a two-lane, two-way road which was replaced by a four-lane divided road on a different alignment (shown in red). Braess' paradox occurred and including the new road resulted in an increase in the total travel time of 6 vehicle-hours.

The volume-capacity ratios of the two networks were compared and the results are shown in Figure 5.9 where red indicates an increase in V/C and green shows a decrease. Increasing the capacity by adding an extra lane on the short section of road to the left of the new road

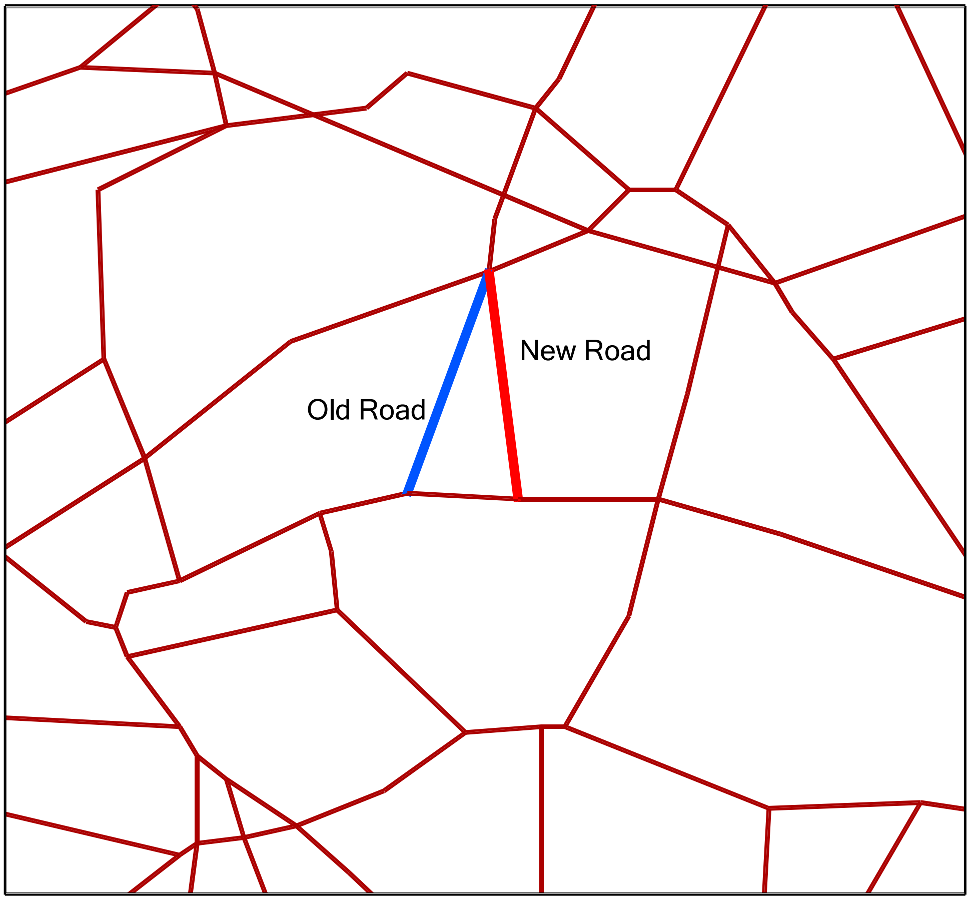


Figure 5.8: Changes in Network for Project 18 (1989)

where the V/C ratio increased by 0.65 results in the elimination of Braess' paradox. The total travel time is now reduced by 3 vehicle-hours. Adding a lane to the section of road to the right of the new road reduces the travel time by a further 4 vehicle-hours. The sections where the lanes were added in the model were on the schedule to be widened a couple of years later so Braess' paradox could be eliminated by including them in the network at an earlier date.

The project that was selected in the 1993 network was project number 76. This is shown in Figure 5.10.

This project was described in some detail in the previous section. A new section of freeway (shown in red) is included between two existing freeways and because of spacing restrictions an existing interchange is closed with a new interchange providing access to the new section of freeway.

In this case the magnitude of the Braess' paradox is 40 vehicle-hours. In other words, the inclusion of project 76 in the network causes the total travel time on the network to increase by 40 vehicle-hours. Once again the volume/capacity ratios on the links were compared

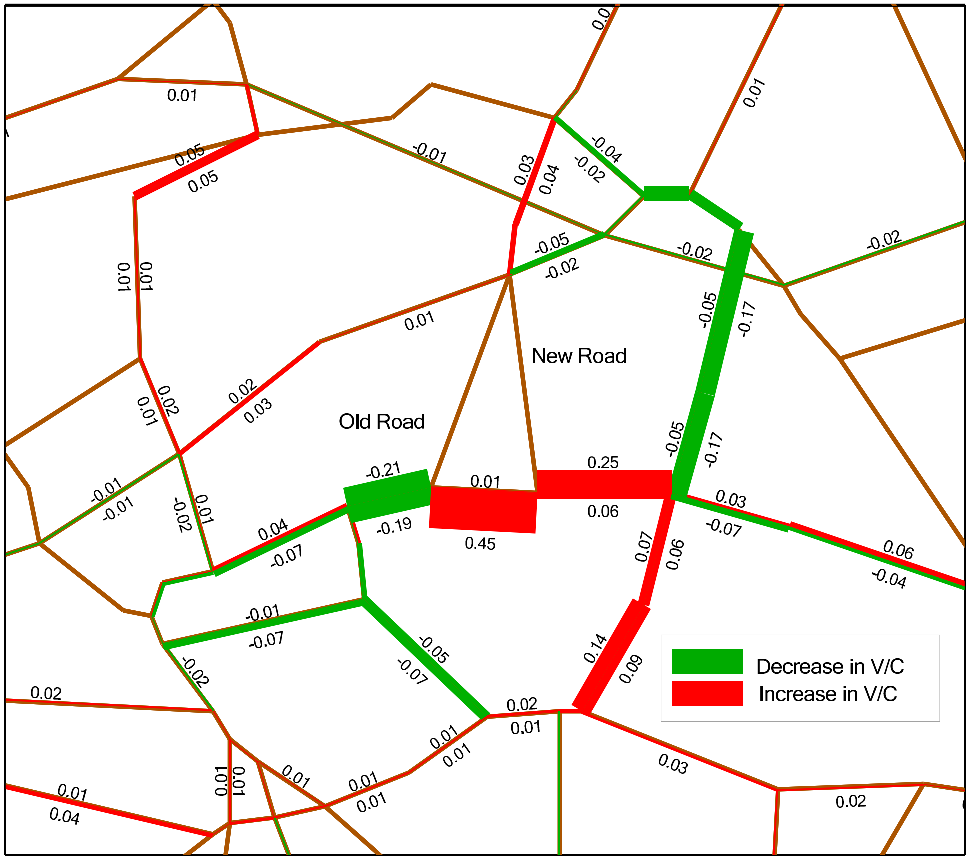


Figure 5.9: Changes in V/C Ratios with Project 18 Included in the 1989 Network

with and without project 76 in the network. The differences in the V/C ratios on the links in the network are shown in Figure 5.11.

If one adds an extra lane for 6.3 kilometres in each direction to the section of road to the south of the new interchange where the the new freeway joins the existing one, then the Braess' paradox is eliminated. Including project 76 in the network now results in a reduction of 8 vehicle-hours in the total travel time. Adding an extra lane in each direction for a further 4.8 kilometres improves the situation even more. The inclusion of the new road (project 76) would then result in a decrease of the total travel time of 19 vehicle-hours.

This widening of the road was actually scheduled for 1995 which explains why Table 5.8 shows that Braess' paradox for this project no longer occurs from 1995 onwards.

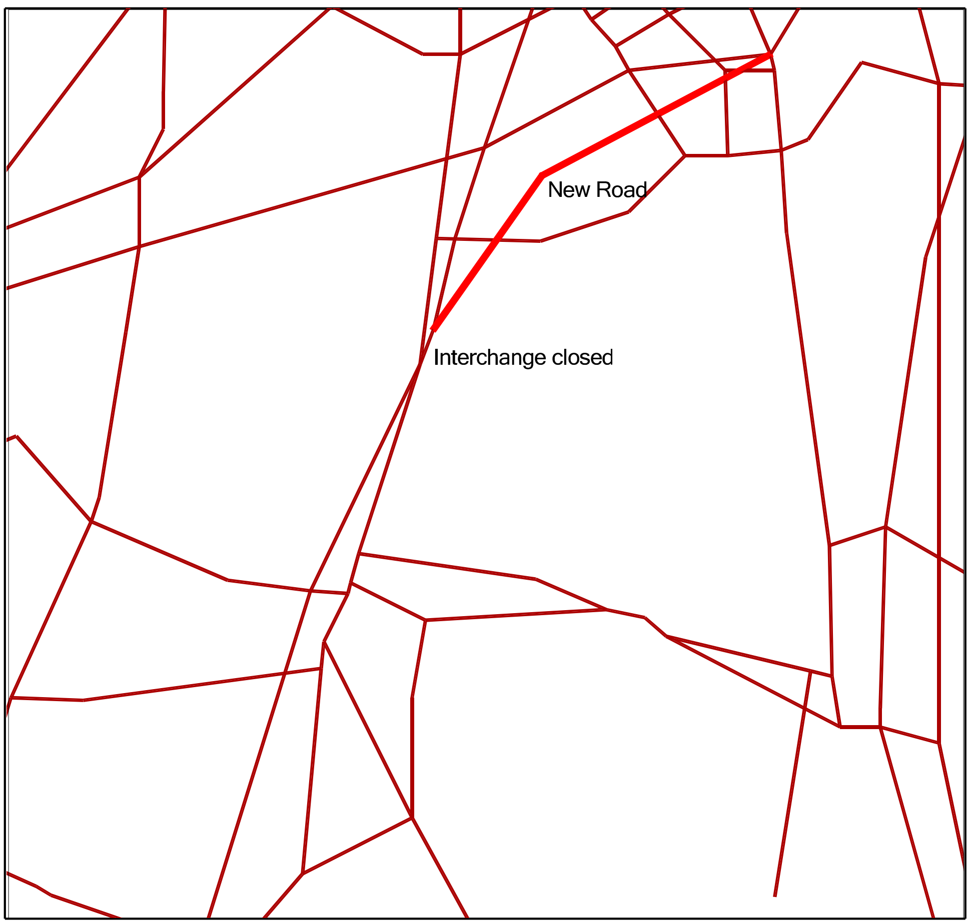


Figure 5.10: Changes in Network for Project 76 (1993)

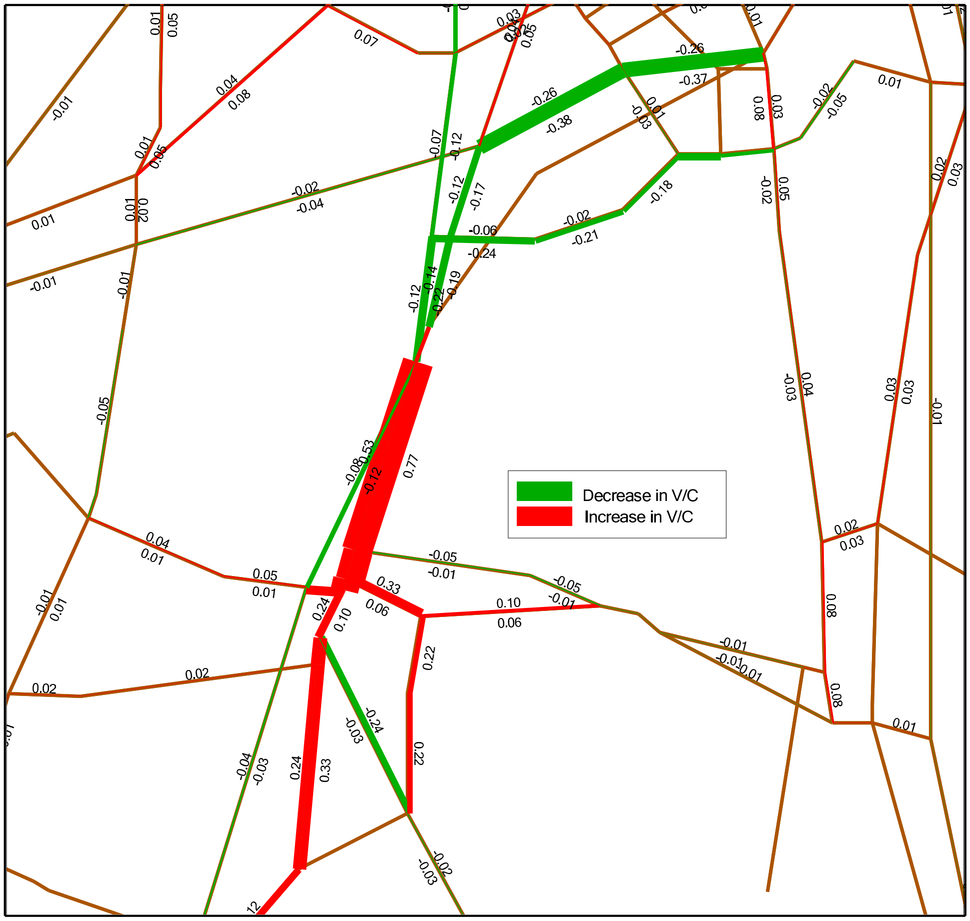


Figure 5.11: Changes in V/C Ratios with Project 76 Included in the 1993 Network

Figure 5.12 shows project 58 in the 1999 network. In this case the improvement to the network consisted of an additional lane in each direction of an existing road and was to be constructed in 1992 between the points A and B.

It is interesting to note that although project 58 was to be constructed in 1992, according to Table 5.8, Braess' paradox would only occur for this project in 1998 and 1999. This shows that whether Braess' paradox occurs or not is dependent on the entire network and the addition of further roads in the future may cause the paradox to occur when there was none initially.

The V/C ratios were again compared and the differences between the two sets of values, with and without the project, are shown in Figure 5.13. When the project is included in the analysis for 1999 it results in an increase in the total travel time of 2 vehicle-hours.

In this case the differences in V/C ratios shown in Figure 5.13 do not indicate where the problem lies as obviously as in the previous two examples. The size of the differences are smaller. However, if the two links immediately adjacent to the road in question at point A

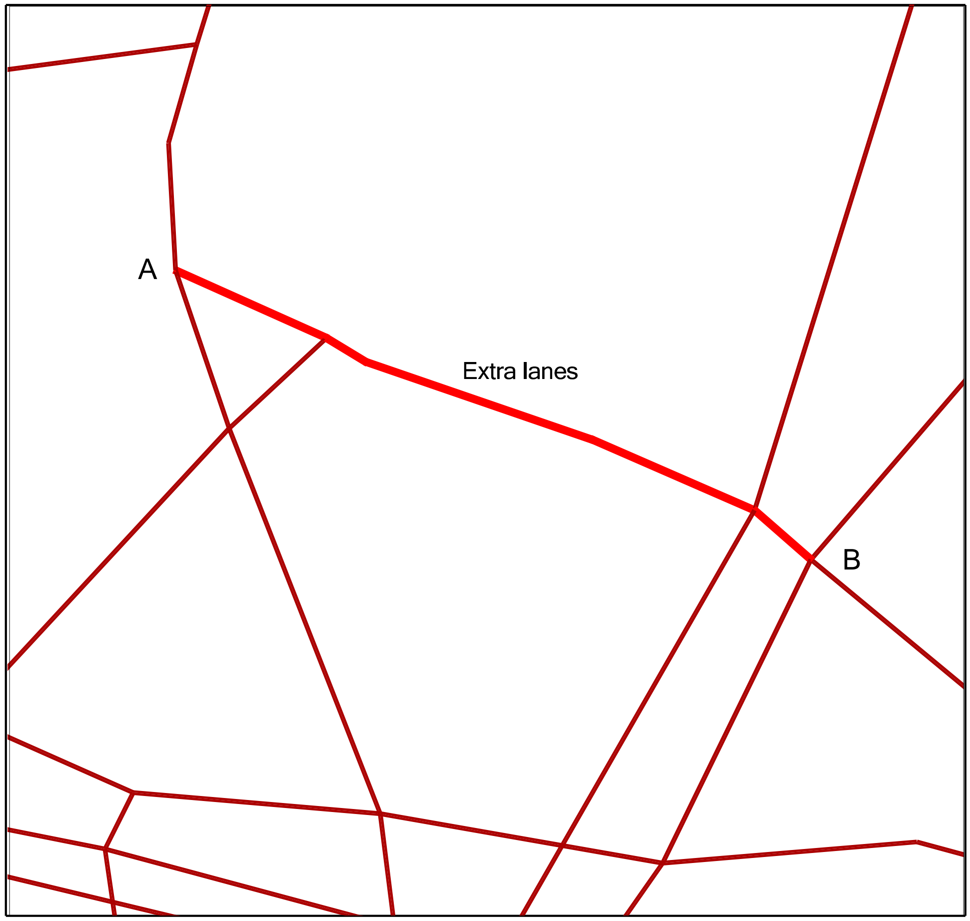


Figure 5.12: Changes in Network for Project 58 (1999)

have an additional lane added in each direction, then the total travel time will decrease by 4 vehicle-hours if project 58 is included.

However, additional lanes are not really needed on this section of road since it had also been widened in 1992. The increases in travel time caused by having the project included in the network are small, 1 vehicle-hour in 1998 and 2 vehicle-hours in 1999. These are more than offset by the reduction in the total travel times for the other years, 6 vehicle-hours in 1996, 5 vehicle-hours in 1997 and 12 vehicle-hours in 2000. Therefore it would probably be best to accept the slight increases in travel times in only two of the years.

This shows that in a changing network, improvements should not only be analysed for Braess' paradox in the year of construction but also in future years.

The examples presented show that in a number of cases it should be possible to eliminate Braess' paradox by adding a short section of the future network.

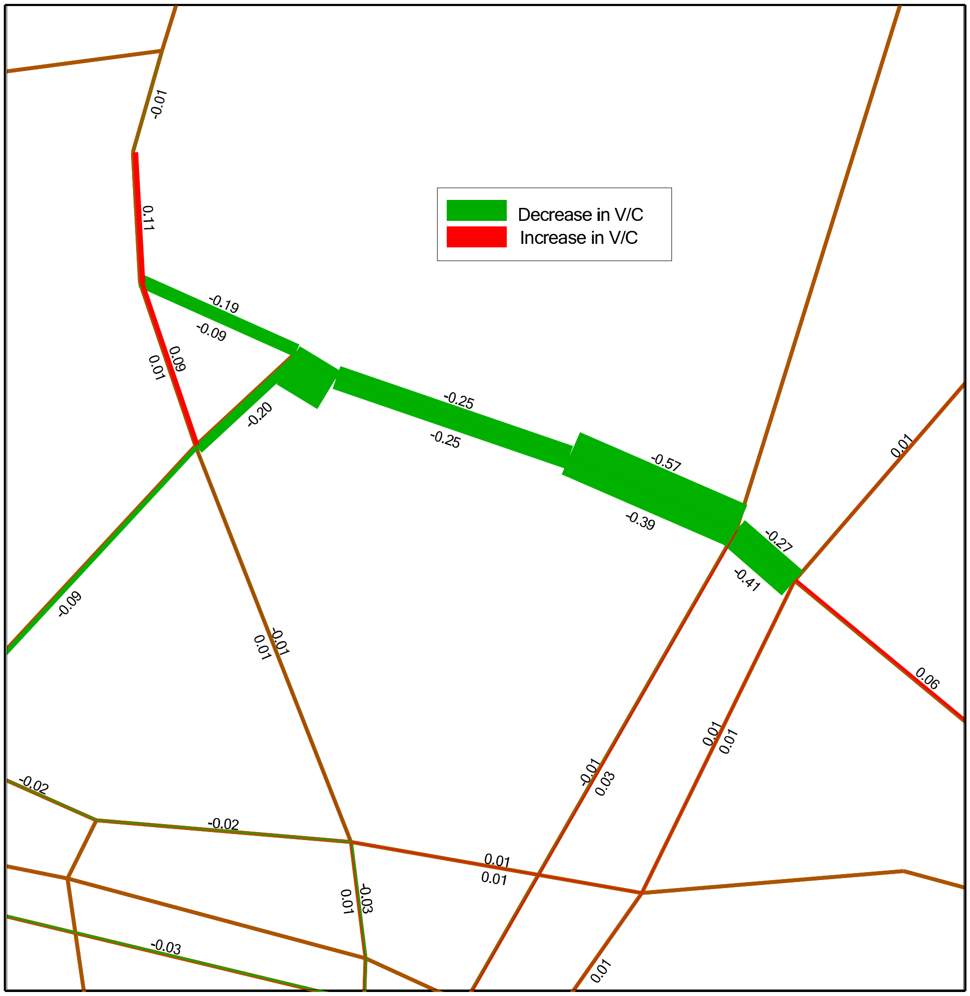


Figure 5.13: Changes in V/C Ratios with Project 58 Included in the 1999 Network

5.4 Removing Projects With Braess' Paradox

The previous section showed how it is often possible to eliminate Braess' paradox by adding capacity upstream or downstream of a project that causes the paradox to occur. This section develops a methodology to remove projects that result in Braess' paradox so as to minimize the total travel time on the network.

When removing projects that cause Braess' paradox from a network one is faced with the following problems:

- Removing one project causing Braess' paradox may result in other projects that showed Braess' paradox initially to no longer show the paradox. Therefore one should not remove all the projects showing Braess' paradox in a single step as this could be counter-productive.

- The removal of a project that results in Braess' paradox may result in other projects, that did not show the paradox initially, showing the paradox. Therefore one should not limit the testing of projects to only those that showed the paradox initially.

Therefore one should test each project that shows Braess' paradox individually and the "sub-network" that is created when removing Braess' paradox projects should also be tested. In this testing, the projects being tested should not be restricted to only those projects that showed the paradox initially.

The procedure for selecting the projets to be removed from the network is shown schematically in Figure 5.14. In this procedure, projects that cause Braess' paradox to occur are removed to form sub-networks which are are also tested and projects showing Braess' paradox are removed creating sub-sub-networks. The process is repeated until there are no further Braess' projects to be removed. The network that results in the minimum travel time is then the optimal network.

The structure of the problem shown in Figure 5.14 suggests that the "branch and bound" method of obtaining an optimal solution may be suitable for finding the optimal set of projects to remove from the network (or the optimal network) in order to minimize the total travel time on the network.

In the branch and bound method for a minimization problem such as we have here, an upper bound for the solution is established. The various branches are then followed and:

- If it can be established that following the branch will result in a solution greater than the existing upper bound the search down the particular branch is discontinued.
- If it appears that a solution less than the existing upper bound can be obtained the search is continued until either a new upper bound is found that is smaller than the existing one or the search is stopped when it is apparent that no such upper bound can be found.

The proposed methodology was developed using the results obtained from the 1985 PWV Update Study [47] which was described previously in this dissertation. In particular, the results obtained with a relative gap of 0.01 and shown in Table 5.8 are used. These results are repeated here as Table 5.18. Since Table 5.18 shows that no projects causing Braess' paradox were found in the 1995 and 1997 networks, these networks will not be considered here.

In order to try and understand how the networks reacted to projects being removed and to establish what the minimum travel times were, exhaustive searches were carried out for the remaining eleven networks. In this process all the projects were tested at each step. In addition, the projects that were included during previous years were also tested in any given year, e.g. in 1991 the projects for 1988, 1989, 1990 and 1990 were tested.

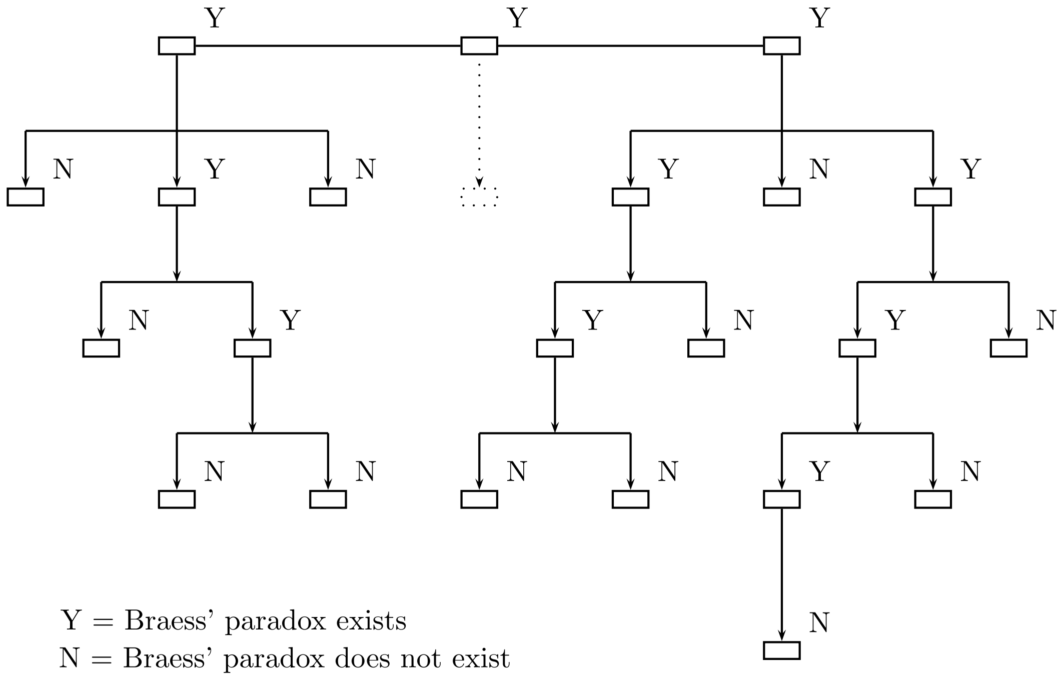


Figure 5.14: Removing Projects with Braess' Paradox to Obtain Minimum Travel Time

In the exhaustive search that was undertaken, all projects that showed Braess' paradox were removed one at a time and the remaining projects were then tested to see if any of them showed Braess' paradox. If any of the projects did show Braess' paradox, the process was repeated by removing the projects one at a time and repeating the process by testing all the remaining projects. This was repeated until no more projects showed Braess' paradox.

The results of the exhaustive search are shown in Table 5.19. Table 5.19 shows the projects that need to be removed in order to minimize the travel time, the amount by which the travel time is reduced and the number of assignments that have to carried out in order to ensure that a minimum travel time has been found.

The last column of Table 5.14 shows that in some cases a large number of assignments have to performed, e.g. for the 1993 network 1066 assignments were done. It would therefore be desirable to find ways to limit the number of assignments that are required.

The results of the exhaustive search were examined and the following observations were made:

- In some cases, projects that do not show Braess' paradox initially, do so after other projects have been removed from the network. An example of this is project 153 in 2000 which reduces the total travel time by one hour when it is in the network initially.

Table 5.18: Braess' Paradox in 1985 Update Study with Relative Gap = 0.01

Year	**No. of Projects**	**No. of Braess' Projects**	**Largest Braess' Paradox**	**Braess' Projects (size of paradox)**
1988	23	1	2	18(2)
1989	31	2	6	18(6); 20(1)
1990	44	4	1	20(1); 29(1); 34(1); 38(1)
1991	48	5	2	20(2); 28(1); 29(2); 34(1); 38(1)
1992	64	2	1	20(1); 29(1)
1993	77	5	40	27(2); 38(1); 52(1); 69(1); 76(40)
1994	91	2	35	20(1); 76(35)
1995	107	0	-	
1996	123	1	1	73(1)
1997	148	0	-	
1998	158	1	1	58(1)
1999	165	1	2	58(2)
2000	186	3	1	27(1); 36(1); 128(1)

However, after project 36 has been removed from the network because including it results in Braess' paradox, project 153 now also shows Braess' paradox and removing it will reduce the travel time by a further one hour.

- Therefore, based on the above point one cannot limit the search for potential projects to remove from the network to only those that showed Braess' paradox initially.

- Some projects show Braess' paradox initially, but after other projects have been removed from the network no longer do so. Examples of this can be found in the 1991 network where projects 29 and 34 were removed from the network so as to eliminate all projects showing Braess' paradox. Initially removing project 28 from the network would reduce the travel time by one hour. However, after projects 29 and 34 have been removed, removing project 28 would increase the travel time by four hours.

- It would therefore be incorrect to remove all the projects that showed Braess' paradox initially since this could have the effect of increasing the travel time on the network.

- The inclusion of some projects in the networks resulted in relatively large reductions in travel times. Although the size of these reductions might decrease when projects with Braess' paradox were removed from the network the reductions were nowhere near large enough to result in these projects showing Braess' paradox.

Based on the results described above it was obvious that while it would be necessary to test all projects for Braess' paradox initially, it would be possible to eliminate a number of them from further consideration. An inspection of the results obtained revealed that the total reduction in travel times that was obtained by removing certain projects from the network was never larger than:

Table 5.19: Obtaining the Optimal Network Through Exhaustive Search

Year	**No. of Projects**	**Braess' Projects (size of paradox)**	**Projects Removed**	**Reduction in Travel Time**	**No. of Assignments**
1988	23	18(2)	18, 20	3	69
1989	31	18(6); 20(1)	18, 20	7	124
1990	44	20(1); 29(1); 34(1); 38(1)	20, 38	2	264
1991	48	20(2); 28(1); 29(2); 34(1); 38(1)	29, 34	3	383
1992	64	20(1); 29(1)	20 or 29	1	193
1993	77	27(2); 38(1); 52(1); 69(1); 76(40)	20, 27, 76	41	1066
1994	91	20(1); 76(35)	76	35	428
1996	123	73(1)	73	1	247
1998	158	58(1)	58	1	317
1999	165	58(2)	58	2	341
2000	186	27(1); 36(1); 128(1)	20, 36 or 36, 89 or 36, 104 or 36, 153 or 36, 175	2	1697

$$\sum B + 2$$

Where B are the amounts by which the travel times decrease when a project showing Braess' paradox is removed from the network (the "size" of the paradox).

Projects which when tested initially reduced the travel time by more than this amount never showed Braess' paradox when other projects were removed from the network and could therefor be excluded from consideration once the initial tests had been completed. That is all projects where:

$$\text{Reduction in travel time} > \sum B + 2$$

are not considered after the initial analysis.

The results that were obtained for the exhaustive searches were then re-analysed and all projects where the reductions in travel times was greater than $\sum B + 2$ were excluded from further analysis. This resulted in a reduced number of assignment being required although the same projects were removed from the networks and the same reductions in travel times were obtained. The results of this analysis are shown in Table 5.20.

Table 5.20 shows that using the reduced search technique results in substantially less assignments being required, of the order of 50 per cent.

One other approach to removing projects showing Braess' paradox was tested in order to see what the effect on the final result and the number of assignments was. This was to remove the project with the largest paradox at each step and to ignore all other Braess' paradox projects at that level. If two or more projects had paradoxes that were equally large, all were tested. The results of this analysis are shown in Table 5.21.

The results shown in Table 5.21 show that following this "largest" strategy can result in a further reduction in the number of assignments required. In only one case was the reduction in the travel time less than what was obtained previously and then only by one hour. It appears that this “largest” strategy could result in a reduction in the number of assignments required with only a chance that the result may be only a local minimum but close to the absolute minimum. However, it should be remembered that this test was carried out on a limited number of networks and further testing with other networks should be carried out.

Table 5.20: Obtaining the Optimal Network Through Reduced Search

Year	**Projects Removed**	**Reduction in Travel Time**	**No. of Assignments Exhaustive Search**	**No. of Assignments Reduced Search**
1988	18, 20	3	69	33
1989	18, 20	7	124	66
1990	20, 38	2	264	116
1991	29, 34	3	383	162
1992	20 or 29	1	193	87
1993	20, 27, 76	41	1066	610
1994	76	35	428	294
1996	73	1	247	142
1998	58	1	317	181
1999	58	2	341	188
2000	20, 36 or 36, 89 or 36, 104 or 36, 153 or 36, 175	2	1697	400

Table 5.21: Obtaining the Optimal Network By Eliminating Largest Paradox at Each Step

Year	**Projects Removed**	**Reduction in Travel Time**	**No. of Assignments Exhaustive Search**	**No. of Assignments Removing Largest**
1988	18, 20	3	69	33*
1989	18, 20	7	124	53
1990	20, 38	2	264	71
1991	29, 34	3	383	96
1992	20 or 29	1	193	87*
1993	76	40^{+}	1066	134
1994	76	35	428	171
1996	73	1	247	142*
1998	58	1	317	181*
1999	58	2	341	188*
2000	20, 36 or 36, 89 or 36, 104 or 36, 153 or 36, 175	2	1697	400*

* - the same result that was obtained using the reduced search

$^{+}$ - the reduction in the travel time is one less than before

Chapter 6

CONCLUSIONS

The work described in this dissertation makes it possible to draw a number of interesting conclusions which are listed below.

- The construction of a network that used the BPR function to describe the delay functions on the links of the network enabled one to examine the level of congestion at which Braess' paradox occurs. It was shown that the paradox occurred over a range of volumes for the demand. It was shown that the paradox occurred at relatively low levels of congestion, approximately level of service C which designates a very acceptable operating condition. The BPR function is the most commonly function used to define the delay occurring on the links of a real-world network.

- It is important to use a stringent stopping criterion when attempting to determine whether Braess' paradox occurs or not. it is recommended that a stopping criterion that will result in an assignment as close as possible to the true equilibrium assignment. It is recommended that a relative gap of 0.01 be used as the stopping criterion. The stopping criterion used should be an objective measurement of the the degree of convergence to the true equilibrium condition and not the number of iterations of the assignment algorithm carried out.

  If less stringent stopping criteria are used it will appear that Braess' paradox is more prevalent than it actually is.

- When it occurs in real-world networks, Braess' paradox also tends to occur over a range of volumes at lower volumes. It is less likely to occur at higher volumes and levels of congestion.

- In real-world networks, whether Braess' paradox occurs or not is dependent on what other links may be added to the network. It is possible that a link may be added in one year and not show the paradox. However, after other links are added in succeeding years Braess' paradox may occur on the link which did not show it previously. The paradox for this link may then disappear in later years when further links are added to the network.

- It was shown that using the Braess' network that had BPR functions for the delay on the links that Braess' paradox could be eliminated by increasing the capacity on the upstream and/or downstream links from the added link. It was also shown how this methodology could be extended to real-world networks where Braess' paradox occurs. This can be done by comparing the volume/capacity (V/C) ratios of the "before" and "after" networks and then increasing the capacity on links in the "after" network where there is an increase in the V/C ratio.

- Where it may not be possible to increase the capacity of adjacent links a methodology was presented to remove links that show Braess' paradox using a limited number of assignments. It was found that in the majority of cases removing the links with the highest value of Braess' paradox in a step-wise manner resulted in the same links being removed as was the case where the search for the links was done in an exhaustive fashion. In those cases where removing the links with the highest values for Braess' paradox did not produce the same results as the exhaustive search, the reduction in the travel time was very close to that obtained with the exhaustive search.

It should be remembered that the conclusions above from real-world networks were based on a limited number of networks that were related to one another. Further tests should be carried out using different networks with different delay functions.

APPENDIX

Table 1: Difference in travel times without specific projects for 1989 with multiples of the trip matrix

	Factors												
Projects	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**
1	1	3	4	3	8	8	12	15	22	25	35	48	63
2	17	22	27	30	36	34	40	41	41	44	60	78	94
3	0	3	2	2	7	7	13	17	16	11	21	31	34
4	-1	0	2	1	2	4	6	6	6	6	11	16	17
5	0	2	3	7	12	15	20	27	35	40	45	62	73
6	18	28	37	40	48	65	74	86	101	127	173	230	284
7	10	19	27	34	47	57	71	90	97	111	150	172	236
8	0	3	5	1	2	7	8	16	29	37	56	81	100
9	14	25	28	35	49	61	80	103	137	177	204	237	318
10	0	3	1	0	2	9	14	19	30	43	66	82	110
11	2	4	5	4	6	8	10	12	19	30	49	55	93
12	9	13	14	17	22	26	31	38	44	49	61	73	82
13	0	1	4	6	11	15	25	33	42	54	77	95	109
14	14	23	29	33	44	48	58	79	84	96	119	136	162
15	5	12	21	30	48	60	74	92	110	151	194	224	274
16	0	0	0	0	1	3	5	6	9	11	13	14	17
17	15	26	42	70	115	188	291	443	653	936	1328	1809	2440
18	-1	0	-3	-4	-3	-6	-6	-7	-9	-8	-13	-15	-12
19	4	6	7	6	10	12	16	22	26	37	46	45	56
20	-1	0	0	0	0	-1	0	-1	0	-1	0	0	-2
21	0	3	5	9	16	25	37	44	55	71	92	118	163
22	4	4	6	12	19	24	38	45	55	66	82	100	121
23	0	0	-1	0	0	2	3	5	8	11	16	23	29
24	19	26	45	71	103	146	194	238	328	423	539	715	923
25	11	12	22	22	21	12	9	37	46	34	35	47	27
26	4	4	8	9	11	19	29	37	46	61	68	112	132
27	0	0	0	0	0	0	1	-1	1	0	0	0	0
28	0	-1	0	1	0	1	0	1	1	2	1	4	2
29	1	0	2	2	1	3	2	1	1	-2	1	2	5
30	17	25	40	61	76	100	119	168	234	353	506	723	953
31	10	15	18	25	30	40	52	64	80	110	141	202	248

Table 2: Difference in travel times without specific projects for 1989 with multiples of the trip matrix - continued

Projects	Factors												
	1.8	**1.9**	**2.0**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3.0**
1	72	71	108	139	172	183	237	300	349	451	543	644	740
2	119	151	221	270	301	397	504	584	677	760	885	1070	1218
3	47	47	89	110	138	175	214	258	304	369	433	532	591
4	22	41	72	74	85	116	143	161	211	244	297	349	403
5	102	123	145	182	221	264	322	390	439	511	625	780	907
6	288	332	411	474	547	681	786	910	1112	1251	1414	1668	1920
7	257	283	346	400	457	557	657	750	904	1022	1189	1429	1680
8	111	164	202	238	282	393	466	548	656	686	804	1020	1229
9	429	495	619	708	823	1019	1206	1469	1663	1903	2319	2736	3135
10	153	168	203	240	295	389	450	527	598	693	866	986	1120
11	127	156	219	263	329	444	558	640	774	896	1054	1275	1504
12	80	87	101	106	131	161	187	237	284	330	389	489	536
13	143	185	218	271	333	425	512	621	699	787	948	1152	1316
14	173	203	233	259	307	353	438	520	594	650	745	811	962
15	355	395	494	563	622	744	935	1068	1175	1414	1711	2092	2431
16	27	29	45	52	64	74	71	106	113	136	166	193	225
17	3236	4253	5454	6979	8781	10977	13560	16621	20218	24381	29215	34851	41246
18	-32	-36	-12	-21	-23	-29	-57	-75	-78	-110	-156	-140	-192
19	66	86	119	139	173	219	259	334	395	453	494	624	698
20	-1	-2	0	0	-4	-3	-2	-1	-6	-4	-2	-13	-4
21	202	223	280	324	366	471	602	658	775	897	1077	1270	1531
22	134	130	159	218	254	311	386	471	585	689	786	963	1097
23	38	53	77	100	110	167	235	285	335	427	532	632	718
24	1156	1435	1753	2135	2635	3211	3923	4694	5659	6804	8158	9662	11365
25	52	77	87	89	131	159	193	209	270	320	366	511	565
26	204	209	246	310	346	388	444	520	585	687	783	930	1113
27	1	-1	0	1	1	0	1	1	2	4	5	9	7
28	9	11	16	25	25	32	54	76	92	91	107	120	143
29	8	7	11	11	18	12	16	27	27	47	65	61	79
30	1300	1731	2242	2929	3727	4722	5896	7251	8863	10755	13095	15702	18644
31	312	400	501	606	773	987	1184	1383	1650	1960	2341	2734	3239

Table 3: Difference in travel times without specific projects for 1993 with multiples of the trip matrix

	Factors												
Projects	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**
1	3	6	10	11	18	24	29	32	48	58	84	109	141
2	23	30	43	50	58	57	58	65	69	87	110	142	194
3	0	1	4	6	8	14	25	23	22	25	39	63	87
4	1	-1	3	4	4	9	14	19	23	28	34	35	29
5	3	5	12	17	23	35	47	67	92	121	151	203	260
6	16	23	30	38	42	48	66	87	112	108	135	208	235
7	12	18	30	41	50	69	98	113	118	148	170	224	270
8	0	4	5	3	3	7	15	28	43	60	78	106	142
9	18	25	33	40	54	74	93	125	180	220	265	349	400
10	-1	2	1	2	6	15	21	33	53	90	120	162	163
11	0	3	6	5	6	8	7	11	16	21	33	53	64
12	7	10	17	17	23	29	34	39	47	56	59	67	83
13	-1	1	6	7	12	21	29	38	54	79	87	112	157
14	16	22	33	41	51	69	82	94	112	136	157	187	207
15	9	21	35	50	69	94	120	147	189	245	271	368	424
16	2	2	6	9	15	24	32	42	49	67	87	108	153
17	20	39	69	122	209	344	544	833	1238	1784	2513	3449	4654
18	-2	-1	2	4	4	10	11	11	22	21	20	28	29
19	6	9	11	14	20	25	30	43	63	77	84	110	137
20	0	-1	3	-2	-1	0	-2	0	0	-1	0	0	1
21	-1	4	8	12	22	35	45	56	72	101	137	188	227
22	2	4	13	17	30	44	54	67	93	109	125	163	196
23	1	1	2	2	2	4	8	14	22	29	44	65	96
24	27	53	88	133	194	258	365	495	663	874	1149	1510	1900
25	10	24	24	16	14	16	18	28	45	33	26	42	39
26	4	12	18	24	34	40	52	71	90	119	152	178	197
27	0	0	1	0	0	-2	0	0	0	0	-1	0	0
28	9	14	18	20	21	26	28	34	39	49	53	71	93
29	0	1	3	3	3	4	4	16	15	9	8	7	5
30	20	34	56	77	90	119	178	289	454	666	929	1320	1802
31	16	20	26	36	48	69	91	134	179	232	308	392	497
33	1	6	6	13	16	14	17	28	43	51	70	97	137
34	0	0	1	0	0	0	1	2	5	9	11	16	17
35	3	8	9	15	21	42	59	82	121	157	207	250	302
36	0	0	0	0	0	0	-1	0	1	-1	-1	0	0
37	3	11	16	13	14	20	26	38	35	42	57	64	81
38	0	-2	-1	-1	-1	-1	-1	0	1	1	1	2	5
39	26	34	40	44	52	64	75	96	105	130	170	192	249

Table 4: Difference in travel times without specific projects for 1993 with multiples of the trip matrix - continued

	Factors												
Projects	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.7**
40	2	12	23	26	40	54	76	108	132	152	180	228	262
41	9	14	21	24	33	47	63	80	120	174	230	333	448
42	10	14	20	29	36	45	60	73	101	102	122	168	221
43	-1	2	6	8	15	20	24	37	61	72	96	134	182
44	13	19	27	42	51	71	81	104	125	125	151	210	226
45	-1	2	9	5	9	20	29	51	85	114	168	241	290
46	-1	2	4	7	10	15	17	28	38	49	64	76	88
47	3	9	19	23	40	63	94	134	164	189	212	256	276
48	7	12	15	15	18	24	32	39	50	59	73	99	108
49	1	1	1	2	2	3	4	10	7	12	23	32	41
50	0	8	12	14	25	32	34	47	65	80	115	161	196
51	1	4	4	4	5	11	14	19	30	32	36	43	60
52	0	0	0	0	-1	-1	0	-1	0	-1	-1	1	-1
53	1	1	2	2	5	8	10	20	31	39	55	74	109
54	4	9	9	13	16	23	22	24	32	36	53	79	101
55	3	1	5	1	3	3	6	14	13	18	24	41	47
56	9	13	17	18	23	27	29	32	38	48	50	65	87
57	3	8	9	6	4	11	11	12	14	19	26	33	37
58	-3	2	0	0	-1	0	0	-1	3	2	-2	1	1
59	2	5	8	10	16	27	31	40	58	77	93	120	136
60	3	6	9	11	15	21	29	52	77	78	97	124	129
61	0	7	14	19	21	31	43	53	65	87	104	122	152
62	9	17	270	36	40	46	56	64	83	114	160	208	247
63	18	31	42	44	65	120	156	208	287	336	448	565	674
64	1	3	4	7	12	19	27	33	50	62	82	92	127
65	58	74	105	114	145	185	240	284	361	451	572	715	893
66	9	13	17	21	23	26	33	38	49	60	64	75	93
67	-2	2	4	6	5	11	15	19	24	30	37	47	59
68	2	2	3	2	1	2	1	2	3	3	3	5	4
69	0	0	-1	0	0	-1	0	0	2	1	5	5	11
70	1	-1	3	0	1	4	6	8	11	19	21	30	37
71	1	-1	1	-1	1	3	4	6	11	12	20	29	43
72	8	13	21	23	33	46	65	65	89	115	148	159	223
73	-2	-1	0	-2	-1	2	7	17	20	35	50	102	126
74	0	0	0	1	0	1	8	16	26	42	53	68	91
75	-1	2	4	8	12	12	18	25	30	39	58	71	86
76	-9	-12	-16	-21	-28	-40	-41	-54	-57	-80	-93	-87	-91
77	4	9	19	29	29	50	83	112	147	195	273	311	392
78	17	28	38	59	83	122	157	202	246	288	328	406	452

Table 5: Difference in travel times without specific projects for 1993 with multiples of the trip matrix - continued

	Factors												
Projects	**1.8**	**1.9**	**2.0**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3.0**
1	171	225	256	266	377	412	513	589	699	879	1020	1150	1298
2	265	283	366	441	526	605	771	968	1167	1466	1756	2077	2479
3	111	120	156	208	250	302	404	481	591	747	903	1078	1315
4	73	75	96	111	121	114	146	176	198	273	307	340	440
5	333	409	542	639	775	946	1134	1322	1602	1920	2274	2681	3140
6	266	280	336	392	445	491	591	734	883	1070	1289	1551	1877
7	337	428	523	588	689	800	978	1215	1453	1660	2011	2341	2866
8	177	227	293	365	459	566	658	816	1025	1185	1412	1612	1921
9	509	669	840	929	1161	1383	1710	2118	2513	2980	3583	4206	4883
10	207	294	350	404	568	619	809	961	1165	1364	1722	2025	2377
11	85	105	120	155	217	258	352	418	536	600	656	790	940
12	97	106	128	156	177	190	225	282	340	383	469	526	635
13	197	242	305	370	456	514	599	772	870	1093	1272	1476	1674
14	261	292	362	409	491	517	591	711	824	935	1116	1291	1513
15	514	618	759	875	1090	1279	1544	1893	2273	2725	3170	3768	4388
16	185	263	352	429	548	672	809	997	1212	1461	1769	2099	2500
17	6175	8097	10473	13341	16821	20984	25953	31858	38811	46806	56094	66807	79167
18	44	47	81	72	88	77	149	176	209	256	309	346	433
19	180	197	239	313	394	436	536	688	868	1030	1227	1409	1741
20	2	-1	1	0	2	1	4	2	9	7	9	12	13
21	299	360	484	569	690	822	1017	1230	1409	1672	2038	2514	2951
22	251	287	367	465	560	619	776	962	1074	1324	1553	1808	2158
23	130	169	251	302	420	495	644	811	969	1196	1438	1719	2002
24	2400	3074	3920	4856	6081	7444	9178	11228	13614	16386	19609	23302	27538
25	83	103	118	176	171	138	229	289	345	477	619	680	809
26	263	299	342	337	336	383	367	398	425	486	483	540	643
27	0	0	3	2	2	3	5	9	8	14	10	19	23
28	110	125	153	180	226	250	290	335	430	526	634	732	849
29	9	12	12	4	7	9	13	14	24	20	21	25	55
30	2455	3253	4270	5507	7027	8781	10957	13555	16522	20028	24145	28803	34226
31	651	838	1073	1345	1668	2038	2461	2935	3554	4295	5007	5860	6768
33	163	180	220	233	283	346	399	522	650	785	964	1178	1353
34	20	31	36	46	64	83	99	145	174	197	253	295	347
35	384	481	634	758	929	1104	1402	1712	2076	2471	2903	3435	4064
36	0	-1	2	4	4	4	5	12	13	19	21	18	22
37	107	126	182	243	323	365	470	582	709	823	1021	1265	1497
38	6	4	10	8	11	8	14	17	23	22	17	26	27
39	302	353	414	519	631	728	901	1074	1193	1420	1613	1926	2364

Table 6: Difference in travel times without specific projects for 1993 with multiples of the trip matrix - continued

	Factors												
Projects	**1.8**	**1.9**	**2.0**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3.0**
40	313	366	452	575	716	865	1029	1256	1423	1637	1963	2295	2701
41	598	788	1041	1257	1559	1860	2237	2699	3233	3860	4553	5447	6358
42	270	342	439	524	656	796	985	1208	1422	1774	2144	2489	3015
43	197	226	260	276	346	369	466	592	643	775	958	1069	1225
44	264	281	305	379	497	415	496	537	629	686	920	1165	1431
45	383	481	597	747	935	1157	1440	1679	2058	2534	3039	3525	4106
46	119	156	194	225	243	270	315	366	436	572	694	823	1013
47	342	382	447	578	683	755	837	966	1072	1264	1572	1856	2211
48	119	142	161	182	226	287	338	381	474	560	652	766	922
49	60	85	110	122	132	186	227	263	297	353	419	477	525
50	235	319	396	415	568	660	810	1000	1216	1559	1870	2177	2523
51	77	98	126	143	193	246	322	373	499	539	626	744	891
52	0	0	1	2	2	0	-1	3	1	2	4	3	8
53	140	177	238	279	325	414	492	574	709	831	993	1160	1338
54	135	154	208	263	299	418	517	651	793	903	1031	1169	1379
55	60	66	80	106	134	184	229	315	335	430	506	577	673
56	98	110	127	140	177	194	222	247	314	378	463	535	626
57	50	62	85	77	89	86	92	93	98	128	135	153	161
58	8	7	17	26	35	36	36	64	65	100	139	145	221
59	181	193	249	317	408	492	589	754	892	1096	1335	1628	1939
60	125	172	228	274	313	340	397	455	559	631	775	883	1032
61	182	222	230	236	243	301	376	424	458	547	584	678	818
62	293	334	444	563	692	800	998	1171	1473	1770	2156	2540	3008
63	881	1102	1378	1665	2027	2365	2820	3466	4081	4832	5768	6835	8101
64	138	171	264	293	404	468	639	784	938	1107	1300	1560	1794
65	1091	1360	1670	2001	2443	2990	3561	4345	5125	6195	7396	8731	10221
66	114	142	167	210	288	375	443	534	644	752	890	1071	1291
67	86	103	124	164	195	279	313	385	454	590	751	928	1215
68	6	5	7	7	8	3	5	11	11	9	11	12	11
69	15	16	29	38	43	64	72	82	109	125	148	183	217
70	51	64	79	100	125	139	176	229	274	338	424	506	595
71	78	112	159	243	369	480	589	721	891	1041	1448	1721	2098
72	270	319	419	504	562	671	872	993	1141	1366	1576	1835	2170
73	164	206	305	361	448	576	712	904	1054	1341	1629	1916	2347
74	126	146	185	236	284	313	387	485	600	724	885	1048	1262
75	102	118	153	195	220	251	316	418	522	607	729	868	1024
76	-96	-114	-89	-86	-66	-125	-81	-107	-224	-322	-329	-357	-436
77	502	623	769	903	1092	1361	1636	1963	2333	2859	3367	3977	4727
78	548	549	598	703	893	979	1090	1043	1436	1624	2128	2552	3031

Table 7: Difference in travel times without specific projects for 1999 with multiples of the trip matrix

	Factors										
Projects	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**
1	6	5	14	19	28	34	44	69	88	81	127
2	14	19	37	52	67	67	61	90	122	157	172
3	2	-3	3	9	11	21	25	26	61	77	80
4	0	-1	2	5	6	10	14	14	22	22	26
5	7	13	24	32	43	59	84	108	143	177	241
6	22	24	35	35	46	67	87	72	89	133	189
7	20	29	45	57	87	110	121	131	196	217	272
8	5	-2	6	5	17	31	45	74	93	123	171
9	20	26	40	52	70	86	123	163	214	266	351
10	0	-2	1	1	16	24	39	66	91	130	183
11	4	3	5	6	7	9	11	21	32	40	57
12	11	12	22	30	37	39	43	52	72	85	105
13	0	-1	6	11	12	18	25	37	57	64	82
14	20	27	41	56	67	78	94	100	147	154	174
15	39	61	91	125	160	207	282	375	501	636	842
16	21	36	50	73	89	105	137	164	194	261	328
17	42	84	174	325	574	958	1535	2355	3521	5072	7131
18	0	-3	3	4	6	10	15	10	10	13	8
19	10	11	18	21	27	40	46	56	91	128	185
20	-1	-4	-2	-2	-1	1	-1	1	0	0	1
21	4	2	11	16	19	26	41	63	98	150	216
22	7	8	11	23	30	42	60	84	106	128	151
23	0	-2	5	13	22	41	63	98	152	225	323
24	88	148	245	351	489	684	974	1362	1950	2670	3649
25	27	24	27	30	41	61	53	71	92	123	144
26	25	38	62	87	112	148	173	197	223	273	325
27	0	0	0	0	0	0	0	0	0	0	1
28	13	12	18	20	26	33	37	45	51	70	83
29	4	1	7	6	9	13	26	38	41	40	30
30	35	58	82	125	215	407	690	1093	1673	2470	3579
31	28	39	57	85	140	147	195	233	322	436	582
33	5	7	17	19	18	34	62	84	120	151	223
34	0	-3	-1	-1	1	2	3	5	10	13	15
35	6	3	16	31	52	80	107	128	189	239	337
36	0	0	0	-1	0	0	-1	0	0	0	0
37	13	8	16	28	33	42	56	71	85	106	138
38	0	-2	0	-1	-1	1	0	3	2	3	7
39	30	36	45	52	65	77	95	119	138	147	157
40	29	45	68	107	147	185	242	273	333	391	495
41	21	30	45	65	98	133	204	300	422	588	797
42	15	16	28	36	44	69	84	90	129	178	224
43	0	2	8	11	20	31	47	69	98	130	178

Table 8: Difference in travel times without specific projects for 1999 with multiples of the trip matrix - continued

	Factors										
Projects	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**
44	28	36	55	58	72	74	72	93	101	126	139
45	11	16	31	63	116	157	231	271	395	553	766
46	7	14	25	39	60	94	131	183	247	327	414
47	18	34	66	104	130	158	200	242	259	269	343
48	9	12	15	17	28	39	47	59	73	91	115
49	1	-1	6	13	22	26	28	33	36	45	61
50	16	27	51	81	123	166	217	275	371	477	628
51	4	3	6	16	22	27	33	42	59	67	103
52	0	0	-1	-2	1	1	0	1	1	0	3
53	1	2	6	6	13	18	27	34	45	47	68
54	5	8	10	13	11	12	17	26	38	47	68
55	1	-3	2	2	7	15	8	17	27	40	58
56	13	12	18	20	25	34	37	39	48	680	79
57	10	10	17	24	37	54	74	82	97	117	136
58	-2	-3	1	5	4	-2	0	1	14	25	31
59	7	7	26	36	64	61	86	78	121	163	198
60	11	11	19	31	48	85	123	185	278	389	475
61	0	-2	2	3	12	17	25	26	27	24	35
62	30	28	44	54	75	98	140	176	216	280	381
63	44	54	83	134	178	231	308	389	505	656	827
64	2	3	12	19	27	40	42	39	45	56	86
65	95	142	207	268	335	455	631	857	1172	1638	2186
66	19	22	28	35	44	61	77	103	118	153	192
67	11	9	20	34	45	71	90	120	167	231	307
68	2	-1	2	2	3	4	4	5	6	7	8
69	0	-3	0	-1	1	1	1	1	3	2	5
70	3	-3	0	1	4	7	11	11	18	18	37
71	4	-2	-2	0	4	6	8	17	23	50	81
72	15	12	24	33	46	59	75	104	131	173	208
73	11	10	20	16	21	31	51	72	90	144	208
74	3	-1	0	10	19	27	53	73	96	124	156
75	5	3	3	5	9	13	22	32	36	39	46
76	-1	-3	8	14	22	21	47	40	77	102	152
77	4	3	16	23	37	54	80	127	161	208	253
78	38	55	82	96	124	138	143	165	194	229	260
79	1	-1	2	2	3	7	9	14	17	22	45
80	3	2	10	12	17	26	44	64	91	123	164
81	19	18	30	37	49	61	78	99	137	192	295
82	20	21	36	46	56	73	106	144	173	232	299
83	0	-3	0	0	1	1	1	2	0	1	3
84	3	-1	0	2	2	4	4	6	4	8	10

Table 9: Difference in travel times without specific projects for 1999 with multiples of the trip matrix - continued

	Factors										
Projects	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**
85	15	13	25	34	47	54	77	114	121	160	168
86	6	5	10	10	13	18	21	22	30	43	74
87	35	53	82	121	164	188	243	288	354	444	579
88	1	-1	1	0	1	0	0	4	11	14	9
89	1	-4	-2	0	1	1	0	3	9	10	10
90	2	1	7	12	21	25	43	51	72	102	131
91	3	-1	4	8	18	23	39	46	57	79	108
92	22	32	42	48	55	65	76	83	103	122	141
93	0	0	0	-2	0	0	4	10	16	24	39
94	6	4	19	32	44	66	98	139	187	234	328
95	26	39	57	73	102	132	183	229	296	367	467
96	11	12	21	27	37	49	72	96	125	155	200
97	0	0	1	2	2	1	4	6	2	5	6
98	7	3	8	10	16	23	27	33	47	62	67
99	9	11	20	31	46	61	85	112	150	202	273
100	14	11	18	17	19	21	28	42	53	73	84
101	8	7	17	24	37	42	64	96	102	134	139
102	3	-1	7	9	10	8	17	21	13	15	22
103	0	0	1	0	4	6	8	12	15	24	29
104	0	0	-1	1	1	0	0	0	0	0	1
105	10	10	16	25	42	64	85	119	149	206	255
106	3	0	12	17	29	36	60	58	78	99	126
107	55	64	94	125	168	192	254	314	363	441	610
108	93	131	192	274	366	489	680	939	1266	1778	2398
109	1	-1	3	4	8	10	18	30	36	50	63
110	7	10	17	17	28	39	55	75	102	115	150
111	2	-2	3	3	5	5	4	3	5	9	9
112	16	19	27	32	42	69	78	82	129	160	227
113	0	-3	0	1	2	4	11	12	10	14	21
114	1	-3	1	3	6	9	16	21	41	48	84
115	3	6	18	37	61	101	146	225	296	388	562
116	0	-2	2	1	3	4	3	5	5	5	7
117	7	6	12	17	21	27	33	42	50	78	107
118	26	32	51	73	105	158	240	351	506	726	1003
119	3	2	2	2	15	19	36	44	54	54	70
120	9	3	8	9	16	37	55	80	125	183	272
121	2	1	8	13	15	36	49	72	111	148	186
122	2	1	6	10	17	31	42	68	105	141	224
123	0	-2	3	5	9	17	35	51	86	131	175
124	27	33	53	76	94	126	159	238	293	392	535
125	2	2	6	11	18	24	33	47	63	92	117

Table 10: Difference in travel times without specific projects for 1999 with multiples of the trip matrix - continued

	Factors										
Projects	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**
126	2	3	6	10	15	27	39	59	84	118	155
127	2	-1	7	7	12	20	33	41	57	80	114
128	1	-3	2	0	2	2	1	3	5	4	2
129	-1	-2	3	0	8	12	19	21	25	31	41
130	0	0	0	-1	0	1	0	2	6	9	8
131	-1	-2	3	4	8	9	2	8	12	19	37
132	0	-3	1	2	8	9	11	4	3	5	10
133	1	-1	2	4	5	8	18	41	39	49	51
134	4	3	7	9	12	23	38	43	60	58	58
135	18	18	26	31	32	48	52	63	75	79	90
136	2	0	3	5	9	11	13	19	30	34	54
137	0	-4	1	2	5	10	16	25	41	59	92
138	2	3	8	11	17	23	32	34	37	45	43
139	0	0	-2	2	7	11	22	29	43	50	71
140	3	1	6	20	34	33	43	59	90	124	165
141	2	3	8	14	19	31	49	61	84	121	153
142	20	21	29	35	44	51	74	92	94	108	118
143	4	0	4	9	18	24	36	56	52	74	100
144	1	-3	0	0	1	3	3	6	6	8	12
145	0	-1	3	6	13	24	40	55	77	109	143
146	4	0	6	6	11	14	28	42	56	82	98
147	3	2	7	12	16	23	31	40	47	66	79
148	0	-2	1	0	0	0	0	1	1	1	2
149	4	2	8	10	11	14	13	17	21	20	31
150	0	-4	-1	-1	3	11	13	17	20	18	19
151	1	1	7	4	4	5	8	8	18	23	34
152	1	-3	4	12	21	51	82	139	210	311	421
153	0	-2	0	-1	0	1	0	0	1	1	1
154	3	0	3	6	9	9	13	11	13	23	31
155	19	19	27	32	41	49	62	76	91	104	152
156	13	18	34	38	51	61	91	126	173	216	281
157	0	0	6	10	22	31	41	59	75	87	101
158	4	11	7	10	20	24	31	44	66	86	118
159	11	13	26	34	59	86	111	152	203	262	311
160	12	15	36	50	56	78	109	149	177	200	223
161	-1	-3	2	4	6	7	10	5	1	5	22
162	0	-3	2	4	11	14	21	27	37	55	87
164	6	5	11	16	29	58	81	109	193	284	382
165	2	1	10	16	30	38	54	73	107	133	177
166	22	27	42	58	69	96	129	188	256	359	470

Table 11: Difference in travel times without specific projects for 1999 with multiples of the trip matrix - continued

	Factors									
Projects	**1.6**	**1.7**	**1.8**	**1.9**	**2.0**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**
1	180	206	260	324	455	502	596	798	973	1166
2	205	248	305	384	490	566	691	855	1030	1228
3	96	129	171	235	336	420	480	617	768	900
4	45	61	70	89	107	96	136	157	209	224
5	297	406	519	651	807	944	1141	1397	1722	2063
6	221	261	324	410	508	567	655	745	897	1069
7	364	483	641	810	974	1253	1513	1846	2292	2829
8	220	268	351	498	618	770	972	1193	1449	1752
9	391	448	569	673	797	1019	1204	1527	1867	2209
10	232	298	376	487	602	806	965	1227	1553	1897
11	83	100	148	183	213	286	365	440	522	636
12	103	137	185	230	264	311	350	422	507	629
13	123	144	197	264	296	379	469	631	773	874
14	205	210	278	365	417	563	639	788	921	1048
15	1013	1245	1684	2096	2605	3302	4103	4973	6101	7502
16	428	549	650	792	1009	1230	1505	1824	2234	2744
17	9834	13323	17715	23227	29991	38261	48225	60204	74425	91223
18	5	9	18	16	12	20	50	73	69	82
19	216	276	337	451	582	727	914	1102	1313	1560
20	0	1	1	3	0	0	8	12	13	19
21	279	346	478	609	780	968	1120	1412	1730	2176
22	187	256	343	421	508	605	725	920	1143	1411
23	435	580	794	1040	1330	1727	2130	2716	3412	4183
24	4897	6611	8715	11334	14602	18597	23431	29208	36030	44076
25	195	258	323	401	520	628	719	888	1061	1291
26	377	458	502	635	737	843	1032	1097	1290	1528
27	1	1	0	1	1	4	8	10	13	12
28	89	125	182	188	246	281	384	415	521	656
29	37	10	50	69	98	124	157	189	229	311
30	4996	6826	9174	12091	15722	20090	25363	31792	39409	48379
31	757	933	1172	1414	1773	2173	2678	3307	4056	4896
33	270	289	397	526	628	804	1016	1221	1465	1923
34	33	44	69	89	111	141	167	207	265	304
35	409	516	663	842	1064	1341	1597	1984	2393	2844
36	0	0	2	4	6	14	22	25	40	44
37	199	250	299	428	526	651	819	1069	1307	1526
38	8	12	12	10	13	21	20	20	36	23
39	198	227	297	338	393	488	568	714	828	992
40	581	727	878	1095	1288	1552	1883	2304	2811	3324
41	1062	1405	1807	2302	2971	3649	4467	5542	6790	8115
42	277	357	464	580	748	948	1189	1517	1846	2196
43	255	305	383	475	624	755	901	1129	1350	1539

Table 12: Difference in travel times without specific projects for 1999 with multiples of the trip matrix - continued

	Factors									
Projects	**1.6**	**1.7**	**1.8**	**1.9**	**2.0**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**
44	139	155	236	283	274	352	480	669	760	940
45	991	1246	1595	2018	2571	3217	4054	5028	6203	7598
46	476	628	746	931	1071	1306	1600	1811	2121	2537
47	406	543	615	801	917	1119	1317	1605	2011	2484
48	138	139	172	223	273	331	404	449	567	678
49	79	101	119	128	123	139	156	187	230	247
50	850	1053	1359	1618	2026	2495	3005	3727	4490	5447
51	136	165	210	264	312	390	497	633	761	916
52	4	3	1	3	6	5	10	7	7	8
53	88	140	196	264	305	372	443	543	641	800
54	82	122	144	182	232	311	379	485	591	718
55	49	69	63	69	86	108	163	202	226	278
56	83	98	136	130	170	191	255	287	347	455
57	174	196	240	278	333	393	440	534	603	644
58	44	62	114	138	189	225	292	408	492	600
59	252	287	429	524	658	836	1076	1283	1607	1951
60	556	805	1019	1204	1528	1956	2370	2893	3520	4409
61	41	37	47	54	58	46	29	55	52	63
62	492	607	856	1126	1399	1750	2188	2701	3349	4182
63	1042	1274	1569	2095	2549	3120	3805	4622	5706	7039
64	116	153	185	213	290	402	481	665	817	1018
65	2884	3702	4765	6013	7573	9412	11570	14276	17372	20915
66	249	337	433	564	709	896	1069	1313	1649	2096
67	376	492	624	781	1008	1217	1497	1796	2270	2875
68	9	11	12	11	14	15	19	26	19	15
69	12	16	17	23	40	51	72	78	90	109
70	50	72	89	118	137	190	231	285	372	424
71	147	241	374	552	724	859	1110	1488	1915	2391
72	2741	340	430	518	664	834	970	1233	1517	1823
73	267	324	491	602	755	958	1253	1591	1962	2476
74	174	227	278	362	441	529	654	758	945	1211
75	52	77	82	95	134	146	176	216	271	329
76	178	206	311	368	410	407	390	488	609	766
77	316	352	418	470	604	716	881	991	1207	1437
78	321	334	415	505	604	771	996	1276	1490	1796
79	67	99	148	190	244	346	434	599	777	913
80	202	248	296	370	436	469	511	558	665	806
81	372	468	596	720	935	1108	1352	1654	2029	2505
82	357	445	542	641	825	964	1149	1417	1720	2141
83	3	6	11	22	19	19	30	45	33	40
84	15	20	20	25	31	35	53	62	83	98
85	207	233	275	328	448	543	708	933	1170	1431

Table 13: Difference in travel times without specific projects for 1999 with multiples of the trip matrix - continued

	Factors									
Projects	**1.6**	**1.7**	**1.8**	**1.9**	**2.0**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**
86	104	139	138	186	239	256	324	462	582	763
87	712	884	1113	1395	1777	2199	2739	3389	4117	4937
88	6	8	9	17	21	25	37	34	33	34
89	7	7	7	12	17	26	27	38	49	54
90	162	177	223	273	312	375	412	531	655	819
91	126	149	216	290	362	440	473	543	660	876
92	157	173	213	269	302	344	424	539	639	739
93	44	65	103	158	241	309	401	485	626	830
94	404	578	691	892	1087	1337	1735	2137	2589	3141
95	601	755	990	1188	1535	1848	2273	2843	3465	4262
96	276	357	451	530	674	799	928	1129	1375	1600
97	4	8	14	5	11	10	15	10	15	8
98	97	156	188	234	295	354	472	618	725	890
99	334	470	609	782	996	1165	1495	1868	2308	2808
100	88	101	142	228	292	362	491	666	792	973
101	160	189	223	271	363	451	570	747	988	1226
102	28	34	90	109	144	152	192	260	304	378
103	33	41	60	62	90	106	113	148	190	214
104	-1	1	-1	0	6	17	8	22	34	39
105	344	468	581	721	975	1158	1447	1862	2230	2692
106	181	212	324	361	471	531	706	896	1077	1294
107	800	970	1263	1638	2023	2532	3112	3845	4642	5639
108	3156	4037	5114	6336	7973	9846	12078	14664	17770	21283
109	90	111	170	215	279	345	360	466	600	777
110	172	234	304	375	464	551	699	829	1072	1318
111	13	20	21	20	29	48	44	58	75	71
112	305	419	586	727	951	1133	1330	1753	2091	2517
113	33	35	54	73	88	117	156	200	239	252
114	111	117	180	249	278	347	432	562	701	842
115	729	936	1234	1580	2014	2409	2985	3703	4634	5727
116	5	8	7	4	4	16	21	35	53	62
117	128	134	161	182	233	283	380	473	605	754
118	1345	1810	2363	3104	3970	5074	6414	7944	9833	12074
119	67	47	87	109	126	154	207	255	288	357
120	300	412	531	749	1045	1359	1866	2377	2925	3616
121	206	328	430	539	683	814	1068	1289	1676	2097
122	344	463	615	824	1080	1373	1677	2141	2618	3148
123	228	336	507	646	800	1023	1309	1656	2075	2596
124	721	937	1260	1686	2064	2593	3117	3862	4734	6761
125	160	197	249	344	419	538	633	804	964	1204
126	195	249	350	470	596	740	864	1087	1358	1616

Table 14: Difference in travel times without specific projects for 1999 with multiples of the trip matrix - continued

Projects	Factors									
	1.6	**1.7**	**1.8**	**1.9**	**2.0**	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**
127	123	176	258	305	358	492	602	797	1023	1236
128	0	-9	-12	-4	2	11	20	30	44	83
129	51	70	81	113	151	169	226	248	298	380
130	16	18	19	24	24	43	47	61	48	67
131	48	47	62	81	103	132	185	208	262	315
132	12	7	8	11	12	16	23	28	36	42
133	73	111	119	186	244	284	381	451	563	613
134	101	124	192	216	251	345	470	613	757	982
135	83	120	153	150	185	234	470	298	414	505
136	69	64	98	121	158	199	275	273	317	400
137	131	142	177	220	331	390	223	570	700	835
138	49	74	79	86	90	114	450	208	240	296
139	80	104	119	155	195	222	141	310	390	518
140	202	275	314	422	527	616	273	944	1196	1411
141	174	220	241	327	380	447	794	664	815	1015
142	132	159	205	306	384	474	582	751	909	1200
143	125	166	221	281	356	432	547	646	768	922
144	17	15	22	28	46	56	62	69	87	92
145	169	225	339	458	568	727	869	1133	1412	1735
146	141	176	201	284	342	438	521	653	758	976
147	103	128	176	233	345	413	503	604	724	843
148	2	5	7	10	24	34	46	54	78	113
149	15	37	47	55	100	111	135	134	183	222
150	26	56	83	100	123	148	200	236	296	398
151	51	76	110	137	186	252	328	399	545	653
152	537	691	919	1142	1429	1730	2103	2584	3167	3943
153	-1	4	6	14	8	19	27	24	33	35
154	29	30	31	23	30	34	39	62	59	78
155	185	223	267	334	436	521	655	892	1136	1292
156	335	405	532	608	705	857	1007	1277	1563	1921
157	133	156	187	201	283	330	435	541	670	767
158	140	188	245	330	440	542	660	832	1004	1136
159	401	513	668	792	1052	1294	1590	1927	2346	2813
160	238	324	384	432	492	592	685	785	936	1147
161	25	26	41	67	86	101	111	149	184	216
162	105	100	127	156	238	287	336	416	512	627
163	100	134	252	306	409	462	525	656	798	924
164	494	733	941	1210	1526	1972	2400	2918	3623	4499
165	220	282	374	497	609	700	876	1122	1388	1651
166	578	770	1017	1336	1656	2008	2437	3048	3750	4501

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