Chapter 1

A general class of distortion operators for pricing contingent claims with applications to CAT bonds

Résumé

Wang (2000) proposa un opérateur de distorsion permettant de récupérer les formules Black-Scholes de tarification d'options. Godin et al. (2012) généralisent cette approche de distorsion à la tarification sans arbitrage à une extension du modèle Black-Scholes basée sur la distribution normale inverse gaussienne pour la sous-classe de mesures martingales correctrices de moyenne. Nous généralisons ces travaux en offrant une classe d'opérateurs de distorsion compatible de façon plus générale avec la valorisation neutre au risque, ce qui ajoute de la flexibilité pour choisir le modèle actif/passif et la mesure neutre au risque. Nous dérivons ensuite plusieurs nouveaux opérateurs de distorsion améliorés permettant de valoriser les risques financiers et d'assurance. Enfin, nous présentons une nouvelle classe de distorsions pour évaluer les obligations catastrophes et offrons une validation empirique.

Abstract

Wang (2000) proposes a distortion operator that recuperates the Black-Scholes option pricing formulas. Godin et al. (2012) extend this distortion-based arbitrage-free pricing approach to a Normal Inverse Gaussian Black-Scholes world for the mean-correcting subclass of risk-neutral measures. We generalize this line of work by offering a class of distortion operators that is compatible with risk-neutral valuation more broadly, adding flexibility to the choices of the asset/liability model and the risk-neutral measure underlying the distortion. We then derive several new and improved distortion operators that can be used to price both financial and insurance risks. Finally, we present a novel class of distortions to price catastrophe bonds and provide an empirical validation.

Keywords : Distortion operator, Wang transform, Distortion risk measure, Arbitrage-free pricing, Insurance pricing, Contingent claim pricing, Pricing of CAT bonds.

1.1 Introduction

Wang (2000) proposes a probability distortion operator $g_\alpha(u) = \Phi(\Phi^{-1}(u) + \alpha)$, $u \in [0,1]$, to price both financial and insurance risks, where Φ is the standard normal cumulative distribution function. In particular, Wang shows that this transform can recover the classical Black-Scholes option pricing formulas. Hamada and Sherris (2003) and Pelsser (2008), among others, study further the applicability of the Wang transform for contingent claims pricing. These authors find the Wang transform consistent with arbitrage-free pricing when the underlying asset follows a geometric Brownian motion, but inadequate under non-Gaussian assumptions. Addressing this limitation, by way of an exponential Normal Inverse Gaussian (NIG) Lévy motion, Godin et al. (2012) propose a distortion operator that can recuperate the arbitrage-free prices under the mean-correcting equivalent martingale measure.

In this paper, we develop a framework for deriving distortion operators that are compatible with risk-neutral valuation under more general assumptions for the underlying asset model and the risk-neutral measure. Using our methodology, we produce new distortions that can recuperate pricing functionals of popular financial and insurance models well beyond the Wang transform and its non-Gaussian extensions. We also present empirical applications of our approach in the characterization of catastrophe (CAT) bond spreads.

Our paper is related to the literature that studies the connection between the Wang transform and other forms of risk pricing. Conditions under which the Bühlmann (1980)'s pricing equilibrium yields the Wang transform are derived by Wang (2003). Multivariate extensions of the Wang transform based on similar lines of reasoning can be found in Kijima (2006), Wang (2007), and Kijima and Muromachi (2008). The equivalence between the Wang transform and the Esscher-Girsanov change of measure proposed by Goovaerts and Laeven (2008) is demonstrated in a static setting by Labuschagne and Offwood (2010). Relative to this literature, our paper is the first to describe the general connection between distortion operators and other pricing principles. The connections between the Wang transform, the Black-Scholes model, the Esscher-Girsanov change of measure, and the Bühlmann's equilibrium are recovered as special cases of our analyses.

Specifically, our contributions are as follows. First, we present the general expression of the distortion operator that recovers risk-neutral pricing functionals and we derive the conditions of applicability. Second, we characterize the change of probability measure applied by our distortion operator, and we show how the connections found in the literature between the Wang transform and other pricing frameworks can be viewed as manifestations of this more fundamental result. Third, we derive new distortion operators that are suitable for financial and insurance risk pricing. Our first distortion extends the NIG distortion of Godin et al. (2012) to the Esscher equivalent martingale measure. The second distortion recuperates the equilibrium prices in Kou (2002)'s jump-diffusion model. The third recovers the size-biased premium principle using the generalized beta of the second kind (GB2) distribution for a loss variable. The fourth distortion operator recuperates the Esscher premium principle for a gamma distributed loss variable. Finally, we propose distortion operators that can depict CAT bond market spreads and show their usefulness in an empirical analysis. Our results provide an interesting evidence that jump-diffusion models are appropriate for pricing CAT bonds, but that investors are averse to natural disasters.

As pointed out by Hamada and Sherris (2003), Pelsser (2008), and Godin et al. (2012), the normality assumption underlying the Wang transform poses significant limitations for practical applications. Our set of new distortion operators extends Godin et al. (2012) in the following manner. We offer a general class of distortion operators that is compatible with risk-neutral valuation, yielding flexibility in the selection of the asset/liability model and the risk-neutral measure. This opens a wide range of possible applications for future research. For instance, our distortion operators could be used to produce new *distortion risk measures*, which are quantile-based measures used by practitioners in finance and insurance, e.g., Dowd and Blake (2006). One advantage for the risk measures based on our proposed distortion operators is that they enable the incorporation of risk-aversion and other considerations that are embedded into the choice of an equivalent martingale measure, as well as risk distribution features (e.g., skewness and kurtosis). In different directions, the previous literature has much attempted to incorporate these features into risk measures (e.g., Bali and Theodossiou, 2008; Gzyl and Mayoral, 2008).

The paper proceeds as follows. Section 1.2 provides some background on distortion operators and other forms of risk pricing. Section 1.3 presents the general expression of our distortion operator and characterizes its applicability to arbitrage-free pricing. Section 1.4 derives new distortion operators consistent with popular financial non-Gaussian option pricing models. Section 1.5 performs the same exercise for insurance pricing models. Section 1.6 presents empirical applications to the pricing of catastrophe bonds. Section 1.7 concludes and discusses other potential applications.

1.2 Background on risk pricing

In this section, we briefly review popular financial and insurance pricing principles.

1.2.1 Arbitrage-free pricing

Let us consider a continuous-time economy where time t takes value within $[0, T]$. This economy stochastic behaviour is characterized by a probability space $(\Omega, \mathcal{F}_T, \mathbb{P})$ equipped with a filtration

$\{\mathcal{F}_t\}_{t\in[0,T]}$ satisfying the usual conditions.

A central result in the theory of asset pricing by arbitrage, the so-called first fundamental theorem of asset pricing, is the equivalence between the absence of (quasi-)arbitrage opportunities and the existence of an equivalent martingale measure $\mathbb{Q}$. See Delbaen and Schachermayer (1994) for the history of this theorem which goes back to the seminal papers of Harrison and Kreps (1979), and Harrison and Pliska (1981). Such a measure is often called a *risk-neutral measure* because the arbitrage-free price process $\{S_t\}_{t\in[0,T]}$ of any traded asset and derivative must satisfy

$$S_t = B_t\,\mathbb{E}^{\mathbb{Q}}\left[\frac{S_s}{B_s}\middle|\mathcal{F}_t\right], \qquad \forall t \le s \le T, \tag{1.2.1}$$

where $\{B_t\}_{t\in[0,T]}$ is the risk-free asset price process.

(Re)insurance contracts (Delbaen and Haezendonck, 1989; Sondermann, 1991) and insurance-linked securities (Vaugirard, 2003) employ arbitrage-free pricing techniques initially developed for financial derivatives. Such frameworks are often characterized by an underlying insurance or catastrophe loss process whose dynamics is punctuated by random jumps, e.g., a jump-diffusion process. It is well-known that with such (non-locally bounded) processes, [1] the market is incomplete, and therefore there exists an infinite number of risk-neutral measures. In this case, the choice of risk-neutral measure is determined by the market, i.e., by the equilibrium resulting from supply and demand which are in turn determined by aggregate risk-aversion, liquidity needs, and other factors. In the realm of incomplete markets, popular modelling assumptions are the minimal martingale measure (Föllmer and Schweizer, 1991), the Esscher martingale measure (Gerber and Shiu, 1994), the variance optimal martingale measure (Schweizer, 1995), the mean-correcting martingale measure, and equilibrium-based martingale measures.

Equilibrium-based martingale measures

General equilibrium models can be viewed as a subset of the arbitrage-free pricing framework in the sense that they are possible approaches for selecting the risk-neutral measure. Indeed, there are no arbitrage opportunities in a rational expectation equilibrium.

For example, consider a continuous-time extension of Lucas (1978)'s model. We assume there is a representative agent possessing endowments who maximizes the agents aggregate utility. Let $U(t, c_t)$ be the aggregate utility at time t for the consumption process $\{c_t\}_{t\in[0,T]}$. Under mild conditions, one can show that this setup produces a pricing kernel that depends only on the aggregate endowment process, denoted by $\{\delta_t\}_{t\in[0,T]}$, such that the price process $\{S_t\}_{t\in[0,T]}$ of

1. The version of the first fundamental theorem of asset pricing for non-locally bounded processes states that the condition of no free lunch with vanishing risk is equivalent to the existence of an equivalent sigma-martingale measure (Delbaen and Schachermayer, 1998). A semi-martingale X is a sigma-martingale if there exists a martingale M and an M-integrable predictable process ϕ such that $X_t = \int_0^t \phi_u dM_u$.

any traded asset and derivative must satisfy the following condition in equilibrium :

$$S_t = \mathbb{E}^{\mathbb{P}}\left[\frac{U_c(s,\delta_s)}{U_c(t,\delta_t)}\, S_s \middle| \mathcal{F}_t\right], \qquad \forall t \leq s \leq T, \tag{1.2.2}$$

where $\mathbb{P}$ is the physical measure, and $U_c \equiv \frac{\partial U}{\partial c}$. This equilibrium condition can be written in the form of equation (1.2.1) by noting that the pricing kernel can be used to define a Radon-Nikodym derivative.

1.2.2 Actuarial premium calculation principles

A prominent problem in actuarial science is to derive premium calculation principles (PCPs) that satisfy a number of desirable properties (see, e.g., Laeven and Goovaerts, 2008). We present below two of such popular approaches. [2]

Distortion operators

The history of distortion operators goes back to Yaari (1987)'s dual theory of choice under risk, in which attitudes toward risks are characterized by a distortion function rather than by an expected utility function. Distortion operators also stem from the axiomatic approach of Wang et al. (1997) to characterize insurance prices.

Let X be a random variable distributed with survival function $\bar{F}_{\mathbb{P}}(x) \equiv \mathbb{P}(X > x)$ under the physical measure $\mathbb{P}$. We introduce a distortion operator g which is an increasing and differentiable function such that $g(0) = 0$, $g(1) = 1$, and $g(u) \in [0,1]$ for all $u \in [0,1]$. It defines a change of measure such that X is distributed with survival function $\bar{F}_g(x) \equiv g(\bar{F}_{\mathbb{P}}(x))$ under the new probability measure.

The price of X is obtained via the expected value under the distorted probability measure. One can show that this expected value has the following Choquet integral representation :

$$H[X;g] \equiv \int_{-\infty}^{0} \left[g\big(\bar{F}_{\mathbb{P}}(x)\big) - 1\right] dx + \int_{0}^{\infty} g\big(\bar{F}_{\mathbb{P}}(x)\big)\, dx, \qquad \bar{F}_{\mathbb{P}}(x) \equiv \mathbb{P}(X > x). \tag{1.2.3}$$

This Choquet integral exhibits monotonicity, translation invariance, positive homogeneity, and is sub-additive if g is concave (Denneberg, 1994). Hence, the functional $H[\cdot;g]$ defines a distortion-based risk measure that is coherent in the sense of Artzner et al. (1999) if g is concave. [3]

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2. There is a plethora of other insurance pricing principles, the reader is referred to Laeven and Goovaerts (2008) and Ai and Brockett (2008) for thorough accounts of these.

3. A more comprehensive treatise on distortion-based risk measures can be found in Dowd and Blake (2006).

Actuarial weighted pricing principles

Furman and Zitikis (2008) propose a broad class of PCPs based on weighted loss distributions. This class can also be viewed as a subclass of the loss function approach (see Remark 1 of Heilmann (1989)). Under this approach, the price of a risk $X \geq 0$ is given by

$$\Pi[X; w] = \frac{\mathbb{E}^{\mathbb{P}}[w(X)X]}{\mathbb{E}^{\mathbb{P}}[w(X)]}, \tag{1.2.4}$$

where $w(x) \geq 0$ for all $x \geq 0$. Thus, the change of probability measure whose Radon-Nikodym derivative is $w(x)/\mathbb{E}^{\mathbb{P}}[w(X)]$ characterizes this pricing principle. Several popular PCPs are contained within the weighted family. For instance, the Esscher principle is obtained with $w(x) = e^{ax}$ and the size-biased premium principle with $w(x) = x^a$, where $a \geq 0$ in both cases.[4]

1.3 A general framework for distortion-based risk-neutral valuation

This section contains our main theoretical results. First, we present the general definition of our distortion operator. Then, we characterize the change of probability measure it applies, and the conditions under which it can be used to compute arbitrage-free prices of derivatives.

1.3.1 A new class of distortion operators

We define below our distortion operator by Definition 1.3.1. This general expression can be reduced to a simpler form with fewer parameters using Proposition 1.3.1 whose proof is in Appendix 1.A.1. This can be crucial to circumvent the parameter identification issues that could otherwise arise in calibration.

Definition 1.3.1. Let X be a continuous random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathbb{Q}$ be a probability measure equivalent to $\mathbb{P}$ on $\mathcal{F}$. Let $\bar{F}_{\mathbb{P}}(x) \equiv \mathbb{P}(X > x)$ and $\bar{F}_{\mathbb{Q}}(x) \equiv \mathbb{Q}(X > x)$. We define the following distortion operator :

$$g_X^{\mathbb{Q},\mathbb{P}}(u) \equiv \bar{F}_{\mathbb{Q}}\big(\bar{F}_{\mathbb{P}}^{-1}(u)\big), \qquad u \in [0,1], \tag{1.3.1}$$

where $\bar{F}_{\mathbb{P}}^{-1}$ is the inverse of $\bar{F}_{\mathbb{P}}$ with the convention $\bar{F}_{\mathbb{P}}^{-1}(0) = +\infty$ and $\bar{F}_{\mathbb{P}}^{-1}(1) = -\infty$.

Proposition 1.3.1. *Let X be a continuous random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathbb{Q}$ be a probability measure equivalent to $\mathbb{P}$ on $\mathcal{F}$. For any continuous and increasing function h,*

$$g_{h(X)}^{\mathbb{Q},\mathbb{P}}(u) = g_X^{\mathbb{Q},\mathbb{P}}(u), \qquad \forall u \in [0,1]. \tag{1.3.2}$$

4. We refer the reader to Table 1 of Furman and Zitikis (2009) for additional examples.

Remark 1.3.1 (Wang transform)**.** Suppose that X is a standard normal $N(0,1)$ random variable under the measure $\mathbb{P}$, and that it is shifted to a $N(\theta,1)$ distribution under the measure $\mathbb{Q}$. In other words, we have $\mathbb{P}(X>x)=1-\Phi(x)$, and $\mathbb{Q}(X>x)=1-\Phi(x-\theta)$. Let h be any continuous and increasing function. By Proposition 1.3.1, $g_{h(X)}^{\mathbb{Q},\mathbb{P}}(u)=g_X^{\mathbb{Q},\mathbb{P}}(u)=\Phi\big(\Phi^{-1}(u)+\theta\big)$.

1.3.2 Change of measure performed by the distortion operator

The change of probability measure performed by the distortion operator $g_X^{\mathbb{Q},\mathbb{P}}$ in Definition 1.3.1 is characterized below by Theorem 1.3.1 whose proof is in Appendix 1.A.2. As stated in Corollary 1.3.1 (proven in Appendix 1.A.3), a key feature of the distortion $g_X^{\mathbb{Q},\mathbb{P}}$ is that it changes the probability measure from $\mathbb{P}$ to $\mathbb{Q}$ when applied on any random variable of the form $h(X)$, where h is any continuous and increasing function.

Definition 1.3.2. Let Z be a continuous random variable on $(\Omega,\mathcal{F},\mathbb{P})$, and let $\mathbb{Q}$ be a probability measure equivalent to $\mathbb{P}$ on $\mathcal{F}$. Define $q_{\mathbb{P}}$ as the probability density function (PDF) of Z under $\mathbb{P}$, and define $q_{\mathbb{Q}}$ as its PDF under $\mathbb{Q}$. We define the likelihood ratio

$$\xi_Z^{\mathbb{Q},\mathbb{P}}(z)\equiv\frac{q_{\mathbb{Q}}(z)}{q_{\mathbb{P}}(z)}. \tag{1.3.3}$$

Theorem 1.3.1. *Let X and Z be continuous random variables on $(\Omega,\mathcal{F},\mathbb{P})$, and let $\mathbb{Q}$ be a probability measure equivalent to $\mathbb{P}$ on $\mathcal{F}$. Denote the survival functions of X and Z under $\mathbb{P}$ by $\bar{F}_{\mathbb{P}}(x)\equiv\mathbb{P}(X>x)$ and $\bar{Q}_{\mathbb{P}}(z)\equiv\mathbb{P}(Z>z)$, and let the PDF of X under $\mathbb{P}$ be denoted by $f_{\mathbb{P}}$. The PDF of X under the probability measure distorted by $g_Z^{\mathbb{Q},\mathbb{P}}$ is given by*

$$f_{g_Z^{\mathbb{Q},\mathbb{P}}}(x)=f_{\mathbb{P}}(x)\,\xi_Z^{\mathbb{Q},\mathbb{P}}\big(\bar{Q}_{\mathbb{P}}^{-1}(\bar{F}_{\mathbb{P}}(x))\big), \tag{1.3.4}$$

where $\bar{Q}_{\mathbb{P}}^{-1}$ is the inverse of $\bar{Q}_{\mathbb{P}}$.

Corollary 1.3.1. *If $X=h(Z)$, where h is continuous and increasing, then the distorted distribution of X coincides with its distribution under $\mathbb{Q}$: $f_{g_Z^{\mathbb{Q},\mathbb{P}}}(x)=f_{\mathbb{Q}}(x)$, for all x in the support.*

1.3.3 Connections between the Wang transform and other pricing frameworks

There is a literature studying the connections between the Wang transform, the Esscher-Girsanov change of measure, and the Bühlmann general equilibrium model. We now show that these results can be recovered by virtue of Theorem 1.3.1. We refer to Wang (2003) and Labuschagne and Offwood (2010) for the original proofs. [5]

5. See Kijima (2006), Wang (2007), and Kijima and Muromachi (2008) for closely related works.

First, let's see how the Wang transform can be related to the Esscher-Girsanov change of measure. Suppose that Z is a $N(0,1)$ random variable under $\mathbb{P}$, and that its distribution is shifted to a $N(\theta,1)$ under $\mathbb{Q}$. As stated in Remark 1.3.1, the distortion operator is the Wang transform $g_Z^{\mathbb{Q},\mathbb{P}}(u) = \Phi\big(\Phi^{-1}(u)+\theta\big)$. Moreover, one can readily show that the Radon-Nikodym derivative is $\xi_Z^{\mathbb{Q},\mathbb{P}}(x) = e^{\theta x-\theta^2/2}$. Therefore, applying Theorem 1.3.1 with $\bar{Q}_{\mathbb{P}}(x) = 1-\Phi(x)$ gives us

$$f_{g_Z^{\mathbb{Q},\mathbb{P}}}(x) = e^{\theta\Phi^{-1}(F_{\mathbb{P}}(x))-\theta^2/2} f_{\mathbb{P}}(x), \tag{1.3.5}$$

where $F_{\mathbb{P}}(x) \equiv 1-\bar{F}_{\mathbb{P}}(x)$. When $\theta = h\nu$, the Definition 3.1 Esscher-Girsanov change of measure recovers the one in Labuschagne and Offwood (2010).

Next, let's see how the Wang transform can yield the Bühlmann (1980)'s pricing equilibrium. The key assumption in Wang (2003) is that X and Z and co-monotone in the sense that under $\mathbb{P}$ they can be expressed as $X = F_{\mathbb{P}}^{-1}(U)$ and $Z = \Phi^{-1}(U)$, where U is uniformly distributed between 0 and 1. Assuming this, using the expression $Z = \Phi^{-1}(F_{\mathbb{P}}(X))$ in (1.3.5) yields

$$f_{g_Z^{\mathbb{Q},\mathbb{P}}}(x) = \mathbb{E}^{\mathbb{P}}\left[e^{\theta Z-\theta^2/2}\middle|\, X=x\right] f_{\mathbb{P}}(x) = \frac{\mathbb{E}^{\mathbb{P}}\left[e^{\theta Z}\middle| X=x\right]}{\mathbb{E}^{\mathbb{P}}\left[e^{\theta Z}\right]} f_{\mathbb{P}}(x). \tag{1.3.6}$$

1.3.4 Risk-neutral pricing using distortion operators

Let X be a random variable representing a financial or insurance risk. For example, X could be the price of a traded asset (as it is for the case of financial derivatives), but it could also be something else like the number of heating degree days (say a weather derivative) or the value of a catastrophe loss index (as for insurance-linked securities). Our goal is to price derivatives on X using a distortion operator. The proposition below whose proof is in Appendix 1.A.4 gives the general solution to this problem.

Proposition 1.3.2. *Let X be a random variable on $(\Omega,\mathcal{F},\mathbb{P})$, and let $\mathbb{Q}$ be a measure equivalent to $\mathbb{P}$ on $\mathcal{F}$. For any continuous and increasing function h, we have*

$$H\Big[h(X); g_X^{\mathbb{Q},\mathbb{P}}\Big] = \mathbb{E}^{\mathbb{Q}}[h(X)]. \tag{1.3.7}$$

In particular, Proposition 1.3.2 also holds for any equivalent martingale measure $\mathbb{Q}$, in which case the arbitrage-free price of a derivative with terminal payoff $h(X)$ is given by the discounted value of $H\Big[h(X); g_X^{\mathbb{Q},\mathbb{P}}\Big]$. Less general versions of this result can be found in the literature. For instance, the proof when X is the terminal value of a geometric Brownian motion (see Remark 1.3.2) can be found in Hamada and Sherris (2003). The proof when X is the terminal value of an exponential NIG Lévy motion and $\mathbb{Q}$ is the mean-correcting martingale measure is in Godin et al. (2012).

Remark 1.3.2 (Black-Scholes model)**.** It follows from Proposition 1.3.2 that the distortion operator that can recuperate the arbitrage-free prices under the Black-Scholes model is $g_{S_T}^{\mathbb{Q},\mathbb{P}}$, where S_T is the terminal value of a geometric Brownian motion, with constant drift μ and volatility σ under the physical measure $\mathbb{P}$, and with drift r and volatility σ under the risk-neutral measure $\mathbb{Q}$, where r is the risk-free rate. Following Remark 1.3.1, one can check that this distortion reduces to the Wang transform $g_{S_T}^{\mathbb{Q},\mathbb{P}}(u) = \Phi\big(\Phi^{-1}(u) + \varphi\big)$ where $\varphi = \big(\frac{r-\mu}{\sigma}\big)\sqrt{T}$, with T denoting the maturity.

1.4 Distortion operators for financial models

Next, we derive distortion operators compatible with arbitrage-free pricing under non-Gaussian extensions of the Black-Scholes model. Empirically, it turns out that such extensions are needed to reproduce well-documented phenomena, such as the “volatility smiles" observed in option markets, and the fact that asset returns exhibit heavier-skewed tails than the normal distribution underpinning the geometric Brownian motion.

We consider a continuous-time market, with time $t \in [0, T]$, containing a liquid asset growing at the risk-free rate r and a (possibly non-traded) underlying risky process $\{S_t\}_{t\in[0,T]}$ defined on a probability space $(\Omega, \mathcal{F}_T, \mathbb{P})$ equipped with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ satisfying the usual assumptions. In Section 1.4.1, we model the underlying asset with an infinite activity Lévy process, and in Section 1.4.2 with a jump-diffusion process. Both approaches, now standard in the literature, have been widely applied in finance (see, e.g., Schoutens, 2003; Cont and Tankov, 2004).

1.4.1 Normal Inverse Gaussian distortion based on the Esscher martingale measure

Godin et al. (2012) propose a distortion operator that can recuperate the arbitrage-free prices for a non-Gaussian extension of the Black-Scholes model based on the Normal Inverse Gaussian (NIG) distribution and the mean-correcting martingale measure. Their distortion operator is proposed in the form of an educated guess in their Definition 2. Here, we take a different approach. We start directly from the general expression of the distortion operator (Definition 1.3.1) and simplify it using Proposition 1.3.1. The latter approach has the advantage of not requiring an *ansatz* for the correct form of the distortion operator. To make this exercise rewarding, we extend the work of Godin et al. (2012) to the Esscher martingale measure, which provides a new NIG-based distortion operator that benefits from the very same advantages.

Dynamics under the physical measure

Under the physical measure, the underlying asset price follows an exponential NIG Lévy process with parameters $(\alpha, \beta, \delta, \mu)$. Its terminal value takes the form (see, e.g., Schoutens, 2003)

$$S_T = S_0 e^{X_T}, \qquad X_T \sim \mathrm{NIG}(\alpha, \beta, \delta T, \mu T), \tag{1.4.1}$$

where $\mathrm{NIG}(\alpha, \beta, \delta, \mu)$ is the Normal Inverse Gaussian distribution as defined in Godin et al. (2012). The distribution is a generalization of the normal distribution that allows for skewness and excess kurtosis. The cumulative distribution function (CDF) and survival function of $\mathrm{NIG}(\alpha, \beta, \delta, \mu)$ are respectively denoted by $\mathbf{NIG}(x; \alpha, \beta, \delta, \mu)$ and $\overline{\mathbf{NIG}}(x; \alpha, \beta, \delta, \mu)$, $x \in \mathbb{R}$. We state below some useful remarks about this distribution.

Remark 1.4.1. If $X \sim \mathrm{NIG}(\alpha, \beta, \delta, \mu)$ we have that $Y = aX + b$, for $a > 0$ and $b \in \mathbb{R}$, is such that $Y \sim \mathrm{NIG}(\alpha/a, \beta/a, a\delta, a\mu + b)$.

Remark 1.4.2. The NIG distribution possesses the symmetry

$$\overline{\mathbf{NIG}}(x; \alpha, \beta, \delta, 0) = \mathbf{NIG}(-x; \alpha, -\beta, \delta, 0).$$

Dynamics under the risk-neutral measure

It turns out that the market described above is incomplete, and therefore there exists an infinite number of equivalent sigma-martingale measures (Eberlein and Jacod, 1997). We choose the Esscher martingale measure as it is a popular choice (e.g., Gerber and Shiu, 1994). Under this measure, the process S is an exponential NIG Lévy process with parameters $(\alpha, \beta + \theta, \delta, \mu)$.[6] Under the Esscher martingale measure $\mathbb{Q}$, the terminal value of the underlying is such that

$$S_T = S_0 e^{X_T^{\mathbb{Q}}}, \qquad X_T^{\mathbb{Q}} \sim \mathrm{NIG}\big(\alpha, \beta + \theta, \delta T, \mu T\big). \tag{1.4.2}$$

Derivation of the distortion operator

Using Remark 1.4.1, we can express (1.4.1) and (1.4.2) more compactly as

$$S_T = h(Z) \equiv S_0 \exp\Big\{\mu T + \sqrt{\delta/\alpha}\, Z\Big\}, \tag{1.4.3}$$

where the random variable Z is such that

$$Z \sim \begin{cases} \mathrm{NIG}\big(\sqrt{\alpha\delta}, \beta\sqrt{\delta/\alpha}, T\sqrt{\alpha\delta}, 0\big), & \text{under } \mathbb{P}, \\ \mathrm{NIG}\big(\sqrt{\alpha\delta}, (\beta+\theta)\sqrt{\delta/\alpha}, T\sqrt{\alpha\delta}, 0\big), & \text{under } \mathbb{Q}. \end{cases} \tag{1.4.4}$$

6. Moreover, if S is the price process of a traded asset, then θ must be determined so that the discounted price process $\{S_t e^{-rt}\}_{t\in[0,T]}$ is a martingale under $\mathbb{Q}$ (see Schoutens, 2003, p. 79).

To obtain the expression of the distortion operator, we start from Definition 1.3.1 and simplify it using Proposition 1.3.1 and the symmetry property of Remark 1.4.2 :

$$g_{S_T}^{\mathbb{Q},\mathbb{P}}(u) = g_{h(Z)}^{\mathbb{Q},\mathbb{P}}(u) = g_{Z}^{\mathbb{Q},\mathbb{P}}(u) = \Phi_{\mathbf{NIG}}^{\mathbb{Q}}\big(\Phi_{\mathbf{NIG}}^{-1}(u)\big), \tag{1.4.5}$$

where the following definitions are used :

$$\Phi_{\mathbf{NIG}}(x) \equiv \mathbf{NIG}\big(x; \xi, \zeta, T\xi, 0\big), \qquad \Phi_{\mathbf{NIG}}^{\mathbb{Q}}(x) \equiv \mathbf{NIG}\big(x; \xi, \zeta^{\mathbb{Q}}, T\xi, 0\big),$$

with $\xi \equiv \sqrt{\alpha\delta}$, $\zeta \equiv -\beta\sqrt{\delta/\alpha}$, $\zeta^{\mathbb{Q}} \equiv -(\beta+\theta)\sqrt{\delta/\alpha}$, and where $\Phi_{\mathbf{NIG}}^{-1}$ is the inverse of $\Phi_{\mathbf{NIG}}$.

By Proposition 1.3.2, it follows that this distortion recovers the arbitrage-free prices under the Esscher martingale measure. In fact, Black-Scholes-type formulas can be recuperated through (1.3.7) in a similar fashion as in Hamada and Sherris (2003). Note that the difference between our NIG distortion and the one proposed by Godin et al. (2012) lies in the choice of the equivalent martingale measure, the later being based on the mean-correcting measure.[7] Since our new Esscher-based NIG distortion exhibits the same improvements over the Wang transform as achieved by Godin et al. (2012), we refer to that work for a thorough account of these improvements.

1.4.2 A distortion operator based on Kou (2002)'s jump-diffusion model

Let us consider now the Kou (2002)'s jump-diffusion option pricing model. This model is based on the equilibrium framework of Section 1.2.1, where the utility function of the representative agent is assumed to be of the form $U(t,c) = e^{-\kappa t}\frac{c^\alpha}{\alpha}$ for $\alpha \in (0,1]$, and $U(t,c) = e^{-\kappa t}\ln c$ for $\alpha = 0$, with κ being a subjective discount factor. Kou's model offers an attractive tradeoff between reality and tractability. It is able to reproduce the leptokurtic feature of the return distribution and the "volatility smile" observed in option markets, yet is simple enough to produce analytical formulas for call/put options, interest rate derivatives, and a variety of path-dependent options. We first describe Kou's model, and then derive its associated distortion.

The endowment process is modelled by the following jump-diffusion process under $\mathbb{P}$:

$$\frac{d\delta_t}{\delta_{t^-}} = \mu_1 dt + \sigma_1 dW_t^{(1)} + d\left[\sum_{i=1}^{N_t}(V_i - 1)\right], \tag{1.4.6}$$

where $\mu_1 \in \mathbb{R}$ and $\sigma_1 > 0$ are constants, $\{W_t^{(1)}\}_{t\in[0,T]}$ is a Wiener process, $\{N_t\}_{t\in[0,T]}$ is a standard Poisson process with intensity $\lambda > 0$, $\{V_i\}_{i\geq 1}$ is a sequence of i.i.d. non-negative random variables, and all sources of randomness are independent.

7. Godin et al. (2012) choose the mean-correcting martingale measure rather than the Esscher martingale measure. Under this measure, S is an exponential NIG Lévy process with parameters $(\alpha, \beta, \delta, \mu + \theta)$. One can check that applying our approach indeed yields the NIG distortion proposed in their Definition 2 : $\Phi_{\mathbf{NIG}}(\Phi_{\mathbf{NIG}}^{-1}(u) + \theta T\sqrt{\alpha/\delta})$.

The underlying follows the jump-diffusion process

$$\frac{dS_t}{S_{t^-}} = \mu dt + \sigma dW_t + d\left[\sum_{i=1}^{N_t}\left(V_i^{\beta} - 1\right)\right], \tag{1.4.7}$$

where $\mu \in \mathbb{R}$, $\sigma > 0$, and $\beta \in \mathbb{R}$ are constants. The new Wiener process $\{W_t\}_{t\in[0,T]}$ has constant correlation $\rho \in [-1, 1]$ to $W^{(1)}$ and is independent from the other sources of randomness.

Dynamics under the physical measure

The distribution of the log-size jumps is modelled by a Asymmetric Double Exponential distribution ADE(η_1, η_2, p). The PDF of this distribution is

$$\mathbf{ade}(x; \eta_1, \eta_2, p) = p\eta_1 e^{-\eta_1 x}\, \mathbb{1}_{x\geq 0} + (1-p)\eta_2 e^{\eta_2 x}\, \mathbb{1}_{x<0}, \qquad x \in \mathbb{R}. \tag{1.4.8}$$

The parameter domain is $\eta_1 > 0$, $\eta_2 > 0$, $p \in [0, 1]$. As stated in Remark 1.4.3, the ADE distribution is closed under scaling.

Remark 1.4.3. If $X \sim \text{ADE}(\eta_1, \eta_2, p)$ then we have that $Y \equiv aX$ is such that

$$Y \sim \begin{cases} \text{ADE}(\eta_1/a, \eta_2/a, p), & a \geq 0, \\ \text{ADE}(\eta_2/|a|, \eta_1/|a|, 1-p), & a < 0. \end{cases}$$

Solving the stochastic differential equation (1.4.7) yields

$$S_T = S_0 \exp\left\{ \left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W_T + \beta \sum_{i=1}^{N_T} Y_i \right\}, \tag{1.4.9}$$

with $\{Y_i \equiv \ln V_i\}_{i\geq 1} \overset{\text{i.i.d.}}{\sim} \text{ADE}(\eta_1, \eta_2, p)$, $W_T \sim N(0, T)$, and $N_T \sim \text{Poisson}(\lambda T)$.

Dynamics under the risk-neutral measure

Theorem 1 of Kou (2002) describes the dynamics under the risk-neutral measure and the conditions for the existence of this measure. From this theorem, it can be shown that under $\mathbb{Q}$ the terminal value of the underlying is such that

$$S_T = S_0 \exp\left\{ \left(\mu - \frac{\sigma^2}{2} - \rho\sigma\sigma_1(1-\alpha)\right)T + \sigma W_T^{\mathbb{Q}} + \beta \sum_{i=1}^{N_T} Y_i \right\}, \tag{1.4.10}$$

with $\{Y_i\}_{i\geq 1} \overset{\text{i.i.d.}}{\sim} \text{ADE}\big(\eta_1^{\mathbb{Q}}, \eta_2^{\mathbb{Q}}, p^{\mathbb{Q}}\big)$, $W_T^{\mathbb{Q}} \sim N(0, T)$, and $N_T \sim \text{Poisson}(\lambda^{\mathbb{Q}} T)$, where

$$\eta_1^{\mathbb{Q}} = \eta_1 - \alpha + 1, \qquad \eta_2^{\mathbb{Q}} = \eta_2 + \alpha - 1, \qquad p^{\mathbb{Q}} = \frac{p}{\zeta}\frac{\eta_1}{\eta_1 - \alpha + 1}, \qquad \lambda^{\mathbb{Q}} = \zeta\lambda, \tag{1.4.11}$$

with $\zeta \equiv \frac{p\eta_1}{\eta_1-\alpha+1} + \frac{(1-p)\eta_2}{\eta_2+\alpha-1}$.

Derivation of the distortion operator

The survival function defined below will be useful in defining the distortion operator. We refer to Appendix B of Kou (2002) for results that ease its numerical implementation.

Definition 1.4.1. Let the following random variables be independent : $Z \sim N(\mu, \sigma^2)$ is a normal variable, $P \sim \text{Poisson}(\lambda)$ is a Poisson variable, and $\{Y_i\}_{i \geq 1} \overset{\text{i.i.d.}}{\sim} \text{ADE}(\eta_1, \eta_2, p)$. The CDF of the *Normal Compound Poisson distribution NCP*$(\lambda, \eta_1, \eta_2, p, \sigma, \mu)$ is defined as

$$\mathbf{NCP}\big(x; \lambda, \eta_1, \eta_2, p, \sigma, \mu\big) \equiv \Pr\left(Z + \sum_{i=1}^{P} Y_i \leq x\right), \qquad x \in \mathbb{R}. \tag{1.4.12}$$

The survival function is denoted by $\overline{\mathbf{NCP}}\big(x; \lambda, \eta_1, \eta_2, p, \sigma, \mu\big) = 1 - \mathbf{NCP}\big(x; \lambda, \eta_1, \eta_2, p, \sigma, \mu\big)$.

Using Remark 1.4.3, it is a straightforward exercise to show that the NCP distribution is closed under affine transformations, as stated below in Remark 1.4.4. Moreover, the CDF and survival function of the NCP distribution are related by the symmetry property stated in Remark 1.4.5.

Remark 1.4.4. If $X \sim \text{NCP}(\lambda, \eta_1, \eta_2, p, \sigma, \mu)$ then we have that $Y = aX + b$ is such that

$$Y \sim \begin{cases} \text{NCP}\big(\lambda, \eta_1/a, \eta_2/a, p, a\sigma, a\mu + b\big), & a \geq 0, \\ \text{NCP}\big(\lambda, \eta_2/|a|, \eta_1/|a|, 1-p, |a|\sigma, a\mu + b\big), & a < 0. \end{cases}$$

Remark 1.4.5. The NCP distribution possesses the following symmetry :

$$\overline{\mathbf{NCP}}(x; \lambda, \eta_1, \eta_2, p, \sigma, \mu) = \mathbf{NCP}(-x; \lambda, \eta_2, \eta_1, 1-p, \sigma, -\mu).$$

We can make use of Remark 1.4.4 to express (1.4.9) and (1.4.10) more compactly as

$$S_T = h(X) \equiv S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}X\right\}, \tag{1.4.13}$$

where the random variable X is such that

$$X \sim \begin{cases} \text{NCP}\big(\lambda T, \tilde{\eta}_1, \tilde{\eta}_2, \tilde{p}, 1, 0\big), & \text{under } \mathbb{P}, \\ \text{NCP}\big(\lambda^{\mathbb{Q}} T, \tilde{\eta}_1^{\mathbb{Q}}, \tilde{\eta}_2^{\mathbb{Q}}, \tilde{p}^{\mathbb{Q}}, 1, -\rho\sigma_1(1-\alpha)\sqrt{T}\big), & \text{under } \mathbb{Q}, \end{cases} \tag{1.4.14}$$

where the tilde parameters are defined in terms of $\nu \equiv \sigma\sqrt{T}/|\beta|$ as follows :

$$\begin{array}{llllllll} (\text{if } \beta \geq 0) & \tilde{\eta}_1 = \eta_1\nu, & \tilde{\eta}_1^{\mathbb{Q}} = \eta_1^{\mathbb{Q}}\nu, & \tilde{\eta}_2 = \eta_2\nu, & \tilde{\eta}_2^{\mathbb{Q}} = \eta_2^{\mathbb{Q}}\nu, & \tilde{p} = p, & \tilde{p}^{\mathbb{Q}} = p^{\mathbb{Q}}, \\ \quad \text{or} \\ (\text{if } \beta < 0) & \tilde{\eta}_1 = \eta_2\nu, & \tilde{\eta}_1^{\mathbb{Q}} = \eta_2^{\mathbb{Q}}\nu, & \tilde{\eta}_2 = \eta_1\nu, & \tilde{\eta}_2^{\mathbb{Q}} = \eta_1^{\mathbb{Q}}\nu, & \tilde{p} = 1-p, & \tilde{p}^{\mathbb{Q}} = 1-p^{\mathbb{Q}}. \end{array} \tag{1.4.15}$$

To obtain the expression of the distortion operator, we start from Definition 1.3.1 and simplify it by using Proposition 1.3.1 and the symmetry property of Remark 1.4.5 :

$$g_{S_T}^{\mathbb{Q},\mathbb{P}}(u) = g_{h(X)}^{\mathbb{Q},\mathbb{P}}(u) = g_X^{\mathbb{Q},\mathbb{P}}(u) = \Phi_{\mathbf{NCP}}^{\mathbb{Q}}\Big(\Phi_{\mathbf{NCP}}^{-1}(u) - \rho\sigma_1(1-\alpha)\sqrt{T}\Big), \tag{1.4.16}$$

where the following definitions are used :

$$\Phi_{\mathbf{NCP}}(x) \equiv \mathbf{NCP}\big(x; \lambda T, \tilde{\eta}_2, \tilde{\eta}_1, 1-\tilde{p}, 1, 0\big), \qquad \Phi^{\mathbb{Q}}_{\mathbf{NCP}}(x) \equiv \mathbf{NCP}\big(x; \lambda^{\mathbb{Q}} T, \tilde{\eta}^{\mathbb{Q}}_2, \tilde{\eta}^{\mathbb{Q}}_1, 1-\tilde{p}^{\mathbb{Q}}, 1, 0\big),$$

with $\Phi^{-1}_{\mathbf{NCP}}$ defined as the inverse of $\Phi_{\mathbf{NCP}}$.

From Proposition 1.3.2, it follows that the distortion operator (1.4.16) recovers the pricing equilibrium described above. For instance, through (1.3.7), the distortion can recuperate the European call formula (20) in Kou (2002). This distortion operator produces a risk-neutralized distribution that can be viewed as capturing premiums for both sources of risks, i.e., jumps and Brownian motion.

1.5 Distortion operators for insurance models

We now derive distortion operators for two insurance pricing models. The first is based on the size-biased pricing principle for the generalized beta of the second kind distribution, and the second is based on the Esscher principle for the gamma distribution. In both cases, the underlying risk is an insurance claim represented by a positive random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$.

We adopt the conventions used in Klugman et al. (2012) for the incomplete beta function and the incomplete gamma function. The incomplete beta function is defined as

$$\beta(x; \tau, \alpha) \equiv \frac{\Gamma(\tau+\alpha)}{\Gamma(\tau)\Gamma(\alpha)} \int_0^x t^{\tau-1}(1-t)^{\alpha-1} dt, \qquad \tau > 0, \quad \alpha > 0, \quad x \in (0,1), \tag{1.5.1}$$

and the (lower) incomplete gamma function is defined as

$$\Gamma(x; \alpha) \equiv \frac{1}{\Gamma(\alpha)} \int_0^x t^{\alpha-1} e^{-t} dt, \qquad \alpha > 0, \quad x > 0, \tag{1.5.2}$$

where $\Gamma(\alpha) \equiv \int_0^\infty t^{\alpha-1} e^{-t} dt$ is the gamma function.

1.5.1 A distortion based on the generalized beta of the second kind distribution

The generalized beta of the second kind (GB2) distribution, sometimes called the transformed beta distribution, is a member of the celebrated Pearson system and was first proposed as a model of the size-of-loss distribution in actuarial sciences by Venter (1983). This large family of heavy-tailed distributions contains the Burr, generalized Pareto, generalized gamma, log-t, and other commonly used distributions. It provides a fairly flexible form that can be used to model highly skewed loss distributions such as those typically observed in non-life insurance (Cummins et al., 1990). Here, we use the GB2 distribution to model the risk X. A thorough account of other size-of-loss distributions can be found in Kleiber and Kotz (2003) and in Klugman et al. (2012).

The GB2 distribution has heavy tails, and therefore only a few of the moments exist (Kleiber, 1997). This implies that some risk-neutral measures, such as the Esscher measure, may exist only for a certain range of the shape parameters. Among the possible risk-neutral measures, the size-biased subclass is an interesting choice as it preserves the shape of the GB2 family, yielding a simple Wang-like distortion operator. We underline that our approach also applies to non shape-preserving changes of measure, although it may require tedious algebra to simplify the form of the distortion function.

Distribution under the physical measure

Under the physical measure $\mathbb{P}$, the risk X follows a GB2$(\alpha, \theta, \gamma, \tau)$ distribution as defined in Klugman et al. (2012). The PDF of this distribution is

$$\mathbf{gb2}(x; \alpha, \theta, \gamma, \tau) = \frac{\Gamma(\alpha + \tau)}{\Gamma(\alpha)\Gamma(\tau)} \frac{\gamma (x/\theta)^{\gamma\tau}}{x[1 + (x/\theta)^{\gamma}]^{\alpha+\tau}}, \qquad x \geq 0, \tag{1.5.3}$$

and its CDF is

$$\mathbf{GB2}(x; \alpha, \theta, \gamma, \tau) = \beta\left(\frac{(x/\theta)^{\gamma}}{1 + (x/\theta)^{\gamma}}; \tau, \alpha\right), \qquad x \geq 0. \tag{1.5.4}$$

The parameter domain of this distribution is $\alpha > 0$, $\theta > 0$, $\gamma > 0$, $\tau > 0$. The parameter controlling the scale is θ. The other parameters control the tail behaviour and the shape in general. It is interesting to note that the log-normal distribution is a limiting case of the GB2 distribution, as stated in the following remark.

Remark 1.5.1 (Log-normal limit)**.** A log-normal distribution with PDF $\mathbf{logN}(x; \mu, \sigma) \equiv \frac{\phi\left(\frac{\ln x - \mu}{\sigma}\right)}{\sigma x}$, $x \geq 0$, is obtained as a limiting case of the GB2$(\alpha, \theta, \gamma, \tau)$ distribution for $\alpha \to \infty$, $\gamma \to 0$, $\theta = (\alpha\gamma^2\sigma^2)^{1/\gamma}$ and $\tau = (\gamma\mu + 1)/(\sigma^2\gamma^2)$ (from McDonald, 1987).

The survival function under $\mathbb{P}$ of the risk X is thus given by

$$\bar{F}_{\mathbb{P}}(x) \equiv \mathbb{P}(X > x) = 1 - \beta\left(\frac{(x/\theta)^{\gamma}}{1 + (x/\theta)^{\gamma}}; \tau, \alpha\right), \qquad x \geq 0. \tag{1.5.5}$$

Distribution under the risk-neutral measure

Under the size-biased risk-neutral measure $\mathbb{Q}$, the PDF of X is given by

$$\frac{x^a}{\mathbb{E}^{\mathbb{P}}[X^a]} \mathbf{gb2}(x; \alpha, \theta, \gamma, \tau) \propto \mathbf{gb2}(x; \tilde{\alpha}, \theta, \gamma, \tilde{\tau}), \tag{1.5.6}$$

where

$$\tilde{\alpha} \equiv \alpha - a/\gamma, \qquad \tilde{\tau} \equiv \tau + a/\gamma. \tag{1.5.7}$$

By normalization, the right-hand side of (1.5.6) implies that $X \sim \text{GB2}(\tilde{\alpha}, \theta, \gamma, \tilde{\tau})$ under $\mathbb{Q}$. Note that the requirement $-\gamma\tau < a < \alpha\gamma$ is needed for the existence of this measure.

The survival function under $\mathbb{Q}$ of X is therefore given by

$$\bar{F}_{\mathbb{Q}}(x) \equiv \mathbb{Q}(X > x) = 1 - \beta\left(\frac{(x/\theta)^\gamma}{1+(x/\theta)^\gamma}; \tilde{\tau}, \tilde{\alpha}\right), \qquad x \geq 0. \tag{1.5.8}$$

Derivation of the distortion operator

To obtain the distortion operator, the first step is to use (1.5.8) in Definition 1.3.1 to obtain

$$g_X^{\mathbb{Q},\mathbb{P}}(u) \equiv \bar{F}_{\mathbb{Q}}\big(\bar{F}_{\mathbb{P}}^{-1}(u)\big) = 1 - \beta\left(\frac{\big(\bar{F}_{\mathbb{P}}^{-1}(u)/\theta\big)^\gamma}{1+\big(\bar{F}_{\mathbb{P}}^{-1}(u)/\theta\big)^\gamma}; \tilde{\tau}, \tilde{\alpha}\right). \tag{1.5.9}$$

Next, we use $x = \bar{F}_{\mathbb{P}}^{-1}(u)$ in (1.5.5) to obtain the following relation :

$$u = 1 - \beta\left(\frac{\big(\bar{F}_{\mathbb{P}}^{-1}(u)/\theta\big)^\gamma}{1+\big(\bar{F}_{\mathbb{P}}^{-1}(u)/\theta\big)^\gamma}; \tau, \alpha\right) \quad \Rightarrow \quad \frac{\big(\bar{F}_{\mathbb{P}}^{-1}(u)/\theta\big)^\gamma}{1+\big(\bar{F}_{\mathbb{P}}^{-1}(u)/\theta\big)^\gamma} = \beta^{-1}\big(1-u; \tau, \alpha\big),$$

where the expression of the right-hand side is obtained by rearranging the terms and taking the inverse incomplete beta function β^{-1}. Using this expression in (1.5.9) gives us the GB2 distortion :

$$g_X^{\mathbb{Q},\mathbb{P}}(u) = 1 - \beta\big(\beta^{-1}(1-u; \tau, \alpha); \tilde{\tau}, \tilde{\alpha}\big), \tag{1.5.10}$$

where $\tilde{\tau} \equiv \tau + a/\gamma$ and $\tilde{\alpha} \equiv \alpha - a/\gamma$.

An interesting special case of the GB2 distortion is obtained for $\tau = \alpha = 1$, for which it is a straightforward exercice to prove that we obtain $g_X^{\mathbb{Q},\mathbb{P}}(u) = 1 - \beta\big(1-u; 1+a/\gamma, 1-a/\gamma\big)$. This distortion is a variant of the beta transform in Wirch and Hardy (1999). In this paper, the authors argue that the corresponding distortion risk measure has advantages over the expected shortfall because it utilizes the whole distribution rather than focusing only the tail.

An even more interesting property of the GB2 distortion is that it reduces to the Wang transform $g_X^{\mathbb{Q},\mathbb{P}}(u) = \Phi\big(\Phi^{-1}(u) + a\sigma\big)$ in the log-normal limiting case stated in Remark 1.5.1.[8] The GB2 distortion can therefore be seen as a very flexible generalization of the Wang transform that can account for heavier tails. It is a well-known fact that correctly modelling the tail behaviour is crucially important for premium calculation and risk measurement. Indeed, the Wang transform must be modified to capture the heavy tail feature.

8. Let X be a log-normal variable distributed with density $\mathbf{logN}(x;\mu,\sigma) \equiv \phi\big(\frac{\ln x - \mu}{\sigma}\big)/(\sigma x)$, $x \geq 0$. It can be shown that $\frac{x^a}{\mathbb{E}^{\mathbb{P}}[X^a]}\mathbf{logN}(x;\mu,\sigma) = \mathbf{logN}(x;\mu + a\sigma^2,\sigma)$. The distortion operator, obtained in a similar fashion as in Section 1.5.1, turns out to be the Wang transform $g_X^{\mathbb{Q},\mathbb{P}}(u) = \Phi\big(\Phi^{-1}(u) + a\sigma\big)$.