Chapter 3

Option pricing under regime-switching models : Novel approaches removing path-dependence

Résumé

Une approche connue pour la tarification des options dans le cadre de modèles à changement de régime consiste à adapter le principe de Girsanov. Une façon d'incorporer l'incertitude de régime consiste alors à calculer les probabilités des régimes sous cette mesure de probabilité neutre au risque. Cet article montre qu'une telle approche, bien que naturelle, engendre des problèmes de dépendance au chemin dans les prix d'options vanilles. Nous argumentons que cette propriété est contre-intuitive et indésirable. Ce travail développe des mesures neutres au risque intuitives pouvant incorporer de manière simple l'aversion au risque de régime et qui n'entraînent pas de tels effets secondaires de dépendance au chemin. Des schémas numériques basés sur la programmation dynamique ainsi que des méthodes de simulations Monte-Carlo pour calculer les prix des options sont présentés pour ces nouvelles mesures neutres au risque.

Abstract

A well-known approach for the pricing of options under regime-switching models is to use the extended Girsanov principle to obtain risk-neutrality. One way to handle regime uncertainty consists in using regime probabilities that are filtered under this risk-neutral measure to compute risk-neutral expected payoffs. The current paper shows that this natural approach creates path-dependence issues within option price dynamics. Indeed, since the underlying asset price can be embedded in a Markov process under the physical measure even when regimes are unobservable, such path-dependence behavior of vanilla option prices is puzzling and may entail non-trivial theoretical features (e.g., time non-separable preferences) in a way that is difficult to characterize. This work develops novel and intuitive risk-neutral measures that can incorporate regime risk-aversion in a simple fashion and which do not lead to such path-dependence side effects. Numerical schemes either based on dynamic programming or Monte-Carlo simulations to compute option prices under the novel risk-neutral dynamics are presented.

Keywords : Option pricing, Regime-switching models, Hidden Markov models, Esscher transform, Path-dependence.

3.1 Introduction

Since their introduction in the economics literature by Hamilton (1989), regime-switching models have received extensive attention in the context of derivatives pricing. This can be explained by the ability of regime-switching models to reproduce stylized facts of financial log-returns such as fat tails, volatility clusters and momentum, see for instance Ang and Timmermann (2012). In particular, regime-switching models are used to price long-dated options such as those embedded in variable annuities, see Hardy (2003). Regime-switching models are sensible choices in such circumstances since the underlying asset of a long-dated option might go through multiple business cycles or varying financial conditions throughout the life of the option. Moreover, regime-switching dynamics allow recovering volatility smiles exhibited by empirical option prices, see Ishijima and Kihara (2005) and Yao et al. (2006).

The usual route to obtain a risk-neutral measure in the context of regime-switching models is to use the extended Girsanov principle (also sometimes referred to as the mean-correcting transform or the regime-switching Esscher transform) which preserves the model specification and shifts the drift to the risk-free rate, see for instance Hardy (2001) and Buffington and Elliott (2002b). Elliott et al. (2005) provide a justification for using the latter transform by showing that it leads to the minimal entropy martingale measure. In previous works, the Girsanov transform is often applied under the assumption of observable regimes. Failing to recognize that latent variables are unobserved can lead to systematic bias in option prices, see Bégin and Gauthier (2017). To handle regime latency, the typical approach found for instance in Liew and Siu (2010) is to compute the filtered risk-neutral distribution of the hidden regimes to obtain weights for derivatives prices associated with each regime which lead to a price in the context of regime uncertainty.

The current paper shows that combining the usual Girsanov transform with the risk-neutral filter in the context of regime-switching models provides price dynamics exhibiting path-dependence even though the underlying asset price can be embedded in a Markov process under the physical measure. Such a feature points toward non-trivial theoretical implications such as time non-separable preferences as in Garcia et al. (2003). Moreover, the interpretation for the time evolution mechanism of risk-neutral regime probabilities in terms of the underlying asset price movements is not very clear. Modeling option prices from a dynamic perspective rather than a static one is very important since such dynamic models are embedded into dynamic hedging performance assessment models, see for instance Trottier et al. (2017).

In the current paper, alternative risk-neutral measures which possess intuitive properties and remove the path-dependence feature are developed. A first approach is a modified version of the regime-switching Girsanov transform that leads to the construction of a wide class of risk-neutral measures by engineering a dynamic transition matrix so as to yield option prices exhibiting the Markov property. Such risk-neutral measures are obtained by the successive alteration of transition probabilities and of the underlying asset drift. A second approach explores two different families of martingale measures whose Radon-Nikodym derivatives are measurable given the partial observable information. For the latter measures, option prices exhibit the Markov property, and furthermore the conditional distribution of the past (unobservable) regime trajectory given the asset full trajectory set is left unaltered. The latter property is consistent with the interpretation of a risk-neutral measure as a representation of aggregate risk-aversion and other determinants of equilibrium prices ; these factors should not distort past risk distributions given the full asset trajectory. Under all of our introduced martingale measures, option prices can be calculated simply either through a dynamic program or a Monte-Carlo simulation.

Several other interesting papers from the regime-switching option pricing literature should be mentioned. Classical regime-switching dynamics were expanded by incorporating jumps, see Naik (1993), Elliott et al. (2007) and Elliott and Siu (2013), or GARCH feedback effects (Duan et al., 2002). European options are priced in a Gaussian regime-switching setting using quadratic global hedging in Rémillard et al. (2017). Multiple types of derivatives were priced such as American options (Buffington and Elliott, 2002a), perpetual American options (Zhang and Guo, 2004), barrier options (Jobert and Rogers, 2006; Ranjbar and Seifi, 2015), and other exotic options such as Asian and lookback options (Boyle and Draviam, 2007). The incorporation to the market of an additional asset providing payoffs at regime switches which allows completing the market is investigated in Guo (2001) and Fuh et al. (2012). The partial differential equations approach to price derivatives in regime-switching markets is presented in Mamon and Rodrigo (2005). Di Masi et al. (1995) investigate mean-variance hedging in the presence of regimes. Various numerical schemes were developed to price options in the regime-switching context, such as trees (Bollen, 1998; Yuen and Yang, 2009), and the fast Fourier transform (Liu et al., 2006). Finally, alternative approaches to pricing such as equilibrium and stochastic games are considered in Garcia et al. (2003) and Shen and Siu (2013).

The paper continues as follows. Section 3.2 introduces the regime-switching market. Section 3.3 illustrates the use of the mean-correcting transform to price options under regime uncertainty. The non-Markov behavior of option prices under such a transform is discussed. Section 3.4 introduces a wide class of risk-neutral measures based on the successive alteration of transition probabilities and of the underlying asset drift. Section 3.5 explores two different families of martingale measures whose Radon-Nikodym derivatives are measurable given the asset trajectory. Section 3.6 concludes.

3.2 Regime-switching market

This section introduces the regime-switching market model. We adopt the shorthand notation $x_{1:n} \equiv (x_1, \ldots, x_n)$, and denote the conditional PDF of random variables X given Y by $f_{X|Y}$.

3.2.1 Regime-switching model

Consider a discrete time space $\mathcal{T} = \{0, \ldots, T\}$ and a probability space $(\Omega, \mathcal{F}_T, \mathbb{P})$. Define a regime process $h = \{h_t\}_{t=0}^{T-1}$ and an innovation process $z^{\mathbb{P}} = \{z_t^{\mathbb{P}}\}_{t=1}^{T}$ which are independent under $\mathbb{P}$. The process $z^{\mathbb{P}}$ is a strong standardized Gaussian white noise. Possible values for regimes are $h_t(\omega) \in \{1, \ldots, H\}$ for all $\omega \in \Omega$, where H is a positive integer. A risk-free asset is introduced and its price is given by $B_t = e^{rt}$ with r being the constant risk-free rate. A risky asset price process is defined by

$$S_t = S_0 \exp\left(\sum_{j=1}^{t} \epsilon_j \right), \qquad t \in \mathcal{T}, \tag{3.2.1}$$

where the asset log-returns are given by

$$\epsilon_{t+1} = \mu_{h_t} + \sigma_{h_t} z_{t+1}^{\mathbb{P}}, \qquad t \in \{0, \ldots, T-1\}, \tag{3.2.2}$$

for some constants μ_i and σ_i, $i \in \{1, \ldots, H\}$.

The filtrations $\mathcal{G} = \{\mathcal{G}_t\}_{t=0}^{T}$, $\mathcal{H} = \{\mathcal{H}_t\}_{t=0}^{T}$ and $\mathcal{F} = \{\mathcal{F}_t\}_{t=0}^{T}$ are defined as

$$\mathcal{G}_t = \sigma(S_0, \ldots, S_t), \qquad \mathcal{H}_t = \sigma(h_0, \ldots, h_t), \qquad \mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t. \tag{3.2.3}$$

$\mathcal{G}$ and $\mathcal{H}$ are sub-σ-algebras of $\mathcal{F}$. The filtration $\mathcal{G}$ is referred to as the partial information whereas $\mathcal{F}$ is called the full information. In practice, investors only have access to information $\mathcal{G}_t$ at time t as regimes are hidden variables.

A standard assumption in the literature is to assume the regime process h is a Markov chain. We therefore assume that for all $j \in \{1, \ldots, H\}$,

$$\mathbb{P}[h_{t+1} = j | \mathcal{G}_{t+1} \vee \mathcal{H}_t] = P_{h_t, j}, \tag{3.2.4}$$

where $P_{k,j}$ represents the probability of a transition $k \to j$ of the Markov chain h. This implies

$$\mathbb{P}[h_{t+1} = j | \mathcal{F}_t] = P_{h_t, j}.$$

This framework is known as a regime-switching (RS) model. The joint mixed PDF of $\big(\epsilon_{1:T}, h_{0:T-1}\big)$ under such model is (proof in Appendix 3.A.1)

$$f^{\mathbb{P}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1}) = f^{\mathbb{P}}_{h_0}(h_0) \prod_{t=2}^{T} P_{h_{t-2},h_{t-1}} \prod_{t=1}^{T} \phi^{\mathbb{P}}_{h_{t-1}}(\epsilon_t), \tag{3.2.5}$$

where we have introduced the functions $\phi^{\mathbb{P}}_i$, $i \in \{1, \dots, H\}$, which are defined as

$$\phi^{\mathbb{P}}_i(x) \equiv \frac{1}{\sigma_i}\phi\bigg(\frac{x-\mu_i}{\sigma_i}\bigg), \qquad x \in \mathbb{R}, \tag{3.2.6}$$

with $\phi(z) \equiv \frac{e^{-z^2/2}}{\sqrt{2\pi}}$ denoting the standard normal PDF. Hence, $\phi^{\mathbb{P}}_i$ is the Gaussian density with mean μ_i and variance σ^2_i.

3.2.2 Regime mass function

Following François et al. (2014), we introduce $\eta^{\mathbb{P}} = \{\eta^{\mathbb{P}}_t\}_{t=0}^{T}$ where $\eta^{\mathbb{P}}_t = \big(\eta^{\mathbb{P}}_{t,1}, \dots, \eta^{\mathbb{P}}_{t,H}\big)$ is defined as the regime mass function process, or filtered density, with respect to the partial information :

$$\eta^{\mathbb{P}}_{t,j} \equiv \mathbb{P}[h_t = j | \mathcal{G}_t], \qquad j \in \{1, \dots, H\}. \tag{3.2.7}$$

The random vector $\eta^{\mathbb{P}}_t = \big(\eta^{\mathbb{P}}_{t,1}, \dots, \eta^{\mathbb{P}}_{t,H}\big)$ determines what are the probabilities at time t that the regime process is currently in each respective possible regime given the observable information.

François et al. (2014) show that the process $\eta^{\mathbb{P}}$ can be computed through the following recursion :

$$\eta^{\mathbb{P}}_{t+1,i} = \frac{\sum_{j=1}^{H} P_{j,i}\, \phi^{\mathbb{P}}_j(\epsilon_{t+1})\eta^{\mathbb{P}}_{t,j}}{\sum_{\ell=1}^{H}\sum_{j=1}^{H} P_{j,\ell}\, \phi^{\mathbb{P}}_j(\epsilon_{t+1})\eta^{\mathbb{P}}_{t,j}} = \frac{\sum_{j=1}^{H} P_{j,i}\, \phi^{\mathbb{P}}_j(\epsilon_{t+1})\eta^{\mathbb{P}}_{t,j}}{\sum_{j=1}^{H} \phi^{\mathbb{P}}_j(\epsilon_{t+1})\eta^{\mathbb{P}}_{t,j}}, \qquad i \in \{1, \dots, H\}. \tag{3.2.8}$$

A direct consequence of this relation is the following proposition.

Proposition 3.2.1 (François et al. 2014)**.** *The joint process $\big\{\big(S_t, \eta^{\mathbb{P}}_t\big)\big\}_{t=0}^{T}$ has the Markov property with respect to the filtration $\mathcal{G}$ under the physical measure $\mathbb{P}$.*

The conditional density of the stock price process under $\mathbb{P}$ is

$$f^{\mathbb{P}}_{S_{t+1}|S_{0:t}}(s|S_{0:t}) = \sum_{k=1}^{H} \eta^{\mathbb{P}}_{t,k} \frac{1}{s\sqrt{2\pi\sigma^2_k}} \exp\left(-\frac{\big[\log(s/S_t) - \mu_k\big]^2}{2\sigma^2_k}\right), \qquad s \geq 0, \tag{3.2.9}$$

which is a mixture of log-normal distributions with mixing weights $\eta^{\mathbb{P}}_t$.

3.3 The RS mean-correcting martingale measure

This section illustrates the traditional approach to option pricing based on a regime-switching version of the mean-correcting martingale measure as in Hardy (2001) and Elliott et al. (2005). This procedure is shown to entail complicated counterintuitive theoretical features such as non-Markovian option price dynamics even though the underlying asset price process can be embedded in a Markov process under the physical measure.

3.3.1 Constructing the RS mean-correcting martingale measure

Consider a European-type contingent claim whose payoff at time T is $\Psi(S_T)$, for some non-negative real function Ψ. For instance, a call option has a payoff $\Psi(S_T) = \max(S_T - K, 0)$ where $K \geq 0$ is the strike price. The problem considered in the current paper is to identify a suitable price process $\Pi = \{\Pi_t\}_{t=0}^{T}$ for the contingent claim, where Π_t represents the contingent claim price at time t. Since regimes are unobservable, only prices Π_t that are $\mathcal{G}_t$-measurable are considered as prices cannot depend on information that is unavailable to investors. This approach is different from the one of Hardy (2001) where the option price depends on the currently prevailing regime.

Define $\mathcal{Q}$ as the set of all probability measures $\mathbb{Q}$ that are equivalent to $\mathbb{P}$ and such that the discounted price process $\{e^{-rt}S_t\}_{t=0}^{T}$ is a $\mathcal{G}$-martingale under the measure $\mathbb{Q}$. Such probability measures are referred to as martingale measures. A well known result from option pricing theory (see, e.g., Harrison and Kreps, 1979, for a proof) is that the set of all pricing processes which do not generate arbitrage opportunities is characterized by

$$\left\{\Pi^{\mathbb{Q}} = \left\{e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\Psi(S_T)|\mathcal{G}_t]\right\}_{t=0}^{T} : \mathbb{Q} \in \mathcal{Q}\right\}.$$

Because the market is incomplete under the regime-switching framework, an infinite number of martingale measures exist and solutions to the option pricing problem are thus not unique.

A common approach is to select a particular martingale measure under which the asset price dynamics remains in the same class of models. This approach is followed for instance by Hardy (2001) who considers a martingale measure under which the risky asset price returns are still a Gaussian regime-switching process with transition probabilities $P_{i,j}$, but where the drift in each regime μ_i is replaced by $r - \frac{1}{2}\sigma_i^2$. Such a martingale measure can be constructed using a regime-switching mean-correcting change of measure following the lines of Elliott et al. (2005) who perform a similar exercise in a continuous-time framework. Replacing μ_i by $r - \frac{1}{2}\sigma_i^2$ in (3.2.6), the joint mixed PDF of returns and regimes under such a risk-neutral measure $\mathbb{Q}$ is

$$f^{\mathbb{Q}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1}) = f^{\mathbb{P}}_{h_0}(h_0)\prod_{t=2}^{T} P_{h_{t-2},h_{t-1}} \prod_{t=1}^{T} \phi^{\mathbb{Q}}_{h_{t-1}}(\epsilon_t), \tag{3.3.1}$$

where the functions $\phi_i^{\mathbb{Q}}$, $i \in \{1, \ldots, H\}$, are defined as

$$\phi_i^{\mathbb{Q}}(x) \equiv \frac{1}{\sigma_i}\phi\left(\frac{x - r + \frac{1}{2}\sigma_i^2}{\sigma_i}\right), \qquad x \in \mathbb{R}. \tag{3.3.2}$$

An assumption implicit to (3.3.1) is that the distribution of the initial regime h_0 is left untouched by the change of measure i.e. $f_{h_0}^{\mathbb{P}} = f_{h_0}^{\mathbb{Q}}$.

The following result (proven in the Online Appendix 3.D.1) shows how to create a new probability measure under which the underlying asset price and regimes dynamics matches the desired one.

Proposition 3.3.1. *Consider any joint mixed PDF for $(\epsilon_{1:T}, h_{0:T-1})$ denoted by $f_{\epsilon_{1:T},h_{0:T-1}}^{\mathbb{Z}}$. Then the measure defined by $\mathbb{Z}[A] \equiv \mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_A \frac{d\mathbb{Z}}{d\mathbb{P}}\right]$, for all $A \in \mathcal{F}_T$, where*

$$\frac{d\mathbb{Z}}{d\mathbb{P}} \equiv \frac{f_{\epsilon_{1:T},h_{0:T-1}}^{\mathbb{Z}}(\epsilon_{1:T}, h_{0:T-1})}{f_{\epsilon_{1:T},h_{0:T-1}}^{\mathbb{P}}(\epsilon_{1:T}, h_{0:T-1})}, \tag{3.3.3}$$

is a probability measure. $\mathbb{Z}$ is equivalent to $\mathbb{P}$ if and only if $f_{\epsilon_{1:T},h_{0:T-1}}^{\mathbb{Z}}(\epsilon_{1:T}, h_{0:T-1})$ is strictly positive almost surely. Furthermore, the joint mixed PDF of $(\epsilon_{1:T}, h_{0:T-1})$ under $\mathbb{Z}$ is $f_{\epsilon_{1:T},h_{0:T-1}}^{\mathbb{Z}}$.

By Theorem 3.3.1, we thus consider the measure $\mathbb{Q}$ generated by the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{f_{\epsilon_{1:T},h_{0:T-1}}^{\mathbb{Q}}(\epsilon_{1:T}, h_{0:T-1})}{f_{\epsilon_{1:T},h_{0:T-1}}^{\mathbb{P}}(\epsilon_{1:T}, h_{0:T-1})}, \tag{3.3.4}$$

where $f_{\epsilon_{1:T},h_{0:T-1}}^{\mathbb{P}}$ and $f_{\epsilon_{1:T},h_{0:T-1}}^{\mathbb{Q}}$ are defined as before; see (3.2.5) and (3.3.1). Simplifying yields (see Online Appendix 3.D.3)

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \prod_{t=1}^{T} \xi_t, \qquad \xi_t = e^{z_t^{\mathbb{P}}\lambda_t - \frac{1}{2}\lambda_t^2}, \tag{3.3.5}$$

where

$$\lambda_t \equiv -\frac{\mu_{h_{t-1}} - r + \frac{1}{2}\sigma_{h_{t-1}}^2}{\sigma_{h_{t-1}}}. \tag{3.3.6}$$

From (3.2.2), defining $z_t^{\mathbb{Q}} \equiv z_t^{\mathbb{P}} - \lambda_t$ yields

$$\epsilon_{t+1} = r - \frac{1}{2}\sigma_{h_t}^2 + \sigma_{h_t} z_{t+1}^{\mathbb{Q}}.$$

By Theorem 3.3.1, the joint PDF of $(\epsilon_{1:T}, h_{0:T-1})$ under $\mathbb{Q}$ is $f_{\epsilon_{1:T},h_{0:T-1}}^{\mathbb{Q}}$. Furthermore,

- $\{z_t^{\mathbb{Q}}\}_{t=1}^{T}$ are independent standard Gaussian random variables under $\mathbb{Q}$,
- $\{z_t^{\mathbb{Q}}\}_{t=1}^{T}$ and $\{h_t\}_{t=0}^{T-1}$ are independent processes under $\mathbb{Q}$,
- $\mathbb{Q}[h_{t+1} = j|\mathcal{G}_{t+1} \vee \mathcal{H}_t] = \mathbb{Q}[h_{t+1} = j|\mathcal{F}_t] = P_{h_t,j}$.

3.3.2 Contingent claim pricing

The joint process $\{(S_t, h_t)\}_{t=0}^{T}$ possesses the Markov property under $\mathbb{Q}$ with respect to the filtration $\mathcal{F}$. The contingent claim price is thus given by

$$\begin{aligned}
\Pi_t^{\mathbb{Q}} &= \mathbb{E}^{\mathbb{Q}}\big[e^{-r(T-t)}\Psi(S_T)\big|\mathcal{G}_t\big], \\
&= \mathbb{E}^{\mathbb{Q}}\Big[\mathbb{E}^{\mathbb{Q}}\big[e^{-r(T-t)}\Psi(S_T)\big|\mathcal{F}_t\big]\Big|\mathcal{G}_t\Big], \\
&= \mathbb{E}^{\mathbb{Q}}\big[g_t(S_t, h_t)\big|\mathcal{G}_t\big], \quad \text{by the Markov property of } \{(S_t, h_t)\}_{t=0}^{T}, \\
&= \sum_{k=1}^{H} \eta_{t,k}^{\mathbb{Q}}\, g_t(S_t, k),
\end{aligned} \tag{3.3.7}$$

where $\eta_{t,j}^{\mathbb{Q}} \equiv \mathbb{Q}\left[h_t = j|\mathcal{G}_t\right]$, and with g_t, $t \in \{0, \dots, T\}$, being real functions characterized by the following dynamic programming scheme starting with $g_T(s,k) = \Psi(s)$:

$$g_t(s,k) = \sum_{\ell=1}^{H} P_{k,\ell} \int_{-\infty}^{\infty} g_{t+1}\Big(se^{r-\sigma_k^2/2+\sigma_k z}, \ell\Big) \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz, \qquad t \in \{0, \dots, T-1\}.$$

For European options, i.e., for $\Psi(s) = \max(s-K, 0)$, Hardy (2001) provides an explicit expression for g_t in the two regimes case.

The formula (3.3.7) illustrates the path-dependence feature generated by the RS mean-correcting transform. At time t, for an investor, $(S_t, \eta_t^{\mathbb{P}})$ completely characterizes the likelihood of every possible future scenarios under the physical measure $\mathbb{P}$ due to the Markov property of $(S, \eta^{\mathbb{P}})$ with respect to the partial information $\mathcal{G}$. Indeed, $f^{\mathbb{P}}_{S_{t+1:T}|\mathcal{G}_t} = f^{\mathbb{P}}_{S_{t+1:T}|S_t,\eta_t^{\mathbb{P}}}$. It would be intuitive to expect that the option price at time t would be measurable with respect to $\sigma(S_t, \eta_t^{\mathbb{P}})$. This is however not the case with the RS mean-correcting transform as $\Pi_t^{\mathbb{Q}}$ is a function of $\eta_t^{\mathbb{Q}}$ which is not $\sigma(S_t, \eta_t^{\mathbb{P}})$-measurable in general since it depends on the whole path $S_0, \dots, S_t$. The option price $\Pi_t^{\mathbb{Q}}$ therefore exhibits path-dependence (non-Markovian behavior) although the underlying asset payoff can be expressed as a function of the last observation of the $\mathcal{G}$-Markov process $(S, \eta^{\mathbb{P}})$ under $\mathbb{P}$. This leads us to question the appropriateness of the RS mean-correcting transform applied to regime-switching models when regimes are latent ; it creates path-dependence in option prices when it would be reasonable to expect these to exhibit the Markov property. The Online Appendix 3.B further illustrates the path-dependence feature in a simplified setting.

3.3.3 Stochastic discount factor representation

The path-dependence feature can be visualized through a *stochastic discount factor* (SDF) representation. As shown in the Online Appendix 3.D.2, prices obey the following relationship :

$$\Pi_t^{\mathbb{Q}} = \mathbb{E}^{\mathbb{P}}\big[\Pi_{t+1}^{\mathbb{Q}} m_{t+1}^{\mathbb{Q}} \big| \mathcal{G}_t\big], \qquad m_{t+1}^{\mathbb{Q}} = e^{-r} \frac{\sum_{i=1}^{H} \eta_{t,i}^{\mathbb{Q}} \, \phi_i^{\mathbb{Q}}(\epsilon_{t+1})}{\sum_{i=1}^{H} \eta_{t,i}^{\mathbb{P}} \, \phi_i^{\mathbb{P}}(\epsilon_{t+1})}. \tag{3.3.8}$$

Therefore, the SDF $m_t^{\mathbb{Q}}$ is not $\sigma(\epsilon_t, \eta_{t-1}^{\mathbb{P}})$-measurable. Pricing under $\mathbb{Q}$ in fact entails weighing prices at time $t+1$ based on the risk-neutral filtered regime probabilities $\eta_t^{\mathbb{Q}}$, and thus in a path-dependent fashion. This could point to complicated theoretical implications such as time non-separable preferences as in Garcia et al. (2003).

3.4 A new family of RS mean-correcting martingale measures

This section shows how the concept of regime-switching mean-correcting change of measure can be adapted to yield a $\sigma(S_t, \eta_t^{\mathbb{P}})$-measurable time-$t$ option price. The key takeaway is that the statistical properties of the regime process must be altered in suitable ways, i.e., so as to remove non-Markovian effects.

3.4.1 General construction of an alternative martingale measure

The joint mixed PDF of $(\epsilon_{1:T}, h_{0:T-1})$ under any probability measure $\mathbb{M}$ can be expressed as

$$\begin{aligned} f^{\mathbb{M}}_{\epsilon_{1:T}, h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1}) \;=\;& f^{\mathbb{M}}_{h_0}(h_0) f^{\mathbb{M}}_{\epsilon_1|h_0}(\epsilon_1|h_0) \times \\ & \prod_{t=2}^{T} f^{\mathbb{M}}_{h_{t-1}|h_{0:t-2},\epsilon_{1:t-1}}\left(h_{t-1}|h_{0:t-2}, \epsilon_{1:t-1}\right) f^{\mathbb{M}}_{\epsilon_t|h_{0:t-1},\epsilon_{1:t-1}}(\epsilon_t|h_{0:t-1}, \epsilon_{1:t-1}). \end{aligned} \tag{3.4.1}$$

To obtain the martingale property, we apply a RS mean correction, i.e., we impose that conditionally on the current regime h_{t-1}, the distribution of the log-return ϵ_t is still Gaussian with a variance equal to the physical one and a mean of $r - \frac{1}{2}\sigma^2_{h_{t-1}}$. Therefore,

$$f^{\mathbb{M}}_{\epsilon_t|h_{0:t-1},\epsilon_{1:t-1}} = \phi^{\mathbb{Q}}_{h_{t-1}}, \qquad t \geq 1, \tag{3.4.2}$$

where $\phi_i^{\mathbb{Q}}$, $i \in \{1, \ldots, H\}$, is defined as before ; see (3.3.2).

Alterations on transition probabilities of the regime process are applied to remove non-Makovian effects on option prices. Consider a multivariate process $\psi = \{\psi_t\}_{t=1}^{T-1}$ where $\psi_t = \left[\psi_t^{(i,j)}\right]_{i,j=1}^{H}$ is a $\mathcal{G}_t$-measurable $H \times H$ random matrix for all $t \in \{0, \ldots, T-1\}$. Transition probabilities of the following form are assumed under $\mathbb{M}$:

$$f^{\mathbb{M}}_{h_{t-1}|h_{0:t-2},\epsilon_{1:t-1}}(h_{t-1}|h_{0:t-2}, \epsilon_{1:t-1}) = P_{h_{t-2},h_{t-1}} \psi_{t-1}^{(h_{t-2},h_{t-1})}, \qquad t \geq 2. \tag{3.4.3}$$

This imposes that for all $t \in \{1, \ldots, T-1\}$ and all $i, j \in \{1, \ldots, H\}$,

$$\psi_t^{(i,j)} > 0 \text{ almost surely}, \qquad \text{and} \qquad \sum_{j=1}^{H} P_{i,j} \psi_t^{(i,j)} = 1 \text{ almost surely}, \tag{3.4.4}$$

to ensure positiveness and normalization. Note also that the initial mass function of the first regime can be modified from $f_{h_0}^{\mathbb{P}}(h_0)$ to $f_{h_0}^{\mathbb{M}}(h_0)$ during the passage from $\mathbb{P}$ to $\mathbb{M}$.

By Theorem 3.3.1, such a measure $\mathbb{M}$ is constructed by the Radon-Nikodym derivative

$$\frac{d\mathbb{M}}{d\mathbb{P}} = \frac{f_{\epsilon_{1:T}, h_{0:T-1}}^{\mathbb{M}}(\epsilon_{1:T}, h_{0:T-1})}{f_{\epsilon_{1:T}, h_{0:T-1}}^{\mathbb{P}}(\epsilon_{1:T}, h_{0:T-1})} = \frac{f_{h_0}^{\mathbb{M}}(h_0)}{f_{h_0}^{\mathbb{P}}(h_0)} \prod_{t=2}^{T} \psi_{t-1}^{(h_{t-2}, h_{t-1})} \prod_{t=1}^{T} \xi_t, \tag{3.4.5}$$

where ξ_t is defined as in (3.3.5).

As shown in Appendix 3.A.2, the risk-neutral mass function of regimes is given by

$$\eta_{t+1,i}^{\mathbb{M}} \equiv \mathbb{M}[h_{t+1} = i | \mathcal{G}_{t+1}] = \frac{\sum_{j=1}^{H} P_{j,i} \psi_{t+1}^{(j,i)} \phi_j^{\mathbb{Q}}(\epsilon_{t+1}) \eta_{t,j}^{\mathbb{M}}}{\sum_{j=1}^{H} \phi_j^{\mathbb{Q}}(\epsilon_{t+1}) \eta_{t,j}^{\mathbb{M}}}, \qquad t \in \{0, \ldots T-1\}, \tag{3.4.6}$$

with $\eta_{0,i}^{\mathbb{M}} = f_{h_0}^{\mathbb{M}}(i)$.

Using (3.4.2) and (3.4.6), it is straightforward to show that

$$f_{\epsilon_{t+1}|\epsilon_{1:t}}^{\mathbb{M}}(\epsilon_{t+1}|\epsilon_{1:t}) = \sum_{i=1}^{H} \eta_{t,i}^{\mathbb{M}} \phi_i^{\mathbb{Q}}(\epsilon_{t+1}). \tag{3.4.7}$$

Hence, provided that $\eta_t^{\mathbb{M}}$ is $\sigma(\eta_t^{\mathbb{P}})$-measurable for all $t \geq 0$, we have that the $\mathcal{G}_t$-conditional distribution of the log-return ϵ_{t+1} under $\mathbb{M}$ depends exclusively on $\eta_t^{\mathbb{P}}$. Furthermore, $\eta_{t+1}^{\mathbb{P}}$ is a function of $(\epsilon_{t+1}, \eta_t^{\mathbb{P}})$, as shown by (3.2.8). Applying this reasoning recursively, it follows that the $\mathcal{G}_t$-conditional distribution of $\epsilon_{t+1:T}$ under $\mathbb{M}$ depends only on $\eta_t^{\mathbb{P}}$. This leads to the following result :

Proposition 3.4.1. *The joint process $\{(S_t, \eta_t^{\mathbb{P}})\}_{t=0}^{T}$ has the Markov property with respect to the filtration $\mathcal{G}$ under the probability measure $\mathbb{M}$ if $\eta_t^{\mathbb{M}}$ is $\sigma(\eta_t^{\mathbb{P}})$-measurable for all $t \geq 0$.*

Under the conditions stated in the above proposition, it follows that the option price

$$\Pi_t^{\mathbb{M}} = \mathbb{E}^{\mathbb{M}}\left[e^{-r(T-t)} \Psi(S_T) \middle| \mathcal{G}_t\right]$$

is $\sigma(S_t, \eta_t^{\mathbb{P}})$-measurable by the Markov property. A simple way of designing a probability measure $\mathbb{M}$ satisfying such conditions is provided next.

3.4.2 A simple construction of an alternative martingale measure

A special case is obtained by specifying the measure $\mathbb{M}$ through the conditions

$$f_{h_0}^{\mathbb{M}} = f_{h_0}^{\mathbb{P}}, \quad \text{and} \quad \psi_t^{(j,i)} = \frac{\eta_{t,i}^{\mathbb{P}}}{P_{j,i}} \quad \text{almost surely}, \qquad i,j \in \{1,\dots,H\}. \tag{3.4.8}$$

Using (3.4.8) in (3.4.6) yields

$$\eta_t^{\mathbb{M}} = \eta_t^{\mathbb{P}} \quad \text{almost surely}. \tag{3.4.9}$$

The condition from Proposition 3.4.1 requiring $\eta_t^{\mathbb{M}}$ to be $\sigma\big(\eta_t^{\mathbb{P}}\big)$-measurable for all $t \geq 0$ is thus trivially satisfied. As stated in Remark 3.4.1, it turns out that the martingale measure $\mathbb{M}$ obtained in this fashion has an interesting interpretation.

Remark 3.4.1. The martingale measure $\mathbb{M}$ obtained with (3.4.8) can be understood as a sequence of two consecutive changes of measure : one from the physical measure $\mathbb{P}$ to an equivalent measure $\tilde{\mathbb{P}}$ under which the statistical properties of returns are preserved, and another from $\tilde{\mathbb{P}}$ to $\mathbb{M}$ which induces the martingale property through a RS mean correction.

Indeed, assume $\tilde{\mathbb{P}}$ is a probability measure such that for all $t \in \{1,\dots,T\}$ and all $j \in \{1,\dots,H\}$,

$$\tilde{\mathbb{P}}[h_0 = j] = f_{h_0}^{\mathbb{P}}(j), \tag{3.4.10}$$

$$\tilde{\mathbb{P}}[h_t = j|\mathcal{G}_t \vee \mathcal{H}_{t-1}] = \eta_{t,j}^{\mathbb{P}}, \tag{3.4.11}$$

$$f^{\tilde{\mathbb{P}}}_{\epsilon_t|h_{0:t-1},\epsilon_{1:t-1}} = f^{\mathbb{P}}_{\epsilon_t|h_{0:t-1},\epsilon_{1:t-1}} = \phi^{\mathbb{P}}_{h_{t-1}}. \tag{3.4.12}$$

In other words, when passing from $\mathbb{P}$ to $\tilde{\mathbb{P}}$, only the transition probabilities are shifted, from P_{h_{t-1},h_t} to $\eta_{t,h_t}^{\mathbb{P}}$. For such a measure $\tilde{\mathbb{P}}$, it can be shown (see Appendix 3.A.3 for a proof) that

$$f^{\tilde{\mathbb{P}}}_{\epsilon_{t+1}|\mathcal{G}_t} = f^{\mathbb{P}}_{\epsilon_{t+1}|\mathcal{G}_t}. \tag{3.4.13}$$

This implies the joint distribution of log-returns is identical under both $\mathbb{P}$ and $\tilde{\mathbb{P}}$, and thus the change of measure from $\mathbb{P}$ to $\tilde{\mathbb{P}}$ preserves the statistical properties of the underlying asset S. Because regime-switching model adequacy and goodness-of-fit statistical tests are characterized by the distribution of the underlying process, there is no reason why $\mathbb{P}$ might be preferred to $\tilde{\mathbb{P}}$ when a regime-switching model is deemed appropriate for the price dynamics of some asset ; both have the same joint distribution. Thus, $\tilde{\mathbb{P}}$ could even be viewed as the physical measure.

Next, let's see how the change of measure can be decomposed. As shown in Appendix 3.A.4, the joint mixed PDF of $(\epsilon_{1:T}, h_{0:T-1})$ under $\tilde{\mathbb{P}}$ is

$$f^{\tilde{\mathbb{P}}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1}) = f^{\mathbb{P}}_{h_0}(h_0) \prod_{t=2}^{T} \eta^{\mathbb{P}}_{t-1,h_{t-1}} \prod_{t=1}^{T} \phi^{\mathbb{P}}_{h_{t-1}}(\epsilon_t). \tag{3.4.14}$$

This implies the following representation of $\mathbb{M}$:

$$\frac{d\mathbb{M}}{d\mathbb{P}} \equiv \frac{d\mathbb{M}}{d\tilde{\mathbb{P}}} \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}, \tag{3.4.15}$$

where

$$\frac{d\mathbb{M}}{d\tilde{\mathbb{P}}} = \frac{f^{\mathbb{M}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1})}{f^{\tilde{\mathbb{P}}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1})} = \prod_{t=1}^{T} \xi_t, \tag{3.4.16}$$

with ξ_t defined as in (3.3.5), and

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \frac{f^{\tilde{\mathbb{P}}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1})}{f^{\mathbb{P}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1})} = \prod_{t=2}^{T} \frac{\eta^{\mathbb{P}}_{t-1,h_{t-1}}}{P_{h_{t-2},h_{t-1}}}. \tag{3.4.17}$$

Therefore, $\mathbb{M}$ can be constructed by applying a regular Girsanov-type change of drift through (3.4.16) to a measure $\tilde{\mathbb{P}}$ under which the risky asset has the same statistical properties as under the physical measure $\mathbb{P}$. This confirms the statement in Remark 3.4.1. In summary, the regime-switching mean-correcting change of measure can be used to yield Markovian option prices, but it must be applied on $\tilde{\mathbb{P}}$, rather than $\mathbb{P}$.

3.4.3 Incorporating regime uncertainty aversion

The condition (3.4.9) implies that regime uncertainty risk is unpriced as the conditional distribution of the hidden regime h_t is left untouched by the passage from $\mathbb{P}$ to $\mathbb{M}$. The current section illustrates a generalization of the previous method which can incorporate regime uncertainty aversion through a so-called *conversion function*. Such a function relates $\eta^{\mathbb{M}}$ to $\eta^{\mathbb{P}}$ by applying a distortion to the regime mass function process.

Definition 3.4.1. Consider functions $\zeta_k : [0,1]^H \to [0,1]$, $k \in \{1,\dots,H\}$, having the property

$$\sum_{k=1}^{H} \zeta_k(\eta_1,\dots,\eta_H) = 1, \quad \text{for all } (\eta_1,\dots,\eta_H) \in [0,1]^H \text{ such that } \sum_{i=1}^{H} \eta_i = 1.$$

The function $\zeta = (\zeta_1,\dots,\zeta_H)$ is referred to as a *conversion function*.

The $\psi_t^{(j,i)}$ from (3.4.3) characterizing the martingale measure $\mathbb{M}$ are determined to enforce the chosen conversion :

$$\eta^{\mathbb{M}}_{t,k} = \zeta_k\big(\eta^{\mathbb{P}}_t\big) \text{ almost surely for all } t \text{ and all } k. \tag{3.4.18}$$

By Proposition 3.4.1, path-dependence problems are purged when such a measure $\mathbb{M}$ is used as a martingale measure for pricing. From (3.4.4) and (3.4.6), the above condition involves using $\psi_t^{(i,j)}$ that are solutions of the following linear system of equations, for all $t \geq 1$:

$$\begin{aligned} \frac{\sum_{j=1}^{H} P_{j,i}\psi_t^{(j,i)}\phi_j^{\mathbb{Q}}(\epsilon_t)\zeta_j\big(\eta^{\mathbb{P}}_{t-1}\big)}{\sum_{j=1}^{H}\phi_j^{\mathbb{Q}}(\epsilon_t)\zeta_j\big(\eta^{\mathbb{P}}_{t-1}\big)} &= \zeta_i\big(\eta^{\mathbb{P}}_t\big), \qquad i \in \{1,\dots,H\}, \\ \sum_{j=1}^{H} P_{i,j}\psi_t^{(i,j)} &= 1, \qquad i \in \{1,\dots,H\}. \end{aligned} \tag{3.4.19}$$

The solutions are characterized in the proposition below whose proof is in Appendix 3.A.5.

Proposition 3.4.2. *The system of equations (3.4.19) admits an infinite number of solutions. The trivial solution is*

$$\psi_t^{(j,i)} = \frac{\zeta_i\big(\eta_t^{\mathbb{P}}\big)}{P_{j,i}}, \qquad i, j \in \{1, \ldots, H\}. \tag{3.4.20}$$

A non-trivial solution to the system (3.4.19) is presented in the Online Appendix 3.C.

Examples of conversion functions could include for instance :

- The identity conversion function :

$$\zeta_k(\eta_1, \ldots, \eta_H) = \eta_k, \tag{3.4.21}$$

- The softmax function : for some real constants a_i, b_i, with $i \in \{1, \ldots, H\}$,

$$\zeta_k(\eta_1, \ldots, \eta_H) = \frac{\exp(a_k + b_k \eta_k)}{\sum_{i=1}^{H} \exp(a_i + b_i \eta_i)}. \tag{3.4.22}$$

The identity conversion function case described in Section 3.4.2 would reflect risk-neutrality with respect to regime uncertainty, whereas the softmax function could reflect risk-aversion to regime uncertainty. Values for parameters (a_k, b_k) of the softmax function could be obtained through calibration using market option prices.

3.4.4 Price computation algorithms

Using martingale measures $\mathbb{M}$ described in the current section, options can be priced by means either of Monte-Carlo simulations or a dynamic programming approach. Both methods are outlined below.

Monte-Carlo simulations

A fairly simple recipe to simulate log-returns ϵ_t within a Monte-Carlo simulation under the measure $\mathbb{M}$ is given : at each $t = 0, \ldots, T-1$,

1. Calculate $\eta_t^{\mathbb{P}}$ from (3.2.8),
2. Calculate $\eta_{t,i}^{\mathbb{M}} = \zeta_i\big(\eta_t^{\mathbb{P}}\big)$, for $i \in \{1, \ldots, H\}$,
3. Draw ϵ_{t+1} from the Gaussian mixture (3.4.7).

Dynamic program

Dynamic programming can be used to price simple contingent claims. By Proposition 3.4.1, the option price is $\sigma\big(S_t, \eta_t^{\mathbb{P}}\big)$-measurable since (3.4.18). Hence

$$\Pi_t^{\mathbb{M}} = \mathbb{E}^{\mathbb{M}}\big[e^{-r(T-t)}\Psi(S_T)\big|\mathcal{G}_t\big] = \pi_t^{\mathbb{M}}\big(S_t, \eta_t^{\mathbb{P}}\big),$$

for some real functions $\pi_0^{\mathbb{M}}, \ldots, \pi_T^{\mathbb{M}}$.

The functions $\pi_0^{\mathbb{M}}, \ldots, \pi_T^{\mathbb{M}}$ can be computed through a simple dynamic program provided by Proposition 3.4.3 which is proven in Appendix 3.A.6.

Proposition 3.4.3. *For $i \in \{1, \ldots, H\}$ and $t \in \{0, \ldots, T-1\}$, define the functions*

$$\chi_{t+1,i}(\eta, \epsilon) \equiv \frac{\sum_{j=1}^{H} P_{j,i}\, \phi_j^{\mathbb{P}}(\epsilon)\eta_j}{\sum_{j=1}^{H} \phi_j^{\mathbb{P}}(\epsilon)\eta_j} \tag{3.4.23}$$

and

$$\chi_{t+1}(\eta_t^{\mathbb{P}}, \epsilon_{t+1}) = \Big(\chi_{t+1,1}(\eta_t^{\mathbb{P}}, \epsilon_{t+1}), \ldots, \chi_{t+1,H}(\eta_t^{\mathbb{P}}, \epsilon_{t+1})\Big). \tag{3.4.24}$$

Then, for any $t \in \{0, \ldots, T-1\}$ and any possible value of S_t and $\eta_t^{\mathbb{P}}$:

$$\pi_t^{\mathbb{M}}(S_t, \eta_t^{\mathbb{P}}) = e^{-r} \sum_{k=1}^{H} \zeta_k(\eta_t^{\mathbb{P}}) \int_{-\infty}^{\infty} \pi_{t+1}^{\mathbb{M}}\Big(S_t e^{r-\sigma_k^2/2+\sigma_k z}, \chi_{t+1}\Big(\eta_t^{\mathbb{P}}, r - \sigma_k^2/2 + \sigma_k z\Big)\Big) \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz, \tag{3.4.25}$$

with $\pi_T^{\mathbb{M}}(S_T, \eta_T^{\mathbb{P}}) = \Psi(S_T)$ where Ψ is the payoff function.

Moreover, the dimension of the pricing functional can be reduced by one as stated below.

Remark 3.4.2. Because $\sum_{k=1}^{H} \eta_{t,k}^{\mathbb{P}} = 1$ almost surely (since they represent probabilities of a sample space partition), the function $\chi_{t+1,i}(\eta, \epsilon)$ only needs to be computed at points where $\eta_1 + \ldots + \eta_H = 1$. Because of this, we can drop $\eta_{t,H}^{\mathbb{P}}$ from the state variables since it is a known quantity when $\eta_{t,1}^{\mathbb{P}}, \ldots, \eta_{t,H-1}^{\mathbb{P}}$ are given. This reduces the dimension of the pricing functional by one since it is possible to write $\pi_t^{\mathbb{M}}(S_t, \eta_t^{\mathbb{P}}) = \bar{g}_t(S_t, \eta_{t,1}^{\mathbb{P}}, \ldots, \eta_{t,H-1}^{\mathbb{P}})$ for some function $\bar{g}_t$, $t \in \mathcal{T}$.

3.5 Martingale measures based on $\mathcal{G}_T$-measurable transforms

There are still some conceptual issues for the approach presented in the previous section. In particular, because the Radon-Nikodym derivative $\frac{d\mathbb{M}}{d\mathbb{P}}$ is not $\mathcal{G}_T$-measurable, there exists events $A \in \mathcal{F}_T$ such that

$$\mathbb{M}[A|\mathcal{G}_T] \neq \mathbb{P}[A|\mathcal{G}_T]. \tag{3.5.1}$$

This means that such risk-neutral measures can alter the likelihood of past regimes given the full asset trajectory. For instance, the most probable regime trajectory could differ significantly under $\mathbb{M}$ (compared to under $\mathbb{P}$). This property might seem counter-intuitive. Indeed, a risk-neutral measure reflects risk-aversion and other considerations that affect equilibrium prices ; as such it might be desirable not to alter the posterior regime distribution when there is no asset risk left, i.e., given $S_{0:T}$.

This section illustrates the construction of martingale measures which leave the $\mathcal{G}_T$-conditional distribution of past regimes unaffected by the change of measure. A first approach relies on the adaptation of the well-known Esscher transform to the latent regimes framework. A second approach, based on a regime-mixture approach, combines features of the Esscher transform and of martingale measures constructed in Section 3.4.

3.5.1 A conditional version of the Esscher transform

The Esscher transform is a popular concept in finance and insurance, and it is therefore relevant to investigate whether it can be adapted to regime-switching models so as to provide a natural solution to path-dependence issues. The Esscher transform presented hereby is a particular case of the general pricing approach under heteroskedasticity of Christoffersen et al. (2009).

The conditional Esscher risk-neutral measure $\widehat{\mathbb{Q}}$ is defined by the Radon-Nikodym derivative

$$\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} = \prod_{t=1}^{T} \widehat{\xi}_t, \qquad \widehat{\xi}_t \equiv e^{-\theta_{t-1}} \left(\frac{S_t}{S_{t-1}} \right)^{\alpha_{t-1}}, \tag{3.5.2}$$

where $\{\theta_t\}_{t=0}^{T}$ and $\{\alpha_t\}_{t=0}^{T}$ are $\mathcal{G}$-adapted processes to be defined. As shown in Appendix 3.A.7, the following condition, which is assumed to hold, ensures that $\widehat{\mathbb{Q}}$ is a probability measure :

$$\theta_t = \log \left(\sum_{k=1}^{H} \eta_{t,k}^{\mathbb{P}} \exp \left(\alpha_t \mu_k + \frac{1}{2} \alpha_t^2 \sigma_k^2 \right) \right). \tag{3.5.3}$$

Moreover, assuming this condition holds, as shown in Appendix 3.A.8, the following condition is necessary and sufficient to ensure that $\widehat{\mathbb{Q}}$ is a risk-neutral measure :

$$\sum_{k=1}^{H} \eta_{t,k}^{\mathbb{P}} \exp \left(\alpha_t \mu_k + \frac{1}{2} \alpha_t^2 \sigma_k^2 \right) \left[1 - \exp \left(\mu_k + \alpha_t \sigma_k^2 + \frac{1}{2} \sigma_k^2 - r \right) \right] = 0. \tag{3.5.4}$$

A solution to this equation always exists since the left hand side tends to minus infinity as $\alpha_t \to \infty$ and to infinity as $\alpha_t \to -\infty$, on top of being a continuous function of α_t. Equation (3.5.4) can be solved numerically to determine α_t ; the solution is a function of $\eta_t^{\mathbb{P}}$, and therefore (θ_t, α_t) is a function of $\eta_t^{\mathbb{P}}$.

Appendix 3.A.9 shows that the distribution of returns under the measure $\widehat{\mathbb{Q}}$ is characterized by

$$\widehat{\mathbb{Q}}[\epsilon_{t+1} \leq x | \mathcal{G}_t] = \sum_{i=1}^{H} \hat{\eta}_{t,i}^{\mathbb{P}} \Phi \left(\frac{x - \mu_i - \alpha_t \sigma_i^2}{\sigma_i} \right), \qquad x \in \mathbb{R}, \tag{3.5.5}$$

where Φ is the standard Gaussian cumulative distribution function, and

$$\hat{\eta}_{t,i}^{\mathbb{P}} = \frac{\eta_{t,i}^{\mathbb{P}} \exp \left(\alpha_t \mu_i + \frac{1}{2} \alpha_t^2 \sigma_i^2 \right)}{\sum_{k=1}^{H} \eta_{t,k}^{\mathbb{P}} \exp \left(\alpha_t \mu_k + \frac{1}{2} \alpha_t^2 \sigma_k^2 \right)}. \tag{3.5.6}$$

The log-returns $\mathcal{G}_t$-conditional distribution under $\widehat{\mathbb{Q}}$ is therefore still a Gaussian mixture with modified mixing weights $\hat{\eta}_t^{\mathbb{P}}$ and means shifted from μ_i to $\mu_i - \alpha_t \sigma_i^2$ for each regime $i \in \{1, \ldots, H\}$. Note that the passage from $\eta_t^{\mathbb{P}}$ to $\hat{\eta}_t^{\mathbb{P}}$ is an instance of a conversion function since α_t is a function of $\eta_t^{\mathbb{P}}$ as shown by (3.5.4).

Equations (3.5.5)-(3.5.6) indicate the $\widehat{\mathbb{Q}}$ distribution of the log-return ϵ_{t+1} given $\mathcal{G}_t$ depends exclusively on $\eta_t^{\mathbb{P}}$ since α_t and $\hat{\eta}_t^{\mathbb{P}}$ are functions of $\eta_t^{\mathbb{P}}$. Furthermore, $\eta_{t+1}^{\mathbb{P}}$ is a function of $(\epsilon_{t+1}, \eta_t^{\mathbb{P}})$; see (3.2.8). Applying this reasoning recursively, it follows that the $\mathcal{G}_t$-conditional distribution of $\epsilon_{t+1:T}$ under $\widehat{\mathbb{Q}}$ depends only on $\eta_t^{\mathbb{P}}$. This leads to the following result :

Proposition 3.5.1. *The joint process $\big\{(S_t, \eta_t^{\mathbb{P}})\big\}_{t=0}^{T}$ has the Markov property with respect to the filtration $\mathcal{G}$ under the probability measure $\widehat{\mathbb{Q}}$.*

This result entails that the option price at time t is $\sigma\big(S_t, \eta_t^{\mathbb{P}}\big)$-measurable as desired. Other desirable theoretical properties satisfied by this measure are outlined in the remark below.

Remark 3.5.1. The risk-neutral measure $\widehat{\mathbb{Q}}$ displays the following desirable properties :

- The option price $\Pi_t^{\widehat{\mathbb{Q}}} = \mathbb{E}^{\widehat{\mathbb{Q}}}\big[e^{-r(T-t)}\Psi(S_T)\big|\mathcal{G}_t\big]$ is $\sigma\big(S_t, \eta_t^{\mathbb{P}}\big)$-measurable.
- $\widehat{\xi}_t$ is $\mathcal{G}_t$-measurable for all $t \in \mathcal{T}$ and therefore $\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} \in \mathcal{G}_T$. Thus, the $\mathcal{G}_T$-conditional distribution of past risks is unaffected by the change of measure : $\widehat{\mathbb{Q}}[A|\mathcal{G}_T] = \mathbb{P}[A|\mathcal{G}_T]$, $\forall A \in \mathcal{F}_T$.
- If the martingale property is already satisfied under $\mathbb{P}$, i.e., $\phi_i^{\mathbb{Q}} = \phi_i^{\mathbb{P}}$ for all $i \in \{1, \ldots, H\}$, then there is no change of measure, i.e., $\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} = 1$ almost surely. [1]
- In the single-regime case ($H = 1$), $\widehat{\mathbb{Q}}$ reduces to the usual Esscher martingale measure $\mathbb{Q}$.

Option pricing schemes

A simple recipe is available to simulate log-returns under the measure $\widehat{\mathbb{Q}}$ within a Monte-Carlo simulation : at each $t = 0, \ldots, T-1$,

1. Calculate $\eta_{t,i}^{\mathbb{P}}$, $i \in \{1, \ldots, H\}$, from (3.2.8),
2. Solve numerically for α_t in (3.5.4),
3. Calculate $\hat{\eta}_{t,i}^{\mathbb{P}}$, $i \in \{1, \ldots, H\}$, from (3.5.6),
4. Draw ϵ_{t+1} from the Gaussian mixture (3.5.5).

Note that the second and third steps can be pre-calculated.

Simple contingent claims can also be priced by dynamic programming. Since the time-t option price is $\sigma\big(S_t, \eta_t^{\mathbb{P}}\big)$-measurable, it follows that for all $t \in \mathcal{T}$ there exists a function $\pi_t^{\widehat{\mathbb{Q}}}$ such that

$$\Pi_t^{\widehat{\mathbb{Q}}} \equiv \mathbb{E}^{\widehat{\mathbb{Q}}}\big[e^{-r(T-t)}\Psi(S_T)\big|\mathcal{G}_t\big] = \pi_t^{\widehat{\mathbb{Q}}}\big(S_t, \eta_t^{\mathbb{P}}\big).$$

1. This is because we then have $\alpha_t = \theta_t = 0$ almost surely for all t.

The dynamic program that enables the recursive computation of the functions $\pi_t^{\widehat{\mathbb{Q}}}$ can be derived following the steps outlined in Section 3.4.4 :

$$\pi_t^{\widehat{\mathbb{Q}}}(S_t, \eta_t^{\mathbb{P}}) = e^{-r} \sum_{k=1}^{H} \hat{\eta}_{t,k}^{\mathbb{P}} \int_{-\infty}^{\infty} \pi_{t+1}^{\widehat{\mathbb{Q}}} \Big(S_t e^{\mu_k - \alpha_t \sigma_k^2 + \sigma_k z}, \chi_{t+1}\Big(\eta_t^{\mathbb{P}}, \mu_k - \alpha_t \sigma_k^2 + \sigma_k z\Big)\Big) \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz, \tag{3.5.7}$$

with $\pi_T^{\widehat{\mathbb{Q}}}(S_T, \eta_T^{\mathbb{P}}) = \Psi(S_T)$ where Ψ is the payoff function, $\hat{\eta}_t^{\mathbb{P}}$ is defined as a function of $\eta_t^{\mathbb{P}}$ through (3.5.6), and χ_{t+1} is defined by (3.4.24).

3.5.2 A regime-mixture transform

We present now a new family of martingale measures based on a regime-mixture approach. A measure from this new family is denoted by $\bar{\mathbb{Q}}$. Similarly to the conditional Esscher transform $\widehat{\mathbb{Q}}$ from Section 3.5.1, the Radon-Nikodym derivative characterizing the new regime-mixture martingale measure $\bar{\mathbb{Q}}$ is $\mathcal{G}_T$-measurable. This implies the $\mathcal{G}_T$-conditional distribution of regimes $h_{0:T-1}$ is left untouched by the change of measure, which can be deemed a desirable property as previously discussed. Moreover, as for RS mean-correcting measures $\mathbb{M}$, the risk-neutral one-period conditional distribution of asset log-returns is a mixture of Gaussian distribution whose mean is the risk-free rate minus the usual convexity correction. The regime-mixture approach therefore combines features of the two families of martingale measures previously considered, namely the new version of the RS mean-correcting measure $\mathbb{M}$ and the conditional Esscher transform $\widehat{\mathbb{Q}}$. We first explain how this measure can be derived.

The PDF of a trajectory under a probability measure $\bar{\mathbb{Q}}$ can be expressed as (see Appendix 3.A.10)

$$f^{\bar{\mathbb{Q}}}_{\epsilon_{1:T}, h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1}) = f^{\bar{\mathbb{Q}}}_{h_{0:T-1}|\mathcal{G}_T}(h_{0:T-1}|\mathcal{G}_T) \prod_{t=1}^{T} f^{\bar{\mathbb{Q}}}_{\epsilon_t|\mathcal{G}_{t-1}}(\epsilon_t|\mathcal{G}_{t-1}). \tag{3.5.8}$$

In comparison, the PDF under $\mathbb{P}$ is given by (see Appendix 3.A.11)

$$f^{\mathbb{P}}_{\epsilon_{1:T}, h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1}) = f^{\mathbb{P}}_{h_{0:T-1}|\mathcal{G}_T}(h_{0:T-1}|\mathcal{G}_T) \prod_{t=1}^{T} \sum_{i=1}^{H} \eta^{\mathbb{P}}_{t-1,i}\, \phi_i^{\mathbb{P}}(\epsilon_t). \tag{3.5.9}$$

The regime-mixture Esscher martingale measure $\bar{\mathbb{Q}}$ is constructed by enforcing

$$f^{\bar{\mathbb{Q}}}_{h_{0:T-1}|\mathcal{G}_T}(h_{0:T-1}|\mathcal{G}_T) \quad = \quad f^{\mathbb{P}}_{h_{0:T-1}|\mathcal{G}_T}(h_{0:T-1}|\mathcal{G}_T), \tag{3.5.10}$$

$$f^{\bar{\mathbb{Q}}}_{\epsilon_t|\mathcal{G}_{t-1}}(\epsilon_t|\mathcal{G}_{t-1}) \quad = \quad \sum_{i=1}^{H} \zeta_i(\eta^{\mathbb{P}}_{t-1}) \phi_i^{\mathbb{Q}}(\epsilon_t), \qquad \forall t \in \{1, \ldots, T\}, \tag{3.5.11}$$

where ζ is the conversion function, and $\phi_i^{\mathbb{Q}}$, $i \in \{1, \ldots, H\}$, is defined as before ; see (3.3.2). The property (3.5.10) states that the $\mathcal{G}_T$-conditional distribution of the regime trajectory is unaltered under $\bar{\mathbb{Q}}$. This is an intuitive feature as previously discussed. The property (3.5.11) states the $\mathcal{G}_{t-1}$-conditional distribution of the log-return ϵ_t under $\bar{\mathbb{Q}}$ is a Gaussian mixture with mixing weights given by the vector $\zeta(\eta^{\mathbb{P}}_{t-1})$, and means shifted from μ_i to $r - \frac{1}{2}\sigma_i^2$ for each regime $i \in \{1, \dots, H\}$. The purpose of the latter condition is to ensure the martingale property is satisfied, and that regime risk is priced according to the chosen conversion function.

As shown in Appendix 3.A.12 the Radon-Nikodym derivative is

$$\frac{d\bar{\mathbb{Q}}}{d\mathbb{P}} = \prod_{t=1}^{T} \bar{\xi}_t, \qquad \bar{\xi}_t \equiv \frac{\sum_{i=1}^{H} \zeta_i(\eta^{\mathbb{P}}_{t-1}) \phi^{\mathbb{Q}}_i(\epsilon_t)}{\sum_{i=1}^{H} \eta^{\mathbb{P}}_{t-1,i}\, \phi^{\mathbb{P}}_i(\epsilon_t)}. \tag{3.5.12}$$

Appendix 3.A.13 shows that the distribution of returns under this measure is characterized by

$$\bar{\mathbb{Q}}[\epsilon_{t+1} \leq x | \mathcal{G}_t] = \sum_{i=1}^{H} \zeta_i(\eta^{\mathbb{P}}_t) \Phi\left(\frac{x - r + \frac{1}{2}\sigma_i^2}{\sigma_i}\right), \qquad x \in \mathbb{R}. \tag{3.5.13}$$

Hence, for any $s = 0, \dots, T-t-1$, the $\mathcal{G}_{t+s}$-conditional distribution of ϵ_{t+1+s} under $\bar{\mathbb{Q}}$ depends only on $\eta^{\mathbb{P}}_{t+s}$. Furthermore, by (3.2.8), $\eta^{\mathbb{P}}_{t+s}$ is a function of $\left(\epsilon_{t+s}, \eta^{\mathbb{P}}_{t+s-1}\right)$. The above reasoning, applied recursively, implies that the $\mathcal{G}_t$-conditional distribution of $\epsilon_{t+1:T}$ under $\bar{\mathbb{Q}}$ depends only on $\eta^{\mathbb{P}}_t$. The next proposition then follows.

Proposition 3.5.2. *The joint process $\left\{(S_t, \eta^{\mathbb{P}}_t)\right\}_{t=0}^{T}$ has the Markov property with respect to the filtration $\mathcal{G}$ under the probability measure $\bar{\mathbb{Q}}$.*

This property entails that the option price $\Pi^{\bar{\mathbb{Q}}}_t = \mathbb{E}^{\bar{\mathbb{Q}}}\left[e^{-r(T-t)}\Psi(S_T)\middle|\mathcal{G}_t\right]$ is $\sigma\left(S_t, \eta^{\mathbb{P}}_t\right)$-measurable. Furthermore, the other properties stated in Remark 3.5.1 also hold for $\bar{\mathbb{Q}}$. Finally, since the underlying asset price joint distribution are identical under $\mathbb{M}$ and $\bar{\mathbb{Q}}$, the pricing algorithms are identical to those given in Section 3.4.4.

3.6 Conclusion

The current work shows that the usual approach to construct martingale measures in a regime-switching framework based on the correction of the drift for each respective regime (i.e., regime-switching mean correction) leads to path-dependence even for vanilla options. More precisely, even if the joint process $(S, \eta^{\mathbb{P}})$ comprising the underlying asset price and the regime mass function given observable information has the Markov property, vanilla derivatives prices at time t would not be a function strictly of the current value of the latter process, i.e., of $(S_t, \eta^{\mathbb{P}}_t)$. The construction of multiple martingale measures possessing intuitive properties and removing the path-dependence feature is illustrated in the current paper.

Our first approach is a modified version of the above concept of RS mean-correcting martingale measure; it also relies on RS mean correction to obtain the martingale property, but with the inclusion of transition probability transforms so as to recuperate the Markov property of option prices. This yields a very wide class of new martingale measures removing the path-dependence. This class includes an interesting special case which can be represented as the successive application of two changes of measures : a first one which allows retaining the exact same underlying asset statistical properties from the physical measure, and then a change of drift on each regime. Obtained generalizations allow for the pricing of regime uncertainty through conversion functions which distort the hidden regime distribution given the currently observed information.

A second approach developed is based on changes of measures whose Radon-Nikodym derivatives are $\sigma(S_0, \ldots, S_T)$-measurable, implying that they do not impact the conditional distribution of the regime hidden trajectory given the full asset trajectory. This approach embeds as a particular case the well-known Esscher transform.

Simple pricing procedures for contingent claims under the developed martingale measures based either on dynamic programming or Monte-Carlo simulations are also provided.

Appendix

3.A Proofs

3.A.1 Proof of Eq. (3.2.5)

$$\begin{aligned}
f^{\mathbb{P}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T},h_{0:T-1}) &= f^{\mathbb{P}}_{\epsilon_1,h_0}(\epsilon_1,h_0)\prod_{t=2}^{T} f^{\mathbb{P}}_{\epsilon_t,h_{t-1}|\epsilon_{1:t-1},h_{0:t-2}}(\epsilon_t,h_{t-1}|\epsilon_{1:t-1},h_{0:t-2}),\\
&= f^{\mathbb{P}}_{h_0}(h_0)f^{\mathbb{P}}_{\epsilon_1|h_0}(\epsilon_1|h_0)\times\\
&\quad \prod_{t=2}^{T} f^{\mathbb{P}}_{\epsilon_t|\epsilon_{1:t-1},h_{0:t-1}}(\epsilon_t|\epsilon_{1:t-1},h_{0:t-1})f^{\mathbb{P}}_{h_{t-1}|\epsilon_{1:t-1},h_{0:t-2}}(h_{t-1}|\epsilon_{1:t-1},h_{0:t-2}),\\
&= f^{\mathbb{P}}_{h_0}(h_0)\prod_{t=2}^{T} P_{h_{t-2},h_{t-1}}\prod_{t=1}^{T}\frac{1}{\sigma_{h_{t-1}}}\phi\left(\frac{\epsilon_t-\mu_{h_{t-1}}}{\sigma_{h_{t-1}}}\right),
\end{aligned}$$

where the last equality follows from (3.2.2) and (3.2.4). Using definition (3.2.6) concludes the proof.

3.A.2 Proof of Eq. (3.4.6)

$$\begin{aligned}
\eta^{\mathbb{M}}_{t+1,i} &= \mathbb{M}\left[h_{t+1}=i|\mathcal{G}_{t+1}\right],\\
&= \sum_{j=1}^{H}\mathbb{M}\left[h_{t+1}=i|\mathcal{G}_{t+1},h_t=j\right]\mathbb{M}\left[h_t=j|\mathcal{G}_{t+1}\right],\\
&= \sum_{j=1}^{H} P_{j,i}\psi^{(j,i)}_{t+1}\frac{f^{\mathbb{M}}_{h_t,\epsilon_{t+1}|\epsilon_{1:t}}(j,\epsilon_{t+1}|\epsilon_{1:t})}{f^{\mathbb{M}}_{\epsilon_{t+1}|\epsilon_{1:t}}(\epsilon_{t+1}|\epsilon_{1:t})}, \qquad \text{from (3.4.3)},\\
&= \sum_{j=1}^{H} P_{j,i}\psi^{(j,i)}_{t+1}\frac{f^{\mathbb{M}}_{h_t|\epsilon_{1:t}}(j|\epsilon_{1:t})f^{\mathbb{M}}_{\epsilon_{t+1}|h_t,\epsilon_{1:t}}(\epsilon_{t+1}|j,\epsilon_{1:t})}{\sum_{k=1}^{H} f^{\mathbb{M}}_{h_t|\epsilon_{1:t}}(k|\epsilon_{1:t})f^{\mathbb{M}}_{\epsilon_{t+1}|h_t,\epsilon_{1:t}}(\epsilon_{t+1}|k,\epsilon_{1:t})},\\
&= \sum_{j=1}^{H} P_{j,i}\psi^{(j,i)}_{t+1}\frac{\eta^{\mathbb{M}}_{t,j}\phi^{\mathbb{Q}}_j(\epsilon_{t+1})}{\sum_{k=1}^{H}\eta^{\mathbb{M}}_{t,k}\phi^{\mathbb{Q}}_k(\epsilon_{t+1})}, \qquad \text{from (3.4.2)}.
\end{aligned}$$

3.A.3 Proof of Eq. (3.4.13)

$$f^{\tilde{\mathbb{P}}}_{\epsilon_{t+1}|\mathcal{G}_t}(x|\mathcal{G}_t) = \sum_{k=1}^{H} f^{\tilde{\mathbb{P}}}_{\epsilon_{t+1},h_t|\mathcal{G}_t}(x,k|\mathcal{G}_t) = \sum_{k=1}^{H} \tilde{\mathbb{P}}[h_t = k|\mathcal{G}_t] f^{\tilde{\mathbb{P}}}_{\epsilon_{t+1}|h_t,\mathcal{G}_t}(x|k,\mathcal{G}_t). \tag{3.A.1}$$

Moreover,

$$\tilde{\mathbb{P}}[h_t = k|\mathcal{G}_t] = \mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{1}_{\{h_t=k\}}|\mathcal{G}_t] = \mathbb{E}^{\tilde{\mathbb{P}}}\Big[\mathbb{E}^{\tilde{\mathbb{P}}}[\mathbb{1}_{\{h_t=k\}}|\mathcal{G}_t,\mathcal{H}_{t-1}]\Big|\mathcal{G}_t\Big] = \mathbb{E}^{\tilde{\mathbb{P}}}\Big[\underbrace{\tilde{\mathbb{P}}[h_t = k|\mathcal{G}_t,\mathcal{H}_{t-1}]}_{=\eta^{\mathbb{P}}_{t,k},\text{ by (3.4.11)}}\Big|\mathcal{G}_t\Big] = \eta^{\mathbb{P}}_{t,k}.$$

Similarly, it can be shown using (3.4.12) that

$$f^{\tilde{\mathbb{P}}}_{\epsilon_{t+1}|h_t,\mathcal{G}_t}(x|k,\mathcal{G}_t) = \phi^{\mathbb{P}}_k(x).$$

Using the above relations in (3.A.1) yields

$$f^{\tilde{\mathbb{P}}}_{\epsilon_{t+1}|\mathcal{G}_t}(x|\mathcal{G}_t) = \sum_{k=1}^{H} \eta^{\mathbb{P}}_{t,k}\phi^{\mathbb{P}}_k(x) = f^{\mathbb{P}}_{\epsilon_{t+1}|\mathcal{G}_t}(x|\mathcal{G}_t),$$

where the last equality is straightforward to prove. Hence, $f^{\tilde{\mathbb{P}}}_{\epsilon_{t+1}|\mathcal{G}_t} = f^{\mathbb{P}}_{\epsilon_{t+1}|\mathcal{G}_t}$.

3.A.4 Proof of Eq. (3.4.14)

$$\begin{aligned}
f^{\tilde{\mathbb{P}}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T},h_{0:T-1}) &= f^{\tilde{\mathbb{P}}}_{\epsilon_1,h_0}(\epsilon_1,h_0)\prod_{t=2}^{T} f^{\tilde{\mathbb{P}}}_{\epsilon_t,h_{t-1}|\epsilon_{1:t-1},h_{0:t-2}}(\epsilon_t,h_{t-1}|\epsilon_{1:t-1},h_{0:t-2}),\\
&= f^{\mathbb{P}}_{h_0}(h_0) f^{\tilde{\mathbb{P}}}_{\epsilon_1|h_0}(\epsilon_1|h_0)\times\\
&\quad \prod_{t=2}^{T} f^{\tilde{\mathbb{P}}}_{\epsilon_t|\epsilon_{1:t-1},h_{0:t-1}}(\epsilon_t|\epsilon_{1:t-1},h_{0:t-1}) f^{\tilde{\mathbb{P}}}_{h_{t-1}|\epsilon_{1:t-1},h_{0:t-2}}(h_{t-1}|\epsilon_{1:t-1},h_{0:t-2}),\\
&= f^{\mathbb{P}}_{h_0}(h_0)\prod_{t=2}^{T}\eta^{\mathbb{P}}_{t-1,h_{t-1}}\prod_{t=1}^{T}\phi^{\mathbb{P}}_{h_{t-1}}(\epsilon_t), \qquad \text{from (3.4.11) and (3.4.12).}
\end{aligned}$$

3.A.5 Proof of Proposition 3.4.2

The system (3.4.19) is equivalent to

$$\sum_{j=1}^{H}\tilde{\psi}^{(j,i)}_t\kappa_{t,j} = 0 \qquad \text{and} \qquad \sum_{j=1}^{H}\tilde{\psi}^{(i,j)}_t = 0, \qquad i\in\{1,\dots,H\},$$

where we have defined

$$\tilde{\psi}^{(j,i)}_t \equiv P_{j,i}\psi^{(j,i)}_t - \zeta_i\big(\eta^{\mathbb{P}}_t\big), \qquad \kappa_{t,j} \equiv \phi^{\mathbb{Q}}_j(\epsilon_t)\zeta_j\big(\eta^{\mathbb{P}}_{t-1}\big).$$

Indeed, the trivial solution is, for all $i, j \in \{1, \ldots, H\}$,

$$\tilde{\psi}_t^{(j,i)} = 0 \qquad \Rightarrow \qquad \psi_t^{(j,i)} = \frac{\zeta_i\big(\eta_t^{\mathbb{P}}\big)}{P_{j,i}}. \tag{3.A.2}$$

The system has H^2 unknown values and $2H$ equations. If $H > 2$, the existence of a solution implies that an infinite number of solutions exist. Even if $H = 2$, we can show there exists an infinite number of solutions.

Indeed, the system can be written as follows for $H = 2$,

$$\underbrace{\begin{bmatrix} \kappa_{t,1} & 0 & \kappa_{t,2} & 0 \\ 0 & \kappa_{t,1} & 0 & \kappa_{t,2} \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{\equiv \mathcal{M}} \begin{bmatrix} \tilde{\psi}_t^{(1,1)} \\ \tilde{\psi}_t^{(1,2)} \\ \tilde{\psi}_t^{(2,1)} \\ \tilde{\psi}_t^{(2,2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since $\det \mathcal{M} = \kappa_{t,1}\kappa_{t,2} - \kappa_{t,1}\kappa_{t,2} = 0$, an infinity of solutions exist by the properties of homogeneous linear systems.

3.A.6 Proof of Proposition 3.4.3

First,

$$\begin{aligned} \pi_t^{\mathbb{M}}\big(S_t, \eta_t^{\mathbb{P}}\big) &= \mathbb{E}^{\mathbb{M}}\Big[e^{-r(T-t)}\Psi(S_T)\Big|\mathcal{G}_t\Big], \\ &= \mathbb{E}^{\mathbb{M}}\Big[e^{-r}\mathbb{E}^{\mathbb{M}}\Big[e^{-r(T-(t+1))}\Psi(S_T)\Big|\mathcal{G}_{t+1}\Big]\Big|\mathcal{G}_t\Big], \\ &= e^{-r}\mathbb{E}^{\mathbb{M}}\Big[\pi_{t+1}^{\mathbb{M}}\big(S_{t+1}, \eta_{t+1}^{\mathbb{P}}\big)\Big|\mathcal{G}_t\Big], \\ &= e^{-r}\sum_{k=1}^{H}\mathbb{M}[h_t = k|\mathcal{G}_t]\,\mathbb{E}^{\mathbb{M}}\Big[\pi_{t+1}^{\mathbb{M}}\big(S_{t+1}, \eta_{t+1}^{\mathbb{P}}\big)\Big|\mathcal{G}_t, h_t = k\Big], \\ &= e^{-r}\sum_{k=1}^{H}\zeta_k\big(\eta_t^{\mathbb{P}}\big)\mathbb{E}^{\mathbb{M}}\Big[\pi_{t+1}^{\mathbb{M}}\big(S_{t+1}, \eta_{t+1}^{\mathbb{P}}\big)\Big|S_t, \eta_t^{\mathbb{P}}, h_t = k\Big], \qquad \text{by (3.4.18)} \end{aligned} \tag{3.A.3}$$

Moreover, from (3.2.8), the definition (3.4.23) implies that

$$\eta_{t+1,i}^{\mathbb{P}} = \chi_{t+1,i}\big(\eta_t^{\mathbb{P}}, \epsilon_{t+1}\big).$$

and thus

$$\eta_{t+1}^{\mathbb{P}} = \chi_{t+1}\big(\eta_t^{\mathbb{P}}, \epsilon_{t+1}\big). \tag{3.A.4}$$

This means

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{M}}\Big[\pi^{\mathbb{M}}_{t+1}(S_{t+1},\eta^{\mathbb{P}}_{t+1})\Big|S_t,\eta^{\mathbb{P}}_t,h_t=k\Big]\\
&=\mathbb{E}^{\mathbb{M}}\Big[\pi^{\mathbb{M}}_{t+1}(S_t e^{\epsilon_{t+1}},\chi_{t+1}(\eta^{\mathbb{P}}_t,\epsilon_{t+1}))\Big|S_t,\eta^{\mathbb{P}}_t,h_t=k\Big],\\
&=\mathbb{E}^{\mathbb{M}}\left[\pi^{\mathbb{M}}_{t+1}\Big(S_t e^{r-\sigma_k^2/2+\sigma_k z^{\mathbb{M}}_{t+1}},\chi_{t+1}\Big(\eta^{\mathbb{P}}_t,r-\sigma_k^2/2+\sigma_k z^{\mathbb{M}}_{t+1}\Big)\Big)\,\Big|S_t,\eta^{\mathbb{P}}_t,h_t=k\right],\\
&=\int_{-\infty}^{\infty}\pi^{\mathbb{M}}_{t+1}\Big(S_t e^{r-\sigma_k^2/2+\sigma_k z},\chi_{t+1}\Big(\eta^{\mathbb{P}}_t,r-\sigma_k^2/2+\sigma_k z\Big)\Big)\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz. \qquad (3.A.5)
\end{aligned}
$$

Combining (3.A.3) and (3.A.5) yields the recursive formula (3.4.25) to obtain the option price $\Pi^{\mathbb{M}}_t=\pi^{\mathbb{M}}_t(S_t,\eta^{\mathbb{P}}_t)$ from $\pi^{\mathbb{M}}_{t+1}$.

3.A.7 Proof of Eq. (3.5.3)

To ensure $\widehat{\mathbb{Q}}$ represents a change of probability measure, the following condition which guarantees that $\mathbb{E}^{\mathbb{P}}\left[\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}\right]=1$ is assumed to hold for all $t\geq 0$:

$$
\begin{aligned}
1 &= \mathbb{E}^{\mathbb{P}}\Big[\widehat{\xi}_{t+1}\Big|\mathcal{G}_t\Big], \qquad (3.A.6)\\
&= e^{-\theta_t}\mathbb{E}^{\mathbb{P}}\left[\left(\frac{S_{t+1}}{S_t}\right)^{\alpha_t}\Big|\mathcal{G}_t\right],\\
&= e^{-\theta_t}\mathbb{E}^{\mathbb{P}}\left[\exp\left(\alpha_t\mu_{h_t}+\alpha_t\sigma_{h_t}z^{\mathbb{P}}_{t+1}\right)\Big|\mathcal{G}_t\right],\\
&= e^{-\theta_t}\sum_{k=1}^{H}\eta^{\mathbb{P}}_{t,k}\exp\left(\alpha_t\mu_k+\frac{1}{2}\alpha_t^2\sigma_k^2\right),\\
\Rightarrow \theta_t &= \log\left(\sum_{k=1}^{H}\eta^{\mathbb{P}}_{t,k}\exp\left(\alpha_t\mu_k+\frac{1}{2}\alpha_t^2\sigma_k^2\right)\right). \qquad (3.A.7)
\end{aligned}
$$

Next, let's prove that $\mathbb{E}^{\mathbb{P}}\left[\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}\right]=1$. The following property will be useful :

$$
\widehat{\xi}_s \text{ is } \mathcal{G}_t\text{-measurable}, \qquad \forall s\leq t. \qquad (3.A.8)
$$

It thus follows that for all $t\geq 1$,

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\prod_{s=t}^{T}\widehat{\xi}_s\Big|\mathcal{G}_{t-1}\right] &= \mathbb{E}^{\mathbb{P}}\left[\prod_{s=t}^{T-1}\widehat{\xi}_s\,\underbrace{\mathbb{E}^{\mathbb{P}}\Big[\widehat{\xi}_T\Big|\mathcal{G}_{T-1}\Big]}_{=1,\text{ by (3.A.6)}}\Big|\mathcal{G}_{t-1}\right], \qquad \text{by (3.A.8)},\\
&\vdots \quad \text{(applying recursively)}\\
&= 1. \qquad (3.A.9)
\end{aligned}
$$

In particular, for $t=1$ the above statement is equivalent to $\mathbb{E}^{\mathbb{P}}\left[\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}\right]=1$.

3.A.8 Proof of Eq. (3.5.4)

To ensure $\widehat{\mathbb{Q}}$ is a martingale measure, the following risk-neutral condition must hold :

$$
\begin{aligned}
e^r &= \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\frac{S_{t+1}}{S_t}\middle|\mathcal{G}_t\right], \\
&= \frac{\mathbb{E}^{\mathbb{P}}\left[\frac{S_{t+1}}{S_t}\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}\middle|\mathcal{G}_t\right]}{\mathbb{E}^{\mathbb{P}}\left[\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}\middle|\mathcal{G}_t\right]}, \\
&= \frac{\prod_{n=1}^{t}\widehat{\xi}_n\,\mathbb{E}^{\mathbb{P}}\left[\frac{S_{t+1}}{S_t}\prod_{n=t+1}^{T}\widehat{\xi}_n\middle|\mathcal{G}_t\right]}{\prod_{n=1}^{t}\widehat{\xi}_n\,\underbrace{\mathbb{E}^{\mathbb{P}}\left[\prod_{n=t+1}^{T}\widehat{\xi}_n\middle|\mathcal{G}_t\right]}_{=1,\text{ by (3.A.9)}}}, \qquad \text{by (3.A.8)}, \\
&= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\frac{S_{t+1}}{S_t}\prod_{n=t+1}^{T}\widehat{\xi}_n\middle|\mathcal{G}_{t+1}\right]\middle|\mathcal{G}_t\right], \\
&= \mathbb{E}^{\mathbb{P}}\left[\frac{S_{t+1}}{S_t}\widehat{\xi}_{t+1}\underbrace{\mathbb{E}^{\mathbb{P}}\left[\prod_{n=t+2}^{T}\widehat{\xi}_n\middle|\mathcal{G}_{t+1}\right]}_{=1,\text{ by (3.A.9)}}\middle|\mathcal{G}_t\right], \qquad \text{by (3.A.8)}, \\
&= \mathbb{E}^{\mathbb{P}}\left[e^{-\theta_t}\left(\frac{S_{t+1}}{S_t}\right)^{\alpha_t+1}\middle|\mathcal{G}_t\right], \\
&= e^{-\theta_t}\mathbb{E}^{\mathbb{P}}\left[\exp\left((\alpha_t+1)\mu_{h_t}+(\alpha_t+1)\sigma_{h_t}z_{t+1}^{\mathbb{P}}\right)\middle|\mathcal{G}_t\right], \\
&= e^{-\theta_t}\sum_{k=1}^{H}\eta_{t,k}^{\mathbb{P}}\exp\left((\alpha_t+1)\mu_k+\frac{1}{2}(\alpha_t+1)^2\sigma_k^2\right). \qquad (3.A.10)
\end{aligned}
$$

Combining (3.5.3) and (3.A.10) yields

$$
\begin{aligned}
&\sum_{k=1}^{H}\eta_{t,k}^{\mathbb{P}}\exp\left(\alpha_t\mu_k+\frac{1}{2}\alpha_t^2\sigma_k^2\right)=\sum_{k=1}^{H}\eta_{t,k}^{\mathbb{P}}\exp\left((\alpha_t+1)\mu_k+\frac{1}{2}(\alpha_t+1)^2\sigma_k^2-r\right), \\
\Rightarrow\quad &\sum_{k=1}^{H}\eta_{t,k}^{\mathbb{P}}\exp\left(\alpha_t\mu_k+\frac{1}{2}\alpha_t^2\sigma_k^2\right)\left[1-\exp\left(\mu_k+\alpha_t\sigma_k^2+\frac{1}{2}\sigma_k^2-r\right)\right]=0.
\end{aligned}
$$

3.A.9 Proof of Eq. (3.5.5)

$$
\begin{aligned}
\widehat{\mathbb{Q}}\left[\epsilon_{t+1} \le x | \mathcal{G}_t\right] &= \frac{\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\epsilon_{t+1}\le x\}}\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}\Big|\mathcal{G}_t\right]}{\mathbb{E}^{\mathbb{P}}\left[\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}\Big|\mathcal{G}_t\right]}, \\
&= \frac{\prod_{n=1}^{t}\widehat{\xi}_n\,\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\epsilon_{t+1}\le x\}}\prod_{n=t+1}^{T}\widehat{\xi}_n\Big|\mathcal{G}_t\right]}{\prod_{n=1}^{t}\widehat{\xi}_n\,\underbrace{\mathbb{E}^{\mathbb{P}}\left[\prod_{n=t+1}^{T}\widehat{\xi}_n\Big|\mathcal{G}_t\right]}_{=1,\text{ by (3.A.9)}}}, \qquad \text{by (3.A.8)}, \\
&= \mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\epsilon_{t+1}\le x\}}\widehat{\xi}_{t+1}\underbrace{\mathbb{E}^{\mathbb{P}}\left[\prod_{n=t+2}^{T}\widehat{\xi}_n\Big|\mathcal{G}_{t+1}\right]}_{=1,\text{ by (3.A.9)}}\Big|\mathcal{G}_t\right], \qquad \text{by (3.A.8)}, \\
&= \mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\epsilon_{t+1}\le x\}}e^{-\theta_t+\alpha_t\epsilon_{t+1}}\Big|\mathcal{G}_t\right], \\
&= e^{-\theta_t}\sum_{i=1}^{H}\eta_{t,i}^{\mathbb{P}}\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\mu_i+\sigma_i z_{t+1}^{\mathbb{P}}\le x\}}e^{\alpha_t\mu_i+\alpha_t\sigma_i z_{t+1}^{\mathbb{P}}}\Big|\mathcal{G}_t, h_t=i\right]. \qquad (3.A.11)
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\mu_i+\sigma_i z_{t+1}^{\mathbb{P}}\le x\}}e^{\alpha_t\mu_i+\alpha_t\sigma_i z_{t+1}^{\mathbb{P}}}\Big|\mathcal{G}_t, h_t=i\right] &= \int_{-\infty}^{(x-\mu_i)/\sigma_i} e^{\alpha_t\mu_i+\alpha_t\sigma_i z}\phi(z)dz, \\
&= \int_{-\infty}^{(x-\mu_i)/\sigma_i} e^{\alpha_t\mu_i+\alpha_t\sigma_i z}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz, \\
&= \int_{-\infty}^{(x-\mu_i)/\sigma_i} e^{\alpha_t\mu_i+\alpha_t^2\sigma_i^2/2}\frac{1}{\sqrt{2\pi}}e^{-(z-\alpha_t\sigma_i)^2/2}dz, \\
&= e^{\alpha_t\mu_i+\alpha_t^2\sigma_i^2/2}\Phi\left(\frac{x-\mu_i}{\sigma_i}-\alpha_t\sigma_i\right). \qquad (3.A.12)
\end{aligned}
$$

Plugging (3.A.7) and (3.A.12) in (3.A.11), we obtain

$$
\begin{aligned}
\widehat{\mathbb{Q}}\left[\epsilon_{t+1} \le x | \mathcal{G}_t\right] &= e^{-\theta_t}\sum_{i=1}^{H}\eta_{t,i}^{\mathbb{P}}e^{\alpha_t\mu_i+\alpha_t^2\sigma_i^2/2}\Phi\left(\frac{x-\mu_i-\alpha_t\sigma_i^2}{\sigma_i}\right), \\
&= \sum_{i=1}^{H}\frac{\eta_{t,i}^{\mathbb{P}}e^{\alpha_t\mu_i+\alpha_t^2\sigma_i^2/2}}{\sum_{k=1}^{H}\eta_{t,k}^{\mathbb{P}}e^{\alpha_t\mu_k+\alpha_t^2\sigma_k^2/2}}\Phi\left(\frac{x-\mu_i-\alpha_t\sigma_i^2}{\sigma_i}\right), \\
&= \sum_{i=1}^{H}\hat{\eta}_{t,i}^{\mathbb{P}}\Phi\left(\frac{x-\mu_i-\alpha_t\sigma_i^2}{\sigma_i}\right).
\end{aligned}
$$

3.A.10 Proof of Eq. (3.5.8)

The PDF of a trajectory $(\epsilon_{1:T}, h_{0:T-1})$ under a generic probability measure $\bar{\mathbb{Q}}$ can be expressed as

$$f^{\bar{\mathbb{Q}}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1}) = f^{\bar{\mathbb{Q}}}_{\epsilon_{1:T}}(\epsilon_{1:T}) f^{\bar{\mathbb{Q}}}_{h_{0:T-1}|\mathcal{G}_T}(h_{0:T-1}|\mathcal{G}_T), \tag{3.A.13}$$

since $\mathcal{G}_T \equiv \sigma(\epsilon_{1:T})$. Moreover,

$$\begin{aligned} f^{\bar{\mathbb{Q}}}_{\epsilon_{1:T}}(\epsilon_{1:T}) &= f^{\bar{\mathbb{Q}}}_{\epsilon_{1:T-1}}(\epsilon_{1:T-1}) f^{\bar{\mathbb{Q}}}_{\epsilon_T|\mathcal{G}_{T-1}}(\epsilon_T|\mathcal{G}_{T-1}), \\ &\vdots \quad \text{(applying recursively)} \\ &= \prod_{t=1}^{T} f^{\bar{\mathbb{Q}}}_{\epsilon_t|\mathcal{G}_{t-1}}(\epsilon_t|\mathcal{G}_{t-1}). \end{aligned} \tag{3.A.14}$$

Combining (3.A.13) and (3.A.14) yields (3.5.8).

3.A.11 Proof of Eq. (3.5.9)

The expression (3.5.8) also holds for $\mathbb{P}$, i.e.,

$$f^{\mathbb{P}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1}) = f^{\mathbb{P}}_{h_{0:T-1}|\mathcal{G}_T}(h_{0:T-1}|\mathcal{G}_T) \prod_{t=1}^{T} f^{\mathbb{P}}_{\epsilon_t|\mathcal{G}_{t-1}}(\epsilon_t|\mathcal{G}_{t-1}). \tag{3.A.15}$$

Plugging the following concludes the proof :

$$f^{\mathbb{P}}_{\epsilon_t|\mathcal{G}_{t-1}}(\epsilon_t|\mathcal{G}_{t-1}) = \sum_{i=1}^{H} \underbrace{\mathbb{P}[h_{t-1} = i|\mathcal{G}_{t-1}]}_{\equiv \eta^{\mathbb{P}}_{t-1,i}} \underbrace{f^{\mathbb{P}}_{\epsilon_t|\mathcal{G}_{t-1},h_{t-1}}(\epsilon_t|\mathcal{G}_{t-1}, i)}_{=\phi^{\mathbb{P}}_i(\epsilon_t)}. \tag{3.A.16}$$

3.A.12 Proof of Eq. (3.5.12)

The Radon-Nikodym derivative is (from Theorem 3.3.1)

$$\frac{d\bar{\mathbb{Q}}}{d\mathbb{P}} \equiv \frac{f^{\bar{\mathbb{Q}}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1})}{f^{\mathbb{P}}_{\epsilon_{1:T},h_{0:T-1}}(\epsilon_{1:T}, h_{0:T-1})}. \tag{3.A.17}$$

Plugging Equation (3.5.8), (3.5.9), (3.5.10) and (3.5.11) yields (3.5.12).

3.A.13 Proof of Eq. (3.5.13)

The following property will be useful :

$$\bar{\xi}_s \text{ is } \mathcal{G}_t\text{-measurable}, \qquad \forall s \leq t. \tag{3.A.18}$$

Also, note that for all $t \geq 1$

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}[\bar{\xi}_t|\mathcal{G}_{t-1}] &= \int_{-\infty}^{\infty} \left\{ \frac{\sum_{i=1}^{H} \zeta_i(\eta_{t-1}^{\mathbb{P}})\phi_i^{\mathbb{Q}}(x)}{\sum_{i=1}^{H} \eta_{t-1,i}^{\mathbb{P}}\, \phi_i^{\mathbb{P}}(x)} f_{\epsilon_t|\mathcal{G}_{t-1}}^{\mathbb{P}}(x|\mathcal{G}_{t-1}) \right\} dx, \\
&= \int_{-\infty}^{\infty} \left\{ \frac{\sum_{i=1}^{H} \zeta_i(\eta_{t-1}^{\mathbb{P}})\phi_i^{\mathbb{Q}}(x)}{\sum_{i=1}^{H} \eta_{t-1,i}^{\mathbb{P}}\, \phi_i^{\mathbb{P}}(x)} \sum_{i=1}^{H} \eta_{t-1,i}^{\mathbb{P}}\, \phi_i^{\mathbb{P}}(x) \right\} dx, \\
&= \sum_{i=1}^{H} \zeta_i(\eta_{t-1}^{\mathbb{P}}) \underbrace{\left[\int_{-\infty}^{\infty} \phi_i^{\mathbb{Q}}(x) dx \right]}_{=1}, \\
&= 1. \qquad (3.A.19)
\end{aligned}
$$

Furthermore, for all $t \geq 1$

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\prod_{s=t}^{T} \bar{\xi}_s \middle| \mathcal{G}_{t-1} \right] &= \mathbb{E}^{\mathbb{P}}\left[\prod_{s=t}^{T-1} \bar{\xi}_s \underbrace{\mathbb{E}^{\mathbb{P}}\left[\bar{\xi}_T \middle| \mathcal{G}_{T-1}\right]}_{=1, \text{ by } (3.A.19)} \middle| \mathcal{G}_{t-1} \right], \qquad \text{by } (3.A.18), \\
&\ \vdots \quad \text{(applying recursively)} \\
&= 1. \qquad (3.A.20)
\end{aligned}
$$

We are now ready to carry out the main proof :

$$\begin{aligned}
\bar{\mathbb{Q}}[\epsilon_{t+1} \leq x|\mathcal{G}_t] &= \mathbb{E}^{\bar{\mathbb{Q}}}\big[\mathbb{1}_{\{\epsilon_{t+1}\leq x\}}\big|\mathcal{G}_t\big], \\
&\equiv \frac{\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\epsilon_{t+1}\leq x\}}\frac{d\bar{\mathbb{Q}}}{d\mathbb{P}}\Big|\mathcal{G}_t\right]}{\mathbb{E}^{\mathbb{P}}\left[\frac{d\bar{\mathbb{Q}}}{d\mathbb{P}}\Big|\mathcal{G}_t\right]}, \\
&= \frac{\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\epsilon_{t+1}\leq x\}}\prod_{s=1}^{T}\bar{\xi}_s\Big|\mathcal{G}_t\right]}{\mathbb{E}^{\mathbb{P}}\left[\prod_{s=1}^{T}\bar{\xi}_s\Big|\mathcal{G}_t\right]}, \\
&= \frac{\prod_{s=1}^{t}\bar{\xi}_s\,\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\epsilon_{t+1}\leq x\}}\prod_{s=t+1}^{T}\bar{\xi}_s\Big|\mathcal{G}_t\right]}{\prod_{s=1}^{t}\bar{\xi}_s\,\underbrace{\mathbb{E}^{\mathbb{P}}\left[\prod_{s=t+1}^{T}\bar{\xi}_s\Bigg|\mathcal{G}_t\right]}_{=1,\text{ by (3.A.20)}}}, \qquad \text{by (3.A.18)}, \\
&= \mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{\epsilon_{t+1}\leq x\}}\bar{\xi}_{t+1}\,\underbrace{\mathbb{E}^{\mathbb{P}}\left[\prod_{s=t+2}^{T}\bar{\xi}_s\Bigg|\mathcal{G}_{t+1}\right]}_{=1,\text{ by (3.A.20)}}\Bigg|\mathcal{G}_t\right], \qquad \text{by (3.A.18)}, \\
&= \mathbb{E}^{\mathbb{P}}\big[\mathbb{1}_{\{\epsilon_{t+1}\leq x\}}\bar{\xi}_{t+1}\big|\mathcal{G}_t\big], \\
&= \int_{-\infty}^{x}\left\{\frac{\sum_{i=1}^{H}\zeta_i(\eta_t^{\mathbb{P}})\phi_i^{\mathbb{Q}}(y)}{\sum_{i=1}^{H}\eta_{t,i}^{\mathbb{P}}\,\phi_i^{\mathbb{P}}(y)}f_{\epsilon_{t+1}|\mathcal{G}_t}^{\mathbb{P}}(y|\mathcal{G}_t)\right\}dy, \\
&= \int_{-\infty}^{x}\left\{\frac{\sum_{i=1}^{H}\zeta_i(\eta_t^{\mathbb{P}})\phi_i^{\mathbb{Q}}(y)}{\sum_{i=1}^{H}\eta_{t,i}^{\mathbb{P}}\,\phi_i^{\mathbb{P}}(y)}\sum_{i=1}^{H}\eta_{t,i}^{\mathbb{P}}\,\phi_i^{\mathbb{P}}(y)\right\}dy, \\
&= \sum_{i=1}^{H}\zeta_i(\eta_t^{\mathbb{P}})\left[\int_{-\infty}^{x}\phi_i^{\mathbb{Q}}(y)dy\right], \\
&= \sum_{i=1}^{H}\zeta_i(\eta_t^{\mathbb{P}})\Phi\left(\frac{x-r+\frac{1}{2}\sigma_i^2}{\sigma_i}\right).
\end{aligned}$$