1.5.2 An extension of the proportional hazards transform

We now turn to the Esscher pricing principle. Because this principle relies on an exponential weight function, the risk-neutral measure does not exist for the heavy-tailed distributions in the GB2 family. It is however applicable to the gamma subfamily, which includes the exponential and chi-squared distributions. Here, we suppose that the insurance claim X is gamma distributed, and we derive the distortion that recuperates the Esscher principle. Interestingly, it turns out that the proportional hazards (PH) transform (Wang, 1995) is a special case of this distortion.

Distribution under the physical measure

Under the physical measure $\mathbb{P}$, the risk X follows a Gamma(α, θ) distribution. The parameter domain of this distribution is $\alpha > 0$, $\theta > 0$. The PDF and CDF are given by

$$\mathbf{gamma}(x;\alpha,\theta) = \frac{(x/\theta)^{\alpha} e^{-x/\theta}}{x\Gamma(\alpha)}, \qquad \mathbf{Gamma}(x;\alpha,\theta) = \Gamma\big(x/\theta;\alpha\big), \qquad x \geq 0. \tag{1.5.11}$$

The survival function of X, under $\mathbb{P}$, is thus $\bar{F}_{\mathbb{P}}(x) = 1 - \Gamma\big(x/\theta;\alpha\big)$, $x \geq 0$.

Distribution under the risk-neutral measure

Under the Esscher risk-neutral measure $\mathbb{Q}$, the PDF of X is given by

$$\frac{e^{ax}}{\mathbb{E}^{\mathbb{P}}[e^{aX}]}\mathbf{gamma}(x;\alpha,\theta) \propto \mathbf{gamma}(x;\alpha,\tilde{\theta}), \qquad \tilde{\theta} \equiv \frac{\theta}{1-a\theta}. \tag{1.5.12}$$

By normalization, the right-hand side indeed implies that $X \sim$ Gamma$(\alpha, \tilde{\theta})$ under $\mathbb{Q}$. Note that $a < 1/\theta$ is required to ensure the existence of this measure.

The survival function of X, under $\mathbb{Q}$, is thus $\bar{F}_{\mathbb{Q}}(x) = 1 - \Gamma\big(x/\tilde{\theta};\alpha\big)$, $x \geq 0$.

Derivation of the distortion operator

The derivation of the distortion operator follows the same steps as in Section 1.5.1. We obtain

$$g_X^{\mathbb{Q},\mathbb{P}}(u) = 1 - \Gamma\big(\Gamma^{-1}(1-u;\alpha)[1-a\theta];\alpha\big), \tag{1.5.13}$$

where Γ^{-1} is the inverse of the incomplete gamma function Γ of (1.5.2).

Note that for the special case $\alpha = 1$ (i.e., the exponential distribution) we obtain $g_X^{\mathbb{Q},\mathbb{P}}(u) = u^{1-a\theta}$, which is the so-called PH transform (Wang, 1995). Our gamma-based distortion can therefore be seen as a natural generalization of the PH transform obtained from a more flexible distribution. Moreover, our analysis clearly deepens and extends the discussions in Wang (1996) regarding the connections between risk-neutral valuation and the PH transform.

1.6 Empirical applications to CAT bonds

Catastrophe (CAT) bonds are insurance-linked securities devised by insurers and reinsurers to shift natural disaster risks to the capital markets.[9] Several authors have proposed to price CAT bonds using the theory of contingent claim pricing for jump-diffusion processes (e.g., Vaugirard, 2003). These CAT bond pricing models usually involve extensive information that are not publicly disclosed (e.g., the trigger level or strike price).[10] It ensues that these types of models have yet to be tested empirically. In fact, most empirical studies on CAT bond spreads have instead relied on pure regression models (e.g., Bodoff and Gan, 2012; Braun, 2015). Meanwhile, explaining CAT bond spreads using distortion operators has been suggested by several works ; calibration of the Wang transform to CAT bond spreads is carried out, for instance, in Wang (2004) and in Galeotti et al. (2013). In this section, we employ our approach to perform further empirical tests on such models as it enables us to reformulate these in terms of the information available to the econometrician (e.g., the expected loss, the probability of first loss, the probability of last loss). Extending this stream of empirical studies, we make several interesting findings, in particular, it turns out that incorporating CAT risk and CAT risk-aversion into the distortion is important for explaining CAT bond spreads. This is established by calibrating a jump-diffusion distortion operator to market spreads. By way of a mixture of normal and exponential distributions, we also offer a simplified approximation of the latter distortion.

1.6.1 Setup

Let the payout to the ceding (re)insurer be contingent upon a risk X breaching a pre-agreed attachment point b, in which case the collateral is liquidated to reimburse the sponsor up to the par amount p paid by the investor at the issue date. If there is no triggering event during the term of the CAT bond, which is typically between one and four years, the principal is returned to the investor plus a coupon payment with spread S above the risk-free rate. Under the distortion premium principle, the spread is given by (from Galeotti et al., 2013)

$$\mathrm{S} = \frac{1}{p}\int_b^{b+p} g\big(\bar{F}_{\mathbb{P}}(x)\big)dx, \qquad \bar{F}_{\mathbb{P}}(x) \equiv \mathbb{P}(X > x). \tag{1.6.1}$$

The integral can be approximated using the trapezoidal rule :

$$\mathrm{S} \approx \big(g(\mathrm{PFL}) + g(\mathrm{PLL})\big)/2, \qquad \mathrm{PFL} \equiv \bar{F}_{\mathbb{P}}(b), \quad \mathrm{PLL} \equiv \bar{F}_{\mathbb{P}}(b+p). \tag{1.6.2}$$

9. CAT bond transactions usually involve a Special-Purpose Vehicle (SPV), located in a tax-efficient jurisdiction, that sells catastrophe protection to a ceding (re)insurer in the form of a reinsurance contract. The SPV then effectively transfers its risk exposure by issuing CAT bond tranches to capital market investors. In order to offer a virtually pure exposure to the natural disaster risk, the proceeds of the issuance are invested by the SPV in highly rated short-term assets that are held in a collateral account.

10. Available databases for CAT bond transactions include Artemis.bm's deal directory, and the reports published by Aon Benfield, Swiss Re, Plenum, and Lane Financial LLC.

If either of the probability of first loss PFL or the probability of last loss PLL is not available, the integral can be approximated using the rectangle method :

$$\mathrm{S} \approx g(\mathrm{EL}), \qquad \mathrm{EL} \equiv \frac{1}{p}\int_b^{b+p} \bar{F}_{\mathbb{P}}(x)dx. \tag{1.6.3}$$

A data set containing 508 CAT bond tranches issued between 1997 and 2016 has been made available to us by Artemis.bm. Roughly half of the CAT bonds in the sample cover the United States, approximatively one quarter are multi-territory, and the rest cover Europe, Japan or other areas. About half of the CAT tranches are multi-peril, and others are mainly wind-specific or earthquake-specific. For each of the 508 transactions, the available information includes the per annum spread S and expected loss EL. The probability of first loss PFL and the probability of last loss PLL are however available only for 284 transactions. Calibration is therefore carried out under the rectangle method (1.6.3), by minimizing the sum of squared pricing errors. Fitting adequacy is investigated using nonparametric regressions to detect systematic pricing errors ; see Azzalini et al. (1989), Zheng (1996), and Li and Racine (2007) for works on nonparametric specification testing.

1.6.2 Pricing CAT bonds using a jump-diffusion distortion

An interesting property of our distortion operators is that they inherit the features of the original root model they are derived from. For instance, the jump-diffusion distortion (1.4.16) can incorporate jump risk, making it an interesting candidate for pricing CAT bonds. Several frameworks based on jump-diffusion processes have in fact been proposed to price such insurance-linked securities (e.g., Vaugirard, 2003). In this section, instead of using the Wang transform as in Wang (2004) and Galeotti et al. (2013), we calibrate our jump-diffusion distortion to CAT bond spreads. Furthermore, we investigate whether jump risk is priced by the market.

Suppose the jumps are used to model natural disasters. Because such events can only lower the aggregate endowment, we are interested in the special case $p = 0$ and $\beta < 0$ of the model presented in Section 1.4.2, where the jump-diffusion process S is used here to represent a catastrophe loss index whose dynamics is affected by positive jumps. For this special case, it is a straightforward exercise to show that the distortion operator (1.4.16) can be simplified and rewritten as follows :

$$g_{\varphi,\lambda,\eta,\nu}(u) \equiv \Upsilon_{\lambda\nu,\frac{\eta}{\nu}}\left(\Upsilon^{-1}_{\lambda,\eta}(u) + \varphi\right), \tag{1.6.4}$$

where $\lambda > 0$, $\eta > 0$, $\nu \geq 1$, $\varphi \in \mathbb{R}$, and $\Upsilon^{-1}_{\lambda,\eta}$ is the inverse of the function defined below. [11]

Definition 1.6.1. Let $\{Y_i\}_{i\geq 1}$ be a sequence of i.i.d. exponential random variables with rate $\eta > 0$, P be a Poisson random variable with rate $\lambda > 0$, and Z be a standard normal random

11. Note that the parameters λ, η, ν, and φ are defined differently than in preceding sections ; the latter definition corresponds to the simplified econometric specification.

variable. We define the following CDF :

$$\Upsilon_{\lambda,\eta}(x) \equiv \Pr\left(Z - \sum_{i=1}^{P} Y_i \leq x\right), \qquad x \in \mathbb{R}. \tag{1.6.5}$$

We calibrate the distortion depicted by (1.6.4) using our CAT bond tranches data set. It turns out that there is an infinite set of solutions that yield the same fitted curve. For example, the solution $(\hat{\varphi}, \hat{\lambda}, \hat{\eta}, \hat{\nu}) = (0.02, 0.20, 4.50, 2.92)$ provides the same distortion as (0.11, 0.23, 2.00, 1.87). We are also interested in determining whether jump risk is priced by the market. We seek an answer to this research question by calibrating the distortion operator under the constraint $\nu = 1$, which states that the jump amplitude and frequency are unchanged under the risk-neutral measure, i.e., that jump risk is not priced by the market. The calibration results are exhibited in Figure 1.1, where we can see that assuming idiosyncratic (i.e., unpriced) jump risk leads to a mis-specified model. This provides, for the first time, interesting evidence that jump-diffusion models are appropriate for pricing CAT bonds, but that investors are risk-averse to natural disasters. We defer to future work for delving further into this question.

1.6.3 Results for other distortions

One potential disadvantage of the jump-diffusion distortion is the use of sophisticated analytical results on the Hh special function from mathematical physics in order to compute it efficiently (see, e.g., Kou, 2002, Appendix B). Hence, it would be interesting to find a readily implementable distortion that also fits well to CAT bond spreads.

Results for calibrating the GB2 distortion (1.5.10) to CAT bond market spreads are exhibited in Figure 1.2. Graph 1.2(b) reveals systematic pricing errors, exacerbated at small values of the expected loss EL. Unreported tests available upon request from the authors show that other distortion operators suffer from similar issues ; such as the two NIG distortions and also Wang (2004)'s distortion based on the Student-t distribution. Hence, such distortions are less suitable for explaining CAT bond spreads. This might appear surprising as the GB2 family is known for its flexibility in modeling insurance losses (Cummins et al., 1990). On the other hand, the theoretical literature on CAT bonds and other insurance-linked securities advocate jump-diffusion processes and compound Poisson processes to model catastrophe loss processes.[12] In the previous sections, we have presented empirical evidence supporting the latter models. The inadequacy of the above distortion functions for explaining CAT bond spreads may therefore be attributed to their inability in capturing and pricing catastrophe risks, in contrast with the jump-diffusion distortion which performs well. Next, we present a general framework addressing this issue.

12. See Vaugirard (2003), Nowak and Romaniuk (2013), Ma and Ma (2013), Perrakis and Boloorforoosh (2013) and Lai et al. (2014) for a non-exhaustive list of such works.

1.6.4 A simple class of distortion operators to price CAT bonds

This section offers a simple class of Esscher-type distortions based on a mixture of distributions to characterize catastrophe risks. It is shown such distortions can provide an accurate representation of CAT bond spreads while being straightforward to implement in practice.

General framework

Let X be a continuous random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ representing a risk. A probability mixture distribution having the following representation is considered :

$$X = e^{\mu + \sigma Z_I}, \tag{1.6.6}$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ are constants, I is a discrete random variable such that

$$\mathbb{P}(I = i) = p_i, \qquad i \in \{1, \ldots, m\}, \qquad \sum_{i=1}^{m} p_i = 1, \tag{1.6.7}$$

and the $Z_1, \ldots, Z_m$ are independent random variables with PDFs denoted by $f_1, \ldots, f_m$. Hence,

$$\mathbb{P}(Z_I \leq z) = \sum_{i=1}^{m} p_i F_i(z), \qquad F_i(z) \equiv \int_{-\infty}^{z} f_i(x)dx, \qquad z \in \mathbb{R}, \tag{1.6.8}$$

as I is presumed independent from the $Z_1, \ldots, Z_m$. Here, $m \geq 1$ is a given constant denoting the number of components in the probability mixture. Such a model can account for a hidden multi-state risk structure and is therefore deemed appropriate for catastrophe-linked risks.

The following notation will be used for the moment generating functions :

$$\zeta_i^{(t)} \equiv \mathbb{E}^{\mathbb{P}}\left[e^{tZ_i}\right], \qquad i \in \{1, \ldots, m\}, \quad t \in \mathbb{R}. \tag{1.6.9}$$

The Esscher risk-neutral measure $\mathbb{Q}$ is considered as it is a common choice in the literature (e.g., Gerber and Shiu, 1994). The Radon-Nikodym derivative associated to this measure is

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{X^a}{\mathbb{E}^{\mathbb{P}}[X^a]} = \frac{e^{a\sigma Z_I}}{\mathbb{E}^{\mathbb{P}}[e^{a\sigma Z_I}]}, \tag{1.6.10}$$

where $a \in \mathbb{R}$ is a constant. The distribution of Z_I under $\mathbb{Q}$ is stated in the following proposition proven in Appendix 1.A.5.

Proposition 1.6.1. *We have*

$$\mathbb{Q}(Z_I \leq z) = \sum_{i=1}^{m} \tilde{p}_i \tilde{F}_i(z), \qquad z \in \mathbb{R}, \tag{1.6.11}$$

where, for all $i \in \{1, \ldots, m\}$,

$$\tilde{p}_i \equiv \frac{p_i \zeta_i^{(a\sigma)}}{\sum_{j=1}^{m} p_j \zeta_j^{(a\sigma)}}, \qquad \tilde{F}_i(z) \equiv \int_{-\infty}^{z} \tilde{f}_i(x)dx, \qquad \tilde{f}_i(x) \equiv \frac{e^{a\sigma x}}{\zeta_i^{(a\sigma)}} f_i(x). \tag{1.6.12}$$

The distortion operator g that applies the Esscher change of measure for the risk X, i.e., such that $g\big(\mathbb{P}(X>x)\big)=\mathbb{Q}(X>x)$ for all $x\in\mathbb{R}$, is given in the proposition below whose proof is a direct application of Corollary 1.3.1.

Proposition 1.6.2. *The distortion that performs the Esscher change of measure for X is*

$$g_{Z_I}^{\mathbb{Q},\mathbb{P}}(u)\equiv\bar{F}_{\mathbb{Q}}\big(\bar{F}_{\mathbb{P}}^{-1}(u)\big),\qquad u\in[0,1],\tag{1.6.13}$$

where $\bar{F}_{\mathbb{Q}}(z)\equiv 1-\mathbb{Q}(Z_I\leq z)$, and $\bar{F}_{\mathbb{P}}^{-1}$ is the inverse of the function $\bar{F}_{\mathbb{P}}(z)\equiv 1-\mathbb{P}(Z_I\leq z)$.

Recuperating distortions from previous literature

The above framework is very general as it encompasses several distortion operators encountered in the previous literature. Some examples are given below for the case $m=1$; detailed proofs are available from the authors upon request.

1. The PH transform (from Wang, 1995) $g_{Z_1}^{\mathbb{Q},\mathbb{P}}(u)=u^{1-a\sigma}$ is obtained if Z_1 is a standard exponential variable under $\mathbb{P}$.
2. The Wang transform $g_{Z_1}^{\mathbb{Q},\mathbb{P}}(u)=\Phi\big(\Phi^{-1}(u)+a\sigma\big)$ is obtained if $Z_1\sim N(0,1)$ under $\mathbb{P}$.
3. The GB2 distortion $g_{Z_1}^{\mathbb{Q},\mathbb{P}}(u)=1-\beta\big(\beta^{-1}(1-u;\tau,\alpha);\tau+a\sigma,\alpha-a\sigma\big)$ is obtained if e^{Z_1} follows a GB2$(\alpha,1,1,\tau)$ distribution (as defined in Klugman et al., 2012) under $\mathbb{P}$.

The multi-state extensions of the above distortions are also directly derived from the above general framework.

A normal-exponential mixture distortion

To approximate the jump-diffusion distortion (1.6.4), we now consider a special case. The mixture consists of $m=2$ states such that $Z_1\sim N(0,1)$ and $Z_2\sim\text{Exp}(\eta)$ under $\mathbb{P}$, where $\text{Exp}(\eta)$ is the exponential distribution with inverse scale parameter $\eta>0$.

It follows that

$$\mathbb{P}(Z_I\leq z)=p_1\Phi(z)+(1-p_1)(1-e^{-\eta z})\mathbb{1}_{\{z\geq 0\}},\qquad z\in\mathbb{R}.\tag{1.6.14}$$

As shown in Appendix 1.A.6, provided that $\eta>a\sigma$,

$$\begin{aligned}\mathbb{Q}(Z_I\leq z)&=\tilde{p}_1\Phi(z-a\sigma)+(1-\tilde{p}_1)\big(1-e^{-(\eta-a\sigma)z}\big)\mathbb{1}_{\{z\geq 0\}},\qquad z\in\mathbb{R},\\ \tilde{p}_1&=\frac{p_1e^{\frac{1}{2}(a\sigma)^2}}{p_1e^{\frac{1}{2}(a\sigma)^2}+(1-p_1)\frac{\eta}{\eta-a\sigma}}.\end{aligned}\tag{1.6.15}$$

The corresponding distortion function is given by Proposition 1.6.2. It is characterized by the parameters (p_1, η, ξ) where $\xi \equiv a\sigma$, with domain $p_1 \in [0, 1]$, $\eta > 0$, $\xi < \eta$. The computation of this distortion is quite straightforward as it involves only basic functions. Moreover, it provides a more accurate explanation of CAT bond spreads as illustrated in Figure 1.2. In particular, residuals in Figure 1.2(c) do not display systematic pricing errors in contrast with the previous distortions. Hence, our mixed distribution approach is successful in providing a more accurate description of observed CAT bond spreads while being simple to implement in practice.

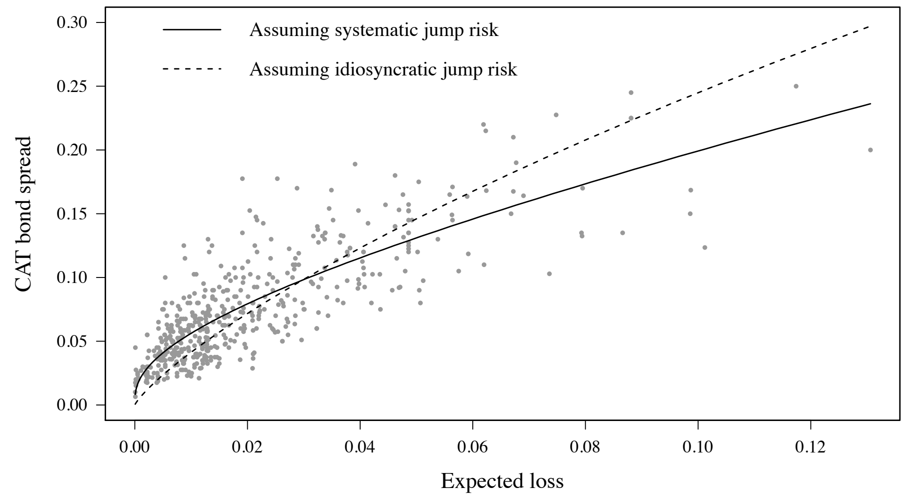
1.7 Conclusion

We propose a general class of probability distortion operators consistent with arbitrage-free pricing. Previous attempts in this direction are the Wang (2000) transform and the NIG distortion of Godin et al. (2012). To illustrate our approach, we derive several new distortions that improve upon the original Wang transform in a similar fashion as the more recent NIG distortion, while generalizing the latter by bringing flexibility in choosing the equivalent martingale measure and the underlying distribution. In fact, our framework makes it a straightforward exercise to derive new distortion operators, for instance from standard non-Gaussian financial theory (see, e.g., Schoutens, 2003). Our research also provides new twists to existing works that investigate the connections between distortion risk measures and other forms of risk pricing (e.g., Wang, 2003; Labuschagne and Offwood, 2010). We effectively characterize the change of measure performed by our distortion operators and show that this offers a deeper understanding of such connections.

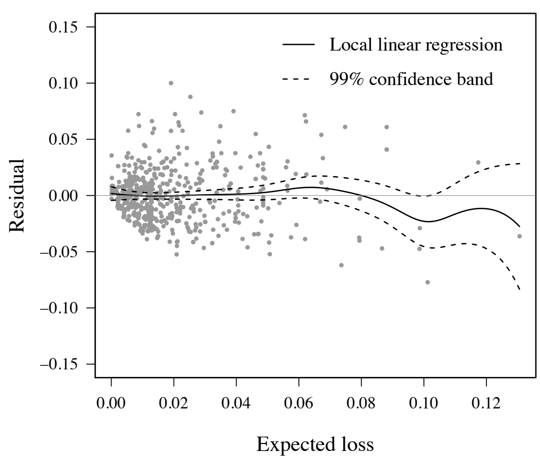
An important area of research opened by our work is the testing of catastrophe (CAT) bond pricing models from publicly available transaction information. Following the lead of Wang (2004) and Galeotti et al. (2013), we provide a explanatory empirical study which indicates that an exponential jump-diffusion distortion is adequate for explaining CAT bond spreads, but only if we allow the distortion to incorporate risk-aversion to natural disasters. Also, a general yet simple class of probability distortion operators based on the Esscher change of measure for multi-state structured risks is proposed to price CAT risks. This new class of distortions provides an accurate depiction of observed CAT bond spreads while being straightforward to implement in practice.

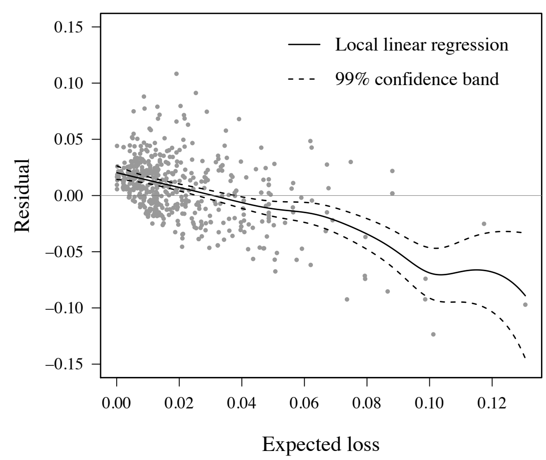
Another potential application of our distortion operator is to produce, by way of equation (1.2.3), distortion-based risk measures that can be used for a wide range of applications, including capital allocation and optimal reinsurance (see Dowd and Blake, 2006, for an account on the applications of distortion risk measures). We also contribute to the discussion in the literature regarding the connection between risk measures, heavy-tailed skewed distributions, and risk pricing. Indeed, our new distortion operator expressed by equation (1.3.1) is directly defined in terms of the physical distribution of the underlying risk and the choice of the equivalent martingale measure. As such, it provides a parametric approach to produce risk measures that incorporate stylized features of financial (or insurance) risk distributions (e.g., skewness and kurtosis) as well as risk-aversion and other determinants of market prices of risks. In fact, there have been several attempts to incorporate these features into risk measures (e.g., Bali and Theodossiou, 2008; Gzyl and Mayoral, 2008). In particular, it is well-known that accounting for the heavy tail feature of the loss distributions observed in non-life insurance is critically important for premium calculation and risk measurement. Our new distortion operator based on the generalized beta of the second kind distribution is a flexible generalization of the Wang transform that can capture the heavy tail feature.

Regarding extensions, it has been shown that the Wang transform can be used to obtain the Black-Scholes prices of exotic options (Labuschagne and Offwood, 2013). The Wang transform has also been extended to a multivariate setting (see, e.g., Kijima, 2006; Wang, 2007). It seems reasonable to presume that our framework can be extended in similar directions. We leave these questions open for future research.



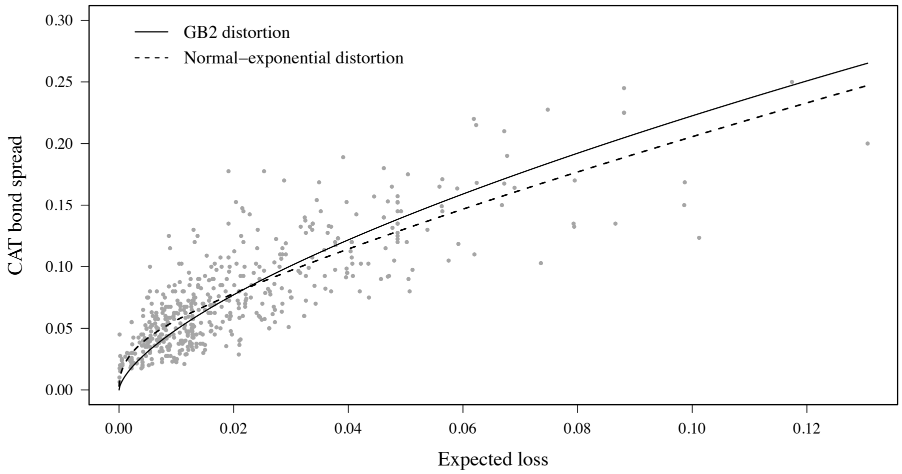
(a) Illustration of the calibrated distortion operators



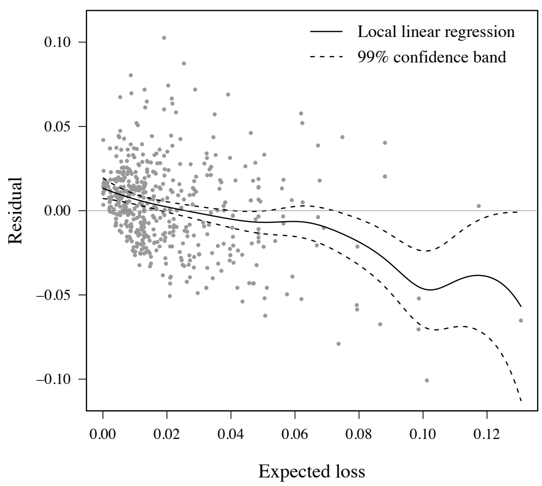


(b) Residuals : Systematic jump risk ($\nu \geq 1$) **(c)** Residuals : Idiosyncratic jump risk ($\nu = 1$)

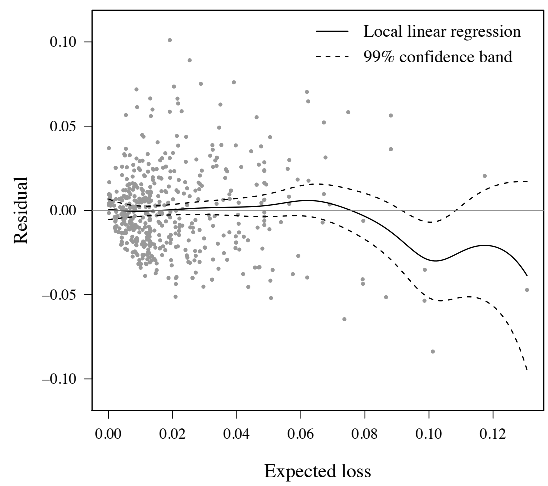
Figure 1.1 – Calibration results for our Artemis.bm data set consisting of 508 CAT bond tranches issued between 1997 and 2016. The calibration is done by minimizing the sum of squared pricing errors. Graph **(a)** illustrates the empirical data and the calibrated distortion operator $g_{\varphi,\lambda,\eta,\nu}$ of (1.6.4). The dashed line is obtained under the constraint $\nu = 1$ (idiosyncratic jump risk), and the solid line is obtained without this constraint (i.e., with $\nu \geq 1$). The residuals are plotted against the expected loss in Graph **(b)** for the unconstrained case, and in Graph **(c)** for the constrained case. A local linear regression of the residuals against the expected loss is carried out to detect systematic pricing errors.



(a) Illustration of the calibrated distortion operators



(b) Residuals : GB2 distortion



(c) Residuals : Normal-exponential distortion

FIGURE 1.2 – Calibration results for our Artemis.bm data set consisting of 508 CAT bond tranches issued between 1997 and 2016. The calibration is done by minimizing the sum of squared pricing errors. Graph **(a)** illustrates the empirical data and the calibrated distortion operators. The solid line corresponds to the GB2 distortion operator of (1.5.10), and the dashed line corresponds to the normal-exponential mixture distortion proposed in Section 1.6.4 of this paper. The residuals are plotted against the expected loss in Graph **(b)** for the GB2 distortion, and in Graph **(c)** for the normal-exponential distortion. A local linear regression of the residuals against the expected loss is carried out to detect systematic pricing errors.

Appendix

1.A Proofs

1.A.1 Proof of Proposition 1.3.1

Let the survival function of X under $\mathbb{P}$ and $\mathbb{Q}$ be denoted by $\bar{F}_{\mathbb{P}}(x) \equiv \mathbb{P}(X > x)$ and $\bar{F}_{\mathbb{Q}}(x) \equiv \mathbb{Q}(X > x)$. Similarly for $h(X)$, define $\bar{Q}_{\mathbb{P}}(x) \equiv \mathbb{P}(h(X) > x)$ and $\bar{Q}_{\mathbb{Q}}(x) \equiv \mathbb{Q}(h(X) > x)$. Since h is continuous and increasing, we have $\bar{Q}_{\mathbb{Q}}(x) = \bar{F}_{\mathbb{Q}}\big(h^{-1}(x)\big)$ and $\bar{Q}_{\mathbb{P}}^{-1}(u) = h\big(\bar{F}_{\mathbb{P}}^{-1}(u)\big)$. Using these last two identities in Definition 1.3.1 gives us

$$g_{h(X)}^{\mathbb{Q},\mathbb{P}}(u) \equiv \bar{Q}_{\mathbb{Q}}\big(\bar{Q}_{\mathbb{P}}^{-1}(u)\big) = \bar{F}_{\mathbb{Q}}\big(\bar{F}_{\mathbb{P}}^{-1}(u)\big) \equiv g_X^{\mathbb{Q},\mathbb{P}}(u).$$

1.A.2 Proof of Theorem 1.3.1

X follows a survival distribution function $\bar{F}_{g_Z^{\mathbb{Q},\mathbb{P}}}(x) = g_Z^{\mathbb{Q},\mathbb{P}}\big(\bar{F}_{\mathbb{P}}(x)\big)$ under the distorted measure. Using the chain rule to take the derivative of this equation with respect to x gives us

$$f_{g_Z^{\mathbb{Q},\mathbb{P}}}(x) = \dot{g}_Z^{\mathbb{Q},\mathbb{P}}\big(\bar{F}_{\mathbb{P}}(x)\big) f_{\mathbb{P}}(x), \tag{1.A.1}$$

where $\dot{g}_Z^{\mathbb{Q},\mathbb{P}}(u) \equiv \frac{d}{du} g_Z^{\mathbb{Q},\mathbb{P}}(u)$.

By Definition 1.3.1, we have $g_Z^{\mathbb{Q},\mathbb{P}}(u) \equiv \bar{Q}_{\mathbb{Q}}\big(\bar{Q}_{\mathbb{P}}^{-1}(u)\big)$. Taking the derivative with respect to u yields

$$\dot{g}_Z^{\mathbb{Q},\mathbb{P}}(u) = -q_{\mathbb{Q}}\big(\bar{Q}_{\mathbb{P}}^{-1}(u)\big) \frac{d\bar{Q}_{\mathbb{P}}^{-1}(u)}{du}, \tag{1.A.2}$$

where $q_{\mathbb{Q}}$ is the PDF of Z under $\mathbb{Q}$. Next, we note that

$$1 = \frac{d}{du} u = \frac{d}{du} \bar{Q}_{\mathbb{P}}\big(\bar{Q}_{\mathbb{P}}^{-1}(u)\big) = -q_{\mathbb{P}}\big(\bar{Q}_{\mathbb{P}}^{-1}(u)\big) \frac{d\bar{Q}_{\mathbb{P}}^{-1}(u)}{du} \quad \Rightarrow \quad \frac{d\bar{Q}_{\mathbb{P}}^{-1}(u)}{du} = \frac{-1}{q_{\mathbb{P}}\big(\bar{Q}_{\mathbb{P}}^{-1}(u)\big)}.$$

Using this last equality in (1.A.2) yields

$$\dot{g}_Z^{\mathbb{Q},\mathbb{P}}(u) = \frac{q_{\mathbb{Q}}\big(\bar{Q}_{\mathbb{P}}^{-1}(u)\big)}{q_{\mathbb{P}}\big(\bar{Q}_{\mathbb{P}}^{-1}(u)\big)} \equiv \xi_Z^{\mathbb{Q},\mathbb{P}}\big(\bar{Q}_{\mathbb{P}}^{-1}(u)\big), \tag{1.A.3}$$

where we have used Definition 1.3.2. Using (1.A.3) in (1.A.1) concludes the proof.

1.A.3 Proof of Corollary 1.3.1

Because $X = h(Z)$, where h is continuous and increasing, we have $\bar{F}_{\mathbb{P}}(x) = \bar{Q}_{\mathbb{P}}(h^{-1}(x))$. Then, by Theorem 1.3.1 it follows that $f_{g_Z^{\mathbb{Q},\mathbb{P}}}(x) = f_{\mathbb{P}}(x)\xi_Z^{\mathbb{Q},\mathbb{P}}(h^{-1}(x))$. Using standard results on the transformation of random variables, one can show that $f_{\mathbb{Q}}(x) = f_{\mathbb{P}}(x)\xi_Z^{\mathbb{Q},\mathbb{P}}(h^{-1}(x))$. [13]

1.A.4 Proof of Proposition 1.3.2

Because h is a continuous and increasing function, it follows from Corollary 1.3.1 that

$$g_X^{\mathbb{Q},\mathbb{P}}(\mathbb{P}(h(X) > x)) = \mathbb{Q}(h(X) > x), \qquad \forall x \in \mathbb{R}.$$

Using this in the definition (1.2.3) of the functional H gives us

$$H\Big[h(X); g_X^{\mathbb{Q},\mathbb{P}}\Big] = \int_{-\infty}^{0} \big[\mathbb{Q}(h(X) > x) - 1\big]dx + \int_0^{\infty} \mathbb{Q}(h(X) > x)dx = \mathbb{E}^{\mathbb{Q}}[h(X)],$$

where the last equality is a well-known identity.

1.A.5 Proof of Proposition 1.6.1

From (1.6.7),

$$\mathbb{E}^{\mathbb{P}}\big[e^{a\sigma Z_I}\big] = \sum_{i=1}^{m} \mathbb{P}(I = i)\mathbb{E}^{\mathbb{P}}\big[e^{a\sigma Z_I}\big|I = i\big] = \sum_{i=1}^{m} p_i\mathbb{E}^{\mathbb{P}}\big[e^{a\sigma Z_i}\big] = \sum_{i=1}^{m} p_i\zeta_i^{(a\sigma)},$$

where definition (1.6.9) is used. Using this in (1.6.10) yields $\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{e^{a\sigma Z_I}}{\sum_{i=1}^m p_i\zeta_i^{(a\sigma)}}$. Hence,

$$\begin{aligned}
\mathbb{Q}(Z_I \le z) \equiv \mathbb{E}^{\mathbb{Q}}\big[\mathbb{1}_{\{Z_I \le z\}}\big] \equiv \mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\mathbb{1}_{\{Z_I \le z\}}\right] &= \frac{\mathbb{E}^{\mathbb{P}}\big[e^{a\sigma Z_I}\mathbb{1}_{\{Z_I \le z\}}\big]}{\sum_{j=1}^m p_j\zeta_j^{(a\sigma)}}, \\
&= \sum_{i=1}^{m} \frac{p_i\mathbb{E}^{\mathbb{P}}\big[e^{a\sigma Z_i}\mathbb{1}_{\{Z_i \le z\}}\big]}{\sum_{j=1}^m p_j\zeta_j^{(a\sigma)}}, \\
&= \sum_{i=1}^{m} \frac{p_i\zeta_i^{(a\sigma)}}{\sum_{j=1}^m p_j\zeta_j^{(a\sigma)}}\mathbb{E}^{\mathbb{P}}\left[\frac{e^{a\sigma Z_i}}{\zeta_i^{(a\sigma)}}\mathbb{1}_{\{Z_i \le z\}}\right],
\end{aligned}$$

from which one can indeed conclude (1.6.11).

13. Define $q_{\mathbb{P}}$ as the PDF of Z under $\mathbb{P}$. Since h is continuous and increasing, it follows that the PDF of $X = h(Z)$ is given by $f_{\mathbb{P}}(x) = q_{\mathbb{P}}(h^{-1}(x))/h'(h^{-1}(x))$, where h' is the derivative of h. Similarly, under $\mathbb{Q}$ we have $f_{\mathbb{Q}}(x) = q_{\mathbb{Q}}(h^{-1}(x))/h'(h^{-1}(x))$. Therefore : $f_{\mathbb{Q}}(x) = f_{\mathbb{P}}(x)\xi_Z^{\mathbb{Q},\mathbb{P}}(h^{-1}(x))$, where $\xi_Z^{\mathbb{Q},\mathbb{P}}(z) \equiv q_{\mathbb{Q}}(z)/q_{\mathbb{P}}(z)$.

1.A.6 Proof of Eq. (1.6.15)

Under $\mathbb{P}$, $Z_1 \sim N(0,1)$ and $Z_2 \sim \text{Exp}(\eta)$. Hence, the PDFs of Z_1 and Z_2 are respectively

$$f_1(x) = \phi(x) \equiv \frac{e^{-x^2/2}}{\sqrt{2\pi}}, \qquad f_2(x) = \eta e^{-\eta x}\mathbb{1}_{\{x\geq 0\}}, \qquad x \in \mathbb{R}.$$

Furthermore,

$$\zeta_1^{(a\sigma)} \equiv \mathbb{E}^{\mathbb{P}}\big[e^{a\sigma Z_1}\big] = e^{\frac{1}{2}(a\sigma)^2}, \qquad \zeta_2^{(a\sigma)} \equiv \mathbb{E}^{\mathbb{P}}\big[e^{a\sigma Z_2}\big] = \frac{\eta}{\eta - a\sigma},$$

where the latter holds if $\eta > a\sigma$. It follows that

$$\tilde{f}_1(x) \equiv \frac{e^{a\sigma x}}{\zeta_1^{(a\sigma)}} f_1(x) = \phi(x - a\sigma), \qquad \tilde{f}_2(x) \equiv \frac{e^{a\sigma x}}{\zeta_2^{(a\sigma)}} f_2(x) = (\eta - a\sigma)e^{-(\eta - a\sigma)x}\mathbb{1}_{\{x\geq 0\}}, \qquad x \in \mathbb{R}.$$

Then, applying Proposition 1.6.1 yields (1.6.15).