# Chapter 2

# Local hedging of variable annuities in the presence of basis risk

**Résumé**

Une méthode de couverture des rentes variables en présence de risque de base est développée. Un modèle à changement de régime est considéré pour la dynamique des actifs du marché. L'approche est basée sur une optimisation locale du risque et est donc très flexible. Le critère d'optimisation locale est lui-même optimisé pour minimiser les exigences de fonds propres associées aux rentes variables, ces dernières étant quantifiées par la mesure de risque CVaR. Par rapport aux benchmarks, notre méthode réussit à réduire simultanément les exigences de fonds propres et à augmenter la rentabilité. En effet, le schéma de couverture locale proposé bénéficie d'une exposition plus élevée au risque de marché et d'une diversification temporelle du risque pour gagner un rendement excédentaire et faciliter l'accumulation de capital. Une version robuste des stratégies de couverture couvrant le risque de modèle et l'incertitude des paramètres est également offerte.

**Abstract**

A method to hedge variable annuities in the presence of basis risk is developed. A regime-switching model is considered for the dynamics of market assets. The approach is based on a local optimization of risk and is therefore very tractable and flexible. The local optimization criterion is itself optimized to minimize capital requirements associated with the variable annuity policy, the latter being quantified by the CVaR risk metric. In comparison to benchmarks, our method is successful in simultaneously reducing capital requirements and increasing profitability. Indeed the proposed local hedging scheme benefits from a higher exposure to equity risk and from time diversification of risk to earn excess return and facilitate the accumulation of capital. A robust version of the hedging strategies addressing model risk and parameter uncertainty is also provided.

*Keywords :* Basis Risk, Hedging, Segregated Funds, Variable Annuities, Risk Measures, Risk Management, Regime-Switching Models.

## 2.1 Introduction

Variable annuity policies issued by life insurance companies are hybrid contracts involving both savings and insurance features. Indeed, such contracts allow the policyholder investing its account value in a mutual fund and obtaining variable returns tied to equity market performance. Moreover, those policies also offer guarantees taking various possible forms : a minimal rate of return on investments, a minimal benefit amount upon the death of the policyholder, etc.

The hedging of variable annuity guarantees by insurers presents multiple specific challenges since such products involve several features which are not present for vanilla options : mortality risk, lapse risk, periodic management charges to policyholder, fancy guarantee structures (e.g., ratchet features or GMWBs[1]), long maturities, and basis risk. Basis risk stems from the fact that insurers apply in practice a cross-hedge with liquid futures to mitigate risk associated with variable annuity liabilities due to the inconvenience or impossibility of shorting shares of the underlying mutual fund. Basis risk therefore refers to the imperfect correlation between returns of funds underlying variable annuity guarantees and returns of futures used to perform the hedge.

The current work aims at studying hedging schemes applicable to variable annuities by placing a special emphasis on the presence of basis risk, which is known to have a substantial impact on hedging residual risk in practice. As indicated in Zhang (2010), during the 2008 financial crisis, basis risk was one the most important sources of losses among insurers which implemented a dynamic hedging schemes to hedge guarantees associated with variable annuities. Although this risk has a material impact on hedging efficiency, basis risk has not received extensive attention within the literature in the context of variable annuities. An exception to this is the work of Ankirchner et al. (2014) who study the impact of variable annuities product design (proportional versus fixed charges) on the magnitude of basis risk and liquidity risk faced by the insurer. Basis risk in the context of option hedging is also studied in Zhang et al. (2017) who provide an analytical solution to a global mean-variance dynamic hedging problem under a bivariate Itô diffusion framework.

The current paper provides with three main contributions. First, a tractable and efficient hedging scheme making use of futures contracts is designed to hedge equity risk related to variable annuities issued by an insurer in the presence of basis risk. The optimization of the hedge is done through a local criterion which is itself optimized to minimize capital requirements associated with a given policy. Using a local criterion provides with sufficient tractability to consider realistic regime-switching asset price dynamics. The latter model is sufficiently realistic

1. A Guaranteed Minimum Withdrawal Benefit (GMWB) is a guarantee attached to a variable annuity which provides to the policyholder the right to withdraw from its policy a minimal amount each month until his initial investment is recouped.

to replicate stylized facts of financial markets (see Augustyniak and Boudreault, 2012, who show that regime-switching models can reproduce the thick left tail of financial returns), but also sufficiently parsimonious to retain tractability. Regime-switching models in the context of variable annuities hedging were considered by Wang and Yin (2012) and Qian et al. (2011) who applied respectively quantile hedging and local-risk minimization to perform the hedge. However, their framework does not include basis risk which is a desirable addition provided by the current paper. A key observation stemming from the simulation experiments illustrated in the current paper is that the omission of basis risk leads to severe risk under-estimation.

Our work mainly focuses on equity risk, which is the only source of uncertainty in the developed hedging scheme. The mutual fund underlying the variable annuity is assumed to be fully invested in equity, and therefore the hedge is performed with an equity futures. Extensions to our model handling stochastic interest rates, dynamic lapses and stochastic mortality improvement will be developed in upcoming papers from the authors. For instance, the inclusion of mortality related hedging instruments such as longevity bonds within the hedging scheme would be very relevant. Including additional sources of risk within our framework would have the effect of increasing capital requirements and potentially reducing the proportion of risk generated by basis risk since the latter would be diluted among other sources of risk. Additionally, the inclusion of dynamic lapses might increase the fair fees level and the magnitude of tail losses ; policyholders are expected to act in an adversarial manner and keep the policy in-force in scenarios where markets go down and large losses are incurred. The impact of embedding stochastic mortality and lapses within hedging schemes is investigated in Gaillardetz et al. (2012), Kling et al. (2014) and Boudreault and Augustyniak (2015) among others.

The second contribution of the current paper is the benchmarking of our method against approaches commonly found in the literature. Our local approach is shown in numerical experiments to greatly outperform the local minimal variance strategy simultaneously in terms of profitability and capital requirements, which is a significant contribution. In absence of model risk, the optimal local hedging approach even outperforms the no-hedging approach in terms of profitability. This is surprising due to the traditional premise stipulating that reducing risk through hedging comes at the expense of lower expected returns. However, this last finding is shown to be sensitive to model risk and parameter uncertainty ; when considering a robust version of the hedging strategy which addresses model risk, the no-hedging is much riskier but slightly more profitable in average than the optimal mean-variance hedge. Our approach benefits from time diversification of risk as it increases its local exposure to equity risk to generate higher expected returns and facilitate the accumulation of capital through time. Many commonly used approaches such as delta hedging attempt minimizing the exposure to equity risk and thus are unable to benefit from time diversification of risk. The sub-optimality of delta hedging in terms of optimization of global risk (as measured for instance by capital requirements) is well documented, see for instance Brandt (2003), Godin

(2016) and Augustyniak et al. (2016).

The third contribution relates to the numerical implementation of the proposed hedging methodology. Approximations based on Taylor expansions are applied on guarantee values to achieve a dimension reduction which substantially increases computational speed and convenience. This approach leads to a representation of the optimal hedging strategy which shows our method is a generalization of Greek-based methods implemented in practice such as delta hedging.

The current paper is divided as follows. Section 2.2 details the mathematical representation of cash flows involved in a dynamic hedging scheme for variable annuity guarantees. Local hedging optimization criteria are discussed in this section. Section 2.3 presents the regime-switching market model and outlines the application of the local hedging methodology to this particular market. Simulation based numerical experiments are presented in Section 2.4. Section 2.5 illustrates the implementation of a robust version of the hedging strategies addressing model risk and parameter uncertainty. Section 2.6 concludes.

## 2.2 Variable annuities hedging mechanics

This section outlines the mathematical model representing cash flows involved during the hedging of variable annuity guarantees by insurers.

### 2.2.1 Cash flows to the insurer

Consider a discrete set of monthly time steps $\mathcal{T} = \{0, \ldots, T\}$ and a probability space $(\Omega, \mathcal{F}_T, \mathbb{P})$ equipped with a filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\in\mathcal{T}}$. The insurance company issues variable annuities to policyholders at time $t = 0$ and hedges risk pertaining to these contracts. Without loss of generality, a Guaranteed Minimal Maturity Benefit (GMMB) policy is considered.

#### Cash flows

The policy account is invested in a mutual fund whose value is a $\mathcal{F}$-adapted process denoted by $F = \{F_t\}_{t\in\mathcal{T}}$. Fees are periodically charged to policyholders and withdrawn from the policy account. Fees apply to all policyholders alive at the beginning of the period, but are only charged at the end of the period. Although fees are sometimes charged at the beginning of the period in practice, the impact of this fees timing assumption is very limited. Indeed, the discount factor applying to a single monthly time period is very close to 1. The total fee rate charged to policyholders is denoted by $\omega_{tot}$. For a policyholder active at time $t-1$, the fees charged at time $t$ are given by $\omega_{tot} A_{t-1} \frac{F_t}{F_{t-1}}$, where $A_{t-1}$ is the post-fee policy account value at time $t-1$. The dynamics of $A = \{A_t\}_{t\in\mathcal{T}}$ is thus given by

$$A_{t+1} = A_t(1-\omega_{tot})\frac{F_{t+1}}{F_t}, \qquad t \in \{0,\dots,T-1\}. \tag{2.2.1}$$

$$\Rightarrow A_t = F_t\frac{A_0}{F_0}(1-\omega_{tot})^t, \qquad t \in \mathcal{T}. \tag{2.2.2}$$

A policyholder who dies during the period $(t, t+1]$ receives the amount $A_{t+1}$ at time $t+1$. Moreover, at any time before maturity, the policyholder has the possibility to withdraw its investment from the variable annuity, which is called a lapse. In practice, insurers can diversify away a large proportion of idiosyncratic mortality and lapse risks by insuring a large number of policyholders. In the current framework, it is assumed that these risks can be fully diversified in this manner and are thus deterministic. The constant lapse rate on each period is denoted by $b$. Furthermore, ${}_tp_x$ is defined as the probability that a policyholder aged $x$ months at time 0 survives $t$ months. The proportion of policies that are still active at time $t$ is thus given by

$$\ell_t = (1-b)^t {}_tp_x, \qquad t \in \mathcal{T}. \tag{2.2.3}$$

Surrender charges are not considered for simplicity, and therefore a policyholder who lapses during the period $(t, t+1]$ receives the amount $A_{t+1}$ at time $t+1$. Such a simplification is acceptable in our framework due to the constant lapse rate assumption. However, extensions embedding dynamic lapses would definitely require the inclusion of surrender charges since such charges have a significant impact on policyholder surrender incentives as shown for instance by MacKay et al. (2017).

For a GMMB contract, in the case of a policyholder lapse or death, the benefit provided to the policyholder is fully funded by its policy account ; the insurer does not incur an outflow in this case. However, due to the guarantee, the insurer needs to pay the difference between the benefit and the policy account value at maturity if the policyholder is alive and the policy remains in-force until maturity. The benefit in excess of the account value paid to a GMMB policyholder whose policy is still active at the maturity $T$ is given by $\max(0, K - A_T)$, where the guaranteed amount $K$ is considered to be deterministic for simplicity.

Only a portion of fees collected from policyholders is allocated to the hedging portfolio for the guarantee, as the rest is allocated to profits and expenses. The fee rate which relates to fees allocated to hedging is denoted by $\omega_{opt}$. Hence, if the policyholder is active at time $t-1$, the amount $\omega_{opt}A_{t-1}\frac{F_t}{F_{t-1}} = \frac{\omega_{opt}}{1-\omega_{tot}}A_t$ is received at time $t$ and used by the insurance company for hedging. Therefore, the net cash outflow for the insurer at time $t$ is given by

$$CF_t = -\frac{\omega_{opt}}{1-\omega_{tot}}A_t\ell_{t-1} + \mathbb{1}_{\{t=T\}}\max(0, K - A_T)\ell_T, \qquad t \in \{1,\dots,T\}, \tag{2.2.4}$$

with $CF_0 = 0$ as no immediate cash flows are involved at time $t = 0$. Defining

$$\tilde{K} \equiv \frac{KF_0}{A_0(1-\omega_{tot})^T}, \qquad \gamma_t \equiv \frac{A_0}{F_0}(1-\omega_{tot})^t\ell_t, \qquad t \in \mathcal{T}, \tag{2.2.5}$$

the relationship (2.2.2) can be used to express the cash flows directly in terms of the fund's value :

$$CF_t = -\omega_{opt}\gamma_{t-1}F_t + \mathbb{1}_{\{t=T\}}\gamma_T \max\big(0, \tilde{K} - F_T\big), \qquad t \in \{1, \dots, T\}. \tag{2.2.6}$$

### Pricing

Since variable annuity option liabilities are not openly traded in markets, their fair value must be modeled. A possibility would be to use a value that is endogenous to the hedging strategy, such as the value that would allow optimizing the hedge according to some predefined criterion. Such price could be obtained through the quadratic global hedging approach ; see Rémillard and Rubenthaler (2013) for a general framework and Rémillard et al. (2017) for the Gaussian regime-switching specialization. However, prices are not endogenized to the hedging strategy in practice since insurance companies already have a pricing model which they use to determine the appropriate level of management expenses (MER) charged to clients.

The fair value of liabilities is obtained through risk-neutral valuation using an equivalent martingale measure $\mathbb{Q}$. We consider a risk-free asset whose price at time $t$ is denoted $B_t = e^{rt}$ where $r$ is the periodic risk-free rate. Since the discount price process $\{F_t/B_t\}_{t\in\mathcal{T}}$ is a martingale under $\mathbb{Q}$, it is straightforward to show that the time-$t$ GMMB contract value is

$$\Pi_t = B_t\,\mathbb{E}^{\mathbb{Q}}\left[\sum_{j=t+1}^{T}\frac{CF_j}{B_j}\middle|\mathcal{F}_t\right] = -\omega_{opt}F_t\sum_{j=t+1}^{T}\gamma_{j-1} + \mathbb{1}_{\{t<T\}}\gamma_T G_t, \qquad t \in \mathcal{T}, \tag{2.2.7}$$

where

$$G_t \equiv B_t\,\mathbb{E}^{\mathbb{Q}}\left[\frac{\max\big(0, \tilde{K} - F_T\big)}{B_T}\middle|\mathcal{F}_t\right]. \tag{2.2.8}$$

Under this convention, the price $\Pi_t$ excludes the cash flow $CF_t$ from the pricing. In particular, $\Pi_T = 0$. The fee rate $\omega_{opt}$ is assumed to be a fair fee rate, i.e., the amount of fees which leads to a null initial value for the guarantee. Setting $\Pi_0 = 0$ in (2.2.7) leads to $\omega_{opt} = \frac{\gamma_T G_0}{F_0\sum_{j=1}^{T}\gamma_{j-1}}$.

Note that if the profitability provided by the fair fee rate $\omega_{opt}$ is deemed inadequate, the insurer might decide to adjust the total fee rate $\omega_{tot}$ provided that it remains competitive. Thus $\omega_{tot}$ and $\omega_{opt}$ would be related through a complex non-linear optimization in this case. However, we do not investigate such mechanisms in the current work since the pricing parameter $\omega_{tot}$ is assumed to be given.

### Hedging

To mitigate risks embedded in guarantees provided by variables annuities, insurers perform a cross hedge based on a different hedging asset $S$, which creates basis risk since the assets $F$

and $S$ are not perfectly correlated. The insurer sets up a hedging portfolio taking positions in two assets : the risk-free asset $B = \{B_t\}_{t\in\mathcal{T}}$ and a risky equity futures contract $S = \{S_t\}_{t\in\mathcal{T}}$, where $S_t$ denotes the futures price at time $t$. The use of futures as hedging instruments is consistent with insurers practices, see for instance Chopra et al. (2009), iA Financial Group (2016) and Manulife Financial Corporation (2016). It is justified by the impossibility of taking short positions on many index funds and by the high liquidity of index futures. The number of long positions within the hedging portfolio during the time interval $(t, t+1]$ are respectively denoted by $\theta^{(B)}_{t+1}$ and $\theta^{(S)}_{t+1}$, with the convention $\theta^{(B)}_0 = \theta^{(S)}_0 = 0$. The insurer performs periodic injections or withdrawals of liquidities from the hedging portfolio at each time step. The injection at time $t$ is denoted by $I_t$ (negative amounts correspond to withdrawals). Define $V^{\theta}_{t-}$ and $V^{\theta}_{t+}$ as the value of the hedging portfolio respectively before and after the injection $I_t$ and the cash flow $CF_t$ at time $t$. This leads to

$$I_t = V^{\theta}_{t+} - V^{\theta}_{t-} + CF_t, \qquad t \in \mathcal{T}. \tag{2.2.9}$$

To ensure the hedging portfolio value tracks the guarantee value, at all steps $t \in \mathcal{T}$ the cash flow injection (or withdrawal) $I_t$ is performed such that the post-injection portfolio value is equal to the guarantee value :

$$V^{\theta}_{t+} = \Pi_t, \qquad t \in \mathcal{T}. \tag{2.2.10}$$

In particular, the time 0 injection is nil, $I_0 = 0$, because $V^{\theta}_{0-} \equiv 0$, $CF_0 = 0$ and $\Pi_0 = 0$.

After the post-injection portfolio value $V^{\theta}_{t+} = \Pi_t$ is observed, the hedging portfolio is rebalanced. A new futures position $\theta^{(S)}_{t+1}$ is decided based on the selected hedging strategy, and then the whole portfolio value is invested at the risk-free rate ; entering positions on futures does not involve immediate cash flows besides margin requirements, and liquidities deposited inside the margin are assumed to accrue at the risk-free rate :

$$\theta^{(B)}_{t+1} = \frac{V^{\theta}_{t+}}{B_t} = \frac{\Pi_t}{B_t}. \tag{2.2.11}$$

The new pre-injection portfolio value at time $t+1$ is then obtained by summing the amount accrued at risk-free rate and profits/losses from futures positions :

$$V^{\theta}_{(t+1)-} = \theta^{(B)}_{t+1} B_{t+1} + \theta^{(S)}_{t+1}(S_{t+1} - S_t). \tag{2.2.12}$$

The following proposition proven in Appendix 2.A.1 gives an explicit expression for injections.

**Proposition 2.2.1.** *For $t \in \{0, \ldots, T-1\}$, define $\delta F_t \equiv F_{t+1} - F_t$, $\delta S_t \equiv S_{t+1} - S_t$, and $\delta G_t = G_{t+1} - G_t$. Cash flow injections are given by*

$$I_{t+1} = \Pi_t(1 - e^r) - \delta F_t \, \omega_{opt} \sum_{j=t+1}^{T} \gamma_{j-1} + \gamma_T \delta G_t - \theta^{(S)}_{t+1} \delta S_t. \tag{2.2.13}$$

**Remark 2.2.1.** The no-hedging (or unhedged) injection $I^{\text{NH}}_{t+1}$ is defined as the injection value that would occur with $\theta^{(S)}_{t+1} = 0$. It follows from Proposition 2.2.1 that $I_{t+1} = I^{\text{NH}}_{t+1} - \theta^{(S)}_{t+1} \delta S_t$.

### 2.2.2 Capital requirements

Insurers must hold reserves and capital to meet future variable annuity guarantee liabilities. In Canada, insurers issuing segregated fund policies (which are the Canadian equivalent of variable annuities) are required by the Office of the Superintendent of Financial Institutions (OSFI) to hold a Total Gross Capital Required (TGCR) at time $t = 0$ represented by [2]

$$\mathrm{TGCR} = \mathrm{CVaR}^{\mathbb{P}}_{0.95}\left[\sum_{t=1}^{T} e^{-rt} I_t\right]. \tag{2.2.14}$$

The CVaR risk measure is defined rigorously in Rockafellar and Uryasev (2002). For a continuous random variable $X$, the $\mathrm{CVaR}_\alpha[X]$ can be interpreted as the average of the worse $100(1-\alpha)\%$ scenarios. Note that U.S. recommendations for determining capital requirements for variable annuities are also based on the CVaR risk measure. [3]

### 2.2.3 Selection of the hedging strategy

The current paper's approach for the selection of the hedging strategy $\theta^{(S)} = \{\theta^{(S)}_t\}_{t\in\mathcal{T}}$ is to use a local criterion based on risk measures to optimize the risk. An $\mathcal{F}_t$ risk measure is a mapping $\mathcal{R}_t : \mathcal{X}_{\mathcal{F}_T} \to \mathcal{X}_{\mathcal{F}_t}$, where $\mathcal{X}_{\mathcal{G}}$ is the set of $\mathcal{G}$-measurable random variables for some sigma-algebra $\mathcal{G}$. The number of futures positions in the hedging portfolio is chosen by minimizing the risk related to the next cash injection :

$$\theta^{(S)*}_{t+1} = \underset{\theta^{(S)}_{t+1}}{\arg\min}\, \mathcal{R}_t(I_{t+1}), \qquad t \in \{0, \dots, T-1\}, \tag{2.2.15}$$

for a given dynamic risk measure $\{\mathcal{R}_t\}_{t=0}^{T-1}$, where $\mathcal{R}_t$ is an $\mathcal{F}_t$ risk measure for each $t$. Local procedures in hedging were pioneered by Ederington (1979) who uses $\mathcal{R}_t(\bullet) = \mathrm{Var}^{\mathbb{P}}[\bullet|\mathcal{F}_t]$. However, an important drawback associated with the variance is that it focuses purely on risk in general, penalizing upside risk and failing to incorporate expected costs in the tradeoff. Rockafellar and Uryasev (2000) use the CVaR measure to optimize their hedge which allows reducing the magnitude and frequency of extreme losses. A plethora of other classes of risk measures that were developed in the literature could also be considered, for instance coherent risk measures (Artzner et al., 1999), deviation measures (Rockafellar et al., 2002) and distortion measures (Wang, 2000).

2. See Section 11 of the instruction guide by the OSFI on the "Use of Internal Models for Determining Required Capital for Segregated Fund Risks" which instructs using $\mathrm{CVaR}^{\mathbb{P}}_{0.95}$ to determine the TGCR for segregated funds. See also Chapter 6 of the "Capital Adequacy Guideline" by the *Autorité des Marchés Financiers* for the Province of Québec.

3. See page 11 of the report "The Application of C-3 Phase II and Actuarial Guideline XLIII" by the American Academy of Actuaries which recommends using the $\mathrm{CVaR}^{\mathbb{P}}_{0.90}$ to determine the TGCR for variable annuities.

## 2.3 A model involving regime-switching equity risk

This section presents the regime-switching price dynamics model for risky assets $S$ and $F$.

### 2.3.1 Market model

Over a long period of time, markets go through various periods of either prosperity or turbulence. To model such dynamics, regime-switching processes have become very popular in the actuarial literature, see for instance Hardy (2001) and Hardy (2003). In these models, the overall state of the market is represented by the state of a Markov chain. Asset return distributions are then presumed to be a function of the current market state.

A regime process $h = \{h_t\}_{t\in\mathcal{T}}$ characterizes the state of the market, where $h_t \in \{1, 2\}$ can only take two possible values. Regimes $h_t$ are latent variables, i.e., they are not directly observable. The following regime-switching dynamics is assumed for the risky asset prices :

$$
\begin{gathered}
R^{(F)}_{t+1} \equiv \log\left(\frac{F_{t+1}}{F_t}\right) = \mu^{(F)}_{h_t} + \sigma^{(F)}_{h_t} z^{(F)}_{t+1}, \qquad R^{(S)}_{t+1} \equiv \log\left(\frac{S_{t+1}}{S_t}\right) = \mu^{(S)}_{h_t} + \sigma^{(S)}_{h_t} z^{(S)}_{t+1}, \\
z_{t+1} \equiv \begin{bmatrix} z^{(F)}_{t+1} \\ z^{(S)}_{t+1} \end{bmatrix} \sim N_2\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho_{h_t} \\ \rho_{h_t} & 1 \end{bmatrix}\right),
\end{gathered}
\tag{2.3.1}
$$

where $N_2(\mu, \Sigma)$ is the bivariate normal distribution with mean vector $\mu$ and covariance matrix $\Sigma$, $z = \{z_t\}_{t\in\mathcal{T}}$ is a strong standardized Gaussian bivariate white noise, and the remaining parameters are constants to be estimated. Under this model, asset prices evolve according to

$$
F_{t+1} = F_t e^{\mu^{(F)}_{h_t} + \sigma^{(F)}_{h_t} z^{(F)}_{t+1}}, \qquad S_{t+1} = S_t e^{\mu^{(S)}_{h_t} + \sigma^{(S)}_{h_t} z^{(S)}_{t+1}}. \tag{2.3.2}
$$

The market information at time $t$ is $\mathcal{F}_t \equiv \sigma\big(S_u, F_u : u = 0, \ldots, t\big)$. Following the lines of François et al. (2014), a full information filtration $\mathcal{G} \equiv \{\mathcal{G}_t\}_{t\in\mathcal{T}}$, where $\mathcal{G}_t \equiv \mathcal{F}_t \vee \sigma\big(h_u : u = 0, \ldots, t\big)$ is introduced. The regime process $h$ is assumed to have the Markov property with respect to $\mathcal{G}$, i.e., for some transition matrix

$$
P = \begin{bmatrix} P_{1,1} & P_{1,2} \\ P_{2,1} & P_{2,2} \end{bmatrix}, \tag{2.3.3}
$$

with $P_{1,2} = 1 - P_{1,1}$ and $P_{2,2} = 1 - P_{2,1}$, the following relationship holds for $i, j \in \{1, 2\}$ :

$$
\mathbb{P}\big(h_{t+1} = j \big| h_t = i, \{h_u\}_{u=0}^{t-1}, \{(F_u, S_u)\}_{u=0}^{t}\big) = \mathbb{P}\big(h_{t+1} = j \big| h_t = i\big) = P_{i,j}. \tag{2.3.4}
$$

As in François et al. (2014), the regime mass functions given partial information $\mathcal{F}_t$ are defined as follows :

$$
\eta^{\mathbb{P}}_{i,t} \equiv \mathbb{P}(h_t = i|\mathcal{F}_t), \qquad i \in \{1, 2\}, \tag{2.3.5}
$$

and can be computed recursively through

$$\eta_{i,t+1}^{\mathbb{P}} = \frac{\sum_{j=1}^{2} \phi_{\mu_j,\Sigma_j}\Big(R_{t+1}^{(F)}, R_{t+1}^{(S)}\Big)\eta_{j,t}^{\mathbb{P}} P_{j,i}}{\sum_{j=1}^{2} \phi_{\mu_j,\Sigma_j}\Big(R_{t+1}^{(F)}, R_{t+1}^{(S)}\Big)\eta_{j,t}^{\mathbb{P}}}, \qquad i \in \{1,2\}, \tag{2.3.6}$$

where $\phi_{\mu_j,\Sigma_j}$ denotes the bivariate Gaussian probability density function with mean vector and covariance matrix given by

$$\mu_j \equiv \begin{bmatrix} \mu_j^{(F)} \\ \mu_j^{(S)} \end{bmatrix}, \qquad \Sigma_j \equiv \begin{bmatrix} \left(\sigma_j^{(F)}\right)^2 & \rho_j \sigma_j^{(F)} \sigma_j^{(S)} \\ \rho_j \sigma_j^{(F)} \sigma_j^{(S)} & \left(\sigma_j^{(S)}\right)^2 \end{bmatrix}. \tag{2.3.7}$$

The following result deriving from (2.3.2) is useful to develop offline schemes for computing the local hedging strategies.

**Proposition 2.3.1.** *The $\mathcal{F}_t$-conditional distribution of $\left(\frac{F_{t+1}}{F_t}, \frac{S_{t+1}}{S_t}\right)$ depends only on $\eta_{1,t}^{\mathbb{P}}$ under the physical measure $\mathbb{P}$.*

Sojourn time conditional probabilities are quite useful to obtain analytical option pricing formulas (see, e.g., Hardy, 2001). For $t \in \{0, \dots, T-1\}$, $\tau \in \{0, \dots, T-t\}$, and $i \in \{1,2\}$, we define the following :

$$H_{t,\tau,i} \equiv \mathbb{P}\big(Y_t = \tau \big| h_t = i\big), \qquad Y_t \equiv \sum_{u=t}^{T-1} \mathbb{1}_{\{h_u = 1\}}, \tag{2.3.8}$$

where $Y_t$ is the sojourn time in regime 1 between time $t$ and time $T$.

A recursive algorithm is available for computing them. Initialize with $H_{T-1,1,1} = H_{T-1,0,2} = 1$ and $H_{T-1,j,1} = H_{T-1,k,2} = 0$ for all $j \neq 1$ and all $k \neq 0$. Starting at $t = T-2$, the following backward induction formulas from Hardy (2001) are used :

$$H_{t,k,1} = \sum_{i=1}^{2} P_{1,i}\, H_{t+1,k-1,i}, \qquad H_{t,k,2} = \sum_{i=1}^{2} P_{2,i}\, H_{t+1,k,i}.$$

### 2.3.2 Valuing the guarantee

The above market model is incomplete, and therefore there exists an infinite number of risk-neutral measures. For analytical tractability, the chosen $\mathbb{Q}$ is such that the dynamics remain a regime-switching model of the same form and with the same transition matrix, but with the drift parameters $\left[\mu_j^{(F)}, \mu_j^{(S)}\right]$ replaced by $\left[r - \frac{1}{2}\left(\sigma_j^{(F)}\right)^2, -\frac{1}{2}\left(\sigma_j^{(S)}\right)^2\right]$ for $j \in \{1,2\}$. Note that the risk-free rate $r$ does not appear in the risk-neutral drift of $S$ since it is a futures contract. The usual Girsanov-type change of measures could be applied to show the existence of such a measure, see for instance Elliott et al. (2005) for analogous work in continuous-time.

An analytical formula for the price $G_t$, see (2.2.8), of the option embedded within the guarantee can be obtained following the lines of Hardy (2001). However, the option price in the latter paper depends on the regime currently prevailing in the economy. In the current work, regimes are unobservable and the option price is therefore a weighted average of the prices associated with each regime where weights are the respective risk-neutral probabilities of currently being in each regime, namely $\eta_{i,t}^{\mathbb{Q}} \equiv \mathbb{Q}\big(h_t = i | \mathcal{F}_t\big)$.

The option price $G_t$ is represented by a function $g\big(t, F_t, \eta_{1,t}^{\mathbb{Q}}\big)$, which for $t < T$ is given by

$$g(t,F,\eta) = \sum_{\tau=0}^{T-t} \Big(\eta H_{t,\tau,1} + (1-\eta) H_{t,\tau,2}\Big)\Big(\tilde{K} e^{-r(T-t)} \Phi\big(-d_2(\tau)\big) - F\Phi\big(-d_1(\tau)\big)\Big), \quad (2.3.9)$$

where $\Phi$ is the standard normal cumulative distribution function, and

$$\begin{aligned} d_1(\tau) &\equiv \frac{\log\big(F/\tilde{K}\big) + (T-t)r + \frac{1}{2}\left[\tau\Big(\sigma_1^{(F)}\Big)^2 + (T-t-\tau)\Big(\sigma_2^{(F)}\Big)^2\right]}{\left[\tau\Big(\sigma_1^{(F)}\Big)^2 + (T-t-\tau)\Big(\sigma_2^{(F)}\Big)^2\right]^{1/2}}, \\ d_2(\tau) &\equiv d_1(\tau) - \left[\tau\Big(\sigma_1^{(F)}\Big)^2 + (T-t-\tau)\Big(\sigma_2^{(F)}\Big)^2\right]^{1/2}. \end{aligned} \quad (2.3.10)$$

The Delta is given by

$$\frac{\partial g}{\partial F}(t,F,\eta) = -\sum_{\tau=0}^{T-t} \big(\eta H_{t,\tau,1} + (1-\eta) H_{t,\tau,2}\big)\Phi\big(-d_1(\tau)\big), \qquad t < T, \quad (2.3.11)$$

and $\frac{\partial g}{\partial F}(T,F,\eta) = -\mathbb{1}_{\{\tilde{K} > F\}}$, i.e., for $t = T$.

### 2.3.3 Taylor expansion on injections

The cash flow injections (or withdrawals) involved in the monthly rebalancing of the hedging portfolio are characterized in Proposition 2.2.1, where the expression of the injection $I_{t+1}$ at time $t+1$ involves the monthly change in value of the guarantee : $\delta G_t \equiv G_{t+1} - G_t$. In order to simplify the solution to the hedging optimization problem, a Taylor approximation of $\delta G_t$ can be applied using the option Greeks. The pricing function $g(t,F,\eta)$ given in (2.3.9) is only defined for discrete values of $t$, and thus the expansion cannot be centered on the time $t$ as it would require a well-defined time sensitivity. Our solution to this issue is based on the following delta-type approximation :

$$\begin{aligned} \delta G_t &= \underbrace{g\big(t+1, F_{t+1}, \eta_{1,t+1}^{\mathbb{Q}}\big) - g\big(t+1, F_t, \eta_{1,t}^{\mathbb{Q}}\big)}_{\approx \delta F_t \frac{\partial g}{\partial F}\left(t+1, F_t, \eta_{1,t}^{\mathbb{Q}}\right)} + g\big(t+1, F_t, \eta_{1,t}^{\mathbb{Q}}\big) - g\big(t, F_t, \eta_{1,t}^{\mathbb{Q}}\big), \\ \Rightarrow \delta G_t &\approx \delta F_t \frac{\partial g}{\partial F}\big(t+1, F_t, \eta_{1,t}^{\mathbb{Q}}\big) + g\big(t+1, F_t, \eta_{1,t}^{\mathbb{Q}}\big) - g\big(t, F_t, \eta_{1,t}^{\mathbb{Q}}\big), \end{aligned} \quad (2.3.12)$$

where $\delta F_t = F_{t+1} - F_t$. Note that the impact of the variation $\delta\eta^{\mathbb{Q}}_{1,t} = \eta^{\mathbb{Q}}_{1,t+1} - \eta^{\mathbb{Q}}_{1,t}$ and of higher order variations, e.g., $(\delta F_t)^2$, have been omitted from the approximation. Although our hedging methodology can be extended to incorporate such corrections, unreported tests, available from the authors upon request, showed that their impact is not material.

Next, define the following Greek letters for $t \in \{0, \ldots, T-1\}$ :

$$\begin{aligned} \Theta_t &\equiv \Pi_t(1-e^r) + \gamma_T\Big[g\big(t+1, F_t, \eta^{\mathbb{Q}}_{1,t}\big) - g\big(t, F_t, \eta^{\mathbb{Q}}_{1,t}\big)\Big], \\ \Delta_t &\equiv -\omega_{opt}\sum_{j=t+1}^{T}\gamma_{j-1} + \gamma_T\frac{\partial g}{\partial F}\big(t+1, F_t, \eta^{\mathbb{Q}}_{1,t}\big). \end{aligned} \tag{2.3.13}$$

Using the approximation (2.3.12) in the injection formula of Proposition 2.2.1 provides us with a delta-type approximation of the injection : $I_{t+1} \approx \tilde{I}_{t+1}$, where

$$\tilde{I}_{t+1} \equiv \Theta_t + \Delta_t \delta F_t - \theta^{(S)}_{t+1}\delta S_t. \tag{2.3.14}$$

The Greek $\Theta_t$ represents the value of the injection at time $t+1$ if $F$ and $S$ are unchanged from time $t$ to time $t+1$. The Greek $\Delta_t$ measures the sensitivity of the injection to $F$.

In the current work, the injection approximation $\tilde{I}_{t+1}$ defined in (2.3.14) is used to tackle a simplified version of the hedging problem (2.2.15) that is more tractable and more easily solved. Since this approximation embeds the bulk of the risk related to the injection $I_{t+1}$, such a formulation is deemed a very good approximation of the hedging problem. Moreover, it provides a generalization of the delta-based approach to hedging and, as such, it is therefore in line with industry practices.

### 2.3.4 Local hedging based on the mean-variance risk measure

Here, a local hedging strategy based on the mean-variance family of risk measures is considered :

$$\theta^{(S)*}_{t+1} = \underset{\theta^{(S)}_{t+1}}{\arg\min}\Big\{\mathrm{Var}^{\mathbb{P}}\big[\tilde{I}_{t+1}\big|\mathcal{F}_t\big] + 2\lambda\mathbb{E}^{\mathbb{P}}\big[\tilde{I}_{t+1}\big|\mathcal{F}_t\big]\Big\}, \qquad t \in \{0, \ldots, T-1\}, \tag{2.3.15}$$

where $\tilde{I}_{t+1}$ is the injection approximation, see (2.3.14), and $\lambda \geq 0$ is a chosen constant quantifying the mean-variance tradeoff. The mean-variance risk measure offers a flexible parametrization of the risk-return tradeoff while benefiting from convenient analytical properties. The solution to the above minimization problem is given in Proposition 2.3.2. The proof of this proposition is omitted since it is straightforward. The explicit formulas for the variance, covariance, and expectation involved in this proposition are provided in Appendix 2.A.2.

**Proposition 2.3.2.** *The mean-variance hedging strategy* (2.3.15) *is given by*

$$\theta^{(S)*}_{t+1} = \Delta_t\frac{\mathrm{Cov}^{\mathbb{P}}[F_{t+1}, S_{t+1}|\mathcal{F}_t]}{\mathrm{Var}^{\mathbb{P}}[S_{t+1}|\mathcal{F}_t]} + \lambda\frac{\mathbb{E}^{\mathbb{P}}[S_{t+1}|\mathcal{F}_t] - S_t}{\mathrm{Var}^{\mathbb{P}}[S_{t+1}|\mathcal{F}_t]}. \tag{2.3.16}$$

The strategy obtained under the choice $\lambda = 0$ is referred to as the *minimal variance strategy*. Interestingly, in the absence of basis risk the minimal variance strategy coincides with the usual form of delta hedging. The mean-variance strategy can be thought of as a generalization of delta hedging which can account for basis risk in addition of bringing expected costs into the tradeoff.

### Properties of the mean-variance strategy

The proposition below characterizes the upper bound to the reduction of local risk, as measured by the variance, that is attainable in the presence of basis risk. It states, for instance, that if the correlation between returns of $S$ and $F$ is 90%, then the proportion of the standard deviation that can be eliminated is $1 - \sqrt{1 - 0.9^2} \approx 56.4\%$. The proof of Proposition 2.3.3 is obtained by combining (2.3.16) and (2.3.14).

**Proposition 2.3.3.** *Let $\tilde{I}^{\text{NH}}_{t+1}$ and $\tilde{I}^{\text{MV}}_{t+1}$ be the injection approximation* (2.3.14) *respectively for $\theta^{(S)}_{t+1} = 0$ (no hedging) and for $\theta^{(S)}_{t+1}$ given by* (2.3.16) *with $\lambda = 0$ (minimal variance). Then*

$$\frac{\text{Var}^{\mathbb{P}}\left[\tilde{I}^{\text{MV}}_{t+1} \middle| \mathcal{F}_t\right]}{\text{Var}^{\mathbb{P}}\left[\tilde{I}^{\text{NH}}_{t+1} \middle| \mathcal{F}_t\right]} = 1 - \text{Corr}^{\mathbb{P}}\left[\delta F_t, \delta S_t \middle| \mathcal{F}_t\right]^2. \tag{2.3.17}$$

The following remark relates the injection under the minimal variance strategy to the injection under the more general mean-variance strategy. This relation will prove to be useful when analyzing the simulation results. To prove it, one simply has to use Proposition 2.3.2 in the injection approximation formula (2.3.14).

**Remark 2.3.1.** Let $\tilde{I}_{t+1}$ and $\tilde{I}^{\text{MV}}_{t+1}$ be the injection approximation (2.3.14) respectively under the mean-variance strategy for some $\lambda \geq 0$ and the minimal variance strategy ($\lambda = 0$). Then

$$\tilde{I}_{t+1} = \tilde{I}^{\text{MV}}_{t+1} - \lambda \frac{\mathbb{E}^{\mathbb{P}}[S_{t+1}|\mathcal{F}_t] - S_t}{\text{Var}^{\mathbb{P}}[S_{t+1}|\mathcal{F}_t]} \delta S_t.$$

### Optimizing the mean-variance tradeoff

For the mean-variance family of risk measures, the free parameter to be optimized is $\lambda$, which characterizes the risk-reward tradeoff. The objective is thus to find the value of that parameter which minimizes the capital (2.2.14) that the insurer is required to hold at time $t = 0$. We solve

$$\lambda^* \equiv \arg\min_{\lambda} \text{CVaR}^{\mathbb{P}}_{0.95}\left[\sum_{t=1}^{T} e^{-rt} I_t\right], \tag{2.3.18}$$

over hedging strategies of the form outlined in Proposition 2.3.2. Note that this optimization problem is based on the exact injections $I_t$ rather than on their approximations $\tilde{I}_t$; this is because there are no numerical incentives not to rely on an exact formulation here.

A simple Monte Carlo simulation procedure is described to compute the value of $\lambda^*$. Consider a hedging strategy of the from

$$\theta_{t+1}^{(S)} = \alpha_t + \lambda \beta_t, \tag{2.3.19}$$

where $\lambda \in \mathbb{R}$, and $(\alpha_t, \beta_t)$ are some given $\mathcal{F}_t$-measurable random variables. Note that the mean-variance hedging strategy outlined in Proposition 2.3.2 has this form. For such strategies, it follows from Remark 2.2.1 that the injection value at time $t+1$ can be written as

$$I_{t+1} = I_{t+1}^{\text{NH}} - \alpha_t \delta S_t - \lambda \beta_t \delta S_t,$$

where $I_{t+1}^{\text{NH}}$ is the injection value that would occur in the absence of hedging. We therefore have

$$\sum_{t=1}^{T} e^{-rt} I_t = \xi_1 - \lambda \xi_2, \tag{2.3.20}$$

where

$$\xi_1 \equiv \sum_{t=1}^{T} e^{-rt} [I_t^{\text{NH}} - \alpha_{t-1} \delta S_{t-1}], \qquad \xi_2 \equiv \sum_{t=1}^{T} e^{-rt} \beta_{t-1} \delta S_{t-1}. \tag{2.3.21}$$

Since $\xi_1$ and $\xi_2$ do not depend on $\lambda$, a single Monte Carlo simulation of the random variables $(\xi_1, \xi_2)$ is required to optimize $\lambda$ through

$$\lambda^* = \arg\min_{\lambda} \text{CVaR}_{0.95}^{\mathbb{P}} \big[\xi_1 - \lambda \xi_2\big]. \tag{2.3.22}$$

We refer to Rockafellar and Uryasev (2000) for a study on the numerical optimization of CVaR risk measures.

### 2.3.5 Local hedging based on a general class of risk measures

The current section characterizes the solution of the local hedging problem for a very large class of risk measures. A hedging strategy which minimizes capital requirements is also presented.

#### A general class of dynamic risk measures

A general class of risk-measures called the $\mathcal{F}_t$-reducible risk measures is introduced.

**Definition 2.3.1.** $\mathcal{R}_t$ is a $\mathcal{F}_t$-*reducible risk measure* if it satisfies the following :

1. $\mathcal{R}_t$ is a law-invariant $\mathcal{F}_t$ risk measure.
2. There exists a real function $f_1$ such that $\mathcal{R}_t(Y_t + X) = f_1(Y_t) + \mathcal{R}_t(X)$ for any risk $X$ and any $\mathcal{F}_t$-measurable random variable $Y_t$.
3. There exists a nonnegative real function $f_2$ such that $\mathcal{R}_t(Y_t X) = f_2(Y_t)\mathcal{R}_t(X)$ for any admissible risk $X$ and any $\mathcal{F}_t$-measurable random variable $Y_t \geq 0$ a.s.

The $\mathcal{F}_t$-reducible class of risk measures is quite large. In particular, it includes the $\mathcal{F}_t$-conditional counterparts of all coherent risk measures in the sense of Artzner et al. (1999), such as the Conditional Value-at-Risk, $\mathrm{CVaR}_\alpha[\bullet|\mathcal{F}_t]$. It is however more general as it also includes the $\mathcal{F}_t$-conditional variance, $\mathrm{Var}[\bullet|\mathcal{F}_t]$, and the $\mathcal{F}_t$-conditional Value-at-Risk, $\mathrm{VaR}_\alpha[\bullet|\mathcal{F}_t]$. Furthermore, as shown in the proposition below, this class includes a large family of risk-reward tradeoffs, e.g., $\mathrm{Std}[\bullet|\mathcal{F}_t] + \lambda\mathbb{E}^{\mathbb{P}}[\bullet|\mathcal{F}_t]$, $\mathrm{VaR}_\alpha[\bullet|\mathcal{F}_t] + \lambda\mathbb{E}^{\mathbb{P}}[\bullet|\mathcal{F}_t]$, and $\mathrm{CVaR}_\alpha[\bullet|\mathcal{F}_t] + \lambda\mathbb{E}^{\mathbb{P}}[\bullet|\mathcal{F}_t]$.

**Proposition 2.3.4.** *Let $\mathcal{R}_t$ be a $\mathcal{F}_t$-reducible risk measure such that $\mathcal{R}_t(Y_t + Z_tX) = f_1(Y_t) + Z_t\mathcal{R}_t(X)$ for all admissible risk $X$ and all $\mathcal{F}_t$-measurable random variables $(Y_t, Z_t)$ where $Z_t \geq 0$ a.s. Then, for any constant $\lambda \in \mathbb{R}$, the risk-measure $\mathcal{M}_t(\bullet) \equiv \mathcal{R}_t(\bullet) + \lambda\mathbb{E}^{\mathbb{P}}[\bullet|\mathcal{F}_t]$ is also $\mathcal{F}_t$-reducible.*

The proof of the above statement is rather straightforward and therefore omitted.

For the purpose of a dynamic hedging strategy, sequences of risk measures that satisfy the temporal law-invariance property of Definition 2.3.2 are considered.

**Definition 2.3.2.** Let $\{\mathcal{R}_t\}_{t=0}^{T-1}$ be a sequence where $\mathcal{R}_t$ is a $\mathcal{F}_t$ risk measure for each $t$. This sequence is said to have the *temporal law-invariance* property if the following condition is satisfied : for all $t_1, t_2 \geq 0$ and all random variables $X_1, X_2$, if the conditional distribution of $X_1$ given $\mathcal{F}_{t_1}$ is the same that the conditional distribution of $X_2$ given $\mathcal{F}_{t_2}$, then $\mathcal{R}_{t_1}(X_1) = \mathcal{R}_{t_2}(X_2)$.

### Offline calculation of the hedging strategies

Here, a local hedging strategy of the following form is considered :

$$\theta_{t+1}^{(S)*} = \arg\min_{\theta_{t+1}^{(S)}} \mathcal{R}_t\big(\tilde{I}_{t+1}\big), \qquad t \in \{0, \dots, T-1\}, \tag{2.3.23}$$

where $\tilde{I}_{t+1}$ is the injection approximation, see (2.3.14), and $\{\mathcal{R}_t\}_{t=0}^{T-1}$ a dynamic risk measure such that $\mathcal{R}_t$ is $\mathcal{F}_t$-reducible for each $t$.

For temporal law-invariant sequences of reducible risk measures, it turns out that efficient pre-calculation of such hedging strategies is made possible by a trick that reduces the dimension of the associated optimization problem. This result is presented in Theorem 2.3.1 whose proof is in Appendix 2.A.2.

**Theorem 2.3.1.** *For any $\mathcal{F}_t$-reducible risk measure $\mathcal{R}_t$, the hedging strategy (2.3.23) is*

$$\theta_{t+1}^{(S)*} = \psi_{t+1}^* \frac{\Delta_t F_t}{S_t}, \qquad \psi_{t+1}^* \equiv \arg\min_{\psi} \mathcal{R}_t\Big(\psi\tfrac{\delta S_t}{S_t} - \tfrac{\delta F_t}{F_t}\Big). \tag{2.3.24}$$

*Moreover, the solution can be expressed as $\psi_{t+1}^* = \Psi_{\mathcal{R}_t}\big(\eta_{1,t}^{\mathbb{P}}\big)$ for some function $\Psi_{\mathcal{R}_t} : [0,1] \to \mathbb{R}$. Furthermore, for a temporal law-invariant sequence $\{\mathcal{R}_t\}_{t=0}^{T-1}$, one has $\Psi_{\mathcal{R}_0} = \cdots = \Psi_{\mathcal{R}_{T-1}}$.*

Theorem 2.3.1 has several consequences worthy of noticing. First, note from (2.3.14) that the $\mathcal{F}_t$-conditional distribution of $\tilde{I}_{t+1}$ depends on the state variables $(\Theta_t, \Delta_t, F_t, S_t, \eta^{\mathbb{P}}_{1,t})$ and the control parameter $\theta^{(S)}_{t+1}$. Hence, it might appear that the solution to the hedging problem is a function of five state variables, making it difficult to use an offline approach to efficiently pre-calculate the solution over a grid of points and then use interpolation methods, especially because the state variables $F_t$ an $S_t$ can cover a large range when the time index $t$ is high, which is the case for variable annuities having a maturity of several years. However, due to Theorem 2.3.1, it is actually possible to reduce the dimensionality of the hedging problem to the single state variable $\eta^{\mathbb{P}}_{1,t} \in [0,1]$. The solution can therefore be pre-calculated offline for a small grid of points over the domain [0,1] to build a continuous function using linear interpolation, reducing computational time by several orders of magnitude.

### Finding the strategy that minimizes capital requirements

The objective of this section is to show how to design a hedging strategy that attains capital requirements at least as low as the best-performing hedging strategy based on a reducible risk measure. This strategy is referred to as the the *minimal TGCR strategy.*

For the class of dynamic reducible risk measures considered in Theorem 2.3.1, the main result is that each hedging strategy has the form $\theta^{(S)*}_{t+1} = \Psi(\eta^{\mathbb{P}}_{1,t})\frac{\Delta_t F_t}{S_t}$ for some function $\Psi : [0,1] \to \mathbb{R}$. Rather than optimizing over the choice of the risk measure, the approach we propose is to directly optimize over the latter function to minimize the capital (2.2.14) that the insurer is required to hold at time $t = 0$ to meet future variable annuity guarantee liabilities. The optimization problem is

$$\Psi^* \equiv \underset{\Psi:[0,1]\to\mathbb{R}}{\arg\min}\ \mathrm{CVaR}^{\mathbb{P}}_{0.95}\left[\sum_{t=1}^{T} e^{-rt} I_t\right], \tag{2.3.25}$$

where $\theta^{(S)}_t = \Psi(\eta^{\mathbb{P}}_{1,t-1})\frac{\Delta_{t-1}F_{t-1}}{S_{t-1}}$. Note that this optimization problem is based on the exact injections $I_t$ rather than on their approximations $\tilde{I}_t$. This is because there are no reasons not to rely on the exact formulation here, in contrast with the local hedging optimization problem for which the delta approximation leads to a tremendous reduction in computational time.

**Remark 2.3.2.** It is not guaranteed that there exists a risk measure which corresponds to the optimal function $\Psi^*$. In fact, such a risk measure does not even need to exist.

To solve the above optimization problem, a parametric approximation of the solution is considered by employing a polynomial of degree $n$ :

$$\Psi(\eta) \equiv \sum_{i=0}^{n} a_i \eta^i, \tag{2.3.26}$$

where $\{a_i\}_{i=0}^n$ are constants to be determined. For the strategy $\theta_{t+1}^{(S)} = \Psi(\eta_{1,t}^{\mathbb{P}})\frac{\Delta_t F_t}{S_t}$, it follows from Remark 2.2.1 that the injection value at time $t+1$ can be expressed as

$$I_{t+1} = I_{t+1}^{\text{NH}} - \frac{\Delta_t F_t}{S_t}\delta S_t \sum_{i=0}^{n} a_i \left(\eta_{1,t}^{\mathbb{P}}\right)^i,$$

where $I_{t+1}^{\text{NH}}$ is the injection value that would occur if no hedging were performed, i.e., with $\theta_{t+1}^{(S)} = 0$. We thus have

$$\sum_{t=1}^{T} e^{-rt} I_t = \xi_1 - \sum_{i=0}^{n} a_i \xi_{2,i}, \tag{2.3.27}$$

where

$$\xi_1 \equiv \sum_{t=1}^{T} e^{-rt} I_t^{\text{NH}}, \qquad \xi_{2,i} \equiv \sum_{t=1}^{T} e^{-rt} \Delta_{t-1} F_{t-1} \frac{\delta S_{t-1}}{S_{t-1}} \left(\eta_{1,t-1}^{\mathbb{P}}\right)^i. \tag{2.3.28}$$

The solution can therefore be formulated as

$$(a_0^*, \ldots, a_n^*) \equiv \underset{(a_0,\ldots,a_n)}{\arg\min}\ \text{CVaR}_{0.95}^{\mathbb{P}} \left[ \xi_1 - \sum_{i=0}^{n} a_i \xi_{2,i} \right]. \tag{2.3.29}$$

This problem can be solved numerically by first simulating a sufficiently large sample of the random variables $(\xi_1, \xi_{2,0}, \ldots, \xi_{2,n})$ to estimate the CVaR and minimize it using standard algorithms.

To choose the degree $n$, one can fix some pre-determined value $n_{\max}$ and simulate $(\xi_1, \xi_{2,0}, \ldots, \xi_{2,n_{\max}})$. This simulated sample suffices to test all degrees below $n_{\max}$. In practice, $n$ is chosen as the smallest value for which the choice $n+1$ yields no further improvements.

## 2.4 Simulation experiments

The numerical simulations presented here form the basis of several analyses that allow making important findings about the properties of the hedging strategies and of the optimal mitigation of risks embedded in variable annuities under the presence of basis risk.

### 2.4.1 Setup

It is assumed that $F$ is the Great-West Life Canadian Equity (GWLIM) BEL fund. The vast majority of the fund wealth (roughly 90%) is invested in Canadian Equity (the remainder being cash investments). The constant risk-free rate assumption is reasonable in this context ; if the mutual fund invested in fixed income, including interest shocks would have been necessary to impact the fluctuation of the fund value. Since the mutual fund is invested in Canadian equity, this justifies using a Canadian equity index to perform the hedge. Thus, $S$ is presumed to be the futures prices of the TSX 60 index.

### Estimation results

The bivariate regime-switching model (2.3.1) is estimated using maximum likelihood. The log-likelihood function is computed through Hamilton (1989)'s filter and maximized using standard global optimization routines. For the first 66 months, the estimation methodology is based on the marginal likelihood function of the Great-West fund as the TSX 60 futures index was launched later (in September, 1999). For all subsequent months the joint density of both time series is considered. Maximum likelihood estimation results are presented in Table 2.1, where it can be seen that the first regime is characterized by positive expected returns and low volatility (bull market), whereas the second regime describes a state of higher volatility and negative expected returns (bear market). It is interesting to note that the correlation of the mutual fund and index futures returns is high in each regime ; it is $\rho_1 = 94.39\%$ in the bull market regime, which is slightly higher than in the bear market regime where it is $\rho_2 = 90.68\%$. Nevertheless, basis risk is not nil. The higher correlation in bull markets is surprising since all asset values are usually expected to depreciate simultaneously during financial crises. A possible explanation could be that, as stated in Robidoux (2015), the tactical (short-term) asset allocation within the mutual fund could significantly differ from the strategic (long-term) asset allocation target during specific market circumstances, for instance a flight to quality during a crisis, and thus the correlation structure could be altered in this situation. This observation entails that insurers should not rely on the expectation that basis risk will dampen during stress periods when hedging is most needed.

**Table 2.1** – Maximum likelihood estimation results for the bivariate lognormal two-state regime-switching model of (2.3.1). The first component is Great-West Life Canadian Equity (GWLIM) BEL fund and the second is the TSX 60 index futures.

| $\mu_1^{(j)}$ | $\sigma_1^{(j)}$ | $\mu_2^{(j)}$ | $\sigma_2^{(j)}$ |
|---|---|---|---|
| *Great-West Life Canadian Equity (GWLIM) BEL* ($j = F$) | | | |
| 0.0084 (0.0024) | 0.0330 (0.0019) | -0.0080 (0.0104) | 0.0734 (0.0081) |
| *TSX 60 index futures* ($j = S$) | | | |
| 0.0085 (0.0026) | 0.0348 (0.0022) | -0.0134 (0.0126) | 0.0858 (0.0097) |
| *Correlations* | | *Transition matrix* | |
| $\rho_1$ | $\rho_2$ | $P_{1,1}$ | $P_{2,1}$ |
| 0.9439 (0.0090) | 0.9068 (0.0269) | 0.9767 (0.0137) | 0.0850 (0.0527) |

*Note* : Standard errors are given in parentheses.

### Baseline parameters

The baseline parameters of the simulation study are presented in Table 2.1 and Table 2.2. Policyholders aged 55 years at time $t = 0$ purchasing an at-the-money GMMB variable annuity with maturity of $T = 120$ months (10 years) are considered. The survival probabilities are obtained following the methodology recommended by the Canadian Institute of Actuaries ; base mortality rates are obtained from table CPM2014, see CIA (2014), and mortality improvements are projected with their proposed rates, see Appendix C of CIA (2010). Monthly mortality rates are obtained from annual rates by assuming that the force of mortality is constant within a given year. The parameters related to asset dynamics are taken from the maximum likelihood estimation results of Table 2.1. Other parameters are deemed representative of real-life practice. For instance, the utilized lapse rate whose annualized value is roughly 4% is consistent with lapse rates presented in Ledlie et al. (2008) which range between 2% and 6%.

### Performance measures

The various local hedging strategies are benchmarked in terms of capital requirements. As explained in Section 2.2.2, the Total Gross Capital Required (TGCR) that must be held at time $t = 0$ by insurance companies in Canada can be modeled by the $\text{CVaR}_{0.95}$ of the discounted sum of injections. Moreover, the $\text{CVaR}_{0.80}$ is recommended to determine reserves :

$$\text{TGCR} = \text{CVaR}^{\mathbb{P}}_{0.95}\left[\sum_{t=1}^{T} e^{-rt} I_t\right], \qquad \text{Reserve} = \text{CVaR}^{\mathbb{P}}_{0.80}\left[\sum_{t=1}^{T} e^{-rt} I_t\right].$$

The main performance metric used in this work is the TGCR as defined above. In particular, hedging strategies that can be optimized to minimize capital requirements (see Sections 2.3.4 and 2.3.5) are implemented under this definition. Note however that such approaches can be generalized to any capital measurement criterion that is based on the discounted sum of injections.

**Table 2.2** – Baseline parameters in monthly frequency.

| | | |
|---|---|---|
| Maturity (in months) | $T$ | 120 |
| Survival probability | ${}_tp_{660}$ | Projected CPM2014 |
| Lapse rate | $b$ | 0.34% |
| Total fee rate | $\omega_{tot}$ | 0.29% |
| Risk-free rate | $r$ | 0.25% |
| GMMB guarantee | $K$ | 100 |
| Initial value of $F$ | $F_0$ | 100 |
| Initial value of $S$ | $S_0$ | 100 |

In the U.S., according to AAA (2011), the $\mathrm{CVaR}^{\mathbb{P}}_{0.90}$ and the $\mathrm{CVaR}^{\mathbb{P}}_{0.70}$ are recommended to quantify capital requirements. These values are therefore also presented. Moreover, the $\mathrm{CVaR}^{\mathbb{P}}_{0.99}$ is given to detect potential flaws of the hedging strategies in terms of heavy tail risk.

### Hedging strategies

This section presents the hedging strategies considered in the simulation experiment of Table 2.3, which will be analyzed in the next section.

The "mean-variance" strategy entails the minimization of a tradeoff between the variance and the expected value of the next cash injection. It is based on the local hedging problem of (2.3.15), the solution to which is given by Proposition 2.3.2. The value of the tradeoff parameter $\lambda$ minimizing the TGCR is determined with the approach of Section 2.3.4 and is marked by a star ($\star$). The minimal variance strategy corresponds to the special case $\lambda = 0$.

The "minimal $\mathrm{VaR}^{\mathbb{P}}_{\alpha}$" and the "minimal $\mathrm{CVaR}^{\mathbb{P}}_{\alpha}$" strategies entails minimizing the risk concerning the next injection, based on the risk measures they are named after. These strategies solve the hedging problem of (2.3.23) where the risk measure $\mathcal{R}_t(\bullet)$ is $\mathrm{VaR}^{\mathbb{P}}_{\alpha}[\bullet|\mathcal{F}_t]$ and $\mathrm{CVaR}^{\mathbb{P}}_{\alpha}[\bullet|\mathcal{F}_t]$, respectively. Their solutions are calculated offline using Theorem 2.3.1. For both strategies, the parameter $\alpha \in [0,1]$ controls the tradeoff between risk and cost minimization ; higher values of $\alpha$ encourage a pure risk reduction. The chosen values of $\alpha$ in the table are restricted to the range within which these strategies are well-behaved ; lower values of $\alpha$ can lead to a non-finite number of futures positions (unbounded solutions) and are therefore avoided.

The "minimal TGCR" strategy refers to the one described in Section 2.3.5. Contrary to the other hedging strategies, this one does not involve minimizing the risk of the next injection. Instead, it is based on the TGCR minimization problem of (2.3.25) under the polynomial model (2.3.26), which yields the numerical optimization problem (2.3.29). A polynomial of degree $n = 8$ is deemed satisfactory for the example currently considered ; in unreported tests, performance results (i.e., see Table 2.3) obtained using polynomials of degree higher than 8 are virtually identical to those obtained with the degree 8 polynomial.

For comparison, the results when no hedging is used are also given : $\theta^{(S)}_t = 0$ for all $t \in \mathcal{T}$, which is equivalent to the absence of a hedging portfolio. Furthermore, each hedging strategy is implemented in an hypothetical ideal case in which there is no basis risk. This is done by supposing that there exist futures contracts on the underlying mutual fund $F$. Although this is not the case in real life, such strategies are nevertheless implemented for the sake of our numerical study which aims at quantifying the impact of basis risk.

### 2.4.2 Results

The results for the setup outlined in the preceding section are presented in Table 2.3. These are obtained from 50,000 Monte Carlo simulation runs of the hedging strategies.

First, let's analyse the case of hedging under no basis risk. It can be seen that the strategies that are best-performing in terms of TGCR are those characterized by a small standard deviation, i.e., those entailing pure dispersion minimization. In particular, the minimal variance strategy ($\lambda = 0$) virtually coincides with minimal VaR and minimal CVaR strategies. Note that under the assumption of absence of basis risk, the minimal variance hedging strategy collapses to the delta-hedging approach widely used in the industry. Moreover, hedging strategies designed to minimize the TGCR do not substantially further improve the results obtained with the minimal variance strategy. For instance, the optimal mean-variance ($\lambda = 1.5$) and the minimal TGCR strategies both attain a TGCR of 3.7, which is quite close to the value of 4.7 obtained under the minimal variance strategy. These strategies constitute an important improvement over the no-hedging strategy, for which the TGCR is 25.3. Compared to the minimal variance strategy, mean-variance strategies with a higher value of the tradeoff parameter $\lambda$ lead to higher TGCR values. As discussed below Proposition 2.3.2, the minimal variance strategy actually corresponds to standard delta hedging as there is no basis risk. Hence, the above results show that delta hedging is quite efficient in the absence of basis risk.

The results for the case of hedging under basis risk are richer and more subtle. Minimal variance, minimal VaR, and minimal CVaR strategies do not coincide with each other as they did in the absence of basis risk. Moreover, the strategies which perform the best in terms of TGCR are not necessarily those which attain a small standard deviation ; the minimal variance strategy in fact yields the worst TGCR. The minimal TGCR strategy and the optimal mean-variance ($\lambda = 7$) respectively attain a TGCR of 8.7 and 8.6. Note that the TGCR of the mean-variance strategy can be lower than the one pertaining to the minimal TGCR strategy as the mean-variance risk measure is not included in the family of reducible risk measures. This result shows that the optimal mean-variance strategy performs at least as well as the best strategy based on reducible risk measures. This is a surprising result as the latter class of hedging strategies is very large and encompasses other forms of risk-return tradeoffs such as the mean-standard deviation, mean-VaR and mean-CVaR. Moreover the optimal mean-variance strategy leads to a value of the $\text{CVaR}^{\mathbb{P}}_{0.99}$ that is smaller than for the minimal variance strategy (17.2 vs. 18.7). The results therefore show no evidence that reducing capital requirements comes at the expense of higher tail risk for levels beyond 95%. Nevertheless, care should be applied when interpreting this result as it is not impossible that risk could be increased in the far tail at levels higher than 99%. It is also interesting to note that the optimal mean-variance strategy leads to both a smaller CVaR and a better expected value ; only the variance is worse compared to the minimal variance hedging strategy. Indeed, the CVaR is also a form of risk-reward tradeoff, and therefore it is possible that it can be lowered simultaneously with the expected value even if this entails a higher variance. The variance increase observed when applying the optimal mean-variance hedge instead of the minimal variance hedge is caused by an increase in upside risk for the insurer, which is a desirable feature.

Another consideration that is worthy of emphasizing is the large difference in capital requirements obtained when comparing the respective cases where basis risk is absent or present. For instance, the TGCR of the minimal variance strategy is 4.7 in the absence of basis risk versus 14.8 when basis risk is considered, which is a 215% increase. Even for an optimized hedging strategy, capital requirements jump from 3.7 to 8.6 (optimal mean-variance strategy) or 8.7 (minimal TGCR strategy) when basis risk is incorporated. Indeed, Proposition 2.3.3 shows that even a small amount of basis risk can lead to a substantial loss of hedging performance. Omitting basis risk in hedging schemes performance assessments could therefore lead to severe risk under-estimation. Results of this nature could have been obtained through a poor hedging instrument choice ; basis risk will be very important if the mutual fund behaves very differently than the hedging asset. However, the statistics presented in Table 2.1 (i.e., the correlation coefficients) indicate that this is not the case here. Such high correlations make it difficult to believe that the insurer could significantly improve upon the presented hedge by choosing different hedging instruments.

The above discussion can be summarized briefly. In the absence of basis risk, conditional variance minimization coincides with delta hedging and is very efficient at reducing the TGCR. In contrast, under basis risk, the strategies attaining the lowest TGCR values are those which put some weight on expected cost minimization. These results ask for further analysis to shed light on the mechanics and determinants of the optimal hedging strategy under basis risk.

### Unveiling the risk mitigation mechanics

With the exception of the mean-variance strategy, the hedging strategies in Table 2.3 can be expressed as $\theta^{(S)}_{t+1} = \Psi(\eta^{\mathbb{P}}_{1,t})\frac{\Delta_t F_t}{S_t}$, and are therefore fully characterized by their function $\Psi$. These are illustrated in Figure 2.1 and allow for a straightforward interpretation of the risk mitigation mechanics. For the minimal variance strategy, the function $\Psi$ is roughly constant and positive, implying the use of short positions only (because $\Delta_t \leq 0$ a.s.). The minimal $\mathrm{VaR}^{\mathbb{P}}_{\alpha}$ and minimal $\mathrm{CVaR}^{\mathbb{P}}_{\alpha}$ strategies also rely almost exclusively on short positions. In contrast, the minimal TGCR strategy uses long positions when the conditional probability of the bull market regime is above a certain threshold. In other words, this strategy uses the futures as an investment vehicle, as opposed to a hedging instrument, in bull market time periods. This behavior is also found in the mean-variance strategy, as shown in Figure 2.2 for a simulated trajectory of the hedging portfolio composition. These observations highlight the fact that there are actually two ways for the insurer to meet futures variable annuity liabilities :

1. The insurer can use the futures contract as a hedging asset, which entails shorting it to cover the long position in the underlying mutual fund. Doing this reduces the conditional variance of the cash flow injections as shown in Proposition 2.3.3. This strategy is however costly in bull markets because it involves shorting an asset whose price grows on average. Moreover, hedging should intuitively be less needed in bull market because the underlying mutual fund also grows on average.
2. The insurer can also invest money in capital markets through long positions in the TSX 60 futures contract. Such risky investments benefit from time diversification of risk, making them smarter choices than the risk-free asset if the time horizon is sufficiently long. Time diversification refers to the imperfect correlation between all remaining futures log-returns until maturity ; investing through a futures over a long horizon therefore reduces risk associated with the latter position when compared to a short-term investment.

Note that other investment vehicles than long equity futures positions could have been considered. Such positions could be replaced by long positions in general investment portfolios targeting long-term growth. Low-volatility funds and risk-managed funds could for instance be considered as these are designed to provide decent returns with low downside risk, which are in line with the use of investment in the current hedging framework. Such extensions are left as further work.

#### Time diversification

This section explains why a large (small) value of the mean-variance tradeoff parameter $\lambda$ is optimal when basis risk is present (absent). The starting point is Remark 2.3.1, which states that the injection under the mean-variance strategy with parameter $\lambda$ can be expressed as

$$\tilde{I}_{t+1} = \tilde{I}^{\text{MV}}_{t+1} + \lambda J_{t+1}, \qquad J_{t+1} \equiv -\frac{\mathbb{E}^{\mathbb{P}}[\delta S_t | \mathcal{F}_t]}{\text{Var}^{\mathbb{P}}[S_{t+1}|\mathcal{F}_t]} \delta S_t, \tag{2.4.1}$$

where $\tilde{I}^{\text{MV}}_{t+1}$ is the injection under the minimal variance strategy. Hence, the mean-variance injection can be represented as a departure “$+\lambda J_{t+1}$" from the minimal variance injection. The above definition implies that $\mathbb{E}^{\mathbb{P}}[J_{t+1}] \leq 0$, so this departure indeed implies a reduction in the expected injection value. Note that it can also be expressed as

$$J_{t+1} \equiv -\frac{\mathbb{E}^{\mathbb{P}}[\delta S_t / S_t | \mathcal{F}_t]}{\text{Var}^{\mathbb{P}}[S_{t+1}/S_t|\mathcal{F}_t]} \frac{\delta S_t}{S_t},$$

which shows that the $\mathcal{F}_t$-conditional distribution of $J_{t+1}$ depends only on $\eta^{\mathbb{P}}_{1,t}$ ; this is because the $\mathcal{F}_t$-conditional distribution of $\frac{\delta S_t}{S_t}$ depends only on $\eta^{\mathbb{P}}_{1,t}$, as it can be seen from (2.3.2). Furthermore, one can show from the above equation that

$$\mathbb{P}(J_{t+1} > 0 | \mathcal{F}_t) = \sum_{i=1}^{2} \eta^{\mathbb{P}}_{i,t} \left[ \Phi\left( \frac{\mu_i^{(S)}}{\sigma_i^{(S)}} \right) \mathbb{1}_{\left\{ \mu_i^{(S)} + \frac{1}{2}\left(\sigma_i^{(S)}\right)^2 < 0 \right\}} + \Phi\left( -\frac{\mu_i^{(S)}}{\sigma_i^{(S)}} \right) \mathbb{1}_{\left\{ \mu_i^{(S)} + \frac{1}{2}\left(\sigma_i^{(S)}\right)^2 > 0 \right\}} \right].$$

In particular, with parameters presented in Table 2.1,

$$\mathbb{P}(J_{t+1} > 0|\mathcal{F}_t) \in [40.0\%, 45.7\%],$$

which means that the departure from the minimal variance strategy entails at least a 40% conditional probability of ending up with a higher injection value. A myopic risk manager could be temped to stick to the minimal variance strategy to reduce the injections volatility. Potential advantages of a mean-variance tradeoff are however revealed when the entire time horizon is considered ; under the baseline parameters of Tables 2.1 and 2.2, one can show through Monte Carlo simulations that

$$\mathbb{P}\left(\sum_{t=1}^{T} J_t e^{-rt} > 0\right) = 2.8\%, \tag{2.4.2}$$

which is because downside risk is reduced through time diversification.

A more formal discussion can be drawn up based on what insurers are interested in : discounted sum of injections. From (2.4.1),

$$\sum_{t=1}^{T} \tilde{I}_t e^{-rt} = \sum_{t=1}^{T} \tilde{I}_t^{\text{MV}} e^{-rt} + \lambda \sum_{t=1}^{T} J_t e^{-rt}. \tag{2.4.3}$$

This provides the last piece required to explain the simulation results :

- Suppose the insurer performing the hedge is not confronted with basis risk ; futures on the fund $F$ are available for hedging. The conditional variance of the minimal variance injection approximation $\tilde{I}_{t+1}^{\text{MV}}$ is completely eliminated, as shown by Proposition 2.3.3. Moreover, this is done at no net cost because the price of risk is the same for the underlying fund and the hedging instrument. This explains why the first sum on the right-hand side of (2.4.3) is distributed around zero with a very small dispersion, and why a near-zero TGCR is thus obtained for the choice $\lambda = 0$ (see Table 2.3). This makes it optimal to use such a small value of $\lambda$, as the TGCR would be increased by the second sum despite the fact that its downside risk is reduced through time diversification.
- Suppose the insurer is confronted with basis risk ; futures $S$ are used to hedge the GMMB contract with the underlying fund $F$. The minimal variance injection still contains an important portion of the no-hedging conditional standard deviation. In fact, Proposition 2.3.3 shows that for a correlation of around 90%, only about 56% of the no-hedging standard deviation is eliminated by the minimal variance strategy. The first sum on the right-hand side of (2.4.3) is therefore not distributed with a very small dispersion as it would be in the absence of basis risk. Hence, it can be optimal to use a large value of $\lambda$ as the second sum can now help in reducing the CVaR through time diversification, as shown by (2.4.2).

The above explanation can be summarized as follows : the minimal variance strategy is less efficient under basis risk, as shown by Proposition 2.3.3, and moreover it neglects the risk reduction offered by time diversification through the long maturity of variable annuities. This explains why a large value of the mean-variance tradeoff parameter $\lambda$ is optimal when there is basis risk.

## 2.5 Robustification against drift uncertainty

A surprising observation from Table 2.3 is that the optimal mean-variance hedge is more profitable in average than the no-hedging strategy (see the expected error of $-22.8$ vs $-6.5$). This is at odds with the traditional premise that hedging reduces risk at the expense of a lower average return. The lower expected return under the absence of hedging results from the optimal hedging strategy which promotes an aggressive investment into the index futures in order to take advantage of the excess equity growth rate of the asset (under $\mathbb{P}$) over the risk-free rate that is used to discount the capital injections, at least when the market is very likely to be in the bull market regime. This is illustrated by a simulated hedging portfolio composition path plotted in Figure 2.2. The use of long futures positions within the hedging portfolio during bull markets raises several concerns. The hedging strategy exhibits regime-timing behavior which could be deemed undesirable, both from the point of view of risk management and of the regulator. Such a strategy relies on the theoretical ability to forecast the conditional drift through the regime mass function $\eta$. In practice, model risk and parameter uncertainty could be very detrimental to the performance of the presented hedging scheme due to the inability of efficiently forecasting drifts, which is a notoriously difficult problem. For instance, the standard errors of the bear market drifts $\mu_2^{(S)}$ and $\mu_2^{(F)}$ displayed in Table 2.1 are very large. Moreover, the statistical uncertainty related to the regime transition probability $P_{2,1}$ in the bear market regime is substantial as indicated by its large standard error provided in Table 2.1. Hence, it is important to investigate whether risk-reward tradeoff based strategies remain useful in such a context of drift uncertainty.

The current section presents a robust version of the risk-reward tradeoff strategies developed in the previous section which does not rely on the ability to accurately forecast the time-variation of the drifts. Although there exists multiple approaches to embed model risk and estimation risk into the hedging strategy, a thorough investigation of other schemes are left as further work. In the robust version, during the optimization of the hedging strategies (see (2.3.16)-(2.3.18) for the mean-variance approach and (2.3.25) for the minimal TGCR approach), the physical measure $\mathbb{P}$ is replaced by a new probability measure $\mathbb{Z}$ under which the drift is time-invariant. More precisely, the constrained probability measure $\mathbb{Z}$ is such that the asset dynamics remains the same lognormal two-state regime-switching model than under $\mathbb{P}$, but with the following drift parameters : $\tilde{\mu}_1^{(S)} = \tilde{\mu}_2^{(S)} = \bar{\mu}^{(S)}$ and $\tilde{\mu}_1^{(F)} = \tilde{\mu}_2^{(F)} = \bar{\mu}^{(F)}$, where

$$\bar{\mu}^{(S)} \equiv \pi_1 \, \mu_1^{(S)} + (1-\pi_1)\mu_2^{(S)} = 0.0037, \qquad \bar{\mu}^{(F)} \equiv \pi_1 \, \mu_1^{(F)} + (1-\pi_1)\mu_2^{(F)} = 0.0048 \quad (2.5.1)$$

are the stationary expected returns, and $\pi_1 = 0.78$ is the stationary probability associated with regime 1.

Calculating the hedging strategies under $\mathbb{Z}$ entails that such strategies do not have the ability to forecast expected returns, in line with drift uncertainty concerns as discussed above. Such strategies are referred to as the *drift-constrained* counterparts. Note that the ability to forecast volatility is preserved. Indeed, there is a large literature documenting the ability of various frameworks to forecast volatility under model/parameter uncertainty (see, e.g., Ardia et al., 2017). The robustification procedure therefore focuses on drift uncertainty.

A realization of the drift-constrained mean-variance strategy is illustrated in Figure 2.3. The reducible drift-constrained strategies are illustrated in Figure 2.4. The drift-constrained strategies almost exclusively use net short positions in the futures, in contrast with their unconstrained counterparts. They represent an under-hedge compared to the minimal variance strategy, i.e., they rely on a long temporal horizon to use temporal diversification so as to save on the cost of shorting an asset whose value grows on average. Such strategies therefore address the concern of model risk since they rely on a more conservative investment component than the unconstrained strategies.

The performance of the robust strategies was assessed by simulating the hedging process by applying the drift-constrained strategies optimized under $\mathbb{Z}$ over underlying asset paths simulated with the physical data-generation measure $\mathbb{P}$. The hedging simulation results obtained are presented in Table 2.4. The optimal value of $\lambda$ is smaller for the drift-constrained mean-variance strategy than for the unconstrained version optimized under $\mathbb{P}$ (4.4 vs. 7). The robust version of the strategy is therefore less aggressive and puts less weight on the mean component than its unconstrained counterpart. Nevertheless, even when considering the robust version of the hedging strategy, departing from the minimal variance strategy still provides the opportunity to both increase profitability through a lower injections mean and reduce capital requirements through a lower TGCR. The robust version of the hedging strategy should therefore be very attractive to practitioners concerned with model risk who wish to increase expected returns without increasing capital requirements. However, for the robust strategy, the expected hedging error is higher than for the no-hedging strategy (-4.7 vs -6.5). This result is therefore consistent with the traditional premise stipulating that hedging reduces capital requirements at the expense of lesser profitability.

## 2.6 Conclusion

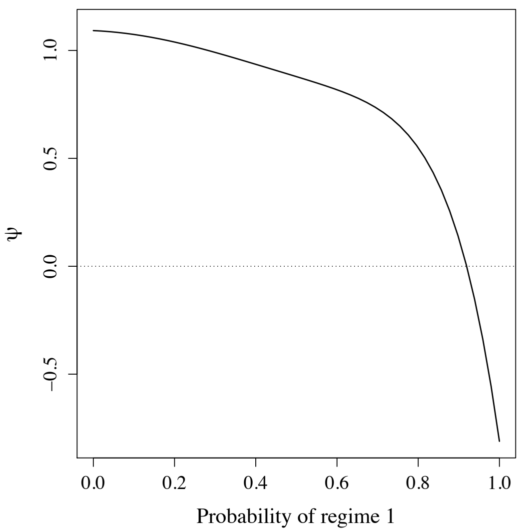
An efficient and tractable methodology is developed for insurers hedging equity risk related to guarantees associated with variable annuity polices in the presence of basis risk. Although the optimization criterion is local, a global flavor is given to the hedge as the local risk measure is optimized to minimize capital requirements. This enables the local hedging approach incorporating time diversification of risk into its design. Taylor expansions on liquidity injections are applied to simplify the implementation of the methodology. Such approximations lead to a family of solutions to the hedging problem encompassing multiple hedging approaches found in the literature such as delta hedging and minimal local variance hedges. Multivariate regime-switching models are considered for the joint dynamics of the guarantee underlying asset and the index futures.

Within simulation experiments, our method is compared to benchmarks drawn from the literature such as minimal variance hedging. The outperformance of our method versus benchmarks in terms of both capital requirements reduction and expected return enhancement is explained by the time diversification of risk associated to additional exposure to equity risk resulting from the optimal local criterion; an additional exposure to equity risk proves beneficial in the long-term since a higher expected return facilitates the accumulation of capital and since the impact of downturns is dampened over a long horizon. The omission of basis risk within hedging performance assessment simulations is shown to lead to a severe under-estimation of risk. For instance, in the simulation experiment presented the inclusion of basis risk within the hedging scheme leads to a drastic increase of 215% of the Total Gross Capital Required (TGCR) associated with a minimal variance strategy.

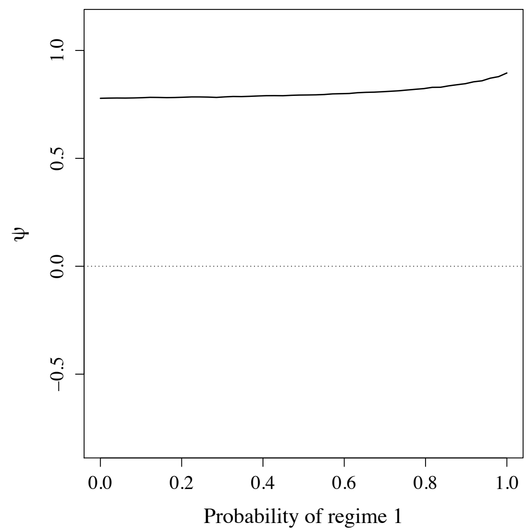
**Table 2.3 –** Comparison of various hedging strategies for the simulation study of Section 2.4.1. The statistics are for the discounted sum of injections : $\sum_{t=1}^{T} e^{-rt} I_t$.

| | Mean | Std.Dev. | $\text{CVaR}^{\mathbb{P}}_{0.70}$ | $\text{CVaR}^{\mathbb{P}}_{0.80}$ | $\text{CVaR}^{\mathbb{P}}_{0.90}$ | $\text{CVaR}^{\mathbb{P}}_{0.95}$ | $\text{CVaR}^{\mathbb{P}}_{0.99}$ |
|---|---|---|---|---|---|---|---|
| **No hedging** | | | | | | | |
| | -6.5 | 11.7 | 8.7 | 14.0 | 20.7 | 25.3 | 32.2 |
| **Hedging under basis risk** | | | | | | | |
| *Mean-variance* | | | | | | | |
| $\lambda = 0$ | 2.9 | 4.9 | 8.9 | 10.5 | 12.8 | 14.8 | 18.7 |
| $\lambda = 2$ | -4.4 | 6.7 | 3.8 | 5.7 | 8.5 | 10.8 | 15.1 |
| $\lambda = 5$ | -15.4 | 11.6 | -1.7 | 1.2 | 5.4 | 8.9 | 15.5 |
| $\lambda = 7\,\star$ | -22.8 | 15.3 | -4.9 | -1.2 | 4.1 | 8.6 | 17.2 |
| $\lambda = 10$ | -33.8 | 21.0 | -9.3 | -4.4 | 2.8 | 8.9 | 20.5 |
| *Minimal TGCR* | | | | | | | |
| $n = 8$ | -15.1 | 10.0 | -3.2 | -0.1 | 4.6 | 8.7 | 16.9 |
| *Minimal* $\text{VaR}^{\mathbb{P}}_{\alpha}$ | | | | | | | |
| $\alpha = 0.95$ | 2.4 | 4.9 | 8.4 | 9.9 | 12.3 | 14.3 | 18.4 |
| $\alpha = 0.60$ | -7.3 | 6.8 | 0.8 | 3.0 | 6.3 | 9.2 | 15.2 |
| *Minimal* $\text{CVaR}^{\mathbb{P}}_{\alpha}$ | | | | | | | |
| $\alpha = 0.95$ | 2.4 | 5.0 | 4.3 | 6.3 | 9.2 | 11.7 | 16.0 |
| $\alpha = 0.15$ | -3.7 | 5.6 | 3.1 | 4.9 | 7.7 | 10.2 | 15.1 |
| **Hedging without basis risk** | | | | | | | |
| *Mean-variance* | | | | | | | |
| $\lambda = 0$ | 0.3 | 1.7 | 2.4 | 2.9 | 3.8 | 4.7 | 6.6 |
| $\lambda = 1.5\,\star$ | -2.5 | 5.9 | 0.9 | 1.6 | 2.7 | 3.7 | 6.0 |
| $\lambda = 5$ | -9.2 | 7.4 | -0.7 | 1.0 | 3.5 | 5.6 | 9.6 |
| $\lambda = 7$ | -13.1 | 10.1 | -1.5 | 0.9 | 4.2 | 7.1 | 12.5 |
| $\lambda = 10$ | -18.9 | 14.1 | -2.5 | 0.7 | 5.5 | 9.5 | 17.1 |
| *Minimal TGCR* | | | | | | | |
| $n = 8$ | -1.7 | 2.3 | 1.2 | 1.8 | 2.8 | 3.7 | 5.7 |
| *Minimal* $\text{VaR}^{\mathbb{P}}_{\alpha}$ | | | | | | | |
| $\alpha = 0.95$ | 0.3 | 1.7 | 2.4 | 2.9 | 3.8 | 4.7 | 6.6 |
| $\alpha = 0.60$ | 0.3 | 1.7 | 2.4 | 2.9 | 3.8 | 4.7 | 6.6 |
| *Minimal* $\text{CVaR}^{\mathbb{P}}_{\alpha}$ | | | | | | | |
| $\alpha = 0.95$ | 0.3 | 1.7 | 2.4 | 2.9 | 3.8 | 4.7 | 6.6 |
| $\alpha = 0.15$ | 0.3 | 1.7 | 2.4 | 2.9 | 3.8 | 4.7 | 6.6 |

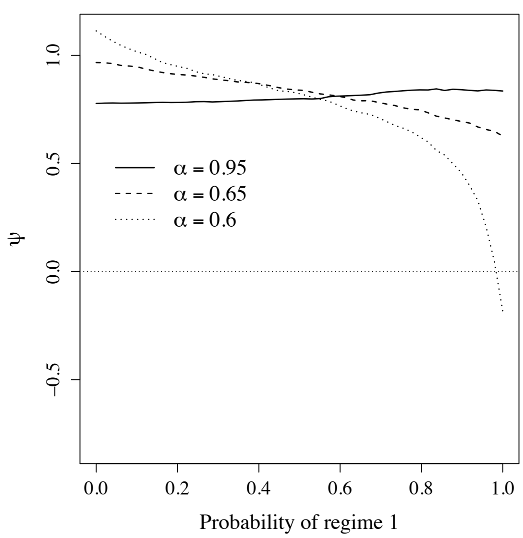
*Notes* : The "mean-variance" strategy is characterized by Proposition 2.3.2. The value of $\lambda$ minimizing capital requirements is marked by a star ($\star$) and is determined with the approach of Section 2.3.4. The "minimal TGCR" strategy refers to the one described in Section 2.3.5. The "minimal $\text{VaR}^{\mathbb{P}}_{\alpha}$" and the "minimal $\text{CVaR}^{\mathbb{P}}_{\alpha}$" strategies are characterized by Theorem 2.3.1 where the risk measure $\mathcal{R}_t(\bullet)$ is $\text{VaR}^{\mathbb{P}}_{\alpha}[\bullet|\mathcal{F}_t]$ and $\text{CVaR}^{\mathbb{P}}_{\alpha}[\bullet|\mathcal{F}_t]$, respectively. The "Hedging without basis risk" pannel refers to a ficticious case where the futures underlying asset is the mutual fund whereas the "Hedging under basis risk" pannel integrates imperfect correlation between the futures underlying asset and the mutual fund.



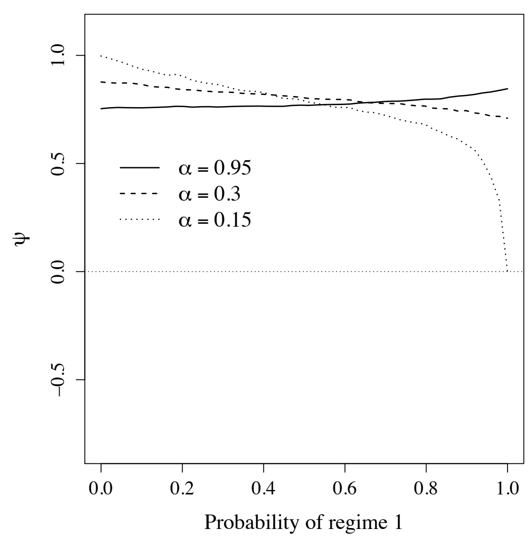

**(a)** Minimal TGCR




**(b)** Minimal variance




**(c)** Minimal $\mathrm{VaR}_\alpha^{\mathbb{P}}$




**(d)** Minimal $\mathrm{CVaR}_\alpha^{\mathbb{P}}$

**FIGURE 2.1** – Representation of hedging strategies for the simulation study of Section 2.4.1. The "minimal TGCR" strategy refers to the one described in Section 2.3.5. The "minimal variance", "minimal $\mathrm{VaR}_\alpha^{\mathbb{P}}$", and "minimal $\mathrm{CVaR}_\alpha^{\mathbb{P}}$" strategies are characterized by Theorem 2.3.1 where the risk measure $\mathcal{R}_t(\bullet)$ is respectively $\mathrm{Var}^{\mathbb{P}}[\bullet|\mathcal{F}_t]$, $\mathrm{VaR}_\alpha^{\mathbb{P}}[\bullet|\mathcal{F}_t]$, and $\mathrm{CVaR}_\alpha^{\mathbb{P}}[\bullet|\mathcal{F}_t]$. Each strategy can be expressed as $\theta_{t+1}^{(S)} = \Psi(\eta_{1,t}^{\mathbb{P}})\frac{\Delta_t F_t}{S_t}$ where the functions $\Psi$ are those illustrated. Curves above the zero level corresponds to short positions (because $\Delta_t \leq 0$ a.s.).

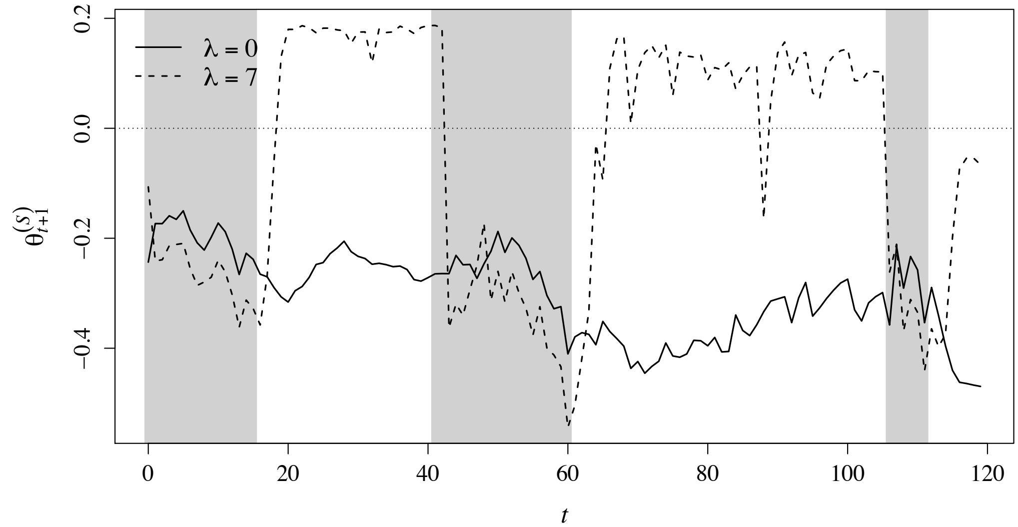



**FIGURE 2.2 –** Example of mean-variance hedging simulation in the setup of Section 2.4.1. The number of long positions in the futures is illustrated as a function of time (in months). Shaded areas correspond to bear market regime periods. One can see that the mean-variance strategy with $\lambda = 7$ combines both long and short positions. In contrast, the minimal variance strategy ($\lambda = 0$) relies solely on short positions.

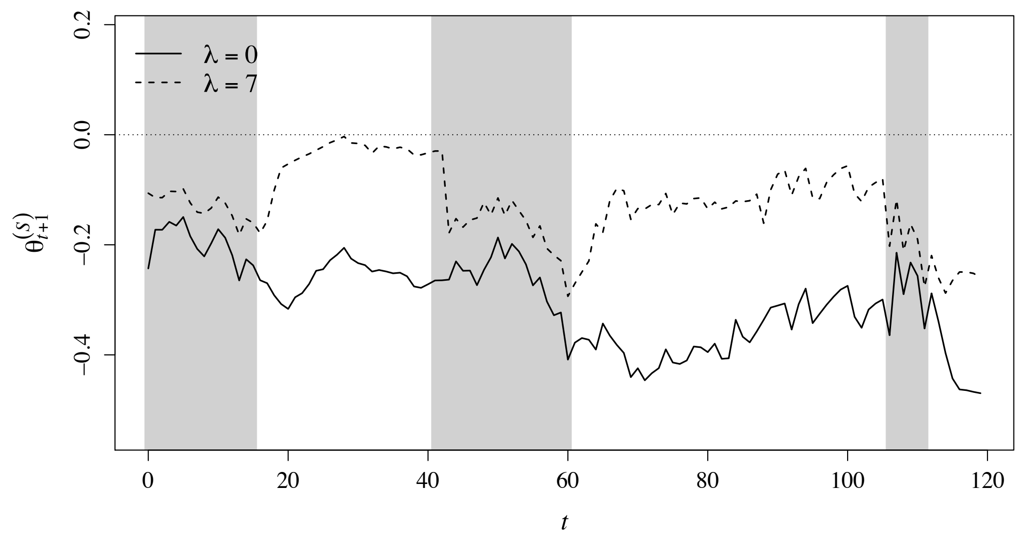
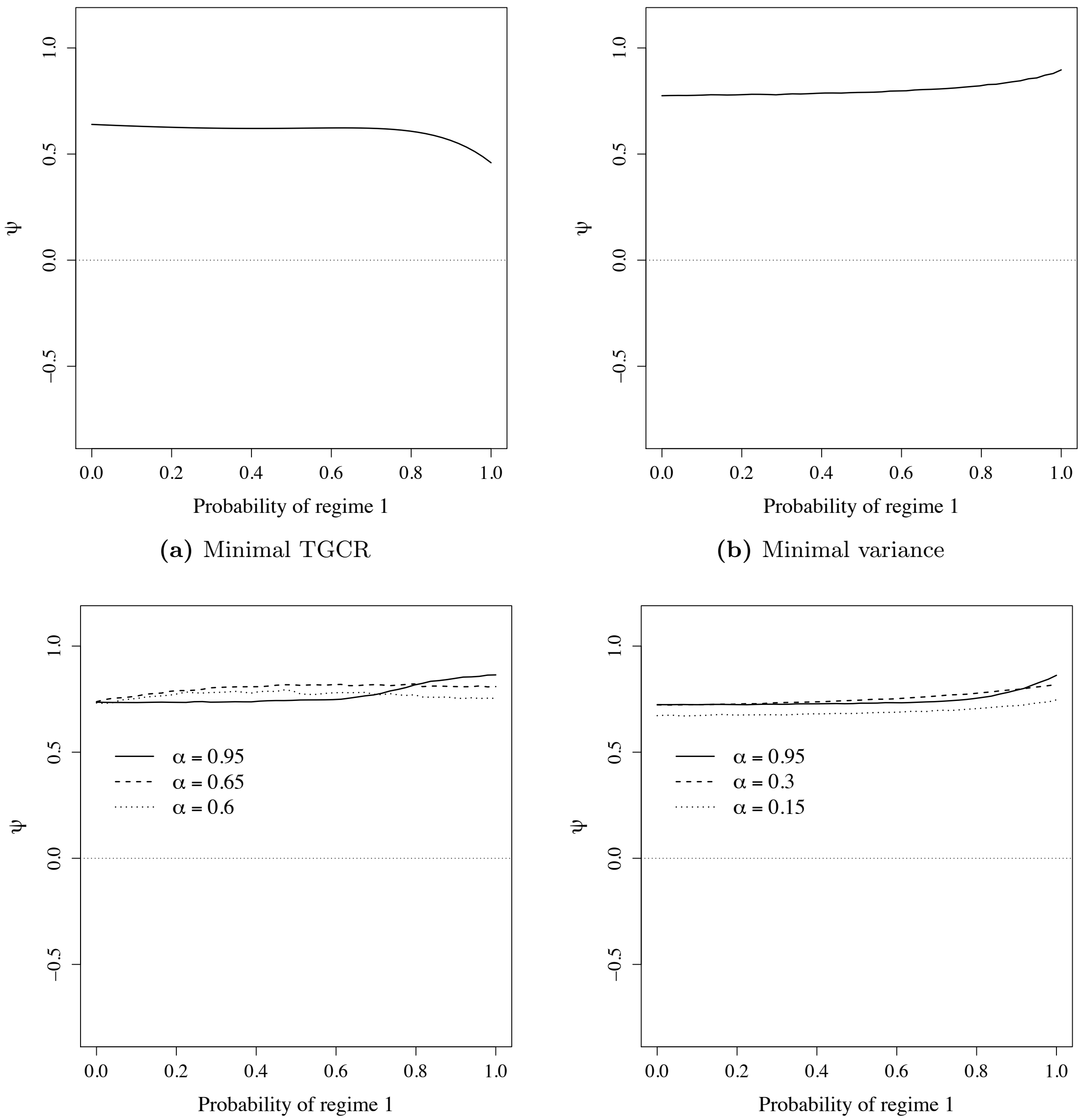



**FIGURE 2.3 –** Example of drift-constrained (see Section 2.5) mean-variance hedging simulation in the setup of Section 2.4.1. The number of positions in the futures is illustrated as a function of time (in months). Shaded areas correspond to bear market regime periods. One can see that the constrained mean-variance strategy represents an under-hedge compared to minimal variance hedging. Both strategies rely only on short positions, in contrast with Figure 2.2.




**(a)** Minimal TGCR

**(b)** Minimal variance

**(c)** Minimal $\text{VaR}_\alpha^{\mathbb{Z}}$

**(d)** Minimal $\text{CVaR}_\alpha^{\mathbb{Z}}$

**FIGURE 2.4** – Representation of the drift-constrained hedging strategies for the simulation study of Section 2.4.1. The strategies are all calculated under the measure $\mathbb{Z}$ described in Section 2.5. The "minimal TGCR" strategy refers to the one described in Section 2.3.5, but where the $\text{CVaR}_{0.95}^{\mathbb{P}}$ risk measure is replaced by $\text{CVaR}_{0.95}^{\mathbb{Z}}$. The "minimal variance", "minimal $\text{VaR}_\alpha^{\mathbb{Z}}$", and "minimal $\text{CVaR}_\alpha^{\mathbb{Z}}$" strategies are characterized by Theorem 2.3.1 where the risk measure $\mathcal{R}_t(\bullet)$ is respectively $\text{Var}^{\mathbb{Z}}[\bullet|\mathcal{F}_t]$, $\text{VaR}_\alpha^{\mathbb{Z}}[\bullet|\mathcal{F}_t]$, and $\text{CVaR}_\alpha^{\mathbb{Z}}[\bullet|\mathcal{F}_t]$. Each strategy can be expressed as $\theta_{t+1}^{(S)} = \Psi(\eta_{1,t}^{\mathbb{Z}})\frac{\Delta_t F_t}{S_t}$ where the functions $\Psi$ are those illustrated. We note that only short positions are used for the drift-constrained strategies, in contrast with their basic counterparts (see Figure 2.1).

**Table 2.4** – Comparison of drift-constrained hedging strategies (see Section 2.5) for the simulation study of Section 2.4.1. The statistics are for the discounted sum of injections : $\sum_{t=1}^{T} e^{-rt} I_t$.

| | Mean | Std.Dev. | $\text{CVaR}^{\mathbb{P}}_{0.70}$ | $\text{CVaR}^{\mathbb{P}}_{0.80}$ | $\text{CVaR}^{\mathbb{P}}_{0.90}$ | $\text{CVaR}^{\mathbb{P}}_{0.95}$ | $\text{CVaR}^{\mathbb{P}}_{0.99}$ |
|---|---|---|---|---|---|---|---|
| **No hedging** | | | | | | | |
| | -6.5 | 11.7 | 8.7 | 14.0 | 20.7 | 25.3 | 32.2 |
| **Hedging under basis risk** | | | | | | | |
| *Mean-variance* | | | | | | | |
| $\lambda = 0$ | 3.0 | 5.0 | 9.0 | 10.6 | 12.9 | 15.0 | 19.0 |
| $\lambda = 2$ | -0.5 | 5.9 | 7.0 | 8.8 | 11.2 | 13.2 | 17.1 |
| $\lambda = 4.4 \star$ | -4.7 | 8.0 | 5.3 | 7.5 | 10.3 | 12.6 | 16.8 |
| $\lambda = 7$ | -9.2 | 10.7 | 3.9 | 6.6 | 10.3 | 13.2 | 18.7 |
| $\lambda = 10$ | -14.5 | 14.1 | 2.6 | 6.1 | 10.8 | 14.7 | 22.2 |
| *Minimal TGCR* | | | | | | | |
| $n = 8$ | -1.6 | 5.7 | 5.7 | 7.8 | 10.6 | 12.9 | 16.4 |
| *Minimal* $\text{VaR}^{\mathbb{Z}}_{\alpha}$ | | | | | | | |
| $\alpha = 0.95$ | 2.9 | 5.1 | 9.1 | 10.7 | 13.1 | 15.1 | 19.2 |
| $\alpha = 0.60$ | 1.1 | 5.0 | 7.2 | 8.9 | 11.3 | 13.4 | 17.4 |
| *Minimal* $\text{CVaR}^{\mathbb{Z}}_{\alpha}$ | | | | | | | |
| $\alpha = 0.95$ | 2.4 | 5.1 | 8.7 | 10.3 | 12.7 | 14.7 | 18.6 |
| $\alpha = 0.15$ | 1.3 | 5.2 | 7.9 | 9.6 | 12.1 | 14.2 | 18.0 |
| **Hedging without basis risk** | | | | | | | |
| *Mean-variance* | | | | | | | |
| $\lambda = 0$ | 0.3 | 1.7 | 2.4 | 2.9 | 3.8 | 4.7 | 6.6 |
| $\lambda = 1.3 \star$ | -0.8 | 2.3 | 2.1 | 2.7 | 3.5 | 4.3 | 6.2 |
| $\lambda = 4.4$ | -3.4 | 4.5 | 2.1 | 3.2 | 4.6 | 5.8 | 8.1 |
| $\lambda = 7$ | -5.6 | 6.6 | 2.4 | 3.9 | 6.1 | 7.9 | 11.5 |
| $\lambda = 10$ | -8.2 | 9.1 | 2.7 | 4.9 | 7.9 | 10.6 | 15.8 |
| *Minimal TGCR* | | | | | | | |
| $n = 8$ | -0.8 | 2.3 | 2.2 | 2.9 | 3.7 | 4.4 | 5.8 |
| *Minimal* $\text{VaR}^{\mathbb{Z}}_{\alpha}$ | | | | | | | |
| $\alpha = 0.95$ | 0.3 | 1.7 | 2.4 | 2.9 | 3.8 | 4.7 | 6.6 |
| $\alpha = 0.60$ | 0.3 | 1.7 | 2.4 | 2.9 | 3.8 | 4.7 | 6.6 |
| *Minimal* $\text{CVaR}^{\mathbb{Z}}_{\alpha}$ | | | | | | | |
| $\alpha = 0.95$ | 0.3 | 1.7 | 2.4 | 2.9 | 3.8 | 4.7 | 6.6 |
| $\alpha = 0.15$ | 0.3 | 1.7 | 2.4 | 2.9 | 3.8 | 4.7 | 6.6 |

*Notes* : The strategies are all calculated under the measure $\mathbb{Z}$ described in Section 2.5. The "mean-variance" strategy is characterized by Proposition 2.3.2 where the measure $\mathbb{P}$ is replaced by $\mathbb{Z}$. The value of $\lambda$ minimizing capital requirements is marked by a star ($\star$) and is determined with the approach of Section 2.3.4, with $\mathbb{P}$ replaced by $\mathbb{Z}$. The "minimal TGCR" strategy refers to the one described in Section 2.3.5, again with $\mathbb{P}$ replaced by $\mathbb{Z}$. The "minimal $\text{VaR}^{\mathbb{Z}}_{\alpha}$" and the "minimal $\text{CVaR}^{\mathbb{Z}}_{\alpha}$" strategies are characterized by Theorem 2.3.1 where the risk measure $\mathcal{R}_t(\bullet)$ is $\text{VaR}^{\mathbb{Z}}_{\alpha}[\bullet|\mathcal{F}_t]$ and $\text{CVaR}^{\mathbb{Z}}_{\alpha}[\bullet|\mathcal{F}_t]$, respectively. The "Hedging without basis risk" pannel refers to a ficticious case where the futures underlying asset is the mutual fund whereas the "Hedging under basis risk" pannel integrates imperfect correlation between the futures underlying asset and the mutual fund.

# Appendix

## 2.A Proofs

### 2.A.1 Proof of Proposition 2.2.1

Define $\delta F_t = F_{t+1} - F_t$. From the GMMB pricing formula (2.2.7), for $t \in \{0, \ldots, T-1\}$,

$$\Pi_{t+1} - \Pi_t = -\omega_{opt}\delta F_t \sum_{j=t+1}^{T} \gamma_{j-1} + \omega_{opt}\gamma_t F_{t+1} + \mathbb{1}_{\{t+1<T\}}\gamma_T G_{t+1} - \mathbb{1}_{\{t<T\}}\gamma_T G_t. \quad (2.A.1)$$

Rewriting the injection formula (2.2.9) for $I_{t+1}$ using (2.2.6), (2.2.10) and (2.2.12) yields

$$\begin{aligned} I_{t+1} &= V^{\theta}_{(t+1)+} + CF_{t+1} - V^{\theta}_{(t+1)-}, \\ &= \Pi_{t+1} - \omega_{opt}\gamma_t F_{t+1} + \mathbb{1}_{\{t+1=T\}}\gamma_T G_T - \theta^{(B)}_{t+1} B_{t+1} - \theta^{(S)}_{t+1}\delta S_t, \end{aligned}$$

where $G_T \equiv \max\bigl(0, \tilde{K} - F_T\bigr)$, and $\delta S_t = S_{t+1} - S_t$. Next, substitute $\Pi_{t+1}$ and $\theta^{(B)}_{t+1}$ by the expressions prescribed by (2.A.1) and (2.2.11), respectively. This gives

$$\begin{aligned} I_{t+1} &= \Pi_t - \omega_{opt}\delta F_t \sum_{j=t+1}^{T} \gamma_{j-1} + \omega_{opt}\gamma_t F_{t+1} + \mathbb{1}_{\{t+1<T\}}\gamma_T G_{t+1} - \mathbb{1}_{\{t<T\}}\gamma_T G_t \\ &\quad -\omega_{opt}\gamma_t F_{t+1} + \mathbb{1}_{\{t+1=T\}}\gamma_T G_T - \frac{\Pi_t}{B_t} B_{t+1} - \theta^{(S)}_{t+1}\delta S_t, \\ &= \Pi_t(1 - e^r) - \omega_{opt}\delta F_t \sum_{j=t+1}^{T} \gamma_{j-1} - \theta^{(S)}_{t+1}\delta S_t + \gamma_T\bigl(\mathbb{1}_{\{t+1<T\}} G_{t+1} - \mathbb{1}_{\{t<T\}} G_t + \mathbb{1}_{\{t+1=T\}} G_T\bigr), \end{aligned}$$

where $\frac{B_{t+1}}{B_t} = e^r$ was used. At last, the identities

$$\mathbb{1}_{\{t+1=T\}} G_T = \mathbb{1}_{\{t+1=T\}} G_{t+1}, \qquad \mathbb{1}_{\{t+1<T\}} + \mathbb{1}_{\{t+1=T\}} = \mathbb{1}_{\{t+1\leq T\}} = \mathbb{1}_{\{t<T\}},$$

are used to obtain

$$I_{t+1} = \Pi_t(1 - e^r) - \omega_{opt}\delta F_t \sum_{j=t+1}^{T} \gamma_{j-1} - \theta^{(S)}_{t+1}\delta S_t + \gamma_T \mathbb{1}_{\{t<T\}}\delta G_t, \quad (2.A.2)$$

where $\delta G_t = G_{t+1} - G_t$. This concludes the proof.

### 2.A.2 Proof of Theorem 2.3.1

First, note that the approximation of the injection, see (2.3.14), can be written as

$$\tilde{I}_{t+1} = \Theta_t - \Delta_t F_t\Big[\psi_{t+1}\tfrac{\delta S_t}{S_t} - \tfrac{\delta F_t}{F_t}\Big], \qquad \psi_{t+1} \equiv \frac{\theta^{(S)}_{t+1} S_t}{\Delta_t F_t}. \tag{2.A.3}$$

Because the embedded option is a put, $-\Delta_t > 0$ almost surely. Hence, for any $\mathcal{F}_t$-reducible risk measure $\mathcal{R}_t$, there exists a real function $f_1$ and a nonnegative function $f_2$ such that

$$\mathcal{R}_t\big(\tilde{I}_{t+1}\big) = f_1(\Theta_t) + f_2(-\Delta_t F_t)\mathcal{R}_t\Big(\psi_{t+1}\tfrac{\delta S_t}{S_t} - \tfrac{\delta F_t}{F_t}\Big).$$

It follows that

$$\psi^*_{t+1} \equiv \underset{\psi_{t+1}}{\arg\min}\, \mathcal{R}_t\big(\tilde{I}_{t+1}\big) = \underset{\psi_{t+1}}{\arg\min}\, \mathcal{R}_t\Big(\psi_{t+1}\tfrac{\delta S_t}{S_t} - \tfrac{\delta F_t}{F_t}\Big), \tag{2.A.4}$$

from which (2.3.24) can indeed be concluded.

As stated in Proposition 2.3.1, the $\mathcal{F}_t$-conditional distribution of $\big(\frac{F_{t+1}}{F_t}, \frac{S_{t+1}}{S_t}\big)$ under $\mathbb{P}$ depends only on the first regime conditional likelihood, $\eta^{\mathbb{P}}_{1,t}$. It follows that the solution to the minimization problem (2.A.4) can be expressed as a function of the form $\psi^*_{t+1} = \Psi_{\mathcal{R}_t}\big(\eta^{\mathbb{P}}_{1,t}\big)$. Finally, from the temporal law-invariance property of the sequence $\{\mathcal{R}_t\}_{t=0}^{T-1}$, it follows that $\Psi_{\mathcal{R}_0} = \cdots = \Psi_{\mathcal{R}_{T-1}}$.

#### Explicit formulas

Deriving the following formulas under the regime-switching model of Section 2.3.1 is a straightforward exercice that involves conditioning on the state of the regime and using the moment generating function of the normal distribution :

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[S_{t+1}|\mathcal{F}_t] &= S_t \sum_{j=1}^{2} \eta^{\mathbb{P}}_{j,t} e^{\mu^{(S)}_j + \frac{1}{2}\left(\sigma^{(S)}_j\right)^2}. \\
\mathrm{Var}^{\mathbb{P}}[S_{t+1}|\mathcal{F}_t] &= S_t^2 \left( \sum_{j=1}^{2} \eta^{\mathbb{P}}_{j,t} e^{2\mu^{(S)}_j + 2\left(\sigma^{(S)}_j\right)^2} - \left[\sum_{j=1}^{2} \eta^{\mathbb{P}}_{j,t} e^{\mu^{(S)}_j + \frac{1}{2}\left(\sigma^{(S)}_j\right)^2}\right]^2 \right). \quad (2.A.5) \\
\mathrm{Cov}^{\mathbb{P}}[F_{t+1}, S_{t+1}|\mathcal{F}_t] &= S_t F_t \Bigg\{ \sum_{j=1}^{2} \eta^{\mathbb{P}}_{j,t} \left[ e^{\mu^{(S)}_j + \mu^{(F)}_j + \frac{1}{2}\left(\sigma^{(S)}_j\right)^2 + \rho_j \sigma^{(S)}_j \sigma^{(F)}_j + \frac{1}{2}\left(\sigma^{(F)}_j\right)^2} \right] \quad (2.A.6) \\
&\qquad - \left( \sum_{j=1}^{2} \eta^{\mathbb{P}}_{j,t} e^{\mu^{(S)}_j + \frac{1}{2}\left(\sigma^{(S)}_j\right)^2} \right) \left( \sum_{j=1}^{2} \eta^{\mathbb{P}}_{j,t} e^{\mu^{(F)}_j + \frac{1}{2}\left(\sigma^{(F)}_j\right)^2} \right) \Bigg\}.
\end{aligned}$$