AN INTRODUCTION TO COMBINATORIAL HOPF ALGEBRAS AND RENORMALISATION

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Abstract. We give an account of renormalisation in connected graded Hopf algebras.

MSC Classification: 05C05, 16T05, 16T10, 16T15, 16T30.

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Contents

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Introduction

Acknowledgements. First of all, I would like to express my gratitude to Darij Grinberg for the very detailed erratum[1] he sent me a couple of years ago about my previous notes [21] on combinatorial Hopf algebras and renormalisation. This definitely contributed to greatly improve the present text. These notes grew up from two series of lectures in Madrid (ICMAT, July 2016) and in Gabes, Tunisia (Faculté des Sciences, December 2016). I thank Kurusch Ebrahimi-Fard and Hedi Rejaiba for their kind invitations as well as all the participants for their pertinent questions and remarks.

1. Preliminaries

In this preliminary section, we review some basic notions in group theory and linear algebra. Some basic results are rephrased in order to get slowly used to bialgebra and Hopf algebra techniques.

[1]Available on his homepage: http://web.mit.edu/∼darij/www/algebra/manchon-errata-update.pdf

1.1. Semigroups, monoids and groups.

Definition 1.1.1. *A **semigroup** is a set E together with a product*

$$\begin{aligned} m : E \times E &\longrightarrow E \\ (x, y) &\longmapsto xy \end{aligned}$$

*such that, for any $x, y, z \in E$ the following **associativity property** holds:*

$$(xy)z = x(yz). \tag{1.1.1}$$

*The semigroup is moreover **commutative** if the commutativity relation $xy = yx$ holds for any $x, y \in E$.*

Note that the associativity property is equivalent to the fact that the following diagram is commutative:

$$\begin{array}{ccc} E \times E \times E & \xrightarrow{\mathrm{Id}_E \times m} & E \times E \\ {\scriptstyle m\times \mathrm{Id}_E}\downarrow & & \downarrow{\scriptstyle m} \\ E \times E & \xrightarrow[m]{} & E \end{array}$$

In the same spirit, the semigroup E is commutative if and only if the following diagram commutes:

$$\begin{array}{ccc} E \times E & \xrightarrow{\tau} & E \times E \\ & {\scriptstyle m}\searrow \quad \swarrow{\scriptstyle m} & \\ & E & \end{array}$$

where τ is the *flip*, defined by $\tau(x, y) = (y, x)$.

Definition 1.1.2. *A **monoid** is a semigroup M together with a **unit element** $e \in M$ such that $ex = xe$ for any $x \in M$.*

Let $\{*\}$ be any set with one single element denoted by $*$, and let $u : \{*\} \to E$ defined by $u(*) = e$. The fact that e is a unit is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccccc} \{*\} \times M & \xrightarrow{u\times \mathrm{Id}_M} & M \times M & \xleftarrow{\mathrm{Id}_M \times u} & M \times \{*\} \\ & {\scriptstyle \sim}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle \sim} & \\ & & M & & \end{array}$$

Proposition 1.1.1. *The unit element of a monoid M is unique.*

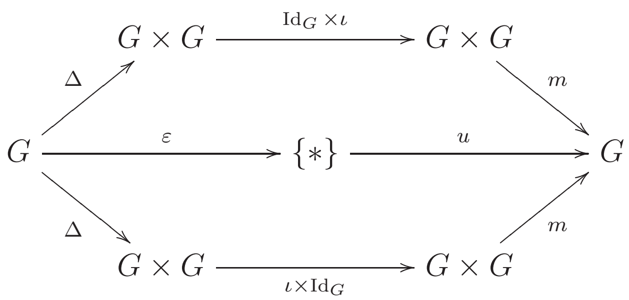
Proof. Let e, e' be two unit elements in a semigroup M. Then $ee' = e = e'$. □

Definition 1.1.3. *A* ***group*** *is a monoid G together with a map*

$$\begin{array}{rcl}\iota : G & \longrightarrow & G \\ x & \longmapsto & x^{-1}\end{array}$$

such that $x^{-1}x = xx^{-1} = e$ for any $x \in G$.

Let $\varepsilon : G \to \{*\}$ be the unique possible map, which sends any element of G to the element $*$. The fact that ι is the inverse map is equivalent to the commutativity of the following diagram:



where $\Delta : G \to G \times G$ is the *diagonal embedding* defined by $\Delta(g) := (g, g)$ for any $g \in G$.

Proposition 1.1.2. *The inverse map in a group is unique.*

Proof. Suppose that x' and x'' are both an inverse for $x \in G$. Then:

$$x' = x'e = x'(xx'') = (x'x)x'' = ex'' = x''.$$

□

1.2. **Rings and fields.**

Definition 1.2.1. *A* ***ring*** *is a triple $(R, +, \cdot)$ where:*

1. *$(R, +)$ is an abelian (i.e. commutative) group, with unit element denoted by 0.*
2. *$(R, \cdot)$ is a monoid, with unit element denoted by 1.*
3. *The following* ***distributivity property*** *holds for any $x, y, z \in R$:*

$$\begin{array}{rcl} x(y+z) & = & xy + xz \text{ (left distributivity)}, \\ (x+y)z & = & xz + yz \text{ (right distributivity)}. \end{array}$$

Note that the left and right distributivity properties are respectively equivalent to the commutativity of the two following diagrams, with $s(x, y) := x + y$, $m(x, y) := xy$ and $\tau_{23} := \mathrm{Id}_R \times \tau \times \mathrm{Id}_R$.

$$\begin{array}{ccc} R^3 & \xrightarrow{\Delta\times \mathrm{Id}_R\times \mathrm{Id}_R} & R^4 \\ \Big\downarrow{\scriptstyle \mathrm{Id}_R\times s} & & \Big\downarrow{\scriptstyle \tau_{23}} \\ R\times R & & R^4 \\ \Big\downarrow{\scriptstyle m} & & \Big\downarrow{\scriptstyle m\times m} \\ R & \xleftarrow[s]{} & R\times R \end{array} \qquad \begin{array}{ccc} R^3 & \xrightarrow{\mathrm{Id}_R\times \mathrm{Id}_R\times\Delta} & R^4 \\ \Big\downarrow{\scriptstyle s\times\mathrm{Id}_R} & & \Big\downarrow{\scriptstyle \tau_{23}} \\ R\times R & & R^4 \\ \Big\downarrow{\scriptstyle m} & & \Big\downarrow{\scriptstyle m\times m} \\ R & \xleftarrow[s]{} & R\times R \end{array}$$

In particular, 0 is an absorbing element in a ring: for any $x \in R$ we have:

$$x.0 = x.(1-1) = x.1 - x.1 = x - x = 0.$$

A *commutative ring* is defined the same way, except that the commutativity assumption for the product $\cdot$ is added. Left and right distributivity properties coincinde in that case.

Remark 1.2.1. Unless specified, we always consider *unital* rings. Non-unital rings R, where $(R, .)$ is only a semigroup and not necessarily a monoid, will be considered in Paragraph 2.1.

We stick to commutative rings for the moment. A *divisor of zero* is an element $x \neq 0$ such that there exists $y \neq 0$ with $xy = 0$. A commutative ring without divisors of zero is an *integral domain*.

Example 1.2.1. The set $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ endowed with usual addition and multiplication is an integral domain.

Example 1.2.2. Let R be a commutative ring. The ring $R[X]$ of *polynomials in one indeterminate with coefficients in* R is defined as:

$$R[X] := \left\{ \sum_{j=0}^{+\infty} r_j X^j,\ r_j \in R,\ r_j = 0 \text{ for } j >> \right\}.$$

The addition is performed termwise: for any $P = \sum_{j\geq 0} r_j X^j$ and $Q = \sum_{j\geq 0} s_j X^j$ in $R[X]$, one has:

$$P + Q := \sum_{j\geq 0} (r_j + s_j) X^j.$$

The product is determined by the rule $X^i X^j = X^{i+j}$ extended by the distributivity relation. Explicitly:

$$PQ = \sum_{j\geq 0} \left(\sum_{a,b\geq 0,\, a+b=j} r_a s_b \right) X^j.$$

The *degree* of the polynomial P is given by

$$\deg P := \sup\{j \geq 0,\, r_j \neq 0\},$$

and the *valuation* of the polynomial P is given by

$$\operatorname{val} P := \inf\{j \geq 0,\, r_j \neq 0\}.$$

A *monomial* is a polynomial P such that $\deg P = \operatorname{val} P$. It is given by $P = rX^j$ for some $r \in R - \{0\}$ and $j \geq 0$.

Example 1.2.3. Let R be a commutative ring. The ring $R[[X]]$ of *formal series in one indeterminate with coefficients in* R is defined as:

$$R[[X]] := \left\{ \sum_{j=0}^{+\infty} r_j X^j,\ r_j \in R \right\},$$

without any other condition on the coefficients. The sum and product are defined the same way as for polynomials. The notion of degree does not make sense anymore, but the notion of valuation still does.

Definition 1.2.2. *An* ***ideal*** *in a commutative ring R is a nonempty subset $J \subseteq R$ such that $xy \in J$ for any $x \in R$ and $y \in J$. In short,*

$$RJ \subseteq J.$$

If the ring R is not commutative, one distinguishes between ***left*** *ideals,* ***right*** *ideals and* ***two-sided*** *ideals, respectively characterized by the properties*

$$RJ \subseteq J, \qquad JR \subseteq J, \qquad RJ \cup JR \subseteq J.$$

The intersection of two left ideals is a left ideal. The same holds for right ideals and two-sided ideals.

Definition 1.2.3. *A field is a commutative ring $\boldsymbol{k}$ which has no ideal except $\{0\}$ and $\boldsymbol{k}$.*

Proposition 1.2.1. *Any nonzero element in a field is invertible.*

Proof. Let $a \in \boldsymbol{k} - \{0\}$, and let J be the smallest ideal (for the inclusion) containing a. We have $J = \boldsymbol{k}$ by definition of a field. In particular $1 \in J$, hence there exists $b \in \boldsymbol{k}$ such that $ab = 1$. □

Examples of fields:

- the field $\mathbb{Q}$ of rational numbers,
- *number fields*, i.e. finite extensions of $\mathbb{Q}$,
- the field $\mathbb{R}$ of real numbers,
- the field $\mathbb{C}$ of complex numbers,
- the field $\boldsymbol{k}(X)$ of rational fractions over a given field $\boldsymbol{k}$,
- the field $\boldsymbol{k}[X^{-1}, X]]$ of Laurent series over a given field $\boldsymbol{k}$, i.e

$$k[X^{-1}, X]] = \left\{ \sum_{j \in \mathbb{Z}} a_j X^j,\ a_j \in \boldsymbol{k},\ a_j = 0 \text{ for } j << \right\},$$

- The field $\mathbb{Q}_p$ of p-adic numbers, p being a prime number,
- The finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, p being a prime number.

Definition 1.2.4. *Let* $\boldsymbol{k}$ *be a field. Its* ***characteristic*** *is the smallest nonzero number* p *such that, for any* $a \in \boldsymbol{k}$, $pa := \underbrace{a + \cdots + a}_{p \text{ times}} = 0$. *If such a* p *does not exist, the field* $\boldsymbol{k}$ *is of* ***characteristic zero***.

The characteristic of a field is always zero or a prime number.

1.3. Modules over a ring.

Definition 1.3.1. *A module over a commutative ring* R *is an abelian group* $(M, +)$ *together with a binary product*

$$\begin{aligned} m : R \times M &\longrightarrow M \\ (\lambda, x) &\longmapsto \lambda x \end{aligned}$$

such that:

- $0x = 0_M$ *for any* $x \in M$,
- $1x = x$ *for any* $x \in M$,
- $\lambda(\mu x) = (\lambda\mu)x$ *for any* $x \in M$ *and* $\lambda, \mu \in R$,
- $\lambda(x - y) = \lambda x - \lambda y$ *for any* $\lambda \in R$ *and* $x, y \in X$.

The term *vector space* is used for modules over a field. Over a noncommutative ring R, one has to distinguish between *left* R*-modules* and *right* R*-modules*.

1.4. Linear algebra.

Let $\boldsymbol{k}$ be a field, and let V be a $\boldsymbol{k}$-vector space. Let us recall that a subset $F \subset V$ is *free* if, for any $(x_1, \ldots, x_p) \in F^p$ and any $(\lambda_1, \ldots, \lambda_p) \in \boldsymbol{k}^p$, if $\lambda_1 x_1 + \cdots + \lambda_p x_p = 0$ then $\lambda_1 = \cdots = \lambda_p = 0$. On the other hand, a subset $F \subset V$ *generates* V if any element of V can be written as a linear combination $\lambda_1 x_1 + \cdots + \lambda_n x_n$ of elements $x_j \in F$, with coefficients $\lambda_j \in \boldsymbol{k}$. A *basis* is a subset which is both free and generating. In this case, any element of V can be written as a linear combination of elements of F in a unique way. Recall that all bases have the same cardinality: the *dimension* of the vector space V.

Proposition 1.4.1. *Any vector space admits a basis*

Proof. (sketch) This is a consequence of the axiom of choice: any free subset can be completed in order to form a basis. □

Definition 1.4.1. *Let* V *and* W *be two vector spaces over the same field* $\boldsymbol{k}$. *A morphism of* $\boldsymbol{k}$*-vector spaces*

$$\varphi : V \longrightarrow W.$$

is called a ***linear map***. *In other words, a linear map* φ *is a morphism of abelian groups which moreover commutes with multiplication by scalars, i.e.*

$$\varphi(\lambda x + \mu y) = \lambda\varphi(x) + \mu\varphi(y)$$

for any $\lambda, \mu \in \boldsymbol{k}$ *and any* $x, y \in V$.

The set of linear maps from V onto W is denoted by $\mathcal{L}(V, W)$. It is naturally endowed with a $\boldsymbol{k}$-vector space structure.

Definition 1.4.2. *Let V be a $\boldsymbol{k}$-vector space, and $W \subseteq V$ a **vector subspace**, i.e. a subset of V stable by addition and multiplication by scalars. The **quotient** V/W is the set of classes in V under the equivalence relation defined by:*

$$x \sim y \Longleftrightarrow x - y \in W.$$

The quotient V/W is a vector space: denoting by $\widetilde{x}$ the class of $x \in V/W$, the vector space operations on V/W are defined by $\widetilde{x} + \widetilde{y} := \widetilde{x+y}$ and $\lambda\widetilde{x} := \widetilde{\lambda x}$. The canonical projection $V \twoheadrightarrow V/W$ is a linear map.

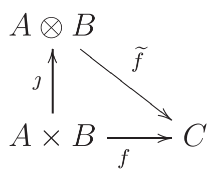
Definition 1.4.3. *Let V_1 V_2 and W be three $\boldsymbol{k}$-vector spaces. A map $\varphi : V_1 \times V_2 \to W$ is **bilinear** if it is linear in each of its argument when the other one is fixed, namely,*

$$f(ax + by, a'x' + b'y') = abf(x, y) + ab'f(x, y') + a'bf(x', y) + a'b'f(x', y'). \tag{1.4.1}$$

1.5. **Tensor product.** Let A and B be two vector spaces over the same field $\boldsymbol{k}$. The *tensor product* $A \otimes B$ is a $\boldsymbol{k}$-vector space which satisfies the following *universal property*: there exists a bilinear map

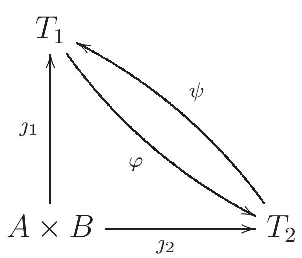
$$\begin{aligned} \jmath : A \times B &\longrightarrow A \otimes B \\ (a, b) &\longmapsto a \otimes b \end{aligned}$$

such that, for any $\boldsymbol{k}$-vector space C and for any bilinear map $f : A \times B \to C$ there is a unique *linear* map $\widetilde{f} : A \otimes B \to C$ such that $f = \widetilde{f} \circ \jmath$, i.e. such that the following diagram commutes:



Proposition 1.5.1. *The tensor product $A \otimes B$ exists and is unique up to isomorphism.*

Proof. Let us show uniqueness first: if $(T_1, \jmath_1)$ and $(T_2, \jmath_2)$ are two candidates for playing the role of a tensor product, the universal property applied to both tells us that there exist linear maps $\varphi : T_1 \to T_2$ and $\psi : T_2 \to T_1$ such that $\jmath_2 = \varphi \circ \jmath_1$ and $\jmath_1 = \psi \circ \jmath_2$:



Applying the universal property twice again shows that $\psi\circ\varphi = \mathrm{Id}_{T_1}$ and $\varphi\circ\psi = \mathrm{Id}_{T_2}$, hence the tensor product is unique up to linear isomorphism.

Now let us prove the existence, which needs the axiom of choice in the infinite-dimensional case: we start from choosing a basis $(e_i)_{i\in I}$ of A and a basis $(f_j)_{j\in J}$ of B. The vector space $A\otimes B$ is then defined as the vector space freely generated by the symbols c_{ij}, $i\in I$, $j\in J$. Explicitly,

$$A\otimes B := \left\{ \sum_{i\in I, j\in J} \lambda_{ij}c_{ij},\ \lambda_{ij}\in \boldsymbol{k},\ \lambda_{ij}=0 \text{ except for a finite number of ordered pairs } (i,j) \right\},$$

and we define the bilinear map $\jmath$ by $\jmath(e_i, e_j) = c_{ij}$, hence $c_{ij} = e_i\otimes e_j$. For any bilinear map $f : A\times B\to C$ (where C is another $\boldsymbol{k}$-vector space), the linear map $\widetilde{f} : A\otimes B\to C$ uniquely determined by $\widetilde{f}(c_{ij}) = f(e_i, e_j)$ is the only one such that $\widetilde{f}\circ\jmath = f$. Hence the space $A\otimes B$ just constructed verifies the universal property. □

The elements $a\otimes b\in A\otimes B$, with $a\in A$ and $b\in B$, generate $A\otimes B$. Tensor products $\boldsymbol{k}\otimes A$ and $A\otimes\boldsymbol{k}$ are canonically identified with A via $1\otimes a\simeq a\otimes 1\simeq a$ for any $a\in A$. Whenever three vector spaces A, B, and C are involved, there is an isomorphism

$$\begin{aligned} \alpha : (A\otimes B)\otimes C &\xrightarrow{\ \sim\ } A\otimes(B\otimes C)\\ (a\otimes b)\otimes c &\longmapsto a\otimes(b\otimes c). \end{aligned}$$

This isomorphism is *not* canonical, because $A\otimes B$ and $B\otimes C$ are themselves defined only up to isomorphism. We shall denote by $A\otimes B\otimes C$ any of these two versions of the iterated tensor product.

Let A and B be two $\boldsymbol{k}$-vector spaces. The flip $\tau : A\otimes B\to B\otimes A$ defined by $\tau(a\otimes b) = b\otimes a$ is a vector space isomorphism. This generalises to a finite collection $(A_1,\ldots,A_n)$ of vector spaces: any permutation $\sigma\in S_n$ yields a linear isomorphism

$$\begin{aligned} \tau_\sigma : A_1\otimes\cdots\otimes A_n &\xrightarrow{\ \sim\ } A_{\sigma_1^{-1}}\otimes\cdots\otimes A_{\sigma_n^{-1}}\\ a_1\otimes\cdots\otimes a_n &\longmapsto a_{\sigma_1^{-1}}\otimes\cdots\otimes a_{\sigma_n^{-1}}, \end{aligned}$$

and we have $\tau_{\omega\sigma} = \tau_\omega\tau_\sigma$ for any $\omega,\sigma\in S_n$. This yields an action of the symmetric group S_n on the direct sum $\bigoplus_{\sigma\in S_n} A_{\sigma_1}\otimes\cdots\otimes A_{\sigma_n}$. The quotient under the action of S_n is called the *unordered tensor product* of $A_1,\ldots,A_n$, denoted by

$$\bigotimes_{j\in\{1,\ldots,n\}} A_j.$$

Proposition 1.5.2. *Let A_1, B_1, A_2, B_2 be $\boldsymbol{k}$-vector spaces. There is a natural injection*

$$\widetilde{\jmath} : \mathcal{L}(A_1, A_2)\otimes\mathcal{L}(B_1, B_2)\longrightarrow \mathcal{L}(A_1\otimes B_1, A_2\otimes B_2)$$

given by

$$\big(\widetilde{\jmath}(f\otimes g)\big)(a\otimes b)=f(a)\otimes g(b).$$

When the vector spaces are finite-fimensional, this embedding is an isomorphism.

Proof. The space $\mathcal{L}(A_1\otimes B_1, A_2\otimes B_2)$ together with the bilinear map:

$$\begin{aligned}\jmath:\mathcal{L}(A_1,A_2)\times\mathcal{L}(B_1,B_2) &\longrightarrow \mathcal{L}(A_1\otimes B_1, A_2\otimes B_2)\\ (f,g) &\longrightarrow \big(a\otimes b\mapsto f(a)\otimes g(b)\big)\end{aligned}$$

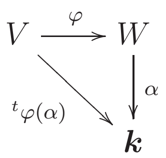
yields the map $\widetilde{\jmath}$ by universal property. It is manifestly injective. In the finite-dimensional case, bijectivity od $\widetilde{\jmath}$ can be proved either by dimension-counting, or by proving that $\mathcal{L}(A_1\otimes B_1, A_2\otimes B_2)$ together with the bilinear map $\jmath$ fulfills the universal property. Details are left to the reader. □

1.6. **Duality.**

Definition 1.6.1. *For any $\boldsymbol{k}$-vector space V, the **dual** V^* is defined by*

$$V^*:=\mathcal{L}(V,\boldsymbol{k}).$$

*For any linear map $\varphi:V\to W$ where W is another vector space, the **transpose** ${}^t\varphi:W^*\to V^*$ of φ is defined by ${}^t\varphi(\alpha):=\alpha\circ\varphi$, namely:*



Proposition 1.6.1. *Ther is a canonical embedding $\Lambda:V\to V^{**}$, which is an isomorphism if and only if V is finite-dimensional.*

Proof. Any $v\in V$ gives rise to $\Lambda(v)=\widetilde{v}\in V^{**}$ defined by $\widetilde{v}(\xi):=\xi(v)$ for any $\xi\in V^*$. The map $\Lambda:V\to V^{**}$ thus defined is clearly injective. Now let $(e_i)_{i\in I}$ be a basis of V. Any finite linear combination $\sum_i\lambda_i\widetilde{e_i}$ belongs to V^{**}. For any $i\in I$, let $\varepsilon_i\in V^*$ be defined by $\varepsilon_i(e_j)=\delta_i^j$. If V is finite-dimensional, then I is finite, and any $\boldsymbol{v}\in V^{**}$ can be written as:

$$\boldsymbol{v}=\sum_{i\in I}\boldsymbol{v}(\varepsilon_i)\widetilde{e_i},$$

hence $\boldsymbol{v}=\widetilde{v}=\Lambda(v)$ with $v:=\sum_{i\in I}\boldsymbol{v}(\varepsilon_i)e_i$. If I is infinite, the family

$$F:=\{(\varepsilon_i)_{i\in I},\,\xi\}$$

is free in V^*, where $\xi\in V^*$ is defined by $\xi(e_i)=1$ for any $i\in I$. We can complete F in a basis of V^*. Now let $\beta\in V^{**}$ defined by $\beta(\xi)=1$ and $\beta(\eta)=0$ for any other η in this basis. In particular, $\beta(\varepsilon_i)=0$ for any $i\in I$, hence β cannot be any $\Lambda(v)$ with $v\in V$. □

Remark 1.6.1. When V is finite-dimensional with a choice of basis $(e_i)_{i\in I}$, the family $(\varepsilon_i)_{i\in I}$ defined above is a basis of V^*, the *dual basis*, characterized by $\varepsilon_i(e_j)=\delta_i^j$.

1.7. **Graded vector spaces.** A vector space is $\mathbb{N}_0$-*graded* if:

$$V = \bigoplus_{n \geq 0} V_n. \tag{1.7.1}$$

The $\boldsymbol{k}$-vector space V_n is the n^{th} *homogeneous component on* V. We shall be interested in the case when the homogeneous components are finite-dimensional. The *Poincaré-Hilbert series of* V is given by:

$$f_V(x) := \sum_{n \geq 0} (\dim V_n) x^n. \tag{1.7.2}$$

The *graded dual* of V is defined by:

$$V^\circ := \bigoplus_{n \geq 0} (V_n)^*. \tag{1.7.3}$$

Proposition 1.7.1. *The graded dual V° is a subspace of the dual V^*, and the graded bidual $V^{\circ\circ}$ is canonically isomorphic to V as a graded vector space.*

Proof. Easy and left as an exercise. □

Definition 1.7.1. *A $\mathbb{N}_0$-graded vector space V is **connected** if* $\dim V_0 = 1$.

1.8. **Filtrations and the functor Gr.**

Definition 1.8.1. *Let V be a $\boldsymbol{k}$-vector field, where $\boldsymbol{k}$ is a field. An **increasing** $\mathbb{N}_0$**-filtration** is a family $(V^n)_{n \geq 0}$ of $\boldsymbol{k}$-vector spaces, with $V^j \subset V^{j+1}$ for any $j \geq 0$ and $\bigcup_{j \geq 0} V^j = V$.*

It will be also convenient to set $V_k = \{0\}$ for any integer $k \leq -1$.

Definition 1.8.2. *Let V be a $\mathbb{N}_0$-filtered vector space. The **associated graded vector space of** V is given by* $\operatorname{Gr} V := \bigoplus_{j \geq 0} V^j / V^{j-1}$.

There is a canonical linear isomorphism $\pi_V : V \to \operatorname{Gr} V$ defined as follows: $\pi(x)$ is the image of x by the projection $\pi_n : V^n \twoheadrightarrow V^n / V^{n-1}$, where n is the *degree* of x, namely:

$$n = |x| = \inf \{j \in \mathbb{N}_0,\, x \in V^j\}. \tag{1.8.1}$$

Proposition 1.8.1. *Let V and W be two filtered vector spaces as above. Any linear map $\varphi : V \to W$ such that $\varphi(V^n) \subset W^n$ for any $n \geq 0$ gives rise to a unique linear map $\operatorname{Gr} \varphi : \operatorname{Gr} V \to \operatorname{Gr} W$ making the following diagram commute:*

$$\begin{array}{ccc} V & \xrightarrow{\ \varphi\ } & W \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \operatorname{Gr} V & \xrightarrow[\operatorname{Gr} \varphi]{} & \operatorname{Gr} W \end{array}$$

Moreover the correspondence Gr *thus defined is a covariant functor: for any three vector spaces V, W, X and any two linear maps $\varphi : V \to W$ and $\psi : W \to X$, we have*

$$\operatorname{Gr}(\psi \circ \varphi) = \operatorname{Gr} \psi \circ \operatorname{Gr} \varphi. \tag{1.8.2}$$

Exercices for Section 1.

Exercise 1.1. Let M be a monoid. Suppose that any element $x \in M$ admits a *right inverse* x'', i.e. such that $xx'' = e$, and a *left inverse* x', i.e. such that $x'x = e$. Show that the left and the right inverse coincide, and hence that M is a group.

Exercise 1.2. Let A and B be two sets. Describe a natural bijective map from $A \times B$ onto $B \times A$. Deduce from that the commutativity of the multiplication of natural numbers.

Exercise 1.3. Let A, B and C three sets. Describe a natural bijective map from $(A \times B) \times C$ onto $A \times (B \times C)$. Deduce from that the associativity of the multiplication of natural numbers.

Exercise 1.4. Imagine two exercises in the spirit of Exercices 1.2 and 1.3, in order to show commutativity and associativity of the addition of natural numbers.

Exercise 1.5. Let R be an integral domain. Show that $R[X]$ is an integral domain. Same question for the ring $R[[X]]$ of formal series.

Exercise 1.6. Let R be a ring not necessarily commutative), and let $R[[X]]$ be its ring of formal series. Define a distance on $R[[X]]$ by the formula:

$$d(f,g) := 2^{-\operatorname{val}(f-g)}. \tag{1.8.3}$$

Show that d is a distance, making $R[[X]]$ a metric space, and show that this metric space is complete.

Exercise 1.7. Let $\boldsymbol{k}$ be a field. Prove that the ring $\boldsymbol{k}[X^{-1}, X]]$ of Laurent series with coefficients in $\boldsymbol{k}$ is a field.

Exercise 1.8. Prove that the characteristic of a field is a prime number (*Hint*: consider the kernel of the multiplication by a prime numer p: what can you say about it?).

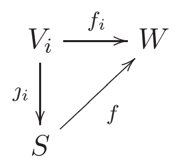
Exercise 1.9. Prove that the operations described in Definition 1.4.2 are well-defined and do endow the quotient V/W with a $\boldsymbol{k}$-vector space structure.

Exercise 1.10. Let $(V_i)_{i\in I}$ be a collection of $\boldsymbol{k}$-vector spaces, indexed by some set I. The *direct sum*:

$$S := \bigoplus_{i\in I} V_i$$

is the set of *finite* formal linear combinations $\sum_{i\in I} \lambda_i v_i$ with $\lambda_i \in \boldsymbol{k}$ and $v_i \in V_i$. Here, finite means that the λ_i's vanish except a finite number of those.

- Prove that S is a $\boldsymbol{k}$-vector space, and prove that the direct sum solves the following universal property: for any vector space W and for any collection $(f_i)_{i\in I}$ of linear maps $f_i : V_i \to W$, there is a unique linear map $f : S \to W$ making the following diagram commute:



Decribe the maps $\jmath_i$ and prove that they are injective.

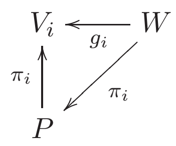
- Prove that the direct sum, when abstractly defined by the universal property above, is unique up to isomorphism.

Exercise 1.11. Let $(V_i)_{i\in I}$ be a collection of $\boldsymbol{k}$-vector spaces, indexed by some set I. The *direct product*:

$$P := \prod_{i\in I} V_i$$

is the set of formal linear combinations $\sum_{i\in I} \lambda_i v_i$ with $\lambda_i \in \boldsymbol{k}$ and $v_i \in V_i$. Contrarily to the direct sum, no finiteness condition is required here.

- Prove that P is a $\boldsymbol{k}$-vector space, and prove that the direct sum solves the following universal property: for any vector space W and for any collection $(g_i)_{i\in I}$ of linear maps $g_i : W \to V_i$, there is a unique linear map $g : W \to P$ making the following diagram commute:



  Decribe the maps π_i and prove that they are surjective.
- Prove that the direct product, when abstractly defined by the universal property above, is unique up to isomorphism.

Exercise 1.12. Prove Proposition 1.7.1.

Exercise 1.13. Let $(V_i)_{i\in I}$ be a collection of $\boldsymbol{k}$-vector spaces, with I finite. Define the unordered tensor product:

$$T := \bigotimes_{i\in I} V_i$$

by means of a universal property (*Hint*: use the product $P := \prod_{i\in I} V_i$ and multilinear maps). Show uniqueness up to isomorphism, and show that this concrete definition matches the definition given in Paragraph 1.5.

2. Hopf algebras: an elementary introduction

2.1. Algebras and modules.

Definition 2.1.1. *Let R be a commutative ring. An R-**algebra** is a ring $\mathcal{A}$, not necessarily commutative nor unital, together with a compatible R-module structure. The compatibility condition is:*

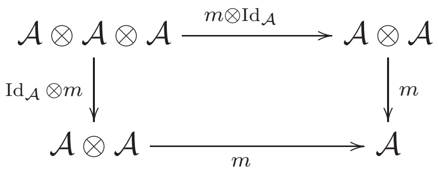
$$\lambda(xy) = (\lambda x)y = x(\lambda y) \tag{2.1.1}$$

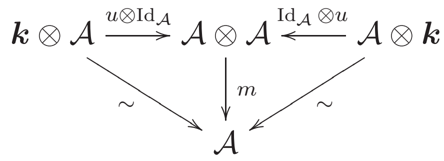
for any $\lambda \in R$ and $x, y \in \mathcal{A}$.

If $\mathcal{A}$ is unital, we denote the unit by $\mathbf{1}_{\mathcal{A}}$ or simply by a boldface $\mathbf{1}$, to distinguish it from the unit 1 of the base ring R.

Proposition 2.1.1. *Let $\boldsymbol{k}$ be a field, and let $\mathcal{A}$ be a $\boldsymbol{k}$-algebra. Then*

(1) *$\mathcal{A}$ is a $\boldsymbol{k}$-vector space.*
(2) *The product naturally defines a linear map $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ via $m(a \otimes b) := ab$.*
(3) *Whenever $\mathcal{A}$ is unital, the map $u : \boldsymbol{k} \to \mathcal{A}$ defined by $u(\lambda) := \lambda \mathbf{1}_{\mathcal{A}}$ is linear.*
(4) *The associativity of the product and the unit property of $\mathbf{1}_{\mathcal{A}}$ are respectively equivalent to the commutativity of the two following diagrams:*





Proof. Straightforward and left to the reader. □

Left ideals, right ideals and two-sided ideals are defined the same way for an R-algebra $\mathcal{A}$ as for a general ring, except that they must be r-submodules of $\mathcal{A}$. A *subalgebra* of $\mathcal{A}$ is a subring which is also an R-submodule.

Example 2.1.1. Let V be a $\boldsymbol{k}$-vector space. The *tensor algebra* of V is defined as:

$$T(V) := \bigoplus_{n \geq 0} V^{\otimes n}, \tag{2.1.2}$$

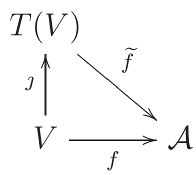
with $V^{\otimes 0} = \boldsymbol{k}$, $V^{\otimes 1} = V$, $V^{\otimes 2} = V \otimes V$, etc. The product is given by concatenation:

$$m(v_1 \cdot v_2 \cdot \cdots \cdot v_p \otimes v_{p+1} \cdot v_{p+2} \cdot \cdots \cdot v_{p+q} := v_1 \cdot v_2 \cdot \cdots \cdot v_{p+q}. \tag{2.1.3}$$

Here we denote the tensor product internal to $T(V)$ with a $\cdot$, to distinguish it with the "external" tensor product $\otimes$ in $T(V) \otimes T(V)$. The unit is the empty word $\mathbf{1} \in \boldsymbol{k} \simeq V^{\otimes 0}$.

Proposition 2.1.2. *Let V be any $\boldsymbol{k}$-vector space. The tensor algebra $T(V)$ is the free unital associative algebra generated by V.*

Proof. We have to show the following universal property: for any unital $\boldsymbol{k}$-algebra $\mathcal{A}$ and for any linear map $f : V \to A$, there exists a unique unital algebra morphism $\widetilde{f} : T(V) \to \mathcal{A}$ such that the following diagram commutes:



where $\jmath$ is the canonical embedding of V in $T(V)$. The map $\widetilde{f}$ is obviously defined by:

$$\widetilde{f}(v_1 \cdot \cdots \cdot v_p) := f(v_1) \cdots f(v_p),$$

where the product takes place in $\mathcal{A}$ on the right-hand side. For $p = 0$ this of course boils down to $\widetilde{f}(\mathbf{1}) = \mathbf{1}_{\mathcal{A}}$. □

Example 2.1.2. Let V be a $\boldsymbol{k}$-vector space. Let J be the two-sided ideal generated by (i.e. the smallest two-sided ideal containing) $\{x \cdot y - y \cdot x,\ x, y \in V\}$. The *symmetric algebra of* V is defined as the quotient $S(V) = T(V)/J$.

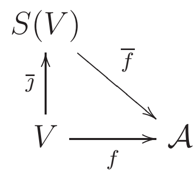
Proposition 2.1.3. *$S(V)$ is the free commutative algebra generated by V.*

Proof. Let us first show that $S(V)$ is a commutative algebra. Let $v = x_1 \cdot \cdots \cdot v_p$ and $w = y_1 \cdot \cdots \cdot y_q$. Let us show that $[v, w] = v \cdot w - w \cdot v$ belongs to J. This is easily seen by induction on $p+q$: indeed the initial cases $p+q = 1$ and $p+q = 2$ are obvious. If $p+q \geq 3$, then $p \geq 2$ or $q \geq 2$. Suppose $p \geq 2$. Then

$$\begin{aligned}
[v, w] &= x_1 \cdot \cdots \cdot x_p \cdot y_1 \cdot \cdots \cdot y_q - y_1 \cdot \cdots \cdot y_q \cdot x_1 \cdot \cdots \cdot x_p \\
&= x_1 \cdot \cdots \cdot x_p \cdot y_1 \cdot \cdots \cdot y_q - x_1 \cdot y_1 \cdot \cdots \cdot y_q \cdot x_2 \cdot \cdots \cdot x_p \\
&\quad + x_1 \cdot y_1 \cdot \cdots \cdot y_q \cdot x_2 \cdot \cdots \cdot x_p - y_1 \cdot \cdots \cdot y_q \cdot x_1 \cdot \cdots \cdot x_p \\
&= x_1 \cdot [x_2 \cdot \cdots \cdot x_p,\, y_1 \cdot \cdots \cdot y_q] + [x_1,\, y_1 \cdot \cdots \cdot y_q] \cdot x_2 \cdot \cdots \cdot x_p
\end{aligned}$$

belongs to J by the induction hypothesis. The case $q \geq 2$ is treated similarly.

Now let us prove the universal property: let $f : V \to \mathcal{A}$ be any linear map, and let $\widetilde{f} : T(V) \to \mathcal{A}$ be the unique extension of f to a unital algebra morphism. As $\mathcal{A}$ is commutative, we obviously have $\widetilde{f}_{|J} = 0$. Hence $\widetilde{f}$ factorizes itself through $S(V) = T(V)/J$, giving rise to $\overline{f} : S(V) \to \mathcal{A}$ such that the following diagram commutes:



where $\overline{\jmath} = \pi \circ \jmath$ and where $\pi : T(V) \twoheadrightarrow V$ is the canonical projection. Such an $\overline{f}$ is unique as $\overline{\jmath}(V)$ generates the algebra $S(V)$. □

Definition 2.1.2. *Let $\mathcal{A}$ be a unital algebra on the field $\boldsymbol{k}$. A* ***left module*** *on $\mathcal{A}$ is a $\boldsymbol{k}$-vector space M together with a linear map*

$$\begin{aligned}
\alpha : \mathcal{A} \otimes M &\longrightarrow M \\
a \otimes x &\longmapsto ax,
\end{aligned}$$

such that $\mathbf{1}_{\mathcal{A}} x = x$ for any $x \in M$, and $a(by) = (ab)y$ for any $a, b \in \mathcal{A}$ and $y \in M$. This amounts to the commutativity of the two following diagrams:

$$\begin{array}{ccc} \mathcal{A}\otimes\mathcal{A}\otimes M & \xrightarrow{\mathrm{Id}_{\mathcal{A}}\otimes\alpha} & \mathcal{A}\otimes M \\ {\scriptstyle m_{\mathcal{A}}\otimes \mathrm{Id}_M}\big\downarrow & & \big\downarrow{\scriptstyle\alpha} \\ \mathcal{A}\otimes M & \xrightarrow[\alpha]{} & M \end{array} \qquad \begin{array}{ccc} \boldsymbol{k}\otimes M & \xrightarrow{u\otimes \mathrm{Id}_M} & \mathcal{A}\otimes M \\ & {\scriptstyle\sim}\searrow & \big\downarrow{\scriptstyle\alpha} \\ & & M \end{array}$$

Right $\mathcal{A}$-modules are defined similarly, replacing $\mathcal{A}\otimes M$ by $M\otimes\mathcal{A}$.

A left $\mathcal{A}$-module M is *simple* if it does not contain any submodule different from $\{0\}$ of M. A left module is called *semi-simple* if it can be written as a direct sum of simple modules.

2.2. Coalgebras and comodules.

Definition 2.2.1. *Let $\boldsymbol{k}$ be a field. A $\boldsymbol{k}$-coalgebra is a $\boldsymbol{k}$-vector space $\mathcal{C}$ together with a linear map $\Delta:\mathcal{A}\to\mathcal{C}\otimes\mathcal{C}$ which is **co-associative**, i.e. such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{C}\otimes\mathcal{C}\otimes\mathcal{C} & \xleftarrow{\Delta\otimes \mathrm{Id}_{\mathcal{C}}} & \mathcal{C}\otimes\mathcal{C} \\ {\scriptstyle \mathrm{Id}_{\mathcal{C}}\otimes\Delta}\big\uparrow & & \big\uparrow{\scriptstyle\Delta} \\ \mathcal{C}\otimes\mathcal{C} & \xleftarrow[\Delta]{} & \mathcal{C} \end{array}$$

*The coalgebra is **co-unital** if moreover there exists a co-unit $\varepsilon:\mathcal{C}\to\boldsymbol{k}$ making the following diagram commute:*

$$\begin{array}{ccccc} \boldsymbol{k}\otimes\mathcal{C} & \xleftarrow{\varepsilon\otimes \mathrm{Id}_{\mathcal{C}}} & \mathcal{C}\otimes\mathcal{C} & \xrightarrow{\mathrm{Id}_{\mathcal{C}}\otimes\varepsilon} & \mathcal{C}\otimes\boldsymbol{k} \\ & {\scriptstyle\sim}\nwarrow & {\scriptstyle\Delta}\big\uparrow & \nearrow{\scriptstyle\sim} & \\ & & \mathcal{C} & & \end{array}$$

A coalgebra is *co-commutative* if moreover $\tau\circ\Delta=\Delta$, where $\tau:\mathcal{C}\otimes\mathcal{C}$ is the *flip*, defined by $\tau(x\otimes y):=y\otimes x$.

Definition 2.2.2. *Let $\mathcal{C}$ be a $\boldsymbol{k}$-coalgebra. A vector subspace $J\subseteq\mathcal{C}$ is:*

- *a **subcoalgebra** if $\Delta(J)\subseteq J\otimes J$,*
- *a **left coideal** if $\Delta(J)\subseteq\mathcal{C}\otimes J$,*
- *a **right coideal** if $\Delta(J)\subseteq J\otimes\mathcal{C}$,*
- *a **two-sided coideal** if $\Delta(J)\subseteq\mathcal{C}\otimes J+J\otimes\mathcal{C}$.*

Proposition 2.2.1. *Let $\boldsymbol{k}$ be a field. The dual $\mathcal{C}^*$ of a co-unital $\boldsymbol{k}$-coalgebra is a unital $\boldsymbol{k}$-algebra. The product (resp. the unit) is given by the transpose of the coproduct (resp. the co-unit).*

Proof. We have $\widetilde{m}={}^t\Delta=(\mathcal{C}\otimes\mathcal{C})^*\to\mathcal{C}^*$. The product m is given by the restriction of $\widetilde{m}$ to $\mathcal{C}^*\otimes\mathcal{C}^*$ (which is strictly contained in $(\mathcal{C}\otimes\mathcal{C})^*$ unless $\mathcal{C}$ is finite-dimensional). Checking associativity is easy and left to the reader, as well as checking the unit axioms for the transposed co-unit $u={}^t\varepsilon:\mathcal{C}^*\to\boldsymbol{k}$. □

For any x in a colgebra $\mathcal{C}$, the coproduct $\Delta x \in \mathcal{C} \otimes \mathcal{C}$ is a finite sum of indecomposable elements. This is emphasized by the widely used *Sweedler notation*:

$$\Delta x = \sum_{(x)} x_1 \otimes x_2. \tag{2.2.1}$$

Equation (2.2.1) must be handled with care, because the decomposition on the right-hand side is by no means unique. It can however be very useful in computations. For example, the iterated coproducts display in Sweedler's notation:

$$\begin{aligned}(\Delta \otimes \mathrm{Id})\Delta x &= \sum_{(x)} x_{1:1} \otimes x_{1:2} \otimes x_2, \\ (\mathrm{Id} \otimes \Delta)\Delta x &= \sum_{(x)} x_1 \otimes x_{2:1} \otimes x_{2:2}.\end{aligned}$$

Coassociativity yields equality of both expressions, which can thus be written in the following simpler form:

$$(\Delta \otimes \mathrm{Id})\Delta x = (\mathrm{Id} \otimes \Delta)\Delta x = \sum_{(x)} x_1 \otimes x_2 \otimes x_3. \tag{2.2.2}$$

The co-commutativity property $\tau \circ \Delta = \Delta$ translates itself in Sweedler's notation as:

$$\sum_{(x)} x_1 \otimes x_2 = \sum_{(x)} x_2 \otimes x_1. \tag{2.2.3}$$

Example 2.2.1 (the coalgebra of a set)**.** Let E be any set, and let $\mathcal{C}$ be the vector space freely generated by E:

$$\mathcal{C} := \left\{ \sum_{a \in E} \lambda_a a,\ \lambda_a = 0 \text{ except for a finite number of them} \right\}$$

The comultiplication is defined by

$$\Delta \left(\sum_{a \in E} \lambda_a a \right) := \sum_{a \in E} \lambda_a a \otimes a. \tag{2.2.4}$$

The co-unit is given by $\varepsilon(a) = 1$ for any $a \in E$ and extended linearly. Note that $\Delta a = a \otimes a$ for any $a \in E$.

Example 2.2.2 (the tensor coalgebra)**.** *Let V be a vector space over a field $\boldsymbol{k}$. The tensor colagebra is defined as:*

$$T^c(V) := \bigoplus_{n \geq 0} V^{\otimes n} \tag{2.2.5}$$

It is isomorphic to the tensor algebra $T(V)$ as a graded vector space. The comultiplication is given by deconcatenation:

$$\Delta(v_1 \cdots v_n) := \sum_{p=0}^{n} v_1 \cdots v_p \otimes v_{p+1} \cdots v_n. \tag{2.2.6}$$

Definition 2.2.3. *Let* $\boldsymbol{k}$ *be a field. A* ***left comodule*** *on the c-unital* $\boldsymbol{k}$*-coalgebra* $\mathcal{C}$ *is a* $\boldsymbol{k}$*-vector space* M *together with a* ***coaction map*** $\Phi : M \to \mathcal{C} \otimes M$ *such that the following diagrams commute:*

$$\begin{array}{ccc} \mathcal{C}\otimes\mathcal{C}\otimes M & \xleftarrow{\Delta\otimes \mathrm{Id}_{\mathcal{C}}} & \mathcal{C}\otimes M \\ {\scriptstyle \mathrm{Id}_{\mathcal{C}}\otimes\Phi}\uparrow & & \uparrow{\scriptstyle \Phi} \\ \mathcal{C}\otimes M & \xleftarrow[\Phi]{} & M \end{array} \qquad\qquad \begin{array}{ccc} \boldsymbol{k}\otimes M & \xleftarrow{\varepsilon\otimes \mathrm{Id}_M} & \mathcal{C}\otimes M \\ & \nwarrow{\scriptstyle \sim} & \uparrow{\scriptstyle \Phi} \\ & & M \end{array}$$

The notion of ***right comodule*** *is defined similarly, with* $M\otimes\mathcal{C}$ *instead of* $\mathcal{C}\otimes M$*.*

Sweedler's notation for a coaction is:

$$\Phi(m) = \sum_{(m)} m_1 \otimes m_0. \tag{2.2.7}$$

We have then:

$$(\Delta\otimes \mathrm{Id}_{\mathcal{C}})\Phi(m) = (\mathrm{Id}_{\mathcal{C}}\otimes\Phi)\Phi(m) = \sum_{(m)} m_1\otimes m_2\otimes m_0. \tag{2.2.8}$$

Theorem 2.2.1 (fundamental theorem of comodule structure theory)**.** *Let* M *be a left comodule over a co-unital coalgebra* $\mathcal{C}$*. For any* $m \in M$*, the subcomodule generated by* m *is finite-dimensional.*

Proof. One can find a finite collection $c_1, \ldots c_s$ of linearly independent elements of $\mathcal{C}$ and a collection $m_1, \ldots, m_s$ of elements of M such that:

$$\Phi(m) = \sum_{i=1}^{s} c_i \otimes m_i.$$

Let N be the vector subspace of M generated by $m_1, \ldots, m_s$. Using the co-unit axiom we see that m belongs to N. Indeed,

$$m = (\varepsilon\otimes \mathrm{Id}_{\mathcal{C}})\circ\Phi(m) = \sum_{i=1}^{s} \varepsilon(c_j)m_j.$$

Now let us prove that N is a subcomodule of M. Let us choose linear forms $f_1, \ldots, f_s$ of $\mathcal{C}$ such that $f_i(c_j) = \delta_i^j$. Then we compute:

$$\begin{aligned}
\Phi(m_i) &= (f_i \otimes \mathrm{Id}_{\mathcal{C}} \otimes \mathrm{Id}_{\mathcal{C}})\big(c_i \otimes \Phi(m_i)\big) \\
&= (f_i \otimes \mathrm{Id}_{\mathcal{C}} \otimes \mathrm{Id}_{\mathcal{C}}) \left(\sum_{i=1}^{s} c_j \otimes \Phi(m_j)\right) \\
&= (f_i \otimes \mathrm{Id}_{\mathcal{C}} \otimes \mathrm{Id}_{\mathcal{C}}) \circ (\mathrm{Id}_{\mathcal{C}} \otimes \Phi) \circ \Phi(m) \\
&= (f_i \otimes \mathrm{Id}_{\mathcal{C}} \otimes \mathrm{Id}_{\mathcal{C}}) \circ (\Delta \otimes \mathrm{Id}_{\mathcal{C}}) \circ \Phi(m) \\
&= (f_i \otimes \mathrm{Id}_{\mathcal{C}} \otimes \mathrm{Id}_{\mathcal{C}}) \circ (\Delta \otimes \mathrm{Id}_{\mathcal{C}}) \left(\sum_{j=1}^{s} c_j \otimes m_j\right) \\
&= \sum_{j=1}^{s} (f_i \otimes \mathrm{Id}_{\mathcal{C}})(\Delta c_j) \otimes m_j.
\end{aligned}$$

Hence $\Phi(m_i) \in \mathcal{C} \otimes N$, which proves Theorem 2.2.1. □

Corollary 2.2.1. *Let M be a left comodule over a co-unital coalgebra $\mathcal{C}$. Any left subcomodule generated by a finite set is finite-dimensional.*

Proof. remark that if $P = \{m_1, \ldots, m_n\}$, the left subcomodule generated by P is the sum of the left comodules generated by the m_j's, and then apply Theorem 2.2.1. □

Theorem 2.2.2 (fundamental theorem of coalgebra structure theory)**.** *Let $\boldsymbol{k}$ be a field, and let $\mathcal{C}$ be a $\boldsymbol{k}$-coalgebra. Then the subcoalgebra generated by one single element x is finite-dimensional.*

Proof. The coalgebra $\mathcal{C}$ is a left comodule over itself. Let N be the left subcomodule, i.e. the left coideal here, generated by x. According to Theorem 2.2.1, N is finite-dimensional. Then the orthogonal $N^\perp$ has finite codimension in $\mathcal{C}^*$, equal to $\dim N$. The dual $\mathcal{C}^*$ is an algebra (see Exercise 2.3), and $N^\perp$ is a left ideal therein, see Exercise 2.4. The quotient space $E = \mathcal{C}^*/N^\perp$ is a finite-dimensional left module over $\mathcal{C}^*$. Let K be the annihilator of this left module. As kernel of the associated representation $\rho : \mathcal{C}^* \to \mathrm{End}\, E$ it has clearly finite codimension, and it is a two-sided ideal.

Now the orthogonal $K^\top$ of K in $\mathcal{C}$ is a subcoalgebra of $\mathcal{C}$, see Exercise 2.5. Moreover it is finite-dimensional, as $\dim K^\top = \mathrm{codim}\, K^{\top\perp} \le \mathrm{codim}\, K$. Finally $K \subset N^\perp$ implies that $N^{\perp\top} \subset K^\top$, so x belongs to $K^\top$. The subcoalgebra generated by x is then included in the finite-dimensional subcoalgebra $K^\top$, which proves the theorem. □

Definition 2.2.4. *A coalgebra $\mathcal{C}$ is said to be **irreducible** if two nonzero subcoalgebras of $\mathcal{C}$ have always nonzero intersection. A **simple** coalgebra is a coalgebra which does not contain any proper subcoalgebra. A coalgebra $\mathcal{C}$ will be called **pointed** if any simple subcoalgebra of $\mathcal{C}$ is one-dimensional.*

Lemma 2.2.1. *Any coalgebra $\mathcal{C}$ contains a simple subcoalgebra.*

Proof. According to theorem 2.2.2 we may suppose that $\mathcal{C}$ is finite-dimensional, and the lemma is immediate in this case. □

Proposition 2.2.2. *A coalgebra $\mathcal{C}$ is irreducible if and only if it contains a unique simple subcoalgebra.*

Proof. Suppose $\mathcal{C}$ irreducible, and suppose that $\mathcal{D}_1$ and $\mathcal{D}_2$ are two simple subcoalgebras. The intersection $\mathcal{D}_1 \cap \mathcal{D}_2$ is nonzero, and hence, by simplicity, $\mathcal{D}_1 = \mathcal{D}_2$. Conversely suppose that $\mathcal{E}$ is the only simple subcoalgebra of $\mathcal{C}$, and let $\mathcal{D}$ any subcoalgebra. According to lemma 2.2.1 we have $\mathcal{E} \subset \mathcal{D}$, hence $\mathcal{E}$ is included in any intersection of subcoalgebras, which proves that $\mathcal{C}$ is irreducible. □

2.3. **Convolution product.** Let $\mathcal{A}$ be an algebra and $\mathcal{C}$ be a coalgebra (over the same field $\boldsymbol{k}$). Then there is an associative product on the space $\mathcal{L}(\mathcal{C}, \mathcal{A})$ of linear maps from $\mathcal{C}$ to $\mathcal{A}$ called the *convolution product*. It is given by:

$$\varphi * \psi = m_{\mathcal{A}} \circ (\varphi \otimes \psi) \circ \Delta_{\mathcal{C}}.$$

In Sweedler's notation it reads:

$$\varphi * \psi(x) = \sum_{(x)} \varphi(x_1)\psi(x_2).$$

The associativity is a direct consequence of both associativity of $\mathcal{A}$ and coassociativity of $\mathcal{C}$.

2.4. **Intermezzo: Lie algebras.** Let $\boldsymbol{k}$ be a field, with characteristic different from 2. A *Lie algebra* on $\boldsymbol{k}$ is a $\boldsymbol{k}$-vector field $\mathfrak{g}$ endowed with a bilinear map $(X, Y) \mapsto [X, Y]$ such that:

(1) For any $X, Y \in \mathfrak{g}$ one has $[Y, X] = -[X, Y]$ (antisymmetry).
(2) For any $X, Y, Z \in \mathfrak{g}$ one has $\big[[X, Y], Z\big] + \big[[Y, Z], X\big] + \big[[Z, X], Y\big] = 0$ (Jacobi identity).

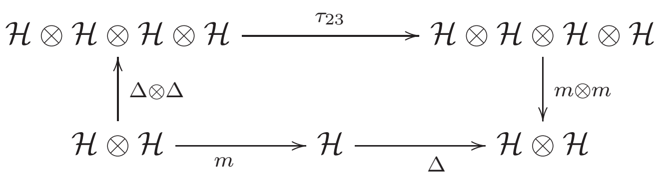
Axiom (1) is equivalent to $[X, X] = 0$ for any $X \in \mathfrak{g}$. Denoting by $\operatorname{ad} X$ the linear endomorphism $Y \mapsto [X, Y] : \mathfrak{g} \to \mathfrak{g}$, The Jacobi identity is equivalent to the fact that $\operatorname{ad} X$ is a derivation of the Lie algebra $\mathfrak{g}$, namely:

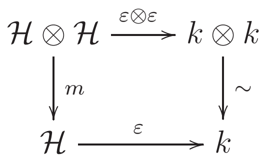
$$\operatorname{ad} X.[Y, Z] = [\operatorname{ad} X.Y,\ Z] + [Y,\ \operatorname{ad} X.Z] \tag{2.4.1}$$

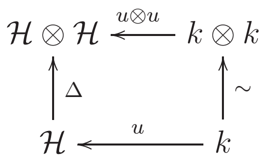
for any $X, Y, Z \in \mathfrak{g}$.

2.5. **Bialgebras and Hopf algebras.**

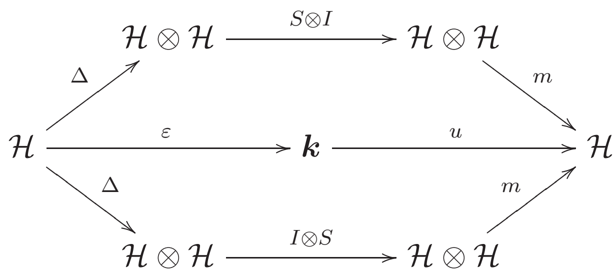
Definition 2.5.1. *A (unital and co-unital)* ***bialgebra*** *is a vector space $\mathcal{H}$ endowed with a structure of unital algebra (m, u) and a structure of co-unital coalgebra (Δ, ε) which are compatible. The compatibility requirement is that Δ is an algebra morphism (or equivalently that m is a coalgebra morphism), ε is an algebra morphism and u is a coalgebra morphism. It is expressed by the commutativity of the three following diagrams:*







Definition 2.5.2. *A Hopf algebra is a bialgebra $\mathcal{H}$ together with a linear map $S : \mathcal{H} \to \mathcal{H}$ called the **antipode**, such that the following diagram commutes:*



In Sweedler's notation it reads:

$$\sum_{(x)} S(x_1)x_2 = \sum_{(x)} x_1 S(x_2) = (u \circ \varepsilon)(x).$$

In other words the antipode is an inverse of the identity I for the convolution product on $\mathcal{L}(H, H)$. The unit for the convolution is the map $u \circ \varepsilon$.

Definition 2.5.3. *A **primitive element** in a bialgebra $\mathcal{H}$ is an element x such that $\Delta x = x \otimes \mathbf{1} + \mathbf{1} \otimes x$. A **grouplike element** is a nonzero element x such that $\Delta x = x \otimes x$. Note that grouplike elements make sense in any coalgebra.*

A *bi-ideal* in a bialgebra $\mathcal{H}$ is a two-sided ideal which is also a two-sided co-ideal. A *Hopf ideal* in a Hopf algebra $\mathcal{H}$ is a bi-ideal J such that $S(J) \subset J$.

Example 2.5.1 (The Hopf algebra of a group)**.** Let G be a group, and let $\boldsymbol{k}G$ be the group algebra (over the field $\boldsymbol{k}$). It is by definition the vector space freely generated by the elements of G: the product of G extends uniquely to a bilinear map from $\boldsymbol{k}G \times \boldsymbol{k}G$ into $\boldsymbol{k}G$, hence a multiplication $m : \boldsymbol{k}G \otimes \boldsymbol{k}G \to \boldsymbol{k}G$, which is associative. The neutral element of G gives the unit for m. The space $\boldsymbol{k}G$ is also endowed with a the co-unital coalgebra structure of the set G defined in Example 2.2.1.

Proposition 2.5.1. *The vector space $\boldsymbol{k}G$ endowed with the algebra and coalgebra structures defined above is a Hopf algebra. The antipode is given by:*

$$S(g) = g^{-1}, g \in G.$$

Proof. The compatibility of the product and the coproduct is an immediate consequence of the following computation: for any $g, h \in G$ we have:

$$\Delta(gh) = gh \otimes gh = (g \otimes g)(h \otimes h) = \Delta g.\Delta h.$$

Now $m(S \otimes I)\Delta(g) = g^{-1}g = e$ and similarly for $m(I \otimes S)\Delta(g)$. But $e = u \circ \varepsilon(g)$ for any $g \in G$, so map S is indeed the antipode. □

Remark 2.5.1. If G were only a semigroup, the same construction would lead to a bialgebra structure on $\boldsymbol{k}G$: the Hopf algebra structure (i.e. the existence of an antipode) reflects the group structure (the existence of the inverse). We have $S^2 = I$ in this case, but involutivity of the antipode is not true for general Hopf algebras.

Example 2.5.2 (Tensor algebras)**.** There is a natural structure of cocommutative Hopf algebra on the tensor algebra $T(V)$ of any vector space V. Namely we define the coproduct Δ as the unique algebra morphism from $T(V)$ into $T(V) \otimes T(V)$ such that:

$$\Delta(1) = 1 \otimes 1, \qquad \Delta(x) = x \otimes 1 + 1 \otimes x,\ x \in V.$$

We define the co-unit as the algebra morphism such that $\varepsilon(1) = 1$ and $\varepsilon_{|V} = 0$ This endows $T(V)$ with a cocommutative bialgebra structure. We claim that the principal anti-automorphism:

$$S(x_1 \otimes \cdots \otimes x_n) = (-1)^n x_n \otimes \cdots \otimes x_1$$

verifies the axioms of an antipode, so that $T(V)$ is indeed a Hopf algebra. For $x \in V$ we have $S(x) = -x$, hence $S * I(x) = I * S(x) = 0$. As V generates $T(V)$ as an algebra it is easy to conclude.

Example 2.5.3 (Enveloping algebras)**.** Let $\mathfrak{g}$ be a Lie algebra. The universal enveloping algebra is the quotient of the tensor algebra $T(\mathfrak{g})$ by the ideal J generated by $x \otimes y - y \otimes x - [x, y],\ x, y \in \mathfrak{g}$.

Lemma 2.5.1. *J is a Hopf ideal, i.e. $\Delta(J) \subset \mathcal{H} \otimes J + J \otimes \mathcal{H}$ and $S(J) \subset J$.*

Proof. The ideal J is generated by primitive elements (according to proposition 2.5.3 below), and any ideal generated by primitive elements is a Hopf ideal (very easy and left to the reader). □

The quotient of a Hopf algebra by a Hopf ideal is a Hopf algebra. Hence the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is a cocommutative Hopf algebra.

We summarise in the proposition below the main properties of the antipode in a Hopf algebra:

Proposition 2.5.2. (cf. [30, Proposition 4.0.1]) *Let $\mathcal{H}$ be a Hopf algebra with multiplication m, comultiplication Δ, unit $u : 1 \mapsto \mathbf{1}$, co-unit ε and antipode S. Then:*

(1) *$S \circ u = u$ and $\varepsilon \circ S = \varepsilon$.*

(2) *S is an algebra antimorphism and a coalgebra antimorphism, i.e. if τ denotes the flip we have:*

$$m \circ (S \otimes S) \circ \tau = S \circ m, \qquad \tau \circ (S \otimes S) \circ \Delta = \Delta \circ S.$$

(3) *If $\mathcal{H}$ is commutative or cocommutative, then $S^2 = I$.*

For a detailed proof, see Chr. Kassel in [19].

Proposition 2.5.3. (1) *If x is a primitive element then $S(x) = -x$.*

(2) *The linear subspace* $\operatorname{Prim}\mathcal{H}$ *of primitive elements in $\mathcal{H}$ is a Lie algebra.*

Proof. If x is primitive, then $(\varepsilon \otimes \varepsilon) \circ \Delta(x) = 2\varepsilon(x)$. On the other hand, $(\varepsilon \otimes \varepsilon) \circ \Delta(x) = \varepsilon(x)$, so $\varepsilon(x) = 0$. Then:

$$0 = (u \circ \varepsilon)(x) = m(S \otimes I)\Delta(x) = S(x) - x.$$

Now let x and y be primitive elements of $\mathcal{H}$. Then we can easily compute:

$$\begin{aligned} \Delta(xy - yx) &= (x \otimes \mathbf{1} + \mathbf{1} \otimes x)(y \otimes \mathbf{1} + \mathbf{1} \otimes y) - (y \otimes \mathbf{1} + \mathbf{1} \otimes y)(x \otimes \mathbf{1} + \mathbf{1} \otimes x) \\ &= (xy - yx) \otimes \mathbf{1} + \mathbf{1} \otimes (xy + yx) + x \otimes y + y \otimes x - y \otimes x - x \otimes y \\ &= (xy - yx) \otimes \mathbf{1} + \mathbf{1} \otimes (xy - yx). \end{aligned}$$

□

Exercises for Section 2.

Exercise 2.1. Let $\mathcal{A}$ be a $\boldsymbol{k}$-algebra, where $\boldsymbol{k}$ is a field. A left $\mathcal{A}$-module M is *simple* if it contains no submodule except $\{0\}$ and M. A left $\mathcal{A}$-module M is *semi-simple* if it is isomorphic to a finite direct sum of simple $\mathcal{A}$-modules. Prove that for any maximal left ideal of $\mathcal{A}$, the quotient $\mathcal{A}/J$ is a simple left $\mathcal{A}$-module. Prove that, conversely, any simple left $\mathcal{A}$-module is the quotient $\mathcal{A}/J$ where J is a maximal left ideal.

Exercise 2.2. [Jacobson's density theorem] Let $\mathcal{A}$ be a $\boldsymbol{k}$-algebra, where $\boldsymbol{k}$ is a field. For any left $\mathcal{A}$-module M we denote by $\mathcal{A}'_M$ the algebra of the $\mathcal{A}$-module endomorphisms of M, and we denote by $\mathcal{A}''_M$ the algebra of the $\mathcal{A}'_M$-module endomorphisms of M.

(1) Describe a natural map $\iota : \mathcal{A} \to \mathcal{A}''_M$.

(2) Now let M be a semi-simple $\mathcal{A}$-module, and let $x_1, \ldots, x_n$ be a finite collection of elements of M. Then for any $a'' \in \mathcal{A}''_M$, there exists an element $a \in \mathcal{A}$ such that $a''x_j = ax_j$ for any $j = 1, \ldots, n$. (*Hint*: use semi-simplicity to prove that any $\mathcal{A}$-submodule of M is a $\mathcal{A}''_M$-submodule.)

Exercise 2.3. Prove that the linear dual $\mathcal{C}^*$ of a co-unital coalgebra $\mathcal{C}$ is a unital algebra, with product (resp. unit map) the transpose of the coproduct (resp. of the co-unit). Is the dual $\mathcal{A}^*$ of a co-unital algebra $\mathcal{A}$ a co-unital coalgebra ?

Exercise 2.4. Let $\mathcal{C}$ be a co-unital coalgebra, and let J be a linear subspace of $\mathcal{C}$. Denote by $J^\perp$ the orthogonal of J in $\mathcal{C}^*$. Prove that:

- J is a two-sided coideal if and only if $J^\perp$ is a subalgebra of $\mathcal{C}^*$.
- J is a left coideal if and only if $J^\perp$ is a left ideal of $\mathcal{C}^*$.
- J is a right coideal if and only if $J^\perp$ is a right ideal of $\mathcal{C}^*$.
- J is a subcoalgebra if and only if $J^\perp$ is a two-sided ideal of $\mathcal{C}^*$.

Exercise 2.5. Let $\mathcal{C}$ be a co-unital coalgebra, and let K be a linear subspace of $\mathcal{C}^*$. Denote by $K^\top$ the orthogonal of K in $\mathcal{C}$. Show that $K^\top = K^\perp \cap \mathcal{C}$ where $K^\perp$ is the orthogonal of K in the bidual $\mathcal{C}^{**}$. Prove that:

- $K^\top$ is a two-sided coideal if and only if K is a subalgebra of $\mathcal{C}^*$.
- $K^\top$ is a left coideal if and only if K is a left ideal of $\mathcal{C}^*$.
- $K^\top$ is a right coideal if and only if K is a right ideal of $\mathcal{C}^*$.
- $K^\top$ is a subcoalgebra if and only if K is a two-sided ideal of $\mathcal{C}^*$.

Exercise 2.6. Let $\mathcal{A}$ be an associative algebra on a field $\boldsymbol{k}$, and let $[a, b] := ab - ba$ for any $a, b \in \mathcal{A}$. Prove that $(\mathcal{A}, [,])$ is a Lie algebra.

Exercise 2.7. [Pre-Lie algebras] A *left pre-Lie algebra* on a field $\boldsymbol{k}$ is a $\boldsymbol{k}$-vector space $\mathcal{A}$ together with a bilinear map $\rhd : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that:

$$a \rhd (b \rhd c) - (a \rhd b) \rhd c = b \rhd (a \rhd c) - (b \rhd a) \rhd c \tag{2.5.1}$$

for any $a, b, c \in \mathcal{A}$.

- Prove that any associative algebra is a left pre-Lie algebra.
- Let $(\mathcal{A}, \rhd)$ be a left pre-Lie algebra. Prove that $[a, b] := a \rhd b - b \rhd a$ is a Lie bracket on $\mathcal{A}$.

Exercise 2.8. [Sweedler's four-dimensional Hopf algebra] Let $\widetilde{\mathcal{H}}$ be the unital algebra generated by three elements g, g^{-1}, x with the relation $gg^{-1} = g^{-1}g = \mathbf{1}$.

(1) Prove that there exists a unique coproduct $\Delta : \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}} \otimes \widetilde{\mathcal{H}}$ such that:

$$\Delta g = g \otimes g, \qquad \Delta x = \mathbf{1} \otimes x + x \otimes g.$$

Prove that the unital algebra $\widetilde{\mathcal{H}}$ together with coproduct Δ, co-unit given by

$$\varepsilon(g) = 1, \qquad \varepsilon(x) = 0$$

and antipode given by

$$S(g) = g^{-1}, \qquad S(x) = -xg^{-1}$$

is a Hopf algebra.

(2) Prove that the ideal J generated by x^2, $g^2 - 1$ and $xg + gx$ is a Hopf ideal. Prove that the quotient $\mathcal{H} := \widetilde{\mathcal{H}}/J$ is four-dimensional, with basis $(\mathbf{1}, x, g, gx)$, where we still denote by g and x their images in the quotient.

(3) Compute the square S^2 of the antipode in $\mathcal{H}$.

3. Gradings, filtrations, connectedness

We introduce the crucial property of connectedness for bialgebras. The main interest resides in the possibility to implement recursive procedures in connected bialgebras, the induction taking place with respect to a filtration (e.g. the coradical filtration) or a grading. An important example of these techniques is the recursive construction of the antipode, which then "comes for free", showing that any connected bialgebra is in fact a connected Hopf algebra.

3.1. **Connected graded bialgebras.** Let $\boldsymbol{k}$ be a field with characteristic zero. We shall denote by $\boldsymbol{k}[[t]]$ the ring of formal series on $\boldsymbol{k}$, and by $\boldsymbol{k}[t^{-1},t]]$ the field of Laurent series on $\boldsymbol{k}$. A *graded Hopf algebra* on $\boldsymbol{k}$ is a graded k-vector space:

$$\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$$

endowed with a product $m : \mathcal{H}\otimes\mathcal{H} \to \mathcal{H}$, a coproduct $\Delta : \mathcal{H} \to \mathcal{H}\otimes\mathcal{H}$, a unit $u : k \to \mathcal{H}$, a co-unit $\varepsilon : \mathcal{H} \to k$ and an antipode $S : \mathcal{H} \to \mathcal{H}$ fulfilling the usual axioms of a Hopf algebra, and such that:

$$\mathcal{H}_p.\mathcal{H}_q \subset \mathcal{H}_{p+q} \tag{3.1.1}$$

$$\Delta(\mathcal{H}_n) \subset \bigoplus_{p+q=n} \mathcal{H}_p\otimes\mathcal{H}_q. \tag{3.1.2}$$

$$S(\mathcal{H}_n) \subset \mathcal{H}_n \tag{3.1.3}$$

Lemma 3.1.1. *The unit* $\mathbf{1}$ *belongs to* $\mathcal{H}_0$*, and* $\varepsilon(\mathcal{H}_n) = \{0\}$ *for any* $n \geq 1$*.*

Proof.[2] Suppose that $\mathbf{1} = \sum_{j\geq 0} a_j$ with $a_j \in \mathcal{H}_j$. Equality $\mathbf{1}a_0 = a_0\mathbf{1} = a_0$ yields

$$a_0a_j = a_ja_0 = 0 \tag{3.1.4}$$

for any $j \geq 1$. Now $\mathbf{1}.\mathbf{1} = \mathbf{1}$ implies:

$$\begin{aligned} a_0^2 &= a_0,\\ a_1 &= a_1a_0 + a_0a_1,\\ a_2 &= a_2a_0 + a_1^2 + a_2a_0,\\ &\vdots\\ a_n &= a_na_0 + a_0a_n + \sum_{i+j=n,\, i\neq 0, j\neq 0} a_ia_j, \end{aligned}$$

which, together with (3.1.4), recursively implies $a_n = 0$ for any $n \geq 1$. Hence $\mathbf{1} = a_0 \in \mathcal{H}_0$. The second assertion is proved similarly by a duality argument. Indeed, the transpose ${}^t\varepsilon$ of

[2] From a Mathoverflow note by E. Wofsey.

the co-unit is a unit map in the algebra $\mathcal{H}^* = \prod_{j\geq 0}(\mathcal{H}_j)^*$. The corresponding unit $\boldsymbol{e} \in \mathcal{H}^*$ is written as the (possibly infinite) following sum:

$$\boldsymbol{e} = \sum_{j\geq 0} \boldsymbol{e}_j$$

with $\boldsymbol{e}_j \in (\mathcal{H}_j)^*$. We now have to prove that $\boldsymbol{e}$ belongs to $(\mathcal{H}_0)^*$, i.e. $\boldsymbol{e}_n = 0$ for any $n \geq 1$. The end of the proof is similar to the one of the first assertion, due to the inclusion

$${}^t\Delta\big((\mathcal{H}_p)^* \otimes (\mathcal{H}_q)^*\big) \subset (\mathcal{H}_{p+q})^*.$$

□

If we do not ask for the existence of an antipode $\mathcal{H}$ we get the definition of a *graded bialgebra*. In a graded bialgebra $\mathcal{H}$ we shall consider the increasing filtration:

$$\mathcal{H}^n = \bigoplus_{p=0}^{n} \mathcal{H}_p.$$

Suppose moreover that $\mathcal{H}$ is *connected*, i.e. $\mathcal{H}_0$ is one-dimensional (see Definition 1.7.1). Then we have:

$$\operatorname{Ker}\varepsilon = \bigoplus_{n\geq 1} \mathcal{H}_n.$$

Proposition 3.1.1. *For any $x \in \mathcal{H}^n, n \geq 1$ we can write:*

$$\Delta x = x \otimes \mathbf{1} + \mathbf{1} \otimes x + \widetilde{\Delta} x, \qquad \widetilde{\Delta} x \in \bigoplus_{p+q=n, p\neq 0, q\neq 0} \mathcal{H}_p \otimes \mathcal{H}_q.$$

The map $\widetilde{\Delta}$ is coassociative on $\operatorname{Ker}\varepsilon$ *and $\widetilde{\Delta}_k = (I^{\otimes k-1} \otimes \widetilde{\Delta})(I^{\otimes k-2} \otimes \widetilde{\Delta})...\widetilde{\Delta}$ sends $\mathcal{H}^n$ into $(\mathcal{H}^{n-k})^{\otimes k+1}$.*

Proof. Thanks to connectedness we clearly can write:

$$\Delta x = u \otimes \mathbf{1} + \mathbf{1} \otimes v + \widetilde{\Delta} x$$

with $u, v \in \mathcal{H}$ and $\widetilde{\Delta} x \in \operatorname{Ker}\varepsilon \otimes \operatorname{Ker}\varepsilon$. The co-unit property then tells us that, with $k \otimes \mathcal{H}$ and $\mathcal{H} \otimes k$ canonically identified with $\mathcal{H}$:

$$x = (\varepsilon \otimes I)(\Delta x) = v, \qquad x = (I \otimes \varepsilon)(\Delta x) = u, \tag{3.1.5}$$

hence $u = v = x$. We shall use the following two variants of Sweedler's notation:

$$\Delta x = \sum_{(x)} x_1 \otimes x_2, \tag{3.1.6}$$

$$\widetilde{\Delta} x = \sum_{(x)} x' \otimes x'', \tag{3.1.7}$$

the second being relevant only for $x \in \operatorname{Ker} \varepsilon$. if x is homogeneous of degree n we can suppose that the components x_1, x_2, x', x'' in the expressions above are homogeneous as well, and we have then $|x_1| + |x_2| = n$ and $|x'| + |x''| = n$. We easily compute:

$$\begin{aligned}(\Delta \otimes I)\Delta(x) &= x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x \\ &+ \sum_{(x)} x' \otimes x'' \otimes 1 + x' \otimes 1 \otimes x'' + 1 \otimes x' \otimes x'' \\ &+ (\widetilde{\Delta} \otimes I)\widetilde{\Delta}(x)\end{aligned}$$

and

$$\begin{aligned}(I \otimes \Delta)\Delta(x) &= x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x \\ &+ \sum_{(x)} x' \otimes x'' \otimes 1 + x' \otimes 1 \otimes x'' + 1 \otimes x' \otimes x'' \\ &+ (I \otimes \widetilde{\Delta})\widetilde{\Delta}(x),\end{aligned}$$

hence the co-associativity of $\widetilde{\Delta}$ comes from the one of Δ. Finally it is easily seen by induction on k that for any $x \in \mathcal{H}^n$ we can write:

$$\widetilde{\Delta}_k(x) = \sum_x x^{(1)} \otimes \cdots \otimes x^{(k+1)}, \tag{3.1.8}$$

with $|x^{(j)}| \geq 1$. The grading imposes:

$$\sum_{j=1}^{k+1} |x^{(j)}| = n,$$

so the maximum possible for any degree $|x^{(j)}|$ is $n - k$. □

3.2. **Connected filtered bialgebras.** A *filtered Hopf algebra* on a field $\boldsymbol{k}$ is a $\boldsymbol{k}$-vector space together with an increasing $\mathbb{N}_0$-indexed filtration:

$$\mathcal{H}^0 \subset \mathcal{H}^1 \subset \cdots \subset \mathcal{H}^n \subset \cdots, \qquad \bigcup_n \mathcal{H}^n = \mathcal{H}$$

endowed with a product $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$, a coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, a unit $u : k \to \mathcal{H}$, a co-unit $\varepsilon : \mathcal{H} \to \boldsymbol{k}$ and an antipode $S : \mathcal{H} \to \mathcal{H}$ fulfilling the usual axioms of a Hopf algebra, and such that:

$$\mathcal{H}^p.\mathcal{H}^q \subset \mathcal{H}^{p+q} \tag{3.2.1}$$

$$\Delta(\mathcal{H}^n) \subset \sum_{p+q=n} \mathcal{H}^p \otimes \mathcal{H}^q \tag{3.2.2}$$

$$S(\mathcal{H}^n) \subset \mathcal{H}^n. \tag{3.2.3}$$

If we do not ask for the existence of an antipode $\mathcal{H}$ we get the definition of a *filtered bialgebra*. For any $x \in \mathcal{H}$ we set:

$$|x| := \min\{n \in \mathbb{N},\ x \in \mathcal{H}^n\}. \tag{3.2.4}$$

Any graded bialgebra or Hopf algebra is obviously filtered by the canonical filtration associated to the grading:

$$\mathcal{H}^n := \bigoplus_{i=0}^{n} \mathcal{H}_i, \tag{3.2.5}$$

and in that case, if x is a nonzero homogeneous element, x is of degree n if and only if $|x| = n$.

Lemma 3.2.1. *The unit of a filtered algebra $\mathcal{H}$ belongs to $\mathcal{H}^0$.*

Proof. Consider the associated graded algebra $\mathrm{Gr}\,\mathcal{H}$. Applying the functor Gr (see Paragraph 1.8 and also Exercise 3.5 below) to the product and the unit, we get the product $\overline{m} = \mathrm{Gr}\,m : \mathrm{Gr}\,\mathcal{H} \otimes \mathrm{Gr}\,\mathcal{H} \to \mathrm{Gr}\,\mathcal{H}$ as well as the unit $\overline{u} : \mathrm{Gr}\,u : \boldsymbol{k} \to \mathrm{Gr}\,\mathcal{H}$. Here $\boldsymbol{k}$ is concentrated in degree zero and coincides with $\mathrm{Gr}\,\boldsymbol{k}$. If the unit u does not take its values in $\mathcal{H}^0$, then its graduate counterpart $\overline{u}$ does not take its values in $(\mathrm{Gr}\,\mathcal{H})_0$, which contradicts Lemma 3.1.1. The conclusion follows from $\mathcal{H}^0 = (\mathrm{Gr}\,\mathcal{H})_0$. □

We say that the filtered bialgebra $\mathcal{H}$ is connected if $\mathcal{H}^0$ is one-dimensional. There is an analogue of Proposition 3.1.1 in the connected filtered case, the proof of which is very similar[3]:

Proposition 3.2.1. *For any $x \in \mathcal{H}^n \cap \mathrm{Ker}\,\varepsilon$, $n \geq 1$, we can write:*

$$\Delta x = x \otimes \mathbf{1} + \mathbf{1} \otimes x + \widetilde{\Delta} x, \qquad \widetilde{\Delta} x \in \sum_{p+q=n,\, p\neq 0,\, q\neq 0} \mathcal{H}^p \otimes \mathcal{H}^q. \tag{3.2.6}$$

The map $\widetilde{\Delta}$ is coassociative on $\mathrm{Ker}\,\varepsilon$ and $\widetilde{\Delta}_k = (I^{\otimes k-1} \otimes \widetilde{\Delta})(I^{\otimes k-2} \otimes \widetilde{\Delta}) \cdots \widetilde{\Delta}$ sends $\mathcal{H}^n$ into $(\mathcal{H}^{n-k})^{\otimes k+1}$.

Proof. First of all we have $\mathbf{1} \in \mathcal{H}^0$. Indeed, suppose $|\mathbf{1}| = d > 0$. Then

$$\Delta \mathbf{1} \in \sum_{k=0}^{d} \mathcal{H}^k \otimes \mathcal{H}^{d-k} \subset \mathcal{H}^d \otimes \mathcal{H}^{d-1} + \mathcal{H}^{d-1} \otimes \mathcal{H}^d,$$

hence $\pi \otimes \pi(\Delta \mathbf{1}) = 0$, where π is the projection from $\mathcal{H}^d$ onto $\mathcal{H}^d/\mathcal{H}^{d-1}$. But $\Delta \mathbf{1} = \mathbf{1} \otimes \mathbf{1}$, hence $\pi \otimes \pi(\Delta \mathbf{1}) = \pi(d) \otimes \pi(d) \neq 0$, which yields a contradiction. Next, for any $n \geq 1$ we have:

$$\begin{aligned}
\Delta(\mathcal{H}^n) &\subset \mathcal{H}^0 \otimes \mathcal{H}^n + \mathcal{H}^n \otimes \mathcal{H}^0 + \sum_{p+q=n,\, p\neq 0,\, q\neq 0} \mathcal{H}^p \otimes \mathcal{H}^q \\
&\subset \mathcal{H}^0 \otimes \mathcal{H}^n + \mathcal{H}^n \otimes \mathcal{H}^0 + \sum_{p+q=n,\, p\neq 0,\, q\neq 0} \mathcal{H}^p_+ \otimes \mathcal{H}^q_+,
\end{aligned}$$

with $\mathcal{H}^p_+ := \mathcal{H}_p \cap \mathrm{Ker}\,\varepsilon$, and similarly for $\mathcal{H}^q_+$. This comes immediately from the fact that $\mathcal{H}^p = \mathcal{H}^0 + \mathcal{H}^p_+$ for any $p \geq 1$. Thus for any $n \geq 1$ and for any $x \in \mathcal{H}^n_+$ we have (here using connectedness for the first time):

$$\Delta x = u \otimes \mathbf{1} + \mathbf{1} \otimes v + \widetilde{\Delta} x \tag{3.2.7}$$

[3]The proof below has been suggested to me by Darij Grinberg.

with $u, v \in \mathcal{H}^n$ and $\widetilde{\Delta}x \in \mathcal{H}^1_+ \otimes \mathcal{H}^{n-1}_+ + \cdots + \mathcal{H}^{n-1}_+ \otimes \mathcal{H}^1_+$. Now using the co-unit property we get:

$$x = (\varepsilon \otimes \mathrm{Id})(\Delta x) = v + \varepsilon(u)\mathbf{1} = (\mathrm{Id} \otimes \varepsilon)(\Delta x) = u + \varepsilon(v)\mathbf{1}.$$

Hence we get:

$$\Delta x = u \otimes \mathbf{1} + \mathbf{1} \otimes v + \widetilde{\Delta}x = x \otimes \mathbf{1} + \mathbf{1} \otimes x + \big(\varepsilon(u) + \varepsilon(v)\big)\mathbf{1} + \widetilde{\Delta}x. \tag{3.2.8}$$

Applying $\varepsilon \otimes \mathrm{Id}$ to (3.2.8), and in view of $\Delta x = 0$, we get $\varepsilon(u) + \varepsilon(v) = 0$, hence:

$$\Delta x = x \otimes \mathbf{1} + \mathbf{1} \otimes x + \widetilde{\Delta}x. \tag{3.2.9}$$

The end of the proof is analogous to the graded case (Proposition 3.1.1). □

The main example of filtration is given by the *coradical filtration*, defined on any co-unital coalgebra $\mathcal{C}$ as follows:

- $\mathcal{C}^0 = 0$ and $\mathcal{C}^1$ is the *coradical*, i.e. the sum of its simple subcoalgebras, where a simple coalgebra stands for a coalgebra which does not contain any nontrivial subcoalgebra. This notion is dual to the notion of *Jacobson radical* of a unital algebra $\mathcal{A}$, i.e. the intersection of all nontrivial maximal left ideals $\mathcal{A}$.
- The $\mathcal{C}^n$'s are inductively defined by:

$$\mathcal{C}^n := \{x \in \mathcal{C},\, \Delta x \in \mathcal{C}^{n-1} \otimes \mathcal{C} + \mathcal{C} \otimes \mathcal{C}^0\}. \tag{3.2.10}$$

One can prove that the coradical filtration is compatible with the coalgebra structure, i.e. $\Delta\mathcal{C}^n \subset \sum_{p+q=n} \mathcal{C}_p \otimes \mathcal{C}_q$. The following theorem is due to S. Montgomery [25, Lemma 1.1].

Theorem 3.2.1. *Let $\mathcal{H}$ be any pointed Hopf algebra, i.e. a Hopf algebra such that any simple subcoalgebra of it is one-dimensional. Then the coradical filtration endows $\mathcal{H}$ with a structure of filtered Hopf algebra.*

Remark 3.2.1. The image of $\boldsymbol{k}$ under the unit map u is a one-dimensional simple subcoagebra of $\mathcal{H}$. If $\mathcal{H}$ is an irreducible coalgebra, by proposition 2.2.2 it is the unique one, and then the coradical is $\mathcal{H}^0 = k.\mathbf{1}$. Any irreducible Hopf algebra is then pointed, and connected with respect to the coradical filtration.

3.3. **Characters and infinitesimal characters.**

Definition 3.3.1. *Let $\mathcal{H}$ and $\mathcal{A}$ be two unital $\boldsymbol{k}$-algebras. A **character of $\mathcal{H}$ with values in $\mathcal{A}$** is a unital algebra morphism from $\mathcal{H}$ to $\mathcal{A}$.*

Definition 3.3.2. *Let $\mathcal{H}$ and $\mathcal{A}$ be two unital $\boldsymbol{k}$-algebras, and suppose that there is a unital algebra morphism $\varepsilon : \mathcal{H} \to \boldsymbol{k}$ (the **augmentation**). Let $e = u_{\mathcal{A}} \circ \varepsilon : \mathcal{H} \to \mathcal{A}$. An **infinitesimal character of $\mathcal{H}$ with values in $\mathcal{A}$** is a linear map $\alpha : \mathcal{H} \to \mathcal{A}$ such that:*

$$\alpha(xy) = \alpha(x)e(y) + e(x)\alpha(y) \text{ for any } x, y \in \mathcal{H}. \tag{3.3.1}$$

The target algebra $\mathcal{A}$ will be commutative in general, and $\mathcal{H}$ will be a co-unital bialgebra or a Hopf algebra, the augmentation being the co-unit.

Proposition 3.3.1. *Let $\mathcal{H}$ be a $\boldsymbol{k}$-bialgebra, and let $\mathcal{A}$ be a commutative unital $\boldsymbol{k}$-algebra. The set $G_{\mathcal{A}}$ of $\mathcal{A}$-valued characters of $\mathcal{H}$ is a group with unit e, and with inverse given by $\varphi \mapsto \varphi \circ S$, where S is the antipode. The set $\mathfrak{g}_{\mathcal{A}}$ of $\mathcal{A}$-valued infinitesimal characters is a Lie algebra.*

Proof. Let $f, g \in \mathcal{L}(\mathcal{H}, \mathcal{A})$. Using the fact that Δ is an algebra morphism we have for any $x, y \in \mathcal{H}$:

$$f * g(xy) = \sum_{(x)(y)} f(x_1 y_1) g(x_2 y_2).$$

If $\mathcal{A}$ is commutative and if f and g are $\mathcal{A}$-valued characters we get:

$$\begin{aligned} f * g(xy) &= \sum_{(x)(y)} f(x_1) f(y_1) g(x_2) g(y_2) \\ &= \sum_{(x)(y)} f(x_1) g(x_2) f(y_1) g(y_2) \\ &= (f * g)(x)(f * g)(y). \end{aligned}$$

The unit $e = u_{\mathcal{A}} \circ \varepsilon$ a unital algebra morphism. The formula for the inverse of a character is easily checked: for any $x \in \mathcal{H}$ we get

$$\begin{aligned} f * (f \circ S)(x) &= \sum_{(x)} f(x_1)(f \circ S)(x_2) \\ &= \sum_{(x)} f\big(x_1 S(x_2)\big) \\ &= f\left(\sum_{(x)} x_1 S(x_2)\right) \\ &= f\big(u \circ \varepsilon(x)\big) = e(x). \end{aligned}$$

□

Note that the computation above works only if f is a character. The relation between the Lie algebra $\mathfrak{g}_{\mathcal{A}}$ and the group $G_{\mathcal{A}}$ can be made more precise under supplementary hypotheses:

Proposition 3.3.2. *Suppose that the base field $\boldsymbol{k}$ is of characteristic zero, and that $\mathcal{H}$ is connected filtered. Then the exponential*

$$\begin{aligned} \exp^* : \mathfrak{g}_{\mathcal{A}} &\longrightarrow G_{\mathcal{A}} \\ \alpha &\longmapsto \exp^* \alpha = \sum_{k=0}^{\infty} \frac{\alpha^{*k}}{k!} \end{aligned}$$

is a bijection. Its inverse is given by:

$$\begin{array}{rcl}\log^* : G_{\mathcal{A}} & \longrightarrow & \mathfrak{g}_{\mathcal{A}} \\ \varphi & \longmapsto & \log^* \varphi = \displaystyle\sum_{k=0}^{\infty} \frac{(e-\varphi)^{*k}}{k}\end{array}$$

Proof. Let m be a positive integer. For any $x \in \mathcal{H}^m$ we have:

$$\Delta(x) = x \otimes \mathbf{1}_{\mathcal{H}} + \mathbf{1}_{\mathcal{H}} \otimes x + \sum_{(x)} x' \otimes x'',$$

where $x', x'' \in \mathcal{H}^{m-1}$. As a consequence, for any linear map $\alpha : \mathcal{H} \to \mathcal{A}$ such that $\alpha(\mathbf{1}_{\mathcal{H}}) = 0$, we have $\alpha^{*n}(x) = 0$ for any $n \geq m+1$. This applies in particular when α is an infinitesimal character. Hence the exponential of α is well-defined, as the sum which defines it is locally finite. The same argument applies to the logarithm. The fact that the exponential of an infinitesimal character is a character is checked by direct inspection. Finally let us consider any character $\varphi \in G_{\mathcal{H}}(\mathcal{A})$. The powers φ^{*m} are also characters for any positive integer m. Now let us define for any $\lambda \in \boldsymbol{k}$:

$$\varphi^{*\lambda} := \exp^* \big(\lambda \log^*(\varphi)\big).$$

For any $x, y \in \mathcal{H}$, the expression $\varphi^{*\lambda}(x)\varphi^{*\lambda}(y) - \varphi^{*\lambda}(xy)$ is polynomial in λ and vanishes at any positive integer, hence vanishes identically, namely

$$\varphi^{*\lambda}(x)\varphi^{*\lambda}(y) = \varphi^{*\lambda}(xy) \tag{3.3.2}$$

It follows that $\varphi^{*\lambda}$ is a character for any $\lambda \in \boldsymbol{k}$. Differentiating (3.3.2) with respect to λ at $\lambda = 0$ immediately gives the infinitesimal character equation for $\log^*(\varphi)$. A standard direct computation then shows that the logarithm and the exponential are mutually inverse. □

Exercises for Section 3.

Exercise 3.1. Show that there is a unique grouplike element in a connected filtered coalgebra.

Exercise 3.2. Let $\mathcal{H}$ be a connected filtered bialgebra and let $\mathcal{A}$ be a unital algebra (not necessarily commutative). Show that

$$\widetilde{G}_{\mathcal{A}} := \{\varphi \in \mathcal{L}(\mathcal{H}, \mathcal{A}),\, \varphi(\mathbf{1}_{\mathcal{H}}) = \mathbf{1}_{\mathcal{A}}\}$$

is a group, and that

$$\widetilde{\mathfrak{g}}_{\mathcal{A}} := \{\varphi \in \mathcal{L}(\mathcal{H}, \mathcal{A}),\, \varphi(\mathbf{1}_{\mathcal{H}}) = 0\}$$

is a Lie algebra. Prove that, if $\boldsymbol{k}$ is of characteristic zero, the exponential map $\exp^*$ is a bijection from $\widetilde{\mathfrak{g}}_{\mathcal{A}}$ onto $\widetilde{G}_{\mathcal{A}}$ with inverse given by $\log^*$.

Exercise 3.3. Deduce from Exercise 3.2 that any connected filtered bialgebra is a Hopf algebra. Give a recursive procedure to compute the antipode.

Exercise 3.4. Let $\mathcal{H}$ be a connected graded Hopf algebra, such that the homogeneous components $\mathcal{H}_n$ are finite-dimensional. Prove that the graded dual $\mathcal{H}^{\circ}$ is also a connected graded Hopf algebra.

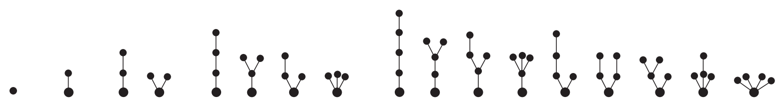
Exercise 3.5. Let $\mathcal{H}$ be a filtered bialgebra. Show that $\operatorname{Gr}\mathcal{H}$ is a graded bialgebra. *Hint:* apply the functor Gr to the product, coproduct, unit and counit. Same question with a filtered Hopf algebra.

Exercise 3.6. Let $\mathcal{H}$ be a Hopf algebra and let $\mathcal{A}$ be a unital algebra (not necessarily commutative). An element $\varphi \in \mathcal{L}(\mathcal{H}, \mathcal{A})$ is a *cocycle* if $\varphi(xy) = \varphi(yx)$ for any $x, y \in \mathcal{H}$. Show that the set of cocycles φ such that $\varphi(\mathbf{1}_{\mathcal{H}}) = \mathbf{1}_{\mathcal{A}}$ is a subgroup $\overline{G}_{\mathcal{A}}$ of the group $\widetilde{G}_{\mathcal{A}}$ defined in Exercise 3.2, and that the set of cocycles φ such that $\varphi(\mathbf{1}_{\mathcal{H}}) = 0$ is a Lie subalgebra $\overline{\mathfrak{g}}_{\mathcal{A}}$ of $\widetilde{\mathfrak{g}}_{\mathcal{A}}$. Prove that, if $\boldsymbol{k}$ is of characteristic zero and if $\mathcal{H}$ is connected filtered, the exponential map $\exp^*$ is a bijection from $\overline{\mathfrak{g}}_{\mathcal{A}}$ onto $\overline{G}_{\mathcal{A}}$ with inverse given by $\log^*$.

4. Examples of graded bialgebras

We review a few important examples of graded bialgebras, connected or not. Most of them have a strong combinatorial flavour: to be precise, they are defined by means of a basis given by combinatorial objects: words, rooted forests, graphs, partially ordered sets... Such bases are often multiplicative, which means that the product of two elements of the basis still belongs to it.

4.1. **The Hopf algebra of rooted forests.** A *rooted tree* is a class of oriented (non planar) graphs with a finite number of vertices, among which one is distinguished and called the *root*, such that any vertex admits exactly one outgoing edge, except the root which has no outgoing edges. Any tree yields a poset structure on the set of its vertices: two vertices x and y verify $x \leq y$ if and only if there is a path from a root to y passing through x. Two graphs are equivalent (hence define the same rooted tree) if and only if the two underlying posets are isomorphic. Here is the list of rooted trees up to five vertices, with edges oriented downwards:



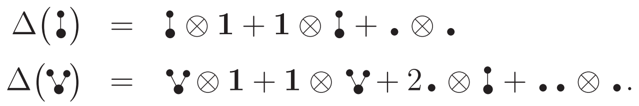
A *rooted forest* is a finite collection of rooted trees. The empty set is the forest with containing no trees, and is denoted by $\mathbf{1}$.

Definition 4.1.1. *The* ***grafting operator*** B_+ *takes any forest and returns the tree obtained by grafting all components onto a common root. In particular,* $B_+(\mathbf{1}) = \bullet$.

Let $\mathcal{T}$ denote the set nonempty rooted trees and let $\mathcal{H} = \boldsymbol{k}[\mathcal{T}]$ be the free commutative and unital algebra generated by the elements of $\mathcal{T}$. We identify a product of trees with the forest consisting of these trees. Therefore the vector space underlying $\mathcal{H}$ is the linear span of rooted forests. This algebra is a graded and connected Hopf algebra, called the *Hopf algebra of rooted forests*, with the following structure. The grading is given by the number of vertices of trees. The coproduct on a rooted forest u (i.e. a product of rooted trees) is described as follows: the set $\mathcal{V}(u)$ of vertices of a forest u is endowed with the partial order defined by $x \leq y$ if and only if there is a path from a root to y passing through x. Any subset W of the set of vertices $\mathcal{V}(u)$ of u defines a *subforest* $u_{|W}$ of u in an obvious manner, i.e. by keeping the edges of u which link two elements of W. The poset structure is given by restriction of the partial order to W, and the minimal elements are the roots of the subforest. The coproduct is then defined by:

$$\Delta(u) = \sum_{\substack{V \amalg W = \mathcal{V}(u) \\ W < V}} u_{|V} \otimes u_{|W}. \tag{4.1.1}$$

Here the notation $W < V$ means that $y \not\leq x$ for any vertex x in V and any vertex y in W. Note that both $\emptyset < V$ and $V < \emptyset$. Such a couple (V, W) is also called an *admissible cut*, with *crown* (or pruning) $u_{|V}$ and *trunk* $u_{|W}$. We have for example:



The counit is $\varepsilon(\mathbf{1}) = 1$ and $\varepsilon(u) = 0$ for any non-empty forest u. The coassociativity of the coproduct is easily checked using an iterated formula for the restricted coproduct

$$\widetilde{\Delta}(u) = \Delta(u) - u \otimes \mathbf{1} - \mathbf{1} \otimes u = \sum_{\substack{V \amalg W = \mathcal{V}(u) \\ W < V,\, V, W \neq \emptyset}} u_{|V} \otimes u_{|W},$$

where the restriction that V and W are nonempty means that V and W give rise to an ordered partition of $\mathcal{V}(u)$ into two blocks. In fact, the iterated restricted coproduct writes in terms of ordered partitions of $\mathcal{V}(u)$ into n blocks:

$$\widetilde{\Delta}^{n-1}(u) = \sum_{\substack{V_1 \amalg \cdots \amalg V_n = \mathcal{V}(u) \\ V_n < \cdots < V_1,\, V_j \neq \emptyset}} u_{|V_1} \otimes \cdots \otimes u_{|V_n},$$

and we get the full iterated coproduct $\Delta^{n-1}(u)$ by allowing empty blocks in the formula above. Note that the relation $<$ on subsets of vertices is not transitive. The notation $V_n < \cdots < V_1$ is to be understood as $V_i < V_j$ for any $i > j$, with $i, j \in \{1, \ldots, n\}$.

This Hopf algebra first appeared in the work of A. Dür in 1986 [12], as an incidence Hopf algebra. It has been rediscovered and intensively studied by D. Kreimer in 1998 [20], as the Hopf algebra describing the combinatorial part of the BPHZ renormalization procedure of Feynman graphs in a scalar φ^3 quantum field theory. D. Kreimer and A. Connes also proved in [9] that the operator B_+ satisfies the property

$$\Delta\big(B_+(t_1 \cdots t_n)\big) = B_+(t_1 \cdots t_n) \otimes \mathbf{1} + (\mathrm{Id} \otimes B_+) \circ \Delta(t_1 \cdots t_n), \tag{4.1.2}$$

for any $t_1, ..., t_n \in \mathcal{T}$. This means that B_+ is a 1-cocycle in the Hochschild cohomology of $\mathcal{H}$ with values in $\mathcal{H}$, and the couple $(\mathcal{H}, B_+)$ is then proved to be universal among commutative Hopf algebras endowed with a 1-cocycle.

For any commutative and unital algebra A, the group $G_{\mathcal{A}}$ of $\mathcal{H}$ can be identified with the set of *formal series expanded over rooted trees with coefficients in* A, i.e. the set of maps form the set of nonempty rooted trees to A. These *B-series* are by now widely used in the study of approximate solutions of nonlinear differential equations[4][16]. Such a map extends multiplicatively in a unique way to an A-valued character of $\mathcal{H}$.

4.2. **Shuffle and quasi-shuffle Hopf algebras.** Let $\boldsymbol{k}$ be a field, and let V be a commutative $\boldsymbol{k}$-algebra (not necessarily unital). Let $\mathcal{H}=T^c(V)=\bigoplus_{n\ge 0}V^{\otimes n}$ be the tensor coalgebra of V (see Example 2.2.2), where we denote by Δ the deconcatenation coproduct. The *quasi-shuffle product* $ш$ is recursively defined on $T^c(V)$ as follows:

- $\mathbf{1}\,ш\, v=v\,ш\,\mathbf{1}=v$ for any $v\in T^c(V)$,
- $av\,ш\,bw=a(v\,ш\,bw)+b(av\,ш\,w)+[ab](v\,ш\,w)$ for any $a,b\in V$ and $v,w\in T^c(V)$, where $[ab]$ is the product of a and b inside V, not to be confused with the word $ab\in V^{\otimes 2}$.

For example, for three letters $a,b,c\in V$, we have:

$$ab\,ш\,c=abc+acb+cab+[ac]b+a[bc].$$

Proposition 4.2.1. $\mathcal{H}=\big(T^c(V),ш\,,\Delta\big)$ *is a connected filtered commutative Hopf algebra.*

Proof. The filtration is given by:

$$\mathcal{H}^n:=\bigoplus_{j=0}^{n}V^{\otimes j}. \tag{4.2.1}$$

Connectedness as well as compatibility of Δ and $ш$ with respect to the filtration are obvious. It remains to show that $ш$ is commutative, compatible with Δ and associative. Commutativity is easily checked by induction. We check the commutativity relation $v'\,ш\,w'=w'\,ш\,v'$ and the compatibility condition $\Delta(v'\,ш\,w')=\Delta(v')\,ш\,\Delta(w')$ for any words v',w' by induction on the sum $|v'|+|w'|$ of the lengths. Indeed, writing $v'=av$ and $w'=bw$ with $a,b\in V$ we can compute:

$$\begin{aligned}
v'\,ш\,w' &= av\,ш\,bw\\
&= a(v\,ш\,bw)+b(av\,ш\,w)+[ab](v\,ш\,w)\\
&= a(bw\,ш\,v)+b(w\,ш\,av)+[ba](w\,ш\,v)\\
&= bw\,ш\,av\\
&= w'\,ш\,v'
\end{aligned}$$

as well as:

$$\begin{aligned}
\Delta(v'\,ш\,w') &= \Delta(av\,ш\,bw)=\Delta\big(a(v\,ш\,bw)+b(av\,ш\,w)+[ab](v\,ш\,w)\big)\\
&= \mathbf{1}\otimes a(v\,ш\,bw)+(a\otimes\mathbf{1})\Delta(v\,ш\,bw)+\mathbf{1}\otimes b(av\,ш\,w)+(b\otimes\mathbf{1})\Delta(av\,ш\,w)\\
&\qquad +\mathbf{1}\otimes[ab](v\,ш\,w)+([ab]\otimes\mathbf{1})\Delta(v\,ш\,w)\\
&= \mathbf{1}\otimes(av\,ш\,bw)+(a\otimes\mathbf{1})\big(\Delta v\,ш\,\Delta(bw)\big)+(b\otimes\mathbf{1})\big(\Delta(av)\,ш\,\Delta w\big)
\end{aligned}$$

[4] A B-series is map from the set of rooted trees, including the empty one, to the base field $\mathbb{R}$ or $\mathbb{C}$ of real or complex numbers. A B-series can be identified with an element of the Butcher group if and only if the coefficient of the empty tree is equal to one.

$$
\begin{aligned}
&\qquad\qquad +([ab]\otimes \mathbf{1})(\Delta v \mathbin{\overline{\text{ш}}} \Delta w)\\
=\;& \mathbf{1}\otimes (av \mathbin{\overline{\text{ш}}} bw) + (a\otimes\mathbf{1})\big(\Delta v \mathbin{\overline{\text{ш}}} (\mathbf{1}\otimes bw)\big) + (a\otimes \mathbf{1})\big(\Delta v \mathbin{\overline{\text{ш}}} (b\otimes \mathbf{1})\Delta w\big)\\
& +(b\otimes\mathbf{1})\big((\mathbf{1}\otimes av)\mathbin{\overline{\text{ш}}} \Delta w\big) + (b\otimes \mathbf{1})\big((a\otimes\mathbf{1})\Delta v \mathbin{\overline{\text{ш}}} \Delta w\big) + ([ab]\otimes\mathbf{1})(\Delta v \mathbin{\overline{\text{ш}}} \Delta w)\\
=\;& (\mathbf{1}\otimes av)\mathbin{\overline{\text{ш}}} (\mathbf{1}\otimes bw) + (\mathbf{1}\otimes av)\mathbin{\overline{\text{ш}}} \big((b\otimes\mathbf{1})\Delta w\big) + (\mathbf{1}\otimes bw)\mathbin{\overline{\text{ш}}} \big((a\otimes \mathbf{1})\Delta v\big)\\
&\qquad\qquad +\big((a\otimes\mathbf{1})\Delta v\big)\mathbin{\overline{\text{ш}}} \big((b\otimes\mathbf{1})\Delta w\big)\\
=\;& \big(\mathbf{1}\otimes av + (a\otimes\mathbf{1})\Delta v\big)\mathbin{\overline{\text{ш}}} \big(\mathbf{1}\otimes bw + (b\otimes \mathbf{1})\Delta w\big)\\
=\;& \Delta(av)\mathbin{\overline{\text{ш}}} \Delta(bw) = \Delta v' \mathbin{\overline{\text{ш}}} \Delta w'.
\end{aligned}
$$

Associativity condition $(v' \mathbin{\overline{\text{ш}}} w') \mathbin{\overline{\text{ш}}} x' = v' \mathbin{\overline{\text{ш}}} (w' \mathbin{\overline{\text{ш}}} x')$ is checked in a similar way, by induction on the sum $|v'|+|w'|+|x'|$. □

A particular case arises when the internal product of V vanishes identically, i.e. when $[ab]=0$ for any $a,b\in V$. The quasi-shuffle product in this case is the *shuffle product* denoted by ш. The recursive definition takes the simpler form:

$$av \mathbin{\text{ш}} bw = a(v \mathbin{\text{ш}} bw) + b(av \mathbin{\text{ш}} w). \tag{4.2.2}$$

4.3. **Incidence Hopf algebras.** Incidence Hopf algebras are Hopf algebras built from suitable families of partially ordered sets. They have been elaborated by W. R. Schmitt in 1994 [28], following the track opened by S. A. Joni and G.-C. Rota when they defined incidence algebras and coalgebras ([27, 18], see also [29]). They form a large family of Hopf algebras, which includes those on trees and the Faà di Bruno one. We quickly describe here the subfamily of "standard reduced" incidence Hopf algebras, which are always commutative.

Definition 4.3.1. *Let P be a partially ordered set (**poset** for short), with order relation denoted by $\le$. For any $x,y\in P$, the **interval** $[x,y]$ is the subset of P formed by the elements z such that $x\le z\le y$.*

Let $\mathcal{P}$ be a family of finite posets P such that there exists a unique minimal element 0_P and a unique maximal element 1_P in P (hence P coincides with the interval $P=[0_P,1_P]$). This family is called *interval closed* if for any poset $P\in\mathcal{P}$ and for any $x\le y\in P$, the interval $[x,y]$ is an element of $\mathcal{P}$. Let $\overline{\mathcal{P}}$ be the quotient $\mathcal{P}/\sim$, where $P\sim Q$ if and only if P and Q are isomorphic as posets[5]. The equivalence class of any poset $P\in\mathcal{P}$ is denoted by $\overline{P}$ (notation borrowed from [13]). The *standard reduced incidence coalgebra* of the family of posets $\mathcal{P}$ is the $\boldsymbol{k}$-vector space freely generated by $\overline{\mathcal{P}}$, with coproduct given by

$$\Delta\overline{P} = \sum_{x\in P} \overline{[0_P,x]}\otimes\overline{[x,1_P]},$$

and counit given by $\varepsilon(\overline{\{*\}})=1$ and $\varepsilon(\overline{P})=0$ if P contains two elements or more.

[5] W. Schmitt allows more general equivalence relations, called *order-compatible relations*.

Definition 4.3.2. *Given two posets P and Q, the* ***direct product*** *$P \times Q$ is the set-theoretic cartesian product of the two posets, with partial order given by $(p, q) \leq (p', q')$ if and only if $p \leq p'$ and $q \leq q'$. A family of finite posets $\mathcal{P}$ is called* ***hereditary*** *if the product $P \times Q$ belongs to $\mathcal{P}$ whenever $P, Q \in \mathcal{P}$.*

The quotient $\overline{\mathcal{P}}$ of a hereditary family is a commutative semigroup generated by the set $\overline{\mathcal{P}_0}$ of classes of *indecomposable posets*, i.e. posets $R \in \mathcal{P}$ such that for any $P, Q \in \mathcal{P}$ of cardinality ≥ 2, $P \times Q$ is not isomorphic to R. The commutativity comes from the obvious isomorphism $P \times Q \sim Q \times P$ for any $P, Q \in \mathcal{P}$. The unit element $\mathbf{1}$ is the class of any poset with only one element.

Proposition 4.3.1. [28, Theorem 4.1] *If $\mathcal{P}$ is a hereditary family of finite posets, the standard reduced coalgebra $\mathcal{H}(\mathcal{P})$ of $\mathcal{P}$ is a Hopf algebra.*

Proof. The semigroup product extends bilinearly to an associative product on $\mathcal{H}(\mathcal{P})$. The compatibility of the coproduct with this product is obvious. The unit is a coalgebra morphism and the counit is an algebra morphism. The existence of the antipode comes from the fact that for any poset $P \in \mathcal{P}$ of cardinal, say, n, we obviously have:

$$\Delta\overline{P} = \overline{P} \otimes \mathbf{1} + \mathbf{1} \otimes \overline{P} + \sum_{(\overline{P})} \overline{P'} \otimes \overline{P''},$$

where P' and P'' contain strictly less than n elements (note that the fact that P is the interval $[0_P, 1_P]$ is crucial here). Considering the reduced coproduct $\widetilde{\Delta}(\overline{P}) = \Delta(\overline{P}) - \overline{P} \otimes \mathbf{1} - \mathbf{1} \otimes \overline{P}$, the iterated reduced coproduct $\widetilde{\Delta}^m(\overline{P})$ therefore vanishes for $m > n$. We have seen (see Exercice 3.2) that this conilpotence property allows us to define the convolution inverse of the identity, and even of any linear map $\varphi : \mathcal{H}(\mathcal{P}) \to \mathcal{H}(\mathcal{P})$ with $\varphi(\mathbf{1}) = \mathbf{1}$. $\square$

Many of the Hopf algebras encountered so far are incidence Hopf algebras. We give three examples, all of them borrowed from [28].

Example 4.3.1 (The binomial and the divided power Hopf algebras)**.** Let $\mathcal{B}$ be the family of finite boolean algebras. An element of $\mathcal{B}$ is any poset isomorphic to the set $\mathcal{P}(A)$ of all subsets of a finite set A. The partial order on $\mathcal{P}(A)$ is given by the inclusion. If B and C are two subsets of A with $B \subset C$, the interval $[B, C]$ in $\mathcal{P}(A)$ is isomorphic to $\mathcal{P}(C \backslash B)$, hence $\mathcal{B}$ is interval-closed. Moreover the obvious property:

$$\mathcal{P}(A) \times \mathcal{P}(B) \sim \mathcal{P}(A \amalg B) \tag{4.3.1}$$

implies that $\mathcal{B}$ is hereditary. The incidence Hopf algebra $\mathcal{H}(\mathcal{B})$ is the so-called *binomial Hopf algebra*, because of the expression of the coproduct on generators. In fact, as a vector space $\mathcal{H}(\mathcal{B})$ is clearly spanned by $\{x_0, x_1, x_2, \ldots\}$, where x_n stands for the isomorphism class of $\mathcal{P}(\{1, \ldots, n\})$. The product is obviously given by $x_m x_n = x_{m+n}$, the unit is $\mathbf{1} = x_0$, the counit is given by $\varepsilon(x_n) = 0$ for $n \geq 1$, and the coproduct is entirely defined by $\Delta(x_1) = x_1 \otimes \mathbf{1} + \mathbf{1} \otimes x_1$. Explicitly we have:

$$\Delta(x_n) = \sum_{k=0}^{n} \binom{n}{k} x_k \otimes x_{n-k}.$$

The graded dual Hopf algebra $\mathcal{H}(\mathcal{B})^*$ is known as the *divided power algebra*: it can be represented as the vector space $\boldsymbol{k}\{y_0, y_1, y_2, \ldots\}$ with multiplication

$$y_m \star y_n = \binom{m+n}{m} y_{m+n}$$

and unit $\mathbf{1} = y_0$. The counit is $\varepsilon(y_n) = 0$ if $n > 0$, and the coproduct is given by

$$\Delta(y_n) = \sum_{p+q=n} y_p \otimes y_q.$$

Example 4.3.2 (The Faà di Bruno Hopf algebra)**.** Let $\mathcal{SP}$ be the family of posets isomorphic to the set $\mathcal{SP}(A)$ of all partitions of some nonempty finite set A. The partial order on set partitions is given by refinement. We denote by 0_A or 0 the partition by singletons, and by 1_A or 1 the partition with only one block. Let $\mathcal{Q}$ be the family of posets isomorphic to the cartesian product of a finite number of elements in $\mathcal{SP}$. If S and T are two partitions of a finite set A with $S \leq T$ (i.e. S is finer than T), the partition S restricts to a partition of any block of T. Denoting by W/S the set of those blocks of S which are included in some block W of T, any partition U such that $S \leq U \leq T$ yields a partition of the set W/S for any block W of T. This in turn yields the following obvious poset isomorphism:

$$[S,T] \sim \prod_{W \in A/T} \mathcal{SP}(W/S). \tag{4.3.2}$$

This shows that $\mathcal{Q}$ is interval closed (and hereditary by definition).

Definition 4.3.3. [28, Example 14.1] *The incidence Hopf algebra $\mathcal{H}(\mathcal{Q})$ is the **Faà di Bruno Hopf algebra**.*

Denote by X_n the isomorphism class of $\mathcal{SP}(\{1, \ldots, n+1\})$. Note that X_0 is the unit of the Hopf algebra. In view of (4.3.2), we have:

$$\begin{aligned}\Delta(X_n) &= \sum_{S \in \mathcal{SP}(\{1,\ldots,n+1\})} \overline{[0,S]} \otimes \overline{[S,1]} \\ &= \sum_{S \in \mathcal{SP}(\{1,\ldots,n+1\})} \left(\prod_{W \in \{1,\ldots,n+1\}/S} \overline{\mathcal{SP}(W)} \right) \otimes \overline{\mathcal{SP}(\{1,\ldots,n+1\}/S)}. \end{aligned} \tag{4.3.3}$$

The coefficient in front of $X_1^{k_1} \cdots X_n^{k_n} \otimes X_m$ in (4.3.3) is equal to the number of partitions of $\{1, \cdots, n+1\}$ with k_j blocks of size $j+1$ (for $j = 1$ to n), $m+1$ blocks altogether, and $k_0 = m + 1 - k_1 - \cdots - k_n$ blocks of size 1. We have then:

$$\Delta(X_n) = \sum_{m=0}^{n} B_{m+1,n+1}(X_0, X_1, X_2, \ldots) \otimes X_m, \tag{4.3.4}$$

where the $B_{m+1,n+1}$'s are the *partial Bell polynomials* [28, Example 14.1].

Example 4.3.3 (The Hopf algebra of rooted forests as an incidence Hopf algebra)**.** Let P be a finite poset not assumed to be isomorphic to an interval. As an example, we can take as poset P the vertex set $\mathcal{V}(F)$ of a rooted forest F, in which $v \le w$ if and only if there is a path from one root to w through v. An *order ideal* (or *initial segment*) in P is a subset I of P such that for any $w \in I$, if $v \le w$, then $v \in I$. For a rooted forest F, an initial segment in $\mathcal{V}(F)$ is a subforest such that any connected component of it contains a root of F. For any finite poset P, we denote by $J(P)$ the poset of all initial segments of P, ordered by inclusion [28, Paragraph 16]. The minimal element $0_{J(P)}$ is the empty set, and the maximal element $1_{J(P)}$ is P. For two finite posets P and Q one obviously has:

$$J(P \amalg Q) \sim J(P) \times J(Q). \tag{4.3.5}$$

The isomrphism class of a poset P is uniquely determined by the isomorphism class of the poset $J(P)$. To see this, consider two posets P and Q, and suppose there is an isomorphism $\Phi : J(P) \to J(Q)$. For any $x \in P$, consider the initial segment $P_{\le x} := \{y \in P,\ y \le x\}$. It has x as unique maximal element. Now, $\Phi(P_{\le x})$ has a unique maximal element which we denote by $\varphi(x)$, and it is not hard to see that the map $\varphi : P \to Q$ thus constructed is a poset isomorphism.

For a poset P and two initial segments $I_1 \subset I_2 \subset P$ with I_1 fixed, the correspondence $I_2 \mapsto I_2 \backslash I_1$ defines a poset isomorphism:

$$[I_1, I_2] \subset J(P) \longrightarrow J(I_2 \backslash I_1). \tag{4.3.6}$$

Differences $Q = I_2 \backslash I_1$ are *convex subsets* of P, i.e. such that for any $x, y \in Q$, we have $[x, y] \le Q$. Conversely, any convex subset $Q \subset P$ can be written as a difference $P_{\le Q} \backslash P_{<Q}$ of two unique initial segments:

$$\begin{aligned} P_{\le Q} &:= \{x \in P,\ \exists y \in Q,\ x \le y\}, \\ P_{<Q} &:= \{x \in P,\ \forall y \in Q,\ x < y\}. \end{aligned}$$

Now let $\mathcal{F}$ be a family of finite posets which is closed by disjoint unions and such that for any poset $P \in \mathcal{F}$, convex subsets of P also belong to $\mathcal{F}$. Then the corresponding family:

$$J(\mathcal{F}) := \{J(P),\ P \in \mathcal{F}\}$$

is hereditary by virtue of isomorphisms (4.3.5) and (4.3.6).

Proposition 4.3.2. *The family $\mathcal{F}$ of rooted forests is stable by taking disjoint unions and convex subposets, and the associated incidence Hopf algebra $\mathcal{H}\big(J(\mathcal{F})\big)$ is isomorphic to the Hopf algebra of rooted forests defined in Paragraph 4.1.*

Proof. Via the isomorphism Φ defined above, the vector space $\mathcal{H}\big(J(\mathcal{F})\big)$ can be identified with the vector space freely generated be the rooted forests. By (4.3.5), the product is then given by disjoint union, and the coproduct writes:

$$\Delta(P) = \sum_{I \in J(P)} (P \backslash I) \otimes I,$$

which is just the coproduct (4.1.1) modulo flipping the terms (we have denoted by the same letter a forest and its underlying poset). The counit is given by $\varepsilon(\mathbf{1}) = 1$ and $\varepsilon(P) = 0$ for any nontrivial forest P. □

This example can be extended, still in the context of incidence Hopf algebras, to oriented cycle-free graphs [22].

4.4. The extraction-contraction forest bialgebra.

Definition 4.4.1. *A **covering subforest** of a rooted forest u is a collection of disjoint subtrees of u such that any vertex of u belongs to one (unique) tree of the collection. For any covering subforest s of u, the **contracted forest** u/s is obtained from u by shrinking each tree to a single vertex.*

We adopt the notation $s \subseteq u$ for "s is a covering subforest of u".

Lemma 4.4.1. *Let u be a rooted forest, let t be a covering subforest of u. Then*

1. *The correspondence $s \mapsto s/t$ is a bijection from the set of covering subforests s such that $t \subseteq s$, onto covering subforests of u/t.*
2. *$(u/t)/(s/t) = u/s$.*

Let $\mathcal{H}$ be the Hopf algebra of rooted forests introduced in Paragraph 4.1. We introduce the following coproduct on $\mathcal{H}$:

$$\Gamma(u) := \sum_{s \text{ covering subforest of } u} s \otimes u/s. \tag{4.4.1}$$

Theorem 4.4.1. *$(\mathcal{H}, \cdot, \Gamma)$ is a graded bialgebra, the grading being given by the number of edges.*

Proof. The multiplicativity of Γ is obvious, as well as compatibility with respect to the edge-grading. Coassociativity comes from the formula:

$$(\Gamma \otimes \mathrm{Id})\Gamma(u) = (\mathrm{Id} \otimes \Gamma)\Gamma(u) = \sum_{t \subseteq s \subseteq u} t \otimes s/t \otimes u/s,$$

which is a direct consequence of Lemma 4.4.1. □

One can remark that $(\mathcal{H}, \cdot, \Gamma)$ is not a Hopf algebra: indeed, the one-vertex tree $\bullet$ is of degree zero, and is grouplike with no inverse.

Exercises for Section 4.

Exercise 4.1. Prove that (4.1.2) together with $\Delta\mathbf{1} = \mathbf{1} \otimes \mathbf{1}$ entirely determines the coproduct Δ (*Hint*: proceed by induction on the degree).

Exercise 4.2. Let D be a set. A *D-decorated rooted forest* is a pair (u, φ) where u is a rooted forest and φ is a map from $\mathcal{V}(u)$ to D. Build up a connected graded Hopf algebra structure on the linear span $\mathcal{H}^D$ of D-decorated rooted forests. Prove that B_+^d verifies (4.1.2) for any $d \in D$, where B_+^d is the grafting operator on a common root decorated by d.

Exercise 4.3. Let p, q, r be three non-negative integers. A *(p,q)-quasi-shuffle of type r* is a surjective map

$$\sigma : \{1, \ldots, p+q\} \twoheadrightarrow \{1, \ldots, p+q-r\}$$

such that $\sigma_1 < \cdots < \sigma_p$ and $\sigma_{p+1} < \cdots < \sigma_{p+q}$. Prove the following explicit formula for the quasi-shuffle product:

$$v_1 \cdots v_p \,⧢\, v_{p+1} \cdots v_{p+q} = \sum_{r \geq 0} \sum_{\sigma \in \mathrm{qsh}(p,q;r)} v_1^\sigma \cdots v_{p+q-r}^\sigma, \tag{4.4.2}$$

where $\mathrm{qsh}(p,q;r)$ stands for the set of (p,q)-quasi-shuffles of type r, and where $v_j^\sigma = \prod_{k,\,\sigma_k = j} v_k$. The product is understood with respect to the internal multiplication of V, and contains one or two terms. Give the explicit formula for the shuffle product ш.

Exercise 4.4. Let J be the ideal of the extraction-contraction bialgebra $(\mathcal{H}, \cdot, \Gamma)$ generated by $\bullet - \mathbf{1}$. Prove that J is a bi-ideal, and that $\mathcal{H}/J$ is a connected graded Hopf algebra.

5. Birkhoff decomposition and renormalisation

5.1. **Birkhoff decomposition.** We consider here the situation where the algebra $\mathcal{A}$ admits a *renormalisation scheme*, i.e. a splitting into two subalgebras:

$$\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$$

with $\mathbf{1}_\mathcal{A} \in \mathcal{A}_+$. Let us denote by $\pi : \mathcal{A} \to \mathcal{A}$ the projection on $\mathcal{A}_-$ parallel to $\mathcal{A}_+$.

Theorem 5.1.1.

(1) *Let $\mathcal{H}$ be a connected filtered Hopf algebra and let $\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$ be a commutative unital algebra endowed with a renormalisation scheme. Any $\varphi \in G_\mathcal{A}$ admits a unique* ***Birkhoff decomposition****:*

$$\varphi = \varphi_-^{*-1} * \varphi_+, \tag{5.1.1}$$

where φ_- sends $\mathbf{1}$ to $\mathbf{1}_\mathcal{A}$ and $\operatorname{Ker}\varepsilon$ *into $\mathcal{A}_-$, and where φ_+ sends $\mathcal{H}$ into $\mathcal{A}_+$. The maps φ_- and φ_+ are given on* $\operatorname{Ker}\varepsilon$ *by the following recursive formulas:*

$$\varphi_-(x) = -\pi\Big(\varphi(x) + \sum_{(x)} \varphi_-(x')\varphi(x'')\Big) \tag{5.1.2}$$

$$\varphi_+(x) = (I - \pi)\Big(\varphi(x) + \sum_{(x)} \varphi_-(x')\varphi(x'')\Big). \tag{5.1.3}$$

(2) *The components* φ_- *and* φ_+ *occurring in the Birkhoff decomposition of* φ *are* $\mathcal{A}$*-valued characters as well.*

Proof. The proof goes along the same lines as the proof of Theorem 4 of [10]: for the first assertion it is immediate from the definition of π that φ_- sends $\operatorname{Ker}\varepsilon$ into $\mathcal{A}_-$, and that φ_+ sends $\operatorname{Ker}\varepsilon$ into $\mathcal{A}_+$. It only remains to check equality $\varphi_+ = \varphi_- * \varphi$, which is an easy computation:

$$\begin{aligned}\varphi_+(x) &= (I-\pi)\Big(\varphi(x)+\sum_{(x)}\varphi_-(x')\varphi(x'')\Big).\\ &= \varphi(x)+\varphi_-(x)+\sum_{(x)}\varphi_-(x')\varphi(x'')\\ &= (\varphi_- * \varphi)(x).\end{aligned}$$

The proof of the second assertion goes exactly as in [10] and relies on the following *Rota-Baxter relation* in $\mathcal{A}$:

$$\pi(a)\pi(b)=\pi\big(\pi(a)b+a\pi(b)\big)-\pi(ab), \tag{5.1.4}$$

which is easily verified by decomposing a and b into their $\mathcal{A}_\pm$-parts. Let φ be a character of $\mathcal{H}$ with values in $\mathcal{A}$. Suppose that we have $\varphi_-(xy)=\varphi_-(x)\varphi_-(y)$ for any $x,y\in\mathcal{H}$ such that $|x|+|y|\le d-1$, and compute for x,y such that $|x|+|y|=d$:

$$\varphi_-(x)\varphi_-(y)=\pi(X)\pi(Y),$$

with $X=\varphi(x)-\sum_{(x)}\varphi_-(x')\varphi(x'')$ and $Y=\varphi(y)-\sum_{(y)}\varphi_-(y')\varphi(y'')$. Using the formula:

$$\pi(X)=-\varphi_-(x),$$

we get:

$$\varphi_-(x)\varphi_-(y)=-\pi\big(XY+\varphi_-(x)Y+X\varphi_-(y)\big),$$

hence:

$$\begin{aligned}\varphi_-(x)\varphi_-(y) &= -\pi\Big(\varphi(x)\varphi(y)+\varphi_-(x)\varphi(y)+\varphi(x)\varphi_-(y)\\ &\quad+\sum_{(x)}\varphi_-(x')\varphi(x'')\big(\varphi(y)+\varphi_-(y)\big)+\sum_{(y)}\big(\varphi(x)+\varphi_-(x)\big)\varphi_-(y')\varphi(y'')\\ &\quad+\sum_{(x)(y)}\varphi_-(x')\varphi(x'')\varphi_-(y')\varphi(y'')\Big).\end{aligned}$$

We have to compare this expression with:

$$\begin{aligned}\varphi_-(xy) &= -\pi\Big(\varphi(xy)+\varphi_-(x)\varphi(y)+\varphi_-(y)\varphi(x)\\ &\quad+\sum_{(x)}\big(\varphi_-(x'y)\varphi(x'')+\varphi_-(x')\varphi(x''y)\big)+\sum_{(y)}\big(\varphi_-(xy')\varphi(y'')+\varphi_-(y')\varphi(xy'')\big)\\ &\quad+\sum_{(x)(y)}\varphi_-(x'y')\varphi(x''y'')\Big).\end{aligned}$$

These two expressions are easily seen to be equal using the commutativity of the algebra $\mathcal{A}$, the character property for φ and the induction hypothesis. □

Remark 5.1.1. Define the *Bogoliubov preparation map* as the map $b : G_{\mathcal{A}} \to \mathcal{L}(\mathcal{H}, \mathcal{A})$ recursively given by:

$$b(\varphi)(x) = \varphi(x) + \sum_{(x)} \varphi_-(x')\varphi(x''). \tag{5.1.5}$$

Then the components of φ in the Birkhoff decomposition read:

$$\varphi_- = -\pi \circ b(\varphi), \qquad \varphi_+ = (I - \pi) \circ b(\varphi). \tag{5.1.6}$$

Bogoliubov preparation map can also be written in a more concise form:

$$b(\varphi) = \varphi_- * (\varphi - e). \tag{5.1.7}$$

Plugging equation (5.1.7) inside (5.1.6) and setting $\alpha := \varphi - e$ we get the following expression for φ_-:

$$\varphi_- = e - P(\varphi_- * \alpha) \tag{5.1.8}$$

$$= e - P(\alpha) + P\big(P(\alpha) * \alpha\big) + \cdots + (-1)^n \underbrace{P\Big(P\big(...P(}_{n \text{ times}}\alpha) * \alpha\big) \cdots * \alpha\Big) + \cdots \tag{5.1.9}$$

where $P : \mathcal{L}(\mathcal{H}, \mathcal{A}) \to \mathcal{L}(\mathcal{H}, \mathcal{A})$ is the projection defined by $P(\alpha) = \pi \circ \alpha$.

5.2. **A short account of renormalisation in physics.** Systems in interaction are most common in physics. When parameters (such as mass, electric charge, acceleration, etc.) characterising the system are considered, it is crucial to distinguish between *bare parameters*, which are the values they would take if the interaction were switched off, and the actually observed parameters. Renormalisation can be defined as any procedure able to transform the bare parameters into the actually observed ones (i.e. with interaction taken into account), which will therefore be called *renormalised*. Consider (from [10]) the following example: the initial acceleration of a spherical balloon is given by:

$$g = \frac{m_0 - M}{m_0 + \frac{M}{2}} g_0 \tag{5.2.1}$$

where $g_0 \simeq 9,81\, m.s^{-2}$ is the gravity acceleration at the surface of the Earth, m_0 is the mass of the balloon, and M is the mass of the volume of the air occupied by it. Note that this acceleration decreases from g_0 to $-2g_0$ when the interaction (represented here by the air mass M) increases from 0 to $+\infty$. The total force $F = mg$ acting on the balloon is the sum of the gravity force $F_0 = m_0 g_0$ and Archimedes' force $-Mg_0$. The bare parameters (i.e. in the absence of air) are thus m_0, F_0, g_0 (mass, force and acceleration respectively), whereas the renormalised parameters are:

$$m = m_0 + \frac{M}{2}, \qquad F = (1 - \frac{M}{m_0})F_0, \qquad g = \frac{m_0 - M}{m_0 + \frac{M}{2}} g_0. \tag{5.2.2}$$

In perturbative quantum field theory an extra difficulty arises: the bare parameters are usually infinite, reflecting the fact that the idealised "isolated system" definitely cannot exist, and in particular cannot be observed. Bare parameters are typically given by divergent integrals[6] such as:

$$\int_{\mathbb{R}^4} \frac{1}{1+\|p\|^2}\,dp. \tag{5.2.3}$$

One must then subtract another infinite quantity to the bare parameter to recover the renormalised parameter, which is finite, as this one can be actually measured! Such a process takes place in two steps:

- a *regularisation procedure*, which replaces the bare infinite parameter by a function of one variable z which tends to infinity when z tends to some z_0.

- the *renormalisation procedure* itself, of combinatorial nature, which extracts an appropriate finite part from the function above when z tends to z_0. When this procedure can be carried out, the theory is called *renormalisable*.

There is usually considerable freedom in the choice of a regularisation procedure. Let us mention, among many others, the *cut-off regularisation*, which amounts to consider integrals like (5.2.3) over a ball of radius z (with $z_0 = +\infty$), and *dimensional regularisation* which consists, roughly speaking, in "integrating over a space of complex dimension z", with $z_0 = d$, the actual space dimension of the physical situation (for example $d = 4$ for the Minkowski space-time). In this case the function which appears is meromorphic in z with a pole at z_0 ([8],[17]).

5.3. **Renormalisation from Birkhoff decomposition.** We focus on a particular example: let $\mathcal{H}$ be a connected graded Hopf algebra over the complex numbers. Let $\mathcal{A}$ be the algebra of germs of meromorphic functions at some $z_0 \in \mathbb{C}$. The algebra $\mathcal{A}$ admits a splitting into two subalgebras:

$$\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-,$$

where $\mathcal{A}_+$ is the algebra of germs of holomorphic functions at $z = 0$, and where $\mathcal{A}_- = z^{-1}\mathbb{C}[z^{-1}]$ is the algabra of polynomials in the variable z^{-1} without constant terms. This splitting is known as the *minimal subtraction scheme*.

Definition 5.3.1. *Let $\varphi : \mathcal{H} \to \mathcal{A}$ be an $\mathcal{A}$-valued character. In Physics, such a character is typically obtained from divergent integrals by a regularization procedure, for example dimensional regularisation. The* ***renormalised character*** *of $\mathcal{A}$ is the complex-valued character*

[6] To be precise, the physical parameters of interest are given by a *series* each term of which is a divergent integral. We do not approach here the question of convergence of this series once each term has been renormalised.

defined by:

$$\varphi^R(x) := \varphi_+(x)(z)_{|z=0}. \tag{5.3.1}$$

*Note that φ takes values in meromorphic functions which have no pole at $z = 0$, hence the definition makes sense. The $\mathcal{A}$-valued character φ_- is called the **counterterm character**.*

Exercises for Section 5.

6. Comodule-bialgebras

The notion of comodule-bialgebra has been studied by R. K. Molnar [24, Definition 2.1.(e)].

6.1. **Definition.**

Definition 6.1.1. *Let $\mathcal{B}$ be a unital c-unital bialgebra over a field $\boldsymbol{k}$. A **comodule-bialgebra on** $\mathcal{B}$ is a unital counital bialgebra in the category of $\mathcal{B}$-comodules.*

To be precise, a comodule-bialgebra on $\mathcal{B}$ is a unital counital bialgebra $\mathcal{H}$ endowed with a linear map

$$\Phi : \mathcal{H} \to \mathcal{B} \otimes \mathcal{H}$$

such that:

- Φ is a left coaction, i.e. the following diagrams commute:

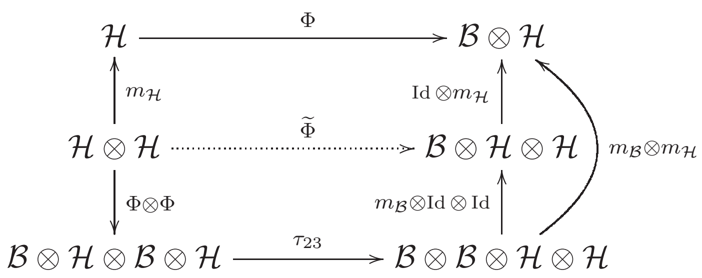
$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\ \Phi\ } & \mathcal{B}\otimes\mathcal{H} \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Delta_{\mathcal{B}}\otimes \mathrm{Id}} \\ \mathcal{B}\otimes\mathcal{H} & \xrightarrow{\ \mathrm{Id}\otimes\Phi\ } & \mathcal{B}\otimes\mathcal{B}\otimes\mathcal{H} \end{array} \qquad \begin{array}{ccc} \mathcal{H} & \xrightarrow{\ \Phi\ } & \mathcal{B}\otimes\mathcal{H} \\ & \searrow^{\sim} & \downarrow{\scriptstyle\varepsilon_{\mathcal{B}}\otimes \mathrm{Id}} \\ & & k\otimes\mathcal{H} \end{array}$$

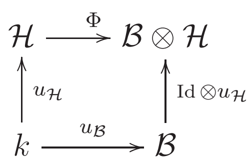
- The coproduct $\Delta_{\mathcal{H}}$ and the counit $\varepsilon_{\mathcal{H}}$ are morphisms of left $\mathcal{B}$-comodules, where the comodule structure on k is given by the unit map $u_{\mathcal{B}}$, and the comodule structure on $\mathcal{H}\otimes\mathcal{H}$ is given by $\widetilde{\Phi} = (m_{\mathcal{B}}\otimes \mathrm{Id}\otimes \mathrm{Id})\circ\tau_{23}\circ(\Phi\otimes\Phi)$. This amounts to the commutativity of the two following diagrams:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\ \Phi\ } & \mathcal{B}\otimes\mathcal{H} \\ {\scriptstyle\Delta_{\mathcal{H}}}\downarrow & & \downarrow{\scriptstyle\mathrm{Id}\otimes\Delta_{\mathcal{H}}} \\ \mathcal{H}\otimes\mathcal{H} & \xrightarrow{\ \widetilde{\Phi}\ } & \mathcal{B}\otimes\mathcal{H}\otimes\mathcal{H} \\ {\scriptstyle\Phi\otimes\Phi}\downarrow & & \uparrow{\scriptstyle m_{\mathcal{B}}\otimes\mathrm{Id}\otimes\mathrm{Id}} \\ \mathcal{B}\otimes\mathcal{H}\otimes\mathcal{B}\otimes\mathcal{H} & \xrightarrow{\ \tau_{23}\ } & \mathcal{B}\otimes\mathcal{B}\otimes\mathcal{H}\otimes\mathcal{H} \end{array} \qquad \begin{array}{ccc} \mathcal{H} & \xrightarrow{\ \Phi\ } & \mathcal{B}\otimes\mathcal{H} \\ {\scriptstyle\varepsilon_{\mathcal{H}}}\downarrow & & \downarrow{\scriptstyle\mathrm{Id}\otimes\varepsilon_{\mathcal{H}}} \\ k & \xrightarrow{\ u_{\mathcal{B}}\ } & \mathcal{B} \end{array}$$

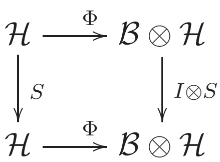
  where τ_{23} stands for the flip of the two middle factors.

- $m_{\mathcal{H}}$ and $u_{\mathcal{H}}$ are morphisms of left $\mathcal{B}$-comodules. This amounts to say that Φ is a unital algebra morphism. In other words, the two following diagrams commute:





The comodule-bialgebra $\mathcal{H}$ is a *comodule-Hopf algebra* if moreover $\mathcal{H}$ is a Hopf algebra with antipode S such that the following diagram commutes:



6.2. **The comodule-Hopf algebra structure on the rooted forests.**

Theorem 6.2.1. *The Hopf algebra $(\mathcal{H},\cdot,\Delta)$ of rooted forests defined in Paragraph 4.1 is a comodule-Hopf algebra over the extraction-contraction bialgebra $(\mathcal{H},\cdot,\Gamma)$ defined in Paragraph 4.4. The coaction map Φ is simply given by the coproduct Γ.*

Proof. One has to check that the following identity of linear maps from $\mathcal{H}$ into $\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}$ holds:

$$(\mathrm{Id}_{\mathcal{H}}\otimes\Delta)\circ\Phi = m^{1,3}\circ(\Gamma\otimes\Gamma)\circ\Delta, \tag{6.2.1}$$

where $m^{1,3}:\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}\longrightarrow\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}$ is defined by:

$$m^{1,3}(a\otimes b\otimes c\otimes d) = ac\otimes b\otimes d. \tag{6.2.2}$$

The verification is immediate for the empty forest. Recall that we denote by $\mathrm{Adm}(t)$ the set of admissible cuts of a forest. We have then for any nonempty forest:

$$\begin{aligned}(\mathrm{Id}_{\mathcal{H}}\otimes\Delta)\circ\Gamma\big(t\big) &= (\mathrm{Id}_{\mathcal{H}}\otimes\Delta)\sum_{\substack{s \text{ subforest}\\ \text{of } t}} s\otimes t/s\\ &= \sum_{\substack{s\text{ subforest}\\ \text{of } t}}\ \sum_{A\sqcup B=\mathcal{V}(t/s),\, A\cap B=\emptyset,\, B<A} s\otimes (t/s)_{|A}\otimes (t/s)_{|B}.\end{aligned}$$

On the other hand we compute:

$$\begin{aligned}m^{1,3}\circ(\Gamma\otimes\Gamma)\circ\Delta(t) &= m^{1,3}\circ(\Gamma\otimes\Gamma)\left(\sum_{A'\sqcup B'=\mathcal{V}(t),\, A'\cap B'=\emptyset,\, B'<A'} t_{|A'}\otimes t_{|B'}\right)\\ &= m^{1,3}\left(\sum_{A'\sqcup B'=\mathcal{V}(t),\, A'\cap B'=\emptyset,\, B'<A'}\ \sum_{\substack{s' \text{ subforest}\\ \text{of } A'}}\ \sum_{\substack{s'' \text{ subforest}\\ \text{of } B'}} s'\otimes A'/s'\otimes s''\otimes B'/s''\right)\end{aligned}$$

$$= \sum_{A'\sqcup B'=\mathcal{V}(t),\, A'\cap B'=\emptyset,\, B'<A'} \ \sum_{\substack{s' \text{ subforest} \\ \text{of } A'}} \ \sum_{\substack{s'' \text{ subforest} \\ \text{of } B'}} s's'' \otimes A'/s' \otimes B'/s''$$

$$= \sum_{A'\sqcup B'=\mathcal{V}(t),\, A'\cap B'=\emptyset,\, B'<A'} \ \sum_{\substack{s \text{ subforest of } t \text{ with} \\ \text{connected components contained in } A' \text{ or } B'}} s \otimes A'\Big/ s\cap A' \otimes B'\Big/ s\cap B'$$

$$= \sum_{\substack{s \text{ subforest} \\ \text{of } t}} \ \sum_{A\sqcup B=\mathcal{V}(t/s),\, A\cap B=\emptyset,\, B<A} s \otimes (t/s)_{|A} \otimes (t/s)_{|B},$$

which proves the theorem. □

Exercises for Section 6.

7. Regularity structures

7.1. **Coloured forests.** Let $(F, \widehat{F})$ be a coloured forest. To be precise,

- F is a rooted forest with set of vertices $N(F)$ and set of edges $E(F)$,
- $\widehat{F} : N(F) \sqcup E(F) \to \mathbb{N} = \{0, 1, 2, \ldots\}$ such that $\widehat{F}_i := \widehat{F}^{-1}(\{i\})$ is a subforest of F for any $i \geq 1$.

The index i can be thought of as a colour, $i = 0$ corresponding to black. The second condition amounts to the following: for any $x, y \in N(F)$ such that $\widehat{F}(x) = \widehat{F}(y) \geq 1$, then $\widehat{F}([x, y]) = \widehat{F}(x) = \widehat{F}(y)$, where $[x, y]$ stands for the edge joining x and y in F. Now for any $i \geq 1$, let $\mathcal{U}_i : (F, \widehat{F}) \mapsto \mathcal{U}_i(F, \widehat{F})$ be a collection of subforests of the argument $(F, \widehat{F})$ such that:

Assumption 7.1. *For any $i \geq 1$, for any coloured forest $(F, \widehat{F})$ and for any subforest $A \in \mathcal{U}_i(F, \widehat{F})$,*

1. *$\widehat{F}_j \cap A = \emptyset$ for any $j > i$,*
2. *$\widehat{F}_i \subset A$,*
3. *either $T \subset A$ or $T \cap A = \emptyset$ for any connected component T of $\widehat{F}_j$ whenever $0 < j < i$.*

For any coloured forest $(F, \widehat{F})$ and for any subforest A of F, we denote by $\widehat{F}_{|A}$ the restriction of the colour map $\widehat{F}$ to A, and we denote by $\widehat{F} \cup_i A : N(F) \sqcup E(F) \to \mathbb{N}$ the colour map defined by $\widehat{F} \cup_i A(x) := i$ if $x \in N(A) \sqcup E(A)$ and $\widehat{F} \cup_i A(x) = \widehat{F}(x)$ otherwise.

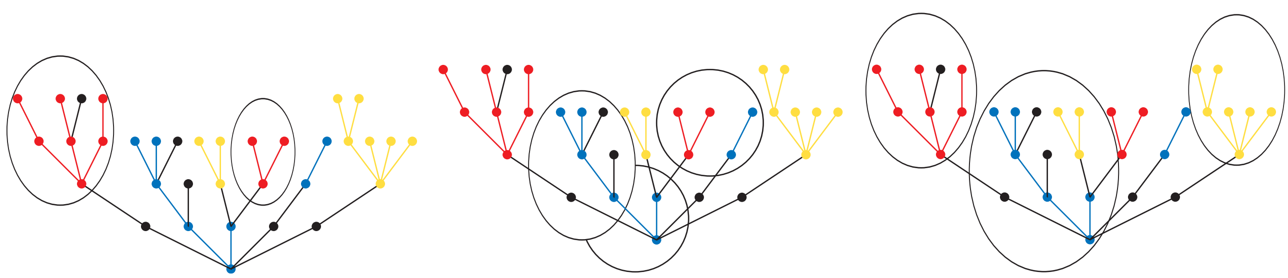


Figure 1: The picture above shows three times the same forest F (which is a tree here) with three colours: red stands for 1, blue for 2 and yellow for 3. Black, which is not considered as a genuine colour, stands for 0. The blobs encompass a subforest A_i subject to Assumption 7.1 for $i = 1, 2, 3$ from left to right. Hence A_i could be an example of element of $\mathcal{U}_i(F, \widehat{F})$ for any $i = 1$, 2 or 3.

Lemma 7.1.1. *Let the map $\mathcal{U}_i$ be subject to Assumption 7.1. Then for any coloured forest $(F,\widehat{F})$ and for any $A\in\mathcal{U}_i(F,\widehat{F})$, both ordered pairs $(A,\widehat{F}_{|A})$ and $(F,\widehat{F}\cup_i A)$ are coloured forests.*

Proof. This is obvious for $(A,\widehat{F}_{|A})$. Setting $\widehat{G}:=\widehat{F}\cup_i A$, one has to check that $\widehat{G}_i=\widehat{G}^{-1}(\{i\})$ is a subforest of F for any $i\geq 1$. This is a simple consequence of Assumption 7.1. □

To sum up, the coloured forest $(F,\widehat{F}\cup_i A)$ is obtained from $(F,\widehat{F})$ by repainting all the edges and vertices of $A\setminus\widehat{F}_i$ with colour number i, thus extending this colour to the whole A. It is now natural to look at the comultiplication:

$$\Delta_i(F,\widehat{F}):=\sum_{A\in\mathcal{U}_i(F,\widehat{F})}(A,\widehat{F}_{|A})\otimes(F,\widehat{F}\cup_i A), \tag{7.1.1}$$

and to look for properties of the collection $\mathcal{U}_i$ such that the corresponding Δ_i is coassociative. It turns out that this procedure will work in a more complicated situation, namely for *typed and decorated* coloured forests.

7.2. **Contractions.** Let $(F,\widehat{F})$ be a coloured forest. The contraction $\mathcal{K}(F,\widehat{F})$ is defined by shrinking each connected component of $\widehat{F}_i$, for $i\geq 1$, on a single node which is also given the colour i. The map $\mathcal{K}$ thus defined is idempotent (i.e. $\mathcal{K}\circ\mathcal{K}=\mathcal{K}$), and $(F,\widehat{F})=\mathcal{K}(F,\widehat{F})$ if and only if the subforests $\widehat{F}_i$ are made of single isolated nodes for any $i\geq 1$.

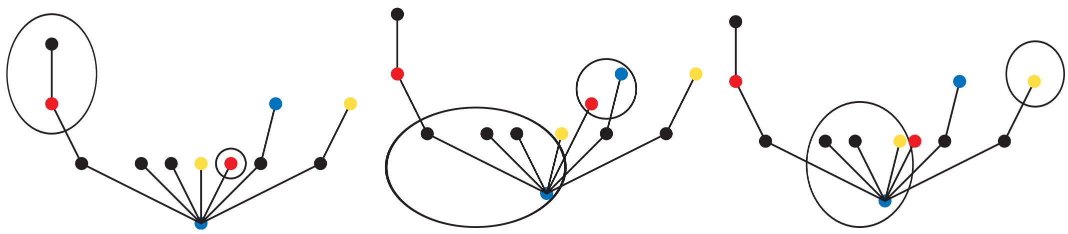


FIGURE 2: The contraction of Figure 1. The blobs encompass subforests $\mathcal{K}(A_i)$ for $i=1,2,3$ from left to right. The subforests $\mathcal{K}(A_i)$ also verify Assumption 7.1.

Let $(F,\widehat{F})$ be a coloured forest. A subforest $A\subset F$ is *admissible* if for any $i\geq 1$ and for any connected component C of $\widehat{F}_i$, either $C\subset A$ of $C\cap A=\emptyset$. The contraction map $\mathcal{K}$ induces a bijection between admissible subforests of $(F,\widehat{F})$ and subforests of $\mathcal{K}(F,\widehat{F})$, the latter being necessarily admissible. In that respect, it is natural to make the following supplementary assumption on the $\mathcal{U}_i$'s:

Assumption 7.2. *For any $i\geq 1$ and for any coloured forest $(F,\widehat{F})$, the contraction map $\mathcal{K}$ induces a bijection from $\mathcal{U}_i(F,\widehat{F})$ onto $\mathcal{U}_i\big(\mathcal{K}(F,\widehat{F})\big)$.*

Note that the elements of $\mathcal{U}_i(F,\widehat{F})$ are admissible by Assumption 7.1.

7.3. **The Chu-Vandermonde identity.** Let S be any finite set, and let $k, \ell : S \to \mathbb{N}$ with $\mathbb{N} := \{0, 1, 2, \ldots\}$. Factorials and binomial coefficients are defined by:

$$\ell! := \prod_{x\in S} \ell(x)!, \qquad \binom{k}{\ell} := \prod_{x\in S} \binom{k(x)}{\ell(x)}, \tag{7.3.1}$$

where $\binom{a}{b} := \dfrac{a!}{b!(a-b)!}$ if $a, b \in \mathbb{N}$ and $a \geq b$, with the convention $\binom{a}{b} = 0$ if $a, b \in \mathbb{N}$ and $a < b$. now let $\widetilde{S}$ be another finite set, and let $\pi : S \to \widetilde{S}$ be a map (not necessarily surjective nor injective). For any map $\ell : S \to \mathbb{N}$ we define $\pi_*\ell : \widetilde{S} \to \mathbb{N}$ by:

$$\pi_*\ell(x) = \begin{cases} 0 \text{ if x is not in the image of } \pi, \\ \displaystyle\sum_{y\in S,\, \pi(y)=x} \ell(y) \text{ if x is in the image of } \pi. \end{cases} \tag{7.3.2}$$

Proposition 7.3.1. *For any* $\widetilde{\ell} : \widetilde{S} \to \mathbb{N}$ *and for any* $k : S \to \mathbb{N}$ *the following holds:*

$$\binom{\pi_* k}{\widetilde{\ell}} = \sum_{\ell,\, \pi_*\ell=\widetilde{\ell}} \binom{k}{\ell}. \tag{7.3.3}$$

Proof. Consider the obvious equality between polynomials with variables indexed by $\widetilde{S}$:

$$\prod_{s\in S} \big(1 + X_{\pi(s)}\big)^{k(s)} = \prod_{\widetilde{s}\in\widetilde{S}} (1 + X_{\widetilde{s}})^{\pi_* k(\widetilde{s})}, \tag{7.3.4}$$

and pick the coefficient of the monomial $\prod_{\widetilde{s}\in\widetilde{S}} X_{\widetilde{s}}^{\widetilde{\ell}(\widetilde{s})}$ on both sides. □

7.4. **Playing with decorations.** We suppose that the edges of our coloured forests are of different types, which is illustrated by a map $\mathfrak{t} : E(F) \to \mathcal{L}$, where $\mathcal{L}$ is a finite set. For later use, $\mathbb{Z}(\mathcal{L})$ is the free abelian group generated by the types. The type map is subintended and given once for all: it should not be confused with the colouring. Let d be a given fixed positive integer, the *dimension*.

Definition 7.4.1. *A* ***decorated forest*** *is a* 5*-uple* $(F, \widehat{F})_{\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}}$, *where*

$$\mathfrak{n} : N(F) \to \mathbb{N}^d, \qquad \mathfrak{o} : N(F) \to \mathbb{Z}^d \oplus \mathbb{Z}(\mathcal{L}), \qquad \mathfrak{e} : E(F) \to \mathbb{N}^d. \tag{7.4.1}$$

The product of two decorated forests is given by their concatenation together with the sum of their corresponding decorations, namely:

$$(F, \widehat{F})_{\mathfrak{e}}^{\mathfrak{n},\mathfrak{o}}.(G, \widehat{G})_{\mathfrak{e}'}^{\mathfrak{n}',\mathfrak{o}'} := (F \sqcup G, \widehat{F} + \widehat{G})_{\mathfrak{e}+\mathfrak{e}'}^{\mathfrak{n}+\mathfrak{n}',\mathfrak{o}+\mathfrak{o}'}.$$

Let us introduce some more notations: for any subforest A of a forest F, the *boundary* of A in F is defined as:

$$\partial(A, F) := \{[v, w] \in E(F) \setminus E(A),\, v \in N(A)\}. \tag{7.4.2}$$

Let $\pi : E(F) \to N(F)$ be the bottom map, defined by $\pi([v,w]) = v$. For any $\varepsilon : E(F) \to \mathbb{Z}^d$, the map $\pi_*\varepsilon : N(F) \to \mathbb{Z}^d$ is defined by:

$$\pi_*\varepsilon(v) := \sum_{a=[v,w]\in E(F)} \varepsilon(a). \tag{7.4.3}$$

We also use the shortand notation ε_A^B for $\mathbf{1}_{E(B)\setminus E(A)}\varepsilon$ for any $A \subset B \subset F$. We are now ready for the definition of our family of coproducts:

$$\Delta_i(F,\widehat{F})^{\mathfrak{n},\mathfrak{o}}_{\mathfrak{e}} := \sum_{A\in\mathcal{U}_i(F,\widehat{F})} \sum_{\varepsilon,\mathfrak{n}'} \frac{1}{\varepsilon!}\binom{\mathfrak{n}}{\mathfrak{n}'} \left(A, \widehat{F}_{|A}\right)^{\mathfrak{n}'+\pi_*\varepsilon,\, \mathfrak{o}_{|N(A)}}_{\mathfrak{e}_{|E(A)}} \otimes \left(F, \widehat{F}\cup_i A\right)^{\mathfrak{n}-\mathfrak{n}',\, \mathfrak{o}+\mathfrak{n}'+\pi_*(\varepsilon-\mathfrak{e}_\emptyset^A)}_{e_A^F+\varepsilon}. \tag{7.4.4}$$

The inner sum runs over the maps $\mathfrak{n}' : N(F) \to \mathbb{N}^d$ with $\mathfrak{n}'(v) = 0$ if $v \notin A$, and over the maps $\varepsilon : E(F) \to \mathbb{N}^d$ with $\varepsilon(a) = 0$ for $a \notin \partial(A,F)$. This sum is *infinite*, and thus takes place in a suitable completion of the tensor product. This can be made precise in terms of *bigraded spaces*, see [4, Section 2]. Now let us introduce another assumption of the collection $\mathcal{U}_i$:

Assumption 7.3. *The correspondences* $\mathcal{U}_i : (F,\widehat{F}) \mapsto \mathcal{U}_i(F,\widehat{F})$ *verify (for* $i \geq 1$*):*

1. $\mathcal{U}_i(F \sqcup G, \widehat{F} + \widehat{G}) = \{C \sqcup D,\ C \in \mathcal{U}_i(F,\widehat{F}) \text{ and } D \in \mathcal{U}_i(G,\widehat{G})\}$.
2. *The following equivalence holds:*

$$\Big(A \in \mathcal{U}_i(F,\widehat{F}) \text{ and } B \in \mathcal{U}_i(F,\widehat{F}\cup_i A)\Big) \Longleftrightarrow \Big(B \in \mathcal{U}_i(F,\widehat{F}) \text{ and } A \in \mathcal{U}_i\big(B,\widehat{F}_{|B}\big)\Big).$$

Theorem 7.4.1. *Under Assumptions 7.1 and 7.3, the coproducts* Δ_i *are multiplicative and coassociative.*

Proof. Multiplicativity is a direct consequence of item (1) of Assumption 7.3 and of the definition of the product of two decorated forests, thanks to the fact that factorials (resp. binomial coefficients) verify $(f+g)! = f!g!$, resp. $\binom{f+g}{f'+g'} = \binom{f}{f'}\binom{g}{g'}$, for two functions f and g with disjoint supports. Coassociativity is proved by a careful direct check: first of all we compute:

$$\begin{aligned}
&(\Delta_i \otimes I)\Delta_i(F,\widehat{F})^{\mathfrak{n},\mathfrak{o}}_{\mathfrak{e}} \\
&= (\Delta_i \otimes I)\left(\sum_{B\in\mathcal{U}_i(F,\widehat{F})} \sum_{\varepsilon,\mathfrak{n}''} \frac{1}{\varepsilon!}\binom{\mathfrak{n}}{\mathfrak{n}''} \left(B, \widehat{F}_{|B}\right)^{\mathfrak{n}''+\pi_*\varepsilon,\, \mathfrak{o}_{|N(B)}}_{\mathfrak{e}_{|E(B)}} \otimes \left(F, \widehat{F}\cup_i B\right)^{\mathfrak{n}-\mathfrak{n}'',\, \mathfrak{o}+\mathfrak{n}''+\pi_*(\varepsilon-\mathfrak{e}_\emptyset^B)}_{e_B^F+\varepsilon}\right) \\
&= \sum_{B\in\mathcal{U}_i(F,\widehat{F})} \sum_{\substack{\varepsilon:E(F)\to\mathbb{N}^d,\, \mathrm{supp}\,\varepsilon\subset\partial(B,F)\\ \mathfrak{n}'':N(F)\to\mathbb{N}^d,\, \mathrm{supp}\,\mathfrak{n}''\subset N(B)}} \sum_{A\in\mathcal{U}_i(B,\widehat{F}_{|B})} \sum_{\substack{\eta:E(B)\to\mathbb{N}^d,\, \mathrm{supp}\,\eta\subset\partial(A,B)\\ \mathfrak{n}':N(B)\to\mathbb{N}^d,\, \mathrm{supp}\,\mathfrak{n}'\subset N(A)}} \frac{1}{\varepsilon!\eta!}\binom{\mathfrak{n}}{\mathfrak{n}''}\binom{\mathfrak{n}''+\pi_*\varepsilon}{\mathfrak{n}'} \\
&\quad \left(A, \widehat{F}_{|A}\right)^{\mathfrak{n}'+\pi_*\eta,\, \mathfrak{o}_{|N(A)}}_{\mathfrak{e}_{|E(A)}} \otimes \left(B, \widehat{F}_{|B}\cup_i A\right)^{\mathfrak{n}''-\mathfrak{n}'+\pi_*\varepsilon,\, \mathfrak{o}_{|N(B)}+\mathfrak{n}'+\pi_*(\eta-\mathfrak{e}_\emptyset^A)}_{\mathfrak{e}_A^B+\eta} \otimes \left(F, \widehat{F}\cup_i B\right)^{\mathfrak{n}-\mathfrak{n}'',\, \pi_*(\varepsilon-\mathfrak{e}_\emptyset^B)+\mathfrak{o}+\mathfrak{n}''}_{\mathfrak{e}_B^F+\varepsilon}.
\end{aligned} \tag{7.4.5}$$

On the other hand,

$$
\begin{aligned}
&(I\otimes\Delta_i)\Delta_i(F,\widehat{F})^{\mathfrak{n},\mathfrak{o}}_{\mathfrak{e}}\\
= \ &(I\otimes\Delta_i)\left(\sum_{A\in\mathcal{U}_i(F,\widehat{F})}\sum_{\widetilde{\varepsilon},\widetilde{\mathfrak{n}}'}\frac{1}{\widetilde{\varepsilon}!}\binom{\mathfrak{n}}{\widetilde{\mathfrak{n}}'}\Big(A,\widehat{F}_{|A}\Big)^{\widetilde{\mathfrak{n}}'+\pi_*\widetilde{\varepsilon},\,\mathfrak{o}_{|N(A)}}_{\mathfrak{e}_{|E(A)}}\otimes\Big(F,\widehat{F}\cup_i A\Big)^{\mathfrak{n}-\widetilde{\mathfrak{n}}',\,\mathfrak{o}+\widetilde{\mathfrak{n}}'+\pi_*(\widetilde{\varepsilon}-\mathfrak{e}^A_\emptyset)}_{\mathfrak{e}^F_A+\widetilde{\varepsilon}}\right)\\
= \ &\sum_{A\in\mathcal{U}_i(F,\widehat{F})}\ \sum_{\substack{\widetilde{\varepsilon}:E(F)\to\mathbb{N}^d,\,\mathrm{supp}\,\widetilde{\varepsilon}\subset\partial(A,F)\\ \widetilde{\mathfrak{n}}':N(F)\to\mathbb{N}^d,\,\mathrm{supp}\,\widetilde{\mathfrak{n}}'\subset N(A)}}\ \sum_{B\in\mathcal{U}_i(F,\widehat{F}\cup_i A)}\ \sum_{\substack{\widetilde{\eta}:E(A)\to\mathbb{N}^d,\,\mathrm{supp}\,\widetilde{\varepsilon}\subset\partial(B,F)\\ \widetilde{\mathfrak{n}}'':N(F)\to\mathbb{N}^d,\,\mathrm{supp}\,\widetilde{\mathfrak{n}}''\subset N(B)}}\frac{1}{\widetilde{\varepsilon}!\widetilde{\eta}!}\binom{\mathfrak{n}}{\widetilde{\mathfrak{n}}'}\binom{\mathfrak{n}-\widetilde{\mathfrak{n}}'}{\widetilde{\mathfrak{n}}''}
\end{aligned}
$$

$$
\Big(A,\widehat{F}_{|A}\Big)^{\widetilde{\mathfrak{n}}'+\pi_*\widetilde{\varepsilon},\,\mathfrak{o}_{|N(A)}}_{\mathfrak{e}_{|E(A)}}\otimes\Big(B,(\widehat{F}\cup_i A)_{|B}\Big)^{\widetilde{\mathfrak{n}}''+\pi_*\widetilde{\eta},\,[\mathfrak{o}+\widetilde{\mathfrak{n}}'+\pi_*(\widetilde{\varepsilon}-\mathfrak{e}^A_\emptyset)]_{|N(B)}}_{\mathfrak{e}^B_A+\widetilde{\varepsilon}^B_A}\otimes\Big(F,\widehat{F}\cup_i B\Big)^{\mathfrak{n}-\widetilde{\mathfrak{n}}'-\widetilde{\mathfrak{n}}'',\,\mathfrak{o}+\widetilde{\mathfrak{n}}'+\widetilde{\mathfrak{n}}''+\pi_*(\widetilde{\varepsilon}^F_B+\widetilde{\eta}+\mathfrak{e}^B_\emptyset)}_{\mathfrak{e}^F_B+\widetilde{\varepsilon}^F_B+\widetilde{\eta}}. \tag{7.4.6}
$$

Here we have used the equality $\widehat{F}\cup_i A)\cup_i B=\widehat{F}\cup_i B$ which holds because $A\subset B$ here. Remark moreover that we have $\widetilde{\varepsilon}^F_A=\widetilde{\varepsilon}^B_A+\widetilde{\varepsilon}^F_B$, and the support of the two components are disjoint. We make the following change of indices:

$$\varepsilon:=\widetilde{\varepsilon}^F_B+\widetilde{\eta},\qquad \eta:=\widetilde{\varepsilon}^B_A,\qquad \eta':=\widetilde{\varepsilon}^F_B=\varepsilon-\widetilde{\eta}.$$

It is injective, with image given by the constraint $\varepsilon-\eta'\geq 0$. Its inverse, defined on this image, is given by:

$$\widetilde{\varepsilon}=\eta+\eta',\qquad \widetilde{\eta}=\varepsilon-\eta'.$$

The combinatorial prefactor in (7.4.6) can then be rewritten as:

$$\frac{1}{\widetilde{\eta}!}\frac{1}{\widetilde{\eta}'!}=\frac{1}{\eta!}\frac{1}{\eta'!}\frac{1}{(\varepsilon-\eta')!}=\frac{1}{\eta!}\frac{1}{\varepsilon!}\binom{\varepsilon}{\eta'}.$$

Hence,

$$
\begin{aligned}
&(I\otimes\Delta_i)\Delta_i(F,\widehat{F})^{\mathfrak{n},\mathfrak{o}}_{\mathfrak{e}}\\
= \ &\sum_{A\in\mathcal{U}_i(F,\widehat{F})}\ \sum_{B\in\mathcal{U}_i(F,\widehat{F}\cup_i A)}\ \sum_{\varepsilon,\eta,\eta'}\ \sum_{\widetilde{\mathfrak{n}}',\widetilde{\mathfrak{n}}''}\frac{1}{\eta!}\frac{1}{\varepsilon!}\binom{\varepsilon}{\eta'}\binom{\mathfrak{n}}{\widetilde{\mathfrak{n}}'}\binom{\mathfrak{n}-\widetilde{\mathfrak{n}}'}{\widetilde{\mathfrak{n}}''}
\end{aligned}
$$

$$
\Big(A,\widehat{F}_{|A}\Big)^{\widetilde{\mathfrak{n}}'+\pi_*(\eta+\eta'),\,\mathfrak{o}_{|N(A)}}_{\mathfrak{e}_{|E(A)}}\otimes\Big(B,(\widehat{F}\cup_i A)_{|B}\Big)^{\widetilde{\mathfrak{n}}''+\pi_*(\varepsilon-\eta'),\,[\mathfrak{o}+\widetilde{\mathfrak{n}}'+\pi_*(\eta+\eta'-\mathfrak{e}^A_\emptyset)]_{|N(B)}}_{\mathfrak{e}^B_A+\eta}\otimes\Big(F,\widehat{F}\cup_i B\Big)^{\mathfrak{n}-\widetilde{\mathfrak{n}}'-\widetilde{\mathfrak{n}}'',\,\mathfrak{o}+\widetilde{\mathfrak{n}}'+\widetilde{\mathfrak{n}}''+\pi_*(\varepsilon+\mathfrak{e}^B_\emptyset)}_{\mathfrak{e}^F_B+\varepsilon}, \tag{7.4.7}
$$

so that the constraint $\varepsilon-\eta'\geq 0$ is encoded in the combinatorial prefactor. Now we make another change of indices, namely,

$$\mathfrak{n}'':=\widetilde{\mathfrak{n}}'+\widetilde{\mathfrak{n}}'',\qquad \mathfrak{n}':=\widetilde{\mathfrak{n}}'+\pi_*\eta'.$$

It is again injective. The inverse (defined on the image) is given by:

$$\widetilde{\mathfrak{n}}'=\mathfrak{n}'-\pi_*\eta',\qquad \widetilde{\mathfrak{n}}''=\mathfrak{n}''+\pi_*\eta'.$$

We thus have the following identities between binomial coefficients:

$$\binom{\mathfrak{n}}{\widetilde{\mathfrak{n}}'}\binom{\mathfrak{n}-\widetilde{\mathfrak{n}}'}{\widetilde{\mathfrak{n}}''} = \frac{\mathfrak{n}!(\mathfrak{n}-\widetilde{\mathfrak{n}}')!}{\widetilde{\mathfrak{n}}'!(\mathfrak{n}-\widetilde{\mathfrak{n}}')!\widetilde{\mathfrak{n}}''!(\mathfrak{n}-\widetilde{\mathfrak{n}}'-\widetilde{\mathfrak{n}}'')!} = \frac{\mathfrak{n}!(\widetilde{\mathfrak{n}}'+\widetilde{\mathfrak{n}}'')!}{\widetilde{\mathfrak{n}}'!(\widetilde{\mathfrak{n}}'+\widetilde{\mathfrak{n}}'')!\widetilde{\mathfrak{n}}''!(\mathfrak{n}-\widetilde{\mathfrak{n}}'-\widetilde{\mathfrak{n}}'')!} = \binom{\mathfrak{n}}{\widetilde{\mathfrak{n}}'+\widetilde{\mathfrak{n}}''}\binom{\widetilde{\mathfrak{n}}'+\widetilde{\mathfrak{n}}''}{\widetilde{\mathfrak{n}}'}$$
$$= \binom{\mathfrak{n}}{\mathfrak{n}''}\binom{\mathfrak{n}''}{\mathfrak{n}'-\pi_*\eta'}.$$

Plugging this into (7.4.7) we get:

$$\begin{aligned}&(I\otimes\Delta_i)\Delta_i(F,\widehat{F})^{\mathfrak{n},\mathfrak{o}}_{\mathfrak{e}}\\ &= \sum_{A\in\mathcal{U}_i(F,\widehat{F})}\ \sum_{B\in\mathcal{U}_i(F,\widehat{F}\cup_i A)}\ \sum_{\varepsilon,\eta,\eta'}\ \sum_{\mathfrak{n}',\mathfrak{n}''}\frac{1}{\eta!}\frac{1}{\varepsilon!}\binom{\varepsilon}{\eta'}\binom{\mathfrak{n}}{\mathfrak{n}''}\binom{\mathfrak{n}''}{\mathfrak{n}'-\pi_*\eta'}\\ &\Big(A,\widehat{F}\big|_A\Big)^{\mathfrak{n}'+\pi_*\eta,\,\mathfrak{o}|_{N(A)}}_{\mathfrak{e}|_{E(A)}}\otimes\Big(B,(\widehat{F}\cup_i A)\big|_B\Big)^{\mathfrak{n}''-\mathfrak{n}'+\pi_*\varepsilon,\,[\mathfrak{o}+\mathfrak{n}'+\pi_*(\eta-\mathfrak{e}^A_\emptyset)]|_{N(B)}}_{\mathfrak{e}^B_A+\eta}\otimes\Big(F,\widehat{F}\cup_i B\Big)^{\mathfrak{n}-\mathfrak{n}'',\,\mathfrak{o}+\mathfrak{n}''+\pi_*(\varepsilon+\mathfrak{e}^B_\emptyset)}_{\mathfrak{e}^F_B+\varepsilon}.\end{aligned}\tag{7.4.8}$$

The generic term in the right-hand side of (7.4.8) depends on η' only by means of the binomial coefficient. By means of the Chu-Vandermonde identity (7.3.3) we have:

$$\begin{aligned}\sum_{\eta'}\binom{\varepsilon}{\eta'}\binom{\mathfrak{n}''}{\mathfrak{n}'-\pi_*\eta'} &= \sum_{\alpha}\ \sum_{\eta',\,\pi_*\eta'=\alpha}\binom{\varepsilon}{\eta'}\binom{\mathfrak{n}''}{\mathfrak{n}'-\alpha}\\ &= \sum_{\alpha}\binom{\pi_*\varepsilon}{\alpha}\binom{\mathfrak{n}''}{\mathfrak{n}'-\alpha}\\ &= \binom{\mathfrak{n}''+\pi_*\varepsilon}{\mathfrak{n}'}.\end{aligned}\tag{7.4.9}$$

In view of Assumption 7.3, plugging (7.4.9) into (7.4.8) and comparing with (7.4.5) proves the coassociativity of the coproducts Δ_i.

7.5. **Decorations and contractions.** We extend the contraction operator $\mathcal{K}$ of Paragraph 7.2 to decorated forests as follows:

$$\mathcal{K}\left((F,\widehat{F})^{\mathfrak{n},\mathfrak{o}}_{\mathfrak{e}}\right) := \mathcal{K}(F,\widehat{F})^{[\mathfrak{n}],\,[\mathfrak{o}]}_{[\mathfrak{e}]}\tag{7.5.1}$$

with:

$$[\mathfrak{n}](x) := \sum_{v\in x}\mathfrak{n}(v),\qquad [\mathfrak{o}](x) := \sum_{v\in x}\mathfrak{o}(v)+\sum_{a\in x^2}\mathfrak{t}(a),\qquad [\mathfrak{e}] := \mathfrak{e}\big|_{E(F)\setminus\widehat{E}}.$$

□

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