CHAPTER 1

Introduction

Boundary value problems model many phenomena in applied sciences and engineering, that's why their study is an important research area despite it's difficult as long as there is no general method to apply, and then to discuss the qualitative and quantitative properties of solutions for a given boundary value problem, different methods are used such the upper and lower solutions method, Mawhin theory, numerical methods... [25],

The objective of this thesis is the study of the existence of solutions of boundary value problems generated by a class of second order nonlinear differential equations with boundary conditions of integral and multipoint type, by applying the degree of coincidence of Mawhin.

The Mawhin theory permits the use of an approach of topological degree type to problems which can be written as an abstract operator equation of the form $Lx = Nx$, where L is a linear noninvertible operator and N is a nonlinear operator acting on a given Banach space.

In 1972, Mawhin has developed a method to solve this equation in his famous paper "Topological degree and boundary value problems for nonlinear differential equations" [40], he assumed that L is a Fredholm operator of index zero. Hence he has developed a new theory of topological degree known as the degree of coincidence for (L, N), that is also known as Mahwin's coincidence degree theory in honor of him.

A boundary value problem is said to be at resonance if the corresponding linear homogenous problem has nontrivial solution, otherwise it's said to be at resonance.

Many authors studied ordinary boundary value problems at resonance using Mawhin coincidence degree theory, we can cite Feng and Webb [11], Guezane-Lakoud and Frioui [14], Mahin and Ward [42], Infante [26], and the references [4,10,12,15,18-20,30-34,40,42,46].

Let us consider the following second order differential equation:

$$u''(t) = f(t, u(t), u'(t)) + e(t),\ t \in (0, 1), \tag{1.1}$$

jointly with the multipoint boundary conditions of type

$$u(0) = 0,\ u(1) = \alpha u(\xi), \tag{1.2}$$

or

$$u'(0) = 0,\ u(1) = \alpha u(\xi) \tag{1.3}$$

where $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and $e \in L^1(0, 1)$, $\xi \in (0, 1)$, $\alpha \in \mathbb{R}$.

In [24], Gupta, Ntouyas and Tsamatos used the Leray-Schauder continuation theorem to prove the existence of solutions of problem (1.1),(1.2) in the case $\alpha \neq 1$ and for the problem (1.1),(1.3), under the assumption $\alpha < \frac{1}{\xi}$, in both cases the problems are at nonresonance.

Later Feng and Webb in [11], considered the problems (1.1),(1.2) and (1.1),(1.3) in the resonance case that is when $\alpha = 1$ for problem (1.1),(1.2) and $\alpha = \frac{1}{\xi}$ for problem (1.1),(1.3). The authors proved by using Mawhin coincidence degree theory, that these problems have at least one solution.

In [26], Infante and Zima studied the existence of positive solutions for the following boundary value problem at resonance

$$u''(t) = f(t, u(t)),\ t \in (0, 1),$$

$$u'(0) = 0,\ u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i),$$

where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be a function satisfying Caratheodory's conditions, $\alpha_i > 0$, $i = 1, 2, ..., m-2$, $0 < \xi_1 \leq \xi_2 \leq ... \leq \xi_{m-2} < 1$, $\sum\limits_{i=1}^{m-2} \alpha_i = 1$. Their approach is based on the Leggett-Williams norm-type theorem for coincidences obtained by O'Regan and Zima, see [47].

Recently in [14], by the help of Mawhin coincidence degree, Guezane-Lakoud and Frioui, proved the existence of solutions for a third order multipoint boundary value problem at resonance of the form

$$u'''(t) = f(t, u(t), u'(t)),\ t \in (0, 1),$$

$$u(0) = u''(0) = 0,\ u(1) = \frac{2}{\eta^2} \int_0^{\eta} u(t)dt,$$

where f is Caratheodory's function and $0 < \eta < 1$.

Ordinary differential equations with multipoint boundary conditions occur naturally arise in some applications such in population dynamics model, in semiconductor problems, thermal conduction problems, hydrodynamic problems..., see [7, 8, 27, 50]. For example, if a dynamic system has m degrees of freedom, then we have exactly m states observed at m different times, and consequently we obtain an m-point boundary value problem. The discretization of some boundary value problems for partial differential equations on irregular domains with the line method leads to multipoint boundary value problems.

The aim of this thesis is the study of a resonance boundary value problem generated by a second order nonlinear differential equation with multipoint and integral boundary conditions:

$$u''(t) = f(t, u(t), u'(t)),\ t \in (0,1),$$

$$u(0) = 0,\ u(1) = \sum_{k=1}^{m} \lambda_k \int_0^{\eta_k} u(t)dt$$

where f is a Caratheodory's function, $0 < \eta_k < 1$, $\lambda_k > 0$, $k = 0, ..., m$, $\sum_{k=1}^{m} \lambda_k \eta_k^2 = 2$. We apply the Mawhin coincidence degree to prove the existence of at least one solution in a Banach space that we will define later.

Differential equations with nonlocal conditions, especially integral conditions, plays an important role in both theory and applications. The study of these problems is motivated by various applications, including thermoelasticity, chemical engineering, plasma physics.....

Let us give a brief outline of each chapter of the thesis.

The first chapter presents a review of some fixed point theorems, notably the Banach contraction principle, Leray-Schauder's nonlinear alternative, notion of homotopy, Fredholm's operators of zero index, the concept of the topological degree and its properties is discussed; two degrees are defined: the degree of Brouwer in finite dimension then the degree of Leray-Schauder in infinite dimension. We cite the theorem of Mawhin coincidence degree and its proof.

In the second chapter, we collect interesting results for some classes of second-order boundary value problems at resonance. For this purpose we will summarize the basic results in the literature and present the main ideas of some research on these problems.

The third chapter is devoted to the study of a second order boundary value problem at resonance. In fact we propose to establish the existence of the solution of a differential equation with multipoint conditions of integral type. The proofs are based on Mahwin's theory of coincidence. The results of this chapter are published:

R. Khaldi, M. Kouidri, Solvability of multipoint value problems with integral condition at resonance, International Journal of Analysis and Applications, Vol 16, Number 3 (2018), 306–316.

The thesis is clotured by some interesting references.