On a relation between the turning point locus of a meromorphic connection and its irregularity sheaf

Sommaire

Let us consider a smooth complex algebraic variety X, a smooth hypersurface $i : Z \hookrightarrow X$ and let $\mathcal{M}$ be a meromorphic connection on X with poles along Z. The aim of this part is to prove the following

Theorem 12.2.7. *The set of good semi-stable points of $\mathcal{M}$ is a subset of the smooth locus of* $\mathrm{Irr}^*_Z(\mathcal{M})$.

The strategy can be described as follows: we first prove 12.2.7 in the case where $\mathcal{M}$ is a sum of modules appearing in the right hand side of (0.0.1). Since by 13.2.1 the local Euler-Poincaré characteristic $\chi(\mathrm{Irr}^*_Z(\mathcal{M}))$ of $\mathrm{Irr}^*_Z(\mathcal{M})$ only depends on the formalization of $\mathcal{M}$ along Z, we deduce that $\chi(\mathrm{Irr}^*_Z(\mathcal{M}))$ is constant in the neighbourhood of a good semi-stable point. To conclude, we combine the perversity of $\mathrm{Irr}^*_Z(\mathcal{M})$ with a general theorem 13.1.6 stating that on a smooth variety, the perverse sheaves with constant local Euler-Poincaré characteristic are smooth.

As an immediate corollary of 12.2.7, we get the following

Theorem 12.2.8. *The stable point locus of $\mathcal{M}$ is included in the intersection of the smooth locus of* $\mathrm{Irr}^*_Z(\mathcal{M})$ *and* $\mathrm{Irr}^*_Z(\mathrm{End}\,\mathcal{M})$.

The converse of 12.2.8 will be discussed in 15.1. After a reduction 15.2 to the two dimensional case resting upon André's criterion [5, 3.4.1] for stable points, we will prove it in 15.4 in the special case where only one ϕ occurs in (0.0.1).

13 The protagonists and some preleminary results

13.1 The Irregularity sheaf

The irregularity sheaf has been introduced in [39]. For a detailed treatment of its fundamental properties, one can refer to [40]. As for useful references concerning perverse sheaves, one can mention [21], [41], [11].

In this subsection, X will be a complex manifold, $i : Z \hookrightarrow X$ a closed analytic subvariety of X, $\mathcal{I}_Z$ its defining ideal, $\mathcal{M} \in D^b_h(\mathcal{D}_X)$ a complex of $\mathcal{D}_X$-modules with bounded holonomic cohomology and $\mathrm{Char}(\mathcal{M})$ its characteristic cycle. $D^b_c(\mathbb{C}_X)$ will denote the derived category of complex of sheaves of $\mathbb{C}$-vector spaces with bounded constructible cohomology and if $\mathcal{F} \in D^b_c(\mathbb{C}_X)$, we will denote by

$$\chi(\mathcal{F}) : x \longrightarrow \sum_k (-1)^k \,\mathrm{rk}\, \mathcal{H}^k \mathcal{F}_x$$

the Euler-Poincaré characteristic of $\mathcal{F}$.

The local algebraic cohomology functor

$$\mathrm{alg}\,\Gamma_Z(\mathcal{M}) := \varinjlim \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^k_Z, \mathcal{M})$$

and the localization functor

$$\mathcal{M}(*Z) := \varinjlim \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}^k_Z, \mathcal{M})$$

give rise to the following distinguished triangle of $D^b(\mathcal{D}_X)$

$$\mathbf{R}\operatorname{alg}\Gamma_Z(\mathcal{M}) \longrightarrow \mathcal{M} \longrightarrow \mathbf{R}\mathcal{M}(*Z) \xrightarrow{+1} \tag{13.1.1}$$

and one can prove [19] by using Bernstein-Sato polynomial that (13.1.1) lies in $D^b_h(\mathcal{D}_X)$. By applying the solution functor $\mathbf{S} = R\mathcal{H}om_{\mathcal{D}_X}(\ \ ,\mathcal{O}_X)$, Kashiwara's constructibility theorem [17] asserts that the distinguished triangle of bounded complex of sheaves

$$i^{-1}\mathbf{S}(\mathbf{R}\mathcal{M}(*Z)) \longrightarrow i^{-1}\mathbf{S}(\mathcal{M}) \longrightarrow \mathbf{S}(\mathbf{R}\operatorname{alg}\Gamma_Z(\mathcal{M})) \xrightarrow{+1} \tag{13.1.2}$$

lies in $D^b_c(\mathbb{C}_X)$. Let us define [14]

$$\operatorname{Irr}^*_Z(\mathcal{M}) := i^{-1}\mathbf{S}(\mathbf{R}\mathcal{M}(*Z))[1] \tag{13.1.3}$$

Hence $\operatorname{Irr}^*_Z(\mathcal{M})$ is the cone of

$$i^{-1}\mathbf{S}(\mathcal{M}) \longrightarrow \mathbf{S}(\mathbf{R}\operatorname{alg}\Gamma_Z(\mathcal{M})) \tag{13.1.4}$$

Of fundamental importance about the irregularity sheaf is the following theorem of Mebkhout [39, 2.1.6]

Theorem 13.1.5. *If Z is an hypersurface, $\operatorname{Irr}^*_Z(\mathcal{M})$ is a perverse sheaf on Z.*

This result will have a crucial role, due to the following general

Theorem 13.1.6. *Let Z be a smooth complex manifold, and let $\mathcal{F}$ be a perverse sheaf on Z with constant Euler-Poincaré characteristic. Then $\mathcal{F}$ is a local system concentrated in degree 0.*

Proof. For the sake of this proof, let us denote by n the dimension of Z. Since $\mathcal{F}$ is perverse, it is generically smooth and the constancy of $\chi(\mathcal{F})$ reads

$$\dim \mathcal{H}^0\mathcal{F}_z - \dim \mathcal{H}^1\mathcal{F}_z + \cdots + (-1)^n \dim \mathcal{H}^n\mathcal{F}_z = \dim \mathcal{H}^0\mathcal{F}_\eta \tag{13.1.7}$$

for every $z \in Z$, with η denoting the generic point of Z. Let us argue on the dimension of Z.

If Z is one dimensional, (13.1.7) simplifies for a given $z \in Z$ into the following equality

$$\dim \mathcal{H}^0\mathcal{F}_z = \dim \mathcal{H}^1\mathcal{F}_z + \dim \mathcal{H}^0\mathcal{F}_\eta$$

Since the perversity condition on $\mathcal{F}$ implies that $\mathcal{H}^0\mathcal{F}$ has no section punctually supported at z [21, 10.3.3], we have

$$\dim \mathcal{H}^0\mathcal{F}_z \leq \mathcal{H}^0\mathcal{F}_\eta,$$

from which we deduce the vanishing of $\mathcal{H}^1\mathcal{F}$, and 13.1.6 follows from the constructibility of $\mathcal{H}^0\mathcal{F}$.

Let us suppose that $n \geq 2$, and take $z_0 \in Z$. According to [39, 2.2.1.5], one can

14. We could either work with $\operatorname{Irr}_Z(\mathcal{M})$ or with its dual $\operatorname{Irr}^*_Z(\mathcal{M})$. This is harmless for our purpose since by combining Verdier biduality theorem [53] and the hypercohomology spectral sequence, one can see that a bounded constructible complex of sheaves is smooth if and only if its dual is smooth.

choose a neighbourhood U of z_0 with the property that for every $z \in U$ different from z_0, there exists a smooth hypersurface $i : Z' \longrightarrow Z$ passing through z and such that $i^{-1}\mathcal{F}$ is perverse. From the induction hypothesis applied to $(Z', i^{-1}\mathcal{F})$, one gets that $\mathcal{H}^0\mathcal{F}_{|U}$ is a local system away from z_0, and for every $k \in [\![1, n]\!]$, the sheaf $\mathcal{H}^k\mathcal{F}_{|U}$ has support included in $\{z_0\}$. Since the statement 13.1.7 is local, we will abuse notation by using Z for U in the sequel. Let us prove the vanishing of $\mathcal{H}^k\mathcal{F}_{z_0}$ for $k \in [\![1, n-1]\!]$ [15].

From the following canonical identification in $D^b_c(Z, \mathbb{C})$

$$R\mathcal{H}om(\mathbb{C}_{z_0}, \mathbb{C}) \xrightarrow{\sim} \mathbb{C}_{z_0}[-2n], \tag{13.1.8}$$

we deduce that the a priori non-zero terms of the hypercohomology spectral sequence [16]

$$E_2^{pq} = R^p\mathcal{H}om(\mathcal{H}^{-q}\mathcal{F}, \mathbb{C}) \Longrightarrow R^{p+q}\mathcal{H}om(\mathcal{F}, \mathbb{C}) \tag{13.1.9}$$

are the terms E_2^{pq} with $(p = 0, \dots, 2n$ and $q = 0)$ or $(p = 2n$ and $q = 0, \dots, -n)$.

Let us prove the degeneracy at sheet 2 of (13.1.9). We start with the following

Lemma 13.1.10. *The sheaves* $E_2^{p,0}$ *are* 0 *for* $p = 1, \dots, 2n-2$.

Proof. By choosing V a small enough ball centered at z_0, one can suppose that the cohomology sheaves of $R\mathcal{H}om(\mathcal{H}^0\mathcal{F}, \mathbb{C})$ are acyclic for $\Gamma(V, \ \ )$. Thus we have

$$\Gamma(V, E_2^{2n-k,0}) \simeq \mathrm{Ext}^{2n-k}(\mathcal{H}^0\mathcal{F}_{|V}, \mathbb{C}),$$

for every k. Hence, Poincaré duality gives a canonical isomorphism

$$\Gamma(V, E_2^{2n-k,0}) \simeq H^k_c(V, \mathcal{H}^0\mathcal{F}_{|V})^\vee \tag{13.1.11}$$

Let us denote by $j : V \longrightarrow \overline{V}$, $i : \partial\overline{V} \longrightarrow \overline{V}$ the canonical immersions. By applying the functor $R\Gamma(\overline{V}, \ \ )$ to the short exact sequence

$$0 \longrightarrow j_!\mathcal{H}^0\mathcal{F}_{|V} \longrightarrow \mathcal{H}^0\mathcal{F}_{|\overline{V}} \longrightarrow i_*\mathcal{H}^0\mathcal{F}_{|\partial\overline{V}} \longrightarrow 0$$

we end up with an exact triangle

$$R\Gamma(\overline{V}, j_!\mathcal{H}^0\mathcal{F}_{|V}) \longrightarrow R\Gamma(\overline{V}, \mathcal{H}^0\mathcal{F}_{|\overline{V}}) \longrightarrow R\Gamma(\partial\overline{V}, \mathcal{H}^0\mathcal{F}_{|\partial\overline{V}}) \xrightarrow{+1}$$

The associated long exact sequence reads

$$\cdots \to H^{k-1}(\partial\overline{V}, \mathcal{H}^0\mathcal{F}_{|\partial\overline{V}}) \to H^k_c(V, \mathcal{H}^0\mathcal{F}_{|V}) \to H^k(\overline{V}, \mathcal{H}^0\mathcal{F}_{|\overline{V}}) \to H^k(\partial\overline{V}, \mathcal{H}^0\mathcal{F}_{|\partial\overline{V}}) \to \cdots$$

Since $\partial\overline{V}$ is a sphere of dimension $2n-1$ and $\mathcal{H}^0\mathcal{F}_{|\partial\overline{V}}$ is constant on it, we have

$$H^{k-1}(\partial\overline{V}, \mathcal{H}^0\mathcal{F}_{|\partial\overline{V}}) \simeq 0$$

15. This does not follow immediately from the perversity condition [21, (10.3.3)], since this condition implies the vanishing of $H^j_{\{z_0\}}(\mathcal{F})_{z_0}$ for $j < n$, but says a priori nothing about the $H^j_{\{z_0\}}(\mathcal{H}^k\mathcal{F})_{z_0}$ unless $j = 0$ and $k = 0$.

16. Let us recall that this spectral sequence is designed for left exact functors. To turn $R\mathcal{H}om(\ \ , \mathbb{C})$ into such a functor, one has to consider $\mathcal{H}om(\ \ , \mathbb{C})$ as a functor from the opposite category of sheaves on Z to the category of sheaves on Z. From this viewpoint, $\mathcal{F}$ is cohomologically concentrated in degrees ranging from $-n$ to 0.

for $k = 2, \dots, 2n-1$. As a result, $H^k_c(V, \mathcal{H}^0\mathcal{F}_{|V})$ injects in $H^k(\overline{V}, \mathcal{H}^0\mathcal{F}_{|\overline{V}})$ if $k = 2, \dots, 2n-1$. By the constructibility property of $\mathcal{H}^0\mathcal{F}$, the cohomology $H^k(\overline{V}, \mathcal{H}^0\mathcal{F}_{|\overline{V}})$ vanishes for $k > 0$ for V chosen small enough. So from (13.1.11), we deduce that for V small enough, we have

$$\Gamma(V, E_2^{p,0}) \simeq 0$$

for $p = 1, \dots, 2n-2$.

Hence the stalk of $E_2^{p,0}$ at z_0 is 0 for $k = 1, \dots, 2n-2$. Since $\mathcal{H}^0\mathcal{F}$ is constant away from z_0, the sheaf $E_2^{p,0} = R^p\mathcal{H}om(\mathcal{H}^0\mathcal{F}, \mathbb{C})$ is also 0 away from z_0, and lemma 13.1.10 is proved. □

From 13.1.10, we deduce that the only terms $E_2^{p,q}$ which could be a priori non zero in (13.1.9) are $E_2^{0,0}, E_2^{2n-1,0}$ and the $E_2^{2n,q}$ for $q = 0, \dots, -n$. We deduce from this that the only differential of (13.1.9) which could be a priori non zero is $d_k : E_k^{0,0} \longrightarrow E_k^{k,-k+1}$ with $k = 2n$. But $-k+1 = -2n+1 < -n$ because $n > 1$, so $E_k^{k,-k+1} \simeq 0$ and then $d_k : E_k^{0,0} \longrightarrow E_k^{k,-k+1}$ has to be 0. Thus, the spectral sequence (13.1.9) degenerates at sheet 2.

We deduce that for $q = 2, \dots, n-1$, the only contribution to $R^{2n-q}\mathcal{H}om(\mathcal{F}, \mathbb{C})$ on the sheet at infinity is $R^{2n}\mathcal{H}om(\mathcal{H}^q\mathcal{F}, \mathbb{C})$. Then for $q = 2, \dots, n-1$, we have

$$R^{2n}\mathcal{H}om(\mathcal{H}^q\mathcal{F}, \mathbb{C}) \simeq R^{2n-q}\mathcal{H}om(\mathcal{F}, \mathbb{C}) \simeq 0$$

Since the $\mathcal{H}^q\mathcal{F}$ are skyscraper sheaves for $q > 0$, we deduce $\mathcal{H}^q\mathcal{F} \simeq 0$ for $q = 2, \dots, n-1$.

On the other hand, the only contribution to $R^{2n-1}\mathcal{H}om(\mathcal{F}, \mathbb{C})$ on the sheet at infinity are $R^{2n}\mathcal{H}om(\mathcal{H}^1\mathcal{F}, \mathbb{C})$ and $R^{2n-1}\mathcal{H}om(\mathcal{H}^0\mathcal{F}, \mathbb{C})$. From

$$R^{2n-1}\mathcal{H}om(\mathcal{F}, \mathbb{C}) \simeq 0$$

we deduce

$$R^{2n}\mathcal{H}om(\mathcal{H}^1\mathcal{F}, \mathbb{C}) \simeq 0$$

so we also have $\mathcal{H}^1\mathcal{F} \simeq 0$.

Then we have proved that $\mathcal{H}^k\mathcal{F} \simeq 0$ for $k = 1, \dots, n-1$, so to conclude the proof of 13.1.6, we are left to prove the vanishing of the germ of $\mathcal{H}^n\mathcal{F}$ at z_0. Since we have

$$\dim \mathcal{H}^0\mathcal{F}_{z_0} = (-1)^{n+1} \dim \mathcal{H}^n\mathcal{F}_{z_0} + \dim \mathcal{H}^0\mathcal{F}_\eta$$

it is enough to show that $\mathcal{H}^0\mathcal{F}$ is a local system. For this, we remind [11, 5.1.19] that $\mathcal{F}$ can always be supposed to satisfy $\mathcal{F}^k = 0$ for $k < 0$ and $k > n$. Let us then consider the truncation

$$\tau_n(\mathcal{F}) := 0 \longrightarrow \mathcal{F}^0 \longrightarrow \cdots \longrightarrow \mathcal{F}^{n-1} \longrightarrow \operatorname{Im} d_{\mathcal{F},n-1} \longrightarrow 0$$

It is the kernel of an exact sequence of complexes

$$0 \longrightarrow \tau_n(\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow \mathcal{H}^n\mathcal{F}[-n] \longrightarrow 0 \tag{13.1.12}$$

and the canonical inclusion of complexes $\mathcal{H}^0\mathcal{F}[0] \longrightarrow \tau_n(\mathcal{F})$ is a quasi-isomorphism. Then, (13.1.12) gives rise to a distinguished triangle in $D^b_c(Z, \mathbb{C})$

$$\mathcal{H}^0\mathcal{F}[0] \longrightarrow \mathcal{F} \longrightarrow \mathcal{H}^n\mathcal{F}[-n] \xrightarrow{+1} \tag{13.1.13}$$

where the second and last complex are perverse. Hence $\mathcal{H}^0\mathcal{F}[0]$ is perverse and we are reduced to prove the following

Lemme 13.1.14. *Let $\mathcal{F}$ be a constructible sheaf on a polydisc D of $\mathbb{C}^n$ centered at the origine, with $n \geq 2$. Suppose that $\mathcal{F}$ is perverse and restricts to a local system on the complementary $j : D^* \hookrightarrow D$ of the origin. Then $\mathcal{F}$ is constant.*

Proof. Since $n \geq 2$, $\mathcal{F}_{|D^*}$ is constant. If we denote by m its rank, we have $j_*j^{-1}\mathcal{F} \simeq \mathbb{C}^m_D$. Since $\mathcal{F}$ is perverse, it has no sections supported at 0 and we deduce the following short exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathbb{C}^m_D \longrightarrow \mathcal{G} \longrightarrow 0 \tag{13.1.15}$$

with $\mathcal{G}$ supported at 0. From the perversity of $\mathcal{F}$ and $\mathbb{C}^m_D$, we get that $\mathcal{G}$ is perverse. Thus it has no non-zero sections supported at the origine, so it is the 0 sheaf, and 13.1.14 is proved. □

Remark 13.1.16. Theorem 13.1.6 can be easily obtained with the use of the Riemann-Hilbert correspondence [16, 7.2.1] combined with Kashiwara-Schapira's identification [21, 11.3.3] of the characteristic variety of an holonomic $\mathcal{D}$-module $\mathcal{M}$ with the micro-support of $\mathbf{S}(\mathcal{M})$. Both of these results are highly non-trivial, while the proof given here only relies on standard facts about perverse sheaves and is purely topological.

13.2 The Euler-Poincaré characteristic of $\mathrm{Irr}_Z(\mathcal{M})$

In this subsection, X, Z and $\mathcal{M}$ are supposed to be algebraic. As noted in the introduction, $\mathrm{Irr}^*_Z(\mathcal{M}) := \mathrm{Irr}^*_{Z^{\mathrm{an}}}(\mathcal{M}^{\mathrm{an}})$ is an analytic invariant. However, we have the following

Theorem 13.2.1. *The value of $\chi(\mathrm{Irr}^*_Z(\mathcal{M}))$ at $x \in Z^{\mathrm{an}}$ only depends on the formalization $\widehat{\mathcal{M}_x} := \widehat{\mathcal{O}_{X,x}} \otimes_{\mathcal{O}_{X,x}} \mathcal{M}_x$.*

Proof. By induction on the number of equation defining Z, the Mayer-Vietoris distinguished triangle [40, 4.2-1]

$$\mathrm{Irr}^*_{Z_1\cap Z_2}(\mathcal{M}) \longrightarrow \mathrm{Irr}^*_{Z_1}(\mathcal{M}) \oplus \mathrm{Irr}^*_{Z_2}(\mathcal{M}) \longrightarrow \mathrm{Irr}^*_{Z_1\cup Z_2}(\mathcal{M})$$

allows us to suppose that Z is an hypersurface. In that case, the localization functor is exact and is simply $\mathcal{O}_X(*Z)\otimes_{\mathcal{O}_X}$. Thus, one can suppose that $\mathcal{M} = \mathcal{M}(*Z)$.

From the hypercohomology spectral sequence

$$E_2^{pq} = \mathcal{H}^p\mathbf{S}(\mathcal{H}^q\mathcal{M}) \Longrightarrow \mathcal{H}^{p+q}\mathbf{S}(\mathcal{M}),$$

we get

$$\chi(\mathrm{Irr}^*_Z(\mathcal{M})) = \sum(-1)^k\chi(\mathrm{Irr}^*_Z(\mathcal{H}^k\mathcal{M})),$$

so it is enough to prove 13.2.1 in the case where $\mathcal{M}$ is an actual holonomic $\mathcal{D}_X$-module.

Let us write

$$\mathrm{Char}(\mathcal{M}) = \sum m_\alpha(\mathcal{M})\overline{T^*_{X_{\alpha,\mathrm{reg}}}X},$$

where $m_\alpha(\mathcal{M})$ is a positive integer, X_α is a closed irreducible subvariety of X and $X_{\alpha,\mathrm{reg}}$ its regular part. Then the local index theorem [18], [20], [35] asserts that

$$\chi(i^{-1}\mathbf{S}(\mathcal{M}^{\mathrm{an}}))(x) = \sum(-1)^{d_\alpha}e(x, X_\alpha)m_\alpha(\mathcal{M}),$$

where d_α denotes the codimension of X_α in X and $e(x, X_\alpha)$ stands for the Euler obstruction defined by Mac-Pherson [32]. This latter invariant is of topological nature and is equal to 0 in case $x \notin X_\alpha$. Then, if X_α is a closed irreducible subvariety of X containing x, we need to show the following

Lemma 13.2.2. *The integer $m_\alpha(\mathcal{M})$ only depends on $\widehat{\mathcal{M}_x}$.*

Let us first recall the definition of $m_\alpha(\mathcal{M})$. Let $F_\mathcal{M}$ be a good filtration for $\mathcal{M}$ defined in a neighbourhood of x and consider its graded module $\mathrm{Gr}_{F_\mathcal{M}}(\mathcal{M})$. If $p : T^*X \longrightarrow X$ denotes the canonical projection, $\mathrm{Gr}_{F_\mathcal{M}}(\mathcal{M})$ is a coherent sheaf of $p_*\mathcal{O}_{T^*X}$-modules on X. Thus, using the adjunction $\mathcal{O}_{T^*X} \longrightarrow p^*p_*\mathcal{O}_{T^*X}$, $p^*\,\mathrm{Gr}_{F_\mathcal{M}}(\mathcal{M})$ canonically defines a coherent sheaf on T^*X. Then if η_α denotes the generic point of $T^*_{X_{\alpha,\mathrm{reg}}}X$, $m_\alpha(\mathcal{M})$ is the length of the restriction of $p^*\,\mathrm{Gr}_{F_\mathcal{M}}(\mathcal{M})$ to $\mathcal{O}_{T^*X,\eta_\alpha}$.

By formally following this construction, one could define a notion of characteristic variety for coherent $\widehat{\mathcal{D}_{X,x}}$-modules as a cycle in the formalization $\widehat{T^*X}$ of T^*X along T^*_xX in such a way that $\widehat{F_{\mathcal{M},x}}$ defines a good $\widehat{\mathcal{D}_{X,x}}$-filtration for $\widehat{\mathcal{M}_x}$. Hence, $\mathrm{Char}(\widehat{\mathcal{M}_x})$ is the cycle associated to $\widehat{p}^*\,\mathrm{Gr}_{\widehat{F_{\mathcal{M},x}}}(\widehat{\mathcal{M}_x})$, with $\widehat{p}$ defined by the following cartesian diagram

$$\begin{tikzcd}
\widehat{T^*X} \arrow[r] \arrow[d, "\widehat{p}"'] & T^*X \arrow[d, "p"] \\
\operatorname{Spec}\widehat{\mathcal{O}_{X,x}} \arrow[r] & X
\end{tikzcd}$$

By faithfullness of $\widehat{\mathcal{O}_{X,x}}$ over $\mathcal{O}_{X,x}$, this is also the cycle associated to $\widehat{p}^*\widehat{\mathrm{Gr}_{F_\mathcal{M}}(\mathcal{M})_x}$.

Coming back to the proof of 13.2.2, if $\mathcal{N}$ is a holonomic $\mathcal{D}_X$-module satisfying $\widehat{\mathcal{M}_x} \simeq \widehat{\mathcal{N}_x}$, we deduce from the previous discussion that

$$\mathrm{Char}(\widehat{p}^*\,\widehat{\mathrm{Gr}_{F_\mathcal{M}}(\mathcal{M})_x}) = \mathrm{Char}(\widehat{p}^*\,\widehat{\mathrm{Gr}_{F_\mathcal{N}}(\mathcal{N})_x}). \tag{13.2.3}$$

Since X_α contains x, the going-down theorem [38, 9.5] applied in an affine neighbourhood of x for $(\overline{T^*_{X_{\alpha,\mathrm{reg}}}X}, x)$ asserts that one can find an irreducible variety Y_α of $\widehat{T^*X}$ dominating $\overline{T^*_{X_{\alpha,\mathrm{reg}}}X}$. If one denotes by η_{Y_α} its generic point, the flatness of the morphism of local rings $\mathcal{O}_{T^*X,\eta_\alpha} \longrightarrow \mathcal{O}_{\widehat{T^*X},\eta_{Y_\alpha}}$ implies the following equality [17]

$$m_\alpha(\mathcal{M}) = \mathrm{length}_{\eta_\alpha}(p^*\,\mathrm{Gr}_{F_\mathcal{M}}(\mathcal{M})) = \mathrm{length}_{\eta_{Y_\alpha}}(\widehat{p}^*\widehat{\mathrm{Gr}_{F_\mathcal{M}}(\mathcal{M})_x})$$

and 13.2.2 results from the equality of multiplicies along Y_α coming from (13.2.3). □

13.3 The notion of good semi-stable points

In this section X is algebraic, Z is a smooth hypersurface of X with function field K and $\mathcal{M}$ is a connection on X with meromorphic poles along Z. Since the notion of semi-stable point is local, take X to be affine, Z to be given by a regular section z, and denote by A the function ring of Z. Let us take K' and e as in (0.0.1).

17. If A is a commutative ring, M a finitely generated A-module and if $P \in \operatorname{Spec} A$, we denote $\mathrm{length}_P(M)$ for $\mathrm{length}(A_P \otimes_A M)$.

Definition 13.3.1. One says that a closed point P of Z is a *good semi-stable point* if it satisfies the following conditions:

1. the normalization A' of A in K' is etale above P.
2. the coefficients of lowest degree of the non zero ϕ's occuring in (1.0.3) are units in the semi-local ring A'_P.
3. the decomposition (0.0.1) descends on $A'_P((z^{1/e}))$.

Remark 13.3.2. The notion of semi-stable point appears in [5, 3.2.4]. The terminology good here is meant to express the unit condition, with reference to the notion of good formal decomposition appearing in [46]. As for the etaleness condition, it is automatic at a stable point [5, 3.4.1 2)].

As a preleminary step to the proof of 12.2.7, let us consider the simplest situation

Lemma 13.3.3. *Let $P \in Z$, $z = 0$ a local defining equation for Z and ϕ a regular function defined in a neighbourhood of P which does not vanish at P. Let $\mathcal{R}$ be an algebraic connexion on X with regular singularities along Z. For every $k \geq 0$, one defines $\mathcal{M}_{k,\phi,\mathcal{R}} = \mathcal{E}^{\phi/z^k} \otimes \mathcal{R}$. Then $\mathrm{Irr}^*_Z(\mathcal{M}_{k,\phi,\mathcal{R}})$ is a local system of rank k on Z^{an} in a neighbourhood of P.*

Proof. Since ϕ does not vanish at P, by the change of variable $z' = z/\sqrt[k]{\phi}$, one can suppose that $\phi = 1$. Since every regular singular connexion is locally a successive extension of rank one regular singular connexion [9], the exactness of Irr^*_Z allows us to reduce the problem to the case where $\mathcal{R}$ has rank 1. In that case, one can see that the characteristic cycle of $\mathcal{M}_{k,1,\mathcal{R}}$ is contained in the union of $T^*_Z X$ and $T^*_X X$, so every curve passing through a point Q closed enough to P and transversed to Z is non-characteristic for $\mathcal{M}_{k,1,\mathcal{R}}$. Let $f : C \hookrightarrow X$ be such a curve. By Cauchy-Kovalevska theorem [16, 4.3.2], the canonical morphism $f^{-1}\mathbf{S}(\mathcal{M}_{k,1,\mathcal{R}}) \longrightarrow \mathbf{S}(f^+\mathcal{M}_{k,1,\mathcal{R}})$ is an isomorphism. Hence, the germ of $\mathrm{Irr}^*_Z(\mathcal{M}_{k,1,\mathcal{R}})$ at Q is that of $\mathrm{Irr}^*_Z(f^+\mathcal{M}_{k,1,\mathcal{R}})$ at Q, so the complex $\mathrm{Irr}^*_Z(\mathcal{M}_{k,1,\mathcal{R}})_Q$ is concentrated in degree 0, and from [6, 3.3.6], we obtain that its 0^{th}-cohomology has dimension k. □

14 The proof of 12.2.7

Consider X, Z, $\mathcal{M}$ as in the introduction. Let $P \in Z$ be a good semi-stable point for $\mathcal{M}$. We start with the following reduction

Lemma 14.0.4. *If the theorem 12.2.7 is true in the case where $K' = K$ and $e = 1$, then it is true in general.*

Proof. Choose a local analytic chart $U = V \times D(0,1)$ of X^{an} with coordinates (t,z) centered at P into which Z^{an} is locally given by $z = 0$. Denote by $p : V' \longrightarrow V$ the normalization of V in K'. By the very definition of good semi stable point, we can suppose by shrinking V enough that p is etale trivial with group G. The map

$$\begin{array}{rcl} \pi : V' \times D(0,1) & \longrightarrow & U \\ (t, z') & \longrightarrow & (p(t), z'^e) \end{array}$$

is proper. Since the canonical morphism

$$\mathcal{M} \longrightarrow \pi_+\pi^+\mathcal{M}(*Z^{\mathrm{an}})$$

induced by the adjunction map identifies $\mathcal{M}$ with a direct factor of $\pi_+\pi^+\mathcal{M}(*Z^{\mathrm{an}})$, namely the invariants under $G \times \mathbb{Z}/e\mathbb{Z}$, it is enough to prove that

$$\mathrm{Irr}_Z(\pi_+\pi^+\mathcal{M}(*Z^{\mathrm{an}})) = \mathrm{Irr}_Z(\pi_+\pi^+\mathcal{M})$$

is a local system. The compatibility of Irr_Z with proper morphisms [40, 3.6-6] gives (in a neighbourhood of P) a canonical isomorphism

$$\mathrm{Irr}_Z(\pi_+\pi^+\mathcal{M}) \xrightarrow{\sim} R\pi_* \mathrm{Irr}_{(z'=0)}(\pi^+\mathcal{M}).$$

Since π is finite, the functor π_* is exact. Then,

$$\mathcal{H}^k \mathrm{Irr}_Z(\pi_+\pi^+\mathcal{M}) \simeq \pi_*\mathcal{H}^k \mathrm{Irr}_{(z'=0)}(\pi^+\mathcal{M})$$

is the zero sheaf for $k \geq 1$, and is a local system for $k = 0$. □

This reduction being done, theorem 13.2.1 implies that in a neighbourhood of P, the function $\chi(\mathrm{Irr}^*_Z(\mathcal{M}))$ is a sum of Euler-Poincaré characteristic coming from modules of the form appearing in 13.3.3. Hence, it is constant in a neighbourhood of P. Since $\mathrm{Irr}^*_Z(\mathcal{M})$ is perverse, the theorem 12.2.7 follows from 13.1.6.

15 Some thoughts about the converse of 12.2.8

The goal of this subsection is to discuss the following

Conjecture 15.0.5. *The intersection of the smooth locus of* $\mathrm{Irr}^*_Z(\mathcal{M})$ *and* $\mathrm{Irr}^*_Z(\mathrm{End}\,\mathcal{M})$ *is a subset of the stable point locus of* $\mathcal{M}$.

In the sequel, let us fix once for all a point P in the smooth locus of $\mathrm{Irr}^*_Z(\mathcal{M})$ and $\mathrm{Irr}^*_Z(\mathrm{End}\,\mathcal{M})$. We recall that η stands for the generic point of Z and that $\mathrm{Irr}^*_Z(\mathcal{M})$ is endowed with a locally finite increasing $\mathbb{Q}_{\geq 1}$-filtration by perverse sheaves $\mathrm{Irr}^*_Z(\mathcal{M})\{r\}$ [39, 6.3.3]. We will denote by $\mathrm{Gr}^r \mathrm{Irr}^*_Z(\mathcal{M})$ its r^{th}-graded piece. Since we have

$$\mathrm{Char}(\mathrm{Irr}^*_Z(\mathcal{M})) = \sum \mathrm{Char}(\mathrm{Gr}^r \mathrm{Irr}^*_Z(\mathcal{M})),$$

the characteristic cycle of $\mathrm{Gr}^r \mathrm{Irr}^*_Z(\mathcal{M})$ is a multiple of $T_{Z^{\mathrm{an}}}Z^{\mathrm{an}}$ in a neighbourhood of P. By 13.1.6, $\mathrm{Gr}^r \mathrm{Irr}^*_Z(\mathcal{M})$ is a local system in a neighbourhood of P. The same holds for $\mathrm{Gr}^r \mathrm{Irr}^*_Z(\mathrm{End}\,\mathcal{M})$.

15.1 André's criterion and micro-caractericity

To establish 15.0.5, one could try to apply André's criterion [5, 3.4.1] for stable points. As a consequence of *loc. cit.*, $P \in Z$ is stable if for every germ of analytic curve $i : C \hookrightarrow X$ cutting Z transversally at P, the Newton polygons[18] $\mathrm{NP}(\mathcal{M}_C)$ and $\mathrm{NP}(\mathrm{End}(\mathcal{M}_C))$ of $i^+\mathcal{M}$ and $i^+\,\mathrm{End}(\mathcal{M})$ are respectively $\mathrm{NP}(\mathcal{M}_\eta)$ and $\mathrm{NP}(\mathrm{End}(\mathcal{M}_\eta))$.

By the work of Ramis [45], the height of the segment of slope $1/(r-1)$ of $\mathrm{NP}(\mathcal{M}_C)$ is equal to the dimension of the r^{th}-graded piece of the irregularity space $\mathrm{Irr}^*_P(\mathcal{M}_C)$. In the same way, the height of the segment of slope $1/(r-1)$ of $\mathrm{NP}(\mathcal{M}_\eta)$ is equal to the dimension of the r^{th}-graded piece of the irregularity space $\mathrm{Irr}^*_{C_{\mathrm{gen}}\cap Z}(\mathcal{M}_{C_{\mathrm{gen}}})$ where C_{gen} is a sufficiently generic germ of curve transverse to Z.

Since in a neighbourhood of P, $\mathrm{Gr}^r \mathrm{Irr}^*_Z(\mathcal{M})$ and $\mathrm{Gr}^r \mathrm{Irr}^*_Z(\mathrm{End}\,\mathcal{M})$ are local systems, one can apply André's criterion if one proves that the formation of $\mathrm{Irr}^*_Z(\mathcal{M})$ and $\mathrm{Irr}^*_Z(\mathrm{End}\,\mathcal{M})$ as $\mathbb{Q}_{\geq 1}$-filtered perverse sheaves commutes with the restriction to hypersurfaces transverse to Z and passing through P, that is to say that in a neighbourhood of P, we are in a non (r)-micro characteristic situation for every $r \geq 1$ in the sense of [31].

As pointed out to me by C. Sabbah, this last condition does not follow from the smoothness of the graded pieces of $\mathrm{Irr}^*_Z(\mathcal{M})$ and $\mathrm{Irr}^*_Z(\mathrm{End}\,\mathcal{M})$ in a neighbourhood of P. Indeed, the work of Laurent and Mebkhout [30] expresses $\mathrm{Char}(\mathrm{Gr}^r \mathrm{Irr}^*_Z(\mathcal{M}))$ as an alternate sum of cycles $C_1^+(r), C_2^+(r), C_1^-(r), C_2^-(r)$ of T^*Z canonically defined from the r-micro characteristic cycle $\mathrm{Char}_r(\mathcal{M})$ of $\mathcal{M}$. The (r)-micro characteristic condition has to do with the union of the support of the $C^{\pm}_{1,2}(r)$. This support can be a priori obscenely complicated, and still the $C^{\pm}_{1,2}(r)$ can give rise through various miraculous cancellations to the most simple cycle for $\mathrm{Gr}^r \mathrm{Irr}^*_Z(\mathcal{M})$.

15.2 Reduction to the two-dimensional case

Let us suppose that Z is given by the equation $z = 0$, and let us denote by $\kappa(P)$ the residue field of P. Then $\kappa(P)[\![z]\!]$ is the ring of functions of a formal curve C_P on X passing through P and transverse to Z. We will note $\mathcal{M}_P$ for $\mathcal{M}_{C_P}$ and following André [5, 3.4.1], we recall that to get the stability of P, it is enough to check the preservation of the generic Newton Polygons of $\mathcal{M}$ and $\mathrm{End}(\mathcal{M})$ by specialization to C_P.

Let us reduce the proof of 15.0.5 to the case where X is two-dimensional[19]. We proceed by induction on the dimension of X and take X to be of dimension ≥ 3 while supposing that 15.0.5 is true in dimension $< \dim X$.

The smoothness assumption on $\mathrm{Irr}^*_Z(\mathcal{M})$ implies that the characteristic variety of $\mathcal{M}$ in a neighbourhood of P is contained in the union of $T^*_Z X$ and $T^*_X X$. Then, any algebraic hypersurface $i : X' \hookrightarrow X$ passing through P and transverse to Z is non characteristic for $\mathcal{M}$ in a neighbourhood of P. Thus by Cauchy-Kovalevska theorem [16, 4.3.2], the canonical morphism $i^{-1}\mathbf{S}(\mathcal{M}) \longrightarrow \mathbf{S}(i^+\mathcal{M})$ is an isomorphism and the same holds for $\mathrm{End}\,\mathcal{M}$, so $\mathrm{Irr}^*_Z(i^+\mathcal{M})$ and $\mathrm{Irr}^*_Z(i^+\,\mathrm{End}\,\mathcal{M})$ are smooth in a neighbourhood of P. From the induction hypothesis applied to $(X', Z', i^+\mathcal{M})$ at P with $Z' := X' \cap Z$, we deduce that P is a stable point of Z' for $i^+\mathcal{M}$. Then, if η' denotes the generic point of Z', we have

$$\mathrm{NP}(\mathcal{M}_P) = \mathrm{NP}(\mathcal{M}_{\eta'}) \quad \text{and} \quad \mathrm{NP}(\mathrm{End}(\mathcal{M}_P)) = \mathrm{NP}(\mathrm{End}(\mathcal{M}_{\eta'})). \tag{15.2.1}$$

18. We will follow André's convention according to which the Newton polygon of a differential module M has $(\mathrm{rk}\,M, 0)$ for higher vertice.

19. In that case, one may hope that the multiplicity of $T^*_P Z$ in $C_1^+(r) + C_2^+(r) + C_1^-(r) + C_2^-(r)$ is computable.

Since $\dim X' \geq 2$, we have $\dim Z' \geq 1$ and so one can deform X' if necessary so that η' is stable for $\mathcal{M}$. In that case, we have

$$\mathrm{NP}(\mathcal{M}_{\eta'}) = \mathrm{NP}(\mathcal{M}_{\eta}) \quad \text{and} \quad \mathrm{NP}(\mathrm{End}(\mathcal{M}_{\eta'})) = \mathrm{NP}(\mathrm{End}(\mathcal{M}_{\eta})) \tag{15.2.2}$$

From André's criterion [5, 3.4.1], the combination of (15.2.1) and (15.2.2) implies the stability of P for $\mathcal{M}$.

15.3 An unconditional consequence of 15.0.5

If 15.0.5 was to hold, we would have $\mathrm{Gr}^r \mathrm{Irr}^*_Z(\mathcal{M}) \simeq 0$ in a neighbourhood of P for every rationnal r such that $1/(r-1)$ is not a slope for $\mathrm{NP}(\mathcal{M}_\eta)$. This can be proved unconditionaly. Suppose that $1/(r-1)$ is not a slope for $\mathrm{NP}(\mathcal{M}_\eta)$. Then $\mathrm{Gr}^r \mathrm{Irr}^*_Z(\mathcal{M})$ is generically 0 and since it is perverse, it is concentrated in degrees ranging from 1 to $\dim Z$. Since in a neighbourhood of P, it is a subsheaf of an actual sheaf placed in degree 0, it has no other choice than to be the 0 sheaf.

15.4 A simple case

Let us suppose here that only one ϕ occurs in (0.0.1) and let us prove that P is a stable point. This situation is easier to handle, and this for two reasons:

1. In the one slope case, Cauchy-Kovalevska compatibility theorem between the solution functor and non-characteristic immersions will be enough for our purpose. We will not have to check if the (r)-micro characteristic condition is fulfilled.
2. The meromorphic connection $\mathrm{End}\,\mathcal{M}$ is generically regular singular along Z, so by a theorem of Deligne [9, 4.1], it is regular singular along Z and then $\mathrm{Irr}^*_Z(\mathrm{End}\,\mathcal{M}) \simeq 0$. Thus, $\mathrm{End}\,\mathcal{M}$ plays no role in that case.

The smoothness assumption on $\mathrm{Irr}^*_Z(\mathcal{M})$ implies that the characteristic variety of $\mathcal{M}$ in a neighbourhood of P is contained in the union of $T^*_Z X$and $T^*_X X$, so every curve passing through P and transversed to Z is non-characteristic for $\mathcal{M}$. Let $f : C \hookrightarrow X$ be such a curve. By Cauchy-Kovalevska theorem [16, 4.3.2], the canonical morphism $f^{-1}\mathbf{S}(\mathcal{M}) \longrightarrow \mathbf{S}(f^+\mathcal{M})$ is an isomorphism. As a result, the generic irregularity of $\mathcal{M}$ is that of $\mathcal{M}_C$. Hence by a theorem of Malgrange [36], $\mathrm{NP}(\mathcal{M}_\eta)$ and $\mathrm{NP}(\mathcal{M}_C)$ have the same height. By a general result of André [5, A.1], we have the following inclusion

$$\mathrm{NP}(\mathcal{M}_C) \subset \mathrm{NP}(\mathcal{M}_\eta). \tag{15.4.1}$$

Since $\mathrm{NP}(\mathcal{M}_\eta)$ has only one edge, (15.4.1) is an equality and André's theorem [5, 3.4.1] applies. □