Programmes Matlab

Méthode de Kozlov-Maz'ya

```
%(* ::Package:: *)

function [er2]= kozlovmazya(N,epsi,it,c1)
N=0;epsi=10^(-3);it=4;c1=0.6;
x=linspace(0,1,N+1);
h=1/(N+1);
f1=sin(pi*x');
g=(1/pi^(4))*(1-(1+pi^(2))*exp(-pi^(2)))*(sin(pi*x'));
gn=g+epsi*randn(size(g));
g=gn;
A=(1/h^2)*(diag(2*ones((N-1),1))-diag(ones(N-2,1),1)-diag(ones(N-2,1),-1));
K=A^(-2)*(eye(N-1)-((eye(N-1)+A)*expm(-(A))));
c=4*((N+1)*(sin(pi/(2*(N+1)))))^2;
gamma=c1*(c^2/(1 -(1+c)*exp(-c)));
gamma,
f(:,1)=zeros(N+1,1);
for k=2:it
    F=(eye(N-1)-gamma*K)*f(2:N,k-1);
    f(:,k)=[0;F;0]+gamma*g;
end
er2=(norm(f1-f(:,it),2))/norm(f1,2);
er2,
plot(x,f1,'r',x,f(:,it),'b*--')
grid
legend('sol. approch\[EAcute]e', 'sol. exacte'),
```

Méthode de Troncature spectrale

```
function [er1, er2,fa1,fa2 ] = Methode_troncature(M,N,epsi,f1,f2,g1,g2)
M=20;
N=4;
epsi=10^-2;h=1/M;
x=0:h:1;
f1=(pi^2)*sin(pi*x);
f2=(pi^2)*sin(2*pi*x);
g1=(1/3)*(2-3.*exp(-2.*pi^2)).*sin(pi.*x)+(1/12).*(-1+3.*exp(-8.*pi^2)-2.*exp(-12.*pi^2)).*sin
g2=(1/6)*(1-3*exp(-2*pi^2)+2*exp(-3*pi^2))*sin(pi*x)+(1/24)*(1+3*exp(-8*pi^2)-4*exp(-12*pi^2))
g1n=g1+epsi*randn(size(g1));
g2n=g2+epsi*randn(size(g2));
g1=g1n;
g2=g2n;
S2=0;
for  n=1:N
a2n= (2*(n^2)*(pi^2))/ (1-exp(-2*(n^2)*(pi^2)));
a3n=(3*(n^2)*(pi^2))/(1-exp(-3*(n^2)*(pi^2)));
S1=g1.*sin(n*pi*x);
S1=sum(S1);
S2=S2+(2*a2n-a3n)*S1*sin(n*pi*x);
end
S4=0;
for  n=1:N
a2n=(2*(n^2)*(pi^2))/ (1-exp(-2*(n^2)*(pi^2)));
a3n=(3*(n^2)*(pi^2))/(1-exp(-3*(n^2)*(pi^2)));
S3=g2.*sin(n*pi*x);
S3=sum(S3);
S4=S4+((-2)*a2n+(2)*a3n)*S3*sin(n*pi*x);
end
fa1=2*h*(S2+S4);
er1=(norm(f1-fa1,2))/norm(f1,2);
l2=0;
for  n=1:N;
a2n= (2*(n^2)*(pi^2))/ (1-exp(-2*(n^2)*(pi^2)));
a3n=(3*(n^2)*(pi^2))/(1-exp(-3*(n^2)*(pi^2)));
l1=g1.*sin(n*pi*x);
l1=sum(l1);
l2=l2+(a2n-a3n)*l1*sin(n*pi*x);
end
l4=0;
for  n=1:N;
a2n=(2*(n^2)*(pi^2))/ (1-exp(-2*(n^2)*(pi^2)));
a3n=(3*(n^2)*(pi^2))/(1-exp(-3*(n^2)*(pi^2)));
l3=g2.*sin(n*pi*x);
l3=sum(l3);
l4=l4+((-1)*a2n+(2)*a3n)*l3*sin(n*pi*x);
end
```

```
fa2=2*h*(l2+l4);
er2=(norm(f2-fa2,2))/norm(f2,2)
subplot (1,2,1);
plot (x,f1,'b',x, fa1,'r*-');grid;
title('f1(x) et son approximation');
legend('sol.exacte', 'sol.approchée');
subplot (1,2,2);
plot (x,f2,'b', x, fa2,'r*-');grid;
title('f2(x) et son approximation');
legend('sol. exacte', ' sol.approchée' );
end
```

Exemple2

```
function  y=funct_cre1(t)
if t>=0 && t<=1/4
    y=0;
else if t>=1/4 && t<=1/2
    y=((4)*t-1);
    else if t>=1/2 && t<=3/4
y=(3-(4)*t);
        else
            y=0;
end

    end
end
```

```
function  y=funct_cre2(t)
if t>=0 && t<=1/5
    y=0;
else if t>=1/5 && t<=2/5
    y=5*t-1;
    else if t>=2/5 && t<=3/5
y=-5*t+3;
        else
            y=0;
end

    end
end
```

Exemple3

```
%function  y=functdisc1(t)
t=0:0.2:1;
if  (t>=0 &t<=1/3);
    y=0;
end
else
    if (t>=1/3 & t<=2/3);
    y=1;
    end
    else
if (t>=2/3)
            y=0;
end
```

```
function  y=functdisc2(t)
if  t>=0 && t<=1/4
    y=0;
else if t>=1/4 && t<=2/4
    y=1;
    else
            y=0;
end

    end
end
```

Conclusion et perspectives

Dans le présent travail on a traité trois problèmes inverses, en utilisant différentes méthodes de régularisation.
Dans l'étude du premièr problème, on a utilisé une méthode itérative basée sur l'algorithme de Kozlov-Mazia pour identifier le terme source dans une équation différentielle du second ordre. Dans le deuxième problème on a déterminé la condition initiale pour un sytème de diffusion en utilisant la méthode des valeurs aux limites auxiliaires, dans le troixième on a identifié le terme source pour le même système, où la méhode de troncature a été introduite pour la construction d'une solution régularisée.
Dans ces études des résultats de convergence ont été établis et des estimations d'erreur ont été obtenus en vertu d'une estimation a priori de la solution exacte . Certains tests numériques ont été illustrés pour vérifier la validité de chaque méthode proposée.

Comme perspectives on se propose de démontrer des inégalités de Carleman globales pour un système linéaire composé de deux équations paraboliques couplées avec des coefficients de diffusion non réguliers dans un domaine borné de $\mathbb{R}^2$. Ces inégalités seront utilisées pour démontrer des résultats de stabilité et d'unicité pour cetrains problèmes inverses.

Publications

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 713403, 9 pages
http://dx.doi.org/10.1155/2015/713403

Research Article

An Iterative Regularization Method for Identifying the Source Term in a Second Order Differential Equation

Fairouz Zouyed and Sebti Djemoui

Applied Math Lab, University Badji Mokhtar Annaba, P.O. Box. 12, 23000 Annaba, Algeria

Correspondence should be addressed to Fairouz Zouyed; fzouyed@gmail.com

Received 28 May 2015; Accepted 21 September 2015

Academic Editor: Peter Dabnichki

Copyright © 2015 F. Zouyed and S. Djemoui. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper discusses the inverse problem of determining an unknown source in a second order differential equation from measured final data. This problem is ill-posed; that is, the solution (if it exists) does not depend continuously on the data. In order to solve the considered problem, an iterative method is proposed. Using this method a regularized solution is constructed and an a priori error estimate between the exact solution and its regularized approximation is obtained. Moreover, numerical results are presented to illustrate the accuracy and efficiency of this method.

1. Introduction

Let H be a separable Hilbert space with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$. Consider the problem of finding the source term $f \in H$ in the following system:

$$\begin{aligned} u''(t) + 2Au'(t) + A^2u(t) &= f, \quad 0 < t < T, \\ u(0) &= 0, \\ u'(0) &= 0, \end{aligned} \tag{1}$$

with the additional data

$$u(T) = g, \tag{2}$$

where $A : D(A) \subset H \to H$ is a positive self-adjoint linear operator with a compact resolvent; we denote by $\sigma(A)$ the spectrum of the operator A.

The problem (1) is an abstract version of the system

$$\begin{aligned} u_{tt}(x,t) - 2\Delta u_t(x,t) + \Delta^2 u(x,t) &= f(x), \\ & 0 < t < T,\ x \in \Omega, \\ u(x,t) = \Delta u(x,t) &= 0, \\ & 0 \le t \le T,\ x \in \partial\Omega, \\ u(x,0) = u_t(x,0) &= 0, \quad x \in \Omega, \end{aligned} \tag{3}$$

which arises in the mathematical study of structural damped vibrations of string or a beam [1–3]. Also this problem can be considered as a biparabolic problem in the abstract setting. For physical motivation we cite the biparabolic model proposed in [4] for more adequate mathematical description of heat and diffusion processes than the classical heat equation. For other models we refer the reader to [5–7].

For most classical partial differential equations, the reconstruction of source functions from the final data or a partial boundary data is an inverse problem with many applications in several branches of sciences and engineering, such as geophysical prospecting and pollutant detection [8–12].

The main difficulty of inverse source identification problems is that they are ill-posed, that is, even if a solution exists, it does not depend continuously on the data; in other words, small error in the data measurement can induce enormous error to the solution. Thus, special regularization methods that restore the stability with respect to measurements errors are needed. In the present work, we focus on an iterative method proposed by Kozlov and Maz'ya [13, 14] for solving the problem; it is based on solving a sequence of well-posed boundary value problems such that the sequence of solutions converges to the solution for the original problem. It has been successfully used for solving various classes of ill-posed elliptic, parabolic, and hyperbolic problems [5, 15–21].

We note that although the interest in inverse problem has rapidly increased during this decade, the literature devoted to the class of problems (1) is quite scarce.

The paper is organized as follows. Section 2 gives some tools which are useful for this study; in Section 3 we introduce some basic results and we show the ill-posedness of the inverse problem; Section 4 gives a regularization solution and error estimation between the approximate solution and the exact one; the numerical implementation is described in Section 5 to illustrate the accuracy and efficiency of this method.

2. Preliminaries

Let $(\varphi_n)_{n\geq 1} \subset H$ be an orthonormal eigenbasis corresponding to the eigenvalues $(\lambda_n)_{n\geq 1}$ such that

$$\begin{gathered} A\varphi_n = \lambda_n \varphi_n, \quad n \in \mathbb{N}^*, \\ 0 < \lambda_1 \leq \lambda_2 \cdots \leq \cdots, \quad \lim_{n\to\infty} \lambda_n = +\infty. \\ \xi = \sum_{n=1}^{\infty} E_n \xi, \\ E_n \xi = (\xi, \varphi_n)\,\varphi_n, \quad \forall \xi \in H. \end{gathered} \tag{4}$$

We denote by $\{T(t) = e^{-tA}\}_{t\geq 0}$ the analytic semigroup generated by $-A$ on H,

$$T(t)\,\xi = \sum_{n=1}^{\infty} e^{-\lambda_n t} E_n \xi, \quad \forall \xi \in H. \tag{5}$$

For $\alpha > 0$, the space H^α is given by

$$H^\alpha = \left\{ \xi \in H : \sum_{n=1}^{\infty} \left(1 + \lambda_n^2\right)^\alpha \|E_n \xi\|^2 < \infty \right\}, \tag{6}$$

with the norm

$$\|\xi\|_{H^\alpha} = \left(\sum_{n=1}^{\infty} \left(1 + \lambda_n^2\right)^\alpha \|E_n \xi\|^2 \right)^{1/2}, \quad \xi \in H^\alpha. \tag{7}$$

We achieve this section by a result concerning nonexpansive operators.

Definition 1. A linear bounded operator $L : H \to H$ is called nonexpansive if $\|L\| \leq 1$.

Let L be an nonexpansive operator; to solve the equation

$$(I - L)\,\varphi = \psi, \tag{8}$$

we state a convergence theorem for a successive approximation method.

Theorem 2 (see [22], p. 66). *Let L be a nonexpansive, self-adjoint positive operator on H. Let $\psi \in H$ be such that (8) has a solution. If 1 is not eigenvalue of L, then the successive approximations*

$$\varphi_{n+1} = L\varphi_n + \psi, \quad n = 0, 1, 2, \ldots \tag{9}$$

converge to a solution to (8) for any initial data $\varphi_0 \in H$. Moreover, $L^n \varphi \to 0$ for every $\varphi \in H$, as $n \to \infty$.

3. Basic Results

3.1. The Direct Problem. Let $Z = D(A) \times H$ with the norm $\|U\|_Z^2 = \|A\xi_1\|^2 + \|\xi_2\|^2$, $U = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in Z$.

For a given $f \in H$, consider the direct problem

$$\begin{gathered} w''(t) + 2Aw'(t) + A^2 w(t) = f, \quad 0 < t < T, \\ w(0) = 0, \\ w'(0) = 0. \end{gathered} \tag{10}$$

Making the change of variable $w' = v$, we can write the second order equation in (10) as a first order system in the space Z as follows:

$$\begin{aligned} &z'(t) = \mathcal{A} z(t) + F, \quad 0 < t < T, \\ &z(0) = 0, \end{aligned} \tag{11}$$

where $z = \begin{pmatrix} w \\ v \end{pmatrix}$, $F = \begin{pmatrix} 0 \\ f \end{pmatrix}$, and $\mathcal{A} = \begin{pmatrix} 0 & I \\ -A^2 & -2A \end{pmatrix}$.

The linear operator $\mathcal{A}$ is unbounded with the domain $D(\mathcal{A}) = D(A^2) \times D(A)$ and it is the infinitesimal generator of strongly continuous semigroup $\{S(t) = e^{t\mathcal{A}}\}_{t\geq 0}$. Moreover $\{S(t)\}_{t\geq 0}$ is analytic (see [1]) and it admits the following explicit form:

$$S(t)\,U = \sum_{n=1}^{\infty} e^{tB_n} P_n U, \quad U = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in Z, \tag{12}$$

where $B_n = \begin{pmatrix} 0 & 1 \\ -\lambda_n^2 & -2\lambda_n \end{pmatrix}$ and $\{P_n\}_{n\geq 1}$ is a complete family of orthogonal projections in Z given by $P_n = \operatorname{diag}(E_n, E_n)$.

Using matrix algebra, we obtain

$$e^{tB_n} = \begin{pmatrix} e^{-\lambda_n t} + \lambda_n t e^{-\lambda_n t} & t e^{-\lambda_n t} \\ -\lambda_n^2 t e^{-\lambda_n t} & -\lambda_n t e^{-\lambda_n t} + e^{-\lambda_n t} \end{pmatrix}. \tag{13}$$

From the semigroup theory (see [23]), the problem (11) admits a unique solution $z \in C([0, T), Z)$ given by

$$z = \int_0^t S(t - s)\,F\,ds. \tag{14}$$

Hence,

$$\begin{aligned} z &= \int_0^t \sum_{n=1}^{\infty} e^{(t-s)B_n} P_n F\,ds \\ &= \int_0^t \sum_{n=1}^{\infty} \begin{pmatrix} \sigma_n^1(t,s) & \sigma_n^2(t,s) \\ \sigma_n^3(t,s) & \sigma_n^4(t,s) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ (f, \varphi_n)\,\varphi_n \end{pmatrix} ds, \end{aligned} \tag{15}$$

such that

$$\begin{aligned} \sigma_n^1(t,s) &= e^{-\lambda_n(t-s)} + \lambda_n (t-s)\, e^{-\lambda_n(t-s)}, \\ \sigma_n^2(t,s) &= (t-s)\, e^{-\lambda_n(t-s)}, \\ \sigma_n^3(t,s) &= -\lambda_n^2 (t-s)\, e^{-\lambda_n(t-s)}, \\ \sigma_n^4(t,s) &= -\lambda_n (t-s)\, e^{-\lambda_n(t-s)} + e^{-\lambda_n(t-s)}. \end{aligned} \tag{16}$$

As a consequence, we obtain the following theorem.

Theorem 3. *The problem (10) admits a unique solution $w \in C([0,T), D(A)) \cap C^1([0,T), H)$ given by*

$$w(t) = K(t) f = A^{-2}\left(I - (I + tA) e^{-tA}\right) f = \sum_{n=1}^{\infty} \frac{\left(1 - (1 + t\lambda_n) e^{-t\lambda_n}\right)}{\lambda_n^2} (f, \varphi_n) \varphi_n. \tag{17}$$

3.2. Ill-Posedness of the Inverse Problem. Now, we wish to solve the inverse problem, that is, find the source term f in the system (1). Making use of the supplementary condition (2) and defining the operator $K(T) : f \to g$, we have

$$g = u(T) = K(T) f = \sum_{n=1}^{\infty} \sigma_n E_n f, \tag{18}$$

where $\sigma_n = (1 - (1 + T\lambda_n)e^{-T\lambda_n})/\lambda_n^2$.

It is easy to see that $K(T)$ is a self-adjoint compact linear operator. On the other hand,

$$g = \sum_{n=1}^{\infty} E_n g = \sum_{n=1}^{\infty} \sigma_n E_n f, \tag{19}$$

so

$$\sigma_n E_n f = E_n g, \tag{20}$$

which implies

$$E_n f = \frac{1}{\sigma_n} E_n g, \tag{21}$$

and therefore

$$f = K(T)^{-1} g = \sum_{n=1}^{\infty} \frac{1}{\sigma_n} E_n g. \tag{22}$$

Note that $1/\sigma_n \to \infty$ as $n \to \infty$, so the inverse problem is ill-posed; that is, the solution does not depend continuously on the given data. Hence this problem cannot be solved by using classical numerical methods.

Remark 4. As many boundary inverse value problems for partial differential equations which are ill-posed, the study of the problem (1) is reduced to the study of the equation $K(T)f = g$, where $K(T)$ is a compact self-adjoint operator in the Hilbert space H. This equation can be rewritten in the following way:

$$f = (I - \gamma K(T)) f + \gamma g = Lf + \gamma g, \tag{23}$$

where γ is a positive number satisfying $\gamma < 1/\|K(T)\|$.

In the next section, we will show that the operator L is nonexpansive and 1 is not eigenvalue of L, so it follows from Theorem 2 that $(f_n)_{n\in\mathbb{N}^*}$ converges and $(I - \gamma K(T))^n f \to 0$, for every $f \in H$, as $n \to \infty$.

4. Iterative Procedure and Convergence Results

The alternating iterative method is based on reducing the ill-posed problem (1) to a sequence of well-posed boundary value problems and consists of the following steps.

First, we start by letting $f_0 \in H$ be arbitrary; the initial approximation u_0 is the solution to the direct problem

$$\begin{aligned} u_0'' + 2Au_0' + A^2 u_0 &= f_0, \quad 0 < t < T, \\ u_0(0) &= 0, \\ u_0'(0) &= 0. \end{aligned} \tag{24}$$

Then, if the pair (f_k, u_k) has been constructed, let

$$f_{k+1} = f_k - \gamma (u_k(T) - g), \tag{25}$$

where γ is such that

$$0 < \gamma < \frac{1}{\|K(T)\|}, \tag{26}$$

and $\|K(T)\| = \sup_{n\in\mathbb{N}^*}(1 - (1 + T\lambda_n)e^{-\lambda_n T})/\lambda_n^2$.

Finally, we get u_{k+1} by solving the problem

$$\begin{aligned} u_{k+1}'' + 2Au_{k+1}' + A^2 u_{k+1} &= f_{k+1}, \quad 0 < t < T, \\ u_{k+1}(0) &= 0, \\ u_{k+1}'(0) &= 0. \end{aligned} \tag{27}$$

Let us iterate backwards in (25) to obtain

$$\begin{aligned} f_{k+1} &= f_k - \gamma K(T) f_k + \gamma g = (I - \gamma K(T)) f_k + \gamma g \\ &= (I - \gamma K(T))^{k+1} f_0 + \gamma \sum_{j=0}^{k} (I - \gamma K(T))^j g. \end{aligned} \tag{28}$$

Now, we introduce some properties and tools which are useful for our main theorems.

Lemma 5. *The norm of the operator $K(t)$ is given by*

$$\|K(t)\| = \sup_{n\in\mathbb{N}^*} \frac{\left(1 - (1 + t\lambda_n) e^{-\lambda_n t}\right)}{\lambda_n^2} = \frac{\left(1 - (1 + t\lambda_1) e^{-\lambda_1 t}\right)}{\lambda_1^2}. \tag{29}$$

Proof. We aim to find the supremum of the function $(1 - (1 + t\lambda_n)e^{-\lambda_n t})/\lambda_n^2$, $n \in \mathbb{N}^*$, and for this purpose, fix t, let $\mu = \lambda t$, and define the function

$$G_1(\mu) = \frac{(1 - (1 + \mu) e^{-\mu})}{\mu^2}, \quad \text{for } \mu \geq \mu_1 = \lambda_1 t. \tag{30}$$

We compute

$$G_1'(\mu) = \frac{\left(\mu^2 + 2\mu + 2\right) e^{-\mu} - 2}{\mu^3}. \tag{31}$$

Put

$$h(\mu) = \left(\mu^2 + 2\mu + 2\right) e^{-\mu} - 2. \quad (32)$$

Hence,

$$G_1'(\mu) = \frac{h(\mu)}{\mu^3}. \quad (33)$$

To study the monotony of G_1, it suffices to determine the sign of h. We have

$$h'(\mu) = -\mu^2 e^{-\mu} < 0, \quad \forall \mu > 0, \quad (34)$$

and then h is decreasing; moreover $h(\mu) \subset]-2, 0[$, $\forall \mu > 0$. Hence $G_1'(\mu) < 0$, $\forall \mu \geq \mu_1$, which implies that G_1 is decreasing and

$$\sup_{\mu \geq \mu_1} G_1(\mu) = G_1(\mu_1). \quad (35)$$

Therefore,

$$\sup_{n \geq 1} \frac{\left(1 - (1 + \lambda_n t) e^{-\lambda_n t}\right)}{\lambda_n^2} = \frac{\left(1 - (1 + \lambda_1 t) e^{-\lambda_1 t}\right)}{\lambda_1^2}. \quad (36)$$

□

Proposition 6. *For the linear operator $L = I - \gamma K(T)$, one has the following properties:*

(1) *L is positive and self-adjoint,*

(2) *L is nonexpansive,*

(3) *1 is not an eigenvalue of L.*

Proof. Form properties of operator A and the definition of L it follows that L is self-adjoint and nonexpansive positive operator and from the inequality

$$0 < 1 - \gamma \frac{\left(1 - (1 + T\lambda) e^{-\lambda T}\right)}{\lambda^2} < 1, \quad \text{for } \lambda \in \sigma(A), \quad (37)$$

it follows that the point spectrum of L, $\sigma_p(L) \subset]0, 1[$. Then 1 is not eigenvalue of the operator L. □

Lemma 7. *If $\lambda > 0$, one has the estimates*

$$\frac{1}{1 + \lambda^2} \leq \max\left(\frac{3}{T^2}, 1\right) \frac{\left(1 - (1 + T\lambda) e^{-\lambda T}\right)}{\lambda^2}, \quad (38)$$

$$0 < \frac{\left(1 - (1 + t\lambda) e^{-\lambda t}\right)}{\lambda^2} < T^2, \quad \forall t \in [0, T]. \quad (39)$$

Proof. To establish (38), let us first prove that

$$\frac{1}{3 + \mu^2} \leq \frac{\left(1 - (1 + \mu) e^{-\mu}\right)}{\mu^2}, \quad \forall \mu > 0, \quad (40)$$

which is equivalent to prove that

$$G_2(\mu) = 3 - \left(3 + \mu^2\right)(1 + \mu) e^{-\mu} \geq 0, \quad \forall \mu > 0. \quad (41)$$

We have

$$G_2'(\mu) = \mu(\mu - 1)^2 e^{-\mu} \geq 0, \quad \forall \mu > 0. \quad (42)$$

Then, G_2 is nondecreasing and it follows that $G_2(\mu) \subset]0, 3[$. So $G_2(\mu) \geq 0$, $\forall \mu > 0$.

Choosing $\mu = T\lambda$ in (40), we obtain

$$\frac{1}{3 + (T\lambda)^2} \leq \frac{\left(1 - (T\lambda + 1) e^{-T\lambda}\right)}{(T\lambda)^2}. \quad (43)$$

So,

$$\frac{T^2}{\max(3, T^2)(1 + \lambda^2)} \leq \frac{\left(1 - (1 + T\lambda) e^{-T\lambda}\right)}{\lambda^2}. \quad (44)$$

From (44), we deduce (38).

Now, we prove the estimate (39). It is easy to verify that

$$G_3(\mu) = \left(1 - (1 + \mu) e^{-\mu}\right) - \mu^2 < 0, \quad \forall \mu > 0. \quad (45)$$

Then, if we choose $\mu = t\lambda$, we get

$$\left(1 - (1 + t\lambda) e^{-t\lambda}\right) < t^2 \lambda^2, \quad \forall \lambda > 0, \ \forall t \in [0, T]. \quad (46)$$

Hence, from (46), (39) follows. □

Theorem 8. *Let u be a solution to the inverse problem (1). Let $f_0 \in H$ be an arbitrary initial data element for the iterative procedure proposed above and let u_k be the kth approximate solution. Then*

(i) *The method converges; that is,*

$$\sup_{t \in [0,T]} \left\| u_k(t) - u(t) \right\| \longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \quad (47)$$

(ii) *Moreover, if, for some $\alpha = 1 + \theta$, $\theta > 0$, $f_0 - f \in H^\alpha$, that is, $\|f_0 - f\|_{H^\alpha} \leq E$, then the rate of convergence of the method is given by*

$$\sup_{t \in [0,T]} \left\| u_k(t) - u(t) \right\| \leq T^2 C E k^{-\alpha/2}, \quad (48)$$

where C is a positive constant independent of k.

Proof. (i) From (28), we get

$$\begin{aligned} f_k &= (I - \gamma K(T))^k f_0 \\ &\quad + \left(I - (I - \gamma K(T))^k\right)(K(T))^{-1} g, \end{aligned} \quad (49)$$

and then

$$f_k = (I - \gamma K(T))^k (f_0 - f) + f, \quad (50)$$

which implies that

$$\begin{aligned} u_k(t) - u(t) &= K(t)(f_k - f) \\ &= K(t)(I - \gamma K(T))^k (f_0 - f). \end{aligned} \quad (51)$$

Hence,

$$\left\|u_k(t)-u(t)\right\| \le \|K(t)\| \left\|(I-\gamma K(T))^k (f_0-f)\right\|. \quad (52)$$

From Lemma 5 and (39) we have

$$\sup_{t\in[0,T]} \|K(t)\| = \sup_{t\in[0,T]} \frac{\left(1-(1+t\lambda_1)e^{-t\lambda_1}\right)}{\lambda_1^2} < T^2. \quad (53)$$

Combining (52) and (53) and passing to the supremum with respect to $t\in[0,T]$, we obtain

$$\sup_{t\in[0,T]} \left\|u_k(t)-u(t)\right\| \le T^2 \left\|(I-\gamma K(T))^k (f_0-f)\right\| \longrightarrow 0, \quad \text{as } k\longrightarrow\infty. \quad (54)$$

(ii) By part (i), we have

$$\begin{aligned}\left\|u_k(t)-u(t)\right\|^2 &\le T^4 \sum_{n=1}^{\infty} \left(1-\gamma\left(\frac{1-(1+\lambda_n T)e^{-\lambda_n T}}{\lambda_n^2}\right)\right)^{2k} \\ &\cdot \left|(f_0-f,\varphi_n)\right|^2,\end{aligned} \quad (55)$$

and hence

$$\begin{aligned}\left\|u_k(t)-u(t)\right\|^2 &\le T^4 \sum_{n=1}^{\infty} \left(1-\gamma\left(\frac{1-(1+\lambda_n T)e^{-\lambda_n T}}{\lambda_n^2}\right)\right)^{2k} \\ &\cdot \left(1+\lambda_n^2\right)^{-\alpha}\left(1+\lambda_n^2\right)^{\alpha} \left|(f_0-f,\varphi_n)\right|^2.\end{aligned} \quad (56)$$

Using the inequality (38), we obtain

$$\begin{aligned}\left\|u_k(t)-u(t)\right\|^2 &\le T^4 \left(\max\left(\frac{3}{T^2},1\right)\right)^{\alpha} \\ &\cdot \sum_{n=1}^{\infty} (1-\gamma\beta_n)^{2k} \beta_n^{\alpha} \left(1+\lambda_n^2\right)^{\alpha} \left|(f_0-f,\varphi_n)\right|^2,\end{aligned} \quad (57)$$

where $\beta_n = ((1-(1+\lambda_n T)e^{-\lambda_n T})/\lambda_n^2)$.

So, it follows that

$$\begin{aligned}\left\|u_k(t)-u(t)\right\|^2 &\le T^4 \left(\max\left(\frac{3}{T^2},1\right)\right)^{\alpha} \\ &\cdot \sup_{0\le\beta_n\le T^2} (1-\gamma\beta_n)^{2k} \beta_n^{\alpha} \left\|f_0-f\right\|_{H^\alpha}^2.\end{aligned} \quad (58)$$

Put

$$\phi(\beta) = (1-\gamma\beta)^{2k}\beta^{\alpha}, \quad 0\le\beta\le T^2. \quad (59)$$

We compute

$$\phi'(\beta) = (1-\gamma\beta)^{2k-1}\beta^{\alpha-1}(-\gamma(2k+\alpha)\beta+\alpha). \quad (60)$$

Setting $\phi'(\beta)=0$, it follows that $\beta^* = \alpha/(2k+\alpha)\gamma$ is the critical point of ϕ. It is easy to see that the maximum of ϕ is attained at β^*. So

$$\begin{aligned}\sup_{0\le\beta\le T^2} \phi(\beta) &\le \phi(\beta^*) = (1-\gamma\beta^*)^{2k}(\beta^*)^{\alpha} \le (\beta^*)^{\alpha} \\ &= \left(\frac{\alpha}{(2k+\alpha)\gamma}\right)^{\alpha},\end{aligned} \quad (61)$$

and hence

$$\sup_{0\le\beta\le T^2} \phi(\beta) \le \left(\frac{\alpha}{2\gamma}\right)^{\alpha} k^{-\alpha}. \quad (62)$$

Combining (58) and (62), we obtain

$$\begin{aligned}&\sup_{t\in[0,T]} \left\|u_k(t)-u(t)\right\|^2 \\ &\quad\le T^4 \left(\frac{\alpha}{2\gamma}\max\left(\frac{3}{T^2},1\right)\right)^{\alpha} \left(\frac{1}{k}\right)^{\alpha} E^2.\end{aligned} \quad (63)$$

Since in practice the measured data g is never known exactly but only up to an error of, say, $\delta>0$, it is our aim to solve the equation $K(T)f=g$ from the knowledge of a perturbed right-hand side g^δ satisfying

$$\left\|g-g^\delta\right\| < \delta, \quad (64)$$

where $\delta>0$ denotes a noise level. In the following theorem, we consider the case of inexact data. □

Theorem 9. *Let $\alpha=1+\theta$, $(\theta>0)$, f_0 be an arbitrary initial data element for the iterative procedure proposed above such that $(f_0-f)\in H^\alpha$, let u_k be the kth approximations solution for the exact data g, and let u_k^δ be the kth approximations solution corresponding to the perturbed data g^δ such that (64) holds. Then one has the following estimate:*

$$\sup_{t\in[0,T]} \left\|u_k(t)-u(t)\right\| \le T^2\left(\delta\gamma k + CE\left(\frac{1}{k}\right)^{\alpha/2}\right). \quad (65)$$

Proof. Let

$$\begin{aligned}f_k &= (I-\gamma K(T))^k f_0 + \gamma\sum_{j=0}^{k-1}(I-\gamma K(T))^j g, \\ u_k(t) &= K(t) f_k, \\ f_k^\delta &= (I-\gamma K(T))^k f_0 + \gamma\sum_{j=0}^{k-1}(I-\gamma K(T))^j g^\delta, \\ u_k^\delta(t) &= K(t) f_k^\delta.\end{aligned} \quad (66)$$

Using the triangle inequality, we obtain

$$\left\|u_k^\delta-u\right\| \le \left\|u_k^\delta-u_k\right\| + \left\|u_k-u\right\|. \quad (67)$$

From Theorem 8, we have

$$\sup_{t\in[0,T]} \left\| u_k(t) - u(t) \right\| \le T^2 CE \left(\frac{1}{k}\right)^{\alpha/2}. \tag{68}$$

On the other hand,

$$\begin{aligned} \left\| u_k^\delta(t) - u_k(t) \right\| &= \left\| K(t)\left(f_k^\delta - f_k\right) \right\| \\ &\le T^2 \gamma \left\| \sum_{j=0}^{k-1} (I - \gamma K(T))^j \left(g^\delta - g\right) \right\| \\ &\le T^2 \delta\gamma \left\| \sum_{j=0}^{k-1} (I - \gamma K(T))^j \right\| \\ &\le T^2 \delta\gamma \sum_{j=0}^{k-1} \left\| (I - \gamma K(T)) \right\|^j . \end{aligned} \tag{69}$$

Since

$$\left\| (I - \gamma K(T)) \right\| \le 1, \tag{70}$$

it follows that

$$\sup_{t\in[0,T]} \left\| u_k^\delta(t) - u_k(t) \right\| \le T^2 \delta\gamma k. \tag{71}$$

Combining (68) and (71) and passing to the supremum with respect to $t \in [0,T]$, we obtain the estimate (65). □

Remark 10. If we choose the number of the iterations $k(\delta)$ so that $k(\delta) \to 0$ as $\delta \to 0$, we obtain

$$\sup_{t\in[0,T]} \left\| u_k^\delta(t) - u(t) \right\| \longrightarrow 0, \quad \text{as } k \longrightarrow +\infty. \tag{72}$$

5. Numerical Implementation

In this section, an example is devised for verifying the effectiveness of the proposed method. Consider the problem of finding a pair of functions $(u(x,t), f(x))$, in the system

$$\begin{aligned} &\frac{\partial^2}{\partial t^2} u(x,t) - 2\frac{\partial^2}{\partial x^2}\left(\frac{\partial}{\partial t} u(x,t)\right) + \frac{\partial^4}{\partial x^4} u(x,t) \\ &\quad = f(x), \quad (t,x) \in (0,1)\times(0,1), \\ &u(0,t) = u(1,t) = 0, \quad t \in (0,1), \\ &u(x,0) = u_t(x,0) = 0, \quad x \in (0,1), \\ &u(x,1) = g(x), \quad x \in (0,1). \end{aligned} \tag{73}$$

Denote

$$\begin{aligned} &A = -\frac{\partial^2}{\partial x^2}, \\ &\text{with } \mathscr{D}(A) = H_0^1(0,1) \cap H^2(0,1) \subset H = L^2(0,1). \\ &\lambda_n = n^2\pi^2, \\ &\varphi_n = \sqrt{2}\sin(n\pi x), \quad n = 1,2,\ldots \end{aligned} \tag{74}$$

are eigenvalues and orthonormal eigenfunctions, which form a basis for H.

The solution of the above problem is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{1 - \left(1 + (n\pi)^2 t\right) e^{-(n\pi)^2 t}}{(n\pi)^4} \right) f_n \varphi_n, \tag{75}$$

where $f_n = (f, \varphi_n) = \sqrt{2}\int_0^1 f(s)\sin(n\pi s)ds$, $n = 1,2,\ldots$

Now, to solve the inverse problem, making use of the supplementary condition and defining the operator $K : f \to g$, we have

$$\begin{aligned} g(x) &= u(x,1) = Kf(x) \\ &= 2\sum_{n=1}^{\infty} \left(\frac{1 - \left(1 + (n\pi)^2\right) e^{-(n\pi)^2}}{(n\pi)^4} \right) \\ &\quad \cdot \left(\int_0^1 f(s)\sin(n\pi s)\,ds \right) \sin(n\pi x). \end{aligned} \tag{76}$$

Example 11. In the following, we first selected the exact solution $f(x)$ and obtained the exact data function $g(x)$ through solving the forward problem. Then we added a normally distributed perturbation to each data function and obtained vectors $g^\delta(x)$. Finally we obtained the regularization solutions through solving the inverse problem with noisy data $g^\delta(x)$ satisfying

$$\left\| g - g^\delta \right\|_{(L_2(0,1))^2} \le \delta. \tag{77}$$

It is easy to see that if $f(x) = \sin\pi x$, then

$$u(x,t) = \frac{\left(1 - \left(1 + \pi^2 t\right) e^{-\pi^2 t}\right)}{\pi^4} \sin(\pi x) \tag{78}$$

is the exact solution of the problem (73). Consequently, $g(x) = ((1 - (1+\pi^2)e^{-\pi^2})/\pi^4)\sin(\pi x)$.

Now, we propose to approximate the first and second space derivatives by using central difference and we consider an equidistant grid points to a spatial step size $x_0 = 0 < x_1 < \cdots < x_{N+1} = 1$, $(h = 1/(N+1))$, where N is a positive integer. We get the following semidiscrete problem:

$$\begin{aligned} &u''(x_i,t) + 2A_h u'(x_i,t) + A_h^2 u(x_i,t) = f(x_i), \\ &\qquad x_i = ih,\ i = 1,\ldots,N,\ 0 < t < 1, \\ &\qquad u(0,t) = u(1,t) = 0, \\ &\qquad 0 < t < 1, \\ &\qquad u(x_i,0) = u'(x_i,0) = 0, \\ &\qquad x_i = ih,\ i = 1,\ldots,N, \\ &\qquad u(x_i,1) = g(x_i), \\ &\qquad x_i = ih,\ i = 1,\ldots,N, \end{aligned} \tag{79}$$

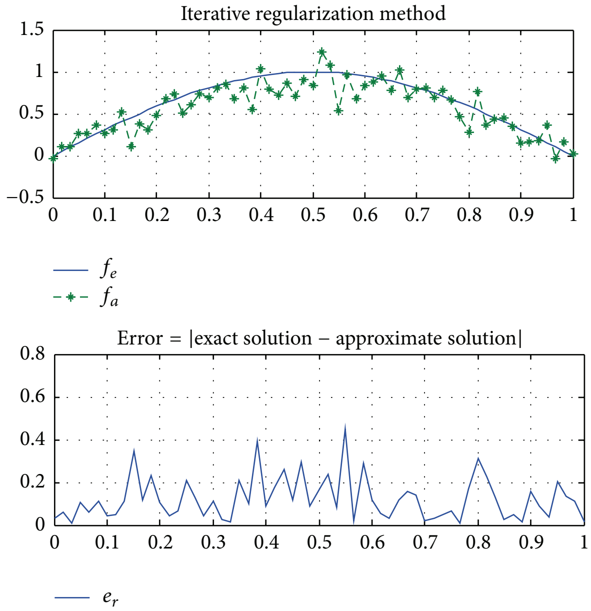

Figure 1: The comparison between the exact solution f_e and its computed approximations f_a for $N = 60$ $k = 4$ and noisy level $\varepsilon = 10^{-3}$.

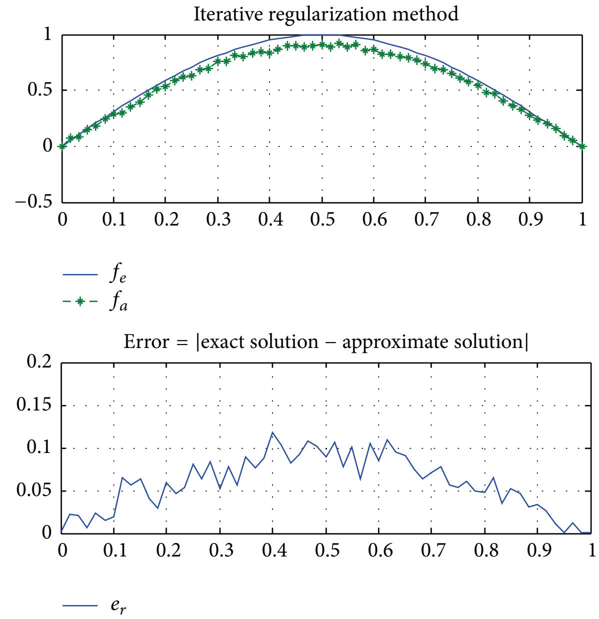

Figure 2: The comparison between the exact solution f_e and its computed approximations f_a for $N = 60$ $k = 4$ and noisy level $\varepsilon = 10^{-4}$.

where A_h is the discretisation matrix stemming from the operator $A = -d^2/dx^2$, and

$$A_h = \frac{1}{h^2} \text{Tridiag}(-1, 2, -1) \tag{80}$$

is a symmetric, positive definite matrix, with eigenvalues

$$\mu_j = 4(N+1)^2 \sin^2 \frac{j\pi}{2(N+1)}, \quad j = 1, \ldots, N, \tag{81}$$

and orthonormal eigenvalues

$$v_j = \left(\sin \frac{mj\pi}{(N+1)} \right)_{1 \le m \le N}, \quad j = 1, \ldots, N. \tag{82}$$

We assume that it is fine enough so that the discretization errors are small compared to the uncertainty δ of the data; this means that A_h is a good approximation of the differential operator A whose unboundedness is reflected in a large norm of A_h (see [24]).

Adding a random distributed perturbation to each data function, we obtain

$$g^\delta = g + \varepsilon \text{randn}(\text{size}(g)), \tag{83}$$

where ε indicates the noise level of the measurements data and the function randn$(\cdot)$ generates arrays of random numbers whose elements are normally distributed with mean 0, variance $\sigma^2 = 1$, and standard deviation $\sigma = 1$. randn(size(g)) returns an array of random entries that is of the same size as g. The noise level δ can be measured in the sense of root mean square error (RMSE) according to

$$\delta = \left\| g^\delta - g \right\|_{l^2} = \left(\frac{1}{N+1} \sum_{i=0}^{N} \left(g(x_i) - g^\delta(x_i) \right)^2 \right)^{1/2}. \tag{84}$$

Table 1: Relative error RE(f).

N	k	ε	RE(f)
60	4	10^{-3}	0.2039
60	4	10^{-4}	0.0945
60	5	10^{-3}	0.3032
60	5	10^{-4}	0.0305

The relative error is given as follows:

$$\text{RE}(f) = \frac{\left\| f_{\text{approximate}} - f_{\text{exact}} \right\|_{l^2}}{\left\| f_{\text{exact}} \right\|_{l^2}}. \tag{85}$$

The discrete iterative approximation of (66) is given by

$$\begin{aligned} f_k^\delta(x_i) &= (I - \gamma K_h)^k f_0(x_i) \\ &\quad + \gamma \sum_{j=0}^{k-1} (I - \gamma K_h)^j g^\delta(x_i), \quad i = 1, \ldots, N, \end{aligned} \tag{86}$$

where $K_h = A_h^{-2}(I_N - (I_N + A_h)e^{-A_h})$ and $\gamma < 1/\|K_h\| = (\mu_1^2/(1 - (1+\mu_1)e^{-\mu_1}))$.

Figures 1–4 display that as the amount of noise ε decreases, the regularized solutions approximate better the exact solution.

Table 1 shows that for $k = 4$ or $k = 5$ the relative error decreases with the decease of epsilon which is consistent with our regularization.

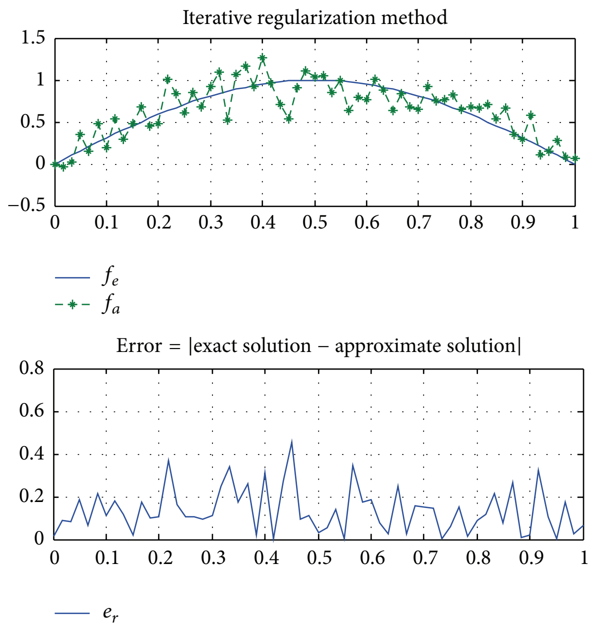

FIGURE 3: The comparison between the exact solution f_e and its computed approximations f_a for $N = 60$ $k = 5$ and noisy level $\varepsilon = 10^{-3}$.

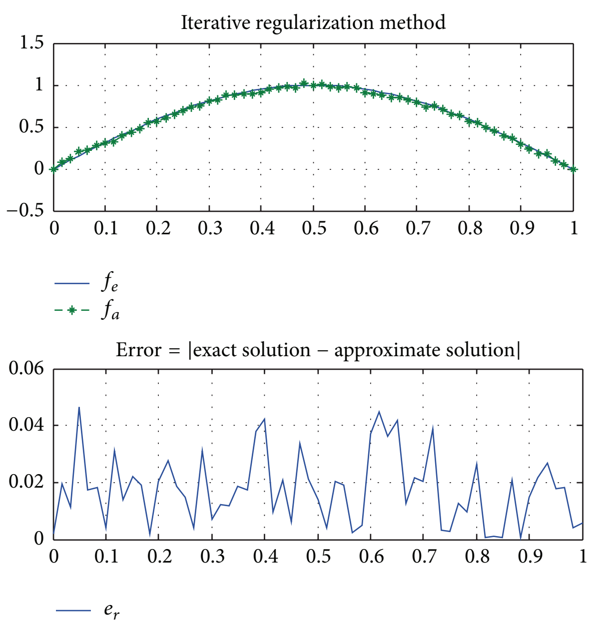

FIGURE 4: The comparison between the exact solution f_e and its computed approximations f_a for $N = 60$ $k = 5$ and noisy level $\varepsilon = 10^{-4}$.

6. Conclusion

In this paper, we have extended the iterative method to identify the unknown source term in a second order differential equation, convergence results were established, and error estimates have been obtained under an a priori bound of the exact solution. Some numerical tests have been given to verify the validity of the method.

Conflict of Interests

The authors declare that they have no conflict of interests.

Authors' Contribution

All authors read and approved the paper.

Acknowledgments

The authors would like to thank the anonymous referees for their suggestions.

References

[1] H. Leiva, "A lemma on C_0-semigroups and applications," *Quaestiones Mathematicae*, vol. 26, no. 3, pp. 247–265, 2003.

[2] H. Leiva, *Linear Reaction-Diffusion Systems*, Notas de Mathematica, No. 185, Mereida, 1999.

[3] L. F. de Oliveira, "On reaction-diffusion systems," *Electronic Journal of Differential Equations*, vol. 1998, no. 24, 10 pages, 1998.

[4] V. L. Fushchich, A. S. Galitsyn, and A. S. Polubinskii, "A new mathematical model of heat conduction processes," *Ukrainian Mathematical Journal*, vol. 42, no. 2, pp. 210–216, 1990.

[5] G. Bastay, "Iterative methods for Ill-posed boundary value problems. Linkoping studies in science and technology," Dissertations 392, Linköping University, Linkoping, Sweden, 1995.

[6] A. S. Carasso, "Bochner subordination, logarithmic diffusion equations, and blind deconvolution of hubble space telescope imagery and other scientific data," *SIAM Journal on Imaging Sciences*, vol. 3, no. 4, pp. 954–980, 2010.

[7] L. Wang, X. Zhou, and X. Wei, *Heat Conduction: Mathematical Models and Analytical Solutions*, Springer, 2008.

[8] M. Andrle and A. El Badia, "Identification of multiple moving pollution sources in surface waters or atmospheric media with boundary observations," *Inverse Problems*, vol. 28, no. 7, Article ID 075009, 2012.

[9] H. W. Engl and C. Groetsch, *Inverse and Ill-Posed Problems*, vol. 4 of *Notes and Reports in Mathematics in Science and Engineering*, Academic press, New York, NY, USA, 1987.

[10] A. Kirsch, *An Introduction to the Mathematical Theory of Inverse Problems*, Springer, Heidelberg, Germany, 1996.

[11] D. Mace and P. Lailly, "Solution of the VSP one-dimensional inverse problem," *Geophysical Prospecting*, vol. 34, no. 7, pp. 1002–1021, 1986.

[12] N. Magnoli and G. A. Viano, "The source identification problem in electromagnetic theory," *Journal of Mathematical Physics*, vol. 38, no. 5, pp. 2366–2388, 1997.

[13] V. A. Kozlov and V. G. Maz'ya, "On iterative procedures for solving ill-posed boundary value problems that preserve differential equations," *Lenningrad Mathematics Journal*, vol. 1, pp. 1207–1228, 1990.

[14] V. A. Kozlov, V. G. Maz'ya, and A. V. Fomin, "An iterative method for solving the Cauchy problem for elliptic equations," *U.S.S.R. Computational Mathematics and Mathematical Physics*, vol. 31, no. 1, pp. 45–52, 1991.

[15] A. B. Bakushinsky and M. Y. Kokurin, *Iterative Methods for Approximate Solution of Inverse Problems*, vol. 577 of *Mathematics and Its Applications*, Springer, Berlin, Germany, 2004.

[16] J. Baumeister and A. Leitao, "On iterative methods for solving ill-posed problems modeled by partial differential equations," *Journal of Inverse and Ill-Posed Problems*, vol. 9, no. 1, pp. 13–29, 2001.

[17] F. Berntsson, V. A. Kozlov, L. Mpinganzima, and B. O. Turesson, "An alternating iterative procedure for the Cauchy problem for the Helmholtz equation," *Inverse Problems in Science and Engineering*, vol. 22, no. 1, pp. 45–62, 2014.

[18] A. Bouzitouna, N. Boussetila, and F. Rebbani, "Two regularization methods for a class of inverse boundary value problems of elliptic type," *Boundary Value Problems*, vol. 2013, article 178, 2013.

[19] A. Lakhdari and N. Boussetila, "An iterative regularization method for an abstract ill-posed biparabolic problem," *Boundary Value Problems*, vol. 2015, article 55, 2015.

[20] J.-G. Wang and T. Wei, "An iterative method for backward time-fractional diffusion problem," *Numerical Methods for Partial Differential Equations*, vol. 30, no. 6, pp. 2029–2041, 2014.

[21] H. W. Zhang and T. Wei, "Two iterative methods for a Cauchy problem of the elliptic equation with variable coefficients in a strip region," *Numerical Algorithms*, vol. 65, no. 4, pp. 875–892, 2014.

[22] M. A. Krasnosel'skii, G. M. Vainikko, P. P. Zabreiko, and Y. U. B. Rutitskii, *Approximate Solutions of Operator Equations*, Wolters-Noordhoff Publishing, Groningen, The Netherlands, 1972.

[23] A. Pazy, *Semigroups of Linear Operators and Application to Partial Differential Equations*, Springer, New York, NY, USA, 1983.

[24] L. Eldén and V. Simoncini, "A numerical solution of a Cauchy problem for an elliptic equation by Krylov subspaces," *Inverse Problems*, vol. 25, no. 6, Article ID 065002, 2009.