CHAPTER 2

Preliminaries

In this chapter we provide the basic notions and results that will be used in the sequel. We will cite some fixed point theorems, such the Banach contraction principle, Leray-Schauder's nonlinear alternative. The notion of the topological degree is also treated in both cases the degree of Brouwer in finite dimension and the degree of Leray-Schauder in infinite dimension, then we will expose their properties. Finally, we give the theorem of Mawhin coincidence degree with the proof.

2.1 Fixed point theorems

The Banach contraction principle, established in 1922 by the Polish mathematician Stefan Banach, is one of the most significant results in analysis and is considered the main source of the metric fixed point theory. The important part of Banach's contraction is to stretch the existence, the uniqueness and the sequence of the successive approximation that converges to a solution of the problem. For more results we refer to [1, 3, 5, 6, 28, 30, 51].

Definition 1 *Let* (X, d) *be a metric space and let* $f : X \to X$ *be a mapping. A point* $x \in X$ *is called a fixed point of* f *if* $x = f(x)$.

Definition 2 *f is called contraction if there exists a fixed constant $k < 1$ such that*

$$d(f(x), f(y)) \leq kd(x, y), \text{ for all } x, y \in X.$$

Theorem 3 *(Banach Contraction Principle) Let (X, d) be a complete metric space, then each contraction map $f : X \to X$ has a unique fixed point.*

Proof. Let x and y be fixed points of f, then $d(x, y) = d(f(x), f(y)) \leq kd(x, y)$. Since $k < 1$, we get $x = y$, that the uniqueness holds.
Now, we will construct explicitly a sequence converging to the fixed point. Let x_0 be an arbitrary but fixed element in X. Define a sequence of iterates $(x_n)_{n\in\mathbb{N}}$ in X by

$$x_n = f(x_{n-1}) = f^n(x_0), \quad \text{for all } n \geq 1.$$

Since f is a contraction, we get

$$d(x_n, x_{n+1}) = d(f(x_{n-1}), f(x_n)) \leq kd(x_{n-1}, x_n), \quad \text{for any } n \geq 1.$$

Thus, we obtain

$$d(x_n, x_{n+1}) \leq k^n d(x_0, x_1), \; for \; all \; n \geq 1.$$

Hence, for any $m > n$, we have

$$d(x_n, x_m) \leq \left(k^n + k^{n+1} + \cdots + k^{m-1}\right) d(x_0, x_1) \leq \frac{k^n}{1-k} d(x_0, x_1).$$

We deduce that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete space X, denote by $x \in X$ its limit. Since f is continuous, we have

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_{n-1}) = f(x).$$

■

Theorem 4 *(Brouwer Fixed Point Theorem) Let B be a closed ball in $\mathbb{R}^n$. Then, any continuous mapping $T : B \to B$ has at least one fixed point.*

Brouwer fixed point theorem is not true in infinite dimensional spaces. The first fixed point theorem in an infinite dimensional Banach space was given by Schauder in 1930.

Theorem 5 *(Leary-schauder Fixed Point Theorem) Let B be the closed unit ball of a Banach E and $f : B \to B$ compact, then f has a fixed point.*

Let X and Y be two normed vector spaces, Ω an open set of X. Let give the definitions of compact map and completely continuous map.

Definition 6 *A continuous mapping $T : \Omega \subset X \to Y$ is called compact if $T(\Omega)$ is relatively compact.*

Lemma 7 *A continuous mapping $T : \Omega \subset X \to Y$ is said to be completely continuous, if the image of any bounded subset B of Ω is relatively compact.*

Theorem 8 *(Ascoli-Arzela) Let $E = C([a, b])$ denotes the space of the continuous functions and $M \subset E$ such that*

1. M is equicontinuous,

2.M is uniformly bounded,

then M is relatively compact in E.

Proposition 9 *Any mapping bounded and of finite rank is completely continuous.*

Remark 10 *Any compact mapping is completely continuous (because for any bounded $B \subset \Omega$ we have $T(B) \subset T(\Omega)$). The converse is true if Ω is bounded.*

Lemma 11 *If $T : X \to Y$ is a linear mapping, with X and Y Banach spaces, for T to be compact it suffices that $T(B(0, 1))$ is precompact. If at least one spaces X or Y is of finite dimension, so T is compact if and only if T is too.*

The principle of continuation is to deform one map into an other simpler one for which we know the existence of a fixed point. This deformation known as homotopy verify certain conditions.

Definition 12 *Let X and Y be two topological spaces. We say that the two continuous applications $f, g : X \to Y$, are homotopic if there exists*

$$H : X \times [0, 1] \to X$$

such that

$$H(x, 0) = f(x) \quad and \quad H(x, 1) = g(x).$$

Example 13 *Let $X = Y = \mathbb{R}^n$, we consider $c : \mathbb{R}^n \to \mathbb{R}^n$ the constant map $c(x) = 0$, and $i : \mathbb{R}^n \to \mathbb{R}^n$ the application $i(x) = x$. Let us show that c and i are homotopes. Let $H : \mathbb{R}^n \times$ [0,1] $\to \mathbb{R}^n$ such that: $H(x, t) = (1 - t)c(x) + ti(x)$, we have*

$$H(x, 0) = (1 - 0) \times 0 + 0 \times x = 0$$

and

$$H(x, 1) = (1 - 1) \times 0 + 1 \times x = x$$

then

$$H(x, t) = tx, \quad H(x, 0) = c(x), \quad H(x, 1) = i(x).$$

Example 14 *Let $X = Y = \mathbb{R}^n - \{0\}$, let $p(x) = \frac{x}{\|x\|}$, and $i(x) = x$. We see that p and i are homotopes by taking:*

$$H : \mathbb{R}^n - \{0\} \times [0, 1] \to \mathbb{R}^n - \{0\}$$

such that:

$$H(x, t) = (1 - t)i(x) + tp(x),$$

we have

$$\begin{aligned} H(x, 0) &= (1 - 0) \times x + 0 \times x\frac{x}{\|x\|} \\ &= xH(x, 1) = (1 - 1) \times x + 1 \times \frac{x}{\|x\|} = \frac{x}{\|x\|} \end{aligned}$$

then

$$\begin{aligned} H(x,t) &= (1-t)x + t\frac{x}{\|x\|}, \\ H(x,0) &= i(x) \quad and \quad H(x,1) = p(x). \end{aligned}$$

Let (X,d) be a complete metric space, and U be an open subset of X.

Definition 15 *Let $F : U \to X$ and $G : U \to X$ be two contractions; here U denotes the closure of U in X. We say that F and G are homotopic if there exists $H : U \times [0,1] \to X$ with the following properties:*

(a) $H(\cdot,0) = g$ and $H(\cdot,1) = f$;

(b) $x = H(x,t)$ for every $x \in \partial U$ and $t \in [0,1]$ (here ∂U denotes the boundary of U in X);

(c) there exists α, $0 \leq \alpha < 1$, such that $d(H(x,t),H(y,t)) \leq \alpha d(x,y)$ for every $x, y \in \overline{u}$ and $t \in [0,1]$,

(d) there exists $M, M \geq 0$, such that $d(H(x,t),H(x,s)) \leq M|t-s|$ for every $x \in \overline{u}$ and $t, s \in [0,1]$.

Theorem 16 *Let (X,d)be a complete metric space and U an open subset of X. Suppose that $F : \overline{U} \to X$and $G : \overline{U} \to X$ are two homotopic contractive maps and G has a fixed point in U. Then F has a fixed point in U.*

Proof. Consider the set

$$A = \{\lambda \in [0,1] : x = H(x,\lambda) \ for \ some \ x \in U\},$$

where H is a homotopy between F and G as described in Definition15. Notice A is nonempty since G has a fixed point, that is, $0 \in A$. We will show that A is both open and closed in $[0,1]$ and hence by connectedness we have that $A = [0,1]$. As a result, F has a fixed point in U. A is closed in $[0,1]$, in fact let

$$\{\lambda_n\}_{n=1}^{\infty} \subseteq A \quad with \quad \lambda_n \to \lambda \in [0,1] \ as \ n \to \infty.$$

Since $\lambda_n \in A$ for $n = 1, 2, ...,$ there exists $x_n \in U$ with $x_n = H(x_n, \lambda_n)$. Also for $n, m \in \{1, 2, ...\}$, we have

$$\begin{aligned} d(x_n, x_m) &= d(H(x_n, \lambda_n), H(x_m, \lambda_m)) \\ &\leq d(H(x_n, \lambda_n), H(x_{n,} \lambda_m)) + d(H(x_{n,} \lambda m), H(x_m, \lambda_m)) \\ &\leq M|\lambda_n - \lambda_m| + \alpha d(x_n, x_m), \end{aligned}$$

that is,

$$d(x_n, x_m) \leq \left(\frac{M}{1-\alpha}\right) |\lambda_n - \lambda_m|.$$

Since $\{\lambda_n\}$ is a Cauchy sequence then $\{x_n\}$is also a Cauchy sequence, and since X is complete there exists $x \in U$ with $\lim_{n\to\infty} x_n = x$. In addition, $x = H(x, \lambda)$, since

$$\begin{aligned} d(x_n, H(x, \lambda)) &= d(H(x_n, \lambda_n), H(x, \lambda)) \\ &\leq M|\lambda_n - \lambda| + \alpha d(x_n, x). \end{aligned}$$

Thus $\lambda \in A$ and A is closed in $[0, 1]$.

Now, we show that A is open in $[0, 1]$. Let $\lambda_0 \in A$, then there exists $x_0 \in U$ with $x_0 = H(x_0, \alpha_0)$. Fix $\varepsilon > 0$ such that

$$\varepsilon \leq \frac{(1-\alpha)r}{M} \quad where \quad r < dist(x_0, \partial U),$$

and where

$$dist(x_0, \partial U) = \inf\{d(x_0, x) : x \in \partial U\}.$$

Fix $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$. Then for

$$\begin{aligned} &x \in B(x_0, r) = \{x : d(x, x_0) \leq r\}, d(x_0, H(x, \lambda)) \\ &\leq d(H(x_0, \lambda_0), H(x, \lambda_0)) + d(H(x, \lambda_0), H(x, \lambda)) \\ &\leq \alpha d(x_0, x) + M|\lambda - \lambda_0| \leq \alpha r + (1-\alpha)r = r. \end{aligned}$$

Thus for each fixed $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$, $H(\cdot, \lambda) : B(x_0, r) \rightarrow B(x_0, r)$. Then we deduce that $H(\ \cdot, \lambda)$ has a fixed point in U. Hence $\lambda \in A$ for any $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ and therefore A is open in $[0, 1]$. ■

2.2 Topological degree

Let Ω be a bounded open of $\mathbb{R}^N$, $f : \overline{\Omega} \rightarrow \mathbb{R}^N$ a continuous function and $b \in \mathbb{R}^N$ such that

$$f(x) = b \tag{2.1}$$

We want to obtain a quantity that give us the number of zeros for the equation (2.1), this quantity should give us the exact number of zeros and should be invariant by small deformations of f. So that we prevent the zeros of f to leave the domain we will impose that $b \notin f(\partial\Omega)$.

2.2.1 Topological degree of Brouwer

Definition 17 *Let Ω be an open bounded $\mathbb{R}^N$, $f \in C(\overline{\Omega}, \mathbb{R}^N)$ and b a regular value of f such that $b \notin f(\partial\Omega)$. Then the degree $deg(f, \Omega, b)$ is defined by*

$$deg(f, \Omega, b) = \sum_{x \in f^{-1}(b)} \mathrm{Si}\, gnJ_f(x)$$

where $J_f(x)$ is the Jacobi matrix of f in b.

Properties of Brouwer degree

The degree satisfies the following properties:

1) If $deg(f, \Omega, b) = 0$, then the equation $f(x) = b$ has at least one solution in Ω.

2) Normalisation: $deg(id, \Omega, b) = 1$, for all $b \in \Omega$, and $deg(id, \Omega, b) = 0$, for all $b \in \mathbb{R}^N / \overline{\Omega}$, where I is the identity on $\overline{\Omega}$.

3) Additivity $deg(f, \Omega, b) = deg(f, \Omega_1, b) + deg(f, \Omega_2, b)$,if Ω_1, Ω_2 are disjoint in $\Omega = \Omega_1 \cup \Omega_2$,and $b \notin f(\overline{\Omega} / (\Omega_1 \cup \Omega_2))$.

4) Homotopy invariance: If f and g are homotopy equivalent via a homotopy $H(t,.)$ such that $H(t,0) = f$, $H(t,1) = g$, $b \notin H(t,\partial\Omega)$ then $deg(f,\Omega,b) = deg(g,\Omega,b)$.

Proposition 18 *Let Ω be a bounded open set of $\mathbb{R}^N$ and two functions $f,g \in C(\overline{\Omega},\mathbb{R}^N)$, Assume that $f = g$ on $\partial\Omega$ and that $b \notin f(\partial\Omega)$, So we have*

$$deg(f,\Omega,b) = deg(g,\Omega,b).$$

Proof. Just use homotopy invariance of the topological degree, considering the homotopy $H(x,t) = tf(x) + (1-t)g(x)$. for all $t \in [0,1]$ we have that

$$deg(H(.,t),\Omega,b) = deg(H(.,0),\Omega,b).$$

and the result follows. For $t = 1$

$$deg(f,\Omega,b) = deg(g,\Omega,b)$$

■

Lemma 19 *(Sard) Let Ω be a bounded open and $f \in C^1(\mathbb{R}^N)$ and*

$$S = \{x \in \Omega, J_f(x) = 0\},$$

the set of singular points of f. Then $f(S)$ is of zero measure.

2.2.2 Topological degree of Leray-Schauder

We will now present a degree having the same role as the degree of Brouwer, but in infinite dimension, ie a tool which makes it possible to ensure that an equation of the form $f(x) = y$, where f is continuous of a Banach E in itself, have at least one solution x. The degree of Leray-Schauder, is built on the mapping which differ from the identity by a compact mapping, ie. the degree of Leray-Schauder is defined for applications that are compact perturbations of the identity of the type $I-T$ where T is compact and I.

Lemma 20 *Let Ω an open bounded set of a Banach space$X$. If $T:\overline{\Omega}\to X$ is a compact operator then for any $\varepsilon>0$ there exist E_ε a subspace of finite dimension and a continuous application $T_\varepsilon:\overline{\Omega}\to E$ such $\|T_\varepsilon u-Tu\|<\varepsilon$, for all $u\in\overline{\Omega}$.*

Definition 21 *Let X be a Banach space, Ω a bounded open set of X, $T:\overline{\Omega}\to X$ a compact operator such that $z\notin(I-T)\,\partial\Omega$. We define the topological degree of Leray-Schauder by*

$$deg_{LS}(I-T,\Omega,z)=deg(I-T_\varepsilon,\Omega\cap E_\varepsilon,z).$$

Remark 22 *In the previous definition $deg_{LS}(I-T,\Omega,z)$ depends only on T and Ω.*

The Leray–Schauder degree conserves the basic properties of Brouwer degree.

Theorem 23 *The Leray-Schauder degree has the following properties*

1) Additivity: $deg_{LS}(I-T,\Omega,z)=deg_{LS}(I-T,\Omega_1,z)+deg_{LS}(I-T,\Omega_2,z)$,if Ω_1, Ω_2 are disjoint in $\Omega=\Omega_1\cup\Omega_2$,and $z\notin(I-T)\,(\partial\Omega_1)\cup(I-T)\,(\partial\Omega_2)$

2)Existence: If $deg_{LS}(I-T,\Omega,z)\neq 0$, then $z\in(I-T)\,(\Omega)$.

3) Homotopy invariance: Let $H\,(t,.)$ such that $H:[0,1]\times\overline{\Omega}$ a compact homotopy such that $z\notin(I-H\,(t,.))\,(\partial\Omega)$,then $deg_{LS}(I-H\,(t,.)\,,\Omega,z)$ is independent of t.

2.3 Mahwin's coincidence degree theory

In 1970, Gaines and Mawhin introduced the theory of the degree of coincidence in the analysis of functional and differential equations. Mawhin has made important contributions since then, and this theory is also known as Mahwin's theory of coincidence. Coincidence theory is considered to be the very powerful technique, especially with regard to questions about the existence of solutions in nonlinear differential equations. Furthermore, many researchers have used it to solve boundary value problems at resonance, see [37, 38, 39, 40, 41, 43, 47].

Let us define the direct sums, projections and topological complement.

Definition 24 *Let E and F be two closed subspaces of a normed vector $\mathbb{R}-spaceX$. We say that E is a topological complement of F if X is the direct sum of F and E (i.e. $X = F \oplus E$).*

Definition 25 *Let X be a vector space. We say that a linear operator $P : X \to X$ is a projection if for all $x \in X$, we have $P(P(x)) = P^2x = P(x)$*

Proposition 26 *Let X be a vector space. A linear operator $P : X \to X$is a projection if and only if $(I - P)$ is a projection. Moreover, if the space X is normed, then P is continuous if and only if $(I - P)$ is continuous.*

Proof. Let P be a projection. So for all $x \in X$

$$\begin{aligned}
(I-P)^2(x) &= (I-P)\left((I-P)(x)\right) \\
&= I(I-P)(x) - P(I-P)(x) \\
&= I(x-Px) - P(x-Px) \\
&= x - p(x) - p(x) + p^2(x) \\
&= x - 2p(x) + p^2(x) \\
&= x - p(x) = (I-P)(x)
\end{aligned}$$

Reciprocally, if $(I - P)$is a projection,$(I - (I - P)) = P$ is too. For the topological framework, as the identity is a continuous mapping and the sum of two continuous mappings is also continuous, then P is continuous if and only if $(I - P)$is. ■

Proposition 27 *If P is a projection in X, then $\ker P = Im(I - P)$ and $ImP = \ker(I - P)$.*

Proof. We prove that $\ker P = Im(I - P)$. If $x \in \ker P \Longrightarrow P(x) = 0$ then

$$(I-P)(x) = x - P(x) = x \Longrightarrow x \in Im(I-P)$$

which implies

$$KerP \subset Im(I-P)$$

Next, if $x \in Im(I-P)$, then

$$\begin{aligned} P((I-P)(x)) &= P(x) - P^2(x) \\ &= P(x) - P(x) = 0 \Longrightarrow (I-P)x \in Ker(P) \end{aligned}$$

hence $Im(I-P) \subset \ker P$ and then

$$KerP = Im(I-P)$$

■

Remark 28 *A topological space X is separated (or Hausdorff) if $\forall x, y \in X : x \neq y$, $\exists Vx$, Vy open as $Vx \cap Vy = \phi$.*

Lemma 29 *The image of any continuous projection in a Hausdorff space is closed. In particular, the images of continuous projections of the Banach spaces are closed.*

Theorem 30 *If P is a continuous projection in a topological vector space of Hausdorff X, then X is the direct sum of ImP and $\ker P$, (ie $X = ImP \oplus \ker P$).*

Definition 31 *If the quotient space X/F is of finite dimension, we say that the subspace closed vector F of X is of finite codimension in X and we write*

$$co\dim(F) = dim(X/F).$$

Proposition 32 *$co\dim(F) = n < \infty$ if, and only if there is a vector subspace closed E of X, as*

$$X = F \oplus E \text{ and } \dim(E) = n$$

2.3.1 Fredholm operators

Definition 33 *Let X and Y be two normed vector spaces; we say that a linear mapping $L : D(L) \subset X \to Y$, is from Fredholm if it satisfies the following conditions*

1. $\ker(L) = L^{-1}(0)$ *is of finite dimension.*

2. $Im(L) = L(D(L))$*is closed and of finite codimension.*

Definition 34 *The index of a Fredholm operator L is the integer*

$$ind(L) = \dim(\ker(L)) - \dim co(Im(L)).$$

Examples.

1. If X and Y are of finite dimensions, then any linear mapping $L : X \to Y$ is from Fredholm with

$$ind(L) = \dim(X) - \dim(Y).$$

If X and Y are Banach spaces and $L : X \to Y$ is a linear mapping bijective, then L is a Fredholm operator of index 0, in fact

$$\dim(\ker(L)) = \dim co(Im(L) = 0.$$

2. The identity is a Fredholm operator of index 0.

Lemma 35 *If L is a Fredholm operator, u is a compact linear application; so $L + u$ is from Fredholm and*

$$ind(L + u) = ind(L).$$

In particular, any perturbation compact identity is a Fredholm index operator 0.

Proposition 36 *If L is a Fredholm operator of zero index, so L is surjective if and only if L is injective.*

2.3.2 Generalized inverse

Let $L : D(L) \subset X \to Y$ be a Fredholm operator of index 0. Let P and Q be two continuous projectors; $P : X \to X$ and $Q : Y \to Y$ such that

$$Im(P) = \ker L \quad \text{and} \quad \ker Q = Im(L).$$

Set

$$X_1 = Im(I - P) = \ker P \quad \text{and} \quad Y_1 = Im(Q),$$

so we can write

$$X = \ker L \oplus X_1, Y = Im(L) \oplus Y_1.$$

Consider an isomorphism,

$$J : \ker L \to Y_1$$

whose existence is ensured by the fact that $\dim \ker L = \dim Y_1 = n$. Note that

$$D(L) = \ker L \oplus (D(L) \cap X_1)$$

and that the restriction of L to $D(L) \cap X_1$ is an isomorphism on $Im(L)$. Denote by Lp this restriction and by $L_p^{-1} : Im(L) \to D(L) \cap X_1$ the inverse of L_p. So the operator

$$J^{-1} \oplus L_p^{-1} : Y = Y_1 \oplus R(L) \to X = \ker L \oplus D(L) \cap X_1,$$

is an isomorphism whose inverse is the operator,

$$L + JP : D(L) \cap Im(I - P) \oplus kerL \to R(L) \oplus Y_1$$

indeed, for every $x \in D(L) \cap Im(I - P) \oplus \ker L$, we write it in the form $x = (I - P)x + Px$,so

$$(L + JP)((I - P)x + Px) = L(I - P)x + JP(Px) = L(I - P)x + JPx,$$

consequently

$$(J^{-1} \oplus L_p^{-1})(L(I-P)x + JPx) = (I-P)x + Px = x.$$

On the other hand, for all $y \in Y$ we have

$$\left(J^{-1} \oplus L_p^{-1}\right) y = \left(J^{-1} \oplus L_p^{-1}\right)(Qy + (I-Q)y) = J^{-1}Qy + L_p^{-1}(I-Q)y,$$

by setting $K_{P,Q} = L_p^{-1}(I-Q)$,($K_{P,Q}$ is the inverse on the right of L associated with P and Q respectively), then we get $(L+JP)^{-1} = J^{-1}Q + K_{P,Q}$.

2.3.3 Perturbations of a Fredholm operator of zero index L-compact

To solve the equation $Lx = y$, we can write $x = Px + (I-P)x$ and $y = Qy + (I-Q)y$ and by substitution of x and y in the previous equation, we obtain

$$L(Px + (I-P)x) = Qy + (I-Q)y,$$

and since $Qy = 0$ and $LPx = 0$ (because $y \in Im(L)$ and $Px \in \ker L$), then

$$L(I-P)x = (I-Q)y,$$

which leads to

$$x - Px = Lp^{-1}(I-Q)y$$

and thus

$$x = Px + J^{-1}Qy + L_p^{-1}(I-Q)y.$$

Now consider the equation $Lx = Nx$, where $N : G \subset X \to Y$ is an operator (usually nonlinear) according to the above result; this last equation with $x \in D(L) \cap G$ is equivalent to

$$\begin{aligned} x &= Px + J^{-1}QNx + K_{P,Q}P, \\ QNx &= Mx, \end{aligned}$$

which is a fixed point problem.

Definition 37 *Let X and Y be two Banach spaces and and $L : D(L) \subset X \to Y$ a Fredholm operator of index 0. Let Ω an open bounded set of X such that $D(L) \cap \Omega \neq 0$.*

1) The map $N : X \to Y$ is $L-$ compact on $\overline{\Omega}$ if only if the operator $QN\left(\overline{\Omega}\right)$ is bounded and $K_{P,Q}N\Omega : \overline{\Omega} \to X$ is compact.

2) The degree of coincidence of L and N on is defined by

$$deg[(L, N), \Omega] = deg_{LS}(I - M, \Omega, 0)$$

where $M = P + J^{-1}QN + K_{P,Q}N$.

2.3.4 Mawhin's theorem

Theorem 38 *Let L be a Fredholm operator of index zero and let N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied.*

Theorem 39 *(i) $Lx \neq \lambda Nx$, for every $(x, \lambda) \in [(D(L) \backslash KerL) \cap \partial\Omega] \times (0, 1)$.*

(ii) $Nx \notin ImL$, for every $x \in KerL \cap \partial\Omega$.

(iii) $\deg(JQN_{|\ker L}, \ker L \cap \partial\Omega, 0) \neq 0$, where $J : ImQ \to \ker L$ is a linear isomorphism, $Q : Y \to Y$ is a projection as above with $ImL = \ker Q$. Then, the equation $Lx = Nx$ has at least one solution in $D(L) \cap \overline{\Omega}$.

Proof. For $\lambda \in [0, 1]$, consider the family of problems

$$x \in D(L) \cap \overline{\Omega}, \; Lx = \lambda Nx + (1 - \lambda)QNx \tag{2.2}$$

Let $M : [0,1] \times \overline{\Omega} \to Y$ be a homotopy defined by

$$M(\lambda, x) = Px + J^{-1}QNx + \lambda K_{P,Q}Nx$$

The problem (2.2) is equivalent to a fixed point problem

$$\begin{aligned} x &= Px + J^{-1}Q(\lambda N + (1-\lambda)QN)x + K_{P,Q}(\lambda N + (1-\lambda)QN)x \\ &= Px + \lambda J^{-1}QNx + (1-\lambda)J^{-1}QNx + \lambda K_{P,Q}Nx + (1-\lambda)K_{P,Q}QNx \\ &= M(\lambda, x). \end{aligned}$$

So this last equation is equivalent to a fixed point problem

$$x = M(\lambda, x), \quad x \in \overline{\Omega}, \tag{2.3}$$

If there exists an $x \in \partial\Omega$ such that $Lx = Nx$, then the proof is completed. Now suppose that

$$Lx \neq Nx \text{ for all } x \in D(L) \cap \Omega \tag{2.4}$$

and on the other hand

$$Lx \neq \lambda Nx + (1-\lambda)QNx \tag{2.5}$$

for all $(\lambda, x) \in]0,1[\times (D(L) \cap \Omega)$. If

$$Lx = Nx + (1-\lambda)QNx$$

for all $(\lambda, x) \in]0,1[\times (D(L) \cap \Omega)$, we obtain by application of Q to both members of the previous equality

$$QNx = 0, \quad Lx = \lambda Nx$$

The first of these equalities and the condition (ii) imply that $x \notin KerL \cap \partial\Omega$ i.e $x \in (D(L) \backslash KerL) \cap \partial\Omega$ and therefore the second equality contradicts (i). By using other times (ii), it follows that

$$Lx \neq QNx, \quad \text{for every} \quad x \in D(L) \cap \partial\Omega. \tag{2.6}$$

using (2.4), (2.5) and (2.6), we deduce that

$$x \neq M(\lambda, x) \text{ for all } (\lambda, x) \in [0,1] \times \partial\Omega \tag{2.7}$$

Since N is L-compact then $M(\lambda, x)$ is compact because. Using the homotopy invariance property of the Leray-Schauder degree, we obtain

$$deg_{LS}(I - M(0,.), \Omega, 0) = deg_{LS}(I - M(1,.), \Omega, 0) \tag{2.8}$$

On the other hand we have

$$deg_{LS}(I - M(0,\lambda), \Omega, 0) = deg_{LS}(I - (P + J^{-1}QN), \Omega, 0) \tag{2.9}$$

Since the image of $P + J^{-1}QN$ is contained in $Ker(L)$, then using the property of reduction of the Leray-Schauder degree and the fact that $P\mid_{KerL} = I\mid_{KerL}$, (since $Ker(L) = Im(P) = Ker(I - P)$), we obtain

$$\begin{aligned} deg_{LS}(I - (P + J^{-1}QN), \Omega, 0) &= deg(I - (P + J^{-1}QN), \Omega \cap KerL, 0) \\ &= deg(J^{-1}QN, \Omega \cap KerL, 0) \end{aligned} \tag{2.10}$$

Thanks to (2.8), (2.9) and (2.10), it follows that $deg_{LS}(I - M(1,.), \Omega, 0) \neq 0$, and so the existence property of the Leray-Schauder degree implies the existence of an $x \in \Omega$ such as $x = M(1, x)$ i.e $x \in D(L) \cap \Omega$, $Lx = Nx$. ■