Contrôle approché explicite par moyennisation

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Contents

5.1 Introduction

5.1.1 Main result

Following [122] we consider a quantum particle in a potential $V(x)$ and an electric field of amplitude $u(t)$. We assume that the dipolar approximation is not valid (see [64, 65]). Then, the particle is represented by its wave function $\psi(t,x)$ solution of the following Schrödinger equation

$$\begin{cases} i\partial_t\psi = (-\Delta + V(x))\psi - u(t)\mu_1(x)\psi - u(t)^2\mu_2(x)\psi, & x \in D, \\ \psi_{|\partial D} = 0, \end{cases} \tag{5.1}$$

with initial condition

$$\psi(0,x) = \psi^0(x), \quad x \in D, \tag{5.2}$$

where $D \subset \mathbb{R}^m$ is a bounded domain with smooth boundary. The functions $V, \mu_1, \mu_2 \in C^\infty(\overline{D}, \mathbb{R})$ are given, μ_1 is the dipolar moment and μ_2 the polarizability moment. For the sake of simplicity, we denote by L^2, H_0^1 and H^2 respectively the usual Lebesgue and Sobolev spaces $L^2(D,\mathbb{C})$, $H_0^1(D,\mathbb{C})$ and $H^2(D,\mathbb{C})$. The following well-posedness result holds (see [42]) by application of the Banach fixed point theorem.

Proposition 5.1. *For any $\psi^0 \in H_0^1 \cap H^2$ and $u \in L^2_{loc}([0,+\infty),\mathbb{R})$, the system (5.1)-(5.2) has a unique weak solution $\psi \in C^0([0,+\infty), H_0^1 \cap H^2)$. Moreover, for all $t > 0$, $||\psi(t,.)||_{L^2} = ||\psi^0||_{L^2}$ and there exists $C = C(\mu_1,\mu_2) > 0$ such that for any $t > 0$,*

$$||\psi(t,\cdot)||_{H^2} \leq ||\psi^0||_{H^2} e^{C\int_0^t |u(\tau)|+|u(\tau)|^2 \mathrm{d}\tau}.$$

Let $\mathcal{S} := \left\{\psi \in L^2(D,\mathbb{C}); ||\psi||_{L^2} = 1\right\}$ and $\langle\cdot,\cdot\rangle$ be the usual scalar product on $L^2(D,\mathbb{C})$

$$\langle f,g\rangle = \int_D f(x)\overline{g(x)}\mathrm{d}x, \quad \text{for } f,g \in L^2(D,\mathbb{C}).$$

We consider the operator A_V defined by

$$A_V\psi := (-\Delta + V(x))\,\psi, \quad D\,(A_V) := H_0^1 \cap H^2, \tag{5.3}$$

and denote by $(\lambda_{k,V})_{k\in\mathbb{N}^*}$ the non-decreasing sequence of its eigenvalues and by $(\varphi_{k,V})_{k\in\mathbb{N}^*}$ the associated eigenvectors in $\mathcal{S}$. The family $(\varphi_{k,V})_{k\in\mathbb{N}^*}$ is a Hilbert basis of L^2. We also define the space $H^4_{(V)} := D(A_V^2)$. As V is fixed, the eigenelements $\varphi_{k,V}$ and $\lambda_{k,V}$ will be denoted by φ_k and λ_k. The operator A_V will be denoted by A.

Our goal is to stabilize the ground state. As the global phase of the wave function is physically meaningless, our target set is

$$\mathcal{C} := \{c\varphi\,;\, c \in \mathbb{C} \text{ and } |c| = 1\}, \tag{5.4}$$

where $\varphi := \varphi_1$.

Let $J_{\neq 0} := \Big\{k \geq 2; \langle\mu_1\varphi,\varphi_k\rangle \neq 0\Big\}$ and $J_0 := \Big\{k \geq 2; \langle\mu_1\varphi,\varphi_k\rangle = 0\Big\}$. We assume that the following hypotheses hold.

Hypothesis 5.1.

i) $\forall k \in J_0, \langle\mu_2\varphi,\varphi_k\rangle \neq 0$ i.e. all coupling are realized either by μ_1 or μ_2,

ii) $Card(J_0) < \infty$ i.e. only a finite number of coupling is missed by μ_1,

iii) $\lambda_1 - \lambda_k \neq \lambda_p - \lambda_q$ for $k,p,q \geq 1$ such that $\{1,k\} \neq \{p,q\}$ and $k \neq 1$.

Remark 5.1. The hypothesis *i)* is weaker than the one in [19] (i.e. $J_0 = \emptyset$). As proved in [111, Section 3.4], we get that generically with respect to μ_1 and μ_2 in $C^\infty(\overline{D}, \mathbb{R})$, the scalar products $\langle \mu_1 \varphi, \varphi_k \rangle$ and $\langle \mu_2 \varphi, \varphi_k \rangle$ are all non-zero. The spectral assumption *iii)* does not hold in every physical situation. For example, it is not satisfied in 1D if $V = 0$. However, it is proved in [111, Lemma 3.12] that if D is the rectangle $[0,1]^n$, Hypothesis 1.1 *iii)* hold generically with respect to V in the set $\mathcal{G} := \{V \in C^\infty(D, \mathbb{R})\,;\, V(x_1, \dots, x_n) = V_1(x_1) + \cdots + V_n(x_n), \text{ with } V_k \in C^\infty([0,1], \mathbb{R})\}$.

As in [57], we use a time-periodic oscillating control of the form

$$u(t, \psi) := \alpha(\psi) + \beta(\psi) \sin\left(\frac{t}{\varepsilon}\right). \tag{5.5}$$

Following classical techniques (see e.g. [124]) of dynamical systems in finite dimension let us introduce the averaged system

$$\begin{cases} i\partial_t \psi_{av} = \big(-\Delta + V(x)\big)\psi_{av} - \alpha(\psi_{av})\mu_1(x)\psi_{av} - \left(\alpha(\psi_{av})^2 + \dfrac{1}{2}\beta(\psi_{av})^2\right)\mu_2(x)\psi_{av}, \\ \psi_{av_{|\partial D}} = 0, \end{cases} \tag{5.6}$$

with initial condition

$$\psi_{av}(0, \cdot) = \psi^0. \tag{5.7}$$

Let $\mathcal{P}$ be the orthogonal projection in L^2 onto the closure of Span $\{\varphi_k; k \geq 2\}$ and γ be a positive constant (to be determined later).

Our stabilization strategy relies on the following Lyapunov function (used in [19]) defined on $\mathcal{S} \cap H_0^1 \cap H^2$ by

$$\mathcal{L}(\psi) := \gamma ||(-\Delta + V)\mathcal{P}\psi||_{L^2}^2 + 1 - |\langle \psi, \varphi \rangle|^2. \tag{5.8}$$

This leads to feedback laws given by

$$\alpha(\psi_{av}(t, \cdot)) := -kI_1(\psi_{av}(t, \cdot)), \qquad \beta(\psi_{av}(t, \cdot)) := g(I_2(\psi_{av}(t, \cdot)), \tag{5.9}$$

with $k > 0$ small enough and

$$g \in C^2(\mathbb{R}, \mathbb{R}^+) \text{ satisfying } g(x) = 0 \text{ if and only if } x \geq 0, \ g' \text{ bounded}, \tag{5.10}$$

and for $j \in \{1, 2\}$, for $z \in H^2$,

$$I_j(z) = \mathrm{Im}\Big[-\gamma\langle(-\Delta + V)\mathcal{P}\,\mu_j z, (-\Delta + V)\mathcal{P}z\rangle + \langle \mu_j z, \varphi\rangle\langle\varphi, z\rangle\Big]. \tag{5.11}$$

We can now state the well-posedness of the averaged closed loop system (5.6).

Proposition 5.2. *Let $R > 0$. There exists $k_0 = k_0(V, \mu_2, R) > 0$ such that for any $\psi^0 \in H^2 \cap H_0^1 \cap \mathcal{S}$ with $\mathcal{L}(\psi^0) < R$ and $k \in (0, k_0)$, the closed-loop system (5.6)-(5.7)-(5.9) has a unique solution $\psi_{av} \in C^0([0, +\infty), H^2 \cap H_0^1)$. There exists $M > 0$ such that*

$$||\psi_{av}(t)||_{H^2} \leq M, \quad \forall t \geq 0. \tag{5.12}$$

Moreover, if $\Delta\psi^0 \in H_0^1 \cap H^2$, then $\Delta\psi_{av} \in C^0([0, +\infty), H_0^1 \cap H^2)$.

For an initial condition $\psi^0 \in \mathcal{S} \cap H^4_{(0)}$, we define the control

$$u^\varepsilon(t) := \alpha(\psi_{av}(t)) + \beta(\psi_{av}(t)) \sin\left(\frac{t}{\varepsilon}\right), \tag{5.13}$$

where ψ_{av} is the solution of (5.6)-(5.7)-(5.9).

The main result of this article is the following one.

Theorem 5.1. *Assume that Hypotheses 5.1 hold. Let $\mathcal{C}$, the target set, be defined by (5.4). There exists $k_0 = k_0(V, \mu_2) > 0$ such that for any $k \in k_0$, for any $s < 2$ and for any $\psi^0 \in \mathcal{S} \cap H^4_{(0)}$ with $0 < \mathcal{L}(\psi^0) < 1$, there exist an increasing time sequence $(T_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^*_+$ tending to $+\infty$ and a decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^*_+$ such that if ψ_ε is the solution of (5.1)-(5.2) associated to the control u^ε defined by (5.13) then for all $n \in \mathbb{N}$, if $\varepsilon \in (0, \varepsilon_n)$,*

$$dist_{H^s}\left(\psi_\varepsilon(t,\cdot), \mathcal{C}\right) \leq \frac{1}{2^n}, \quad \forall t \in [T_n, T_{n+1}].$$

Remark 5.2. Theorem 5.1 gives the semi-global approximate controllability with explicit controls of system (5.1). Hypotheses 5.1 are needed to ensure that the invariant set coincides with the target set. The semi-global aspect comes from the hypothesis $0 < \mathcal{L}(\psi^0) < 1$: by reducing γ (in a way dependant of ψ^0), this condition can be fulfilled as soon as $\psi^0 \notin \mathcal{C}$.

In Theorem 5.1, there is a gap between the H^4 regularity of the initial condition and the approximate controllability in H^s with $s < 2$. The extra-regularity is used in this article to prove an approximation property in H^2 between the oscillating system and the averaged one (see Section 5.3). Weakening this regularity assumption is an open problem for which an alternative strategy is required. The last loss of regularity comes from the application of a weak LaSalle principle instead of a strong one due to lack of compactness in infinite dimension.

5.1.2 A review of previous results

In this section, we recall previous results about quantum systems with bilinear controls. The model (5.1) of an infinite potential well was proposed by Rouchon [122] in the dipolar approximation ($\mu_2 = 0$). A classical negative result was obtained in [5] by Ball, Marsden and Slemrod for infinite dimensional bilinear control systems. This result implies, for system (5.1) with $\mu_2 = 0$, that the set of reachable states from any initial data in $H^2 \cap H^1_0 \cap \mathcal{S}$ with control in $L^2(0,T)$ has a dense complement in $H^2 \cap H^1_0 \cap \mathcal{S}$. However, exact controllability was proved in 1D by Beauchard [10] for $V = 0$ and $\mu_1(x) = x$ in more regular spaces (H^7). This result was then refined in [16] by Beauchard and Laurent for more general μ_1 and a regularity H^3.

The question of stabilization is addressed in [19] where Beauchard and Nersesyan extended previous results from Nersesyan [111]. They proved, under appropriate assumptions on μ_1, the semi-global weak H^2 stabilization of the wave function towards the ground state using explicit feedback control and Lyapunov techniques in infinite dimension.

However sometimes, for example in the case of higher laser intensities, this model is not efficient (see e.g. [64, 65]) and we need to add a polarizability term $u(t)^2\mu_2(x)\psi$ in the model. This term, if not neglected, can also be helpful in mathematical proofs. Indeed the result of [19] only holds if μ_1 couples the ground state to any other eigenstate and then the use of the polarizability enables us to weaken this assumption. Mathematical use of the expansion of the Hamiltonian beyond the dipolar approximation was used by Grigoriu, Lefter and Turinici in [80, 135]. A finite dimension approximation of this model was studied in [57] by Coron, Grigoriu, Lefter and Turinici. The authors proposed discontinuous feedback laws and periodic highly oscillating feedback laws to stabilize the ground state. In this article, we extend in our infinite dimensional framework their idea of using (time-dependent) periodic feedback laws. We also refer to the book by Coron [54] for a comprehensive presentation of the feedback strategy and the use of time-varying feedback laws.

How to adapt the Lyapunov or LaSalle strategy in an infinite dimensional framework is not clear because closed bounded sets are not compact so the trajectories may lack compactness in the considered topology. In this direction we should cite some related works of Mirrahimi and Beauchard [17, 103] where the idea was to prove approximate convergence results. In this article, we will use an adaptation of the LaSalle invariance principle for weak convergence which was used for example in [19] by Beauchard and Nersesyan. There are other strategies to show a strong stabilization property. Coron and d'Andréa-Novel proved in [55] the compactness of the trajectories by a direct method for a beam equation and thus the strong stabilization. Couchouron [61, 62] gave sufficient conditions to obtain the compactness in favorable cases where the control acts diagonally on the state. Another strategy to obtain strong results is to look for a strict Lyapunov function, which is an even trickier question, and was done for example by Coron, d'Andréa-Novel and Bastin [56] for a system of conservation laws.

The question of approximate controllability has been addressed by various authors using various techniques. In [112], Nersesyan uses a Lyapunov strategy to obtain approximate controllability in large time in regular spaces. In [45], Chambrion, Mason, Sigalotti and Boscain proved approximate controllability in L^2 for a wider class of systems using geometric control tools for the Galerkin approximations. The hypotheses needed were weakened in [25] and the approximate controllability was extended to some H^s spaces in [30].

Explicit approximate controllability in large time has also been obtained by Ervedoza and Puel in [70] on a model of trapped ion, using different tools.

5.1.3 Structure of this article

As announced in Section 5.1.1, we study the system (5.1) by introducing a highly oscillating time-periodic control and the corresponding averaged system. Section 5.2 is devoted to the introduction of this averaged system and its weak stabilization using Lyapunov techniques and an adaptation of the LaSalle invariance principle in infinite dimension.

In Section 5.3 we study the approximation property between the solution of the averaged system and the solution of (5.1) with the same initial condition. We prove that on every finite time interval these two solutions remain arbitrarily close provided that the control is oscillating enough. This is an extension of classical averaging results for finite dimension dynamical systems.

Finally gathering the stabilization result of Section 5.2 and the approximation property of Section 5.3, we prove Theorem 5.1 in Section 5.4.

Section 5.5 is devoted to numerical simulations illustrating several aspects of Theorem 5.1 and of the averaging strategy.

5.2 Stabilization of the averaged system

5.2.1 Definition of the averaged system

System (5.1) with feedback law u defined by (5.5) can be rewritten as

$$\begin{cases} \partial_t \psi(t) = -iA\psi(t) + F\left(\frac{t}{\varepsilon}, \psi(t)\right), \\ \psi_{|\partial D} = 0, \end{cases} \tag{5.14}$$

where the operator A is defined by (5.3) and

$$F(s,z) := i\left(\alpha(z) + \beta(z)\sin(s)\right)\mu_1 z + i\left(\alpha(z) + \beta(z)\sin(s)\right)^2 \mu_2 z. \tag{5.15}$$

For any z, $F(.,z)$ is T-periodic (with here $T = 2\pi$). Following classical techniques of averaging, we introduce $F^0(z) := \frac{1}{T}\int_0^T F(t,z)\mathrm{d}t$. We can define the averaged system associated to (5.14) by

$$\begin{cases} \partial_t \psi_{av} = -iA\psi_{av} + F^0(\psi_{av}), \\ \psi_{av_{|\partial D}} = 0. \end{cases} \tag{5.16}$$

Straightforward computations of F^0 show that the system (5.16) can be rewritten as (5.6).

We show by Lyapunov techniques that we can choose α and β such that the solution of the averaged system (5.16) is weakly convergent in H^2 towards our target set $\mathcal{C}$.

5.2.2 Control Lyapunov function and damping feedback laws

Our candidate for the Lyapunov function, $\mathcal{L}$, is defined in (5.8). It is clear that $\mathcal{L}(\psi) \geq 0$ whenever $\psi \in \mathcal{S} \cap H_0^1 \cap H^2$ and that $\mathcal{L}(\psi) = 0$ if and only if $\psi \in \mathcal{C}$.

The main advantage of this Lyapunov function is that it can be used to bound the H^2 norm. In fact, for any $\psi \in \mathcal{S} \cap H_0^1 \cap H^2$,

$$\mathcal{L}(\psi) \geq \gamma||(-\Delta + V)\mathcal{P}\psi||^2_{L^2} \geq \frac{\gamma}{2}||\Delta(\mathcal{P}\psi)||^2_{L^2} - C \geq \frac{\gamma}{4}||\Delta\psi||^2_{L^2} - C,$$

where here, as in all this article, C is a positive constant possibly different each time it appears. This leads to the existence of $\tilde{C} > 0$ satisfying

$$||\psi||^2_{H^2} \leq \tilde{C}(1 + \mathcal{L}(\psi)), \quad \forall \psi \in \mathcal{S} \cap H_0^1 \cap H^2. \tag{5.17}$$

Remark 5.3. Although the idea of using a feedback of the form (5.5) is inspired by [57], the construction of the Lyapunov function and of the controls is here different because we are dealing with an infinite dimensional framework. We follow the strategy used in [111, 19].

Choice of the feedbacks. We would like to choose the feedbacks α and β such that for all $t \geq 0$, $\frac{d}{dt}\mathcal{L}(\psi_{av}(t)) \leq 0$ where ψ_{av} is the solution of (5.6),(5.7).

If $\Delta\psi_{av}(t) \in H_0^1 \cap H^2$ for all $t \geq 0$ then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(\psi_{av}(t)) = 2\gamma\mathrm{Re}\big[\langle(-\Delta+V)\mathcal{P}\partial_t\psi_{av}, (-\Delta+V)\mathcal{P}\psi_{av}\rangle\big] - 2\mathrm{Re}\big[\langle\partial_t\psi_{av},\varphi\rangle\langle\varphi,\psi_{av}\rangle\big]$$
$$= 2\gamma\mathrm{Re}\Big[\langle(-\Delta+V)\mathcal{P}\big(i\Delta\psi_{av} - iV\psi_{av} + i\alpha\mu_1\psi_{av} + i(\alpha^2+\frac{1}{2}\beta^2)\mu_2\psi_{av}\big), (-\Delta+V)\mathcal{P}\psi_{av}\rangle\Big]$$
$$- 2\mathrm{Re}\Big[\langle i\Delta\psi_{av} - iV\psi_{av} + i\alpha\mu_1\psi_{av} + i(\alpha^2+\frac{1}{2}\beta^2)\mu_2\psi_{av},\varphi\rangle\langle\varphi,\psi_{av}\rangle\Big].$$

Then, we perform integration by parts. As $\mathcal{P}$ commutes with $(-\Delta+V)$, V is real and thanks to the following boundary conditions

$$(-\Delta+V)\mathcal{P}\psi_{av|\partial D} = \psi_{av|\partial D} = \varphi_{|\partial D} = 0,$$

we have

$$2\gamma\mathrm{Re}\Big[\langle -i(-\Delta+V)^2\mathcal{P}\psi_{av}, (-\Delta+V)\mathcal{P}\psi_{av}\rangle\Big] - 2\mathrm{Re}\Big[\langle(i\Delta - iV)\psi_{av},\varphi\rangle\langle\varphi,\psi_{av}\rangle\Big]$$
$$= 2\gamma\mathrm{Re}\Big[\langle -i\nabla(-\Delta+V)\mathcal{P}\psi_{av}, \nabla(-\Delta+V)\mathcal{P}\psi_{av}\rangle\Big]$$
$$+ 2\gamma\mathrm{Re}\Big[\langle -iV(-\Delta+V)\mathcal{P}\psi_{av}, (-\Delta+V)\mathcal{P}\psi_{av}\rangle\Big] + 2\lambda_1\mathrm{Re}\Big[\langle i\psi_{av},\varphi\rangle\langle\varphi,\psi_{av}\rangle\Big]$$
$$= 0.$$

This leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(\psi_{av}(t)) = 2\alpha I_1(\psi_{av}(t)) + 2\left(\alpha^2 + \frac{1}{2}\beta^2\right) I_2(\psi_{av}(t)), \tag{5.18}$$

where I_j is defined in (5.11).
In order to have a decreasing Lyapunov function we define the feedback laws α and β as in (5.9). Thus (5.18) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(\psi_{av}(t)) = -2\left(kI_1^2(1-kI_2) - \frac{1}{2}I_2 g^2(I_2)\right). \tag{5.19}$$

If we assume that we can choose the constant k such that $(1-kI_2) > 0$ for all $t \geq 0$ and if $\Delta\psi_{av}(t) \in H_0^1 \cap H^2$ then the feedbacks (5.9) in system (5.6) lead to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(\psi_{av}(t)) \leq 0, \quad \forall t \geq 0. \tag{5.20}$$

Well-posedness and boundedness proofs. Using the previous heuristic on the Lyapunov function, we can state and prove the well-posedness of the closed loop system (5.6)-(5.9) globally in time and derive a uniform bound on the H^2 norm of the solution. Namely, we prove Proposition 5.2.

Proof of Proposition 5.2. By the explicit expression (5.11) of I_2, we get for any $z \in H^2$, $|I_2(z)| \leq f\big(||z||_{H^2}\big)$ where

$$f(x) := ||\mu_2||_{L^\infty} + \gamma(x + ||V||_{L^\infty} + \lambda_1)(||\mu_2||_{\mathcal{C}^2}x + ||V||_{L^\infty}||\mu_2||_{L^\infty} + \lambda_1||\mu_2||_{L^\infty}).$$

Notice that f is increasing on $\mathbb{R}^+$. Let $K := 2f\left(\sqrt{\tilde{C}(1+R)}\right)$ where $\tilde{C}$ is defined by (5.17), $k_0 := \frac{1}{K}$ and $k \in (0, k_0)$.

The local existence and regularity is obtained by a classical fixed point argument : there exists $T^* > 0$ such that the closed loop system (5.6) with initial condition (5.7) and feedback laws (5.9) admits a unique solution defined on $(0, T^*)$ and satisfying either $T^* = +\infty$ or $T^* < +\infty$ and

$$\limsup_{t \to T^*} ||\psi_{av}(t)||_{H^2} = +\infty.$$

We have

$$|I_2(\psi_{av}(0))| \leq f(||\psi^0||_{H^2}) \leq f\left(\sqrt{\tilde{C}(1+\mathcal{L}(\psi^0))}\right) \leq \frac{K}{2},$$

thus, by continuity, $|I_2(\psi_{av}(t))| \leq K$ for t small enough.
Let

$$T_{max} := \sup\left\{t \in (0, T^*); |I_2(\psi_{av}(\tau))| \leq K, \forall \tau \in (0,t)\right\}.$$

We want to prove that $T_{max} = T^* = +\infty$.
For all $t \in [0, T_{max})$, we have $\left(1 - kI_2(\psi_{av}(t))\right) > 0$, which implies (by (5.19)), $\mathcal{L}(\psi_{av}(\cdot))$ is decreasing on $[0, T_{max})$. Estimate (5.17) leads to

$$||\psi_{av}(t)||_{H^2} \leq \sqrt{\tilde{C}(1+\mathcal{L}(\psi_{av}(t))} \leq \sqrt{\tilde{C}(1+\mathcal{L}(\psi^0))}, \quad \forall t \in [0, T_{max}). \tag{5.21}$$

Let us proceed by contradiction and assume that $T_{max} < T^*$. Thus $|I_2(\psi_{av}(T_{max}))| = K$. By definition of K,

$$|I_2(\psi_{av}(t))| \leq f\left(\sqrt{\tilde{C}(1+\mathcal{L}(\psi^0)}\right) \leq \frac{K}{2} \quad \forall t \in [0, T_{max}).$$

This is inconsistent with $|I_2(\psi_{av}(T_{max}))| = K$ so $T_{max} = T^*$ and the solution is bounded in H^2 when it is defined. As no blow-up is possible thanks to (5.21) we obtain that $T_{max} = T^* = +\infty$ and thus the solution is global in time and bounded.

Finally, taking the time derivative of the equation we obtain the announced regularity. □

5.2.3 Convergence Analysis

In all this section we assume that $k \in (0, k_0)$ where k_0 is defined in Proposition 5.2 with $R = 1$. The closed-loop stabilization for the averaged system (5.6) is given by the next statement.

Theorem 5.2. *Assume that Hypotheses 5.1 hold. If $\psi^0 \in \mathcal{S} \cap H^4_{(0)}$ with $0 < \mathcal{L}(\psi^0) < 1$, then the solution ψ_{av} of the closed-loop system (5.6)-(5.9) with initial condition (5.7) satisfies*

$$\psi_{av}(t) \underset{t \to \infty}{\rightharpoonup} \mathcal{C} \quad \text{in } H^2.$$

We prove this theorem by adapting the LaSalle invariance principle to infinite dimension in the same spirit as in [19]. This is done in two steps. First we prove that the invariant set, relatively to the closed-loop system (5.6)-(5.9) and the Lyapunov function $\mathcal{L}$, is $\mathcal{C}$. Here, Hypotheses 5.1 are crucial. Then we prove that every adherent point for the weak H^2 topology of the solution of this closed-loop system is contained in $\mathcal{C}$. This is due to the continuity of the propagator of the closed-loop system for the weak H^2 topology.

5.2.3.1 Invariant set

Proposition 5.3. *Assume that Hypotheses 5.1 hold. Assume that ψ^0 belongs to $\mathcal{S}\cap H_0^1\cap H^2$ and satisfies $\langle\psi^0,\varphi\rangle\neq 0$. If the function $t\mapsto\mathcal{L}(\psi_{av}(t))$ is constant, then $\psi^0\in\mathcal{C}$.*

Proof. Thanks to (5.19), the fact that $\big(1-kI_2(\psi_{av}(t))\big)>0$ for all $t\geq 0$ and (5.10) we get

$$I_1[\psi_{av}(\cdot)]\equiv 0,\quad I_2(\psi_{av}(\cdot))g^2\left(I_2(\psi_{av}(\cdot))\right)\equiv 0 \text{ i.e. } I_2(\psi_{av}(t))\geq 0,\quad \forall t\geq 0.$$

By (5.9) this implies that $\alpha(\psi_{av}(\cdot))\equiv\beta(\psi_{av}(\cdot))\equiv 0$ and then ψ_{av} is solution of the uncontrolled Schrödinger equation. So,

$$\psi_{av}(t)=\sum_{j=1}^{\infty}e^{-i\lambda_j t}\langle\psi^0,\varphi_j\rangle\varphi_j.$$

Recall that $\varphi:=\varphi_1$ is the ground state. Following the idea of [111], we obtain after computations and gathering the terms with different exponential term

$$\begin{aligned}I_1(\psi_{av}(t))=&\sum_{j,k\geq 2}\tilde{P}(\psi^0,j,k,\mu_1)e^{-i(\lambda_j-\lambda_k)t}+\sum_{j\in J_{\neq 0}}\tilde{\tilde{P}}(\psi^0,j,\mu_1)e^{i(\lambda_j-\lambda_1)t}\\&-\sum_{j\in J_{\neq 0}}\langle\psi^0,\varphi_j\rangle\langle\varphi,\psi^0\rangle\langle\mu_1\varphi_j,\varphi\rangle(1+\gamma\lambda_j^2)e^{-i(\lambda_j-\lambda_1)t},\end{aligned}$$

where $\tilde{P}(\psi^0,j,k,\mu_1)$ and $\tilde{\tilde{P}}(\psi^0,j,\mu_1)$ are constants. Then, by [111, Lemma 3.10],

$$\langle\psi^0,\varphi_j\rangle\langle\varphi,\psi^0\rangle\langle\mu_1\varphi_j,\varphi\rangle(1+\gamma\lambda_j^2)=0,\quad\forall j\in J_{\neq 0}.$$

Using the assumption $\langle\varphi,\psi^0\rangle\neq 0$ and Hypotheses 5.1 it comes that for all $j\in J_{\neq 0}$, $\langle\psi^0,\varphi_j\rangle=0$. This leads to

$$\psi_{av}(t)=e^{-i\lambda_1 t}\langle\psi^0,\varphi\rangle\varphi+\sum_{j\in J_0}e^{-i\lambda_j t}\langle\psi^0,\varphi_j\rangle\varphi_j,$$

where by Hypotheses 5.1, J_0 is a finite set. By simple computations we obtain,

$$\begin{aligned}I_2(\psi_{av}(t))=\mathrm{Im}\Big(&-\sum_{k,j\in J_0}\gamma\lambda_j\langle\varphi_j,\psi^0\rangle\langle\psi^0,\varphi_k\rangle\langle(-\Delta+V)\mathcal{P}(\mu_2\varphi_k),\varphi_j\rangle e^{i(\lambda_j-\lambda_k)t}\\&-\sum_{j\in J_0}\gamma\lambda_j\langle\varphi_j,\psi^0\rangle\langle\psi^0,\varphi\rangle\langle(-\Delta+V)\mathcal{P}(\mu_2\varphi),\varphi_j\rangle e^{i(\lambda_j-\lambda_1)t}\\&+\sum_{j\in J_0}\langle\psi^0,\varphi_j\rangle\langle\varphi,\psi^0\rangle\langle\mu_2\varphi_j,\varphi\rangle e^{-i(\lambda_j-\lambda_1)t}+|\langle\psi^0,\varphi\rangle|^2\langle\mu_2\varphi,\varphi\rangle\Big)\geq 0.\end{aligned}\tag{5.22}$$

There exists $N_0\in\mathbb{N}^*$ and $(\omega_n)_{n\in\{0,\dots,N_0\}}$ such that

$$\{\omega_n\,;\,n\in\{0,\dots,N_0\}\}=\{\pm(\lambda_k-\lambda_j)\,;\,(k,j)\in J_0\times(J_0\cup\{1\})\},$$

with $\omega_0 = 0$ and $\omega_j \neq \omega_k$ if $j \neq k$. Thus, (5.22) implies that for any $n \in \{0, \dots, N_0\}$, there exists $\Lambda_n = \Lambda_n(\psi^0, \mu_2) \in \mathbb{C}$ such that

$$\mathrm{Im}\Big(\sum_{j=0}^{N_0} \Lambda_j e^{i\omega_j t}\Big) \geq 0, \quad \forall t \geq 0. \tag{5.23}$$

Straightforward computations give

$$\Lambda_0 = |\langle \psi^0, \varphi\rangle|^2 \langle \mu_2 \varphi, \varphi\rangle - \sum_{j\in J_0} \gamma \lambda_j^2 |\langle \varphi_j, \psi^0\rangle|^2 \langle \mu_2 \varphi_j, \varphi_j\rangle.$$

Thus, $\mathrm{Im}\big(\Lambda_0\big) = 0$ and our inequality (5.23) can be rewritten as

$$\mathrm{Im}\Big(\sum_{j=1}^{N_0} \Lambda_j e^{i\omega_j t}\Big) \geq 0, \quad \forall t \geq 0,$$

with the ω_j being all different and non-zero. Then using the same argument as in [57, Proof of Theorem 3.1], we get that $\Lambda_j = 0$ for $j \geq 1$ and then using (5.22) in particular that the coefficient of $e^{-i(\lambda_j - \lambda_1)t}$ vanishes. It implies $\langle \psi^0, \varphi_j\rangle = 0$ for all $j \in J_0$. Consequently, $\psi^0 = \langle \psi^0, \varphi\rangle \varphi$. As $\psi^0, \varphi \in \mathcal{S}$, we obtain $\psi^0 \in \mathcal{C}$.

□

5.2.3.2 Weak H^2 continuity of the propagator

We denote by $\mathcal{U}_t(\psi^0)$ the propagator of the closed-loop system (5.6)-(5.9). We detail here the continuity property of this propagator and of the feedback laws we need to apply the LaSalle invariance principle.

Proposition 5.4. *Let $z_n \in \mathcal{S} \cap H_0^1 \cap H^2$ be a sequence such that $z_n \rightharpoonup z_\infty$ in H^2. For every $T > 0$, there exists $N \subset (0,T)$ of zero Lebesgue measure verifying for all $t \in (0,T)\backslash N$,*

i) $\mathcal{U}_t(z_n) \underset{n\to\infty}{\rightharpoonup} \mathcal{U}_t(z_\infty)$ in H^2,

ii) $\alpha\big(\mathcal{U}_t(z_n)\big) \underset{n\to\infty}{\to} \alpha\big(\mathcal{U}_t(z_\infty)\big)$ and $\beta\big(\mathcal{U}_t(z_n)\big) \underset{n\to\infty}{\to} \beta\big(\mathcal{U}_t(z_\infty)\big)$.

Proof. Proof of ii). We start by proving that if $(z_n)_{n\in\mathbb{N}} \in H_0^1 \cap H^2$ satisfy

$z_n \underset{n\to\infty}{\rightharpoonup} z_\infty$ in H^2 then $\alpha(z_n) \underset{n\to\infty}{\to} \alpha(z_\infty)$ and $\beta(z_n) \underset{n\to\infty}{\to} \beta(z_\infty)$. Thus *ii*) will be a simple consequence of *i*). As proved in [19, Proposition 2.2], using the fact that the regularity $H^{3/2}$ is sufficient to define the feedback, we get

$$I_j(z_n) \underset{n\to+\infty}{\longrightarrow} I_j(z_\infty), \quad \text{for } j = 1, 2.$$

So by the design of our feedback,

$$\alpha(z_n) \underset{n\to+\infty}{\longrightarrow} \alpha(z_\infty), \quad \beta(z_n) \underset{n\to+\infty}{\longrightarrow} \beta(z_\infty).$$

Proof of i). The exact same proof as in [19, Proposition 2.2] based on extraction in less regular spaces, uniqueness property of the closed loop system and taking into account the polarizability term leads to the announced result.

□

5.2.3.3 LaSalle invariance principle

We now have all the needed tools to prove Theorem 5.2.

Proof of Theorem 5.2. Consider $\psi^0 \in \mathcal{S} \cap H^4_{(0)}$ with $0 < \mathcal{L}(\psi^0) < 1$. Thanks to the bound (5.17), $\mathcal{U}_t(\psi^0)$ is bounded in H^2. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence of times tending to $+\infty$ and $\psi_\infty \in H^2$ be such that $\mathcal{U}_{t_n}(\psi^0) \underset{n\to\infty}{\rightharpoonup} \psi_\infty$ in H^2. We want to show that $\psi_\infty \in \mathcal{C}$.

We prove that $\alpha(\mathcal{U}_t(\psi_\infty)) = 0$ and $\beta(\mathcal{U}_t(\psi_\infty)) = 0$. Indeed, the function $t \mapsto \alpha\big(\mathcal{U}_t(\psi^0)\big)$ belongs to $L^2(0,+\infty)$ (because of (5.19) and (5.9)) so the sequence of functions $(t \in (0,+\infty) \mapsto \alpha\big(\mathcal{U}_{t_n+t}(\psi^0)\big)_n$ tends to zero in $L^2(0,+\infty)$. Then by the Lebesgue reciprocal theorem there exists a subsequence $(t_{n_k})_{k\in\mathbb{N}}$ and $N_1 \subset (0,+\infty)$ of zero Lebesgue measure such that

$$\alpha\big(\mathcal{U}_{t+t_{n_k}}(\psi^0)\big) \underset{k\to\infty}{\to} 0, \quad \forall t \in (0,+\infty)\backslash N_1.$$

Let $T \in (0,+\infty)$. Using Proposition 5.4, there exists $N \subset (0,T)$ of zero Lebesgue measure such that

$$\alpha\big(\mathcal{U}_{t+t_{n_k}}(\psi^0)\big) \underset{k\to\infty}{\to} \alpha\big(\mathcal{U}_t(\psi_\infty)\big), \quad \forall t \in (0,T)\backslash N.$$

Hence, $\alpha\big(\mathcal{U}_t(\psi_\infty)\big) = 0$ for all $t \in (0,T)\backslash(N_1 \cup N)$. The function $t \mapsto \alpha\big(\mathcal{U}_t(\psi_\infty)\big)$ being continuous we get $\alpha\big(\mathcal{U}_t(\psi_\infty)\big) = 0$ for all $t \in [0,T]$, and this for all $T > 0$. Finally $\alpha\big(\mathcal{U}_t(\psi_\infty)\big) = 0$ for all $t \geq 0$.

The same argument holds for β as $\tilde{g} : t \mapsto I_2\big(\mathcal{U}_t(\psi^0)\big)g^2\left(I_2\big(\mathcal{U}_t(\psi^0)\big)\right)$ belongs to $L^1(0,+\infty)$. Then by the proof of Proposition 5.4,

$$\tilde{g}\left(\mathcal{U}_{t+t_{n_k}}(\psi^0)\right) \underset{k\to\infty}{\to} \tilde{g}\left(\mathcal{U}_t(\psi_\infty)\right), \quad \forall t \in (0,T)\backslash N,$$

and $\tilde{g}\big(\mathcal{U}_t(\psi_\infty)\big) = 0$ implies $\beta\big(\mathcal{U}_t(\psi_\infty)\big) \equiv 0$.

These two results lead to the fact that $\mathcal{L}\big(\mathcal{U}_t(\psi_\infty)\big)$ is constant.

By (5.20), $\mathcal{L}(\psi_\infty) \leq \mathcal{L}(\psi^0) < 1$ so $\langle \psi_\infty, \varphi \rangle \neq 0$. All assumptions of Proposition 5.3 are satisfied then $\psi_\infty \in \mathcal{C}$.

This concludes the proof of Theorem 5.2 and the convergence analysis of (5.6).

□

5.3 Approximation by averaging

The method of averaging was mostly used for finite-dimensional dynamical systems (see e.g. [124]). The concept of averaging in quantum control theory has already produced interesting results. For example, in [106] the authors make important use of these averaging properties in finite dimension through what is called in quantum physics the rotating wave approximation. The main idea of using a highly oscillating control is that if it is oscillating enough the initial system behaves like the averaged system. We extend this concept in our infinite dimensional framework : we prove an approximation result on every finite time interval. More precisely we have the following result.

Proposition 5.5. *Let $[s,L]$ be a fixed interval and $\psi^0 \in \mathcal{S} \cap H^4_{(0)}$ with $0 < \mathcal{L}(\psi^0) < 1$. Let ψ_{av} be the solution of the closed loop system (5.6),(5.9) with initial condition $\psi_{av}(s,\cdot) = \psi^0$. For any $\delta > 0$, there exists $\varepsilon_0 > 0$ such that, if ψ_ε is the solution of (5.1) associated to the same initial condition $\psi_\varepsilon(s,\cdot) = \psi^0$ and control $u^\varepsilon(t)$ defined by (5.13) with $\varepsilon \in (0,\varepsilon_0)$ then*

$$||\psi_\varepsilon(t,\cdot) - \psi_{av}(t,\cdot)||_{H^2} \leq \delta, \quad \forall t \in [s,L].$$

Remark 5.4. Notice that the controls α and β were defined using the averaged system in a feedback form but the control u^ε used for the system (5.1) is explicit and is not defined as a feedback control.

Remark 5.5. Due to the infinite dimensional framework, we are facing regularity issues and cannot adapt directly the strategy of [124].

Proof. We define for $(t,z,\tilde{z}) \in \mathbb{R} \times H^2 \times H^2$,

$$\tilde{F}(t,z,\tilde{z}) := i\left(\alpha(\tilde{z}) + \beta(\tilde{z})\sin(t)\right)\mu_1 z + i\left(\alpha(\tilde{z}) + \beta(\tilde{z})\sin(t)\right)^2 \mu_2 z. \tag{5.24}$$

Notice that thanks to (5.15) for any $(t,z) \in \mathbb{R} \times H^2$,

$$\tilde{F}(t,z,z) = F(t,z). \tag{5.25}$$

With these notations the considered system (5.1) with control (5.13) and initial condition $\psi_\varepsilon(s,\cdot) = \psi^0$ can be rewritten as

$$\begin{cases} \partial_t \psi_\varepsilon(t) = -iA\psi_\varepsilon(t) + \tilde{F}\left(\dfrac{t}{\varepsilon}, \psi_\varepsilon(t), \psi_{av}(t)\right), \\ \psi_{\varepsilon_{|\partial D}} = 0, \end{cases}$$

where ψ_{av} is the solution of the closed-loop system (5.6) with initial condition $\psi_{av}(s,\cdot) = \psi^0$.

Denoting by T_A the semigroup generated by $-iA$, we have for any $t \geq s$,

$$\psi_\varepsilon(t) = T_A(t-s)\psi^0 + \int_s^t T_A(t-\tau)\tilde{F}\left(\frac{\tau}{\varepsilon}, \psi_\varepsilon(\tau), \psi_{av}(\tau)\right)\mathrm{d}\tau,$$
$$\psi_{av}(t) = T_A(t-s)\psi^0 + \int_s^t T_A(t-\tau)F^0\big(\psi_{av}(\tau)\big)\mathrm{d}\tau.$$

This implies for any $t \geq s$,

$$\begin{aligned} ||\psi_\varepsilon(t) - \psi_{av}(t)||_{H^2} \leq & \left|\left|\int_s^t T_A(t-\tau)\left[F\left(\frac{\tau}{\varepsilon}, \psi_{av}(\tau)\right) - F^0\left(\psi_{av}(\tau)\right)\right]\mathrm{d}\tau\right|\right|_{H^2} \\ & + \left|\left|\int_s^t T_A(t-\tau)\Big[\tilde{F}\left(\frac{\tau}{\varepsilon}, \psi_\varepsilon(\tau), \psi_{av}(\tau)\right) - F\left(\frac{\tau}{\varepsilon}, \psi_{av}(\tau)\right)\Big]\mathrm{d}\tau\right|\right|_{H^2}. \end{aligned} \tag{5.26}$$

We study separately the two terms of the right-hand side of (5.26).

First step : We show the existence of $C > 0$ such that for any $t \geq s$, for any $\varepsilon > 0$,

$$\begin{aligned} \left|\left|\int_s^t T_A(t-\tau)\left[\tilde{F}\left(\frac{\tau}{\varepsilon}, \psi_\varepsilon(\tau), \psi_{av}(\tau)\right) - F\left(\frac{\tau}{\varepsilon}, \psi_{av}(\tau)\right)\right]\mathrm{d}\tau\right|\right|_{H^2} \\ \leq C\int_s^t \big|\big|\psi_\varepsilon(\tau) - \psi_{av}(\tau)\big|\big|_{H^2}\mathrm{d}\tau. \end{aligned} \tag{5.27}$$

By (5.15),(5.24), it comes that for any $\tau \geq s$, for any $\varepsilon > 0$,

$$\begin{aligned}
&\tilde{F}\left(\frac{\tau}{\varepsilon}, \psi_\varepsilon(\tau), \psi_{av}(\tau)\right) - F\left(\frac{\tau}{\varepsilon}, \psi_{av}(\tau)\right) \\
&\quad = i\left(\alpha(\psi_{av}(\tau)) + \beta(\psi_{av}(\tau)) \sin\left(\frac{\tau}{\varepsilon}\right)\right) \mu_1\left[\psi_\varepsilon(\tau) - \psi_{av}(\tau)\right] \\
&\quad + i\left(\alpha(\psi_{av}(\tau)) + \beta(\psi_{av}(\tau)) \sin\left(\frac{\tau}{\varepsilon}\right)\right)^2 \mu_2\left[\psi_\varepsilon(\tau) - \psi_{av}(\tau)\right].
\end{aligned}$$

As ψ_{av} is bounded in H^2, using (5.9) and (5.11) we get the existence of $M_1 > 0$ such that for all $\tau \geq s$,

$$|\alpha(\psi_{av}(\tau))| + |\beta(\psi_{av}(\tau))| \leq M_1. \tag{5.28}$$

As $|\sin\left(\frac{\tau}{\varepsilon}\right)| \leq 1$, we get the existence of $C > 0$ independent of ε such that for any $\tau \geq s$, for any $\varepsilon > 0$,

$$\left|\left|\tilde{F}\left(\frac{\tau}{\varepsilon}, \psi_\varepsilon(\tau), \psi_{av}(\tau)\right) - F\left(\frac{\tau}{\varepsilon}, \psi_{av}(\tau)\right)\right|\right|_{H^2} \leq C\left|\left|\psi_\varepsilon(\tau) - \psi_{av}(\tau)\right|\right|_{H^2}. \tag{5.29}$$

Then the contraction property of T_A implies (5.27).

Second step : We show that there exists $C > 0$ satisfying for all $t \in [s, L]$, for any $\varepsilon > 0$,

$$\left|\left|\int_s^t T_A(t-\tau)\left[F\left(\frac{\tau}{\varepsilon}, \psi_{av}(\tau)\right) - F^0(\psi_{av}(\tau))\right] \mathrm{d}\tau\right|\right|_{H^2} \leq C\varepsilon. \tag{5.30}$$

We follow computations on the semigroup T_A done in [82]. For $(t, v) \in \mathbb{R}^+ \times C^1([s, L], H^2)$, we define U and H by

$$\begin{aligned}
U(t, v(\cdot)) &:= \int_0^t \left(F(\tau, v(\cdot)) - F^0(v(\cdot))\right) \mathrm{d}\tau, \\
H(t, v) &:= \mathrm{d}_v U(t, v)\dot{v},
\end{aligned}$$

where $\dot{v}$ is the time derivative of v.
Notice that the T-periodicity of $F(\cdot, v)$ and the definition of F^0 imply that $U(\cdot, v)$ is also T-periodic.

Lemma 5.1. *As $\psi_{av} \in C^1([s, L], H_0^1 \cap H^2)$, we have for any $t \in [s, L]$, for any $\varepsilon > 0$,*

$$\begin{aligned}
&\int_s^t T_A(t-\tau)\left[F\left(\frac{\tau}{\varepsilon}, \psi_{av}(\tau)\right) - F^0(\psi_{av}(\tau))\right] \mathrm{d}\tau = \\
&\quad \varepsilon U\left(\frac{t}{\varepsilon}, \psi_{av}(t)\right) - \varepsilon T_A(t-s) U\left(\frac{s}{\varepsilon}, \psi_{av}(s)\right) \\
&\quad - i\varepsilon A \int_s^t T_A(t-\tau) U\left(\frac{\tau}{\varepsilon}, \psi_{av}(\tau)\right) \mathrm{d}\tau - \varepsilon \int_s^t T_A(t-\tau) H\left(\frac{\tau}{\varepsilon}, \psi_{av}(\tau)\right) \mathrm{d}\tau.
\end{aligned}$$

Proof. The proof is done in [82, Lemma 2.2] .

□

We study separately each term of the previous right-hand side.
• With $\kappa = \lfloor \frac{t}{\varepsilon T} \rfloor$, we have $\frac{t}{\varepsilon} - \kappa T \in [0, T]$ and by periodicity

$$\begin{aligned} U\left(\frac{t}{\varepsilon}, \psi_{av}(t)\right) &= \int_0^{t/\varepsilon} \Big(F(\tau, \psi_{av}(t)) - F^0(\psi_{av}(t))\Big) \mathrm{d}\tau \\ &= \int_0^{t/\varepsilon - \kappa T} \Big(F(\tau, \psi_{av}(t)) - F^0(\psi_{av}(t))\Big) \mathrm{d}\tau. \end{aligned}$$

As ψ_{av} is bounded in H^2 and $\alpha(\psi_{av})$, $\beta(\psi_{av})$ are bounded there exists $M_2 > 0$ such that

$$||F(\tau, \psi_{av}(t))||_{H^2} \leq M_2, \quad ||F^0(\psi_{av}(t))||_{H^2} \leq M_2, \quad \forall \tau \geq 0, \forall t \geq s.$$

This leads to

$$\left|\left|U\left(\frac{t}{\varepsilon}, \psi_{av}(t)\right)\right|\right|_{H^2} \leq \int_0^{t/\varepsilon - \kappa T} 2M_2 \mathrm{d}\tau \leq 2M_2 T, \quad \forall t \geq s, \forall \varepsilon > 0.$$

The same computations lead to

$$\left|\left|T_A(t-s)U\left(\frac{s}{\varepsilon}, \psi_{av}(s)\right)\right|\right|_{H^2} \leq 2M_2 T, \quad \forall t \geq s, \forall \varepsilon > 0.$$

Then,

$$\left|\left|\varepsilon U\left(\frac{t}{\varepsilon}, \psi_{av}(t)\right) + \varepsilon T_A(t-s)U\left(\frac{s}{\varepsilon}, \psi_{av}(s)\right)\right|\right|_{H^2} \leq C\varepsilon, \quad \forall t \geq s, \forall \varepsilon > 0. \tag{5.31}$$

• By switching property,

$$A\int_s^t T_A(t-\tau)U\left(\frac{\tau}{\varepsilon}, \psi_{av}(\tau)\right) \mathrm{d}\tau = \int_s^t T_A(t-\tau)AU\left(\frac{\tau}{\varepsilon}, \psi_{av}(\tau)\right) \mathrm{d}\tau,$$

and for any $t \in [s, L]$, for any $\varepsilon > 0$

$$\begin{aligned} AU\left(\frac{t}{\varepsilon}, \psi_{av}(t)\right) &= A\int_0^{t/\varepsilon - \kappa T} \left[F(\tau, \psi_{av}(t)) - F^0(\psi_{av}(t))\right] \mathrm{d}\tau \\ &= \int_0^{t/\varepsilon - \kappa T} \left[AF(\tau, \psi_{av}(t)) - AF^0(\psi_{av}(t))\right] \mathrm{d}\tau. \end{aligned}$$

By definition of F and F^0 we have

$$\begin{aligned} AF(t,z) &= i(\alpha(z) + \beta(z)\sin(t))A(\mu_1 z) + i(\alpha(z) + \beta(z)\sin(t))^2 A(\mu_2 z), \\ AF^0(z) &= i\alpha(z)A(\mu_1 z) + i\left(\alpha(z)^2 + \frac{1}{2}\beta(z)^2\right)A(\mu_2 z). \end{aligned}$$

By regularity hypothesis on μ_1, μ_2 and V there exists $C > 0$ such that

$$||A(\mu_1 z)||_{H^2} \leq C||\Delta z||_{H^2}, \quad ||A(\mu_2 z)||_{H^2} \leq C||\Delta z||_{H^2}.$$

Thus thanks to Proposition 5.2 and the bound (5.28) on $\alpha(\psi_{av})$ and $\beta(\psi_{av})$, we get the existence of $M_3 > 0$ satisfying

$$||AF(\tau, \psi_{av}(t))||_{H^2} \leq M_3, \quad ||AF^0(\psi_{av}(t))||_{H^2} \leq M_3, \quad \forall \tau \geq 0, \forall t \in [s, L].$$

So, for any $t \in [s, L]$, for any $\varepsilon > 0$, $\left|\left|AU\left(\frac{t}{\varepsilon}, \psi_{av}(t)\right)\right|\right|_{H^2} \leq 2M_3T$. Consequently , there exists $C > 0$ such that

$$\left|\left|i\varepsilon A \int_s^t T_A(t-\tau)U\left(\frac{\tau}{\varepsilon}, \psi_{av}(\tau)\right) \mathrm{d}\tau\right|\right|_{H^2} \leq C\varepsilon, \quad \forall t \in [s, L], \forall \varepsilon > 0. \tag{5.32}$$

- For the last term we need to estimate $H\left(\frac{t}{\varepsilon}, \psi_{av}(t)\right)$. We have

$$\begin{aligned} H\left(\frac{t}{\varepsilon}, \psi_{av}(t)\right) &= \mathrm{d}_v U\left(\frac{t}{\varepsilon}, \psi_{av}(t)\right).\partial_t \psi_{av}(t) \\ &= \int_0^{t/\varepsilon - \kappa T} \Big(\mathrm{d}_v F(\tau, \psi_{av}).\partial_t \psi_{av} - \mathrm{d}F^0(\psi_{av}).\partial_t \psi_{av}\Big)\mathrm{d}\tau. \end{aligned}$$

Using (5.9) and (5.11), we have for any $v, w \in C^0([s, L], H_0^1 \cap H^2)$,

$$\mathrm{d}\alpha(v).w = -k\mathrm{d}I_1(v).w, \quad \mathrm{d}\beta(v).w = g'\big(I_2(v)\big)\mathrm{d}I_2(v).w, \tag{5.33}$$

where,

$$\begin{aligned} \mathrm{d}I_j(v).w = \mathrm{Im}\Big[&-\gamma\langle(-\Delta+V)\mathcal{P}(\mu_j w), (-\Delta+V)\mathcal{P}v\rangle \\ &-\gamma\langle(-\Delta+V)\mathcal{P}(\mu_j v), (-\Delta+V)\mathcal{P}w\rangle + \langle\mu_j w, \varphi\rangle\langle\varphi, v\rangle + \langle\mu_j v, \varphi\rangle\langle\varphi, w\rangle\Big]. \end{aligned}$$

Finally, we have

$$\begin{aligned} &\mathrm{d}_v F(t, v).w = i(\alpha(v) + \beta(v)\sin t)\mu_1 w + i(\mathrm{d}\alpha(v).w + \mathrm{d}\beta(v).w \sin t)\mu_1 v \\ &+i(\alpha(v) + \beta(v)\sin t)^2\mu_2 w + 2i(\alpha(v) + \beta(v)\sin t)(\mathrm{d}\alpha(v).w + \mathrm{d}\beta(v).w\sin t)\mu_2 v, \end{aligned} \tag{5.34}$$

and

$$\begin{aligned} \mathrm{d}F^0(v).w &= i\alpha(v)\mu_1 w + i\mathrm{d}\alpha(v).w\mu_1 v + i\left(\alpha(v)^2 + \frac{1}{2}\beta(v)^2\right)\mu_2 w \\ &+i(2\alpha(v)\mathrm{d}\alpha(v).w + \beta(v)\mathrm{d}\beta(v).w\sin t)\mu_2 v. \end{aligned} \tag{5.35}$$

By Proposition 5.2, $\partial_t \psi_{av} \in C^0([0, +\infty), H_0^1 \cap H^2)$ so there exists $M_4 > 0$ such that

$$||\partial_t \psi_{av}(t)||_{H^2} \leq M_4, \quad \forall t \in [s, L].$$

Hence, the same computations as previously lead to the existence of $C > 0$ satisfying

$$|\mathrm{d}\alpha(\psi_{av}(t)).\partial_t \psi_{av}(t)| + |\mathrm{d}\beta(\psi_{av}(t)).\partial_t \psi_{av}(t)| \leq C, \quad \forall t \in [s, L],$$

and thus by (5.34),(5.35), for any $t \in [s, L]$, for any $\tau \geq 0$,

$$||\mathrm{d}_v F(\tau, \psi_{av}(t)).\partial_t \psi_{av}(t)||_{H^2} + ||\mathrm{d}F^0(\psi_{av}(t)).\partial_t \psi_{av}(t)||_{H^2} \leq C.$$

As a consequence,

$$\left|\left|H\left(\frac{\tau}{\varepsilon},\psi_{av}(\tau)\right)\right|\right|_{H^2} \leq CT, \quad \forall \tau \in [s,L], \forall \varepsilon > 0,$$

and then,

$$\left|\left|\varepsilon \int_s^t T_A(t-\tau) H\left(\frac{\tau}{\varepsilon},\psi_{av}(\tau)\right)\right|\right|_{H^2} \mathrm{d}\tau \leq (CLT)\varepsilon. \tag{5.36}$$

We are now able to deal with the remaining term of the right-hand side of (5.26). Gathering inequalities (5.31), (5.32) and (5.36) in Lemma 5.1 we obtain that there exists $C > 0$ such that inequality (5.30) holds.

Third step : Putting together (5.26), (5.27) and (5.30) we obtain that there exists $C > 0$ such that for any $t \in [s, L]$, for any $\varepsilon > 0$,

$$||\psi_\varepsilon(t) - \psi_{av}(t)||_{H^2} \leq C\varepsilon + C\int_s^t ||\psi_\varepsilon(\tau) - \psi_{av}(\tau)||_{H^2}\mathrm{d}\tau.$$

Hence Grönwall's lemma implies

$$||\psi_\varepsilon(t) - \psi_{av}(t)||_{H^2} \leq C\varepsilon e^{C(t-s)} \leq (Ce^{C(L-s)})\varepsilon, \quad \forall t \in [s,L],$$

and Proposition 5.5 is proved with $\varepsilon_0 = \dfrac{\delta}{Ce^{C(L-s)}}$.

□

Remark 5.6. The proof we used is fundamentally based on the boundedness of $\Delta\psi_{av}(t)$ on $[s, L]$ and on Grönwall's lemma so it cannot be extended directly to an infinite time interval $[s, +\infty)$.

5.4 Explicit approximate controllability

The solution ψ_{av} of the averaged system (5.6),(5.9), can be driven in the H^2 weak topology to the target set $\mathcal{C}$. The solution ψ_ε of the system (5.1) associated to the same initial condition, with control u^ε, stays close to ψ_{av} on every finite time interval provided that the control is oscillating enough. Gathering these two results we prove Theorem 5.1.

Proof of Theorem 5.1. We consider $s < 2$ fixed.

By Theorem 5.2, we can construct an increasing time sequence $(T_n)_{n\in\mathbb{N}}$ tending to $+\infty$ such that for any $n \in \mathbb{N}$,

$$\mathrm{dist}_{H^s}(\psi_{av}(t), \mathcal{C}) \leq \frac{1}{2^{n+1}}, \quad \forall t \geq T_n. \tag{5.37}$$

Using Proposition 5.5 on the time interval $[0, T_{n+1}]$ we then construct a decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that for any $n \in \mathbb{N}$,

$$||\psi_\varepsilon(t) - \psi_{av}(t)||_{H^s} \leq \frac{1}{2^{n+1}}, \quad \forall t \in [0, T_{n+1}], \forall \varepsilon \in (0, \varepsilon_n). \tag{5.38}$$

Then (5.37),(5.38) imply that

$$\forall n \in \mathbb{N}, \quad \operatorname{dist}_{H^s}(\psi_\varepsilon(t), \mathcal{C}) \leq \frac{1}{2^n}, \quad \forall t \in [T_n, T_{n+1}], \forall \varepsilon \in (0, \varepsilon_n),$$

which is the statement of Theorem 5.1.

□

5.5 Numerical simulations

This section is dedicated to numerical simulations of system (5.1). First, we detail how we approximate the solutions of (5.1) and (5.6). Then, we check the validity of the implemented code. Finally, we illustrate different aspects of Theorem 5.1 and of the averaging property, Proposition 5.5.

5.5.1 Settings

In all what follows, we set $D = [0, 1]$. As the potential V will vary in this section, the eigenelements of $-\Delta + V$ are denoted $\varphi_{k,V}$ and $\lambda_{k,V}$. Any function $\psi \in L^2((0,1), \mathbb{C})$ is approximated by its first M modes

$$\psi(t) \approx \sum_{k=1}^{M} x_k(t)\varphi_{k,V.}$$

The unknown eigenvectors $\varphi_{k,V}$ are approximated in the following way

$$\varphi_{k,V} \approx \sum_{j=1}^{N} a_j^k \varphi_{k,0}.$$

The equality $(-\Delta + V)\varphi_{k,V} = \lambda_{k,V}\varphi_{k,V}$ leads to $Ba^k = \lambda_{k,V} a^k$ with

$$a^k = (a_1^k, \dots, a_N^k)^t, \quad B = \operatorname{diag}(\lambda_{1,0}, \dots, \lambda_{N,0}) + \big(\langle V\varphi_{i,0}, \varphi_{j,0}\rangle\big)_{1\leq i,j\leq N}.$$

Notice that $\lambda_{k,0} = (k\pi)^2$ and $\varphi_{k,0} = \sqrt{2}\sin(k\pi\cdot)$ are explicit. The scalar products are approximated by the Matlab function `quadl`. The eigenelements a^k and $\lambda_{k,V}$ are then approximated by the Matlab function `eig`.

5.5.2 Approximation of ψ_ε and ψ_{av}

Let

$$H_0 := \operatorname{diag}(\lambda_{1,V}, \dots, \lambda_{M,V}), \quad H_n := \big(\langle \mu_n \varphi_{i,V}, \varphi_{j,V}\rangle\big)_{1\leq i,j\leq M}, \; n \in \{1, 2\}.$$

It follows that the feedback laws (5.11) are approximated, for $X \in \mathbb{R}^M$ and $j \in \{1, 2\}$, by

$$I_j(X) := \operatorname{Im}\Big(-\gamma\big(H_0(0, (H_jX)_2, \dots, (H_jX)_M)^t\big)^t \left(H_0(0, \overline{x_2}, \dots, \overline{x_M})\right) + (H_jX)_1\overline{x_1}\Big),$$

leading to

$$\alpha(X) := -kI_1(X), \quad \beta(X) := -\min(I_2(X), 0).$$

Thus, if we define X_{av}, $X_\varepsilon \in \mathbb{R}^M$, systems (5.1) and (5.6) are approximated by

$$i\frac{\mathrm{d}X_\varepsilon}{\mathrm{d}t} = \big(H_0 - u_\varepsilon(t)H_1 - u_\varepsilon(t)^2 H_2\big)X_\varepsilon, \tag{5.39}$$

and

$$i\frac{\mathrm{d}X_{av}}{\mathrm{d}t} = \left(H_0 - \alpha(X_{av})H_1 - \left(\alpha^2(X_{av}) + \frac{1}{2}\beta(X_{av})^2\right)H_2\right)X_{av}, \tag{5.40}$$

where $u_\varepsilon(t) = \alpha(X_{av}(t)) + \beta(X_{av}(t))\sin(t/\varepsilon)$. Equations (5.39) and (5.40) are solved numerically (simultaneously) using Euler method with a time step $\mathrm{d}t$ and a Strang splitting method.

5.5.3 Validation

We now prove the validity of the implemented code. The eigenvectors $\varphi_{k,V}$ are approximated by $N = 50$ modes. We take, as a test case, $V(x) := (x - 1/2)^2$, $\mu_1(x) := x^2$ and $\mu_2(x) := x$. The considered initial condition is $\psi^0 = \frac{1}{\sqrt{2}}\varphi_{1,V} + \frac{i}{\sqrt{2}}\varphi_{2,V}$. The value of the oscillating parameter is $\varepsilon = 10^{-3}$. The parameter γ is chosen such that $\mathcal{L}(\psi^0) = 3/4$. We compute the discrete Lyapunov function for the averaged system and the H^s norm (with $s = 1.8$) to the ground state for both the oscillating and the averaged system. The time scale is $[0, T]$ with $T = 1000$ and a time step $\mathrm{d}t = 10^{-3}$. For $M = 5$, we get the results presented in Figure 5.1.

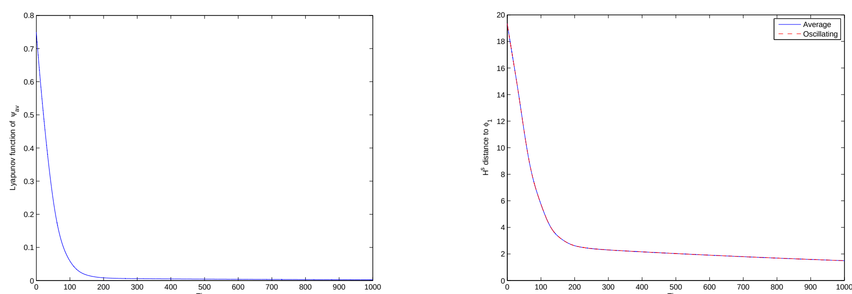

Figure 5.1: Lyapunov function of the averaged system (left). H^s norm to the ground state (right) for the averaged system (continuous line) and the oscillating system (dashed line).

As expected, we observe the convergence of the Lyapunov function to 0. The solutions of (5.39) and (5.40) are driven to the ground state (up to a global phase). To validate the simulations, we have also tested the code for $M = 10$ and $M = 20$. We obtained the same asymptotic behaviour and the same values for the Lyapunov function and the H^s distance to the target.

As the approximate controllability uses the fact that the controls are oscillating, the time step $\mathrm{d}t$ cannot be taken large with respect to the oscillating parameter ε. For $\varepsilon = 10^{-3}$, we obtain the same results with $\mathrm{d}t = 10^{-3}$ and $\mathrm{d}t = 10^{-4}$. However, instabilities appear on the oscillating system for $\mathrm{d}t = 10^{-2}$. Thus, in all what follows the time step will be chosen smaller than ε. We now present several simulations to illustrate various aspects of Theorem 5.1.

5.5.4 Influence of the initial condition

For every other initial condition tested, the asymptotic behaviour is the same. We present here the results for the same parameters as in Figure 5.1 but with the initial condition $\psi^0 = \frac{1}{\sqrt{3}}\varphi_{1,V} + \frac{1}{\sqrt{3}}\varphi_{2,V} + \frac{i}{\sqrt{3}}\varphi_{3,V}$. In this case, the stabilization of the averaged system is slower and we computed it for $T = 5000$.

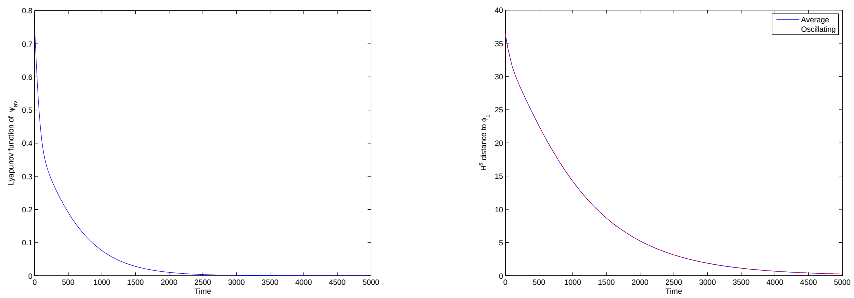

Figure 5.2: Lyapunov function of the averaged system (left). H^s norm to the ground state (right) for the averaged system (continuous line) and the oscillating system (dashed line).

We observe in Figure 5.2 the same asymptotic behaviour as in Figure 5.1.

5.5.5 Averaging strategy

We present numerically the influence of the oscillating parameter ε. First, we consider the same potential, dipolar and polarizability moments as in Figure 5.1. We compute the discrete H^s norm (for $s = 1.8$) to the ground state (up to a global phase) and the discrete H^2 norm of $X_{av} - X_\varepsilon$. Figure 5.3 is obtained with $\varepsilon = 10^{-3}$ while Figure 5.4 is obtained with $\varepsilon = 10^{-4}$. Both are computed with a time step $\mathrm{d}t = \varepsilon$ and final time $T = 500$. For a fixed parameter ε, we observe that the H^2 distance between the solution of (5.1) and the solution of (5.6) with the same initial condition does not increase as the time goes to infinity but rather tends to a limit value. This limit value is of the same order of magnitude as ε. We observe that

$$\frac{||X_{av}(T) - X_{10^{-3}}(T)||_{H^2}}{||X_{av}(T) - X_{10^{-4}}(T)||_{H^2}} \approx 30.$$

This validates numerically the results of Proposition 5.5 and indicates that this averaging property should be valid on an infinite time horizon.

The same behaviour has been obtained with other parameters. We present here the simulations with $\mu_1(x) := cos(x)$ and $\mu_2(x) := cos(2x)$, inspired by the physical situation of alignment dynamic of a HCN molecule as in [65]. Figure 5.5 is obtained with $\varepsilon = 10^{-3}$ while Figure 5.6 is obtained with $\varepsilon = 10^{-4}$. Both are computed with a time step $\mathrm{d}t = \varepsilon$ and final time $T = 1000$ (as the stabilization process seems slower in this case).

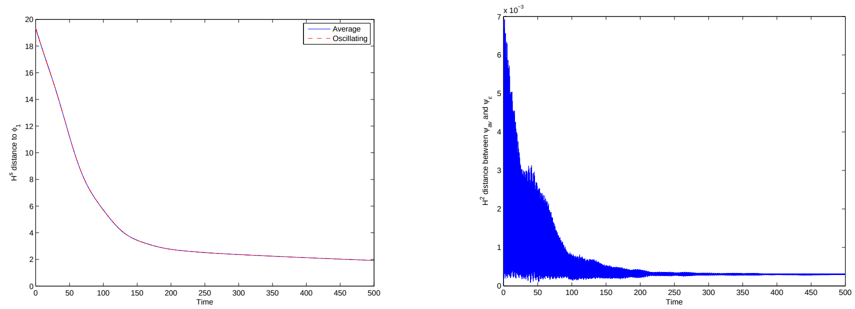

Figure 5.3: H^s norm to the ground state (left) for the averaged system (continuous line) and the oscillating system (dashed line). H^2 gap from the average (right).

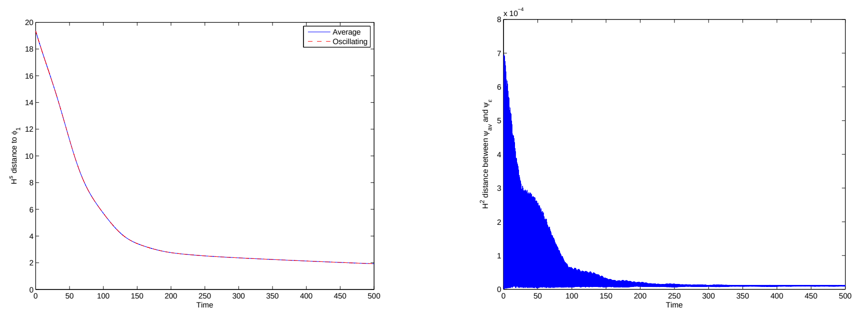

Figure 5.4: H^s norm to the ground state (left) for the averaged system (continuous line) and the oscillating system (dashed line). H^2 gap from the average (right).

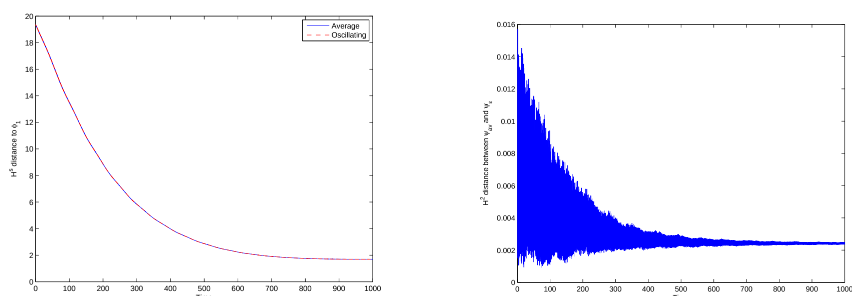

Figure 5.5: H^s norm to the ground state (left) for the averaged system (continuous line) and the oscillating system (dashed line). H^2 gap from the average (right).

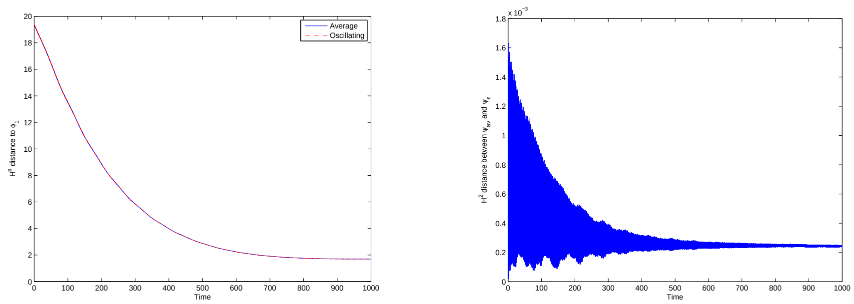

Figure 5.6: H^s norm to the ground state (left) for the averaged system (continuous line) and the oscillating system (dashed line). H^2 gap from the average (right).

Remark 5.7. Although the time scales at stake in these simulations can seem very large, one has to remember that the Schrödinger equation is considered in atomic unity.

5.6 Conclusion, open problems and perspectives

In this article we have defined explicit oscillating controls that drive the solution of our system arbitrarily close to the ground state provided that the control is oscillating enough and the time is large enough. To achieve this we have used and developed tools from the theory of finite dimension dynamical systems and applied them to the considered Schrödinger equation. We managed by adding a mathematically and physically meaningful term to weaken the previous assumptions on the coupling realized by this model. The assumptions that were made are proved to be generic with respect to the functions determining the system (potential, dipolar and polarizability moments). The results presented should be generalizable to a compact manifold with the Laplace-Beltrami operator. We performed numerical simulations to illustrate the approximate controllability. This gives numerical bounds on the time scale and on the values of the oscillating parameter needed to drive any initial condition arbitrarily close to the ground state.

A challenging question would be to prove an approximation property of the averaged system on an infinite time interval $[s, +\infty)$. This would lead to approximate stabilization to the ground state. Based on the numerical simulations, this result seems to hold. Unfortunately the tools developed here are really based on the finite time interval and cannot be extended directly. In [16], Beauchard and Laurent proved the local exact controllability in H^3 around the ground state for the system (5.1) in the dipolar approximation (i.e. $\mu_2 \equiv 0$) under some coupling assumptions in one dimension. If one manages to extend their result to the system (5.1) with suitable assumptions on μ_2, this may lead to a global exact controllability result around the ground state, at least for the one-dimensional case. The main difficulty would be to obtain the approximate convergence in the same functional setting as their local exact controllability result and with coherent assumptions on the polarizability moment.

Chapitre 6

Contrôle exact global

Sommaire

On considère une particule quantique dans un intervalle, modélisée par sa fonction d'onde ψ, dans un potentiel $V(x)$. On contrôle l'évolution de cette fonction d'onde par un champ extérieur d'amplitude $u(t)$ réelle. En prenant en compte le terme dipolaire et le terme de polarisabilité, on considère le système de Schrödinger

$$\begin{cases} i\partial_t\psi = \left(-\partial^2_{xx} + V(x)\right)\psi - u(t)\mu_1(x)\psi - u(t)^2\mu_2(x)\psi, & (t,x)\in(0,T)\times(0,1), \\ \psi(t,0) = \psi(t,1) = 0, & t\in(0,T), \\ \psi(0,x) = \psi_0(x), & x\in(0,1), \end{cases} \tag{6.1}$$

présenté en Section 1.3 page 27. L'objectif de ce Chapitre est de montrer que certaines techniques développées pour le modèle bilinéaire

$$\begin{cases} i\partial_t\psi = \left(-\partial^2_{xx} + V(x)\right)\psi - u(t)\mu_1(x)\psi, & (t,x)\in(0,T)\times(0,1), \\ \psi(t,0) = \psi(t,1) = 0, & t\in(0,T), \\ \psi(0,x) = \psi_0(x), & x\in(0,1), \end{cases} \tag{6.2}$$

présentées en Section 1.2 et détaillées en Partie I, s'appliquent au cas du modèle unidimensionnel avec un terme de polarisabilité (6.1).

On rappelle que, pour $V\in L^2((0,1),\mathbb{R})$, on note $\lambda_{k,V}$ et $\varphi_{k,V}$ les valeurs propres et vecteurs propres de l'opérateur A_V défini par $A_V\psi := \left(-\partial^2_{xx} + V(x)\right)\psi$ avec domaine $D(A_V) := H^2\cap H^1_0((0,1),\mathbb{C})$. Pour $s>0$, l'espace $H^s_{(V)} := D(A_V^{s/2})$ est muni de la norme

$$\|\psi\|_{H^s_{(V)}} := \left(\sum_{k=1}^{+\infty} |k^s\langle\psi,\varphi_{k,V}\rangle|^2\right)^{\frac{1}{2}}.$$

La sphère unité de $L^2((0,1),\mathbb{C})$ est notée $\mathcal{S}$. Le résultat principal de ce chapitre est le suivant.

Théorème 6.1. *Pour tout* $V, \mu_1 \in H^6((0,1),\mathbb{R})$ *le système (6.1) est globalement exactement contrôlable dans* $H^6_{(V)}$*, génériquement par rapport à* $\mu_2 \in H^6((0,1),\mathbb{R})$*. Plus précisément, pour tout* $V, \mu_1 \in H^6((0,1),\mathbb{R})$*, il existe un ensemble* $\mathcal{Q}_{V,\mu_1}$ *résiduel dans* $H^6((0,1),\mathbb{R})$ *tel que si* $\mu_2 \in \mathcal{Q}_{V,\mu_1}$*, pour tout* $\psi_0, \psi_f \in \mathcal{S} \cap H^6_{(V)}$*, il existe* $T > 0$ *et* $u \in H^1_0((0,T),\mathbb{R})$ *tels que la solution associée de (6.1) satisfasse* $\psi(T) = \psi_f$*.*

Ce théorème montre que la prise en compte du terme de polarisabilité permet d'obtenir la contrôlabilité exacte globale en temps grand pour des modèles où la contrôlabilité sans ce terme de polarisabilité est fausse ou ouverte (par exemple $\mu_1 = 0$ ou V quelconque et $\mu_1 \notin \mathcal{Q}_V$ comme défini au Théorème 4.1 page 122). La preuve de ce théorème repose principalement sur trois arguments. En utilisant l'argument de perturbation développé en Section 4.5.2, si l'on considère le contrôle $u(t) = \tilde{u}(t) + 2$, alors le système (6.1) s'écrit

$$\begin{cases} i\partial_t \psi = \left(-\partial^2_{xx} + V(x) - 2\mu_1(x) - 4\mu_2(x)\right)\psi - \tilde{u}(t)(\mu_1 + 4\mu_2)(x)\psi - \tilde{u}(t)^2\mu_2(x)\psi, \\ \psi(t,0) = \psi(t,1) = 0, \\ \psi(0,x) = \psi_0(x). \end{cases} \tag{6.3}$$

Ainsi, on peut distribuer une partie du moment de polarisabilité sur le potentiel et une partie sur le moment dipolaire. Même pour des fonctions V et μ_1 quelconques, on se ramène donc à étudier le système (6.1) avec des hypothèses favorables sur V et μ_1.

La deuxième étape de la preuve consiste à montrer le contrôle approché vers l'état fondamental $\varphi_{1,V}$ pour la norme H^5. Dans le cadre $\mu_2 = 0$, ce résultat est obtenu par V. Nersesyan [112] sous des hypothèses favorables sur V et μ_1. La preuve de ce résultat utilise la fonction de Lyapunov

$$\mathcal{L}(z) := \gamma \| \left(-\Delta + V\right)^3 \mathcal{P} z\|^2_{L^2} + 1 - |\langle z, \varphi_{1,V}\rangle|^2, \quad z \in \mathcal{S} \cap H^6_{(V)}, \tag{6.4}$$

où $\mathcal{P}$ est la projection orthogonale dans L^2 sur l'espace engendré par $\{\varphi_{k,V}\,;\, k \geq 2\}$ et $\gamma > 0$ une constante à déterminer. C'est cette fonction de Lyapunov (déjà utilisée par K. Beauchard et V. Nersesyan [111, 19]) qui a été adaptée au cadre de la contrôlabilité simultanée au Chapitre 4. Dans [112], la décroissance de la fonction de Lyapunov est assurée par un argument variationnel lié aux propriétés du linéarisé du système (6.2) au voisinage de trajectoires associées au contrôle $u \equiv 0$. Les systèmes (6.1) et (6.2) admettant le même linéarisé au voisinage de ces trajectoires, ce résultat est directement étendu au cas $\mu_2 \neq 0$.

Grâce à l'argument de réversibilité en temps classique, la dernière étape de la preuve du Théorème 6.1 consiste à prouver la contrôlabilité exacte locale dans $H^5_{(V)}$ autour de l'état fondamental $\varphi_{1,V}$. Dans le cadre $V = 0$ et $\mu_2 = 0$, ce résultat est prouvé avec des contrôles $H^1_0((0,T),\mathbb{R})$ par K. Beauchard et C. Laurent [16]. Le cas d'un potentiel V non nul nécessite simplement quelques adaptations techniques pour obtenir le caractère bien posé. La preuve de contrôlabilité de [16] étant basée sur le contrôle du linéarisé au voisinage de la trajectoire $(\Phi_1, u \equiv 0)$ dans $H^5_{(0)}$ avec des contrôles $H^1_0((0,T),\mathbb{R})$, ce résultat s'étend directement au cadre μ_2 non nul. En effet, les systèmes (6.1) et (6.2) possèdent le même linéarisé au voisinage de la trajectoire $(\Phi_{1,V}, u \equiv 0)$ et la régularité $H^1_0((0,T),\mathbb{R})$ de u se transfère automatiquement à u^2. Cette dernière remarque justifie le cadre fonctionnel $H^5_{(V)}$

utilisé ici. En effet, une première idée serait d'adapter le résultat de contrôle du linéarisé dans $H^3_{(V)}$ avec des contrôles $L^2((0,T),\mathbb{R})$. Le terme u^2 aurait alors seulement une régularité $L^1((0,T),\mathbb{R})$ insuffisante pour appliquer les résultats de [16] et conclure au caractère bien posé de (6.1) dans $H^3_{(V)}$. Le contrôle du linéarisé dans $H^3_{(V)}$ avec des contrôles $L^4((0,T),\mathbb{R})$ est un problème ouvert.

Structure du chapitre. On commence par préciser en Section 6.1 les résultats de régularité pour les solutions du problème (6.1). En Section 6.2, on montre la contrôlabilité approchée vers l'état fondamental $\varphi_{1,V}$ sous des hypothèses favorables sur V et μ_1 en utilisant la fonction de Lyapunov $\mathcal{L}$ définie en (6.4). La Section 6.3 prouve la contrôlabilité exacte locale dans $H^5_{(V)}$ avec des contrôles $H^1_0((0,T),\mathbb{R})$ au voisinage de l'état fondamental sous des hypothèses favorables sur V et μ_1. On conclut la preuve du Théorème 6.1 en Section 6.4 en utilisant le système perturbé (6.3).

6.1 Régularité des solutions

Cette section est dédiée à l'étude du caractère bien posé et aux résultats de régularité du système (6.1). Le résultat suivant est une adaptation de [16, Proposition 5].

Proposition 6.1. *Soient* $V,\mu_1,\mu_2 \in H^5((0,1),\mathbb{R})$, $T>0$, $\psi_0 \in H^5_{(V)}$, $f \in H^1_0((0,T),H^3\cap H^1_0)$ *et* $u \in H^1_0((0,T),\mathbb{R})$. *Le système*

$$\begin{cases} i\partial_t\psi = \left(-\partial^2_{xx}+V(x)\right)\psi - u(t)\mu_1(x)\psi - u(t)^2\mu_2(x)\psi - f(t,x), & (t,x)\in(0,T)\times(0,1),\\ \psi(t,0)=\psi(t,1)=0, & t\in(0,T),\\ \psi(0,x)=\psi_0(x), & x\in(0,1), \end{cases} \tag{6.5}$$

admet une unique solution faible i.e. une fonction $\psi \in C^1([0,T],H^3_{(V)})$ *telle que l'égalité*

$$\psi(t) = e^{-iA_V t}\psi_0 + i\int_0^t e^{-iA_V(t-\tau)}\left(u(\tau)\mu_1\psi(\tau)+u(\tau)^2\mu_2\psi(\tau)+f(\tau)\right)\mathrm{d}\tau, \tag{6.6}$$

soit vérifiée dans $C^1([0,T],H^3_{(V)})$. *Pour tout* $R>0$, *il existe* $C=C(T,\mu_1,\mu_2,R)>0$ *tel que si* $\|u\|_{H^1_0(0,T)}<R$, *la solution associée vérifie*

$$\|\psi\|_{C^1([0,T],H^3_{(V)})} \leq C\left(\|\psi_0\|_{H^5_{(V)}} + \|f\|_{H^1((0,T),H^3\cap H^1_0)}\right).$$

Si $f\equiv 0$, *alors*

$$\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2}, \quad \textit{pour tout } t\in[0,T],$$

et la solution vérifie $A_V\psi+u(t)\mu_1\psi+u(t)^2\mu_2\psi \in C^0([0,T],H^3_{(V)})$. *En particulier, l'hypothèse* $u(T)=0$ *implique* $\psi(T)\in H^5_{(V)}$.

Dans la suite, la solution de (6.1) avec condition initiale ψ_0 et contrôle u sera notée $\psi(\cdot,\psi_0,u)$.

Démonstration. Comme $u \in H_0^1((0,T),\mathbb{R})$ implique $u^2 \in H_0^1((0,T),\mathbb{R})$, le passage du cas $\mu_2 = 0$ à $\mu_2 \in H^5((0,1),\mathbb{R})$ est immédiat. On précise donc simplement les adaptations nécessaires pour étendre [16, Proposition 5] au cadre V non nul. L'existence et l'unicité de la solution faible dans $C^0([0,T],H^3_{(V)})$ sont assurées par un théorème de point fixe et le lemme suivant.

Lemme 6.1. *Soient $T>0$ et $f \in L^2((0,T),H^3 \cap H_0^1)$. La fonction $G : t \mapsto \int_0^t e^{iA_V s} f(s)\mathrm{d}s$ appartient à $C^0([0,T],H^3_{(V)})$ et vérifie*

$$\|G\|_{L^\infty((0,T),H^3_{(V)})} \leq c_1(T)\|f\|_{L^2((0,T),H^3\cap H_0^1)}$$

où la constante $c_1(T)$ est uniformément bornée pour T dans un intervalle borné.

Pour $V=0$, ce lemme est [16, Lemme 1]. L'adaptation au cas V non nul en suivant la même stratégie est due à V. Nersesyan et H. Nersisyan [113, Proposition 3.1] en utilisant les estimations suivantes (voir par exemple [117, Théorème 4]) pour $V \in L^2((0,1),\mathbb{R})$

$$\lambda_{k,V} = k^2\pi^2 + \int_0^1 V(x)\mathrm{d}x + r_k, \quad \forall k \in \mathbb{N}^* \text{ avec } \sum_{k=1}^{\infty} r_k^2 < \infty, \tag{6.7}$$

$$\|\varphi_{k,V} - \varphi_k\|_{L^\infty} \leq \frac{C}{k}, \quad \forall k \in \mathbb{N}^*, \tag{6.8}$$

$$\|\varphi'_{k,V} - \varphi'_k\|_{L^\infty} \leq C, \quad \forall k \in \mathbb{N}^*. \tag{6.9}$$

Finalement, de même que pour la preuve de [16, Proposition 5], la régularité $C^1([0,T],H^3_{(V)})$ est obtenue en dérivant l'équation par rapport au temps et en utilisant le Lemme 6.1.

□

Pour des conditions initiales plus régulières, on utilise aussi le résultat de régularité suivant.

Proposition 6.2. *Soit $T>0$. Pour tout $u \in C^2([0,T],\mathbb{R})$ tel que $\dot{u}(0)=0$, pour tout $\psi_0 \in H^6_{(V-u(0)\mu_1-u(0)^2\mu_2)}$, (6.1) admet une unique solution faible $C^0([0,T],H^6)$. De plus, si $\dot{u}(T)=0$, alors $\psi(T) \in H^6_{(V-u(T)\mu_1-u(T)^2\mu_2)}$.*

Démonstration. Si $u \in C^2([0,T],\mathbb{R})$ avec $\dot{u}(0)=0$ alors u^2 vérifie les mêmes propriétés. Le résultat est donc impliqué par [112, Lemme 2.1] pour le cas $\mu_2=0$.

□

6.2 Contrôle approché vers l'état fondamental

Cette section est dédiée à la contrôlabilité approchée vers l'état fondamental $\varphi_{1,V}$. On suppose que les fonctions $V, \mu_1 \in H^6((0,1),\mathbb{R})$ vérifient

(C_1) $\langle \mu_1\varphi_{1,V}, \varphi_{k,V}\rangle \neq 0$, pour tout $k \in \mathbb{N}^*$,

(C_2) $\lambda_{1,V} - \lambda_{j,V} \neq \lambda_{p,V} - \lambda_{q,V}$, pour tout $j,p,q \geq 1$ et $j \neq 1$.

Théorème 6.2. *Soient* $V, \mu_1, \mu_2 \in H^6((0,1),\mathbb{R})$ *tels que les Conditions* $(\mathbf{C_1})$ *et* $(\mathbf{C_2})$ *soient vérifiées. Soit* $\psi_0 \in \mathcal{S} \cap H^6_{(V)}$ *tel que* $\langle \psi_0, \varphi_{1,V} \rangle \neq 0$. *Pour tout* $\varepsilon > 0$, *il existe* $T > 0$ *et* $u \in C^2_0((0,T),\mathbb{R})$ *tels que*

$$\|\psi(T,\psi_0,u) - \varphi_{1,V}\|_{H^5} < \varepsilon. \tag{6.10}$$

Démonstration. Dans le cas $\mu_2 = 0$, la preuve est celle de [112, Théorème 2.3]. L'adaptation au cas $\mu_2 \in H^6((0,1),\mathbb{R})$ est immédiate, on rappelle simplement la trame de la preuve.

Pour tout $\psi_0 \in H^6_{(V)}$, les systèmes (6.1) et (6.2) ayant le même linéarisé au voisinage de la trajectoire $\psi(\cdot,\psi_0,0)$ on en déduit immédiatement le lemme suivant qui assure la décroissance de la fonction de Lyapunov (voir [112, Proposition 2.6]).

Lemme 6.2. *Soient* $V, \mu_1, \mu_2 \in H^6((0,1),\mathbb{R})$ *tels que les Conditions* $(\mathbf{C_1})$ *et* $(\mathbf{C_2})$ *soient vérifiées. Pour tout* $\psi_0 \in \mathcal{S} \cap H^6_{(V)}$ *vérifiant* $\langle \psi_0, \varphi_{1,V} \rangle \neq 0$ *et* $\mathcal{L}(\psi_0) > 0$, *il existe un temps* $T > 0$ *et un contrôle* $u \in C^2_0((0,T),\mathbb{R})$ *tels que*

$$\mathcal{L}(\psi(T,\psi_0,u)) < \mathcal{L}(\psi_0).$$

Soit $\psi_0 \in \mathcal{S} \cap H^6_{(V)}$ tel que $\langle \psi_0, \varphi_{1,V} \rangle \neq 0$. On fixe $\gamma > 0$ dans la définition (6.4) tel que $\mathcal{L}(\psi_0) < 1$. Si $\mathcal{L}(\psi_0) > 0$, on définit

$$\mathcal{K} := \left\{ \psi_f \in H^6_{(V)} \,;\, \psi(T_n,\psi_0,u_n) \underset{n\to\infty}{\longrightarrow} \psi_f, \text{ dans } H^5 \text{ où } T_n > 0, u_n \in C^2_0((0,T_n),\mathbb{R}) \right\}.$$

L'infimum de $\mathcal{L}$ sur $\mathcal{K}$ est atteint i.e. il existe $e \in \mathcal{K}$ tel que

$$\mathcal{L}(e) = \inf_{\psi \in \mathcal{K}} \mathcal{L}(\psi). \tag{6.11}$$

En particulier, $\mathcal{L}(e) \leq \mathcal{L}(\psi_0) < 1$ d'où $\langle e, \varphi_{1,V} \rangle \neq 0$. D'après le Lemme 6.2, on obtient que si $\mathcal{L}(e) > 0$ alors il existe $T > 0$ et $u \in C^2_0((0,T),\mathbb{R})$ tels que $\mathcal{L}(\psi(T,e,u)) < \mathcal{L}(e)$. Comme $\psi(T,e,u) \in \mathcal{K}$, on obtient une contradiction avec (6.11). D'où $\mathcal{L}(e) = 0$. Ceci implique $\varphi_{1,V} \in \mathcal{K}$ et conclut la preuve du Théorème 6.2.

□

6.3 Contrôle exact local autour de l'état fondamental

Cette section est dédiée à la contrôlabilité exacte locale autour de l'état fondamental $\varphi_{1,V}$ dans $H^5_{(V)}$ avec des contrôles $H^1_0((0,T),\mathbb{R})$. Dans cette section, on suppose que les fonctions $V, \mu_1 \in H^5((0,1),\mathbb{R})$ vérifient

$(\mathbf{C_3})$ il existe $C > 0$ tel que

$$|\langle \mu_1 \varphi_{1,V}, \varphi_{k,V} \rangle| \geq \frac{C}{k^3}, \text{ pour tout } k \in \mathbb{N}^*.$$

Théorème 6.3. *Soient* $V, \mu_1, \mu_2 \in H^5((0,1),\mathbb{R})$ *tels que la Condition* $(\mathbf{C_3})$ *soit vérifiée. Soit* $T > 0$. *Il existe* $\delta > 0$ *et une application* C^1

$$\Gamma : \mathcal{O}_T \longrightarrow H^1_0((0,T),\mathbb{R})$$

où

$$\mathcal{O}_T := \left\{ \psi_f \in \mathcal{S} \cap H^5_{(V)} \, ; \, \| \psi_f - \Phi_{1,V}(T) \|_{H^5} < \delta \right\},$$

telle que $\Gamma(\Phi_{1,V}(T)) = 0$*, et pour tout* $\psi_f \in \mathcal{O}_T$*, la solution de (6.1) avec condition initiale* $\psi_0 = \varphi_{1,V}$ *et contrôle* $u = \Gamma(\psi_f)$ *satisfasse* $\psi(T) = \psi_f$*.*

Démonstration. Dans le cas $\mu_2 = 0$ et $V = 0$, la preuve est celle de [16, Théorème 2]. L'adaptation au cas $V, \mu_2 \in H^5((0,1),\mathbb{R})$ est directe, on rappelle simplement la trame de la preuve.

Soit $T > 0$. En utilisant la régularité obtenue en Proposition 6.1 et la stratégie de [16, Proposition 6] on obtient la régularité C^1 de l'application

$$\begin{array}{rccc} \Theta_T : & H^1_0((0,T),\mathbb{R}) & \to & \mathcal{H} \\ & u & \mapsto & \mathcal{P}_{\mathcal{H}}\big(\psi(T, \varphi_{1,V}, u)\big) \end{array} \tag{6.12}$$

où

$$\mathcal{H} := \left\{ \psi \in H^5_{(V)} \, ; \, \mathrm{Re}(\langle \psi, \varphi_{1,V} \rangle) = 0 \right\},$$

et $\mathcal{P}_{\mathcal{H}}$ est la projection orthogonale dans $L^2((0,1),\mathbb{C})$ sur $\mathcal{H}$ i.e.

$$\mathcal{P}_{\mathcal{H}}(\psi) = \psi - \mathrm{Re}(\langle \psi, \varphi_{1,V} \rangle)\varphi_{1,V}.$$

Sa différentielle en 0 est donnée par $\mathrm{d}\Theta(0).v = \Psi(T)$, où Ψ est la solution de

$$\begin{cases} i\partial_t \Psi = \left(-\partial^2_{xx} + V(x)\right) \Psi - v(t)\mu_1(x)\Phi_{1,V}, & (t,x) \in (0,T) \times (0,1), \\ \Psi(t,0) = \Psi(t,1) = 0, & t \in (0,T), \\ \Psi(0,x) = 0, & x \in (0,1). \end{cases} \tag{6.13}$$

En utilisant la Condition **($\mathbf{C_3}$)**, l'asymptotique (6.7) et [16, Corollaire 2], on obtient l'existence d'un application linéaire continue

$$\mathcal{M} : \mathcal{H} \mapsto L^2((0,T),\mathbb{R}),$$

telle que pour tout $\Psi_f \in \mathcal{H}$, $w := \mathcal{M}(\Psi_f)$ est solution du problème de moments

$$\begin{cases} \displaystyle\int_0^T w(t)\mathrm{d}t = 0, \\ \displaystyle\int_0^T w(t)(T-t)\mathrm{d}t = \frac{1}{\langle \mu_1 \varphi_{1,V}, \varphi_{1,V} \rangle} \langle \Psi_f, \Phi_{1,V}(T) \rangle, \\ \displaystyle\int_0^T w(t) e^{i(\lambda_{k,V} - \lambda_{1,V})t}\mathrm{d}t = \frac{\lambda_{1,V} - \lambda_{k,V}}{\langle \mu_1 \varphi_{1,V}, \varphi_{k,V} \rangle} \langle \Psi_f, \Phi_{k,V}(T) \rangle, \quad \forall k \geq 2. \end{cases} \tag{6.14}$$

Le choix $v := t \mapsto \int_0^t w(\tau)\mathrm{d}\tau \in H^1_0((0,T),\mathbb{R})$ fournit alors un inverse à droite continu de la différentielle $\mathrm{d}\Theta(0) : H^1_0((0,T),\mathbb{R}) \to \mathcal{H}$.

Finalement, l'application du théorème d'inversion locale à Θ_T au point $u = 0$, la conservation de la norme L^2 et l'hypothèse $\psi_f \in \mathcal{S}$ concluent la preuve du Théorème 6.3.

□

6.4 Contrôle exact global

En combinant la contrôlabilité globale approchée du Théorème 6.2 et la contrôlabilité exacte locale du Théorème 6.3 on obtient la contrôlabilité exacte globale de (6.1) sous des hypothèses favorables sur V et μ_1.

Théorème 6.4. *Soient $V, \mu_1, \mu_2 \in H^6((0,1),\mathbb{R})$ tels que les Conditions* **$(\mathbf{C_2})$** *et* **$(\mathbf{C_3})$** *soient vérifiées. Pour tout $\psi_0, \psi_f \in \mathcal{S} \cap H^6_{(V)}$, il existe $T > 0$ et $u \in H^1_0((0,T),\mathbb{R})$ tels que la solution de (6.1) associée satisfasse $\psi(T) = \psi_f$.*

Démonstration. Première étape. Soient $\psi_0, \psi_f \in \mathcal{S} \cap H^6_{(V)}$ tels que $\langle \psi_0, \varphi_{1,V} \rangle \neq 0$ et $\langle \psi_f, \varphi_{1,V} \rangle \neq 0$. Soient $T_* > 0$ et $\delta > 0$ le rayon de contrôlabilité exacte locale dans $H^5_{(V)}$ en temps T_* donné par le Théorème 6.3. D'après le Théorème 6.2, il existe $T_0, T_f > 0$, $u_0 \in C^2_0((0,T_0),\mathbb{R})$ et $u_f \in C^2_0((0,T_f),\mathbb{R})$ tels que

$$\|\psi(T_0, \psi_0, u_0) - \varphi_{1,V}\|_{H^5} + \|\psi(T_f, \overline{\psi_f}, u_f) - \varphi_{1,V}\|_{H^5} < \delta. \tag{6.15}$$

D'après le Théorème 6.3, il existe $u_* \in H^1_0((0,T_*),\mathbb{R})$ tel que

$$\psi(T_*, \varphi_{1,V}, u_*) = e^{-i\lambda_{1,V} T_*} \overline{\psi(T_0, \psi_0, u_0)}.$$

En utilisant la réversibilité en temps de (6.1), on obtient alors que si u est défini sur $[0, T_0 + T_*]$ par $u(t) = u_0(t)$ pour $t \in [0, T_0]$ et $u(t + T_0) = u(T_* - t)$ pour $t \in [0, T_*]$ alors $u \in H^1_0((0, T_0 + T_*),\mathbb{R})$ et

$$\psi(T_0 + T_*, \psi_0, u) = \Phi_{1,V}(T_*).$$

Ainsi il existe $T^* > 0$ tel que si l'on étend u par 0 sur $[T_0 + T_*, T_0 + T_* + T^*]$ alors,

$$\psi(T_0 + T_* + T^*, \psi_0, u) = \varphi_{1,V}. \tag{6.16}$$

Les mêmes arguments conduisent alors à l'existence de $\tilde{u} \in H^1_0((0, T_f + T_* + T^*),\mathbb{R})$ tel que

$$\psi(T_f + T_* + T^*, \overline{\psi_f}, \tilde{u}) = \varphi_{1,V}.$$

Conjointement à (6.16), la réversibilité en temps de (6.1) implique alors, pour $T := T_0 + T_f + 2T_* + 2T^*$, l'existence de $u \in H^1_0((0,T),\mathbb{R})$ tel que

$$\psi(T, \psi_0, u) = \psi_f.$$

Deuxième étape. On conclut la preuve du Théorème 6.4 en montrant que l'on peut se passer des hypothèses $\langle \psi_0, \varphi_{1,V} \rangle \neq 0$ et $\langle \psi_f, \varphi_{1,V} \rangle \neq 0$.

D'après la réversibilité en temps de (6.1), il est suffisant de montrer que pour tout $\psi_0 \in \mathcal{S} \cap H^6_{(V)}$, il existe $T > 0$ et $u \in C^2_0((0,T),\mathbb{R})$ tels que $\psi(T, \psi_0, u) \in \mathcal{S} \cap H^6_{(V)}$ et $\langle \psi(T, \psi_0, u), \varphi_{1,V} \rangle \neq 0$. Soit $\widehat{\psi_0} \in \mathcal{S} \cap H^6_{(V)}$ tel que $\langle \widehat{\psi_0}, \varphi_{1,V} \rangle \neq 0$ et $\|\psi_0 - \widehat{\psi_0}\|_{L^2} < \sqrt{2}$. D'après la première étape, il existe $T > 0$ et $\hat{u} \in H^1_0((0,T),\mathbb{R})$ tels que $\psi(T, \widehat{\psi_0}, \hat{u}) = \varphi_{1,V}$. La conservation de la norme L^2 implique

$$\|\psi(T, \psi_0, \hat{u}) - \varphi_{1,V}\|_{L^2} = \|\psi_0 - \widehat{\psi_0}\|_{L^2} < \sqrt{2}.$$

Finalement, le choix de $u \in C^2_0((0,T),\mathbb{R})$ suffisamment proche de $\hat{u}$ termine la preuve du Théorème 6.4.

□

En adaptant l'argument de perturbation utilisé en Section 4.5.2 et le Théorème 6.4 on prouve le Théorème 6.1.

Démonstration du Théorème 6.1. Soient $V, \mu_1 \in H^6((0,1),\mathbb{R})$. On définit $\mathcal{Q}_{V,\mu_1}$ l'ensemble des fonctions $\mu_2 \in H^6((0,1),\mathbb{R})$ telles que les Conditions $(\mathbf{C_2})$ et $(\mathbf{C_3})$ soient vérifiées avec les fonctions V et μ_1 remplacées respectivement par $V-2\mu_1-4\mu_2$ et $\mu_1+4\mu_2$ i.e. on définit

- $(\mathbf{C_2'})$ $\lambda_{1,V-2\mu_1-4\mu_2} - \lambda_{j,V-2\mu_1-4\mu_2} \neq \lambda_{p,V-2\mu_1-4\mu_2} - \lambda_{q,V-2\mu_1-4\mu_2}$, pour tout $j,p,q \geq 1$ et $j \neq 1$.
- $(\mathbf{C_3'})$ il existe $C>0$ tel que

$$|\langle(\mu_1+4\mu_2)\varphi_{1,V-2\mu_1-4\mu_2},\varphi_{k,V-2\mu_1-4\mu_2}\rangle| \geq \frac{C}{k^3}, \text{ pour tout } k \in \mathbb{N}^*.$$

et

$$\mathcal{Q}_{V,\mu_1} := \left\{\mu_2 \in H^6((0,1),\mathbb{R})\,;\,\text{Conditions } (\mathbf{C_2'}) \text{ et } (\mathbf{C_3'})\right\}.$$

Première étape : contrôlabilité exacte globale pour $\mu_2 \in \mathcal{Q}_{V,\mu_1}$. On considère le système (6.1) avec les fonctions V et μ_1 remplacées respectivement par $V-2\mu_1-4\mu_2$ et $\mu_1+4\mu_2$ i.e. le système (6.3). On note $\widetilde{\psi}(T,\psi_0,u)$ le propagateur de (6.3) au temps T. On a

$$\widetilde{\psi}(t,\psi_0,u) = \psi(t,\psi_0,u+2), \quad \text{pour } t \in [0,T]. \tag{6.17}$$

Soient $\psi_0,\psi_f \in \mathcal{S}\cap H^6_{(V)}$ et $u_1 \in C^2([0,1],\mathbb{R})$ vérifiant $u_1(0)=\dot{u}_1(0)=\dot{u}_1(1)=0$ et $u_1(1)=2$. D'après la Proposition 6.2, on a

$$\widetilde{\psi}_0 := \psi(1,\psi_0,u_1) \in \mathcal{S}\cap H^6_{(V-2\mu_1-4\mu_2)},$$
$$\overline{\widetilde{\psi}_f} := \psi(1,\overline{\psi_f},u_1) \in \mathcal{S}\cap H^6_{(V-2\mu_1-4\mu_2)}.$$

Comme $\mu_2 \in \mathcal{Q}_{V,\mu_1}$, le Théorème 6.4 implique l'existence de $\tilde{T}>0$ et $\tilde{u} \in H^1_0((0,\tilde{T}),\mathbb{R})$ tels que

$$\widetilde{\psi}(\tilde{T},\widetilde{\psi}_0,\tilde{u}) = \widetilde{\psi}_f.$$

On pose $T = 2+\tilde{T}$ et

$$u(t) = \begin{cases} u_1(t) & \text{pour } t \in [0,1], \\ \tilde{u}(t-1)+2 & \text{pour } t \in [1,\tilde{T}+1], \\ u_1(1-(t-1-\tilde{T})) & \text{pour } t \in [\tilde{T}+1,T]. \end{cases}$$

Ainsi, la réversibilité en temps de (6.1) et (6.17) impliquent

$$\psi(T,\psi_0,u) = \psi_f,$$

avec $u \in H^1_0((0,T),\mathbb{R})$.

Deuxième étape. On conclut la preuve du Théorème 6.1 en montrant que $\mathcal{Q}_{V,\mu_1}$ est résiduel dans $H^6((0,1),\mathbb{R})$.

Pour tout $W \in H^6((0,1),\mathbb{R})$, d'après le Lemme 4.2 page 143 pour $s = 6$, l'ensemble $\mathcal{Q}_W$ des fonctions $\mu \in H^6((0,1),\mathbb{R})$ telles que $\{1, \lambda_{k,W+\mu}\}_{k\in\mathbb{N}^*}$ soient rationnellement indépendants et qu'il existe $C > 0$ tel que

$$|\langle \mu\varphi_{1,W+\mu}, \varphi_{k,W+\mu}\rangle| \geq \frac{C}{k^3}, \quad \forall k \in \mathbb{N}^*,$$

est résiduel dans $H^6((0,1),\mathbb{R})$. On pose $W := V - \mu_1 \in H^6((0,1),\mathbb{R})$. Pour tout $\mu \in \mathcal{Q}_W$, μ_2 défini par $\mu_2 := -\frac{1}{4}(\mu_1 + \mu)$ appartient à $\mathcal{Q}_{V,\mu_1}$. Ceci termine la preuve du Théorème 6.1. $\square$