Continuation unique de l'équation de Grushin singulière 2D

Ce chapitre est inspiré de la prépublication [107].

Contents

7.1 Introduction

7.1.1 Main result

Let us consider for $\gamma > 0$ the following degenerate singular parabolic equation

$$\begin{cases} \partial_t f - \partial_{xx}^2 f - |x|^{2\gamma}\partial_{yy}^2 f + \dfrac{c_\nu}{x^2} f = u(t,x,y)\chi_\omega(x,y), & (t,x,y) \in (0,T)\times\Omega, \\ f(t,-1,y) = f(t,1,y) = 0, & (t,y) \in (0,T)\times(0,1), \\ f(t,x,0) = f(t,x,1), & (t,x) \in (0,T)\times(-1,1), \\ \partial_y f(t,x,0) = \partial_y f(t,x,1), & (t,x) \in (0,T)\times(-1,1), \end{cases} \tag{7.1}$$

with initial condition

$$f(0,x,y) = f^0(x,y), \quad (x,y) \in \Omega. \tag{7.2}$$

The domain is $\Omega := (-1,1) \times (0,1)$ and ω, the control domain, is an open subset of Ω and χ denotes the indicator function. The constant c_ν of the singular potential is defined by $c_\nu := \nu^2 - \frac{1}{4}$, for $\nu \in (0,1)$. The degeneracy set $\{x = 0\}$ coincides with the singularity set ; it separates the domain Ω in two connected components. Due to the singular potential, the first difficulty of this work is to give meaning to solutions of (7.1). Through the study of an associated 1D heat equation, we will design a suitable extension of the considered operator generating a continuous semigroup. The solutions considered in this article will be related to this semigroup. This is detailed in Section 7.2.

In [28], Boscain and Laurent studied the Laplace-Beltrami operator for the Grushin-like metric given by the orthonormal basis $X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 \\ |x|^\gamma \end{pmatrix}$ on $\mathbb{R} \times \mathbb{T}$ with $\gamma > 0$ i.e.

$$Lu := \partial_{xx}^2 u + |x|^{2\gamma} \partial_{yy}^2 u - \frac{\gamma}{x} \partial_x u. \tag{7.3}$$

They proved that this operator with domain $C_0^\infty\big((\mathbb{R} \backslash \{0\}) \times \mathbb{T}\big)$ is essentially self-adjoint on $L^2(\mathbb{R} \times \mathbb{T})$ if and only if $\gamma > 1$. Thus, for the heat equation associated to this Laplace-Beltrami operator, no information passes through the singular set $\{x = 0\}$ when $\gamma > 1$. This prevents controllability from one side of the singularity.

Up to the change of variables $u = |x|^{\gamma/2} v$, the Laplace-Beltrami operator L is equal to

$$\Delta v = \partial_{xx}^2 v + |x|^{2\gamma} \partial_{yy}^2 v - \frac{\gamma}{2}\left(\frac{\gamma}{2} + 1\right) \frac{v}{x^2}.$$

The model (7.1) can then be seen as a heat equation for this Laplace-Beltrami operator. By choosing the coefficient c_ν instead of $\frac{\gamma}{2}\left(\frac{\gamma}{2} + 1\right)$ we authorize a wider class of singular potentials and decouple the effects of the degeneracy and the singularity for a better understanding of each one of these phenomena. Adapting the arguments of [28], one obtains that for any $\gamma > 0$, the operator $-\partial_{xx}^2 - |x|^{2\gamma} \partial_{yy}^2 + \frac{\lambda}{x^2}$ with domain $C_0^\infty(\Omega \backslash \{x = 0\})$ is essentially self-adjoint on $L^2(\Omega)$ if and only if $\lambda \geq \frac{3}{4}$. Thus, our study focuses on the range of constants $c_\nu < \frac{3}{4}$ i.e. $\nu < 1$.

The lower bound $c_\nu > -\frac{1}{4}$ for the range of constants considered comes from well posedness issues linked to the use of the following Hardy inequality (see e.g. [39] for a simple proof)

$$\int_0^1 \frac{z(x)^2}{x^2} \mathrm{d}x \leq 4 \int_0^1 z_x(x)^2 \mathrm{d}x, \quad \forall z \in H^1((0,1), \mathbb{R}) \text{ with } z(0) = 0. \tag{7.4}$$

The critical case $c_\nu = -\frac{1}{4}$ in this inequality is not covered by the technics of this article.

Recall that (7.1) is said to be approximately controllable in time $T > 0$ if for any $(f^0, f^T) \in L^2(\Omega)^2$, for any $\varepsilon > 0$, there exists $u \in L^2((0,T) \times \Omega)$ such that the solution of (7.1)-(7.2) satisfies

$$||f(T) - f^T||_{L^2(\Omega)} \leq \varepsilon.$$

The main result of this article is the following theorem.

Theorem 7.1. *Let $T > 0$, $\gamma > 0$ and $\nu \in (0,1)$. If ω is an open subset of one of the connected components of $\Omega\backslash\{x = 0\}$ then (7.1) is approximately controllable in time T if and only if $\nu \in \left(0, \frac{1}{2}\right]$ i.e. if and only if $c_\nu \in \left(-\frac{1}{4}, 0\right]$.*

This theorem thus fills the gap, for the approximate controllability property, between validity of Hardy inequality ($c_\nu > -\frac{1}{4}$) and the essential self-adjointness property of [28] for $c_\nu \geq \frac{3}{4}$.

Remark 7.1. As it will be noticed during the proof (see Remark 7.4), if ω intersects both connected components of $\Omega\backslash\{x = 0\}$, then approximate controllability holds for any $\nu \in (0,1)$ i.e. any $c_\nu \in \left(-\frac{1}{4}, \frac{3}{4}\right)$.

Theorem 7.1 can be partially adapted to the case of homogeneous Dirichlet boundary conditions in the following way.

Theorem 7.2. *Let $T > 0$, $\gamma > 0$ and $\nu \in (0,1)$. For $\ell > 0$, set $\Omega^\ell := (-1,1) \times (0,\ell)$. Consider the system with homogeneous Dirichlet boundary conditions*

$$\begin{cases} \partial_t f - \partial^2_{xx} f - |x|^{2\gamma}\partial^2_{yy} f + \dfrac{c_\nu}{x^2} f = u(t,x,y)\chi_\omega(x,y), & (t,x,y) \in (0,T) \times \Omega^\ell, \\ f(t,x,y) = 0, & (t,x,y) \in (0,T) \times \partial\Omega^\ell, \\ f(0,x,y) = f^0(x,y), & (x,y) \in \Omega^\ell. \end{cases} \tag{7.5}$$

If $\nu \in \left(0, \frac{1}{2}\right]$, then system (7.5) is approximately controllable in time T, for any $\ell > 0$.
If $\nu \in \left(\frac{1}{2}, 1\right)$ and $\gamma = 1$, there exist values of $\ell > 0$ such that for any $T > 0$ approximate controllability does not hold in time T for system (7.5).

Thus, the positive result of approximate controllability also holds for homogeneous Dirichlet boundary conditions. The negative result based on an explicit counterexample will necessitate special lengths in the y variable (i.e. particular values of ℓ) and only stands in the case $\gamma = 1$. These assumptions are technical and we conjecture that approximate controllability does not hold for system (7.5) for any $\gamma > 0$, any $\ell > 0$ if $\nu \in \left(\frac{1}{2}, 1\right)$. We will focus in the rest of the paper on Theorem 7.1 and detail only the modifications for Theorem 7.2 when necessary.

The model (7.1) can also be seen as an extension of [14] where Beauchard *et al.* studied the null controllability without the singular potential (i.e. in the case $\nu = \frac{1}{2}$). The authors proved that null controllability holds if $\gamma \in (0,1)$ and does not hold if $\gamma > 1$. In the case $\gamma = 1$, for ω a strip in the y direction, null controllability holds if and only if the time is large enough.

The inverse square potential for the Grushin equation has already been taken into account by Cannarsa and Guglielmi in [37] in the case where both degeneracy and singularity are at the boundary. With our notations, they proved null controllability in sufficiently large time for $\Omega = (0,1) \times (0,1)$, $\omega = (a,b) \times (0,1)$, $\gamma = 1$ and any $c_\nu > -\frac{1}{4}$. They also proved that approximate controllability holds for any control domain $\omega \subset \Omega$, any $\gamma > 0$ and any $c_\nu > -\frac{1}{4}$. Thus, the fact that our model presents an internal singularity instead of a boundary singularity deeply affects the approximate controllability property.
By a classical duality argument, Theorem 7.1 will be proved by unique continuation on the adjoint system. Following techniques used in [14] this problem will be studied trough the 1D equations satisfied by the coefficients of the solution in the expansion in Fourier series

in the y variable. As a corollary we will obtain the following approximate controllability result for the 1D heat equation with a singular inverse square potential.

Theorem 7.3. *Let $T > 0$ and $\nu \in (0,1)$. If ω is an open subset of $(-1,0)$ or $(0,1)$, then approximate controllability holds for*

$$\begin{cases} \partial_t f - \partial_{xx}^2 f + \dfrac{c_\nu}{x^2} f = u(t,x)\chi_\omega(x), & (t,x) \in (0,T) \times (-1,1), \\ f(t,-1) = f(t,1) = 0, & t \in (0,T), \end{cases} \tag{7.6}$$

if and only if $\nu \in \left(0, \frac{1}{2}\right]$ i.e. if and only if $c_\nu \in \left(-\frac{1}{4}, 0\right]$.

The null controllability issue for the 1D heat equation with such an internal inverse square singularity remains an open question. Like (7.1), the solutions of (7.6) are related to the semigroup generated by a suitable extension of the Laplace operator with a singular potential.

7.1.2 Structure of this article

We end this introduction by a brief review of previous results concerning degenerate and/or singular parabolic equations.

Due to the internal singularity and the fact that the considered operators admit several self-adjoint extensions, the functional setting and the well posedness are crucial issues in this article. Section 7.2 is dedicated to these questions.

Section 7.3 is dedicated to the study of the unique continuation property. When it holds, unique continuation is proved using tools from the uniformly parabolic case and by adapting Carleman estimates to our setting. When $\nu \in \left(\frac{1}{2}, 1\right)$, explicit counterexamples will be constructed using Bessel functions.

7.1.3 A review of previous results

The first result for a heat equation with an inverse square potential $\frac{\lambda}{|x|^2}$ deals with well posedness issues. In [6], Baras and Goldstein proved complete instantaneous blow-up for positive initial conditions in space dimension N (the singularity being at the boundary of the domain in the one dimensional case) if $\lambda < \lambda^*(N) := -\frac{(N-2)^2}{4}$. Notice that this critical value is the best constant in the Hardy inequality. Cabré and Martel also studied in [35] the relation between blow-up of such equations and the existence of an Hardy inequality. Thus, most of the following studies focus on the range of constants $\lambda \geq \lambda^*(N)$. In this case, well posedness in $L^2(\Omega)$ has been proved in [139] by Vazquez and Zuazua.
The controllability issues were first studied for degenerate equations. In [38, 101, 39, 40], Cannarsa, Martinez and Vancostenoble proved null controllability with a distributed control for a one dimensional parabolic equation degenerating at the boundary. Then, they extended this result to more general degeneracies and in dimension two. These results are based on suitable Hardy inequalities and Carleman estimates. More recently, Gueye proved in [81] null controllability for a class of one dimensional hyperbolic equations degenerating at the boundary (and the corresponding parabolic degenerate equation via transmutation) with control on the degenerate boundary. Its proof relies on appropriate nonharmonic Fourier series.

Meanwhile, these Carleman estimates were adapted for heat equation with an inverse square potential $\frac{1}{|x|^2}$ in dimension $N \geq 3$. Notice that in this case the singularity is the point $\{0\}$. In [138], Vancostenoble and Zuazua proved null controllability in the case where the control domain ω contains an annulus centred on the singularity. Their proof relies on a decomposition in spherical harmonics reducing the problem to the study of a 1D heat equation with an inverse square potential which is singular at the boundary. The geometric assumptions on the control domain were then removed by Ervedoza in [69] using a direct Carleman strategy in dimension $N \geq 3$. Notice that although these results deal with internal singularity they cannot be adapted to our setting. Indeed, in [138] it is crucial that the singularity of the 1D problem obtained by decomposition in spherical harmonics is at the boundary. The Carleman strategy developed in [69] cannot be adapted in this article because our singularity is no longer a point but separates the domain in two connected components.
For null controllability for a one dimensional parabolic equation both degenerate and singular at the boundary we refer to [137] by Vancostenoble. The proof extends the previous Carleman strategy together with an improved Hardy inequality.
As the functional setting for this study is obtained through the design of a suitable self-adjoint extension of our Grushin-like operator, let us mention the work [29] conducted simultaneously to this study. In this paper, Boscain and Prandi studied some extensions of the Laplace-Beltrami operator (7.3) for $\gamma \in \mathbb{R}$. Among other things, they design for a suitable range of constants an extension called bridging extension that allows full communication through the singular set. Even if we authorize in this article a wider class of singular potentials, the approximate controllability from one side of the singularity given by Theorem 7.1 shows full agreement with the bridging extension of [29].

7.2 Well posedness

The previous results of the literature dealing with an inverse square potential were obtained thanks to some Hardy-type inequality. For a boundary inverse square singularity (as in [137]), the condition $z(0) = 0$ needed for (7.4) to hold is contained in the homogeneous Dirichlet boundary conditions considered. Thus, in [137], the appropriate functional setting to study the 1D operator $-\partial_{xx}^2 + \frac{\lambda}{x^2}$ with $\lambda > -\frac{1}{4}$ is

$$\left\{ f \in H^2_{loc}((0,1]) \cap H^1_0(0,1)\,;\, -\partial_{xx}^2 f + \frac{\lambda}{x^2} f \in L^2(0,1) \right\}.$$

For an internal inverse square singularity one still has

$$\int_{-1}^{1} \frac{z(x)^2}{x^2} \mathrm{d}x \leq 4 \int_{-1}^{1} z_x(x)^2 \mathrm{d}x, \quad \forall z \in H^1(-1,1) \text{ such that } z(0) = 0. \tag{7.7}$$

This inequality ceases to be true if $z(0) \neq 0$. Thus, the functional setting must contain some informations on the behaviour of the functions at the singularity.

As announced, we design a suitable self-adjoint extension of the operator $-\partial_{xx}^2 - |x|^{2\gamma}\partial_{yy}^2 + \frac{c_\nu}{x^2}$ on $C_0^\infty(\Omega \backslash \{x = 0\})$. The next subsection deals with an associated one dimensional equation. Subsection 7.2.2 will then relate this one dimensional problem to the original problem in dimension two.

7.2.1 Reduction to a 1D problem

For $n \in \mathbb{Z}$, $\gamma > 0$ and $\nu \in (0,1)$ let us consider the following homogeneous problem

$$\begin{cases} \partial_t f - \partial_{xx}^2 f + \dfrac{c_\nu}{x^2} f + (2n\pi)^2 |x|^{2\gamma} f = 0, & (t,x) \in (0,T) \times (-1,1), \\ f(t,-1) = f(t,1) = 0, & t \in (0,T). \end{cases} \tag{7.8}$$

This equation is inspired by the equation satisfied by the coefficients of the Fourier expansion in the y variable done in [14] and will be linked to (7.1) in Subsection 7.2.2. From now on, we focus on the well posedness of (7.8).

Remark 7.2. A naive functional setting for this equation is the adaptation of [137]

$$\Big\{ f \in L^2(-1,1)\,;\, f_{|[0,1]} \in H^2_{loc}((0,1]) \cap H^1_0(0,1),\ f_{|[-1,0]} \in H^2_{loc}([-1,0)) \cap H^1_0(-1,0) \\ \text{and } -\partial_{xx}^2 f + \frac{c_\nu}{x^2} f \in L^2(-1,1) \Big\}.$$

However, a functional setting where the two problems on $(-1,0)$ and $(0,1)$ are well posed is not pertinent for the control problem from one side of the singularity. It leads to decoupled dynamics on the connected components of $(-1,0) \cup (0,1)$.

We study the differential operator

$$A_n f := -\partial_{xx}^2 f + \frac{c_\nu}{x^2} f(x) + (2n\pi)^2 |x|^{2\gamma} f(x).$$

As $\nu < 1$, the results of [28] imply that A_n defined on $C_0^\infty((-1,0) \cup (0,1))$ admits several self-adjoint extensions. Let us specify the self-adjoint extension that will be used. Let

$$\tilde{H}^2_0(-1,1) := \left\{ f \in H^2(-1,1)\,;\, f(0) = f'(0) = 0 \right\},$$

and

$$\mathcal{F}_s := \Big\{ f \in L^2(-1,1)\,; f = c_1^+ |x|^{\nu+1/2} + c_2^+ |x|^{-\nu+1/2} \text{ on } (0,1) \\ \text{and } f = c_1^- |x|^{\nu+1/2} + c_2^- |x|^{-\nu+1/2} \text{ on } (-1,0) \Big\}.$$

The domain of the operator is defined as

$$D(A_n) := \Big\{ f = f_r + f_s\,;\, f_r \in \tilde{H}^2_0(-1,1),\ f_s \in \mathcal{F}_s \text{ such that } f(-1) = f(1) = 0, \\ c_1^- + c_2^- + c_1^+ + c_2^+ = 0 \text{ and} \\ (\nu + 1/2)c_1^- + (-\nu + 1/2)c_2^- = (\nu + 1/2)c_1^+ + (-\nu + 1/2)c_2^+ \Big\}, \tag{7.9}$$

Notice that for $\nu \in (0,1)$, $D(A_n) \subset L^2(-1,1)$. As this domain is independent of n, it will be denoted by $D(A)$ in the rest of this article. The coefficients of the singular part will be denoted by c_1^+ if there is no ambiguity and $c_1^+(f)$ otherwise. The conditions imposed on these coefficients in (7.9) will be referred to as the "transmission conditions".

Using classical computations (see for instance [2, Proposition 3.1]) we get that for any $f_r \in \tilde{H}_0^2(-1,1)$, $x \mapsto \frac{1}{x^2} f_r(x) \in L^2(-1,1)$. Notice that for any $f_s \in \mathcal{F}_s$,

$$-\partial_{xx}^2 f_s + \frac{c_\nu}{x^2} f_s = 0. \tag{7.10}$$

Thus, for any $f \in D(A)$,

$$A_n f = \Big(-\partial_{xx}^2 f_r + \frac{c_\nu}{x^2} f_r\Big) + (2n\pi)^2 |x|^{2\gamma} f \in L^2(-1,1).$$

This operator satisfies the following properties

Proposition 7.1. *For any $n \in \mathbb{Z}$, the operator $(A_n, D(A))$ is self-adjoint. Moreover, for any $f \in D(A)$,*

$$\langle A_n f, f\rangle \geq m_\nu \int_{-1}^{1} \partial_x f_r(x)^2 \mathrm{d}x + (2n\pi)^2 \int_{-1}^{1} |x|^{2\gamma} f(x)^2 \mathrm{d}x \geq 0,$$

where $m_\nu := \min\{1, 4\nu^2\}$.

This proposition is proved in Appendix 7.A.

Remark 7.3. The construction of this self-adjoint extension of the minimal operator is inspired by [142, Theorem 13.3.1, Case 5] for general self-adjoint extensions of Sturm-Liouville differential operators and by [2] for the explicit characterization of the minimal and maximal domains. We have nevertheless detailed this proof for the sake of clarity. The reader interested in the link between this construction and the general theory of self adjoint extensions of Sturm-Liouville operator should refer to Appendix 7.B.

Using Proposition 7.1, the well posedness of the one dimensional system (7.8) follows from the Hille-Yosida theorem (see e.g. [43, Theorem 3.2.1]).

Proposition 7.2. *For any $n \in \mathbb{Z}$ and any $f^0 \in L^2(-1,1)$, problem (7.8) with initial condition $f(0,\cdot) = f^0$ has a unique solution*

$$f \in C^0([0,+\infty), L^2(-1,1)) \cap C^0((0,+\infty), D(A)) \cap C^1((0,+\infty), L^2(-1,1)).$$

This solution satisfies

$$||f(t)||_{L^2(-1,1)} \leq ||f^0||_{L^2(-1,1)}.$$

In all what follows, we denote by $e^{-A_n t}$ the semigroup generated by $-A_n$ i.e. for any $f^0 \in L^2(-1,1)$, the function $t \mapsto e^{-A_n t} f^0$ is the solution of (7.8) given by Proposition 7.2. We now turn back to the initial problem in dimension two.

7.2.2 Semigroup associated to the 2D problem

Let $f^0 \in L^2(\Omega)$. For almost every $x \in (-1,1)$, $f^0(x,\cdot) \in L^2(0,1)$ and thus can be expanded in Fourier series as follows

$$f^0(x,y) = \sum_{n\in\mathbb{Z}} f_n^0(x)\varphi_n(y), \tag{7.11}$$

where $(\varphi_n)_{n\in\mathbb{Z}}$ is the Hilbert basis of $L^2(0,1)$ of eigenvectors of the Laplace operator on $H^2(0,1)$ with periodic boundary conditions i.e.

$$\varphi_n(y) := \sqrt{2}\sin(2n\pi y),\ \forall n \in \mathbb{N}^*; \quad \varphi_{-n}(y) := \sqrt{2}\cos(2n\pi y),\ \forall n \in \mathbb{N}^*; \quad \varphi_0(y) := 1$$

and

$$f_n^0(x) := \int_{-1}^{1} f^0(x,y)\varphi_n(y)\mathrm{d}y.$$

For any $t \in (0,T)$, we define the following operator

$$(S(t)f^0)(x,y) := \sum_{n\in\mathbb{Z}} f_n(t,x)\varphi_n(y), \tag{7.12}$$

where for any $n \in \mathbb{Z}$, $f_n(t) := e^{-A_n t} f_n^0$. Then, the following proposition holds.

Proposition 7.3. *$S(t)$ defined by (7.12) is a continuous semigroup of contractions in $L^2(\Omega)$.*

Proof of Proposition 7.3. By Proposition 7.2, $S(t)$ is well defined, with value in $L^2(\Omega)$, it is a semigroup and satisfies the contraction property. For any $f^0 \in L^2(\Omega)$, we have

$$||S(t)f^0 - f^0||^2_{L^2(\Omega)} = \sum_{n\in\mathbb{Z}} ||f_n(t,\cdot) - f_n^0||^2_{L^2(-1,1)}.$$

By Proposition 7.2 it comes that

$$||f_n(t,\cdot) - f_n^0||_{L^2(-1,1)} \underset{t\to 0}{\longrightarrow} 0,$$
$$||f_n(t,\cdot) - f_n^0||_{L^2(-1,1)} \leq 2\,||f_n^0||_{L^2(-1,1)}.$$

Thus, by the dominated convergence theorem, $S(t)f^0 \underset{t\to 0}{\longrightarrow} f^0$ in $L^2(\Omega)$.

□

Recall that its infinitesimal generator $\mathcal{A}$ is defined on

$$D(\mathcal{A}) := \left\{ f \in L^2(\Omega)\,;\ \lim_{t\to 0} \frac{S(t)f - f}{t} \text{ exists } \right\},$$

by

$$\mathcal{A}f := \lim_{t\to 0} \frac{S(t)f - f}{t}.$$

The previous limits are related to the L^2 norm. Then, from [116, Theorems 1.3.1 and 1.4.3] it comes that $(\mathcal{A}, D(\mathcal{A}))$ is a closed dissipative densely defined operator and satisfies for any $\lambda > 0$, $R(\lambda I - \mathcal{A}) = L^2(\Omega)$. The following proposition links the system (7.1) and the semigroup $S(t)$.

Proposition 7.4. *The infinitesimal generator $\mathcal{A}$ of $S(t)$ is characterized by*

$$D(\mathcal{A}) = \Big\{ f \in L^2(\Omega)\,;\, f = \sum_{n\in\mathbb{Z}} f_n(x)\varphi_n(y) \ \text{with}\ f_n \in D(A)\ \text{and} \\ \sum_{n\in\mathbb{Z}} ||A_n f_n||^2_{L^2(-1,1)} < +\infty \Big\}, \tag{7.13}$$

and

$$\mathcal{A}f = -\sum_{n\in\mathbb{Z}} (A_n f_n)(x)\varphi_n(y). \tag{7.14}$$

This operator extends the Grushin differential operator in the sense that

$$\mathcal{A}f = \partial^2_{xx} f + |x|^{2\gamma}\partial^2_{yy} f - \frac{c_\nu}{x^2} f, \quad \forall f \in C_0^\infty(\Omega\backslash\{x=0\}). \tag{7.15}$$

Proof of Proposition 7.4. Let $f^0 \in D(\mathcal{A})$. Then, $\mathcal{A}f^0 \in L^2(\Omega)$ and

$$\frac{S(t)f^0 - f^0}{t} \underset{t\to 0}{\longrightarrow} \mathcal{A}f^0, \quad \text{in } L^2(\Omega).$$

As $\mathcal{A}f^0 \in L^2(\Omega)$, it can be decomposed in Fourier series in the y variable i.e.

$$\mathcal{A}f^0(x,y) = \sum_{n\in\mathbb{Z}} (\mathcal{A}f^0)_n(x)\varphi_n(y).$$

Thus,

$$\left|\left| \frac{S(t)f^0 - f^0}{t} - \mathcal{A}f^0 \right|\right|^2_{L^2(\Omega)} = \sum_{n\in\mathbb{Z}} \left|\left| \frac{f_n(t) - f_n^0}{t} - (\mathcal{A}f^0)_n \right|\right|^2_{L^2(-1,1)} \underset{t\to 0}{\longrightarrow} 0.$$

This implies that for any $n \in \mathbb{Z}$, $f_n^0 \in D(A)$ and

$$(\mathcal{A}f^0)_n = -A_n f_n^0.$$

We thus get

$$-\mathcal{A}f^0 = \sum_{n\in\mathbb{Z}} (A_n f_n^0)(x)\varphi_n(y).$$

Conversely, let $g \in L^2(\Omega)$ be such that for any $n \in \mathbb{Z}$, $g_n \in D(A)$ and $\sum_{n\in\mathbb{Z}} ||A_n g_n||^2_{L^2(-1,1)} < +\infty$. Let $f \in D(\mathcal{A})$. Then,

$$|\langle \mathcal{A}f, g\rangle| \leq \sum_{n\in\mathbb{Z}} |\langle A_n f_n, g_n\rangle| \leq \left(\sum_{n\in\mathbb{Z}} ||f_n||^2_{L^2(-1,1)}\right)^{\frac{1}{2}} \left(\sum_{n\in\mathbb{Z}} ||A_n g_n||^2_{L^2(-1,1)}\right)^{\frac{1}{2}}.$$

This implies that $g \in D(\mathcal{A}^*)$. Finally, self-adjointness of $S(t)$ and thus of $\mathcal{A}$ ends the proof of (7.13). Straightforward computations lead to (7.15) and thus ends the proof of Proposition 7.4.

□

Using Proposition 7.4, we rewrite (7.1)-(7.2) in the form

$$\begin{cases} f'(t) = \mathcal{A}f(t) + v(t), & t \in [0,T], \\ f(0) = f^0, \end{cases} \tag{7.16}$$

where $v(t) : (x,y) \in \Omega \mapsto u(t,x,y)\chi_\omega(x,y)$. In the following a solution of (7.1) will mean a solution of (7.16). The following proposition is classical (see e.g. [116]) and ends this well posedness section

Proposition 7.5. *For any $f^0 \in L^2(\Omega)$, any $T > 0$ and $v \in L^2((0,T); L^2(\Omega))$, system (7.16) has a unique mild solution given by*

$$f(t) = S(t)f^0 + \int_0^t S(t-\tau)v(\tau)\mathrm{d}\tau, \quad t \in [0,T].$$

7.3 Unique continuation

Without loss of generality, we assume that $\omega \subset (-1,0) \times (0,1)$. Using the abstract formulation (7.16) we get that the adjoint system of (7.1) is given by

$$\begin{cases} \partial_t g - \partial_{xx}^2 g - |x|^{2\gamma}\partial_{yy}^2 g + \frac{c_\nu}{x^2} g = 0, & (t,x,y) \in (0,T) \times \Omega, \\ g(t,-1,y) = g(t,1,y) = 0, & (t,y) \in (0,T) \times (0,1), \\ g(t,x,0) = g(t,x,1), & (t,x) \in (0,T) \times (-1,1), \\ \partial_y g(t,x,0) = \partial_y g(t,x,1), & (t,x) \in (0,T) \times (-1,1), \\ g(0,x,y) = g^0(x,y), & (x,y) \in \Omega. \end{cases} \tag{7.17}$$

From Section 7.2, it comes that for any $g^0 \in L^2(\Omega)$, system (7.17) has a unique solution given by $S(t)g^0$. Thanks to a classical duality argument, Theorem 7.1 is proved by the following theorem dealing with unique continuation for the adjoint system (7.17).

Theorem 7.4. *Let $T > 0$, $\gamma > 0$ and $\nu \in (0,1)$. Assume that $g^0 \in L^2(\Omega)$ is such that $\chi_\omega S(t)g^0 \equiv 0$ for almost every $t \in [0,T]$. Then, g^0 must be identically zero on Ω if and only if $\nu \in \left(0, \frac{1}{2}\right]$ i.e. $c_\nu \in \left(-\frac{1}{4}, 0\right]$.*

The rest of this section is dedicated to the proof of Theorem 7.4. In Subsection 7.3.1, we prove that $S(t)g^0$ must be identically zero on the connected component of $\Omega\backslash\{x=0\}$ containing ω using unique continuation for uniformly parabolic operators. This will imply that any Fourier component g_n has no singular part and is identically zero on one side of $[-1,1]\backslash\{0\}$. Then, we are left to study a one dimensional equation on the regular part with a boundary inverse square singularity. If $\nu \in \left(0, \frac{1}{2}\right]$, we prove in Subsection 7.3.2 that unique continuation holds thanks to a suitable Carleman-type estimate. Finally, if $\nu \in \left(\frac{1}{2}, 1\right)$, we construct explicit solutions that contradict unique continuation in Subsection 7.3.3.

7.3.1 Reduction to the case of a boundary singularity

The goal of this section is the proof of the following proposition

Proposition 7.6. *Let $T > 0$, $\gamma > 0$, $\nu \in (0,1)$ and ω be an open subset of $(-1,0)\times(0,1)$. Assume that $g^0 \in L^2(\Omega)$ is such that $\chi_\omega S(t)g^0 \equiv 0$ for almost every $t \in [0,T]$. Then $S(t)g^0$ is identically zero on $(-1,0)\times(0,1)$. For any $n \in \mathbb{Z}$, the singular part of the n^{th} Fourier component satisfies $g_{n,s}(t,x) = 0$ for every $(t,x) \in (0,T)\times(-1,1)$.*

Proof of Proposition 7.6. Let $\varepsilon > 0$ be such that

$$\omega \subset \Omega_\varepsilon^- := (-1,-\varepsilon)\times(0,1).$$

For every $t \in [0,T]$,

$$(S(t)g^0)(x,y) = \sum_{n\in\mathbb{Z}} g_n(t,x)\varphi_n(y),$$

where g_n is the solution of (7.8) with initial condition g_n^0.
The solution of (7.17) is defined through an abstract extension operator. We check that on Ω_ε^-, the operator $\mathcal{A}$ is uniformly elliptic. Let $h \in D(\mathcal{A})$ and $\phi \in C_0^\infty(\Omega_\varepsilon^-)$. Then,

$$\begin{aligned}
\langle \mathcal{A}h, \phi\rangle_{L^2(\Omega_\varepsilon^-)} &= \int_{-1}^{-\varepsilon}\int_0^1 \mathcal{A}h(x,y)\phi(x,y)\mathrm{d}y\mathrm{d}x \\
&= -\sum_{n\in\mathbb{Z}} \langle A_n h_n, \phi_n\rangle_{L^2(-1,-\varepsilon)} \\
&= -\sum_{n\in\mathbb{Z}} \langle h_n, A_n \phi_n\rangle_{L^2(-1,-\varepsilon)} \\
&= \langle h, \left(\partial_{xx}^2 + |x|^{2\gamma}\partial_{yy}^2 - \frac{c_\nu}{x^2}\right)\phi\rangle_{L^2(\Omega_\varepsilon^-)}.
\end{aligned}$$

Thus, $h \in D(\mathcal{A})$ implies that

$$\mathcal{A}h \overset{\mathcal{D}'(\Omega_\varepsilon^-)}{=} \left(\partial_{xx}^2 + |x|^{2\gamma}\partial_{yy}^2 - \frac{c_\nu}{x^2}\right) h.$$

As $h \in D(\mathcal{A})$, this equality also holds in $L^2(\Omega_\varepsilon^-)$. In particular, this implies that

$$\partial_{xx}^2 h + |x|^{2\gamma}\partial_{yy}^2 h \in L^2(\Omega_\varepsilon^-),$$

and also that $\mathcal{A}$ is uniformly elliptic on Ω_ε^-. Thus, using classical unique continuation results for uniformly parabolic operators with variable coefficients (see e.g. [125, Theorem 1.1]), it comes that $S(t)g^0 = 0$ for every $t \in (0,T]$ in $L^2(\Omega_\varepsilon^-)$. Then, it comes that $S(t)g^0 = 0$ for every $t \in (0,T]$ in $L^2(\Omega_0^-)$. If, for any $n \in \mathbb{Z}$, we decompose g_n in regular and singular part (respectively $g_{n,r}$ and $g_{n,s}$ as defined in (7.9)) we get

$$c_1^-(g_n(t)) = c_2^-(g_n(t)) = 0, \quad \forall t \in (0,T). \tag{7.18}$$

$$g_{n,r}(t,x) \equiv 0, \quad \forall (t,x) \in (0,T)\times(-1,0), \tag{7.19}$$

Using the transmission conditions in (7.9), it also comes that $c_1^+(g_n(t)) = c_2^+(g_n(t)) = 0$ and thus the singular part is identically zero on $(0,T)\times(-1,1)$. This ends the proof of Proposition 7.6.

□

Remark 7.4. Notice that Proposition 7.6 proves that if ω intersects both connected components of $\Omega\backslash\{x=0\}$, then unique continuation hold for any $\nu \in (0,1)$.

Proposition 7.6 implies that if $\chi_\omega S(t)g^0$ is identically zero then for any $n \in \mathbb{Z}$, $g_n \in C^0((0,T], H^2 \cap H_0^1(0,1)) \cap C^1((0,T], L^2(0,1))$ is solution of

$$\begin{cases} \partial_t g_n - \partial_{xx}^2 g_n + \left(\dfrac{c_\nu}{x^2} + (2n\pi)^2 x^{2\gamma}\right) g_n = 0, & (t,x) \in (0,T) \times (0,1), \\ g_n(t,0) = g_n(t,1) = 0, & t \in (0,T), \\ \partial_x g_n(t,0) = 0, & t \in (0,T). \end{cases} \tag{7.20}$$

For $\nu \in \left(0, \frac{1}{2}\right]$, we prove in Subsection 7.3.2 that $g_n \equiv 0$ using a suitable Carleman estimate. For $\nu \in \left(\frac{1}{2}, 1\right)$ we design in Subsection 7.3.3 explicit non trivial solutions.

7.3.2 Unique continuation for $\nu \in \left(0, \frac{1}{2}\right]$

In this subsection we assume that $\nu \in \left(0, \frac{1}{2}\right]$ and prove the Carleman type inequality stated in Proposition 7.7 below. Let us define the weights that will be used to prove this inequality (see Remark 7.6 for comments on the weights).
Let $\theta : t \in (0,T) \mapsto \frac{1}{t(T-t)}$. Let $p \in C^4([0,1])$ be such that there exist positive constants m_0, m_1, m_2 such that for any $x \in [0,1]$

$$p(x) \geq m_0 > 0, \quad p_x(x) \geq m_1 > 0, \quad -p_{xx}(x) \geq m_2 > 0. \tag{7.21}$$

We set $\sigma(t,x) := \theta(t)p(x)$. For any $n \in \mathbb{Z}$ and any $\gamma > 0$, we introduce the following operator

$$\mathcal{P}_n := \partial_t - \partial_{xx}^2 + \left(\frac{c_\nu}{x^2} + (2n\pi)^2 |x|^{2\gamma}\right).$$

Then, the following proposition holds.

Proposition 7.7. *Let $T > 0$ and $Q_T := (0,T) \times (0,1)$. There exist $R_0, C_0 > 0$ such that for any $R \geq R_0$, any $g \in C^1((0,T], L^2(0,1)) \cap C^0((0,T], H^2 \cap H_0^1(0,1))$ with $\partial_x g(t,0) \equiv 0$ on $(0,T)$ satisfies*

$$C_0 \iint_{Q_T} \left(R^3\theta^3 g^2 + R\theta g_x^2\right) e^{-2R\sigma} \mathrm{d}x\mathrm{d}t \leq \iint_{Q_T} |\mathcal{P}_n g| e^{-2R\sigma} \mathrm{d}x\mathrm{d}t. \tag{7.22}$$

Before proving Proposition 7.7 let us point out that it ends the proof of the "if" assertion of Theorem 7.4. Let $g^0 \in L^2(\Omega)$ be such that $\chi_\omega S(t)g^0 \equiv 0$. Using Proposition 7.6 and the final comment of Subsection 7.3.1, it comes that for any $n \in \mathbb{Z}$, $g_n \in C^1((0,T], L^2(0,1)) \cap C^0((0,T], H^2 \cap H_0^1(0,1))$ with $\partial_x g_n(t,0) \equiv 0$ on $(0,T)$. As, g_n is solution of (7.20), it comes that $\mathcal{P}_n g_n \equiv 0$ on $(0,T) \times (0,1)$. Then, Proposition 7.7 implies that $g_n \equiv 0$ and thus, as $g \in C^0([0,T], L^2(0,1))$, we recover $g^0 = 0$.

Remark 7.5. Contrarily to Carleman estimates proved by Vancostenoble [137], there are no boundary terms in the right-hand side of the inequality. Actually, the homogeneous Neumann boundary condition at $x=0$ is crucial for inequality (7.22) to hold.

Proof of Proposition 7.7. We denote the partial derivative by subscripts: z_x stands for $\partial_x z$. We set for $R > 0$,

$$z(t,x) := e^{-R\sigma(t,x)} g(t,x). \tag{7.23}$$

Thus, for any $x \in (0,1)$, $z(0,x) = z(T,x) = z_t(0,x) = z_t(T,x) = 0$. The boundary conditions on g also imply that for any $t \in (0,T)$, $z(t,0) = z(t,1) = z_x(t,0) = 0$. Notice that these boundary conditions imply that $x \mapsto \frac{z(t,x)}{x^2} \in L^2(0,1)$ which justifies the following computations.
Straightforward computations lead to $e^{-R\sigma}\mathcal{P}_n g = P_R^+ z + P_R^- z$ where

$$\begin{aligned} P_R^+ z &:= (R\sigma_t - R^2\sigma_x^2)z - z_{xx} + \left(\frac{c_\nu}{x^2} + (2n\pi)^2|x|^{2\gamma}\right) z, \\ P_R^- z &:= z_t - 2R\sigma_x z_x - R\sigma_{xx} z. \end{aligned}$$

Then,

$$\iint_{Q_T} P_R^+ z P_R^- z \mathrm{d}x\mathrm{d}t \le \frac{1}{2}\iint_{Q_T} e^{-2R\sigma}|\mathcal{P}_n g|^2 \mathrm{d}x\mathrm{d}t. \tag{7.24}$$

The rest of the proof follows the classical Carleman strategy [87] (see [54] for a pedagogical presentation). We just pay attention to the singular terms.

First step : integrations by part lead to

$$\begin{aligned} \iint_{Q_T} P_R^+ z P_R^- z \mathrm{d}x\mathrm{d}t &= R\int_0^T \sigma_x(t,1) z_x^2(t,1)\mathrm{d}t - 2R\iint_{Q_T} \sigma_{xx} z_x^2 \mathrm{d}x\mathrm{d}t \\ &+ \iint_{Q_T} \left(-\frac{R}{2}\sigma_{tt} + 2R^2\sigma_x\sigma_{xt} - 2R^3\sigma_x^2\sigma_{xx} + \frac{R}{2}\sigma_{xxxx}\right) z^2 \mathrm{d}x\mathrm{d}t \\ &+ R\iint_{Q_T} \left(-2\frac{c_\nu}{x^3} + 2\gamma(2n\pi)^2|x|^{2\gamma-1}\right)\sigma_x z^2 \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{7.25}$$

Performing integrations by parts, it is easily seen that $\langle P_R^+ z, P_R^- z\rangle = I_1 + \cdots + I_5$, where

$$\begin{aligned} I_1 :=& \langle (R\sigma_t - R^2\sigma_x^2)z - z_{xx}, z_t\rangle = \iint_{Q_T} \Big(-\frac{R}{2}\sigma_{tt} + R^2\sigma_x\sigma_{xt}\Big) z^2 \mathrm{d}x\mathrm{d}t, \\ I_2 :=& -R^2\langle \sigma_t z, 2\sigma_x z_x + \sigma_{xx} z\rangle = R^2\iint_{Q_T} \sigma_{xt}\sigma_x z^2 \mathrm{d}x\mathrm{d}t, \\ I_3 :=& R^3\langle \sigma_x^2 z, 2\sigma_x z_x + \sigma_{xx} z\rangle = -R^3\iint_{Q_T} 2\sigma_x^2\sigma_{xx} z^2 \mathrm{d}x\mathrm{d}t, \\ I_4 :=& R\langle z_{xx}, 2\sigma_x z_x + \sigma_{xx} z\rangle \\ =& R\int_0^T \sigma_x(t,1) z_x^2(t,1)\mathrm{d}t - R\iint_{Q_T} 2\sigma_{xx} z_x^2 + \sigma_{xxx} z z_x \mathrm{d}x\mathrm{d}t \\ =& R\int_0^T \sigma_x(t,1) z_x^2(t,1)\mathrm{d}t - R\iint_{Q_T} 2\sigma_{xx} z_x^2 - \frac{1}{2}\sigma_{xxxx} z^2 \mathrm{d}x\mathrm{d}t, \end{aligned}$$

and

$$I_5 := \langle \Big(\frac{c_\nu}{x^2} + (2n\pi)^2|x|^{2\gamma}\Big) z, z_t - 2R\sigma_x z_x - R\sigma_{xx} z\rangle.$$

Integrations by parts with Lemma 7.4 to estimate the boundary terms lead to

$$\begin{aligned} I_5 &= -R\int_0^T \left[\left(\frac{c_\nu}{x^2}+(2n\pi)^2|x|^{2\gamma}\right)\sigma_x z^2\right]_0^1 \mathrm{d}t + R\iint_{Q_T}\left(\frac{c_\nu}{x^2}+(2n\pi)^2|x|^{2\gamma}\right)_x \sigma_x z^2 \mathrm{d}x\mathrm{d}t \\ &= R\iint_{Q_T}\left(-2\frac{c_\nu}{x^3}+2\gamma(2n\pi)^2|x|^{2\gamma-1}\right)\sigma_x z^2\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Summing these terms leads to (7.25). Combining with (7.24) it comes that

$$\begin{aligned} \frac{1}{2}\iint_{Q_T} e^{-2R\sigma}|\mathcal{P}_n g|^2\mathrm{d}x\mathrm{d}t &\geq R\int_0^T \sigma_x(t,1)z_x^2(t,1)\mathrm{d}t - 2R\iint_{Q_T}\sigma_{xx}z_x^2\mathrm{d}x\mathrm{d}t \\ &+ \iint_{Q_T}\left(-\frac{R}{2}\sigma_{tt}+2R^2\sigma_x\sigma_{xt}-2R^3\sigma_x^2\sigma_{xx}+\frac{R}{2}\sigma_{xxxx}\right)z^2\mathrm{d}x\mathrm{d}t \\ &+ R\iint_{Q_T}\left(-2\frac{c_\nu}{x^3}+2\gamma(2n\pi)^2|x|^{2\gamma-1}\right)\sigma_x z^2\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{7.26}$$

Second step : lower bounds on the right-hand side of (7.26). Recall that $\sigma(t,x) = \theta(t)p(x)$. Using (7.21) in inequality (7.26) leads to

$$\begin{aligned} \frac{1}{2}\iint_{Q_T} e^{-2R\sigma}|\mathcal{P}_n g|^2\mathrm{d}x\mathrm{d}t &\geq m_1\int_0^T R\theta(t)z_x^2(t,1)\mathrm{d}t + 2m_2R\iint_{Q_T}\theta z_x^2\mathrm{d}x\mathrm{d}t \\ &+ \iint_{Q_T}\left(-2R^3\theta^3p_x^2p_{xx}+2R^2\theta\theta_t p_x^2-\frac{R}{2}\theta_{tt}p+\frac{R}{2}\theta p_{xxxx}\right)z^2\mathrm{d}x\mathrm{d}t \\ &+ R\iint_{Q_T}\theta\left(-2\frac{c_\nu}{x^3}+2\gamma(2n\pi)^2|x|^{2\gamma-1}\right)p_x z^2\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{7.27}$$

We study these terms separately. As $p_x \geq m_1 > 0$ on $[0,1]$ and $c_\nu \leq 0$, it comes that

$$m_1\int_0^T R\theta(t)z_x^2(t,1)\mathrm{d}t + R\iint_{Q_T}\theta\left(-2\frac{c_\nu}{x^3}+2\gamma(2n\pi)^2|x|^{2\gamma-1}\right)p_x z^2\mathrm{d}x\mathrm{d}t \geq 0, \tag{7.28}$$

each one of these terms being nonnegative. The definition of θ implies the existence of $C>0$ such that

$$|\theta\theta_t|+|\theta_{tt}|+\theta \leq C\theta^3, \quad \text{on } (0,T).$$

Together with (7.21), this leads to the existence of $\tilde{C}>0$ such that for R large enough

$$\iint_{Q_T}\left(-2R^3\theta^3p_x^2p_{xx}+2R^2\theta\theta_t p_x^2-\frac{R}{2}\theta_{tt}p+\frac{R}{2}\theta p_{xxxx}\right)z^2\mathrm{d}x\mathrm{d}t \geq \tilde{C}R^3\iint_{Q_T}\theta^3z^2\mathrm{d}x\mathrm{d}t. \tag{7.29}$$

Using (7.28) and (7.29) in (7.27) it comes that for R large enough

$$\frac{1}{2}\iint_{Q_T} e^{-2R\sigma}|\mathcal{P}_n g|^2\mathrm{d}x\mathrm{d}t \geq 2m_2R\iint_{Q_T}\theta z_x^2\mathrm{d}x\mathrm{d}t + \tilde{C}R^3\iint_{Q_T}\theta^3z^2\mathrm{d}x\mathrm{d}t. \tag{7.30}$$

Thus, (7.23) leads to

$$\frac{1}{2}\iint_{Q_T} e^{-2R\sigma}|\mathcal{P}_n g|^2 \mathrm{d}x\mathrm{d}t \geq 2m_2 R \iint_{Q_T} \theta g_x^2 e^{-2R\sigma}\mathrm{d}x\mathrm{d}t + \iint_{Q_T} (\tilde{C}R^3\theta^3 - 2m_2R^2\theta^2 p_x^2) g^2 e^{-2R\sigma}\mathrm{d}x\mathrm{d}t.$$

The choice of R large enough ends the proof of Proposition 7.7.

□

Remark 7.6. Let us point some of the differences between Proposition 7.7 and the Carleman estimates established in the case of a boundary inverse square singularity in [138, 137]. In both estimates the singular potential appears as

$$\iint_{Q_T} \frac{\sigma_x}{x^3} z^2 \mathrm{d}x\mathrm{d}t.$$

In [138], the weight is defined by $p(x) = 1 - \frac{x^2}{2}$. Thus, the singular potential can be treated with some Hardy type inequalities. In our situation, we choose the weight as an increasing concave positive function(for example, let us take $p(x) = -x^2 + 4x + 1$). This allows to deal easily with the lower bounds for the boundary term and for the potential $x \mapsto x^{2\gamma}$. However the price to pay is that there is for the singular potential a remaining term of the form

$$\iint_{Q_T} \theta \frac{z^2}{x^3} \mathrm{d}x\mathrm{d}t.$$

This term is dealt with in (7.28) (and is finite) thanks to the boundary condition $\partial_x g(t, 0) = 0$.

Adaptation to the case of Dirichlet boundary conditions. Let us explain how we can adapt this unique continuation result to prove the positive result of Theorem 7.2. The Fourier expansion is done in the Hilbert basis of eigenvectors of the Laplace operator on $H^2(0, 1)$ with homogeneous Dirichlet boundary condition i.e. $\big(\varphi_n^D(y) := \sqrt{2}\sin(n\pi y)\big)_{n\in\mathbb{N}^*}$. Similarly, the semigroup associated is

$$(S^D(t)f^0)(x, y) := \sum_{n\in\mathbb{N}^*} f_n(t, x)\varphi_n^D(y). \tag{7.31}$$

As the previous results hold true for any coefficient, we recover the results of Propositions 7.5, 7.6 and 7.7 for system (7.5). This holds for $\ell = 1$ and then for any $\ell > 0$ by an obvious change of variables.

7.3.3 Non unique continuation for $\nu \in \left(\frac{1}{2}, 1\right)$

In all this subsection, we assume that $\nu \in \left(\frac{1}{2}, 1\right)$.

7.3.3.1 Periodic boundary conditions on y

The goal of this section is to prove the following proposition.

Proposition 7.8. *Let $\gamma > 0$ and ω be an open subset of $(-1,0) \times (0,1)$. There exists $g^0 \in L^2(\Omega)$ such that the associated solution of (7.17) is not identically zero on Ω and satisfies $\chi_\omega S(t) g^0 \equiv 0$.*

Let J_ν be the Bessel function of first kind of order ν. The following properties of Bessel functions are classical and can be found for example in [140]. The function J_ν is defined on $[0,+\infty)$ by

$$J_\nu(x) := \left(\frac{x}{2}\right)^\nu \sum_{k\in\mathbb{N}} \frac{(-1)^k}{2^{2k} k!\,\Gamma(k+\nu+1)} x^{2k},$$

and solves the following Bessel equation

$$x^2 y''(x) + x y'(x) + (x^2 - \nu^2) y = 0. \tag{7.32}$$

J_ν possesses an infinite number of positive zeros denoted $j_{\nu,n}$ for $n \in \mathbb{N}^*$. The construction of our explicit counterexample is based on the following lemma.

Lemma 7.1. *For any $\lambda \in \left\{ j_{\nu,n}^2 \,;\, n \in \mathbb{N}^* \right\}$, the function $b_\lambda(x) := x^{1/2} J_\nu(x\sqrt{\lambda})$ satisfies*

$$\begin{cases} -b_\lambda''(x) + \dfrac{c_\nu}{x^2} b_\lambda(x) = \lambda b_\lambda(x), \\ b_\lambda(0) = b_\lambda(1) = 0, \\ b_\lambda'(0) = 0. \end{cases}$$

Proof of Lemma 7.1. Using (7.32) we get

$$\begin{aligned} -b_\lambda''(x) + \frac{c_\nu}{x^2} b_\lambda(x) &= \frac{-1}{x^{3/2}} \left(\lambda x^2 J_\nu''(x\sqrt{\lambda}) + x\sqrt{\lambda} J_\nu'(x\sqrt{\lambda}) - \nu^2 J_\nu(x\sqrt{\lambda}) \right) \\ &= \lambda x^{1/2} J_\nu(x\sqrt{\lambda}). \end{aligned}$$

As $\nu > 0$ it comes that $b_\lambda(0) = 0$. The fact that $\lambda \in \left\{ j_{\nu,n}^2 \,;\, n \in \mathbb{N}^* \right\}$ implies that $b_\lambda(1) = 0$. As,

$$b_\lambda(x) = \lambda^{\nu/2} \frac{x^{\nu+1/2}}{2^\nu} \sum_{k\in\mathbb{N}} \frac{(-1)^k \lambda^k}{2^{2k} k!\,\Gamma(k+\nu+1)} x^{2k},$$

and $\nu > \frac{1}{2}$ it comes that $b_\lambda'(0) = 0$. This ends the proof of Lemma 7.1. □

We now prove Proposition 7.8. This ends the proof of Theorem 7.4.

Proof of Proposition 7.8. Let $\lambda \in \left\{ j_{\nu,n}^2 \,;\, n \in \mathbb{N}^* \right\}$ and b_λ be as in Lemma 7.1. We define

$$g^0 : (x,y) \in \Omega \mapsto b_\lambda(x) \chi_{x \geq 0}(x).$$

Then $g^0 \in L^2(\Omega)$ and for any $n \in \mathbb{Z}\backslash\{0\}$, $g_n^0 \equiv 0$. From Lemma 7.1, it comes that the associated solution of (7.17) is

$$g(t) = e^{-A_0 t} g_0^0 : (x,y) \mapsto e^{-\lambda t} b_\lambda(x) \chi_{x\geq 0}(x).$$

This construction ends the proof of Proposition 7.8.

□

Remark 7.7. Notice that for $\nu \in \left(0, \frac{1}{2}\right]$, the explicit solutions constructed in the previous lemma are still strong solutions but does not satisfy $b'_\lambda(0) = 0$. This enlightens the crucial importance of the functional setting for unique continuation to hold.

As this counterexample is fundamentally based on the coefficient $n = 0$, it does not extend to the case of homogeneous Dirichlet boundary conditions. We design for this case, in the next subsection, a similar counterexample for $\gamma = 1$ and specific values of the length ℓ in the y direction.

Adaptation to the 1D heat equation. Let us point out that the previous study proves Theorem 7.3. Proposition 7.1 for $n = 0$ implies the existence of a mild solution to (7.6) for any initial condition in $L^2(-1, 1)$ and any control $u \in L^2(0, T)$. Proposition 7.2 gives the well posedness of the adjoint system in $D(A)$ for any initial condition in $L^2(-1, 1)$. The arguments developed in Subsection 7.3.1 are automatically adapted to this one dimensional setting. Then, Proposition 7.7 with $n = 0$ gives unique continuation for $\nu \in \left(0, \frac{1}{2}\right]$. The counterexample designed in Proposition 7.8 being based on the one dimensional system for $n = 0$ ends the proof of Theorem 7.3.

7.3.3.2 Homogeneous Dirichlet boundary conditions on y

In all what follows, we assume that $\gamma = 1$. Recall that the semigroup S^D associated to Dirichlet boundary conditions is defined in (7.31). We end the proof of Theorem 7.2 with the following proposition

Proposition 7.9. *There exists $\ell > 0$ such that for any subset ω of $(-1, 0) \times (0, \ell)$, there exists $g^0 \in L^2(\Omega)$ such that $S^D(t)g^0$ is not identically zero on Ω and satisfies $\chi_\omega S^D(t)g^0 \equiv 0$.*

Following the study of Proposition 7.8, we prove that there exists $m > 0$ and $g \not\equiv 0$ such that

$$\begin{cases} \partial_t g - \partial_{xx}^2 g + \dfrac{c_\nu}{x^2} g + m^2 x^2 g = 0, \\ g(t, 0) = g(t, 1) = 0, \\ \partial_x g(t, 0) = 0. \end{cases} \tag{7.33}$$

Let $M_{k,\mu}$ be the Whittaker function of first type with parameters $k \geq 1$ and $\mu > 0$. The properties of Whittaker functions can be found for example in [66, Sections 13.14 and 13.22]. The function $M_{k,\mu}$ is defined on $[0, +\infty)$ by

$$M_{k,\mu}(x) := x^{\mu+1/2} e^{-x/2} \frac{\Gamma(2\mu + 1)}{\Gamma(\mu - k + 1/2)} \sum_{n \in \mathbb{N}} \frac{\Gamma(n + \mu - k + 1/2)}{n!\, \Gamma(n + 2\mu + 1)} x^n,$$

and solves the following Whittaker equation

$$-y''(x) + \left(\frac{1}{4} - \frac{k}{x} + \frac{\mu^2 - 1/4}{x^2}\right) y(x) = 0. \tag{7.34}$$

Lemma 7.2. *Let $\lambda > 0$ be such that $M_{1,\frac{\nu}{2}}\left(\frac{\lambda}{4}\right) = 0$ and $m := \frac{\lambda}{4}$. Then, the function $w_\lambda(x) := x^{-1/2} M_{1,\frac{\nu}{2}}\left(\frac{\lambda}{4}x^2\right)$ satisfies*

$$\begin{cases} -w_\lambda''(x) + \dfrac{c_\nu}{x^2} w_\lambda(x) + m^2 x^2 w_\lambda(x) = \lambda w_\lambda(x), \\ w_\lambda(0) = w_\lambda(1) = 0, \\ w_\lambda'(0) = 0. \end{cases}$$

Thus, as Lemma 7.1 implied Proposition 7.8 it directly comes that if ℓ satisfies $\frac{n\pi}{\ell} = m$ for some $n \in \mathbb{N}^*$, Lemma 7.2 implies Proposition 7.9. Then, there is an infinite number of values of ℓ such that Proposition 7.9 holds.

Proof of Lemma 7.2. By [66, Section 13.22], if $\frac{1}{2} + \mu - k < 0$ and $1 + 2\mu > 0$, then $M_{k,\mu}$ admits a zero in $(0, +\infty)$. As $\nu \in \left(\frac{1}{2}, 1\right)$, there exists $\lambda > 0$ such that $M_{1,\frac{\nu}{2}}\left(\frac{\lambda}{4}\right) = 0$. Straightforward computations lead to

$$w_\lambda''(x) = \frac{3}{4x^{5/2}} M_{1,\frac{\nu}{2}}\left(\frac{\lambda}{4}x^2\right) + \frac{\lambda^2 x^{3/2}}{4} M''_{1,\frac{\nu}{2}}\left(\frac{\lambda}{4}x^2\right).$$

Thus, using (7.34), it comes that

$$-w_\lambda''(x) + \frac{c_\nu}{x^2} w_\lambda(x) + m^2 x^2 w_\lambda(x) = \lambda x^{-1/2} M_{1,\frac{\nu}{2}}\left(\frac{\lambda}{4}x^2\right).$$

Recall that

$$w_\lambda(x) = x^{\nu+\frac{1}{2}} \left(\frac{\lambda}{4}\right)^{\frac{\nu+1}{2}} e^{-\frac{\lambda}{2}x^2} \frac{\Gamma(\nu+1)}{\Gamma((\nu-1)/2)} \sum_{n\in\mathbb{N}} \frac{\Gamma(n+(\nu-1)/2)}{n!\,\Gamma(n+\nu+1)} \left(\frac{\lambda}{4}\right)^n x^{2n}.$$

Thus, as $\nu > \frac{1}{2}$, it comes that $w_\lambda(0) = w_\lambda'(0) = 0$. The choice of λ implies that $w_\lambda(1) = 0$. This ends the proof of Lemma 7.2.

□

Remark 7.8. Notice that for $\nu \in \left(0, \frac{1}{2}\right]$, the explicit solutions constructed in the previous proposition does not satisfy $w_\lambda'(0) = 0$.

As our strategy relies on explicit counterexamples, the restriction $\gamma = 1$ and particular values of ℓ seems only technical and we conjecture that for system (7.5), unique continuation does not hold for any $\gamma > 0$ and any value of $\ell > 0$.

7.4 Conclusion, open problems and perspectives

In this paper we have investigated the approximate controllability properties for a 2D Grushin-like equation which presents both a degeneracy and an inverse square singularity on the internal set $\{x = 0\}$. As the associated operator possesses several self-adjoint extensions, the functional setting in which we study the well posedness and unique continuation for the adjoint system is crucial. This functional setting relies on a precise study of the 1D associated operators.

We prove a necessary and sufficient condition on the coefficient c_ν of the potential $\frac{c_\nu}{x^2}$ for unique continuation to hold. The positive result is proved using classical unique continuation results for uniformly parabolic operators and a 1D Carleman type estimate that holds due to the construction of the functional setting. The negative result is proved by designing an explicit counterexample based on Bessel functions. These results have been extended to homogeneous Dirichlet boundary conditions in the y direction. The negative result in this setting for $\nu \in \left(\frac{1}{2}, 1\right)$, for any $\gamma > 0$ and any $\ell > 0$ remains an open problem.

An interesting open problem coming from this work is the question of null controllability in the case $\nu \in \left(0, \frac{1}{2}\right]$. The classical strategy would be to prove uniform observability for the 1D adjoint systems. This has been done in the case where there is no singular potential in [14] and with a singular potential for the one-side problem in [37]. The Carleman type estimate we proved in this paper might not be directly used as it holds true only for the regular part of the coefficient g_n. Dealing with the singular part in Carleman type estimates is quite tricky as we cannot perform integrations by part on the singular part. The other difficulty relies on the fact that we want these estimates to be uniform with respect to n.

7.A One dimensional operator

This appendix is dedicated to the proof of Proposition 7.1 where we investigate the self-adjointness and positivity properties of the operator associated to the one dimensional problem (7.8). The proof uses the following two lemmas.

Lemma 7.3. *For $f, g \in \tilde{H}_0^2(-1,1) \oplus \mathcal{F}_s$, if we define*

$$[f,g](x) := (fg' - f'g)(x), \quad \forall x \neq 0,$$

then

$$\int_{-1}^{1} \left(-\partial_{xx}^2 f + \frac{c_\nu}{x^2} f\right)(x) g(x) \mathrm{d}x = \int_{-1}^{1} f(x) \left(-\partial_{xx}^2 g + \frac{c_\nu}{x^2} g\right)(x) \mathrm{d}x + [f,g](1) - [f,g](0^+) + [f,g](0^-) - [f,g](-1).$$

Proof of Lemma 7.3. See [142, Lemma 9.2.3]. □

The following lemma characterizes the behaviour of the regular part at the singularity.

Lemma 7.4. *For $f \in \tilde{H}_0^2(-1,1)$,*

$$\lim_{x \to 0} \frac{f(x)}{|x|^{3/2}} = 0 \quad \textit{and} \quad \lim_{x \to 0} \frac{f'(x)}{|x|^{1/2}} = 0.$$

Proof of Lemma 7.4. As $f(0) = f'(0) = 0$, it comes that

$$f(x) = \int_0^x \int_0^t f''(s) \mathrm{d}s \mathrm{d}t.$$

Then, Cauchy-Schwarz inequality implies,

$$|f(x)| \leq \left| \int_0^x \sqrt{t} \left| \int_0^t |f''(s)|^2 \mathrm{d}s \right|^{1/2} \mathrm{d}t \right| \leq \frac{2}{3} \left| \int_0^x |f''(s)|^2 \mathrm{d}s \right|^{1/2} |x|^{3/2}.$$

The proof of the second limit is similar.

□

We now turn to the proof of Proposition 7.1.

Proof of Proposition 7.1. We start by proving that $(A_n, D(A))$ is a symmetric operator. Thus, A_n^* is an extension of A_n and self-adjointness will follow from the equality $D(A_n^*) = D(A_n)$.

First step : we prove that $(A_n, D(A))$ is a symmetric operator.
Let $f, g \in D(A)$. As $f(1) = g(1) = f(-1) = g(-1) = 0$, it comes that

$$[f,g](1) = [f,g](-1) = 0.$$

Lemma 7.4 imply that

$$[f,g](0^+) = [f_s, g_s](0^+) = \big(c_1^+(f)c_2^+(g) - c_2^+(f)c_1^+(g)\big)[|x|^{\nu+1/2}, |x|^{-\nu+1/2}](0^+),$$

and

$$\begin{aligned}[f,g](0^-) &= [f_s, g_s](0^-)\\ &= \big(c_1^-(f)c_2^-(g) - c_2^-(f)c_1^-(g)\big)[|x|^{\nu+1/2}, |x|^{-\nu+1/2}](0^-)\\ &= -\big(c_1^-(f)c_2^-(g) - c_2^-(f)c_1^-(g)\big)[|x|^{\nu+1/2}, |x|^{-\nu+1/2}](0^+).\end{aligned}$$

The transmission conditions on the coefficients of the singular part given in (7.9) can be rewritten as

$$\begin{pmatrix} c_1^+(f) \\ c_2^+(f) \end{pmatrix} = \frac{-1}{2\nu}\begin{pmatrix} -1 & 2\nu-1 \\ 2\nu+1 & 1 \end{pmatrix}\begin{pmatrix} c_1^-(f) \\ c_2^-(f) \end{pmatrix}, \quad \forall f \in D(A). \tag{7.35}$$

Thus, for any $f, g \in D(A)$

$$c_1^+(f)c_2^+(g) - c_2^+(f)c_1^+(g) = -\big(c_1^-(f)c_2^-(g) - c_2^-(f)c_1^-(g)\big).$$

This leads to

$$[f,g](0^+) = [f,g](0^-).$$

Finally, Lemma 7.3 imply that for any $f, g \in D(A)$, $\langle A_n f, g\rangle = \langle f, A_n g\rangle$.

Thus, to prove self-adjointness it remains to prove that $D(A_n^*) = D(A)$. As $D(A)$ is independent of n and $x \mapsto (2n\pi)^2|x|^{2\gamma} \in L^\infty(-1,1)$ it comes that $D(A_n^*) = D(A_0^*)$.

Second step : minimal and maximal domains. First, we explicit the minimal and maximal domains in the case of a boundary singularity. Without loss of generality, we study the operator in $(0,1)$.
Using [2, Proposition 3.1], the minimal and maximal domains associated to the differential expression A_0 in $L^2(0,1)$ are respectively equal to

$$H_0^2([0,1]) := \big\{y \in H^2([0,1])\,;\, y(0) = y(1) = y'(0) = y'(1) = 0\big\}$$

and

$$\big\{y \in H^2([0,1])\,;\, y(0) = y'(0) = 0\big\} \oplus \operatorname{Span}\left\{x^{\nu+1/2},\, x^{-\nu+1/2}\right\}.$$

Then, [142, Lemma 13.3.1] imply that the minimal and maximal domains associated to A_0 on the interval $(-1, 1)$ are given by

$$D_{min} := \left\{ f \in \tilde{H}_0^2(-1,1)\,;\, f(-1) = f(1) = f'(-1) = f'(1) = 0 \right\}, \tag{7.36}$$

and

$$D_{max} := \tilde{H}_0^2(-1,1) \oplus \mathcal{F}_s. \tag{7.37}$$

Besides, the minimal and maximal operators form an adjoint pair

Third step : self-adjointness. The operator A_0 being a symmetric extension of the minimal operator it comes that $D(A_0) \subset D(A_0^*) \subset D_{max}$. Let $g \in D(A_0^*)$ be decomposed as $g = g_r + g_s$ with $g_r \in \tilde{H}_0^2(-1,1)$ and $g_s \in \mathcal{F}_s$. We prove that g satisfy the boundary and transmission conditions. By the definition of $D(A_0^*)$, there exists $c > 0$ such that for any $f \in D(A)$,

$$|\langle A_0 f, g\rangle| \le c||f||_{L^2}.$$

Let $f \in D(A) \cap \tilde{H}_0^2(-1,1)$ be such that $f \equiv 0$ in $(-1, 0)$. Then, Lemma 7.3 implies that

$$\langle A_0 f, g\rangle = \langle f, A_0 g\rangle + [f, g](1) = \langle f, A_0 g\rangle + f'(1)g(1).$$

Thus, $g(1) = 0$. Symmetric arguments imply that $g(-1) = 0$.
We now turn to the transmission conditions. Let $f \in D(A)$ be such that its singular part is given by

$$c_1^+(f) := \frac{1}{2\nu}, \quad c_2^+(f) := -\frac{1}{2\nu}.$$

Then, the transmission conditions imply

$$c_1^-(f) = \frac{1}{2\nu}, \quad c_2^-(f) = -\frac{1}{2\nu}.$$

By Lemma 7.3

$$\langle A_0 f, g\rangle = \langle f, A_0 g\rangle + [f, g](0^-) - [f, g](0^+).$$

Using Lemma 7.4 it comes that $[f, g](0^-) = [f_s, g_s](0^-)$ and $[f, g](0^+) = [f_s, g_s](0^+)$. Straightforward computations lead to

$$[f, g](0^+) = -c_1^+(g) - c_2^+(g), \quad [f, g](0^-) = c_1^-(g) + c_2^-(g).$$

We thus recover the first transmission condition. The second transmission condition follow from the same computations with the choice of a particular $f \in D(A)$ satisfying

$$c_1^+(f) := -\frac{\nu - 1/2}{2\nu}, \quad c_2^+(f) := -\frac{\nu + 1/2}{2\nu}.$$

Finally, this proves that $(A_n, D(A))$ is a self-adjoint operator.

Fourth step : positivity. We end the proof of Proposition 7.1 by proving that for any $f \in D(A)$, $\langle A_n f, f\rangle \ge 0$. Let $f \in D(A)$.

Using Lemma 7.3 and integration by parts it comes that

$$\begin{aligned}\langle A_n f, f\rangle &= \int_{-1}^{1}\Big(-\partial_{xx}^2 f_r + \frac{c_\nu}{x^2} f_r\Big)(x) f(x)\mathrm{d}x + \int_{-1}^{1}(2n\pi)^2|x|^{2\gamma} f^2(x)\mathrm{d}x,\\ &= \int_{-1}^{1}(\partial_x f_r)^2(x) + \frac{c_\nu}{x^2} f_r^2(x)\mathrm{d}x + \int_{-1}^{1}(2n\pi)^2|x|^{2\gamma} f^2(x)\mathrm{d}x + (-\partial_x f_r)(1) f_r(1)\\ &+ \partial_x f_r(-1) f_r(-1) + [f_r, f_s](1) - [f_r, f_s](0^+) + [f_r, f_s](0^-) - [f_r, f_s](-1).\end{aligned}$$

Using Lemma 7.4, it comes that $[f_r, f_s](0^+) = [f_r, f_s](0^-) = 0$. Gathering the boundary terms and using $f(1) = f(-1) = 0$ it comes that

$$\begin{aligned}\langle A_n f, f\rangle = &\int_{-1}^{1}(\partial_x f_r)^2(x) + \frac{c_\nu}{x^2} f_r^2(x)\mathrm{d}x + \int_{-1}^{1}(2n\pi)^2|x|^{2\gamma} f^2(x)\mathrm{d}x\\ &+ f_r(1)\partial_x f_s(1) - f_r(-1)\partial_x f_s(-1).\end{aligned}$$

The transmission conditions on $c_1^+, c_2^+, c_1^-, c_2^-$ in $D(A)$ impose that

$$\begin{aligned}f_r(1)\partial_x f_s(1) &= -\big(c_1^+(f) + c_2^+(f)\big)\left(\left(\nu + \frac{1}{2}\right) c_1^+(f) + \left(-\nu + \frac{1}{2}\right) c_2^+(f)\right)\\ &= f_r(-1)\partial_x f_s(-1).\end{aligned}$$

Thus, using Hardy inequality (7.7)

$$\begin{aligned}\langle A f, f\rangle &\geq \int_{-1}^{1}(\partial_x f_r)^2(x) + \frac{c_\nu}{x^2} f_r^2(x)\mathrm{d}x + \int_{-1}^{1}(2n\pi)^2|x|^{2\gamma} f^2(x)\mathrm{d}x,\\ &\geq m_\nu \int_{-1}^{1}(\partial_x f_r)^2(x)\mathrm{d}x + \int_{-1}^{1}(2n\pi)^2|x|^{2\gamma} f^2(x)\mathrm{d}x, \qquad (7.38)\end{aligned}$$

where $m_\nu := \min\{1, 4\nu^2\}$. This ends the proof of Proposition 7.1.

□

7.B Abstract self adjoint extensions

This appendix is dedicated to enlighten the choices made in the construction of the functional setting leading to the definition (7.9) of $D(A)$.

The question of finding the self-adjoint extensions of a given closed symmetric operator is classical. In [120, Theorem X.2] such extensions are characterized by means of isometries between the deficiency subspaces. The particular case of Sturm-Liouville operators has been widely studied : most of these result are contained in [142]. The self-adjoint extensions are characterized by means of boundary conditions. In our case, we are concerned with the Sturm-Liouville operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{c_\nu}{x^2}$ on the interval $(-1, 1)$. This fits in the setting of [142, Chapter 13]. The number of boundary conditions to impose is given by the deficiency index. Following [2, Proposition 3.1], it comes that our operator on the interval $(0, 1)$ has deficiency index 2. This is closely related to the fact that $\nu \in (0, 1)$. Then, [142, Lemma 13.3.1] implies that the deficiency index for the interval $(-1, 1)$ is 4. We thus get the following proposition which is simply a rewriting of [142, Theorem 13.3.1 Case 5].

Proposition 7.10. *Let u and v in D_{max} be such that their restriction on $(0,1)$ (resp. $(-1,0)$) are linearly independent modulo $H_0^2(0,1)$ (resp. $H_0^2(-1,0)$) and*

$$[u,v](-1) = [u,v](0^-) = [u,v](0^+) = [u,v](1) = 1.$$

Let $M_1, \dots, M_4$ be 4×2 complex matrices. Then every self-adjoint extension of the minimal operator is given by the restriction of D_{max} to the functions f satisfying the boundary conditions

$$M_1 \begin{pmatrix} [f,u](-1) \\ [f,v](-1) \end{pmatrix} + M_2 \begin{pmatrix} [f,u](0^-) \\ [f,v](0^-) \end{pmatrix} + M_3 \begin{pmatrix} [f,u](0^+) \\ [f,v](0^+) \end{pmatrix} + M_4 \begin{pmatrix} [f,u](1) \\ [f,v](1) \end{pmatrix} = 0,$$

where the matrices satisfy $(M_1\, M_2\, M_3\, M_4)$ has full rank and

$$M_1 E M_1^* - M_2 E M_2^* + M_3 E M_3^* - M_4 E M_4^* = 0, \text{ with } E := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Conversely, every choice of such matrices defines a self-adjoint extension.

We end this appendix by giving the choices of such matrices that we made and give another functional setting that would lead to well posedness but that is not adapted to controllability issues. We define on $(0,1)$ u and v to be solutions of

$$-f''(x) + \frac{c_\nu}{x^2} f(x) = 0$$

with $(u(1) = 0, u'(1) = 1)$ and $(v(1) = -1, v'(1) = 0)$ i.e.

$$u(x) = \frac{1}{2\nu} x^{\nu+1/2} - \frac{1}{2\nu} x^{-\nu+1/2},$$
$$v(x) = -\frac{\nu - 1/2}{2\nu} x^{\nu+1/2} - \frac{\nu + 1/2}{2\nu} x^{-\nu+1/2}.$$

Thus for any $f \in D_{max}$, $[f,u](1) = f(1)$ and $[f,v](1) = f'(1)$, and for any $x \in [0,1]$, $[u,v](x) \equiv 1$. We design u and v similarly on $(-1,0)$ i.e.

$$u(x) = -\frac{1}{2\nu} |x|^{\nu+1/2} + \frac{1}{2\nu} |x|^{-\nu+1/2},$$
$$v(x) = -\frac{\nu - 1/2}{2\nu} |x|^{\nu+1/2} - \frac{\nu + 1/2}{2\nu} |x|^{-\nu+1/2}.$$

Due to the choice of functions u and v, the homogeneous Dirichlet conditions at ± 1 are implied by the choice

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \ M_2 = \begin{pmatrix} 0 \quad 0 \\ \tilde{M}_2 \\ 0 \quad 0 \end{pmatrix}, \ M_3 = \begin{pmatrix} 0 \quad 0 \\ \tilde{M}_3 \\ 0 \quad 0 \end{pmatrix}, \ M_4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then, the conditions of Proposition 7.10 are satisfied if and only if the matrix $(\tilde{M}_2\, \tilde{M}_3)$ has rank 2 and $\det(\tilde{M}_2) = \det(\tilde{M}_3)$. Straightforward computations lead to, for any $f \in D_{max}$

$$[f,u](0^+) = c_1^+ + c_2^+, \qquad [f,v](0^+) = \left(\nu + \frac{1}{2}\right) c_1^+ + \left(-\nu + \frac{1}{2}\right) c_2^+,$$
$$[f,u](0^-) = c_1^- + c_2^-, \qquad [f,v](0^-) = -\left(\nu + \frac{1}{2}\right) c_1^- - \left(-\nu + \frac{1}{2}\right) c_2^-.$$

Thus, the choice $\tilde{M}_2 = \tilde{M}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ lead to the definition of $D(A)$ in (7.9). The computations done in the fourth step of the proof of Proposition 7.1 (see (7.38)) prove the positivity and thus, Proposition 7.1 could also be seen as an application of Proposition 7.10.
At this stage, there is another choice that would lead to a self-adjoint positive extension. If, we set $\tilde{M}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\tilde{M}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then the domain with conditions

$$c_1^+ = -c_2^+, \quad c_1^- = -\frac{-\nu + 1/2}{\nu + 1/2} c_2^-,$$

give rise to a self-adjoint positive operator. However, from a point of view of controllability, this domain does not seem interesting as this conditions couple the coefficients on each side on the singularity and there is no transmission of information through the singular set. In particular, we cannot apply the results developed in this article to this functional setting.