MATHEMATICAL AND PHYSICAL BACKGROUND

A.1 Introduction

A short survey of the mathematical and physical background of the thesis is presented in this appendix. The most important aspects are discussed and several references are given for each topic. It is thus intended as a quick reference guide to understand or refresh some deeper technical (and sometimes more obscure) aspects mentioned throughout the thesis.

This appendix is structured in 11 sections, including this introduction. In Section A.2 we present several special functions that appear in mathematical physics and which are closely related to our work. Some notions of functional analysis are introduced in Section A.3, in particular Lax-Milgram's theorem and Fredholm's alternative. The Sobolev spaces are introduced in Section A.4, which constitute the natural function spaces in which the solutions of boundary-value problems are searched. In Section A.5 we present some operators and integral theorems that appear in vector calculus and in elementary differential geometry. The powerful mathematical tool of the theory of distributions is described in Section A.6. In Section A.7 we describe multi-dimensional Fourier transforms and their properties in the framework of the theory of distributions. In Section A.8 a general outline of Green's functions and fundamental solutions is found. In section A.9 we present a brief survey of wave propagation and some related topics. Linear water-wave theory, which is one of the main applications for the Laplace equation, is shown in Section A.10. Finally, in Section A.11 we study some aspects of the linear acoustic theory, which is one of the main applications for the Helmholtz equation.

A.2 Special functions

The special functions of mathematical physics, also known as higher transcendental functions, are functions that play a fundamental role in a great variety of physical and mathematical applications. They can not be described as a composition of a finite number of elementary functions. Elementary functions are functions which are built upon a finite combination of constant functions, elementary field operations (addition, subtraction, multiplication, division, and root extraction), and algebraic, exponential, trigonometric, and logarithmic functions and their inverses under repeated compositions. Elementary functions are divided into algebraic and transcendental functions. An algebraic function is a function which can be constructed using only a finite number of the elementary field operations together with the inverses of functions capable of being so constructed. A transcendental function is a function that is not algebraic, e.g., the exponential and trigonometric functions and their inverses are transcendental. The higher transcendental functions are functions which go even beyond the transcendental functions, and can only be described by means of integral representations and infinite series expansions. Some of them, though, are widely studied due their multiple applications, and are therefore called special functions.

Definitions and some properties of several special functions, which are used throughout this thesis, are presented in this section. We begin with the complex exponential and logarithm. They are only transcendental functions, but they allow to comprehend better the other special functions, particularly their properties in the complex plane. The singularities of the Green's functions studied herein for two-dimensional problems are always of logarithmic type. Afterwards we present the gamma or generalized factorial function. The exponential integral and its related functions appear in the computation of the half-plane Green's function for the Laplace equation. Bessel and Hankel functions play an important role in problems with circular or cylindrical symmetry. They are also known as cylindrical harmonics and appear in the computation of the Green's function for the Helmholtz equation in two dimensions. Closely related to them are the modified Bessel functions. Spherical Bessel and Hankel functions appear in problems with spherical symmetry and, in particular, in the computation of the Green's function for the Helmholtz equation in three dimensions. Struve functions can be seen as some sort of perturbed Bessel and Hankel functions, and appear when taking primitives of them. They also appear in some impedance calculations. Finally we present the Legendre functions, the associated Legendre functions, and the spherical harmonics, which are all closely related, and which appear in problems with spherical symmetry.

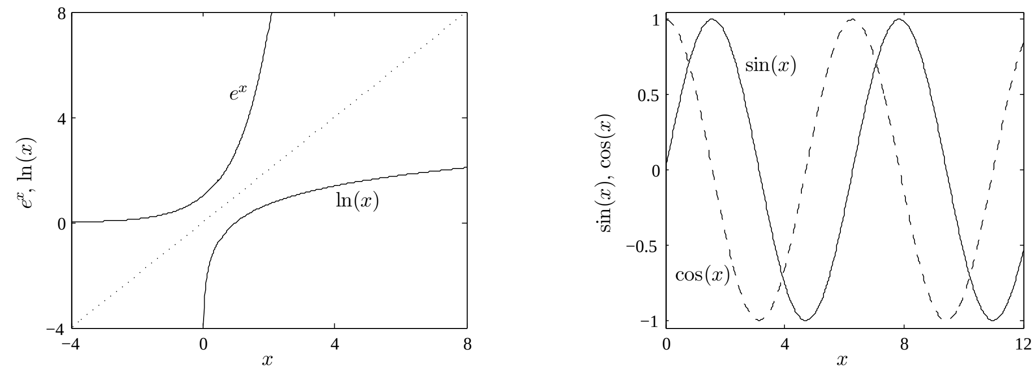
The special functions and their properties are deeply linked with the theory of complex variables. To understand the former, some knowledge is required of the latter, which deals with the complex imaginary unit, $i = \sqrt{-1}$, and with related topics, such as complex integration contours, residue calculus, analytic continuation, etc. Some references for the complex variable theory are Arfken & Weber (2005), Bak & Newman (1997), Dettman (1984), and Morse & Feshbach (1953). Further interesting topics are the theory of asymptotic expansions (Courant & Hilbert 1966, Dettman 1984, Estrada & Kanwal 2002), and the methods of stationary phase and steepest descent (Bender & Orszag 1978, Dettman 1984, Watson 1944). Specific references for special functions are given in each subsection. In particular, some references which are useful for almost all of these special functions are Abramowitz & Stegun (1972), Erdélyi (1953), and Magnus & Oberhettinger (1954). Another somewhat older but still quite interesting reference is Jahnke & Emde (1945).

A.2.1 Complex exponential and logarithm

a) Complex exponential

The complex exponential and logarithm are trascendental functions that play a central role in the theory of complex functions. Even though they are not considered to be special functions, their intrinsic properties allow a far better comprehension of the latter, and are therefore listed herein. Some references are Abramowitz & Stegun (1972), Bak & Newman (1997), Dettman (1984), Jahnke & Emde (1945), and Weisstein (2002). The complex exponential is an analytic function in the entire complex z-plane, being thus an entire function, and it coincides with the usual exponential function for real arguments, which is shown in Figure A.1. It is defined by

$$\exp z = e^z = e^x e^{iy} = e^x \cos y + i\, e^x \sin y, \qquad z = x + iy, \tag{A.1}$$



(a) Exponential and logarithm

(b) Sine and cosine

FIGURE A.1. Exponential, logarithm, and trigonometric functions for real arguments.

where e denotes Euler's number

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \sum_{n=0}^{\infty}\frac{1}{n!} = 2.718281828\ldots, \tag{A.2}$$

which receives its name from the Swissborn Russian mathematician and physicist Leonhard Euler (1707–1783), who is considered one of the greatest mathematicians of all time. Some properties of the complex exponential are

$$e^{z_1}e^{z_2} = e^{z_1+z_2}, \tag{A.3}$$

$$e^{z_1}/e^{z_2} = e^{z_1-z_2}, \tag{A.4}$$

$$|e^z| = e^x, \tag{A.5}$$

$$e^{z+2\pi i} = e^z. \tag{A.6}$$

Property (A.5) implies that $\exp z$ has no zeros, and property (A.6) means that $\exp z$ is periodic with period $2\pi i$. The derivative and the primitive of the complex exponential, omitting the integration constant, is the function itself:

$$\frac{\mathrm{d}}{\mathrm{d}z}e^z = e^z, \qquad \int e^z\,\mathrm{d}z = e^z. \tag{A.7}$$

It has the power series expansion

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}. \tag{A.8}$$

The complex exponential allows us also to define the complex trigonometric functions

$$\sin z = \frac{e^{iz}-e^{-iz}}{2i}, \tag{A.9}$$

$$\cos z = \frac{e^{iz}+e^{-iz}}{2}, \tag{A.10}$$

$$\tan z = \frac{\sin z}{\cos z} = -i\,\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}}, \tag{A.11}$$

and likewise the complex hyperbolic functions

$$\sinh z = \frac{e^z - e^{-z}}{2} = -i\sin(iz), \tag{A.12}$$

$$\cosh z = \frac{e^z + e^{-z}}{2} = \cos(iz), \tag{A.13}$$

$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{e^z - e^{-z}}{e^z + e^{-z}} = -i\tan(iz). \tag{A.14}$$

The sine and cosine trigonometric functions for real arguments are illustrated in Figure A.1.

b) Complex logarithm

The complex logarithm $\ln z$ is an extension of the natural logarithm function for real arguments (vid. Figure A.1) into the whole complex z-plane, and is thus the inverse function of the complex exponential $\exp z$. There is, however, a difficulty in trying to define this inverse function due the periodicity of the exponential, i.e., due the fact that

$$e^{z+i2\pi n} = e^z, \qquad n \in \mathbb{Z}. \tag{A.15}$$

The complex logarithm has to be understood thus as a multi-valued function, which can become properly single-valued when the domain of the exponential is restricted, e.g., to the strip $-\pi < \mathfrak{Im}\, z \leq \pi$. In this specific case, the function is one-to-one and an inverse does exist, called the principal value of the logarithm, which is given by

$$\ln z = \ln|z| + i\arg z, \qquad -\pi < \arg z \leq \pi, \tag{A.16}$$

or, equivalently in polar and cartesian coordinates, by

$$\ln z = \ln r + i\theta, \qquad -\pi < \theta \leq \pi, \tag{A.17}$$

$$\ln z = \ln\sqrt{x^2+y^2} + i\arctan\frac{y}{x}, \qquad -\pi < \arctan\frac{y}{x} \leq \pi, \tag{A.18}$$

where

$$z = re^{i\theta} = x + iy. \tag{A.19}$$

So defined, the logarithm $\ln z$ is holomorphic for all complex numbers which do not lie on the negative real axis including the origin, and has the property

$$e^{\ln z} = z, \qquad z \neq 0. \tag{A.20}$$

We see that it is not defined at $z = 0$ and is discontinuous on the negative real axis, which means that the function cannot be analytic at these points. In fact, the jump across the negative real axis is given by

$$\ln(x + i0) - \ln(x - i0) = i2\pi \qquad \forall x < 0. \tag{A.21}$$

Elsewhere the function is differentiable, and its derivative and primitive, omitting the integration constant, are given by

$$\frac{\mathrm{d}}{\mathrm{d}z}\ln z = \frac{1}{z}, \qquad \int \ln z\,\mathrm{d}z = z\ln z - z, \qquad z \neq 0. \tag{A.22}$$

Particularly, it holds that

$$\ln(i) = \frac{i\pi}{2}. \tag{A.23}$$

It admits also the power series expansions

$$\ln z = \sum_{n=0}^{\infty} \frac{2}{2n+1}\left(\frac{z-1}{z+1}\right)^{2n+1}, \qquad \mathfrak{Re}\, z > 0, \tag{A.24}$$

$$\ln(z+1) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{z^n}{n}, \qquad |z| < 1. \tag{A.25}$$

There exist consequently many logarithm functions depending on the restriction that is placed on the argument $\arg z$ to make the function single-valued. The complex logarithm can be conceived as having many branches, each of which is single-valued and fits the definition of a proper function. If we take the argument $\arg z$ satisfying the above restriction for the principal value, then

$$\mathrm{Ln}\, z = \ln|z| + i(\arg z \pm 2\pi n), \qquad -\pi < \arg z \leq \pi, \qquad n = 0, 1, 2, \ldots, \tag{A.26}$$

is a multi-valued function with infinitely many branches, each for a different integer n, and each single-valued. This general logarithmic function can be defined by

$$\mathrm{Ln}\, z = \int_1^z \frac{\mathrm{d}t}{t}, \tag{A.27}$$

where the integration path does not pass through the origin. Another way to work with the complex logarithm function is using a more complicated surface consisting of infinitely many planes joined together so that the function varies continuously when passing from one plane to the next. Such a surface is called Riemann surface in honor of the German mathematician Georg Friedrich Bernhard Riemann (1826–1866), who made important contributions to analysis and differential geometry. The discontinuity of the complex logarithm at the negative real axis was introduced in a rather arbitrary way as a restriction on the $\arg z$ to make the function single-valued. This line of discontinuity is called a branch cut and can be moved at will by defining different branches of the function. It does not even need to be a straight line, but it must start at $z = 0$, where the logarithm fails to be analytic. This point is called a branch point and is a more basic type of singularity than the points on a particular branch cut. The branch cut connects thus the branch point $z = 0$ with infinity, which is the other branch point. Working with Riemann surfaces avoids the use of branch cuts, but gives up the simplicity of defining a function on a set of points in a single complex plane, which is the reason why we will not use them, and deal with branch cuts instead throughout this work. For the multi-valued complex logarithm $\mathrm{Ln}\, z$ the usual properties of the real logarithm hold, e.g.,

$$\mathrm{Ln}(z_1 z_2) = \mathrm{Ln}\, z_1 + \mathrm{Ln}\, z_2, \tag{A.28}$$

$$\mathrm{Ln}(z_1/z_2) = \mathrm{Ln}\, z_1 - \mathrm{Ln}\, z_2, \tag{A.29}$$

which also holds for the single-valued complex logarithm $\ln z$, provided that care is exercised in selecting the branches. The complex logarithm allows also to define the function z^a, where a is any complex constant, due

$$z^a = e^{a \operatorname{Ln} z}. \tag{A.30}$$

If $a = m$, an integer, then (A.30) is single-valued due the periodicity of the complex exponential. If $a = p/q$, where p and q are integers, then (A.30) has q distinct values. And finally, if a is irrational or complex, then there are infinitely many values of z^a. We have also that, except at the branch point $z = 0$ and on a branch cut, z^a is analytic and, omitting the integration constant,

$$\frac{\mathrm{d}}{\mathrm{d}z} z^a = a z^{a-1}, \qquad \int z^a \,\mathrm{d}z = \frac{z^{a+1}}{a+1}. \tag{A.31}$$

In particular, the complex square root is defined by

$$\sqrt{z} = z^{1/2} = e^{\frac{1}{2} \operatorname{Ln} z}, \tag{A.32}$$

and we characterize its principal value as

$$\sqrt{z} = \sqrt{x+iy} = \sqrt{r}\, e^{i\theta/2} = \sqrt{\frac{r+x}{2}} + \frac{iy}{\sqrt{2(r+x)}} \qquad (-\pi < \theta \leq \pi). \tag{A.33}$$

The complex logarithm allows in the same way to define several other functions, which have branch cuts or have to be considered as multi-valued. Among these are, e.g., the inverse trigonometric functions

$$\arcsin z = -i \operatorname{Ln}\Big(iz + \sqrt{1-z^2}\Big), \tag{A.34}$$

$$\arccos z = -i \operatorname{Ln}\Big(z + \sqrt{z^2-1}\Big) = \frac{\pi}{2} - \arcsin z, \tag{A.35}$$

$$\arctan z = \frac{i}{2}\big(\operatorname{Ln}(1-iz) - \operatorname{Ln}(1+iz)\big), \tag{A.36}$$

and the inverse hyperbolic functions

$$\operatorname{arcsinh} z = \operatorname{Ln}\Big(z + \sqrt{1+z^2}\Big) = -i \arcsin(iz), \tag{A.37}$$

$$\operatorname{arccosh} z = \operatorname{Ln}\Big(z + \sqrt{z^2-1}\Big) = i \arccos z, \tag{A.38}$$

$$\operatorname{arctanh} z = \frac{1}{2}\big(\operatorname{Ln}(1+z) - \operatorname{Ln}(1-z)\big) = -i \arctan(iz). \tag{A.39}$$

Finally we remark that throughout this work, unless it is specifically stated otherwise, always the principal value for the complex logarithm is used, which has a branch cut along the negative real axis, and has the advantage of reducing itself to the usual natural logarithm when z is real and positive. This consideration is applied also to complex functions that are derived from the complex logarithm.

A.2.2 Gamma function

a) Definition

The gamma function is a special function that is defined to be an extension of the factorial function to complex and real number arguments. Some references on this function are the books of Abramowitz & Stegun (1972), Arfken & Weber (2005), Erdélyi (1953), Jahnke & Emde (1945), Magnus & Oberhettinger (1954), Spiegel & Liu (1999), and the one of Weisstein (2002). It is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, \mathrm{d}t \qquad (\mathfrak{Re}\, z > 0). \tag{A.40}$$

It can be also defined by Euler's formula

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)(z+2)\dots(z+n)} \qquad (z \neq 0, -1, -2, -3, \dots). \tag{A.41}$$

A third definition is given by Euler's infinite product formula

$$\frac{1}{\Gamma(z)} = z\, e^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right) e^{-z/n} \right], \tag{A.42}$$

where γ denotes Euler's constant (sometimes also called Euler-Mascheroni constant), which he discovered in 1735 and which is given by

$$\gamma = \lim_{n\to\infty} \left(\sum_{p=1}^{n} \frac{1}{p} - \ln(n) \right) = -\int_0^\infty e^{-t} \ln t\, \mathrm{d}t = 0.5772156649\dots. \tag{A.43}$$

Euler's constant can be also represented as

$$\gamma = \int_0^\infty \frac{1}{t} \left(\frac{1}{t+1} - e^{-t} \right) \mathrm{d}t = \int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t} \right) e^{-t}\, \mathrm{d}t. \tag{A.44}$$

The gamma function is graphically depicted in Figure A.2.

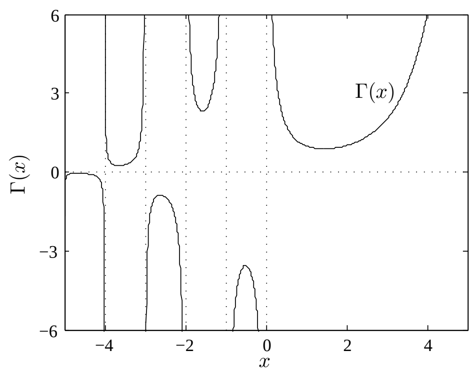


FIGURE A.2. Gamma function for real arguments.

b) Properties

The gamma function $\Gamma(z)$ is single-valued and analytic over the entire complex plane, save for the points $z = -n$ ($n = 0, 1, 2, 3, \ldots$), where it possesses simple poles with residues $(-1)^n/n!$. Its reciprocal $1/\Gamma(z)$ is an entire function possessing simple zeros at the points $z = -n$ ($n = 0, 1, 2, 3, \ldots$). There are no points z where $\Gamma(z) = 0$. The gamma function satisfies the recurrence relation

$$\Gamma(z+1) = z\Gamma(z), \tag{A.45}$$

and the reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \qquad (z \notin \mathbb{Z}). \tag{A.46}$$

The gamma function satisfies also the duplication formula

$$\Gamma(2z) = (2\pi)^{-\frac{1}{2}} 2^{2z-\frac{1}{2}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right), \tag{A.47}$$

and, in general, the Gauss' multiplication formula

$$\Gamma(nz) = (2\pi)^{\frac{1}{2}(1-n)} 2^{nz-\frac{1}{2}} \prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right), \tag{A.48}$$

which receives its name from the German mathematician and scientist of profound genius Carl Friedrich Gauss (1777–1855), who contributed significantly to many fields in mathematics and science. The gamma function is linked with the factorial function, for integer arguments, through

$$\Gamma(n+1) = n! \qquad (n = 0, 1, 2, 3, \ldots), \tag{A.49}$$

where, in particular,

$$\Gamma(1) = 0! = 1. \tag{A.50}$$

Special values for the gamma function are

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \tag{A.51}$$

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{n!\,2^{2n}} \qquad (n = 0, 1, 2, 3, \ldots), \tag{A.52}$$

$$\Gamma\left(-n + \frac{1}{2}\right) = \frac{(-1)^n n!\,2^{2n}\sqrt{\pi}}{(2n)!} \qquad (n = 0, 1, 2, 3, \ldots). \tag{A.53}$$

The derivative of the gamma function is given by

$$\frac{\mathrm{d}}{\mathrm{d}z}\Gamma(z) = -\Gamma(z)\left[\gamma + \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{n+z} - \frac{1}{n}\right)\right], \tag{A.54}$$

and a power series expansion for its logarithm is

$$\ln\Gamma(z) = -\ln(z) - \gamma z - \sum_{n=1}^{\infty}\left[\ln\left(1 + \frac{z}{n}\right) - \frac{z}{n}\right]. \tag{A.55}$$

The Γ function, for large arguments as $|z| \to \infty$, has the asymptotic expansion

$$\Gamma(z) \sim \sqrt{2\pi}\, e^{-z} z^{z-\frac{1}{2}} \left[1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} - \frac{571}{2488320z^4} + \cdots\right], \tag{A.56}$$

which is called Stirling's formula, named in honor of the Scottish mathematician James Stirling (1692–1770).

A.2.3 Exponential integral and related functions

a) Definition

The exponential integral, the cosine integral, and the sine integral functions are special functions that appear frequently in physical problems. Some references for them are Abramowitz & Stegun (1972), Arfken & Weber (2005), Chaudhry & Zubair (2002), Erdélyi (1953), Glaisher (1870), Jahnke & Emde (1945), and Weisstein (2002). The exponential integral is defined by

$$\mathrm{Ei}(z) = -\fint_{-z}^{\infty} \frac{e^{-t}}{t}\,\mathrm{d}t = \fint_{-\infty}^{z} \frac{e^{t}}{t}\,\mathrm{d}t \qquad \big(|\arg z| < \pi\big). \tag{A.57}$$

Analytic continuation of (A.57) yields a multi-valued function with branch points at $z = 0$ and $z = \infty$. It is a single-valued function in the complex z-plane cut along the negative real axis. Since $1/t$ diverges at $t = 0$, the integral has to be understood in terms of the Cauchy principal value (cf., e.g., Arfken & Weber 2005, or vid. Subsection A.6.5), named after the French mathematician and early pioneer of analysis Augustin Louis Cauchy (1789–1857). We introduce also the complementary exponential integral function

$$\mathrm{Ein}(z) = \int_0^z \frac{e^t - 1}{t}\,\mathrm{d}t, \tag{A.58}$$

which is an entire function and whose relation with (A.57) is given by

$$\mathrm{Ein}(z) = \mathrm{Ei}(z) - \gamma - \ln z, \tag{A.59}$$

where γ denotes Euler's constant (A.43). For the cosine integral function, there exist at least three definitions, which are

$$\mathrm{Ci}(z) = \gamma + \ln z + \int_0^z \frac{\cos t - 1}{t}\,\mathrm{d}t \qquad \big(|\arg z| < \pi\big), \tag{A.60}$$

$$\mathrm{ci}(z) = -\int_z^{\infty} \frac{\cos t}{t}\,\mathrm{d}t \qquad \big(|\arg z| < \pi\big), \tag{A.61}$$

$$\mathrm{Cin}(z) = \int_0^z \frac{\cos t - 1}{t}\,\mathrm{d}t. \tag{A.62}$$

The cosine integral $\mathrm{ci}(z)$ is the primitive of $\cos(z)/z$ which is zero for $z = \infty$. In the same manner as the exponential integral (A.57), the cosine integral functions (A.60) and (A.61) have also a branch cut along the negative real axis. They are related by

$$\mathrm{ci}(z) = \mathrm{Ci}(z) \qquad \big(|\arg z| < \pi\big), \tag{A.63}$$

$$\mathrm{Cin}(z) = \mathrm{Ci}(z) - \gamma - \ln z. \tag{A.64}$$

For the sine integral function, two different definitions exist, which are

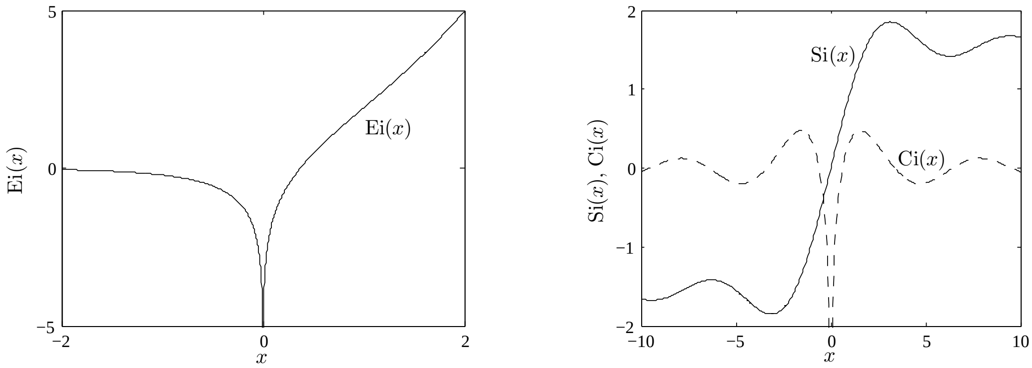
$$\mathrm{Si}(z) = \int_0^z \frac{\sin t}{t}\,\mathrm{d}t, \tag{A.65}$$

$$\mathrm{si}(z) = -\int_z^\infty \frac{\sin t}{t}\,\mathrm{d}t. \tag{A.66}$$

The sine integral $\mathrm{Si}(z)$ is the primitive of $\sin(z)/z$ which is zero for $z = 0$, while $\mathrm{si}(z)$ is the primitive of $\sin(z)/z$ which is zero for $z = \infty$. They are both analytic in the whole complex z-plane, and are related by

$$\mathrm{si}(z) = \mathrm{Si}(z) - \frac{\pi}{2}. \tag{A.67}$$

The exponential integral and its related trigonometric integrals are illustrated in Figure A.3.



(a) Exponential integral

(b) Sine integral and cosine integral

FIGURE A.3. Exponential integral and trigonometric integrals for real arguments.

b) Properties

The exponential integral, the cosine integral, and the sine integral functions satisfy the relations

$$\mathrm{Ei}(iz) = \mathrm{Ci}(z) + i\left(\mathrm{Si}(z) + \frac{\pi}{2}\right) \qquad (\mathfrak{Re}\, z > 0), \tag{A.68}$$

$$\mathrm{Ei}(-iz) = \mathrm{Ci}(z) - i\left(\mathrm{Si}(z) + \frac{\pi}{2}\right) \qquad (\mathfrak{Re}\, z > 0), \tag{A.69}$$

$$\mathrm{Ci}(z) = \frac{1}{2}\big[\,\mathrm{Ei}(iz) + \mathrm{Ei}(-iz)\,\big] \qquad (\mathfrak{Re}\, z > 0), \tag{A.70}$$

$$\mathrm{Si}(z) = \frac{1}{2i}\big[\,\mathrm{Ei}(iz) - \mathrm{Ei}(-iz)\,\big] - \frac{\pi}{2} \qquad (\mathfrak{Re}\, z > 0), \tag{A.71}$$

Their derivatives and primitives, omitting the integration constants, are given by

$$\frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{Ei}(z) = \frac{e^z}{z}, \qquad \int \mathrm{Ei}(z)\,\mathrm{d}z = z\,\mathrm{Ei}(z) - e^z, \tag{A.72}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{Ci}(z) = \frac{\cos z}{z}, \qquad \int \mathrm{Ci}(z)\,\mathrm{d}z = z\,\mathrm{Ci}(z) - \sin z, \tag{A.73}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{Si}(z) = \frac{\sin z}{z}, \qquad \int \mathrm{Si}(z)\,\mathrm{d}z = z\,\mathrm{Si}(z) + \cos z. \tag{A.74}$$

For small arguments z, the exponential, cosine, and sine integral functions have the convergent series expansions

$$\mathrm{Ei}(z) = \gamma + \ln z + \sum_{n=1}^{\infty} \frac{z^n}{n\,n!}, \tag{A.75}$$

$$\mathrm{Ci}(z) = \gamma + \ln z + \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n}}{2n(2n)!}, \tag{A.76}$$

$$\mathrm{Si}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)(2n+1)!}, \tag{A.77}$$

which can be alternatively used to define them. They can be derived from the integral representations. For instance, (A.75) results from considering the primitive of the first expression in (A.72), replacing the exponential function by its series expansion (A.8). Hence

$$\mathrm{Ei}(z) = C + \ln z + \sum_{n=1}^{\infty} \frac{z^n}{n\,n!}. \tag{A.78}$$

To find the remaining integration constant C we can take, in the sense of the principal value for the appearing integrals, the limit

$$\begin{aligned} C &= \lim_{\varepsilon\to 0^+} \left\{ \mathrm{Ei}(\varepsilon) - \ln(\varepsilon) \right\} = \lim_{\varepsilon\to 0^+} \left\{ -\int_{\varepsilon}^{\infty} \frac{e^{-t}}{t}\,\mathrm{d}t + \int_{\varepsilon}^{\infty} \frac{1}{t(t+1)}\,\mathrm{d}t - \ln(1+\varepsilon) \right\} \\ &= \int_0^{\infty} \frac{1}{t}\left(\frac{1}{t+1} - e^{-t}\right)\mathrm{d}t = \gamma, \end{aligned} \tag{A.79}$$

where we considered (A.44) and the fact that

$$\ln(z) = \ln(1+z) - \int_z^{\infty} \frac{1}{t(t+1)}\,\mathrm{d}t. \tag{A.80}$$

For large arguments, as $x \to \infty$ along the real line, these exponential and trigonometric integrals have the asymptotic divergent series expansions

$$\mathrm{Ei}(x) = \frac{e^x}{x} \sum_{n=0}^{\infty} \frac{n!}{x^n}, \tag{A.81}$$

$$\mathrm{Ci}(x) = \frac{\sin x}{x} \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{x^{2n}} - \frac{\cos x}{x} \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)!}{x^{2n+1}}, \tag{A.82}$$

$$\mathrm{Si}(x) = \frac{\pi}{2} - \frac{\cos x}{x} \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{x^{2n}} - \frac{\sin x}{x} \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)!}{x^{2n+1}}. \tag{A.83}$$

Therefore on the imaginary axis, as $|y| \to \infty$ for $y \in \mathbb{R}$, the exponential integral has the asymptotic divergent series expansion

$$\mathrm{Ei}(iy) = i\pi\,\mathrm{sign}(y) + \frac{e^{iy}}{iy}\sum_{n=0}^{\infty}\frac{n!}{(iy)^n}. \tag{A.84}$$

A.2.4 Bessel and Hankel functions

a) Differential equation and definition

Bessel functions, also called cylinder functions or cylindrical harmonics, are special functions that, together with the closely related Hankel functions, appear in a wide variety of physical problems. Some references on them are Abramowitz & Stegun (1972), Arfken & Weber (2005), Courant & Hilbert (1966), Erdélyi (1953), Jackson (1999), Jahnke & Emde (1945), Luke (1962), Magnus & Oberhettinger (1954), Morse & Feshbach (1953), Sommerfeld (1949), Spiegel & Liu (1999), Watson (1944), and Weisstein (2002). We consider the Bessel differential equation of order ν for a function $W : \mathbb{C} \to \mathbb{C}$, given by

$$z^2\frac{\mathrm{d}^2W}{\mathrm{d}z^2}(z) + z\frac{\mathrm{d}W}{\mathrm{d}z}(z) + (z^2 - \nu^2)W(z) = 0, \tag{A.85}$$

where, in general, $\nu \in \mathbb{C}$ is an unrestricted value. The Bessel differential equation is named after the German mathematician and astronomer Friedrich Wilhelm Bessel (1784–1846), who generalized and systemized thoroughly the Bessel functions, although it was the Dutch-born Swiss mathematician Daniel Bernoulli (1700–1782) who in fact first defined them. Independent solutions of this equation are the Bessel functions of the first kind $J_\nu(z)$ and of the second kind $Y_\nu(z)$, the latter also known as Neumann or Weber function, named respectively after the German mathematicians Franz Ernst Neumann (1798–1895) and Heinrich Martin Weber (1842–1913). They are depicted in Figure A.4 and related through

$$Y_\nu(z) = \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}, \qquad \nu \notin \mathbb{Z}, \tag{A.86}$$

$$Y_n(z) = \lim_{\nu\to n}\frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}, \qquad n \in \mathbb{Z}. \tag{A.87}$$

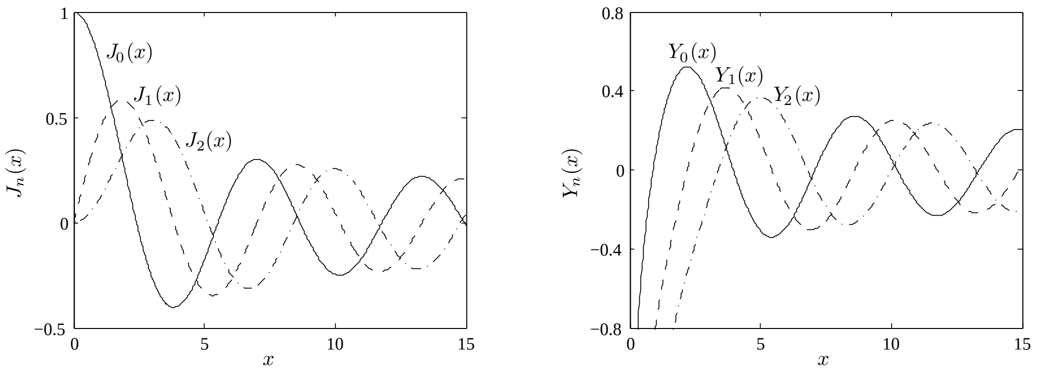
It holds in particular that

$$Y_{n+1/2}(z) = (-1)^{n+1}J_{-n-1/2}(z), \qquad n \in \mathbb{Z}. \tag{A.88}$$

The Hankel functions of the first kind $H_\nu^{(1)}(z)$ and of the second kind $H_\nu^{(2)}(z)$, also known as Bessel functions of the third kind, are also linearly independent solutions of the differential equation (A.85). They receive their name from the German mathematician Hermann Hankel (1839–1873), and are related to the Bessel functions of the first and second kinds through the complex linear combinations

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z), \tag{A.89}$$

$$H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z). \tag{A.90}$$



(a) Bessel function $J_n(x)$ for $n = 0, 1, 2$ (b) Neumann function $Y_n(x)$ for $n = 0, 1, 2$

FIGURE A.4. Bessel and Neumann functions for real arguments.

The three kinds of Bessel functions are holomorphic functions of z throughout the complex z-plane cut along the negative real axis, and for fixed z ($\neq 0$) each is an entire function of ν. When $\nu = n$, for $n \in \mathbb{Z}$, then $J_\nu(z)$ has no branch point and is an entire function of z. It holds that $J_\nu(z)$, for $\mathfrak{Re}\,\nu \geq 0$, is bounded as $z \to 0$ in any bounded range of $\arg z$. The functions $J_\nu(z)$ and $J_{-\nu}(z)$ are linearly independent except when ν is an integer. The functions $J_\nu(z)$ and $Y_\nu(z)$ are linearly independent for all values of ν. The function $H_\nu^{(1)}(z)$ tends to zero as $|z| \to \infty$ in the sector $0 < \arg z < \pi$ and the function $H_\nu^{(2)}(z)$ tends to zero as $|z| \to \infty$ in the sector $-\pi < \arg z < 0$. For all values of ν, $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ are linearly independent. The Bessel functions satisfy also the relations:

$$J_{-n}(z) = (-1)^n J_n(z), \qquad Y_{-n}(z) = (-1)^n Y_n(z), \tag{A.91}$$

$$H_{-\nu}^{(1)}(z) = e^{\nu\pi i} H_\nu^{(1)}(z), \qquad H_{-\nu}^{(2)}(z) = e^{-\nu\pi i} H_\nu^{(2)}(z). \tag{A.92}$$

When using complex conjugate arguments, then for $\nu \in \mathbb{R}$ follows

$$J_\nu(\bar{z}) = \overline{J_\nu(z)}, \qquad Y_\nu(\bar{z}) = \overline{Y_\nu(z)}, \tag{A.93}$$

$$H_\nu^{(1)}(\bar{z}) = \overline{H_\nu^{(2)}(z)}, \qquad H_\nu^{(2)}(\bar{z}) = \overline{H_\nu^{(1)}(z)}. \tag{A.94}$$

b) Ascending series

The Bessel function $J_\nu(z)$ has the power series expansion

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(\nu + m + 1)} \left(\frac{z}{2}\right)^{2m+\nu}, \tag{A.95}$$

where Γ stands for the gamma function (A.40). For an integer order $n \geq 0$, the Bessel function $J_n(z)$ has the power series expansion

$$J_n(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+n)!} \left(\frac{z}{2}\right)^{2m+n}, \tag{A.96}$$

and for the Neumann function $Y_n(z)$ it is given by

$$Y_n(z) = \frac{2}{\pi} J_n(z)\left(\ln\frac{z}{2}+\gamma\right) - \frac{1}{\pi}\sum_{m=0}^{n-1}\frac{(n-m-1)!}{m!}\left(\frac{z}{2}\right)^{2m-n}$$
$$- \frac{1}{\pi}\sum_{m=0}^{\infty}(-1)^m\frac{\psi(m+n)+\psi(m)}{m!\,(m+n)!}\left(\frac{z}{2}\right)^{2m+n}, \qquad \text{(A.97)}$$

where

$$\psi(0)=0, \qquad \psi(m)=\sum_{p=1}^{m}\frac{1}{p} \qquad (m=1,2,\ldots), \qquad \text{(A.98)}$$

and γ denotes Euler's constant (A.43). For $n=0$ the following expansions hold

$$J_0(z) = 1 - \frac{z^2/4}{(1!)^2} + \frac{(z^2/4)^2}{(2!)^2} - \frac{(z^2/4)^3}{(3!)^2} + \ldots, \qquad \text{(A.99)}$$

$$Y_0(z) = \frac{2}{\pi}J_0(z)\left(\ln\frac{z}{2}+\gamma\right)$$
$$+ \frac{2}{\pi}\left\{\frac{z^2/4}{(1!)^2} - \left(1+\frac{1}{2}\right)\frac{(z^2/4)^2}{(2!)^2} + \left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{(z^2/4)^3}{(3!)^2} - \ldots\right\}. \qquad \text{(A.100)}$$

Similarly, if $n=1$, then

$$J_1(z) = \frac{z}{2}\left\{1 - \frac{z^2/4}{2\,(1!)^2} + \frac{(z^2/4)^2}{3\,(2!)^2} - \frac{(z^2/4)^3}{4\,(3!)^2} + \ldots\right\}, \qquad \text{(A.101)}$$

$$Y_1(z) = \frac{2}{\pi}J_1(z)\left(\ln\frac{z}{2}+\gamma\right) - \frac{2}{\pi z}$$
$$+ \frac{1}{\pi}\left\{-\frac{z}{2} + \frac{2\left(1+\frac{1}{2}\right)-\frac{1}{2}}{2\,(1!)^2}\left(\frac{z}{2}\right)^3 - \frac{2\left(1+\frac{1}{2}+\frac{1}{3}\right)-\frac{1}{3}}{3\,(2!)^2}\left(\frac{z}{2}\right)^5 + \ldots\right\}. \qquad \text{(A.102)}$$

c) Generating function and associated series

The Bessel function $J_n(z)$ has the generating function

$$e^{\frac{1}{2}z\left(t-\frac{1}{t}\right)} = \sum_{m=-\infty}^{\infty} J_m(z)\,t^m \qquad (t \neq 0). \qquad \text{(A.103)}$$

This function allows, for an angle θ, the series expansions in terms of Bessel functions:

$$\cos(z\sin\theta) = J_0(z) + 2\sum_{m=1}^{\infty} J_{2m}(z)\cos(2m\theta), \qquad \text{(A.104)}$$

$$\sin(z\sin\theta) = 2\sum_{m=0}^{\infty} J_{2m+1}(z)\sin\big((2m+1)\theta\big), \qquad \text{(A.105)}$$

$$\cos(z\cos\theta) = J_0(z) + 2\sum_{m=1}^{\infty} J_{2m}(z)\cos(2m\theta), \qquad \text{(A.106)}$$

$$\sin(z\cos\theta) = 2\sum_{m=0}^{\infty}(-1)^m J_{2m+1}(z)\cos\big((2m+1)\theta\big). \tag{A.107}$$

By combining (A.106) and (A.107) we obtain the Jacobi-Anger expansion

$$e^{iz\cos\theta} = \sum_{m=-\infty}^{\infty} i^m J_m(z)\, e^{im\theta}, \tag{A.108}$$

named after the Prussian mathematician Carl Gustav Jacob Jacobi (1804–1851) and the German mathematician and astronomer Carl Theodor Anger (1803–1858). It describes the expansion of a plane wave in terms of cylindrical waves. Other related special series are

$$1 = J_0(z) + 2\sum_{m=1}^{\infty} J_{2m}(z), \tag{A.109}$$

$$\cos z = J_0(z) + 2\sum_{m=1}^{\infty}(-1)^m J_{2m}(z), \tag{A.110}$$

$$\sin z = 2\sum_{m=0}^{\infty}(-1)^m J_{2m+1}(z). \tag{A.111}$$

d) Integral representations

The Bessel functions of order zero admit the integral representations

$$J_0(z) = \frac{1}{\pi}\int_0^{\pi}\cos(z\sin\theta)\,\mathrm{d}\theta = \frac{1}{\pi}\int_0^{\pi}\cos(z\cos\theta)\,\mathrm{d}\theta, \tag{A.112}$$

$$Y_0(z) = \frac{4}{\pi^2}\int_0^{\pi/2}\cos(z\cos\theta)\big\{\gamma + \ln(2z\sin^2\theta)\big\}\,\mathrm{d}\theta. \tag{A.113}$$

For arbitrary orders and for $|\arg z| < \pi/2$ we have

$$J_\nu(z) = \frac{1}{\pi}\int_0^{\pi}\cos(z\sin\theta - \nu\theta)\,\mathrm{d}\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^{\infty} e^{-z\sinh t - \nu t}\,\mathrm{d}t, \tag{A.114}$$

$$Y_\nu(z) = \frac{1}{\pi}\int_0^{\pi}\sin(z\sin\theta - \nu\theta)\,\mathrm{d}\theta - \frac{1}{\pi}\int_0^{\infty}\big\{e^{\nu t} + e^{-\nu t}\cos(\nu\pi)\big\}\, e^{-z\sinh t}\,\mathrm{d}t. \tag{A.115}$$

The Hankel functions admit the integral representations

$$H_\nu^{(1)}(z) = \frac{1}{\pi i}\int_{-\infty}^{\infty+\pi i} e^{z\sinh t - \nu t}\,\mathrm{d}t \qquad \big(|\arg z| < \pi/2\big), \tag{A.116}$$

$$H_\nu^{(2)}(z) = -\frac{1}{\pi i}\int_{-\infty}^{\infty-\pi i} e^{z\sinh t - \nu t}\,\mathrm{d}t \qquad \big(|\arg z| < \pi/2\big). \tag{A.117}$$

e) Recurrence relations

If W_ν is used to denote J_ν, Y_ν, $H_\nu^{(1)}$, $H_\nu^{(2)}$, or any linear combination of these functions whose coefficients are independent of z and ν, then the following recurrence relations hold for all of them:

$$\frac{2\nu}{z}W_\nu(z) = W_{\nu-1}(z) + W_{\nu+1}(z), \tag{A.118}$$

$$2\,\frac{\mathrm{d}W_\nu}{\mathrm{d}z}(z) = W_{\nu-1}(z) - W_{\nu+1}(z), \tag{A.119}$$

$$\frac{\mathrm{d}W_\nu}{\mathrm{d}z}(z) = W_{\nu-1}(z) - \frac{\nu}{z}W_\nu(z), \tag{A.120}$$

$$\frac{\mathrm{d}W_\nu}{\mathrm{d}z}(z) = -W_{\nu+1}(z) + \frac{\nu}{z}W_\nu(z), \tag{A.121}$$

$$\frac{\mathrm{d}W_0}{\mathrm{d}z}(z) = -W_1(z). \tag{A.122}$$

Particular cases for the above are

$$\frac{\mathrm{d}W_1}{\mathrm{d}z}(z) = W_0(z) - \frac{1}{z}W_1(z), \tag{A.123}$$

$$W_2(z) = \frac{2}{z}W_1(z) - W_0(z), \tag{A.124}$$

$$\frac{\mathrm{d}W_2}{\mathrm{d}z}(z) = \left(1 - \frac{4}{z^2}\right)W_1(z) + \frac{2}{z}W_0(z) = W_1(z) - \frac{2}{z}W_2(z). \tag{A.125}$$

For the derivatives, considering $m = 0, 1, 2, \ldots$, it also holds that

$$\left(\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^m \big\{z^\nu W_\nu(z)\big\} = z^{\nu-m}W_{\nu-m}(z), \tag{A.126}$$

$$\left(\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^m \big\{z^{-\nu} W_\nu(z)\big\} = (-1)^m z^{-\nu-m}W_{\nu+m}(z). \tag{A.127}$$

Some primitives of Bessel functions, omitting the integration constants, are given by

$$\int W_0(z)\,\mathrm{d}z = \frac{\pi z}{2}\big\{W_0(z)\boldsymbol{H}_{-1}(z) + W_1(z)\boldsymbol{H}_0(z)\big\}, \tag{A.128}$$

$$\int W_1(z)\,\mathrm{d}z = -W_0(z), \tag{A.129}$$

where $\boldsymbol{H}_\nu$ denotes the Struve function of order ν (vid. Subsection A.2.7).

f) Asymptotic behavior

For small arguments, when ν is fixed and $z \to 0$, the Bessel functions behave like

$$J_\nu(z) \sim \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu \qquad (\nu \neq -1, -2, -3, \ldots), \tag{A.130}$$

$$Y_0(z) \sim -iH_0^{(1)}(z) \sim iH_0^{(2)}(z) \sim \frac{2}{\pi}\ln z, \tag{A.131}$$

$$Y_\nu(z) \sim -iH_\nu^{(1)}(z) \sim iH_\nu^{(2)}(z) \sim -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{z}\right)^\nu \qquad (\mathfrak{Re}\,\nu > 0). \tag{A.132}$$

The asymptotic forms of the Bessel functions, when ν is fixed and $|z| \to \infty$, are given by

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \qquad |\arg z| < \pi, \tag{A.133}$$

$$Y_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \qquad |\arg z| < \pi, \tag{A.134}$$

$$H_\nu^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}}\, e^{i\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)}, \qquad -\pi < \arg z < 2\pi, \tag{A.135}$$

$$H_\nu^{(2)}(z) \sim \sqrt{\frac{2}{\pi z}}\, e^{-i\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)}, \qquad -2\pi < \arg z < \pi. \tag{A.136}$$

In particular, the zeroth and first order Hankel functions behave at the origin, for $z \to 0$, as

$$H_0^{(1)}(z) \sim \frac{2i}{\pi} \ln z, \qquad H_0^{(2)}(z) \sim -\frac{2i}{\pi} \ln z, \tag{A.137}$$

$$H_1^{(1)}(z) \sim -\frac{2i}{\pi z}, \qquad H_1^{(2)}(z) \sim \frac{2i}{\pi z}. \tag{A.138}$$

At infinity, for $|z| \to \infty$, they behave like

$$H_0^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}}\, e^{i(z - \frac{\pi}{4})}, \qquad H_0^{(2)}(z) \sim \sqrt{\frac{2}{\pi z}}\, e^{-i(z - \frac{\pi}{4})}, \tag{A.139}$$

$$H_1^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}}\, e^{i(z - \frac{3\pi}{4})}, \qquad H_1^{(2)}(z) \sim \sqrt{\frac{2}{\pi z}}\, e^{-i(z - \frac{3\pi}{4})}. \tag{A.140}$$

g) Addition theorems

If W_ν denotes any linear combination of Bessel, Neumann, or Hankel functions, then Neumann's addition theorem for $u, v \in \mathbb{C}$ asserts that

$$W_\nu(u \pm v) = \sum_{m=-\infty}^{\infty} W_{\nu \mp m}(u) J_m(v) \qquad \big(|v| < |u|\big). \tag{A.141}$$

The restriction $|v| < |u|$ is unnecessary when $W_\nu = J_\nu$ and ν is an integer or zero. We have similarly Graf's addition theorem, which states that

$$W_\nu(w) e^{i\nu\chi} = \sum_{m=-\infty}^{\infty} W_{\nu+m}(u) J_m(v) e^{im\alpha} \qquad \big(|ve^{\pm i\alpha}| < |u|\big), \tag{A.142}$$

where

$$w = \sqrt{u^2 + v^2 - 2uv\cos\alpha}, \tag{A.143}$$

and

$$u - v\cos\alpha = w\cos\chi, \qquad v\sin\alpha = w\sin\chi, \tag{A.144}$$

being the branches chosen so that $w \to u$ and $\chi \to 0$ as $v \to 0$. If u, v are real and positive, and $0 \le \alpha \le \pi$, then w, χ are real and nonnegative, and the geometrical relationship of the variables is shown in Figure A.5. Again, the restriction $|ve^{\pm i\alpha}| < |u|$ is unnecessary when $W_\nu = J_\nu$ and ν is an integer or zero.

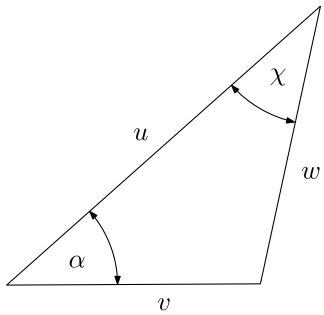


FIGURE A.5. Geometrical relationship of the variables for Graf's addition theorem.

The addition theorem of Graf allows us to establish, for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$ and $k \in \mathbb{C}$, the addition theorem for the Hankel functions

$$H_\nu^{(1)}\big(k|\boldsymbol{x}-\boldsymbol{y}|\big)e^{i\nu\varphi} = \sum_{m=-\infty}^{\infty} H_{\nu+m}^{(1)}\big(k|\boldsymbol{x}|\big)J_m\big(k|\boldsymbol{y}|\big)e^{im\theta} \qquad \big(|\boldsymbol{y}| < |\boldsymbol{x}|\big), \tag{A.145}$$

where

$$\cos\theta = \frac{\boldsymbol{x}\cdot\boldsymbol{y}}{|\boldsymbol{x}|\,|\boldsymbol{y}|} \qquad\qquad \cos\varphi = \frac{\boldsymbol{x}\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}|\,|\boldsymbol{x}-\boldsymbol{y}|}. \tag{A.146}$$

In the particular case when $\nu = 0$, the addition theorem for $|\boldsymbol{y}| < |\boldsymbol{x}|$ becomes

$$H_0^{(1)}\big(k|\boldsymbol{x}-\boldsymbol{y}|\big) = H_0^{(1)}\big(k|\boldsymbol{x}|\big)J_0\big(k|\boldsymbol{y}|\big) + 2\sum_{m=1}^{\infty} H_m^{(1)}\big(k|\boldsymbol{x}|\big)J_m\big(k|\boldsymbol{y}|\big)\cos(m\theta). \tag{A.147}$$

A.2.5 Modified Bessel functions

a) Differential equation and definition

Modified Bessel functions are special functions that appear also in a wide variety of physical problems. Roughly speaking, they correspond to Bessel and Hankel functions (vid. Subsection A.2.4) with a purely imaginary argument and therefore they do not oscillate on the real axis as the former but rather increase or decrease exponentially. Some references for them are Abramowitz & Stegun (1972), Arfken & Weber (2005), Erdélyi (1953), Jackson (1999), Jahnke & Emde (1945), Luke (1962), Magnus & Oberhettinger (1954), Morse & Feshbach (1953), Spiegel & Liu (1999), Watson (1944), and Weisstein (2002). We consider the modified Bessel differential equation of order ν for a function $W : \mathbb{C} \to \mathbb{C}$, which is given by

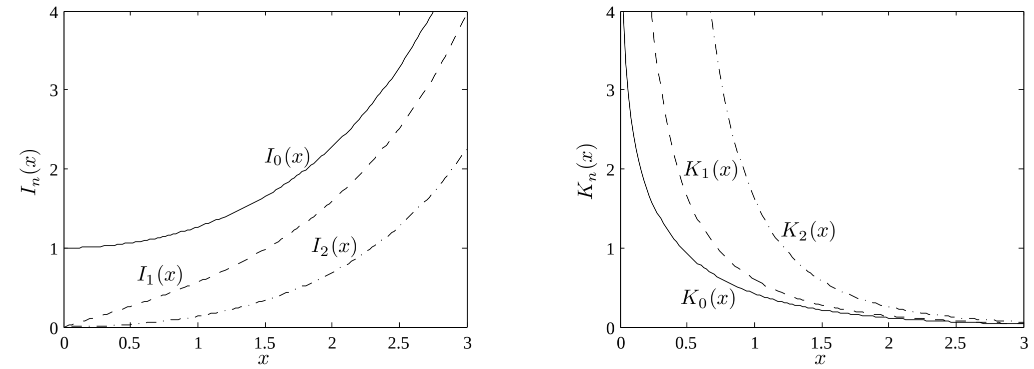
$$z^2\frac{\mathrm{d}^2W}{\mathrm{d}z^2}(z) + z\frac{\mathrm{d}W}{\mathrm{d}z}(z) - (z^2+\nu^2)W(z) = 0, \tag{A.148}$$

where, in general, $\nu \in \mathbb{C}$ is an unrestricted value. Independent solutions of this equation are the modified Bessel functions of the first kind $I_\nu(z)$ and of the second kind $K_\nu(z)$. They are depicted in Figure A.6. Each is a regular function of z throughout the z-plane cut along the negative real axis, and for fixed z ($\neq 0$) each is an entire function of ν. When $\nu = n$, for $n \in \mathbb{Z}$, then $I_\nu(z)$ is an entire function of z. The function $I_\nu(z)$, for $\mathfrak{Re}\,\nu \geq 0$, is bounded as $z \to 0$ in any bounded range of $\arg z$. The functions $I_\nu(z)$ and $I_{-\nu}(z)$

are linearly independent except when ν is an integer. The function $K_\nu(z)$ tends to zero as $|z| \to \infty$ in the sector $|\arg z| < \pi/2$, and for all values of ν, $I_\nu(z)$ and $K_\nu(z)$ are linearly independent. The functions $I_\nu(z)$ and $K_\nu(z)$ are real and positive when $\nu > -1$ and $z > 0$. The function $K_\nu(z)$ is related to $I_\nu(z)$ through

$$K_\nu(z) = \frac{\pi}{2}\left(\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}\right), \qquad \nu \notin \mathbb{Z}, \tag{A.149}$$

$$K_n(z) = \lim_{\nu\to n} \frac{\pi}{2}\left(\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}\right), \qquad n \in \mathbb{Z}. \tag{A.150}$$



(a) Modified Bessel function $I_n(x)$, $n = 0, 1, 2$ (b) Modified Bessel function $K_n(x)$, $n = 0, 1, 2$

FIGURE A.6. Modified Bessel functions for real arguments.

The modified Bessel function $I_\nu(z)$ is related to the Bessel function $J_\nu(z)$ through

$$I_\nu(z) = e^{-i\nu\pi/2} J_\nu\big(z\, e^{i\pi/2}\big), \qquad -\pi < \arg z \le \frac{\pi}{2}, \tag{A.151}$$

$$I_\nu(z) = e^{3i\nu\pi/2} J_\nu\big(z\, e^{-3i\pi/2}\big), \qquad -\frac{\pi}{2} < \arg z \le \pi, \tag{A.152}$$

and $K_\nu(z)$ is related to the Hankel functions $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ through

$$K_\nu(z) = \frac{i\pi}{2} e^{i\nu\pi/2} H_\nu^{(1)}\big(z\, e^{i\pi/2}\big), \qquad -\pi < \arg z \le \frac{\pi}{2}, \tag{A.153}$$

$$K_\nu(z) = -\frac{i\pi}{2} e^{-i\nu\pi/2} H_\nu^{(2)}\big(z\, e^{-i\pi/2}\big), \qquad -\frac{\pi}{2} < \arg z \le \pi. \tag{A.154}$$

For the Neumann function $Y_\nu(z)$ it holds that

$$Y_\nu(z) = e^{i(\nu+1)\pi/2} I_\nu(z) - \frac{2}{\pi} e^{-i\nu\pi/2} K_\nu(z), \qquad -\pi < \arg z \le \frac{\pi}{2}. \tag{A.155}$$

For negative orders it holds also that

$$I_{-n}(z) = I_n(z), \qquad n \in \mathbb{Z}, \tag{A.156}$$

$$K_{-\nu}(z) = K_\nu(z), \qquad \nu \in \mathbb{C}. \tag{A.157}$$

When using complex conjugate arguments, then for $\nu \in \mathbb{R}$ follows

$$I_\nu(\bar{z}) = \overline{I_\nu(z)}, \qquad K_\nu(\bar{z}) = \overline{K_\nu(z)}. \tag{A.158}$$

Most of the properties of modified Bessel functions can be deduced immediately from those of ordinary Bessel functions by the application of these relations.

b) Ascending series

The modified Bessel function $I_\nu(z)$ has the power series expansion

$$I_\nu(z) = \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(\nu+m+1)} \left(\frac{z}{2}\right)^{2m+\nu}, \tag{A.159}$$

where Γ stands for the gamma function (A.40). For an integer order $n \geq 0$, the modified Bessel function $I_n(z)$ has the power series expansion

$$I_n(z) = \sum_{m=0}^{\infty} \frac{1}{m!\,(m+n)!} \left(\frac{z}{2}\right)^{2m+n}, \tag{A.160}$$

and for the function $K_n(z)$ it is given by

$$\begin{aligned} K_n(z) = {} & (-1)^{n+1} I_n(z) \left(\ln\frac{z}{2} + \gamma\right) + \frac{1}{2}\sum_{m=0}^{n-1}(-1)^m \frac{(n-m-1)!}{m!} \left(\frac{z}{2}\right)^{2m-n} \\ & + \frac{(-1)^n}{2} \sum_{m=0}^{\infty} \frac{\psi(m+n)+\psi(m)}{m!\,(m+n)!} \left(\frac{z}{2}\right)^{2m+n}, \end{aligned} \tag{A.161}$$

where

$$\psi(0) = 0, \qquad \psi(m) = \sum_{p=1}^{m} \frac{1}{p} \qquad (m = 1, 2, \ldots), \tag{A.162}$$

and γ denotes Euler's constant (A.43). For $n = 0$ the following expansions hold

$$I_0(z) = 1 + \frac{z^2/4}{(1!)^2} + \frac{(z^2/4)^2}{(2!)^2} + \frac{(z^2/4)^3}{(3!)^2} + \ldots, \tag{A.163}$$

$$\begin{aligned} K_0(z) = {} & -I_0(z)\left(\ln\frac{z}{2} + \gamma\right) \\ & + \frac{z^2/4}{(1!)^2} + \left(1 + \frac{1}{2}\right)\frac{(z^2/4)^2}{(2!)^2} + \left(1 + \frac{1}{2} + \frac{1}{3}\right)\frac{(z^2/4)^3}{(3!)^2} + \ldots. \end{aligned} \tag{A.164}$$

Similarly, if $n = 1$, then

$$I_1(z) = \frac{z}{2}\left\{1 + \frac{z^2/4}{2\,(1!)^2} + \frac{(z^2/4)^2}{3\,(2!)^2} + \frac{(z^2/4)^3}{4\,(3!)^2} + \ldots\right\}, \tag{A.165}$$

$$\begin{aligned} K_1(z) = {} & I_1(z)\left(\ln\frac{z}{2} + \gamma\right) + \frac{1}{z} \\ & - \frac{1}{2}\left\{\frac{z}{2} + \frac{2\left(1+\frac{1}{2}\right) - \frac{1}{2}}{2\,(1!)^2}\left(\frac{z}{2}\right)^3 + \frac{2\left(1+\frac{1}{2}+\frac{1}{3}\right) - \frac{1}{3}}{3\,(2!)^2}\left(\frac{z}{2}\right)^5 + \ldots\right\}. \end{aligned} \tag{A.166}$$

c) Generating function and associated series

The modified Bessel function $I_n(z)$ has the generating function

$$e^{\frac{1}{2}z\left(t+\frac{1}{t}\right)} = \sum_{m=-\infty}^{\infty} I_m(z)\, t^m \qquad (t \neq 0), \tag{A.167}$$

which allows, for an angle θ, the series expansions in terms of modified Bessel functions:

$$e^{z\cos\theta} = I_0(z) + 2\sum_{m=1}^{\infty} I_m(z)\cos(m\theta), \tag{A.168}$$

$$\begin{aligned} e^{z\sin\theta} = I_0(z) &+ 2\sum_{m=0}^{\infty}(-1)^m I_{2m+1}(z)\sin\big((2m+1)\theta\big) \\ &+ 2\sum_{m=1}^{\infty}(-1)^m I_{2m}(z)\cos\big(2m\theta\big). \end{aligned} \tag{A.169}$$

Other related special series are

$$1 = I_0(z) + 2\sum_{m=1}^{\infty}(-1)^m I_{2m}(z), \tag{A.170}$$

$$e^{z} = I_0(z) + 2\sum_{m=1}^{\infty} I_m(z), \tag{A.171}$$

$$e^{-z} = I_0(z) + 2\sum_{m=1}^{\infty}(-1)^m I_m(z), \tag{A.172}$$

$$\cosh z = I_0(z) + 2\sum_{m=1}^{\infty} I_{2m}(z), \tag{A.173}$$

$$\sinh z = 2\sum_{m=0}^{\infty} I_{2m+1}(z). \tag{A.174}$$

d) Integral representations

The modified Bessel functions of order zero admit the integral representations

$$I_0(z) = \frac{1}{\pi}\int_0^{\pi} e^{\pm z\cos\theta}\,\mathrm{d}\theta = \frac{1}{\pi}\int_0^{\pi}\cosh(z\cos\theta)\,\mathrm{d}\theta, \tag{A.175}$$

$$K_0(z) = -\frac{1}{\pi}\int_0^{\pi} e^{\pm z\cos\theta}\big\{\gamma + \ln(2z\sin^2\theta)\big\}\,\mathrm{d}\theta. \tag{A.176}$$

For arbitrary orders and for $|\arg z| < \pi/2$ we have that

$$I_\nu(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta}\cos(\nu\theta)\,\mathrm{d}\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^{\infty} e^{-z\cosh t-\nu t}\,\mathrm{d}t, \tag{A.177}$$

$$K_\nu(z) = \int_0^{\infty} e^{-z\cosh t}\cosh(\nu t)\,\mathrm{d}t. \tag{A.178}$$

e) Recurrence relations

If W_ν is used to denote I_ν, $e^{\nu\pi i}K_\nu$, or any linear combination of these functions whose coefficients are independent of z and ν, then the following recurrence relations hold:

$$\frac{2\nu}{z}W_\nu(z) = W_{\nu-1}(z) - W_{\nu+1}(z), \tag{A.179}$$

$$2\,\frac{\mathrm{d}W_\nu}{\mathrm{d}z}(z) = W_{\nu-1}(z) + W_{\nu+1}(z), \tag{A.180}$$

$$\frac{\mathrm{d}W_\nu}{\mathrm{d}z}(z) = W_{\nu-1}(z) - \frac{\nu}{z}W_\nu(z), \tag{A.181}$$

$$\frac{\mathrm{d}W_\nu}{\mathrm{d}z}(z) = W_{\nu+1}(z) + \frac{\nu}{z}W_\nu(z), \tag{A.182}$$

$$\frac{\mathrm{d}I_0}{\mathrm{d}z}(z) = I_1(z), \qquad \frac{\mathrm{d}K_0}{\mathrm{d}z}(z) = -K_1(z). \tag{A.183}$$

For the derivatives, considering $m = 0, 1, 2, \ldots$, it also holds that

$$\left(\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^m \left\{z^\nu W_\nu(z)\right\} = z^{\nu-m}W_{\nu-m}(z), \tag{A.184}$$

$$\left(\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^m \left\{z^{-\nu} W_\nu(z)\right\} = z^{-\nu-m}W_{\nu+m}(z). \tag{A.185}$$

f) Asymptotic behavior

Modified Bessel functions behave for small arguments, when ν is fixed and $z \to 0$, as

$$I_\nu(z) \sim \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu \qquad (\nu \neq -1, -2, -3, \ldots), \tag{A.186}$$

$$K_0(z) \sim -\ln z, \tag{A.187}$$

$$K_\nu(z) \sim \frac{\Gamma(\nu)}{2}\left(\frac{2}{z}\right)^\nu \qquad (\mathfrak{Re}\,\nu > 0). \tag{A.188}$$

The asymptotic forms of the modified Bessel functions, when ν is fixed and $|z| \to \infty$, are

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}}, \qquad |\arg z| < \frac{\pi}{2}, \tag{A.189}$$

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}}\,e^{-z}, \qquad |\arg z| < \frac{3\pi}{2}. \tag{A.190}$$

A.2.6 Spherical Bessel and Hankel functions

a) Differential equation and definition

Spherical Bessel functions or Bessel functions of fractional order are special functions that play the role of Bessel or cylinder functions for spherical problems. Some references are Abramowitz & Stegun (1972), Arfken & Weber (2005), Erdélyi (1953), Jackson (1999), and Weisstein (2002). They satisfy the spherical Bessel differential equation

$$z^2\frac{\mathrm{d}^2w}{\mathrm{d}z^2}(z) + 2z\frac{\mathrm{d}w}{\mathrm{d}z}(z) + \big(z^2 - \nu(\nu+1)\big)w(z) = 0 \qquad (\nu \in \mathbb{C}), \tag{A.191}$$

which can be obtained by applying separation of spherical variables to the Helmholtz equation. Particular linearly independent solutions of this equation are the spherical Bessel functions of the first kind

$$j_\nu(z) = \sqrt{\frac{\pi}{2z}}J_{\nu+1/2}(z), \tag{A.192}$$

and the spherical Bessel functions of the second kind or spherical Neumann functions

$$y_\nu(z) = \sqrt{\frac{\pi}{2z}}Y_{\nu+1/2}(z), \tag{A.193}$$

where $J_{\nu+1/2}$ and $Y_{\nu+1/2}$ denote respectively the Bessel function of the first kind and the Bessel function of the second kind or Neumann function. They are shown in Figure A.7. Other independent solutions of (A.191) are the spherical Hankel functions of the first and second kinds, also known as spherical Bessel functions of the third kind, given by

$$h_\nu^{(1)}(z) = j_\nu(z) + iy_\nu(z) = \sqrt{\frac{\pi}{2z}}H_{\nu+1/2}^{(1)}(z), \tag{A.194}$$

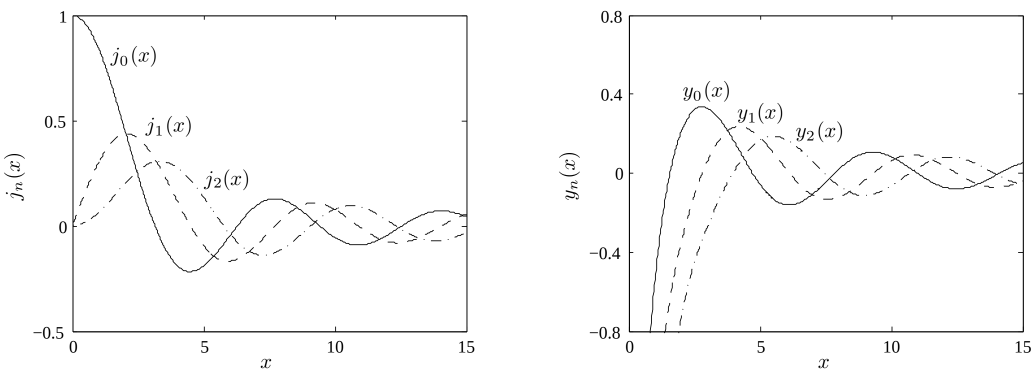
$$h_\nu^{(2)}(z) = j_\nu(z) - iy_\nu(z) = \sqrt{\frac{\pi}{2z}}H_{\nu+1/2}^{(2)}(z), \tag{A.195}$$

where $H_{\nu+1/2}^{(1)}$ and $H_{\nu+1/2}^{(2)}$ denote respectively the Hankel functions of the first and second kinds. The Bessel and Hankel functions are thoroughly discussed in Subsection A.2.4. The spherical Bessel and Hankel functions are most commonly encountered in the case where $\nu = n$, being n a positive integer or zero. They satisfy for $n \in \mathbb{Z}$ the relations

$$y_n(z) = (-1)^{n+1}j_{-n-1}(z), \tag{A.196}$$

and

$$h_{-n-1}^{(1)}(z) = i(-1)^n h_n^{(1)}(z), \qquad h_{-n-1}^{(2)}(z) = -i(-1)^n h_n^{(2)}(z). \tag{A.197}$$



(a) Spherical Bessel function $j_n(x)$, $n = 0, 1, 2$ (b) Spherical Neumann function $y_n(x)$, $n = 0, 1, 2$

FIGURE A.7. Spherical Bessel and Neumann functions for real arguments.

b) Ascending series

The spherical Bessel function $j_\nu(z)$ has the ascending series expansion

$$j_\nu(z) = \frac{\sqrt{\pi}}{2} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(\nu+m+3/2)} \left(\frac{z}{2}\right)^{2m+\nu}, \tag{A.198}$$

where Γ denotes the gamma function (A.40). For the spherical Neumann function $y_\nu(z)$ it is given by

$$y_\nu(z) = \frac{(-1)^{\nu+1}}{2^\nu z^{\nu+1}} \sum_{m=0}^{\infty} \frac{(-1)^m 4^{\nu-m}\sqrt{\pi}}{m!\,\Gamma(m-\nu+1/2)} z^{2m}. \tag{A.199}$$

For an integer order $n \geq 0$ they are given by

$$j_n(z) = 2^n z^n \sum_{m=0}^{\infty} \frac{(-1)^m (m+n)!}{m!\,(2n+2m+1)!} z^{2m}, \tag{A.200}$$

and

$$y_n(z) = \frac{(-1)^{n+1}}{2^n z^{n+1}} \sum_{m=0}^{\infty} \frac{(-1)^m (m-n)!}{m!\,(2m-2n)!} z^{2m}. \tag{A.201}$$

For the spherical Hankel functions we have also the exact formulae

$$h_n^{(1)}(z) = (-i)^{n+1} \frac{e^{iz}}{z} \sum_{m=0}^{n} \frac{i^m}{m!\,(2z)^m} \frac{(n+m)!}{(n-m)!}, \tag{A.202}$$

$$h_n^{(2)}(z) = i^{n+1} \frac{e^{-iz}}{z} \sum_{m=0}^{n} \frac{(-i)^m}{m!\,(2z)^m} \frac{(n+m)!}{(n-m)!}. \tag{A.203}$$

c) Special values

The spherical Bessel function $j_n(z)$ adopts, for $n = 0, 1, 2$, the values

$$j_0(z) = \frac{\sin z}{z}, \tag{A.204}$$

$$j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z}, \tag{A.205}$$

$$j_2(z) = \left(\frac{3}{z^3} - \frac{1}{z}\right) \sin z - \frac{3}{z^2} \cos z. \tag{A.206}$$

For $n = 0, 1, 2$ the spherical Neumann function $y_n(z)$ adopts the values

$$y_0(z) = -j_{-1}(z) = -\frac{\cos z}{z}, \tag{A.207}$$

$$y_1(z) = -j_{-2}(z) = -\frac{\cos z}{z^2} - \frac{\sin z}{z}, \tag{A.208}$$

$$y_2(z) = -j_{-3}(z) = \left(-\frac{3}{z^3} + \frac{1}{z}\right) \cos z - \frac{3}{z^2} \sin z. \tag{A.209}$$

For the spherical Hankel functions, these values are given by

$$h_0^{(1)}(z) = -\frac{i}{z}e^{iz}, \qquad h_0^{(2)}(z) = \frac{i}{z}e^{-iz}, \tag{A.210}$$

$$h_1^{(1)}(z) = \left(-\frac{1}{z} - \frac{i}{z^2}\right)e^{iz}, \qquad h_1^{(2)}(z) = \left(-\frac{1}{z} + \frac{i}{z^2}\right)e^{-iz}, \tag{A.211}$$

$$h_2^{(1)}(z) = \left(\frac{i}{z} - \frac{3}{z^2} - \frac{3i}{z^3}\right)e^{iz}, \qquad h_2^{(2)}(z) = \left(-\frac{i}{z} - \frac{3}{z^2} + \frac{3i}{z^3}\right)e^{-iz}. \tag{A.212}$$

d) Recurrence relations

If w_n is used to denote j_n, y_n, $h_n^{(1)}$, $h_n^{(2)}$, or any linear combination of these functions whose coefficients are independent of z and n, then the following recurrence relations hold:

$$\frac{2n+1}{z}w_n(z) = w_{n-1}(z) + w_{n+1}(z), \tag{A.213}$$

$$(2n+1)\frac{\mathrm{d}w_n}{\mathrm{d}z}(z) = n\,w_{n-1}(z) - (n+1)w_{n+1}(z). \tag{A.214}$$

$$\frac{\mathrm{d}w_n}{\mathrm{d}z}(z) = w_{n-1}(z) - \frac{n+1}{z}w_n(z). \tag{A.215}$$

$$\frac{\mathrm{d}w_n}{\mathrm{d}z}(z) = \frac{n}{z}w_n(z) - w_{n+1}(z). \tag{A.216}$$

$$\frac{\mathrm{d}w_0}{\mathrm{d}z}(z) = -w_1(z). \tag{A.217}$$

Rearranging these relations yields

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{z^{n+1}w_n(z)\right\} = z^{n+1}w_{n-1}(z), \tag{A.218}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{z^{-n}w_n(z)\right\} = -z^{-n}w_{n+1}(z). \tag{A.219}$$

By mathematical induction we can establish also the Rayleigh formulae

$$j_n(z) = (-1)^n z^n \left(\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \left\{\frac{\sin z}{z}\right\}, \tag{A.220}$$

$$y_n(z) = -(-1)^n z^n \left(\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \left\{\frac{\cos z}{z}\right\}, \tag{A.221}$$

$$h_n^{(1)}(z) = -i(-1)^n z^n \left(\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \left\{\frac{e^{iz}}{z}\right\}, \tag{A.222}$$

$$h_n^{(2)}(z) = i(-1)^n z^n \left(\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \left\{\frac{e^{-iz}}{z}\right\}. \tag{A.223}$$

e) Limiting values

The asymptotic limiting values of the spherical Bessel functions for small arguments, i.e., as $z \to 0$ and for fixed n, are given by

$$j_n(z) \sim \frac{2^n n!}{(2n+1)!} z^n, \tag{A.224}$$

$$y_n(z) \sim -\frac{(2n)!}{2^n n!} z^{-n-1}. \tag{A.225}$$

The asymptotic forms of the spherical Bessel and Hankel functions for large arguments, as $|z| \to \infty$ and for fixed n, are, likewise as for the Bessel and Hankel functions, given by

$$j_n(z) \sim \frac{1}{z} \sin\Big(z - \frac{n\pi}{2}\Big), \tag{A.226}$$

$$y_n(z) \sim -\frac{1}{z} \cos\Big(z - \frac{n\pi}{2}\Big), \tag{A.227}$$

$$h_n^{(1)}(z) \sim (-i)^{n+1} \frac{e^{iz}}{z} = -i \frac{e^{i(z-n\pi/2)}}{z}, \tag{A.228}$$

$$h_n^{(2)}(z) \sim i^{n+1} \frac{e^{-iz}}{z} = i \frac{e^{-i(z-n\pi/2)}}{z}. \tag{A.229}$$

f) Addition theorems

The spherical Bessel functions satisfy, for arbitrary complex u, v, λ, θ, the addition theorems

$$j_0(\lambda w) = \sum_{n=0}^{\infty} (2n+1) j_n(\lambda u) j_n(\lambda v) P_n(\cos\theta), \tag{A.230}$$

$$y_0(\lambda w) = \sum_{n=0}^{\infty} (2n+1) y_n(\lambda u) j_n(\lambda v) P_n(\cos\theta) \qquad \big(|ve^{\pm i\theta}| < |u|\big), \tag{A.231}$$

where

$$w = \sqrt{u^2 + v^2 - 2uv\cos\theta}, \tag{A.232}$$

and where $P_n(z)$ denotes the Legendre polynomial of degree n (vid. Subsection A.2.8). Similarly, for the spherical Hankel functions we have that

$$h_0^{(1)}(\lambda w) = \sum_{n=0}^{\infty} (2n+1) h_n^{(1)}(\lambda u) j_n(\lambda v) P_n(\cos\theta) \qquad \big(|ve^{\pm i\theta}| < |u|\big). \tag{A.233}$$

As for cylindrical functions, we have the Jacobi-Anger expansion

$$e^{i\lambda\cos\theta} = \sum_{n=0}^{\infty} i^n (2n+1) j_n(\lambda) P_n(\cos\theta), \tag{A.234}$$

which describes the expansion of a plane wave in terms of spherical waves.

A.2.7 Struve functions

a) Differential equation and definition

Struve functions are special functions that occur in many places in physics and applied mathematics, e.g., in optics, in fluid dynamics, and quite prominently in acoustics for impedance calculations. Some references for Struve functions are Abramowitz & Stegun (1972), Erdélyi (1953), Jahnke & Emde (1945), Magnus & Oberhettinger (1954), and Weisstein (2002). They satisfy for a function $\boldsymbol{W} : \mathbb{C} \to \mathbb{C}$ the following non-homogeneous Bessel differential equation of order ν:

$$z^2 \frac{\mathrm{d}^2 \boldsymbol{W}}{\mathrm{d}z^2}(z) + z \frac{\mathrm{d}\boldsymbol{W}}{\mathrm{d}z}(z) + (z^2 - \nu^2)\boldsymbol{W}(z) = \frac{4\,(z/2)^{\nu+1}}{\sqrt{\pi}\,\Gamma(\nu + 1/2)}, \tag{A.235}$$

where, in general, $\nu \in \mathbb{C}$ is an unrestricted value, and Γ denotes the gamma function (A.40). The general solution of (A.235) is given by

$$\boldsymbol{W}(z) = a\, J_\nu(z) + b\, Y_\nu(z) + \boldsymbol{H}_\nu(z) \qquad (a, b \in \mathbb{C}), \tag{A.236}$$

where $J_\nu(z)$ and $Y_\nu(z)$ are the Bessel and Neumann functions of order ν (cf. Subsection A.2.4), and where $z^{-\nu}\boldsymbol{H}_\nu(z)$ is an entire function of z. The function $\boldsymbol{H}_\nu(z)$ is known as the Struve function of order ν, and is named after the Russian-born German astronomer Karl Hermann Struve (1854–1920), who was part of the famous Struve family of astronomers. It is illustrated in Figure A.8 for real arguments and some integer orders.

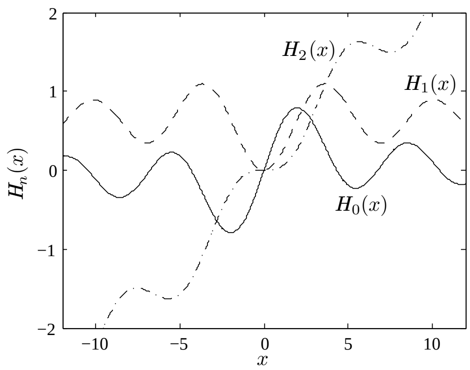


FIGURE A.8. Struve function $\boldsymbol{H}_n(x)$ for real arguments, where $n = 0, 1, 2$.

b) Power series expansion

The Struve function $\boldsymbol{H}_\nu(z)$ admits the power series expansion

$$\boldsymbol{H}_\nu(z) = \left(\frac{z}{2}\right)^{\nu+1} \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m}}{\Gamma(m + 3/2)\Gamma(m + \nu + 3/2)}. \tag{A.237}$$

By considering n as a positive integer, we have for half integer orders that

$$\boldsymbol{H}_{n+1/2}(z) = Y_{n+1/2}(z) + \frac{1}{\pi} \sum_{m=0}^{n} \frac{\Gamma(m + 1/2)}{\Gamma(n - m + 1)} \left(\frac{z}{2}\right)^{-2m+n-1/2}. \tag{A.238}$$

Particular power series expansions are

$$\boldsymbol{H}_0(z) = \frac{2}{\pi}\left\{z - \frac{z^3}{1^2\cdot 3^2} + \frac{z^5}{1^2\cdot 3^2\cdot 5^2} - \ldots\right\}, \tag{A.239}$$

and

$$\boldsymbol{H}_1(z) = \frac{2}{\pi}\left\{\frac{z^2}{1^2\cdot 3} - \frac{z^4}{1^2\cdot 3^2\cdot 5} + \frac{z^6}{1^2\cdot 3^2\cdot 5^2\cdot 7} - \ldots\right\}. \tag{A.240}$$

c) Integral representations

If $\mathfrak{Re}\,\nu > -1/2$, then the Struve function $\boldsymbol{H}_\nu(z)$ has the integral representation

$$\boldsymbol{H}_\nu(z) = \frac{2\,(z/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu+1/2)}\int_0^1 (1-t^2)^{\nu-1/2}\sin(zt)\,\mathrm{d}t. \tag{A.241}$$

Under the same condition, it admits also the integral representations

$$\boldsymbol{H}_\nu(z) = \frac{2\,(z/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu+1/2)}\int_0^{\pi/2} \sin(z\cos\theta)\sin^{2\nu}\theta\,\mathrm{d}\theta, \tag{A.242}$$

and, for $|\arg z| < \pi/2$, also

$$\boldsymbol{H}_\nu(z) = Y_\nu(z) + \frac{2\,(z/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu+1/2)}\int_0^{\infty} e^{-zt}(1+t^2)^{\nu-1/2}\,\mathrm{d}t. \tag{A.243}$$

In particular, it holds that

$$\boldsymbol{H}_0(z) = \frac{1}{\pi}\int_0^{\pi}\sin(z\sin\theta)\,\mathrm{d}\theta = \frac{2}{\pi}\int_0^{\pi/2}\sin(z\cos\theta)\,\mathrm{d}\theta, \tag{A.244}$$

and

$$\boldsymbol{H}_1(z) = \frac{z}{\pi}\int_0^{\pi}\sin(z\sin\theta)\cos^2\theta\,\mathrm{d}\theta = \frac{2z}{\pi}\int_0^{\pi/2}\sin(z\cos\theta)\sin^2\theta\,\mathrm{d}\theta. \tag{A.245}$$

d) Recurrence relations

The Struve function $\boldsymbol{H}_\nu(z)$ satisfies the recurrence relations

$$\boldsymbol{H}_{\nu-1}(z) + \boldsymbol{H}_{\nu+1}(z) = \frac{2\nu}{z}\boldsymbol{H}_\nu(z) + \frac{(z/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu+3/2)}, \tag{A.246}$$

$$\boldsymbol{H}_{\nu-1}(z) - \boldsymbol{H}_{\nu+1}(z) = 2\,\frac{\mathrm{d}\boldsymbol{H}_\nu}{\mathrm{d}z}(z) - \frac{(z/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu+3/2)}, \tag{A.247}$$

$$\frac{\mathrm{d}\boldsymbol{H}_0}{\mathrm{d}z}(z) = \frac{2}{\pi} - \boldsymbol{H}_1(z) = \boldsymbol{H}_{-1}(z). \tag{A.248}$$

For the derivatives it also holds that

$$\frac{\mathrm{d}}{\mathrm{d}z}\big\{z^\nu\boldsymbol{H}_\nu(z)\big\} = z^\nu\boldsymbol{H}_{\nu-1}(z), \tag{A.249}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\big\{z^{-\nu}\boldsymbol{H}_\nu(z)\big\} = \frac{1}{\sqrt{\pi}\,2^\nu\Gamma(\nu+3/2)} - z^{-\nu}\boldsymbol{H}_{\nu+1}(z). \tag{A.250}$$

e) Special properties

For an integer $n \geq 0$ holds

$$\boldsymbol{H}_{-n-1/2}(z) = (-1)^n J_{n+1/2}(z). \tag{A.251}$$

Special values are

$$\boldsymbol{H}_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\,(1-\cos z), \tag{A.252}$$

$$\boldsymbol{H}_{3/2}(z) = \sqrt{\frac{z}{2\pi}}\left(1+\frac{2}{z^2}\right) - \sqrt{\frac{2}{\pi z}}\left(\sin z + \frac{\cos z}{z}\right). \tag{A.253}$$

Struve functions can be be also expanded in terms of Bessel functions according to

$$\boldsymbol{H}_0(z) = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{J_{2m+1}(z)}{2m+1}, \tag{A.254}$$

$$\boldsymbol{H}_1(z) = \frac{2}{\pi} - \frac{2}{\pi}J_0(z) + \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{J_{2m}(z)}{4m^2-1}. \tag{A.255}$$

f) Integrals

The Struve function $\boldsymbol{H}_0(z)$ satisfies

$$\int_z^{\infty} t^{-1}\boldsymbol{H}_0(t)\,\mathrm{d}t = \frac{\pi}{2} - \frac{2}{\pi}\left\{z - \frac{z^3}{1^2\cdot 3^2\cdot 3} + \frac{z^5}{1^2\cdot 3^2\cdot 5^2\cdot 5} - \dots\right\}, \tag{A.256}$$

and in particular

$$\int_0^{\infty} t^{-1}\boldsymbol{H}_0(t)\,\mathrm{d}t = \frac{\pi}{2}. \tag{A.257}$$

Its primitive is given by

$$\int_0^{z} \boldsymbol{H}_0(t)\,\mathrm{d}t = \frac{\pi}{2}\left\{\frac{z^2}{2} - \frac{z^4}{1^2\cdot 3^2\cdot 4} + \frac{z^6}{1^2\cdot 3^2\cdot 5^2\cdot 6} - \dots\right\}. \tag{A.258}$$

We have also that

$$\int_z^{\infty} t^{-2}\boldsymbol{H}_1(t)\,\mathrm{d}t = \frac{1}{2z}\boldsymbol{H}_1(t) + \frac{1}{2}\int_z^{\infty} t^{-1}\boldsymbol{H}_0(t)\,\mathrm{d}t. \tag{A.259}$$

For higher orders we have

$$\int_0^{z} t^{-\nu}\boldsymbol{H}_{\nu+1}(t)\,\mathrm{d}t = \frac{z}{\sqrt{\pi}\,2^{\nu}\Gamma(\nu+3/2)} - z^{-\nu}\boldsymbol{H}_{\nu}(z). \tag{A.260}$$

If $|\mathfrak{Re}\,\mu| < 1$ and $\mathfrak{Re}\,\nu > \mathfrak{Re}\,\mu - 3/2$, then

$$\int_0^{\infty} t^{\mu-\nu-1}\boldsymbol{H}_{\nu}(t)\,\mathrm{d}t = \frac{\Gamma(\mu/2)\,2^{\mu-\nu-1}\tan(\mu\pi/2)}{\Gamma(\nu-\mu/2+1)}. \tag{A.261}$$

If $\mathfrak{Re}\,\nu > -1/2$, then we have also that

$$\int_0^z t^{\nu+1}\boldsymbol{H}_{\nu+1}(t)\,\mathrm{d}t = (2\nu+1)\int_0^z t^{\nu}\boldsymbol{H}_{\nu}(t)\,\mathrm{d}t - z^{\nu+1}\boldsymbol{H}_{\nu}(t) + \frac{z^{2\nu+2}}{(\nu+1)\,2^{\nu+1}\sqrt{\pi}\,\Gamma(\nu+3/2)}. \tag{A.262}$$

g) Asymptotic expansions for large arguments

The Struve functions behave asymptotically for large arguments, as $|z| \to \infty$ and considering $|\arg z| < \pi$, as

$$\boldsymbol{H}_{\nu}(z) - Y_{\nu}(z) = \frac{1}{\pi}\sum_{m=0}^{n-1}\frac{\Gamma(n+1/2)}{\Gamma(\nu-m+1/2)}\left(\frac{2}{z}\right)^{\nu-2m-1} + R_n, \tag{A.263}$$

where $R_n = \mathcal{O}\big(|z|^{\nu-2n-1}\big)$. If ν is real, z positive, and $n+1/2-\nu \geq 0$, then the remainder after n terms is of the same sign and numerically less than the first term neglected. In particular, for $|\arg z| < \pi$, it holds that

$$\boldsymbol{H}_0(z) - Y_0(z) \sim \frac{2}{\pi}\left\{\frac{1}{z} - \frac{1}{z^3} + \frac{1^2\cdot 3^2}{z^5} - \frac{1^2\cdot 3^2\cdot 5^2}{z^7} + \ldots\right\}, \tag{A.264}$$

and

$$\boldsymbol{H}_1(z) - Y_1(z) \sim \frac{2}{\pi}\left\{1 + \frac{1}{z^2} - \frac{1^2\cdot 3}{z^4} + \frac{1^2\cdot 3^2\cdot 5}{z^6} - \ldots\right\}. \tag{A.265}$$

For primitives of $\boldsymbol{H}_0(z)$ we have also, for $|\arg z| < \pi$, that

$$\int_0^z \big\{\boldsymbol{H}_0(t) - Y_0(t)\big\}\,\mathrm{d}t - \frac{2}{\pi}\big\{\ln(2z) + \gamma\big\} \sim \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}(2m)!(2m-1)!}{(m!)^2(2z)^{2m}}, \tag{A.266}$$

and

$$\int_z^{\infty} t^{-1}\big\{\boldsymbol{H}_0(t) - Y_0(t)\big\}\,\mathrm{d}t \sim \frac{2}{\pi z}\sum_{m=0}^{\infty}\frac{(-1)^m\{(2m)!\}^2}{(m!)^2(2m+1)(2z)^{2m}}, \tag{A.267}$$

where γ denotes Euler's constant (A.43).

A.2.8 Legendre functions

a) Differential equation and definition

Legendre functions are special functions that appear in many mathematical and physical situations. They receive their name from the French mathematician Adrien-Marie Legendre (1752–1833). Some references for them are Abramowitz & Stegun (1972), Arfken & Weber (2005), Courant & Hilbert (1966), Erdélyi (1953), Jackson (1999), Jahnke & Emde (1945), Magnus & Oberhettinger (1954), and Morse & Feshbach (1953), and likewise Spiegel & Liu (1999), Sommerfeld (1949), and Weisstein (2002). We use the convention $z = x + iy$, where x, y are reals, and in particular, x always means a real number in the interval $-1 \leq x \leq 1$ with $\cos\theta = x$, where θ is likewise a real number. We consider also $\nu \in \mathbb{C}$ unrestricted and n a positive integer or zero.

Legendre functions of degree ν are the solutions of the Legendre differential equation

$$(1-z^2)\frac{\mathrm{d}^2 P}{\mathrm{d}z^2}(z) - 2z\frac{\mathrm{d}P}{\mathrm{d}z}(z) + \nu(\nu+1)P(z) = 0, \tag{A.268}$$

which can be also rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{(1-z^2)\frac{\mathrm{d}P}{\mathrm{d}z}(z)\right\} + \nu(\nu+1)P(z) = 0. \tag{A.269}$$

The Legendre differential equation has nonessential singularities at $z = 1, -1$, and ∞. Since the Legendre differential equation is a second-order ordinary differential equation, it has two linearly independent solutions. A solution $P_\nu(z)$, which is regular at finite points, is called a Legendre function of the first kind, while a solution $Q_\nu(z)$, which is singular at the points $z = \pm 1$, is called a Legendre function of the second kind.

For an integer degree $\nu = n$ ($n = 0, 1, 2, \ldots$), the Legendre function of the first kind reduces to a polynomial $P_n(z)$, known as the Legendre polynomial. It is a polynomial of n-th degree, and can be represented by the Rodrigues formula

$$P_n(z) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left\{(z^2-1)^n\right\}, \tag{A.270}$$

which is named after the French banker, mathematician, and social reformer Benjamin Olinde Rodrigues (1795–1851).

In a similar way, for an integer degree $\nu = n$ ($n \in \mathbb{N}_0$) and for all z that do not lie on the real line segment $[-1, 1]$, we can represent the Legendre function of the second kind by

$$Q_n(z) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left\{(z^2-1)^n \ln\left(\frac{z+1}{z-1}\right)\right\} - \frac{1}{2}P_n(z)\ln\left(\frac{z+1}{z-1}\right), \tag{A.271}$$

which can be rewritten as

$$Q_n(z) = \frac{1}{2}P_n(z)\ln\left(\frac{z+1}{z-1}\right) - W_{n-1}(z), \tag{A.272}$$

where

$$W_{n-1}(z) = \sum_{m=1}^{n}\frac{1}{m}P_{m-1}(z)P_{n-m}(z), \qquad n \geq 1, \tag{A.273}$$

$$W_{-1}(z) = 0. \tag{A.274}$$

The function $Q_n(z)$ is single-valued and has a branch cut on the real axis between the branch points -1 and $+1$. Values of $Q_n(z)$ on the cut line are customarily assigned by the relation

$$Q_n(x) = \frac{1}{2}\left\{Q_n(x+i0) + Q_n(x-i0)\right\}, \qquad -1 < x < 1, \tag{A.275}$$

where the arithmetic average approaches from both the positive imaginary side and the negative imaginary side. Thus, in formulae like (A.271) and (A.272) we have only to replace

$$\ln\left(\frac{z+1}{z-1}\right) \quad \text{by} \quad \ln\left(\frac{1+x}{1-x}\right) \tag{A.276}$$

to obtain valid expressions that hold on the cut line $-1 < x < 1$. For example, (A.272) has to be replaced in this case by

$$Q_n(x) = \frac{1}{2}P_n(x)\ln\left(\frac{1+x}{1-x}\right) - W_{n-1}(x) \qquad -1 < x < 1. \tag{A.277}$$

For a non-integer degree ν, the Legendre function of the first kind P_ν can be defined by means of the Schläfli integral

$$P_\nu(z) = \frac{1}{2\pi i}\oint_C \frac{(t^2-1)^\nu}{2^\nu(t-z)^{\nu+1}}\,\mathrm{d}t, \tag{A.278}$$

where C is a simple complex integration contour around the points $t = z$ and $t = 1$, but not crossing the cut line -1 to $-\infty$. This integral is named after the Swiss mathematician Ludwig Schläfli (1814–1895), who among other important contributions gave the integral representations of the Bessel and gamma functions.

The Legendre function of the second kind Q_ν, for a non-integer degree ν, is obtained from the Schläfli integral, and defined by

$$Q_\nu(z) = \frac{-1}{4i\sin(\nu\pi)}\oint_D \frac{(t^2-1)^\nu}{2^\nu(z-t)^{\nu+1}}\,\mathrm{d}t, \qquad \nu \notin \mathbb{Z}, \tag{A.279}$$

where the integration contour D has the form of a figure eight and it does not enclose the point $t = z$. Furthermore, we have that $\arg(t^2-1) = 0$ on the intersection of the integration contour D with the positive real axis at the right of $t = 1$. The function Q_ν thus obtained is regular and single-valued in the complex z-plane which has been cut along the real axis from $+1$ to $-\infty$. In case that the real part of $\nu + 1$ is positive, we can contract the path of integration and write (A.279) as

$$Q_\nu(z) = \frac{1}{2^{\nu+1}}\int_{-1}^{1} \frac{(1-t^2)^\nu}{(z-t)^{\nu+1}}\,\mathrm{d}t, \tag{A.280}$$

being this formula now applicable for nonnegative integral ν also.

b) Properties on the complex plane

The Legendre functions P_ν satisfy, for all $z \in \mathbb{C}$ and for unrestricted degree ν, the recurrence relations

$$(2\nu+1)zP_\nu(z) = (\nu+1)P_{\nu+1}(z) + \nu P_{\nu-1}(z), \tag{A.281}$$

$$(2\nu+1)P_\nu(z) = \frac{\mathrm{d}P_{\nu+1}}{\mathrm{d}z}(z) - \frac{\mathrm{d}P_{\nu-1}}{\mathrm{d}z}(z), \tag{A.282}$$

$$(\nu+1)P_\nu(z) = \frac{\mathrm{d}P_{\nu+1}}{\mathrm{d}z}(z) - z\frac{\mathrm{d}P_\nu}{\mathrm{d}z}(z), \tag{A.283}$$

$$\nu P_\nu(z) = z\frac{\mathrm{d}P_\nu}{\mathrm{d}z}(z) - \frac{\mathrm{d}P_{\nu-1}}{\mathrm{d}z}(z), \tag{A.284}$$

$$(z^2-1)\frac{\mathrm{d}P_\nu}{\mathrm{d}z}(z) = \nu zP_\nu(z) - \nu P_{\nu-1}(z), \tag{A.285}$$

$$(z^2-1)\frac{\mathrm{d}P_\nu}{\mathrm{d}z}(z) = (\nu+1)P_{\nu+1}(z) - (\nu-1)zP_\nu(z), \tag{A.286}$$

which hold also for Q_ν and for any linear combination of P_ν and Q_ν. In particular, they hold also on the cut line $-1 < x < 1$. With respect to the degree ν we have the identities

$$P_\nu(z) = P_{-\nu-1}(z), \tag{A.287}$$

$$Q_\nu(z) = Q_{-\nu-1}(z). \tag{A.288}$$

c) Properties on the cut line

On the cut line $-1 < x < 1$ and for an integer degree n, the Legendre polynomials P_n satisfy the recurrence relations

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x), \tag{A.289}$$

$$(2n+1)P_n(x) = \frac{\mathrm{d}P_{n+1}}{\mathrm{d}x}(x) - \frac{\mathrm{d}P_{n-1}}{\mathrm{d}x}(x), \tag{A.290}$$

$$(n+1)P_n(x) = \frac{\mathrm{d}P_{n+1}}{\mathrm{d}x}(x) - x\frac{\mathrm{d}P_n}{\mathrm{d}x}(x), \tag{A.291}$$

$$nP_n(x) = x\frac{\mathrm{d}P_n}{\mathrm{d}x}(x) - \frac{\mathrm{d}P_{n-1}}{\mathrm{d}x}(x), \tag{A.292}$$

$$(x^2-1)\frac{\mathrm{d}P_n}{\mathrm{d}x}(x) = nxP_n(x) - nP_{n-1}(x), \tag{A.293}$$

$$(x^2-1)\frac{\mathrm{d}P_n}{\mathrm{d}x}(x) = (n+1)P_{n+1}(x) - (n-1)xP_n(x), \tag{A.294}$$

which holds also for Q_n and for any linear combination of P_n and Q_n. The Legendre functions P_n and Q_n on the cut line are represented graphically in Figure A.9 for some integer orders. We have similarly for negative arguments that

$$P_n(-x) = (-1)^n P_n(x), \tag{A.295}$$

$$Q_n(-x) = (-1)^{n+1} Q_n(x). \tag{A.296}$$

With respect to the degree n we have the identities

$$P_n(x) = P_{-n-1}(x), \tag{A.297}$$

$$Q_n(x) = Q_{-n-1}(x). \tag{A.298}$$

A generating function for the Legendre polynomials is given by

$$\frac{1}{\sqrt{1-2tx+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \qquad |t| < 1. \tag{A.299}$$

Another generating function is given by

$$e^{tx} J_0\Big(t\sqrt{1-x^2}\Big) = \sum_{n=0}^{\infty} \frac{P_n(x)}{n!}\, t^n, \tag{A.300}$$

where $J_0(x)$ is a zeroth order Bessel function of the first kind (vid. Subsection A.2.4). Expanding the Rodrigues formula (A.270) yields the sum formula

$$P_n(z) = \frac{1}{2^n} \sum_{m=0}^{[n/2]} \frac{(-1)^m (2n-2m)!}{m!\,(n-m)!\,(n-2m)!}\, z^{n-2m}, \tag{A.301}$$

where $[r]$ denotes the floor function of r, i.e., the highest integer smaller than r. Another sum formula is

$$P_n(z) = \frac{1}{2^n} \sum_{m=0}^{n} \left(\frac{n!}{m!\,(n-m)!} \right)^2 (z-1)^{n-m}(z+1)^m. \tag{A.302}$$

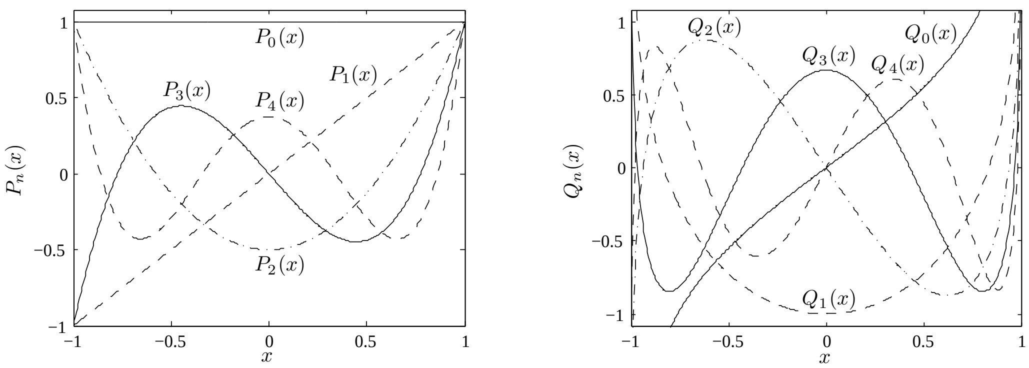
The Legendre polynomials are orthogonal in the interval $[-1, 1]$, and satisfy the relation

$$\int_{-1}^{1} P_n(x) P_m(x)\,\mathrm{d}x = \frac{2}{2n+1}\,\delta_{nm}, \tag{A.303}$$

where δ_{nm} denotes the delta of Kronecker,

$$\delta_{nm} = \begin{cases} 1 & \text{if } \ n = m, \\ 0 & \text{if } \ n \neq m, \end{cases} \tag{A.304}$$

named after the German mathematician and logician Leopold Kronecker (1823–1891).



(a) Legendre polynomials $P_n(x)$, $n = 0, 1, 2, 3, 4$ (b) Legendre functions $Q_n(x)$, $n = 0, 1, 2, 3, 4$

FIGURE A.9. Legendre functions on the cut line.

Some special values of the Legendre polynomials P_n are

$$P_n(1) = 1, \tag{A.305}$$

$$P_n(-1) = (-1)^n. \tag{A.306}$$

On the origin it holds that

$$P_n(0) = \begin{cases} (-1)^{n/2}\,\dfrac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots n} & \text{if } \ n \ \text{even}, \\ 0 & \text{if } \ n \ \text{odd}. \end{cases} \tag{A.307}$$

We have also the bound

$$|P_n(x)| \leq 1, \qquad -1 < x < 1. \tag{A.308}$$

For the Legendre function of the second kind Q_n we have the special values

$$Q_n(1) = \infty, \tag{A.309}$$

$$Q_n(\infty) = 0. \tag{A.310}$$

On the origin it holds that

$$Q_n(0) = \begin{cases} (-1)^{(n+1)/2} \dfrac{2 \cdot 4 \cdot 6 \cdots (n-1)}{1 \cdot 3 \cdot 5 \cdot 7 \cdots n} & \text{if } n \text{ odd}, \\ 0 & \text{if } n \text{ even}, \end{cases} \tag{A.311}$$

being, in particular, $Q_1(0) = -1$.

d) Explicit expressions

Some explicit expressions of Legendre polynomials, for $0 \leq n \leq 4$ and considering respectively $-1 \leq x \leq 1$ and $\cos\theta = x$, are

$$P_0(x) = 1, \qquad P_0(\cos\theta) = 1, \tag{A.312}$$

$$P_1(x) = x, \qquad P_1(\cos\theta) = \cos\theta, \tag{A.313}$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1), \qquad P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1), \tag{A.314}$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x), \qquad P_3(\cos\theta) = \frac{1}{2}\cos\theta(5\cos^2\theta - 3), \tag{A.315}$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \quad P_4(\cos\theta) = \frac{1}{8}(35\cos^4\theta - 30\cos^2\theta + 3). \tag{A.316}$$

For the Legendre functions of the second kind, when considering the values on the branch cut $-1 < x < 1$, we have the expressions

$$Q_0(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \tag{A.317}$$

$$Q_1(x) = \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1, \tag{A.318}$$

$$Q_2(x) = \frac{1}{4}(3x^2 - 1)\ln\left(\frac{1+x}{1-x}\right) - \frac{3x}{2}, \tag{A.319}$$

$$Q_3(x) = \frac{1}{4}(5x^3 - 3x)\ln\left(\frac{1+x}{1-x}\right) - \frac{5x^2}{2} + \frac{2}{3}, \tag{A.320}$$

$$Q_4(x) = \frac{1}{16}(35x^4 - 30x^2 + 3)\ln\left(\frac{1+x}{1-x}\right) - \frac{35x^3}{8} + \frac{55x}{24}. \tag{A.321}$$

We remark that formulae (A.312)–(A.316) can be extended straightforwardly from x to $z \in \mathbb{C}$. To extend formulae (A.317)–(A.321) in such a way, though, we have to consider the replacement done in (A.276).

A.2.9 Associated Legendre functions

a) Differential equation and definition

The associated Legendre functions or Legendre functions of higher order are special functions that can be regarded as a generalization of the Legendre functions (vid. Subsection A.2.8). They are also important for many mathematical and physical situations. Some references for them are Abramowitz & Stegun (1972), Arfken & Weber (2005), Courant & Hilbert (1966), Erdélyi (1953), Jackson (1999), Jahnke & Emde (1945), Magnus & Oberhettinger (1954), Morse & Feshbach (1953), Sommerfeld (1949), Spiegel & Liu (1999), and Weisstein (2002). We use the convention $z = x + iy$, where x, y are reals, and in particular, x always means a real number in the interval $-1 \leq x \leq 1$ with $\cos\theta = x$, where θ is likewise a real number. We consider also $\nu, \mu \in \mathbb{C}$ unrestricted and n, m positive integers or zero. We follow mainly the notation of Abramowitz & Stegun (1972), Jackson (1999), and Magnus & Oberhettinger (1954).

Associated Legendre functions of degree ν and order μ are the solutions of the associated Legendre differential equation

$$(1 - z^2)\frac{\mathrm{d}^2 P}{\mathrm{d}z^2}(z) - 2z\frac{\mathrm{d}P}{\mathrm{d}z}(z) + \left(\nu(\nu+1) + \frac{\mu^2}{1-z^2}\right) P(z) = 0, \tag{A.322}$$

which can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{(1 - z^2)\frac{\mathrm{d}P}{\mathrm{d}z}(z)\right\} + \left(\nu(\nu+1) + \frac{\mu^2}{1-z^2}\right) P(z) = 0. \tag{A.323}$$

The associated Legendre differential equation has nonessential singularities at $z = 1, -1$ and ∞, which are ordinary branch points. Since the associated Legendre differential equation is a second-order ordinary differential equation, it has two linearly independent solutions. A solution $P_\nu^\mu(z)$, which is regular at finite points, is called an associated Legendre function of the first kind, while a solution $Q_\nu^\mu(z)$, which is singular at the points $z = \pm 1$, is called an associated Legendre function of the second kind.

For integer degree $\nu = n$ ($n \in \mathbb{N}_0$), integer order $\mu = m$ ($m \in \mathbb{N}_0$), and for all z that do not lie on the real line segment $[-1, 1]$, we can represent the associated Legendre functions of the first and second kind by the Rodrigues' formulae

$$P_n^m(z) = (z^2 - 1)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}z^m}P_n(z) = \frac{(z^2-1)^{m/2}}{2^n n!}\frac{\mathrm{d}^{m+n}}{\mathrm{d}z^{m+n}}\left\{(z^2-1)^n\right\}, \tag{A.324}$$

and

$$Q_n^m(z) = (z^2 - 1)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}z^m}Q_n(z), \tag{A.325}$$

where $P_n(z)$ and $Q_n(z)$ denote respectively the Legendre functions of the first and second kind. Both functions, $P_n^m(z)$ and $Q_n^m(z)$, are single-valued and have a branch cut on the real axis between the branch points -1 and $+1$. The appearing square roots have to be considered in such a way that

$$(z^2 - 1)^{m/2} = (z-1)^{m/2}(z+1)^{m/2}, \tag{A.326}$$

where

$$|\arg(z \pm 1)| < \pi, \qquad |\arg(z)| < \pi. \tag{A.327}$$

The values of $P_n^m(z)$ and $Q_n^m(z)$ on the cut line $-1 < x < 1$ are customarily assigned by the relations

$$P_n^m(x) = \frac{1}{2}\left\{e^{i\pi m/2}P_n^m(x+i0) + e^{-i\pi m/2}P_n(x-i0)\right\}, \tag{A.328}$$

and

$$Q_n^m(x) = \frac{1}{2} e^{-i\pi m} \big\{ e^{-i\pi m/2} Q_n^m(x+i0) + e^{i\pi m/2} Q_n(x-i0) \big\}. \tag{A.329}$$

These formulae are obtained through the replacement of $z-1$ by $(1-x)e^{\pm i\pi}$, (z^2-1) by $(1-x^2)e^{\pm i\pi}$, and $z+1$ by $1+x$, for $z = x \pm i0$. Thus, on the cut line $-1 < x < 1$, formulae (A.324) and (A.325) have to be taken as

$$P_n^m(x) = (-1)^m (1-x^2)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}x^m} P_n(x), \tag{A.330}$$

and

$$Q_n^m(x) = (-1)^m (1-x^2)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}x^m} Q_n(x). \tag{A.331}$$

We remark that some authors define the associated Legendre functions on the cut line omitting the factor $(-1)^m$.

Further extensions of the associated Legendre functions for a complex degree ν or a complex order μ can be performed by adapting the Schläfli integrals (A.278) and (A.279). They can be also expressed in terms of hypergeometric functions.

b) Properties on the complex plane

The associated Legendre functions P_ν^μ satisfy, for all $z \in \mathbb{C}$ outside the cut line $[-1,1]$, and for unrestricted degree ν and order μ, the recurrence relations

$$(2\nu+1) z P_\nu^\mu(z) = (\nu-\mu+1) P_{\nu+1}^\mu(z) + (\nu+\mu) P_{\nu-1}^\mu(z), \tag{A.332}$$

$$(z^2-1)^{1/2} P_\nu^{\mu+1}(z) = (\nu-\mu) z P_\nu^\mu(z) - (\nu+\mu) P_{\nu-1}^\mu(z), \tag{A.333}$$

$$(z^2-1) \frac{\mathrm{d}P_\nu^\mu}{\mathrm{d}z}(z) = (\nu+\mu)(\nu-\mu+1)(z^2-1)^{1/2} P_\nu^{\mu-1}(z) - \mu z P_\nu^\mu(z), \tag{A.334}$$

$$(z^2-1) \frac{\mathrm{d}P_\nu^\mu}{\mathrm{d}z}(z) = \nu z P_\nu^\mu(z) - (\nu+\mu) P_{\nu-1}^\mu(z), \tag{A.335}$$

$$P_{\nu+1}^\mu(z) = P_{\nu-1}^\mu(z) + (2\nu+1)(z^2-1)^{1/2} P_\nu^{\mu-1}(z), \tag{A.336}$$

$$(z^2-1)^{1/2} P_\nu^{\mu+1}(z) = (\nu+\mu)(\nu-\mu+1)(z^2-1)^{1/2} P_\nu^{\mu-1}(z) - 2\mu z P_\nu^\mu(z), \tag{A.337}$$

which hold also for Q_ν^μ and for any linear combination of P_ν^μ and Q_ν^μ. They hold also on the cut line $-1 < x < 1$, when we replace

$$(z^2-1)^{1/2} \quad \text{by} \quad (1-x^2)^{1/2}. \tag{A.338}$$

The associated Legendre functions of order zero are simply the Legendre functions, i.e.,

$$P_\nu^0(z) = P_\nu(z), \tag{A.339}$$

$$Q_\nu^0(z) = Q_\nu(z). \tag{A.340}$$

With respect to the degree ν we have the identities

$$P_\nu^\mu(z) = P_{-\nu-1}^\mu(z), \tag{A.341}$$

$$Q_\nu^\mu(z) = Q_{-\nu-1}^\mu(z). \tag{A.342}$$

c) Properties on the cut line

For an integer degree n and an integer order m, the associated Legendre functions P_n^m satisfy, on the cut line $-1 < x < 1$, the recurrence relations

$$(2n+1)xP_n^m(x) = (n-m+1)P_{n+1}^m(x) + (n+m)P_{n-1}^m(x), \tag{A.343}$$

$$\sqrt{1-x^2}P_n^{m+1}(x) = (n-m)xP_n^m(x) - (n+m)P_{n-1}^m(x), \tag{A.344}$$

$$(x^2-1)\frac{\mathrm{d}P_n^m}{\mathrm{d}x}(x) = (n+m)(n-m+1)\sqrt{1-x^2}P_n^{m-1}(x) - mxP_n^m(x), \tag{A.345}$$

$$(x^2-1)\frac{\mathrm{d}P_n^m}{\mathrm{d}x}(x) = nxP_n^m(x) - (n+m)P_{n-1}^m(x), \tag{A.346}$$

$$P_{n+1}^m(x) = P_{n-1}^m(x) + (2n+1)\sqrt{1-x^2}P_n^{m-1}(x), \tag{A.347}$$

$$\sqrt{1-x^2}P_n^{m+1}(x) = (n+m)(n-m+1)\sqrt{1-x^2}P_n^{m-1}(x) - 2mxP_n^m(x), \tag{A.348}$$

which hold also for Q_n^m and for any linear combination of P_n^m and Q_n^m. The associated Legendre functions P_n^m and Q_n^m on the cut line are represented graphically in Figure A.10 for some integer orders. On the cut line, the associated Legendre functions of order zero are again the Legendre functions, i.e.,

$$P_n^0(x) = P_n(x), \tag{A.349}$$

$$Q_n^0(x) = Q_n(x). \tag{A.350}$$

With respect to the integer degree n we have the identities

$$P_n^m(x) = P_{-n-1}^m(x), \tag{A.351}$$

$$Q_n^m(x) = Q_{-n-1}^m(x). \tag{A.352}$$

If the order m is higher than the degree n, then the associated Legendre function of the first kind P_n^m is zero, namely

$$P_n^m(x) = 0, \qquad m > n, \tag{A.353}$$

which does not apply to the function Q_n^m. For negative arguments we have that

$$P_n^m(-x) = (-1)^{n+m}P_n^m(x), \tag{A.354}$$

For a negative order $m \in \{0, 1, \ldots, n\}$ it holds that

$$P_n^{-m}(x) = (-1)^m\frac{(n-m)!}{(n+m)!}P_n^m(x), \tag{A.355}$$

$$Q_n^{-m}(x) = (-1)^m\frac{(n-m)!}{(n+m)!}Q_n^m(x). \tag{A.356}$$

Additional identities are

$$P_n^n(x) = (-1)^n\frac{(2n)!}{2^n n!}(1-x^2)^{n/2}, \tag{A.357}$$

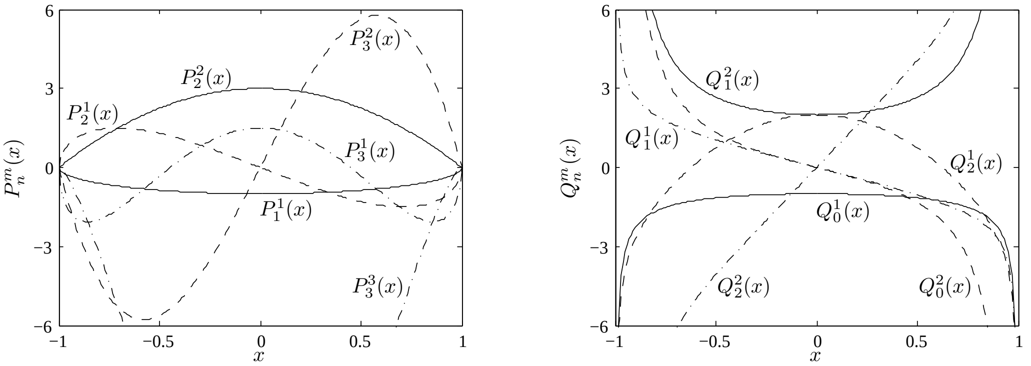
$$P_{n+1}^n(x) = x(2n+1)P_n^n(x), \tag{A.358}$$

$$P_n^{-n}(x) = \frac{1}{2^n n!}(1-x^2)^{n/2}, \tag{A.359}$$

$$P_{n+1}^{-n}(x) = \frac{(-1)^n}{(2n)!} x P_n^n(x). \tag{A.360}$$

A generating function for the associated Legendre functions of the first kind is

$$\frac{(-1)^m (2m)!(1-x^2)^{m/2} t^m}{2^m m!(1-2tx+t^2)^{m+1/2}} = \sum_{n=m}^{\infty} P_n^m(x) t^n, \qquad |t| < 1. \tag{A.361}$$



(a) Associated Legendre functions of the first kind $P_n^m(x)$, for $1 \leq n \leq 3$ and $1 \leq m \leq n$

(b) Associated Legendre functions of the second kind $Q_n^m(x)$, for $0 \leq n \leq 2$ and $m \in \{1, 2\}$

FIGURE A.10. Associated Legendre functions on the cut line.

The associated Legendre functions of the first kind are orthogonal in the interval $[-1, 1]$ with respect to degree, and satisfy the relation

$$\int_{-1}^{1} P_n^m(x) P_l^m(x) \, \mathrm{d}x = \frac{2}{(2n+1)} \frac{(n+m)!}{(n-m)!} \delta_{nl}, \qquad m \in \{0, 1, \ldots, n\}, \tag{A.362}$$

where δ_{nl} denotes the delta of Kronecker. They are also orthogonal in the interval $[-1, 1]$ with respect to order when using the weighting function $(1-x^2)^{-1}$, namely

$$\int_{-1}^{1} \frac{P_n^m(x) P_n^k(x)}{(1-x^2)} \, \mathrm{d}x = \frac{(n+m)!}{m(n-m)!} \delta_{mk}, \qquad m, k \in \{0, 1, \ldots, n\}, \tag{A.363}$$

when m and k are not simultaneously zero.

d) Explicit expressions

Some explicit expressions for associated Legendre functions of the first kind, considering respectively $-1 \leq x \leq 1$ and $\cos\theta = x$, for $1 \leq n \leq 3$ and $1 \leq m \leq n$, are

$$P_1^1(x) = -\sqrt{1-x^2}, \qquad P_1^1(\cos\theta) = -\sin\theta, \tag{A.364}$$

$$P_2^1(x) = -3x\sqrt{1-x^2}, \qquad P_2^1(\cos\theta) = -3\cos\theta\sin\theta, \tag{A.365}$$

$$P_2^2(x) = 3(1-x^2), \qquad P_2^2(\cos\theta) = 3\sin^2\theta, \tag{A.366}$$

$$P_3^1(x) = -\frac{3}{2}(5x^2-1)\sqrt{1-x^2}, \qquad P_3^1(\cos\theta) = -\frac{3}{2}(5\cos^2\theta - 1)\sin\theta, \tag{A.367}$$

$$P_3^2(x) = 15x(1-x^2), \qquad P_3^2(\cos\theta) = 15\cos\theta\sin^2\theta, \tag{A.368}$$

$$P_3^3(x) = -15(1-x^2)^{3/2}, \qquad P_3^3(\cos\theta) = -15\sin^3\theta. \tag{A.369}$$

For the associated Legendre functions of the second kind, considering $0 \le n \le 2$ and $m \in \{1,2\}$, we have that

$$Q_0^1(x) = -\frac{1}{\sqrt{1-x^2}}, \tag{A.370}$$

$$Q_0^2(x) = \frac{2x}{1-x^2}, \tag{A.371}$$

$$Q_1^1(x) = -\frac{1}{2}\sqrt{1-x^2}\ln\left(\frac{1+x}{1-x}\right) - \frac{x}{\sqrt{1-x^2}}, \tag{A.372}$$

$$Q_1^2(x) = \frac{2}{1-x^2}, \tag{A.373}$$

$$Q_2^1(x) = -\frac{3x}{2}\sqrt{1-x^2}\ln\left(\frac{1+x}{1-x}\right) - \frac{3x^2-2}{\sqrt{1-x^2}}, \tag{A.374}$$

$$Q_2^2(x) = \frac{3}{2}(1-x^2)\ln\left(\frac{1+x}{1-x}\right) - \frac{x(3x^2-5)}{1-x^2}. \tag{A.375}$$

We remark that to extend formulae (A.364)–(A.369) from x to $z \in \mathbb{C}$, we have to consider the replacement done in (A.338). For the formulae (A.370)–(A.375), additionally the replacement done in (A.276) has to be taken into account.

A.2.10 Spherical harmonics

a) Differential equation and definition

Spherical harmonics, also known as surface harmonics or tesseral and sectoral harmonics, are special functions that appear when solving Laplace's equation using separation of variables in spherical coordinates. They represent the angular portion of the solution, and are formed by products between trigonometric functions and associated Legendre functions (cf. Subsection A.2.9). The spherical harmonics constitute thus an orthonormal basis over the unit sphere. Some of the references for them are Abramowitz & Stegun (1972), Arfken & Weber (2005), Erdélyi (1953), Jackson (1999), Magnus & Oberhettinger (1954), Nédélec (2001), Sommerfeld (1949), and Weisstein (2002). For the spherical harmonics, we follow mainly the notation of Jackson (1999) and Weisstein (2002).

We consider in $\mathbb{R}^3$ the system of spherical coordinates (r, θ, φ), which is described with the convention normally used in physics, i.e., reversing the roles of θ and φ. Thus, we denote by r the radius ($0 \le r < \infty$), by θ the polar or colatitudinal coordinate ($0 \le \theta \le \pi$), and by φ the azimuthal or longitudinal coordinate ($-\pi < \varphi \le \pi$), as shown in Figure A.11. The spherical coordinates (r, θ, φ) and the cartesian coordinates (x, y, z) are related through

$$r = \sqrt{x^2+y^2+z^2}, \qquad x = r\sin\theta\cos\varphi, \tag{A.376}$$

$$\theta = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right), \qquad y = r\sin\theta\sin\varphi, \tag{A.377}$$

$$\varphi = \arctan\left(\frac{y}{x}\right), \qquad\qquad z = r\cos\theta. \tag{A.378}$$

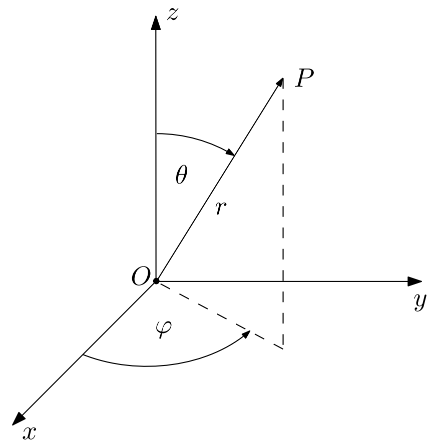


FIGURE A.11. Spherical coordinates.

By considering in $\mathbb{R}^3$ the angular part of Laplace's equation in spherical coordinates, i.e., working on the unit sphere with $r = 1$, we obtain the spherical harmonic differential equation of degree $l = 0, 1, 2, \ldots$, given by

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\,\frac{\partial Y}{\partial\theta}(\theta,\varphi)\right\} + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}(\theta,\varphi) + l(l+1)Y(\theta,\varphi) = 0. \tag{A.379}$$

The solutions of this differential equation are the spherical harmonics

$$Y_l^m(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}\,\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\theta)e^{im\varphi}, \tag{A.380}$$

where $m \in \{-l, -(l-1), \ldots, 0, \ldots, (l-1), l\}$ and $P_l^m(x)$ denotes the associated Legendre function of degree l and order m. Some spherical harmonics are illustrated in Figure A.12.

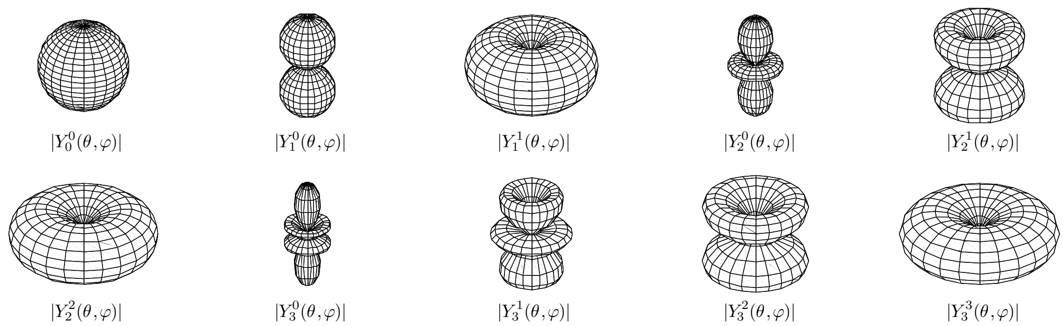


FIGURE A.12. Spherical harmonics in absolute value.

b) Properties

The spherical harmonics form a complete orthogonal set on the surface of the unit sphere in the two indices l, m. Their orthonormality implies that

$$\int_0^{2\pi}\int_0^{\pi} Y_l^m(\theta,\varphi)\,\overline{Y_n^k(\theta,\varphi)}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = \delta_{ln}\delta_{mk}, \tag{A.381}$$

where $\overline{z}$ denotes the complex conjugate of z, and δ_{ln} the delta of Kronecker for the coefficients l and n. For a negative order m it holds that

$$Y_l^{-m}(\theta,\varphi) = (-1)^m\overline{Y_l^m(\theta,\varphi)}. \tag{A.382}$$

Spherical harmonics are bounded by

$$|Y_l^m(\theta,\varphi)| \leq \sqrt{\frac{2l+1}{4\pi}}. \tag{A.383}$$

Some particular cases of spherical harmonics are

$$Y_l^l(\theta,\varphi) = \frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}\,\sin^l\theta\, e^{il\varphi}, \tag{A.384}$$

$$Y_l^0(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}}\,P_l(\cos\theta), \tag{A.385}$$

$$Y_l^{-l}(\theta,\varphi) = \frac{1}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}\,\sin^l\theta\, e^{-il\varphi}, \tag{A.386}$$

where $P_l(x)$ denotes the Legendre polynomial of degree l.

c) Addition theorem

We consider two different directions (θ_1,φ_1) and (θ_2,φ_2) in the spherical coordinate system on the unit sphere, which are separated by an angle β, as shown in Figure A.13. These angles satisfy the trigonometric identity

$$\cos\beta = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2). \tag{A.387}$$

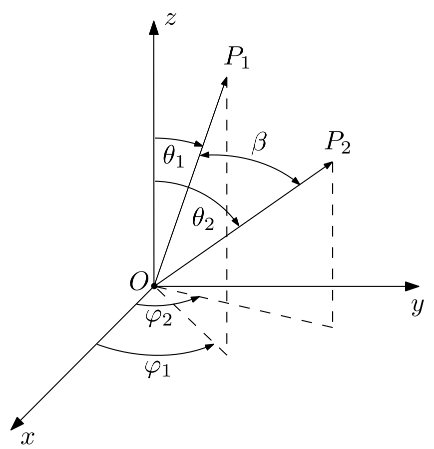


FIGURE A.13. Angles for the addition theorem of spherical harmonics.

The addition theorem for spherical harmonics asserts that

$$P_n(\cos\beta) = \frac{4\pi}{2n+1}\sum_{m=-n}^{n}(-1)^m Y_n^m(\theta_1,\varphi_1)\,Y_n^{-m}(\theta_2,\varphi_2), \tag{A.388}$$

or, equivalently,

$$P_n(\cos\beta) = \frac{4\pi}{2n+1}\sum_{m=-n}^{n} Y_n^m(\theta_1,\varphi_1)\,\overline{Y_n^m(\theta_2,\varphi_2)}. \tag{A.389}$$

In terms of the associated Legendre functions the addition theorem is

$$\begin{aligned} P_n(\cos\beta) &= P_n(\cos\theta_1)P_n(\cos\theta_2) \\ &\quad + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta_1)P_n^m(\cos\theta_2)\cos\big(m(\varphi_1-\varphi_2)\big), \end{aligned} \tag{A.390}$$

being the expression (A.387) the particular case of the theorem when $n=1$.

d) Explicit expressions

Some explicit expressions of spherical harmonics are

$$Y_0^0(\theta,\varphi) = \frac{1}{\sqrt{4\pi}}, \qquad Y_1^{-1}(\theta,\varphi) = \sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{-i\varphi}, \tag{A.391}$$

$$Y_1^0(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\,\cos\theta, \qquad Y_1^1(\theta,\varphi) = -\sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{i\varphi}, \tag{A.392}$$

$$Y_2^{-2}(\theta,\varphi) = \sqrt{\frac{15}{32\pi}}\,\sin^2\theta\, e^{-2i\varphi}, \qquad Y_2^{-1}(\theta,\varphi) = \sqrt{\frac{15}{8\pi}}\,\sin\theta\cos\theta\, e^{-i\varphi}, \tag{A.393}$$

$$Y_2^0(\theta,\varphi) = \sqrt{\frac{5}{16\pi}}\,(3\cos^2\theta-1), \qquad Y_2^1(\theta,\varphi) = -\sqrt{\frac{15}{8\pi}}\,\sin\theta\cos\theta\, e^{i\varphi}, \tag{A.394}$$

$$Y_2^2(\theta,\varphi) = \sqrt{\frac{15}{32\pi}}\,\sin^2\theta\, e^{2i\varphi}, \qquad Y_3^{-3}(\theta,\varphi) = \sqrt{\frac{35}{64\pi}}\,\sin^3\theta\, e^{-3i\varphi}, \tag{A.395}$$

$$Y_3^{-2}(\theta,\varphi) = \sqrt{\frac{105}{32\pi}}\,\sin^2\theta\cos\theta\, e^{-2i\varphi}, \tag{A.396}$$

$$Y_3^{-1}(\theta,\varphi) = \sqrt{\frac{21}{64\pi}}\,\sin\theta(5\cos^2\theta-1)\, e^{-i\varphi}, \tag{A.397}$$

$$Y_3^0(\theta,\varphi) = \sqrt{\frac{7}{16\pi}}\,(5\cos^3\theta-3\cos\theta), \tag{A.398}$$

$$Y_3^1(\theta,\varphi) = -\sqrt{\frac{21}{64\pi}}\,\sin\theta(5\cos^2\theta-1)\, e^{i\varphi}, \tag{A.399}$$

$$Y_3^2(\theta,\varphi) = -\sqrt{\frac{105}{32\pi}}\,\sin^2\theta\cos\theta\, e^{2i\varphi}, \tag{A.400}$$

$$Y_3^3(\theta,\varphi) = -\sqrt{\frac{35}{64\pi}}\,\sin^3\theta\, e^{3i\varphi}, \tag{A.401}$$