Linear acoustic theory

The linear acoustic theory is concerned with the propagation of sound waves considered as small perturbations in a fluid or gas. Consequently the equations of acoustics are obtained by linearization of the equations for the motion of fluids. The two main media for the propagation and scattering of sound waves are air and water (underwater acoustics). A third important medium with properties close to those of water is the human body, i.e., biological tissue (ultrasound). We are herein interested in obtaining the differential equations that govern the acoustic wave propagation, whose linearization yields the scalar wave equation of acoustics. By considering simple-harmonic waves for the wave equation, we obtain finally the Helmholtz equation. When the frequency is zero, this equation turns into the Poisson or the Laplace equation. The corresponding boundary conditions are also developed, in particular the impedance boundary condition. A good and complete reference for the linear acoustic theory is the article by Morse & Ingard (1961), which is closely followed herein. Other references are DeSanto (1992), Elmore & Heald (1969), Howe (2007), Kinsler, Frey, Coppens & Sanders (1999), Kress (2002), and Strutt (1877).

Acoustic motion is, almost by definition, a perturbation. The slow compressions and expansions of materials, studied in thermodynamics, are not thought of as acoustical phenomena, nor is the steady flow of air usually called sound. It is only when the compression is irregular enough so that overall thermodynamic equilibrium may not be maintained, or when the steady flow is deflected by some obstacles so that wave motion is produced, that we consider part of the motion to be acoustical. In other words, we think of sound as a by-product, wanted or unwanted, of slower, more regular mechanical processes. And, whether the generating process be the motion of a violin bow or the rush of gas from a turbo-jet, the part of motion we call sound usually carries but a minute fraction of the energy present in the primary process, which is not considered to be acoustical.

This definition of acoustical motion as being the small, irregular part of some larger, more regular motion of matter, gives rise to difficulties when we try to develop a consistent mathematical representation of its behavior. When the irregularities are large enough, for example, there is no clear-cut way of separating the acoustical from the non-acoustical part of the motion. In fact, only in the cases where the non-steady motions are first-order perturbations of some larger, steady-state motion can one hope to make a self-consistent definition which separates acoustic from non-acoustic motion and, even here, there are ambiguities in the case of some types of near field. Thus it is not surprising that the earliest work in acoustic theory, and even now a vast quantity, has to do with situations where the acoustical part of the motion is small enough so that linear approximations can be used. These are our cases of interest in this thesis. Strictly speaking, the equations to be discussed here are valid only when the acoustical component of the motion is ”sufficiently” small, but it is only in this limit that we can unequivocably separate the total motion into its acoustical and its non-acoustical parts.

Still another limitation of the validity of acoustical theory is imposed by the atomicity of matter. The thermal motions of individual molecules, for instance, are not usually representable by the equations of sound. These equations are meant to represent the average behavior of large assemblies of molecules. Thus, for instance, when we speak of an element of volume we implicitly assume that its dimensions, while being smaller than any wavelength of acoustical motion present, are large compared to inter-molecular spacings.

A.11.1 Differential equations

a) Basic equations of motion

Considering the fluid as a continuous medium, two points of view can be adopted in describing its motion. In the first, the Lagrangian motion, the history of each individual fluid element, or particle, is recorded in terms of its position $\boldsymbol{x}$ as a function of the time t. Each particle is identified by means of a parameter, which is usually chosen to be the position vector $\boldsymbol{x}_0$ of the element at $t = 0$. The Lagrangian description of fluid motion is expressed by the set of functions $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{x}_0, t)$.

In the second, or Eulerian, description, on the other hand, the fluid motion is described in terms of a velocity field $\boldsymbol{V}(\boldsymbol{x}, t)$ in which the position $\boldsymbol{x}$ and the time t are independent variables. The variation of $\boldsymbol{V}$ with time, or of any other fluid property in this description, refers thus to a fixed point in space rather than to a specific fluid element, as is the case with the Lagrangian description.

If a field quantity is denoted by Ψ_L in the Lagrangian and by Ψ_E in the Eulerian description, then the relation between the time derivatives in the two descriptions is

$$\frac{\mathrm{d}\Psi_L}{\mathrm{d}t} = \frac{\partial \Psi_E}{\partial t} + (\boldsymbol{V} \cdot \nabla)\Psi_E. \tag{A.898}$$

We remark that in the case of linear acoustics for a homogeneous medium at rest we need not be concerned about the difference between $(\mathrm{d}\Psi_L/\mathrm{d}t)$ and $(\partial\Psi_E/\partial t)$, since the term $(\boldsymbol{V} \cdot \nabla)\Psi_E$ is then of second order. However, in a moving or inhomogeneous medium the distinction must be maintained even in the linear approximation.

We shall ordinarily use the Eulerian description and, if we ever need the Lagrangian time derivative, we shall express it as the right-hand side of (A.898), omitting the subscripts. We express herein the fluid motion in terms of the three velocity components V_i of the velocity vector $\boldsymbol{V}$. We denote further the velocity amplitude as $V = |\boldsymbol{V}|$. In addition, the state of the fluid is described in terms of two independent thermodynamic variables such as pressure and temperature or density and entropy. We assume that thermodynamic equilibrium is maintained within each volume element. Thus in all we have five field variables: the three velocity components and the two independent thermodynamic variables. In order to determine these functions of $\boldsymbol{x}$ and t we need five equations. These turn out to be conservation laws: conservation of mass (one equation), conservation of momentum (three equations), and conservation of energy (one equation).

If the density of the fluid is denoted by ϱ and $i, j \in \{1, 2, 3\}$, then the mass flow in the fluid can be expressed by the vector components

$$\varrho V_i \tag{A.899}$$

and the total momentum flux by the tensor

$$t_{ij} = P_{ij} + \varrho V_i V_j, \tag{A.900}$$

in which the first term is the contribution from the thermal motion and the second term the contribution from the gross motion of the fluid. The term P_{ij} is the fluid stress tensor

$$P_{ij} = (P - \varepsilon \nabla \cdot \boldsymbol{V})\delta_{ij} - 2\eta U_{ij} = P\delta_{ij} - D_{ij}, \tag{A.901}$$

where P is the total pressure in the fluid, δ_{ij} is the delta of Kronecker, D_{ij} is the viscous stress tensor, ε and η are two coefficients of viscosity, and

$$U_{ij} = \frac{1}{2}\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right) \tag{A.902}$$

is the shear-strain tensor. In this notation the bulk viscosity would be $3\varepsilon + 2\eta$, and if this were zero (as Stokes assumed for an ideal gas), then η would equal $-3\varepsilon/2$. However, acoustical measurement shows that bulk viscosity is not usually zero (in some cases it may be considerably larger than η) so it will be assumed that ε and η are independent parameters of the fluid. In addition, we define the energy density of the fluid as

$$h = \frac{1}{2}\varrho V^2 + \varrho E, \tag{A.903}$$

the sum of its kinetic energy and the internal energy (E denotes the internal energy per unit mass), and the energy flow vector as

$$I_i = hV_i + \sum_j P_{ij}V_j - K\frac{\partial T}{\partial x_i}, \tag{A.904}$$

in which T is the temperature, K is the thermal conductivity constant, and $\partial T/\partial x_i$ is the temperature gradient in the location of interest. Thus $-K(\partial T/\partial x_i)$ corresponds to the heat flow vector. The term $\sum_j P_{ij}V_j$ contains the work done by the pressure as well as the dissipation caused by the viscous stresses.

The basic equations of motion for the fluid, representing the conservation of mass, momentum, and energy, can thus be written in the forms

$$\frac{\partial \varrho}{\partial t} + \sum_i \frac{\partial(\varrho V_i)}{\partial x_i} = Q(\boldsymbol{x}, t), \tag{A.905}$$

$$\frac{\partial(\varrho V_i)}{\partial t} + \sum_j \frac{\partial t_{ij}}{\partial x_j} = F_i(\boldsymbol{x}, t), \tag{A.906}$$

$$\frac{\partial h}{\partial t} + \sum_i \frac{\partial I_i}{\partial x_i} = H(\boldsymbol{x}, t), \tag{A.907}$$

where Q, F_i, and H are source terms representing the time rate of introduction of mass, momentum, and heat energy into the fluid, per unit volume. The energy equation (A.907) can be rewritten in the somewhat different form

$$\varrho\frac{\mathrm{d}E}{\mathrm{d}t} = \varrho\left(\frac{\partial E}{\partial t} + \boldsymbol{V}\cdot\nabla E\right) = K\Delta T + D - P\nabla\cdot\boldsymbol{V} + H, \tag{A.908}$$

which represents the fact that a given element of fluid has its internal energy changed either by heat flow, or by viscous dissipation

$$D = \sum_{ij} D_{ij} U_{ij} = \varepsilon \sum_j U_{jj}^2 + 2\eta \sum_{ij} U_{ij}^2, \tag{A.909}$$

or by direct change of volume, or else by direct injection of heat from outside the system. This last form of energy equation can be obtained directly from the first law of thermodynamics

$$\frac{\mathrm{d}E}{\mathrm{d}t} = T\frac{\mathrm{d}S}{\mathrm{d}t} + \frac{P}{\varrho^2}\frac{\mathrm{d}\varrho}{\mathrm{d}t}, \tag{A.910}$$

if, for the rate of entropy production per unit mass $\mathrm{d}S/\mathrm{d}t$, we introduce

$$T\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{K}{\varrho}\Delta T + \frac{D}{\varrho} + \frac{H}{\varrho}, \tag{A.911}$$

and, for the density change $\mathrm{d}\varrho/\mathrm{d}t$ we use

$$\frac{\mathrm{d}\varrho}{\mathrm{d}t} = \frac{\partial \varrho}{\partial t} + \boldsymbol{V}\cdot\nabla\varrho = -\varrho\nabla\cdot\boldsymbol{V}. \tag{A.912}$$

If we wish to change from one pair of thermodynamic variables to another we usually make use of the equation of state of the gas

$$P = P(\varrho, T). \tag{A.913}$$

For a perfect gas, it is given by

$$P = R\varrho T, \tag{A.914}$$

being $R = 8.314472$ [J/°K/mol] the (ideal) gas constant.

b) Wave equation

Returning to equations (A.905) to (A.907), by elimination of $\partial^2(\varrho V_i)/\partial x_i \partial t$ from the first two, we obtain

$$\frac{\partial^2 \varrho}{\partial t^2} - c_0^2 \Delta\varrho = \frac{\partial Q}{\partial t} - \sum_i \frac{\partial F_i}{\partial x_i} + \Delta(P - c_0^2\varrho) + \sum_{ij}\left(\frac{\partial^2 D_{ij}}{\partial x_i \partial x_j} + \frac{\partial^2(\varrho V_i V_j)}{\partial x_i \partial x_j}\right). \tag{A.915}$$

We have subtracted the term $c_0^2\Delta\varrho$ from both sides of the equation, where c_0 is the space average of the velocity of sound (c_0 can depend on t). The right-hand terms will vanish for a homogeneous, lossless, and source-free medium at rest, in which case we obtain for the density ϱ the familiar wave equation

$$\Delta\varrho - \frac{1}{c_0^2}\frac{\partial^2 \varrho}{\partial t^2} = 0. \tag{A.916}$$

Under all other circumstances the right-hand side of (A.915) will not vanish, but will represent some sort of sound source, either produced by external forces or injections of fluid or by inhomogeneities, motions, or losses in the fluid itself.

The first term, representing the injection of fluid, gives rise to a monopole wave. For air-flow sirens and pulsed-jet engines, for example, it represents the major source term. The second term, corresponding to body forces on the fluid, gives rise to dipole waves. Even when this term is independent of time it may have an effect on sound transmission, as, e.g., in the case of the force of gravity.

The third term represents several effects. The variation of pressure is produced both by a density and an entropy variation. If the fluid changes are isentropic, then the third term corresponds to the scattering or refraction of sound by variations in temperature or composition of the medium. It may also correspond to a source of sound, in the case of a fluctuating temperature in a turbulent medium. If the motion is not isentropic, then the term $\Delta(P - c_0^2\varrho)$ also contains contributions from entropy fluctuations in the medium. These effects will include losses produced by heat conduction and also the generation of sound by heat sources.

The fourth term, the double divergence of D_{ij}, represents the effects of viscous losses and/or the generation of sound by oscillating viscous stresses in a moving medium. If the coefficients of viscosity should vary from point to point, one would have also the effect of scattering from such inhomogeneities, but these are usually quite negligible. Finally, the fifth term, the double divergence of the term $\varrho V_i V_j$, represents the scattering or the generation of sound caused by the motion of the medium. If the two previous terms are thought of as stresses produced by thermal motion, this last term can be considered as representing the Reynolds stress of the gross motion. It is the major source of sound in turbulent flow and produces quadrupole radiation.

c) Linear approximation

After having summarized the possible effects in fluid motion, we shall now consider the problem of linearisation of the equations (A.905) to (A.907) and the interpretation of its results. These equations are non-linear in the variables ϱ and V_i. Not only are there terms where the product ϱV_i occurs explicitly, but also terms such as h and I_i implicitly depend on ϱ and V_i in a non-linear way. Furthermore, the momentum flux t_{ij} is not usually linearly related to the other field variables. In the first place the gross motion of the fluid, if there is one, contributes a stress $\varrho V_i V_j$ and in the second place there is a non-linear relationship between the pressure P and the other thermodynamic variables. For example, in an isentropic motion we have $(P/P_0) = (\varrho/\varrho_0)^\gamma$, and for a non-isentropic motion we have

$$\frac{P}{P_0} = \left(\frac{\varrho}{\varrho_0}\right)^\gamma e^{(S-S_0)/C_v}. \quad \text{(A.917)}$$

A Taylor expansion of this last equation around the equilibrium state (ϱ_0, S_0) yields

$$P - P_0 = c^2(\varrho - \varrho_0) + \frac{P_0}{C_v}(S - S_0) + \frac{1}{2}(\gamma - 1)c^2(\varrho - \varrho_0)^2 + \frac{P_0}{2C_v^2}(S - S_0)^2 + \dots \quad \text{(A.918)}$$

where C_p and C_v are respectively the specific heats at constant pressure and constant volume, $\gamma = C_p/C_v$, and $c^2 = \gamma P_0/\varrho_0$. Thus, only when the deviation of P from the equilibrium value P_0 is small enough is the linear relation

$$P \approx P_0 + c^2(\varrho - \varrho_0) + \frac{P_0}{C_v}(S - S_0) \quad \text{(A.919)}$$

a good approximation. As we already mentioned, in acoustics we are usually concerned with the effects of some small, time-dependent deviations from some equilibrium state of the system. When the equilibrium state is homogeneous and static, the perturbation can easily be separated off and the resulting first order equations are relatively simple. But when the equilibrium state involves inhomogeneities or steady flows the separation is less straightforward. Even here, however, if the inhomogeneities are confined to a finite region of space, the equilibrium state outside this region being homogeneous and static, then the separating out of the acoustic motions in the outer region is not difficult.

In any case, we assume that the medium in the equilibrium state is described by the field quantities $\boldsymbol{V}_0 = \boldsymbol{v}$, P_0, ϱ_0, T_0, and S_0, for example, and define the acoustic velocity, pressure, density, temperature, and entropy as the differences between the actual values and the equilibrium values

$$\begin{gathered} \boldsymbol{u} = \boldsymbol{V} - \boldsymbol{V}_0 = \boldsymbol{V} - \boldsymbol{v}, \qquad p = P - P_0, \qquad \delta = \varrho - \varrho_0, \\ \theta = T - T_0, \qquad \sigma = S - S_0. \end{gathered} \tag{A.920}$$

If $\boldsymbol{u}$, p, etc., are small enough we can obtain reasonably accurate equations, involving these acoustic variables to the first order, in terms of the equilibrium values (not necessarily to the first order). If we have already solved for the equilibrium state, the equilibrium values $\boldsymbol{V}_0 = \boldsymbol{v}$, P_0, etc., may be regarded as known parameters, being $\boldsymbol{u}$, p, etc., the unknowns. Thus the first order relationship between the acoustic pressure, density, and entropy arising from (A.918) is

$$p \approx c^2\delta + \frac{P_0}{C_v}\sigma. \tag{A.921}$$

Our procedure will thus be to replace the quantities ϱ, $\boldsymbol{V}$, T, etc., in equations (A.905) to (A.908) by $(\varrho_0 + \delta)$, $(\boldsymbol{v} + \boldsymbol{u})$, $(T_0 + \theta)$, etc., and to keep only terms in first order of the acoustic quantities δ, $\boldsymbol{u}$, θ, etc. The terms containing only ϱ_0, $\boldsymbol{v}$, T_0 (which we call inhomogeneous terms) need not be considered when we are computing the propagation of sound. On the other hand, in the study of the generation of sound these inhomogeneous terms are often the source terms.

In general, the linear approximations thus obtained will be valid if the mean acoustic velocity amplitude $u = |\boldsymbol{u}|$ is small compared to the wave velocity c. There are exceptions however. In the problem of the diffraction of sound by a semi-infinite screen, for example, the acoustic velocity becomes very large in the regions close to the edge of the screen. In such regions non-linear effects are to be expected.

The linearized forms for the equations of mass, momentum, and energy conservation, and the equation of state (perfect gas), for a moving, inhomogeneous medium, are

$$\frac{\partial\delta}{\partial t} + \delta\sum_i \frac{\partial v_i}{\partial x_i} + \varrho_0\sum_i \frac{\partial u_i}{\partial x_i} + \sum_i u_i\frac{\partial\varrho_0}{\partial x_i} \approx Q, \tag{A.922}$$

$$\frac{\partial}{\partial t}(\varrho_0 u_i + \delta v_i) + \sum_j \frac{\partial}{\partial x_j}\bigl\{\varrho_0(u_i v_j + u_j v_i) + \delta v_i v_j + p_{ij}\bigr\} \approx F_i, \tag{A.923}$$

$$\varrho_0 T_0 \left(\frac{\partial \sigma}{\partial t} + \boldsymbol{u} \cdot \nabla S_0 \right) + \frac{p}{R} \frac{\mathrm{d} S_0}{\mathrm{d}t} \approx K \Delta \theta + 4\eta \sum_{ij} u_{ij} v_{ij} + H, \tag{A.924}$$

$$p \approx R \varrho_0 \theta + R T_0 \delta = c^2 \delta + \frac{P_0}{C_v} \sigma, \tag{A.925}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + (\boldsymbol{V} \cdot \nabla), \tag{A.926}$$

$$u_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \tag{A.927}$$

and

$$p_{ij} = p \delta_{ij} - d_{ij}, \tag{A.928}$$

$$d_{ij} = \varepsilon \operatorname{div}(\boldsymbol{u}) \delta_{ij} + 2\eta u_{ij}, \tag{A.929}$$

are acoustic counterparts of the quantities defined earlier. The source terms Q, $\boldsymbol{F}$, and H are the non-equilibrium parts of the fluid injection, body force, and heat injection. The equilibrium part of Q, for example, has been canceled against $(\partial \varrho_0 / \partial t) + \operatorname{div}(\varrho_0 \boldsymbol{v})$ from the left-hand side of (A.905).

These results are so general as to be impractical to use without further specialization. For example, one has to assume that $\operatorname{div}(\boldsymbol{v}) = 0$ (usually a quite allowable assumption) before one can obtain the linear form of the general wave equation

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla \right)^2 \delta - \Delta p \approx \frac{\partial Q}{\partial t} - \nabla \cdot \boldsymbol{F} + \nabla \cdot \mathfrak{D} \cdot \nabla, \tag{A.930}$$

where the last term is the double divergence of the tensor $\mathfrak{D}$, which has elements d_{ij}. In order to obtain a wave equation in terms of acoustic pressure p alone, we must determine δ and d_{ij} in terms of p. To do this in the most general case is not a particularly rewarding exercise, it is much more useful to do it for a number of specific situations which are of practical interest.

But, before we go to special cases, it is necessary to say a few words about the meaning of such quadratic quantities as acoustic intensity, acoustic energy, density, and the like. For example, the energy flow vector

$$\boldsymbol{I} = \left(\frac{1}{2} \varrho V^2 + \varrho E \right) \boldsymbol{V} + \mathfrak{P} \cdot \boldsymbol{V} - K \nabla T, \tag{A.931}$$

where $\mathfrak{P}$ is the fluid stress tensor, with elements P_{ij}. The natural definition of the acoustic energy flow would be

$$\boldsymbol{i} = (\boldsymbol{I})_{\text{with sound}} - (\boldsymbol{I})_{\text{without sound}} = \boldsymbol{I} - \boldsymbol{I}_0, \tag{A.932}$$

with corresponding expressions for the acoustic energy density, $w = h - h_0$, and mass flow vector, $(\varrho \boldsymbol{V})_{\text{with sound}} - (\varrho_0 \boldsymbol{V}_0)$. Similarly with the momentum flow tensor, from which the acoustic radiation pressure tensor is obtained, i.e.,

$$m_{ij} = (P_{ij} + \varrho V_i V_j)_{\text{with sound}} - (P_{ij} + \varrho V_i V_j)_{\text{without sound}}. \tag{A.933}$$

These quantities clearly will contain second order terms in the acoustic variables, therefore their rigorous calculation would require acoustic equations which are correct to the second order. As with equation (A.930), it is not very useful to perform this calculation in the most general case. It is sufficient to point out here that the acoustic energy flow, etc., correct to second order, can indeed be expressed in terms of products of the first order acoustic variables.

In the general acoustic equations (A.922) to (A.925) we have included the source terms Q, $\boldsymbol{F}$, and H, corresponding to the rate of transfer of mass, momentum, and heat energy from external sources. The sound field produced by these sources can be expressed in terms of volume integrals over these source functions. As mentioned above, we have not included terms, such as $\varrho V_i V_j$ or ΔP_0, which do not include acoustic variables. The justification for this omission is that these terms balance each other locally in the equations of motion, for example fluctuations in velocities are balanced by local pressure fluctuations, and the like. These fluctuations produce sound (i.e., acoustic radiation), but in the region where the fluctuations occur (the near field), the acoustic radiation is small compared to the fluctuations themselves. However, the acoustic radiation produced produced by the fluctuations extends outside the region of fluctuation, into regions where the fluid is otherwise homogeneous and at rest (the far field), and here it can more easily be computed (and, experimentally, more easily measured).

Thus, in the study of the generation of sound by fluctuations in the fluid itself, it is essential to retain in the source terms the terms which do not contain the acoustic variables themselves. Within the region of fluctuation, the differentiation between sound and equilibrium motion is quite artificial (the local fluid motion could be regarded as part of the acoustic near field), and in many cases it is more straightforward to use the original equations (A.905) to (A.908) and (A.915) in their integral form, where the net effect of the sources appears as an integral over the region of fluctuation.

d) Acoustic equations for a fluid at rest

We discuss herein the special forms taken on by equations (A.922) to (A.933) when the equilibrium state of the fluid involves only a few of the various possible effects discussed above. At first we will assume that, in the equilibrium state, the fluid is at rest and that the acoustic changes in density are isentropic ($\sigma = 0$). In this case the relation between the acoustic pressure p and the acoustic density δ, from equation (A.921), is simply

$$p = c^2\delta, \qquad c^2 = \frac{\gamma P}{\varrho}. \tag{A.934}$$

From here on we will omit the subscript 0 from the symbols for equilibrium values in situations like that of equation (A.934), where the difference between P and P_0 or ϱ and ϱ_0 would make only a second-order difference in the equations. We also will use the symbol $=$ instead of $\approx$, since from now on we commit ourselves to the linear equations. The wave equation (A.930) then reduces to the familiar

$$\Delta p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0. \tag{A.935}$$

Once the pressure has been computed, the other acoustic variables follow from the equations defined previously:

$$\text{Velocity} \qquad \boldsymbol{u} = -\frac{1}{\varrho}\int \nabla p \,\mathrm{d}t, \tag{A.936}$$

$$\text{Displacement} \qquad \boldsymbol{d} = \int \boldsymbol{u}\,\mathrm{d}t, \tag{A.937}$$

$$\text{Temperature} \qquad \theta = (\gamma - 1)\frac{T}{\varrho c^2}p, \qquad \left(\gamma = \frac{C_p}{C_v}\right) \tag{A.938}$$

$$\text{Density} \qquad \delta = \frac{p}{c^2}. \tag{A.939}$$

All these variables satisfy a homogeneous wave equation such as (A.935).

Waves with simple-harmonic time dependence are of the form

$$p = p_0 e^{-i\omega t}, \qquad \omega = kc, \tag{A.940}$$

where p_0 does not depend on t, and where i denotes the complex imaginary unit, ω the pulsation, and k the wave number. These are single-frequency waves and have a time factor $e^{-i\omega t}$. For these waves, the acoustic variables of velocity and displacement are given, in particular, by

$$\text{Velocity} \qquad \boldsymbol{u} = -\frac{1}{ik\varrho c}\nabla p, \tag{A.941}$$

$$\text{Displacement} \qquad \boldsymbol{d} = -\frac{1}{k^2\varrho c^2}\nabla p. \tag{A.942}$$

For a plane sound wave, which has the general form

$$p = f(ct - \boldsymbol{n}\cdot\boldsymbol{x}), \tag{A.943}$$

being $\boldsymbol{n}$ a unit vector normal to the wave front, the acoustic velocity is

$$\boldsymbol{u} = \frac{\boldsymbol{n}}{\varrho c} f(ct - \boldsymbol{n}\cdot\boldsymbol{x}). \tag{A.944}$$

The quantity ϱc is called the characteristic acoustic impedance of the medium. Since $\operatorname{div}(\boldsymbol{d})$ is the relative volume change of the medium, we can use equation (A.934) to obtain another relation between $\boldsymbol{d}$ and p, namely

$$p = -\varrho c^2 \operatorname{div}(\boldsymbol{d}), \tag{A.945}$$

which states that the isentropic compressibility of the fluid is equal to $1/(\varrho c^2)$.

The sound energy flow vector (the sound intensity) is

$$\boldsymbol{i} = p\boldsymbol{u} = \varrho c u^2 \boldsymbol{n} = \frac{p^2}{\varrho c}\boldsymbol{n}. \tag{A.946}$$

It is tempting to consider this equation as self-evident, but it should be remembered that $\boldsymbol{i}$ is a second-order quantity, which must be evaluated from equation (A.932). In the special case of a homogeneous medium at rest, the other second-order terms cancel out and equation (A.946) is indeed correct to second order. In a moving medium, the result is not so simple.

The situation is also not so straightforward in regard to the mass flow vector. One might assume that it equals $\delta \boldsymbol{u}$, but this would result in a non-zero, time-average, mass flow for a plane wave, an erroneous result. In this case, the additional second-order terms in the basic equations do contribute, making the mass flow vector zero in the second-order approximation.

On the other hand, the magnitude of the acoustic momentum flux is correctly given by the expression ϱu^2 to the second order. The rate of momentum transfer is equal to the radiation pressure on a perfect absorber.

Generally we are interested in the time average of these quantities. For single-frequency waves (time factor $e^{-i\omega t}$), these are

$$\boldsymbol{i} = \frac{1}{2}\,\mathfrak{Re}\{p\bar{\boldsymbol{u}}\}, \tag{A.947}$$

where $\bar{\boldsymbol{u}}$ denotes the complex conjugate of $\boldsymbol{u}$. For a plane wave, like (A.943), we have

$$\boldsymbol{i} = \frac{1}{2}\varrho c u^2 \boldsymbol{n} = \frac{\boldsymbol{n}}{2\varrho c}|p|^2. \tag{A.948}$$

The acoustic density is

$$w = \frac{1}{2}\varrho u^2 + \frac{1}{2\varrho c^2}|p|^2, \tag{A.949}$$

where the first term is the kinetic energy density and the second term the potential energy density. In a plane wave these are equal. We note that the magnitude of the acoustic radiation pressure is thus equal to the acoustic energy density.

The simple wave equation (A.935) is modified when there are body forces or inhomogeneities present, even though there is no motion of the fluid in the equilibrium state, as two examples will suffice to show. For example, the force of gravity has a direct effect on the wave motion, in addition to the indirect effect produced by the change in density with height. In this case, the body force $\boldsymbol{F}$ is equal to $\varrho\,\boldsymbol{g}$, where $\boldsymbol{g}$ is the acceleration of gravity, being $g = |\boldsymbol{g}|$, and thus the term $\operatorname{div}(\boldsymbol{F})$ in equation (A.930) becomes $\boldsymbol{g}\cdot\nabla\varrho + \varrho\nabla\cdot\boldsymbol{g}$, where the magnitude of the second term is to that of the first as the wavelength is to the radius of the Earth, so the second term can usually be neglected. Therefore the wave equation in the presence of the force of gravity is

$$\frac{\partial^2 p}{\partial t^2} = c^2 \Delta p + \boldsymbol{g}\cdot\nabla p. \tag{A.950}$$

The added term has the effect of making the medium anisotropic. For a simple-harmonic, plane wave $\exp(ik\boldsymbol{n}\cdot\boldsymbol{x} - i\omega t)$, if $\boldsymbol{n}$ is perpendicular to $\boldsymbol{g}$, then $k = \omega/c$, but if $\boldsymbol{n}$ is parallel to $\boldsymbol{g}$, the propagation constant k is

$$k_g = i\frac{g}{2c^2} + \frac{\omega}{c}\sqrt{1 - \frac{g^2}{4c^2\omega^2}}. \tag{A.951}$$

We note that a wave propagating downward (in the direction of the acceleration of gravity $\boldsymbol{g}$) is attenuated at a rate $e^{-\alpha x_3}$, where $\alpha = (g/2c^2)$, independent of frequency, and its phase velocity is $c/\sqrt{1-(g^2/4c^2\omega^2)}$. If the frequency of the wave is less than $(g/4\pi c)$, there will be no wave motion downward.

A similar anisotropy occurs when the anisotropy is not produced by a body force, but is caused by an inhomogeneity in one of the characteristics of the medium. In a solid or liquid medium the elasticity or the density may vary from point to point (as is caused by a salinity gradient in sea-water, for instance). If the medium is a gas, the inhomogeneity must manifest itself by changes in temperature and/or entropy density. For a source-free medium at rest, equation (A.930) shows that $(\partial^2\delta/\partial t^2) = \Delta p$, but this equation reduces to the usual wave equation (A.935) only when the equilibrium entropy density is uniform and the acoustical motions are isentropic. If the equilibrium entropy density S_0 is not uniform the wave equation is modified, even though the acoustic motion is still isentropic.

If the acoustic disturbance is isentropic, then $(\mathrm{d}S/\mathrm{d}t) = (\partial S/\partial t) + \boldsymbol{u}\cdot\nabla S = 0$, and if the equilibrium entropy density S_0 is a function of position but not of time, then

$$\frac{\partial\sigma}{\partial t} + \boldsymbol{u}\cdot\nabla S_0 = 0. \tag{A.952}$$

Referring to equations (A.921) and (A.936), we obtain

$$\frac{\partial\delta}{\partial t} = \frac{1}{c^2}\frac{\partial p}{\partial t} - \frac{\varrho}{C_p}\frac{\partial\sigma}{\partial t} = \frac{1}{c^2}\frac{\partial p}{\partial t} + \frac{\varrho}{C_p}\boldsymbol{u}\cdot\nabla S_0, \tag{A.953}$$

and thus

$$\frac{\partial^2\delta}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \frac{1}{C_p}\nabla p\cdot\nabla S_0, \tag{A.954}$$

which, when inserted into equation (A.930) for a source-free medium at rest finally produces the equation

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = \Delta p + \frac{1}{C_p}\nabla p\cdot\nabla S_0, \tag{A.955}$$

which has the same form as equation (A.950) representing the effect of gravity. Thus an entropy gradient in the equilibrium state will produce anisotropy in sound propagation. As with the solutions for equation (A.950), sound will be attenuated in the direction of entropy increase, will be amplified in the direction of decreasing S_0. However, a much larger effect arises from the fact that a change in entropy will produce a change in c from point to point, so that the coefficient of $(\partial^2 p/\partial t^2)$ in equation (A.955) will depend on position.

Further effects of fluid motion, transport phenomena, and internal energy losses can be appreciated in Morse & Ingard (1961).

e) Simple-harmonic waves

Simple-harmonic waves are used when the sources and fields have a single frequency, or else, when the total field has been analyzed into its frequency components and we are studying one of these components. These waves acquire thus the form of equation (A.940). Here all aspects of the wave have a common time factor $e^{-i\omega t}$ and the space part of the pressure or density wave (vid. equations (A.915) and (A.930)) satisfies the inhomogeneous

Helmholtz equation in the variable $\boldsymbol{x}$, namely

$$\Delta\Psi + k^2\Psi = q(\boldsymbol{x}), \qquad k = \frac{\omega}{c}, \tag{A.956}$$

where Ψ may be the density ϱ, in which case q represents $-(1/c_0^2)$ times the quantities on the right-hand side of equation (A.915), with time factor $e^{-i\omega t}$ divided out, or else, if we are using the linear approximations, Ψ may be the acoustic pressure p, in which case q may be some of the terms on the right-hand side of equation (A.930). Some of these quantities are truly inhomogeneous terms, being completely specified functions of the spatial coordinates $\boldsymbol{x}$, other terms are linear in the unknown Ψ or its derivatives, and still other terms are quadratic in Ψ and its derivatives (the quadratic terms are neglected in our present discussion). From Ψ, of course, we can obtain the other properties of the wave, its fluid velocity, displacement, temperature, etc., by means of the relations given in equations (A.936) to (A.939).

The Helmholtz equation (A.956) can be solved for any wave number k. If we assume, in the equilibrium state, that the fluid is at rest and that the acoustic changes in density are isentropic, then we obtain the familiar homogeneous Helmholtz equation

$$\Delta\Psi + k^2\Psi = 0. \tag{A.957}$$

A particular case of this equation is when the frequency f is zero, being $f = \omega/2\pi$, in which case the Laplace equation appears, namely

$$\Delta\Psi = 0, \qquad k = 0. \tag{A.958}$$

Similarly, if the frequency is zero for the inhomogeneous Helmholtz equation (A.956), then we obtain the Poisson equation

$$\Delta\Psi = q(\boldsymbol{x}), \qquad k = 0. \tag{A.959}$$

A.11.2 Boundary conditions

a) Reaction of the surface to sound

We discuss now the behavior of sound in the neighborhood of a boundary surface, and see whether we can express this behavior in terms of boundary conditions on the acoustic field. It turns out that in many cases the sorts of boundary conditions familiar in the classical theory of boundary-value problems, such as that the ratio of value to normal gradient of pressure is specified at every point on the boundary, is at least approximately valid.

At first sight it may seem surprising that the ratio of pressure to its normal gradient, which to first order equals the ratio of pressure to normal velocity at the surface, could be specified, even approximately, at each point of the surface, independently of the configuration of the incident wave (vid. equation (A.936)). Of course, if the wall is perfectly rigid so that the value of the ratio is infinite everywhere, then the assumption that this ratio is independent of the nature of the incident wave is not so surprising. But many actual boundary surfaces are not very rigid, and in many problems in theoretical acoustics the effect of the yielding of the boundary to the sound pressure is the essential part of the problem. When the boundary does yield, for the classical boundary conditions to be valid would imply that the ratio of incident pressure to normal displacement of the boundary would be a characteristic of each point of the surface by itself, independent of what happens at any other point of the surface. To see what this implies, regarding the acoustic nature of the boundary surface, and when it is likely to be valid, let us discuss the simple case of the incidence of a plane wave of sound on a plane boundary surface.

Suppose the boundary is the x_2-x_3 plane, with the boundary material occupying the region of positive x_1 and the fluid carrying the incident sound wave occupying the region of negative x_1, to the left of the boundary plane. Suppose also that the incident wave has frequency $f = \omega/2\pi$ and that its direction of propagation is at the angle of incidence ϕ to the x_1 axis, the direction normal to the boundary. The incident wave, therefore, has a pressure and fluid velocity distribution, within the fluid (vid. equation (A.944)), given by

$$p = p_i \exp(ikx_1 \cos\phi + ikx_2 \sin\phi - i\omega t), \tag{A.960}$$

$$\boldsymbol{u} = \frac{p}{\varrho c}(\boldsymbol{a}_1 \cos\phi + \boldsymbol{a}_2 \sin\phi), \qquad k = \frac{\omega}{c} = \frac{2\pi}{\lambda}, \tag{A.961}$$

where ϱ is the fluid density, c is the velocity of sound waves, and λ the wavelength of the wave in the fluid in the region $x_1 < 0$.

At $x_1 = 0$ the wave is modified because the boundary surface does not move in response to the pressure in the same way that the free fluid does. In general, the presence of the acoustic pressure p produces motion of the surface, but the degree of motion depends on the nature of the boundary material and its structure. If the fluid viscosity is small, we can safely assume that the tangential component of fluid velocity close to the surface need not be equal to the tangential velocity of the boundary itself, thus a discontinuity in tangential velocity is allowed at the boundary. But there must be continuity in normal velocity through the boundary surface, and there must also be continuity in pressure across the surface.

If the surface is porous, so that the fluid can penetrate into the surface material, then there can be an average fluid velocity into the surface without motion of the boundary material itself. If the pores do not interconnect, then it would be true that the mean normal velocity of penetration of the fluid into the pores would bear a simple ratio to the pressure at the surface, independent of the pressure and velocity of the wave at other points on the surface. In this case we could expect the ratio between pressure and normal velocity at the surface to be a point property of the surface, perhaps dependent on the frequency of the incident wave, but independent of its configuration.

b) Acoustic impedance

The ratio between pressure and velocity normal to a boundary surface is called the normal acoustic impedance z_n of the surface. When it is a point property of the surface, independent of the configuration of the incident wave (and we have indicated that this is the case in practice for many porous surfaces), then the classical type of boundary condition is applicable. For a wave of frequency $f = \omega/2\pi$, the normal fluid velocity just outside the surface is equal to $(1/i\omega\varrho)$ times the normal gradient of the pressure there. Thus the ratio of pressure to its normal gradient at a point of the surface would equal the value of the normal impedance of the surface at the point, divided by $ik\varrho c$, where $k = \omega/c = 2\pi/\lambda$, and where ϱc is the characteristic impedance of the fluid medium (vid. equation (A.944)):

$$\frac{p}{\partial p/\partial n} = \frac{z_n}{ik\varrho c} = \frac{\zeta}{ik} = \frac{1}{ik}(\chi - i\xi), \tag{A.962}$$

where ζ is the dimensionless specific impedance of the surface, and χ and ξ are its resistive and reactive components. If z_n is a point property of the surface, then classical boundary conditions can be used for single-frequency incident waves.

For example, for the conditions of equations (A.960) and (A.961), the ratio between the reflected amplitude p_r and the incident amplitude p_i in in the region $x_1 < 0$, being the total wave

$$p = \left(p_i e^{ikx_1 \cos\phi} + p_r e^{-ikx_1 \cos\phi}\right) e^{ikx_2 \sin\phi - i\omega t}, \tag{A.963}$$

is easily shown from equation (A.962) to be

$$R = \frac{p_r}{p_i} = \frac{-1 + \zeta\cos\phi}{1 + \zeta\cos\phi} = -\frac{(1 - \chi\cos\phi) + i\xi\cos\phi}{(1 + \chi\cos\phi) - i\xi\cos\phi}, \tag{A.964}$$

and the ratio of reflected to incident intensity is

$$|R|^2 = 1 - \alpha = \frac{(1 - \chi\cos\phi)^2 + \xi^2\cos^2\phi}{(1 + \chi\cos\phi)^2 + \xi^2\cos^2\phi}, \tag{A.965}$$

where α is called the absorption coefficient of the surface. If χ and ξ are point properties of the surface, independent of the configuration of the incident wave (independent, in this case, of the angle of incidence ϕ), then the problem is solved. The fraction α of energy absorbed by the surface can be computed from equation (A.965) as a function of the incident angle ϕ, considering χ and ξ to be independent of ϕ. For example, if the specific resistance χ is larger than unity, then the absorption coefficient has a maximum for an angle of incidence $\phi = \arccos(1/\chi)$, dropping to zero at grazing incidence, $\phi = 90°$.

But if $z_n = \varrho c\zeta$ is not a point function of position on the boundary surface, then the problem is not really solved, for the value of z_n will depend on the configuration of the motion of the boundary surface itself, and to obtain the appropriate values of χ and ξ to use in equation (A.965), we will have to investigate the behavior of the sound wave inside the boundary material, an investigation we do not need to undertake when z_n is a point function of position and the classical boundary conditions of equation (A.962) can be used.

c) Exceptions to the classical boundary conditions

To appreciate the nature of difficulties which can arise, let us continue to discuss the simple example of the equations (A.960) and (A.961), that of a plane wave incident on a plane boundary, for the case where we do have to consider the wave motion inside the boundary. To keep the example simple, we suppose the material forming the boundary to fill the region $x_1 > 0$ uniformly. We will also suppose that the material is homogeneous to the extent that we can talk about a mean displacement and velocity of the material. The wave properties of the material may not be isotropic, however, we shall assume that the wave velocity in the x_1 direction is c_n and that in a direction parallel to the boundary plane it is c_t, where both these quantities may be complex and also frequency dependent. In other words, pressure waves are possible in the material, the wave equation and the relation between pressure and material velocity,

$$c_n^2 \frac{\partial^2 p}{\partial x_1^2} + c_t^2 \left(\frac{\partial^2 p}{\partial x_2^2} + \frac{\partial^2 p}{\partial x_3^2} \right) + \omega^2 p = 0, \tag{A.966}$$

$$u_1 = \frac{1}{i\omega \varrho_n} \frac{\partial p}{\partial x_1}, \qquad u_2 = \frac{1}{i\omega \varrho_t} \frac{\partial p}{\partial x_2}, \qquad u_3 = \frac{1}{i\omega \varrho_t} \frac{\partial p}{\partial x_3}, \tag{A.967}$$

serving to define the quantities c_n, c_t, ϱ_n and ϱ_t.

If the pressure inside the boundary ($x_1 > 0$) is to satisfy this wave equation and also to fit the wave form of equation (A.963) at $x_1 = 0$, then the pressure and velocity waves inside the material must be

$$p = p_t \exp \left(ik_n x_1 \sqrt{1 - \left(\frac{c_t}{c}\right)^2 \sin^2\phi} + ikx_2 \sin\phi - i\omega t \right), \tag{A.968}$$

$$\boldsymbol{u} = \frac{p}{\varrho_n c_n} \boldsymbol{a}_1 \sqrt{1 - \left(\frac{c_t}{c}\right)^2 \sin^2\phi} + \frac{p}{\varrho_t c_t} \boldsymbol{a}_2 \frac{c_t}{c} \sin\phi, \tag{A.969}$$

where $k_n = \omega/c_n$, $k = \omega/c$, and c is the sound velocity in the fluid outside the boundary ($x_1 < 0$). Equating p and u_1 at $x_1 = 0$ with those from equation (A.963), we find for the ratio of reflected to incident pressures, outside the boundary surface, that

$$R = \frac{p_r}{p_i} = \frac{-\sqrt{1 - (c_t/c)^2 \sin^2\phi} + (\varrho_n c_n/\varrho c) \cos\phi}{\sqrt{1 - (c_t/c)^2 \sin^2\phi} + (\varrho_n c_n/\varrho c) \cos\phi}. \tag{A.970}$$

The absorption coefficient α is $1 - |R|^2$, as before.

Comparison with equation (A.964) shows that the specific surface impedance in this instance is

$$\zeta(\phi) = \frac{\varrho_n c_n}{\varrho c} \left\{ 1 - \left(\frac{c_t}{c}\right)^2 \sin^2\phi \right\}^{-1/2}, \tag{A.971}$$

which is not independent of ϕ unless c_t, the transverse velocity in the boundary material, is negligibly small compared to c, the wave velocity in the fluid outside the boundary. Unless c_t is small compared to c, the impedance of the surface is not a point property of the surface, independent of the configuration of the incident wave (in the example, independent of ϕ), and to find its value for any specific configuration of incident wave we must work out the corresponding wave configuration inside the boundary material.

From the point of view of the theoretical acoustician, therefore, there are two general types of boundary-value problems which are encountered. The first type is where the boundary material is such that its normal acoustic impedance is a point property of the surface, independent of the configuration of the incident wave. For this type the ratio of pressure to normal gradient of pressure at each point of the boundary is uniquely specified for each frequency, and the well-known methods of the classical theory of boundary-value problems can be employed. The second type is where it is not possible to consider the surface impedance to be independent of the configuration of the incident wave. In these types of problems it is not possible to substitute a surface impedance for an analysis of the wave inside the boundary, here the internal wave must be studied in detail and its reaction to the incident external wave must be calculated for each configuration of incident wave. These types of problems are usually much more difficult to solve than are the first type.

For further effects on the boundary conditions by the relative motion of fluid and boundary, and for viscous and conduction losses near the boundary we refer to the article by Morse & Ingard (1961).