Foreword

In this thesis we are essentially interested in the mathematical modeling of wave propagation phenomena by using Green's functions and integral equation techniques. As some poet from the ancient Roman Empire inspired by the Muses might have said (Hein 2006):

Non fluctus numerare licet iam machinatori,
Invenienda est nam functio Viridii.

This Latin epigram can be translated more or less as "to count the waves is no longer permitted for the engineer, since to be found has the function of Green". An epigram is a short, pungent, and often satirical poem, which was very popular among the ancient Greeks and Romans. It consists commonly of one elegiac couplet, i.e., a hexameter followed by a pentameter. Two possible questions that arise from our epigram are: "why does someone want to count waves?", and even more: "what is a function of Green and for what purpose do we want to find it?" Let us hence begin with the first question.

Since the dawn of mankind have waves, specifically water waves, been a source of wonder and admiration, but also of fear and respect. Giant sea waves caused by storms have drowned thousands of ships and adventurous sailors, who blamed for their fate the wrath of the mighty gods of antiquity. On more quite days, though, it was always a delightful pleasure to watch from afar the sea waves braking against the coast. For the ancient Romans, in fact, the expression of counting sea waves (*fluctus numerare*) was used in the sense of having leisure time (*otium*), as opposed to working and doing business (*negotium*). Therefore the message is clear: the leisure time is over and the engineer has work to be done. In fact, even if it is not specifically mentioned, it is implicitly understood that this premise applies as much to the civil engineer (*machinator*) as to the military engineer (*munitor*). A straight interpretation of the hexameter is also perfectly allowed. To count the waves individually as they pass by before our eyes is usually not the best way to try to comprehend and reproduce the behavior of wave propagation phenomena, so as to be afterwards used for our convenience. Hence, to understand and treat waves, what sometimes can be quite difficult, we need powerful theoretical tools and efficient mathematical methods.

This takes us now to our second question, which is closely related to the first one. A function of Green (*functio Viridii*), usually referred to as a "Green's function", has no direct relationship with the green color as may be wrongly inferred from a straight translation that disregards the little word play lying behind. The word for Green (*Viridii*) is in the genitive singular case, i.e., it stands not for the adjective green (*viridis*), but rather as a (quite rare) singular of the plural neuter noun of the second declension for green things (*viridia*), which usually refers to green plants, herbs, and trees. Its literal translation, when we consider it as a proper noun, is then "of the Green" or "of Green", which in English is equivalent to "Green's". A Green's function is, in fact, a mathematical tool that allows us to solve wave propagation problems, as I hope should become clear throughout this thesis. The first person who used this kind of functions, and after whom they are named, was the British mathematician and physicist George Green (1793–1841), hence the word play with the color of the same name. They were introduced by Green (1828) in his research on potential theory, where he considered a particular case of them. A Green's function helps us also to solve other kinds of physical problems, but is particularly useful when dealing with infinite exterior domains, since it achieves to synthesize the physical properties of the underlying system. It is therefore in our best interest to find (*invenienda est*) such a Green's function.

1.2 Motivation and overview

1.2.1 Wave propagation

Waves, as summarized in the insightful review by Keller (1979), are disturbances that propagate through space and time, usually by transference of energy. Propagation is the process of travel or movement from one place to another. Thus wave propagation is another name for the movement of a physical disturbance, often in an oscillatory manner. The example which has been recognized longest is that of the motion of waves on the surface of water. Another is sound, which was known to be a wave motion at least by the time of the magnificent English physicist, mathematician, astronomer, natural philosopher, alchemist, and theologian Sir Isaac Newton (1643–1727). In 1690 the Dutch mathematician, astronomer, and physicist Christiaan Huygens (1629–1695) proposed that light is also a wave motion. Gradually other types of waves were recognized. By the end of the nineteenth century elastic waves of various kinds were known, electromagnetic waves had been produced, etc. In the twentieth century matter waves governed by quantum mechanics were discovered, and an active search is still underway for gravitational waves. A discussion on the origin and development of the modern concept of wave is given by Manacorda (1991).

The laws of physics provide systems of one or more partial differential equations governing each type of wave. Any particular case of wave propagation is governed by the appropriate equations, together with certain auxiliary conditions. These may include initial conditions, boundary conditions, radiation conditions, asymptotic decaying conditions, regularity conditions, etc. The differential equations together with the auxiliary conditions constitute a mathematical problem for the determination of the wave motion. These problems are the subject matter of the mathematical theory of wave propagation. Some references on this subject that we can mention are Courant & Hilbert (1966), Elmore & Heald (1969), Felsen & Marcuwitz (2003), and Morse & Feshbach (1953).

Maxwell's equations of electromagnetic theory and Schrödinger's equation in quantum mechanics are both usually linear. They are named after the Scottish mathematician and theoretical physicist James Clerk Maxwell (1831–1879) and the Austrian physicist Erwin Rudolf Josef Alexander Schrödinger (1887–1961). Furthermore, the equations governing most waves can be linearized to describe small amplitude waves. Examples of these linearized equations are the scalar wave equation of acoustics and its time-harmonic version, the Helmholtz equation, which receives its name from the German physician and physicist Hermann Ludwig Ferdinand von Helmholtz (1821–1894). Another example is the Laplace equation in hydrodynamics, in which case it is the boundary condition which is linearized and not the equation itself. This equation is named after the French mathematician and astronomer Pierre Simon, marquis de Laplace (1749–1827). Such linear equations with linear auxiliary conditions are the subject of the theory of linear wave propagation. It is this theory which we shall consider.

The classical researchers were concerned with obtaining exact and explicit expressions for the solutions of wave propagation problems. Because the problems were linear, they constructed these expressions by superposition, i.e., by linear combination, of particular solutions. The particular solutions had to be simple enough to be found explicitly and the problem had to be special enough for the coefficients in the linear combination to be found.

One of the devised methods is the image method (cf., e.g., Morse & Feshbach 1953), in which the particular solution is that due to a point source in the whole space. The domains to which the method applies must be bounded by one or several planes on which the field or its normal derivative vanishes. In some cases it is possible to obtain the solution due to a point source in such a domain by superposing the whole space solution due to the source and the whole space solutions due to the images of the source in the bounding planes. Unfortunately the scope of this method is very limited, but when it works it yields a great deal of insight into the solution and a simple expression for it. The image method also applies to the impedance boundary condition, in which a linear combination of the wave function and its normal derivative vanishes on a bounding plane. Then the image of a point source is a point source plus a line of sources with exponentially increasing or decreasing strengths. The line extends from the image point to infinity in a direction normal to the plane. These results can be also extended for impedance boundary conditions with an oblique derivative instead of a normal derivative (cf. Gilbarg & Trudinger 1983, Keller 1981), in which case the line of images is parallel to the direction of differentiation.

The major classical method is nonetheless that of separation of variables (cf., e.g., Evans 1998, Weinberger 1995). In this method the particular solutions are products of functions of one variable each, and the desired solution is a series or integral of these product solutions, with suitable coefficients. It follows from the partial differential equation that the functions of one variable each satisfy certain ordinary differential equations. Most of the special functions of classical analysis arose in this way, such as those of Bessel, Neumann, Hankel, Mathieu, Struve, Anger, Weber, Legendre, Hermite, Laguerre, Lamé, Lommel, etc. To determine the coefficients in the superposition of the product solutions, the method of expanding a function as a series or integral of orthogonal functions was developed. In this way the theory of Fourier series originated, and also the method of integral transforms, including those of Fourier, Laplace, Hankel, Mellin, Gauss, etc.

Despite its much broader scope than the image method, the method of separation of variables is also quite limited. Only very special partial differential equations possess enough product solutions to be useful. For example, there are only 13 coordinate systems in which the three-dimensional Laplace equation has an adequate number of such solutions, and there are only 11 coordinate systems in which the three-dimensional Helmholtz equation does. Furthermore only for very special boundaries can the expansion coefficients be found by the use of orthogonal functions. Generally they must be complete coordinate surfaces of a coordinate system in which the equation is separable.

Another classical method is the one of eigenfunction expansions (cf. Morse & Feshbach 1953, Butkov 1968). In this case the solutions are expressed as sums or integrals of eigenfunctions, which are themselves solutions of partial differential equations. This method was developed by Lord Rayleigh and others as a consequence of partial separation of variables. They sought particular solutions which were products of a function of one variable (e.g., time) multiplied by a function of several variables (e.g., spatial coordinates). This method led to the use of eigenfunction expansions, to the introduction of adjoint problems, and to other aspects of the theory of linear operators. It also led to the use of variational principles for estimating eigenvalues and approximating eigenfunctions, such as the Rayleigh-Ritz method. These procedures are needed because there exists no way for finding eigenvalues and eigenfunctions explicitly in general. However, if the eigenfunction problem is itself separable, it can be solved by the method of separation of variables.

Finally, there is the method of converting a problem into an integral equation with the aid of a Green's function (cf., e.g., Courant & Hilbert 1966). But generally the integral equation cannot be solved explicitly. In some cases it can be solved by means of integral transforms, but then the original problem can also be solved in this way.

In more recent times several other methods have also been developed, which use, e.g., asymptotic analysis, special transforms, among other theoretical tools. A brief account on them can be found in Keller (1979).

1.2.2 Numerical methods

All the previously mentioned methods to solve wave propagation problems are analytic and they require that the involved domains have some rather specific geometries to be used satisfactorily. In the method of variable separation, e.g., the domain should be described easily in the chosen coordinate system so as to be used effectively. The advent of modern computers and their huge calculation power made it possible to develop a whole new range of methods, the so-called numerical methods. These methods are not concerned with finding an exact solution to the problem, but rather with obtaining an approximate solution that stays close enough to the exact one. The basic idea in any numerical method for differential equations is to discretize the given continuous problem with infinitely many degrees of freedom to obtain a discrete problem or system of equations with only finitely many unknowns that may be solved using a computer. At the end of the discretization procedure, a linear matrix system is obtained, which is what finally is programmed into the computer.

a) Bounded domains

Two classes of numerical methods are mainly used to solve boundary-value problems on bounded domains: the finite difference method (FDM) and the finite element method (FEM). Both yield sparse and banded linear matrix systems. In the FDM, the discrete problem is obtained by replacing the derivatives with difference quotients involving the values of the unknown at certain (finitely many) points, which conform the discrete mesh and which are placed typically at the intersections of mutually perpendicular lines. The FDM is easy to implement, but it becomes very difficult to adapt it to more complicated geometries of the domain. A reference for the FDM is Rappaz & Picasso (1998).

The FEM, on the other hand, uses a Galerkin scheme on the variational or weak formulation of the problem. Such a scheme discretizes a boundary-value problem from its weak formulation by approximating the function space of the solution through a finite set of basis functions, and receives its name from the Russian mathematician and engineer Boris Grigoryevich Galerkin (1871–1945). The FEM is thus based on the discretization of the solution's function space rather than of the differential operator, as is the case with the FDM. The FEM is not so easy to implement as the FDM, since finite element interaction integrals have to be computed to build the linear matrix system. Nevertheless, the FEM is very flexible to be adapted to any reasonable geometry of the domain by choosing adequately the involved finite elements. It was originally introduced by engineers in the late 1950's as a method to solve numerically partial differential equations in structural engineering, but since then it was further developed into a general method for the numerical solution of all kinds of partial differential equations, having thus applications in many areas of science and engineering. Some references for this method are Ciarlet (1979), Gockenbach (2006), and Johnson (1987).

Meanwhile, several other classes of numerical methods for the treatment of differential equations have arisen, which are related to the ones above. Among them we can mention the collocation method (CM), the spectral method (SM), and the finite volume method (FVM). In the CM an approximation is sought in a finite element space by requiring the differential equation to be satisfied exactly at a finite number of collocation points, rather than by an orthogonality condition. The SM, on the other hand, uses globally defined functions, such as eigenfunctions, rather than piecewise polynomials approximating functions, and the discrete solution may be determined by either orthogonality or collocation. The FVM applies to differential equations in divergence form. This method is based on approximating the boundary integral that results from integrating over an arbitrary volume and transforming the integral of the divergence into an integral of a flux over the boundary. All these methods deal essentially with bounded domains, since infinite unbounded domains cannot be stored into a computer with a finite amount of memory. For further details on these methods we refer to Sloan et al. (2001).

b) Unbounded domains

In the case of wave propagation problems, and in particular of scattering problems, the involved domains are usually unbounded. To deal with this situation, two different approaches have been devised: domain truncation and integral equation techniques. Both approaches result in some sort of bounded domains, which can then be discretized numerically without problems.

In the first approach, i.e., the truncation of the domain, some sort of boundary condition has to be imposed on the truncated (artificial) boundary. Techniques that operate in this way are the Dirichlet-to-Neumann (DtN) or Steklov-Poincaré operator, artificial boundary conditions (ABC), perfectly matched layers (PML), and the infinite element method (IEM). The DtN operator relates on the truncated boundary curve the Dirichlet and the Neumann data, i.e., the value of the solution and of its normal derivative. Thus, the knowledge of the problem's solution outside the truncated domain, either by a series or an integral representation, allows its use as a boundary condition for the problem inside the truncated domain. Explicit expressions for the DtN operator are usually quite difficult to obtain, except for some few specific geometries. We refer to Givoli (1999) for further details on this operator. In the case of an ABC, a condition is imposed on the truncated boundary that allows the passage only of outgoing waves and eliminates the ingoing ones. The ABC has the disadvantage that it is a global boundary condition, i.e., it specifies a coupling of the values of the solution on the whole artificial boundary by some integral expression. The same holds for the DtN operator, which can be regarded as some sort of ABC. There exist in general only approximations for an ABC, which work well when the wave incidence is nearly normal, but not so well when it is very oblique. Some references for ABC are Nataf (2006) and Tsynkov (1998). In the case of PML, an absorbing layer of finite depth is placed around the truncated boundary so as to absorb the outgoing waves and reduce as much as possible their reflections back into the truncated domain's interior. On the absorbing layer, the problem is stated using a dissipative wave equation. For further details on PML we refer to Johnson (2008). The IEM, on the other hand, avoids the need of an artificial boundary by partitioning the complement of the truncated domain into a finite amount of so-called infinite elements. These infinite elements reduce to finite elements on the coupling surface and are described in some appropriate coordinate system. References for the IEM and likewise for the other techniques are Ihlenburg (1998) and Marburg & Nolte (2008). Interesting reviews of several of these methods can be also found in Thompson (2005) and Zienkiewicz & Taylor (2000). On the whole, once the domain is truncated with any one of the mentioned techniques, the problem can be solved numerically by using the FEM, the FDM, or some other numerical method that works well with bounded domains. This approach has nonetheless the drawback that the discretization of the additional truncated boundary may produce undesired reflections of the outgoing waves back towards the interior of the truncated domain, due the involved numerical approximations.

It is in fact the second approach, i.e., the integral equation techniques, the one that is of our concern throughout this thesis. This approach takes advantage of the fact that the wave propagation problem can be converted into an integral equation with the help of a Green's function. The integral equation is built in such a way that its support lies on a bounded region, e.g., the domain's boundary. Even though we mentioned that this approach may not be so practical to find an analytic solution, it becomes very useful when it is combined with an appropriate numerical method to solve the integral equation. Typically either a collocation method or a finite element method is used for this purpose. The latter is based on a variational formulation and is thus numerically more stable and accurate than the former, particularly when the involved geometries contain corners or are otherwise complicated. At the end, the general solution of the problem is retrieved by means of an integral representation formula that requires the solution of the previously solved integral equation. Of course, integral equation techniques can be likewise used to solve wave propagation problems in bounded domains. A big advantage of these techniques is their simplicity to represent the far field of the solution. Some references on integral equation techniques are the books of Hsiao & Wendland (2008), Nédélec (2001), and Steinbach (2008).

The drawback of integral equation techniques is their more complex mathematical treatment and the requirement of knowing the Green's function of the system. It is the Green's function that stores the information of the system's physics throughout the considered domain and which allows to collapse the problem towards an integral equation. The Green's function is usually problematic to integrate, since it corresponds to the solution of the homogeneous system subject to a singularity load, e.g., the electrical field arising from a point charge. Integrating such singular fields is not easy in general. For simple element geometries, like straight segments or planar triangles, analytical integration can be used. For more general elements it is possible to design purely numerical schemes that adapt to the singularity, but at a great computational cost. When the source point and target element where the integration is done are far apart, then the integration becomes easier due to the smooth asymptotic decay of the Green's function. It is this feature that is typically employed in schemes designed to accelerate the involved computations, e.g., in fast multipole methods (FMM). A reference for these methods is Gumerov & Duraiswami (2004).

In some particular cases the differential problem can be stated equivalently as a boundary integral equation, whose support lies on the bounded boundary. For example, this occurs in (bounded) obstacle scattering, where fields in linear homogeneous media are involved. Some kind of Green's integral theorem is typically used for this purpose. This way, to solve the wave propagation problem, only the calculation of boundary values is required rather than of values throughout the unbounded exterior domain. The technique that solves such a boundary integral equation by means of the finite element method is called the boundary element method (BEM). It is sometimes also known as the method of moments (MoM), specifically in electromagnetics, or simply as the boundary integral equation method (BIEM). The BEM is in a significant manner more efficient in terms of computational resources for problems where the surface versus volume ratio is small. The dimension of a problem expressed in the domain's volume is therefore reduced towards its boundary surface, i.e., one dimension less. The matrix resulting from the numerical discretization of the problem, though, becomes full, and to build it, as already mentioned, singular integrals have to be evaluated. The application of the BEM can be schematically described through the following steps:

1. Definition of the differential problem.
2. Calculation of the Green's function.
3. Derivation of the integral representation.
4. Development of the integral equation.
5. Rearrangement as a variational formulation.
6. Implementation of the numerical discretization.
7. Construction of the linear matrix system.
8. Computational resolution of the problem.
9. Graphical representation of the results.

The BEM is only applicable to problems for which Green's functions can be calculated, which places considerable restrictions on the range and generality of the problems to which boundary elements can be usefully applied. We remark that non-linearities and inhomogeneous media can be also included in the formulation, although they generally introduce volume integrals in the integral equation, which of course require the volume to be discretized before attempting to solve the problem, and thus removing one of the main advantages of the BEM. A good general survey on the BEM can be found in the article of Costabel (1986). Its implementation in obstacle scattering and some notions on FMM can be found in Terrasse & Abboud (2006). Other references for this method are Becker (1992), Chen & Zhou (1992), and Kirkup (2007). We note also the interesting historical remarks on boundary integral operators performed by Costabel (2007).

We mention finally that there is still an active research going on to study these numerical methods more deeply, existing also a great variety of so-called hybrid methods, where two or more of the techniques are combined together. A reference on this subject is the book of Brezzi & Fortin (1991).

1.2.3 Wave scattering and impedance half-spaces

Scattering is a general physical process whereby waves of some form, e.g., light, sound, or moving particles, are forced to deviate from a straight trajectory by one or more localized non-uniformities in the medium through which they pass. These non-uniformities are called scatterers or scattering centers. There exist many types of scatterers, ranging from microscopic particles to macroscopic targets, including bubbles, density fluctuations in fluids, surface roughness, defects in crystalline solids, among many others. In mathematics and physics, the discipline that deals with the scattering of waves and particles is called scattering theory. This theory studies basically how the solutions of partial differential equations without scatterer, i.e., freely propagating waves or particles, change when interacting with its presence, typically a boundary condition or another particle. We speak of a direct scattering problem when the scattered radiation or particle flux is to be determined, based on the known characteristics of the scatterer. In an inverse scattering problem, on the other hand, some unknown characteristic of an object is to be determined, e.g., its shape or internal constitution, from measurement data of its radiation or its scattered particles. Some references on scattering are Felsen & Marcuwitz (2003), Lax & Phillips (1989), and Pike & Sabatier (2002). For inverse scattering we refer to Potthast (2001).

Our concern throughout the thesis is specifically about direct obstacle scattering, where the scatterer (i.e., the obstacle) is given by an impenetrable macroscopic target that is modeled through a boundary condition. For a better understanding of the involved phenomena and due their inherent complexity, we consider only scalar linear wave propagation in time-harmonic regime, i.e., the partial differential equation of our model is given either by the Helmholtz or the Laplace equation. We observe that the latter equation is in fact the limit case of the former as the frequency tends towards zero. The time-harmonic regime implies that the involved system is independent of time and that only a single frequency is taken into account. If desired, time-dependent solutions of the system can be then constructed with the help of the Fourier transform (vid. Section A.7), by combining the solutions obtained for different frequencies. Alternatively, the solutions of a time-dependent system can be directly computed by means of retarded potentials (cf. Barton 1989, Butkov 1968, Felsen & Marcuwitz 2003). Time-dependent scattering is also considered in Wilcox (1975). Once the models for these scalar linear partial differential equations are well understood, then more complex types of waves can be taken into account, e.g., electromagnetic or elastic waves. The Helmholtz and Laplace equations can be thus regarded as a more simplified case of other wave equations.

The resolution of scattering problems for bounded obstacles with arbitrary shape by means of integral equation techniques is in general well-known, particularly when dealing with Dirichlet or Neumann boundary conditions. A Dirichlet boundary condition, named after the German mathematician Johann Peter Gustav Lejeune Dirichlet (1805–1859), specifies the value of the field at the boundary. A Neumann boundary condition, on the other hand, specifies the value of the field's normal derivative at the boundary, and receives its name from the German mathematician Carl Gottfried Neumann (1832–1925), who is considered one of the initiators of the theory of integral equations. The Green's function of the system is of course also well-known, and it is obtained directly from the fundamental solution of the involved wave equation, i.e., the Helmholtz or the Laplace equation. This applies also to the radiation condition to be imposed at infinity, which is known as the Sommerfeld radiation condition in honor of the German theoretical physicist Arnold Johannes Wilhelm Sommerfeld (1868–1951), who made invaluable contributions to quantum theory and to the classical theory of electromagnetism. We remark that in particular the problem of the Laplace equation around a bounded obstacle is not strictly speaking a wave scattering problem but rather a perturbation problem, and likewise at infinity we speak of an asymptotic decaying condition rather than of a radiation condition. Some references that we can mention, among the many that exist, are Kress (2002), Nédélec (2001), and Terrasse & Abboud (2006). We mention also the interesting results about radiation conditions in a rather general framework described by Costabel & Dauge (1997).

In the case of an impedance boundary condition, the general agreement is that the theory for bounded obstacles is well-known, but it is rather scarcely discussed in the literature. An impedance boundary condition specifies a linear combination of the field's value and of its normal derivative at the boundary, i.e., it acts as a weighted combination of Dirichlet and Neumann boundary conditions. It is also known as a third type or Robin boundary condition, after the French mathematical analyst and applied mathematician Victor Gustave Robin (1855–1897). Usually the emphasis is given to Dirichlet and Neumann boundary conditions, probably because they are simpler to treat and because with an impedance boundary condition the existence and uniqueness of the problem can be only ensured almost always, but not always. Some of the references that include the impedance boundary condition are Alber & Ramm (2009), Colton & Kress (1983), Hsiao & Wendland (2008), Filippi, Bergassoli, Habault & Lefebvre (1999), and Kirsch & Grinberg (2008).

When the obstacle in a scattering problem is no longer bounded, then usually a different Green's function and a different radiation condition have to be taken into account to find a solution by means of integral equation techniques. These work well only when the scattering problem is at most a compact perturbation of the problem for which the Green's function was originally determined, i.e., when these problems differ only on a compact portion of their involved domains. An unbounded obstacle, e.g., an infinite half-space, constitutes clearly a non-compact perturbation of the full-space.

We are particularly interested in solving scattering problems either on two- or three-dimensional half-spaces, where the former are also simply referred to as half-planes and the latter just as half-spaces. If Dirichlet or Neumann boundary conditions are considered, then the Green's function is directly found through the image method. Furthermore, the same Sommerfeld radiation condition continues to hold in this case.

For an impedance half-space, i.e., when an impedance boundary condition is used on a half-space, the story is not so straightforward. As we already pointed out, the image method can be also used in this case to compute the Green's function, but the results are far from being explicit and some of the obtained terms are only known in integral form, as so-called Sommerfeld-type integrals (cf. Casciato & Sarabandi 2000, Taraldsen 2005). The difficulties arise from the fact that an impedance boundary condition allows the propagation of surface waves along the boundary, whose relation with a point source is far from simple. Another method that we can mention and that is used to solve this kind of problems is the Wiener-Hopf technique, which yields an exact solution to complex integral equations and is based on integral transforms and analyticity properties of complex functions. Further details can be found in Davies (2002), Dettman (1984), and Wright (2005).

We remark that in scattering problems on half-spaces, or likewise on compact perturbations of them, there appear two different kinds of waves: volume and surface waves. Volume waves propagate throughout the domain and behave in the same manner as waves propagating in free-space. They are linked to the wave equation under consideration, i.e., to the Helmholtz equation, since for the Laplace equation there are no volume waves. Surface waves, on the other hand, propagate only near the boundary and are related to the considered boundary condition. They decrease exponentially towards the interior of the domain and may appear as much for the Helmholtz as for the Laplace equation. They exist only when the boundary condition is of impedance-type, but not when it is of Dirichlet- or Neumann-type, which may explain why the latter conditions are simpler in their treatment.

a) Helmholtz equation

The impedance half-space wave propagation problem for the Helmholtz equation was at first formulated by Sommerfeld (1909), who was strongly motivated by the around 1900 newly established wireless telegraphy of Maxwell, Hertz, Bose, Tesla, and Marconi, among others. Sommerfeld wanted to explain why radio waves could travel long distances across the ocean, and thus overcome the curvature of the Earth. In his work, he undertook a detailed analysis of the radiation problem for an infinitesimal vertical Hertzian dipole over a lossy medium, and as part of the solution he found explicitly a radial Zenneck surface wave, named after the German physicist and electrical engineer Jonathan Adolf Wilhelm Zenneck (1871–1959), who first described them (Zenneck 1907). Thus both Zenneck and

Sommerfeld obtained results that lent considerable credence to the view of the Italian inventor and marchese Guglielmo Marconi (1874–1937), that the electromagnetic waves were guided along the surface. Sommerfeld's solution was later criticized by the German mathematician Hermann Klaus Hugo Weyl (1885–1955), who published on the same subject (Weyl 1919) and who obtained a solution very similar to the one found by Sommerfeld, but without the surface-wave term. Sommerfeld (1926) returned later to the same problem and solved it using a different approach, where he confirmed the correctness of Weyl's solution. The apparent inclusion of a sign error in Sommerfeld's original work, which he never admitted, prompted much debate over several decades on the existence of a Zenneck-type surface wave and its significance in the fields generated by a vertical electric dipole. A more detailed account can be found in Collin (2004). The corrected formulation confirmed the existence of a surface wave for certain values of impedance and observation angles, but showed its contribution to the total field only significant within a certain range of distances, dependent on the impedance of the half-space. Thus, the concept of the surface wave as being the important factor for long-distance propagation lost favor. Further references on this historical discussion can be also found in the articles of Casciato & Sarabandi (2000), Nobile & Hayek (1985), Sarabandi, Casciato & Koh (1992), and Taraldsen (2004, 2005).

Just to finish the story, Kennelly (1902) and, independently, Heaviside (1902), had predicted before the existence of an ionized layer at considerable height above the Earth's surface. It was thought that such a layer could possibly reflect the electromagnetic waves back to Earth. Although it was not until Breit & Tuve (1926) showed experimentally that radio waves were indeed reflected from the ionosphere, that this became finally the accepted mechanism for the long-distance propagation of radio waves. We refer to Anduaga (2008) for a more detailed historical essay on the concept of the ionosphere.

Nonetheless, even if Sommerfeld's explanation proved later to be wrong, its problem remained (and still remains) of great theoretical interest. Since its first publication, it is an understatement to say that this problem has received a significant amount of attention in the literature with literally hundreds of papers published on the subject. Besides electromagnetic waves, the problem is also important for outdoor sound propagation (cf. Morse & Ingard 1961, Embleton 1996) and for water waves in shallow waters near the coast (cf. Mei, Stiassnie & Yue 2005, Herbich 1999).

Thus, as a way to state a brief account on the problem, Sommerfeld (1909), working in the field of electromagnetism, was the first to solve the spherical wave reflection problem, stated as a dipole source on a finitely conducting earth. Weyl (1919) reformulated the problem by modeling the radiation from a point source located above the earth as a superposition of an infinite number of elementary plane waves, propagating in different (complex) directions. Sommerfeld (1926) solved his problem again using integrals that were afterwards called of Sommerfeld-type. Van der Pol (1935) applied several ingenious substitutions that simplified the integrals appearing in the derivations. Norton (1936, 1937) expanded upon these and other results from Van der Pol & Niessen (1930) and, with the aid of equations by Wise (1931), generated the most useful results up to that time. Baños & Wesley (1953, 1954) and Baños (1966) obtained similar solutions by using the double saddle point method. Further developments on the propagation of radio waves can be also found in the book of Sommerfeld (1949). We remark that in electromagnetic scattering, the impedance boundary condition describes an obstacle which is not perfectly conducting, but does not allow the electromagnetic field to penetrate deeply into the scattering domain.

The greatest interest in the problem stemmed nonetheless from the acoustics community, to describe outdoor sound propagation. The acoustical problem of spherical wave reflection was first attacked by Rudnick (1947), who relied heavily on the electromagnetic theories of Van der Pol and Norton. Subsequently, Lawhead & Rudnick (1951*a*,*b*) and Ingard (1951) obtained approximate solutions in terms of the error function. Wenzel (1974) and Chien & Soroka (1975, 1980) obtained solutions containing a surface-wave term. Exhaustive lists of references with other solutions for the problem can be found in Habault & Filippi (1981) and in Nobile & Hayek (1985). We can mention on this behalf also the articles of Briquet & Filippi (1977), Attenborough, Hayek & Lawther (1980), Li, Wu & Seybert (1994), and Attenborough (2002), and more recently also Ochmann (2004) and Ochmann & Brick (2008), among the many others that exist. For the two-dimensional case, in particular, we can refer to the articles of Chandler-Wilde & Hothersall (1995*a*,*b*) and Granat, Tahar & Ha-Duong (1999).

The purpose of these articles is essentially the same: they try to compute in one way or the other the reflection of spherical waves (in three dimensions) or cylindrical waves (in two dimensions) on an impedance boundary. This corresponds to the computation of the Green's function for the problem, since spherical and cylindrical waves are originated by a point source. Books that consider this problem and other aspects of Green's functions are the ones of Greenberg (1971), DeSanto (1992), and Duffy (2001). The great variety of results for the same problem reflects its difficulty and its interest. The expressions found for the Green's function contain typically either complicated integrals, which derive from a Fourier transform or some other kind of integral transform, or unpractical infinite series expansions, which do not hold for all conditions or everywhere. There exists no relatively simple expression in terms of known elementary or special functions. For the treatment of the integrals, special integration contours are taken into account and at the end some parts are approximated by methods of asymptotic analysis like the ones of stationary phase or of steepest descent, the latter also known as the saddle-point approximation. Some references for these asymptotic methods are Bender & Orszag (1978), Estrada & Kanwal (2002), Murray (1984), and Wong (2001).

It is notably on this behalf that using a Fourier transform yields a manageable expression for the spectral Green's function (cf. Durán, Muga & Nédélec 2005*a*,*b*, 2006, 2009). In two dimensions, we considered this expression to compute numerically the spatial Green's function with the help of a fast Fourier transform (FFT) for the regular part, whereas its singular part was treated analytically (Durán, Hein & Nédélec 2007*a*,*b*). Further details of these calculations can be found in Hein (2006, 2007). This method allows to compute effectively the Green's function, without the use of asymptotic approximations, but it can become quite burdensome when building bigger matrixes for the BEM due the multiple evaluations required for the FFT.

Outdoor sound propagation is in fact the classic application for the Helmholtz equation stated in an impedance half-space, where the acoustic waves propagate freely in the upper half-space and interact with the ground, i.e., the impenetrable lower half-space, through an impedance boundary condition on their common boundary. The Helmholtz equation is derived directly from the scalar acoustic wave equation by assuming a time-harmonic regime. The acoustic impedance in this case corresponds to a (complex) proportionality coefficient that relates the normal velocity of the fluid, where the sound propagates, to the excess pressure on the boundary. A real impedance implies that the boundary is non-dissipative, whereas a strictly complex (i.e., non-real) impedance is associated with an absorbing boundary. We remark that the limit cases of the boundary condition of impedance-type, the ones of Dirichlet- and Neumann-type, correspond respectively to sound-soft and sound-hard boundary surfaces. For more details on the physics of the problem, we refer to DeSanto (1992), Embleton (1996), Filippi et al. (1999), and Morse & Ingard (1961). The use of an impedance boundary condition is validated and discussed in the articles of Attenborough (1983) and Bermúdez, Hervella-Nieto, Prieto & Rodríguez (2007).

There exists also some literature on experimental measurements for this topic. Extensive experimental studies of sound propagation horizontally near the ground, mainly over grass, are performed by Embleton, Piercy & Olson (1976), who even suggest the presence of surface waves. Different impedance versus frequency models for various types of ground surface are compared by Attenborough (1985). Studies of acoustic wave propagation over grassland and snow are developed by Albert & Orcutt (1990). In the paper of Albert (2003), experimental evidence is given that confirms the existence of acoustic surface waves in a natural outdoor setting, which in this case is above a snow cover. For a study of sound propagation in forests we refer to Tarrero et al. (2008). Extensive measurement results and theoretical models are also discussed by Attenborough, Li & Horoshenkov (2007).

The use of some BEM to solve the problem has also received some attention in the literature. Further references can be found in De Lacerda, Wrobel & Mansur (1997), De Lacerda, Wrobel, Power & Mansur (1998), and Li et al. (1994). For some two-dimensional applications of the BEM we cite Chen & Waubke (2007), Durán, Hein & Nédélec (2007*a*,*b*), and Granat, Tahar & Ha-Duong (1999). Some integral equations for this case are also treated in Chandler-Wilde (1997) and Chandler-Wilde & Peplow (2005). Integral equations in three dimensions for Dirichlet and Neumann boundary conditions, and the low-frequency case, can be found in Dassios & Kleinman (1999). For the appropriate radiation condition of the problem, and likewise for its existence and uniqueness, we refer to Durán, Muga & Nédélec (2005*a*,*b*, 2006, 2009).

b) Laplace equation

The impedance half-space wave propagation problem for the Laplace equation is particularly of great importance in hydrodynamics, since it describes linear surface waves on water of infinite depth. The interest for this problem can be traced back to December 1813, when the French Académie des Sciences announced a mathematical prize competition on the subject of surface wave propagation on liquid of indefinite depth. The prize was awarded in 1816 to the French mathematician and early pioneer of analysis Augustin Louis Cauchy (1789–1857), who submitted his entry in September 1815 and which was eventually published in Cauchy (1827). Another memoir, to record his independent work, was deposited in October 1815 by the French mathematician, geometer, and physicist Siméon Denis Poisson (1781–1840), one of the judges of the competition, which was published in Poisson (1818). Both memoirs are classical works in the field of hydrodynamics. For a more detailed historical account on the water-wave theory we refer to Craik (2004).

With the passage of time, the interest in the description of wave motion in the presence of submerged or floating bodies increased. The first study of wave motion caused by a submerged obstacle was carried out in the classical (and often reprinted) text of Lamb (1916), who analyzed the two-dimensional wave motion due to a submerged cylinder. Further studies dealing with simple submerged obstacles were done by Havelock (1917, 1927), for spheres and doublets, and by Dean (1945), for plane barriers.

A major breakthrough in the field arrived nonetheless with the classic works on the motion of floating bodies by John (1949, 1950), who showed how the boundary-value problem could be reduced to an integral equation over the wetted portion of the partly immersed body. John studied the problem in general form, stating necessary conditions for the uniqueness of its solution. He also gave expressions in the form of discrete eigenfunction expansions for the Green's functions of the problem, in two and three dimensions, and considering finite and infinite water depth. His work inspired (and still inspires) a vast amount of literature, particularly in the subjects of the existence and uniqueness of solutions, the computation of Green's functions, and the development of integral equation methods.

A standard reference that synthesizes the known theory up to its time is the thorough and insightful article by Wehausen & Laitone (1960). It includes also the known expressions for Green's functions. A closely related article is Wehausen (1971). More recent references on these topics are the books of Mei (1983), Linton & McIver (2001), Kuznetsov, Maz'ya & Vainberg (2002), and Mei, Stiassnie & Yue (2005). The classical representation of these Green's functions, in three dimensions, is in terms of a semi-infinite integral involving a Bessel function (vid. Subsection A.2.4) and a Cauchy principal-value singularity (vid. Subsection A.6.5). Separate expressions exist for infinite and finite (constant) depth of the fluid, but their forms are similar and the infinite-depth limit can be recovered as a special case of the finite-depth integral representation. According to Newman (1985), the principal drawback of these expressions is that they are extremely time-consuming to evaluate numerically. Some articles dealing with the finite-depth Green's function are the ones of Angell, Hsiao & Kleinman (1986), Black (1975), Chakrabarti (2001), Fenton (1978), Linton (1999), Macaskill (1979), Mei (1978), Pidcock (1985), and Xia (2001).

In the case of infinite-depth water in three dimensions, a simpler analytic representation for the source potential or Green's function exists as the sum of a finite integral, with a monotonic integrand involving elementary transcendental functions, and a wave-like term of closed form involving Bessel and Struve functions (vid. Subsection A.2.7). This expression, which was suggested by Havelock (1955), has been rederived or publicized in different forms by Kim (1965), Hearn (1977), Noblesse (1982), Newman (1984*b*, 1985), Pidcock (1985), and Chakrabarti (2001). Other expressions for this Green's function were developed by Moran (1964), Hess & Smith (1967), Dautray & Lions (1987), and Peter & Meylan (2004). Likewise, analogous expressions for the two-dimensional Green's function are considered in the works of Thorne (1953), Kim (1965), Macaskill (1979), and Greenberg (1971). A more general two-dimensional case that takes surface tension into account was considered by Harter, Abrahams & Simon (2007), Harter, Simon & Abrahams (2008), and Motygin & McIver (2009), using potentials expressed in terms of exponential integrals (vid. Subsection A.2.3). Analogous observations to the ones of the Helmholtz equation can be made also for the case of the Laplace equation.

Water-wave motion near floating or submerged bodies is the classic application for the Laplace equation stated in an impedance half-space. The Laplace equation is obtained by considering the dynamic of an incompressible inviscid fluid, as is the case with water. The impedance boundary condition corresponds to the linearized free-surface condition, which allows the propagation of (water) surface waves. The impedance in this case can be regarded as a wave number for the surface waves, which acts in an equivalent manner as the wave number for the Helmholtz equation, but now only along the boundary surface. Again, a real impedance implies that the boundary is non-dissipative, whereas a strictly complex impedance is associated with an absorbing boundary. Further details on the physical aspects of the problem can be found in Kuznetsov, Maz'ya & Vainberg (2002) and Wehausen & Laitone (1960).

Reviews of numerical methods to solve water-wave problems and further references can be found in Mei (1978) and Yeung (1982). A review of ocean waves interacting with ice is done by Squire, Dugan, Wadhams, Rottier & Liu (1995). A computation of a Green's function for this case can be found in Squire & Dixon (2001). Boundary integral equations are developed in Angell, Hsiao & Kleinman (1986) and Sayer (1980). For the use of the BEM we refer to the articles of Hess & Smith (1967), Hochmuth (2001), Lee, Newman & Zhu (1996) and Liapis (1992, 1993). Resonances for water-wave problems are studied in Hazard & Lenoir (1993, 1998, 2002).

1.2.4 Applications

Wave propagation problems in impedance half-spaces, or in compact perturbations of them, have many applications in science and engineering. We already mentioned the applications to outdoor sound propagation (Filippi et al. 1999, Morse & Ingard 1961), to radio wave propagation above the ground (Sommerfeld 1949), and to water waves in shallow waters near the coast (Mei et al. 2005, Herbich 1999), in the case of the Helmholtz equation, and to the motion of water waves near floating or submerged bodies (Kuznetsov et al. 2002, Wehausen & Laitone 1960), in the case of the Laplace equation. Further specific applications include the scattering of light by a photonic crystal (Joannopoulos et al. 2008, Sakoda 2005, Yasumoto 2006, Durán, Guarini & Jerez-Hanckes 2009), the computation of harbor resonances in coastal engineering (Mei et al. 2005, Panchang & Demirbilek 2001), and the treatment of elliptic partial differential equations, specifically the Laplace equation, with an oblique-derivative boundary condition (Gilbarg & Trudinger 1983, Keller 1981, Paneah 2000). This thesis is concerned with the latter two of these applications.

a) Harbor resonances in coastal engineering

A harbor (sometimes also spelled as harbour) is a partially enclosed body of water connected through one or more openings to the sea. Conventional harbors are built along a coast where a shielded area may be provided by natural indentations and/or by breakwaters protruding seaward from the coast. Harbors provide anchorage and a place of refuge for ships. Key features of all harbors include shelter from both long and short period open sea waves, easy safe access to the sea in all types of weather, adequate depth and maneuvering room within the harbor, shelter from storm winds, and minimal navigation channel dredging. A harbor can be sometimes subject to a so-called harbor oscillation or surging, which corresponds to a nontidal vertical water movement. Usually these vertical motions are low, but when oscillations are excited by a tsunami or a storm surge, they may become quite large. Variable winds, air oscillations, or surf beat may also cause oscillations. Nonetheless, the most studied excitation is caused by incident tsunamis, which have typical periods from a few minutes to an hour, and are originated from distant earthquakes. If the total duration of the tsunami is sufficiently long, oscillations excited in the harbor may persist for days, resulting in broken mooring lines, damaged fenders, hazards in berthing and loading or in navigation through the entrance, and so on. Sometimes incoming ships have to wait outside the harbor until oscillations within subside, causing costly delays. Harbor oscillations are discussed in the books of Mei (1983), Mei et al. (2005), and Herbich (1999). For a single and comprehensive technical document about coastal projects we refer to the Coastal Engineering Manual of the U.S. Army Corps of Engineers (2002).

To understand roughly the physical mechanism of these oscillations, we consider a harbor with the entrance in line with a long and straight coastline. Onshore waves are partly reflected and partly absorbed along the coast. A small portion is however diffracted through the entrance into the harbor and reflected repeatedly by the interior boundaries. Some of the reflected wave energy escapes the harbor and radiates again to the ocean, while some of it stays inside. If the wavetrain is of long duration, and the incident wave frequency is close to a standing-wave frequency in the closed basin, then a so-called resonance occurs in the basin, i.e., even a relatively weak incident wave of such characteristics can induce a large response in the harbor. When a harbor is closed and the damping is neglected, the free-wave motion is known to be the superposition of normal modes of standing waves with a discrete spectrum of characteristic frequencies. When a harbor has a small opening and is subject to incident waves we may expect a resonance whenever the frequency of the incident waves is close to a characteristic frequency of the closed harbor.

Resonances are therefore closely related to the phenomena of seiching (in lakes and harbors) and sloshing (in coffee cups and storage tanks), which correspond to standing waves in enclosed or partially enclosed bodies of water. These phenomena have been observed already since very early times. Forel (1895) quotes a vivid description of seiching in the Lake of Constance in 1549 from “Les Chroniques de Cristophe Schulthaiss”, and

Darwin (1899) refers to seiching in the Lake of Geneva in 1600 with a peak-to-peak amplitude of over one meter. Observations in cups and pots doubtless predate recorded history. Scientific studies date from Merian (1828) and Poisson (1828–1829), and especially from the observations in the Lake of Geneva by Forel (1895), which began in 1869. A thorough and historical review of the seiching phenomenon in harbors and further references can be found in Miles (1974).

A resonance of a different type is given by the so-called Helmholtz mode when the oscillatory motion inside the harbor is much slower than each of the normal modes (Burrows 1985). It corresponds to the resonant mode with the longest period, where the water appears to move up and down unison throughout the harbor, which seems to have been first studied by Miles & Munk (1961). This very long period mode appears to be particularly significant for harbors responding to the energy of a tsunami, and for several harbors on the Great Lakes that respond to long-wave energy spectra generated by storms. We remark that from the mathematical point of view, resonances correspond to poles of the scattering and radiation potentials when they are extended to the complex frequency domain (cf. Poisson & Joly 1991). Harbor resonance should be avoided or minimized in harbor planning and operation to reduce adverse effects such as hazardous navigation and mooring of vessels, deterioration of structures, and sediment deposition or erosion within the harbor.

Examples of harbor resonances are the Ciutadella inlet in the Menorca Island on the Western Mediterranean (Marcos, Monserrat, Medina & Lomónaco 2005), the Duluth-Superior Harbor in Minnesota on the Lake Superior (Jordan, Stortz & Sydor 1981), the Port Kembla Harbour on the central coast of New South Wales in Australia (Luick & Hinwood 2008), the Los Angeles Harbor Pier 400 in California (Seabergh & Thomas 1995), and the port of Ploče in Croatia on the Adriatic Sea (Vilibić & Mihanović 2005).

Considerable effort has been devoted to achieving a good understanding of the phenomena of harbor resonance. Lamb (1916) analyzed the free oscillation in closed rectangular and circular basins. His solutions then clarified the natural periods and modes of free surface oscillations related to these special configurations. As the first but important step to approach the practical situation, McNown (1952) studied the forced oscillation in a circular harbor which is connected to the open sea through a narrow mouth. He made the assumption that standing wave conditions are always formed at the harbor entrance when resonance occurs. Since the radiation effect was ruled out, he showed that a resonant harbor behaves the same as a closed basin. Similar research was also carried out by Kravtchenko & McNown (1955) on rectangular harbors.

Since the paper of Miles & Munk (1961), who first treated harbor oscillations by a scattering theory, the study of harbor resonance has been steadily progressing both theoretically and experimentally. Miles & Munk (1961) considered the wave energy radiation effect expanding offshore from the harbor entrance and applied a Green's function to analyze the harbor oscillation. They even found that the wider the harbor mouth, the smaller the amplitude of the resonant oscillation. That is, narrowing the harbor entrance does not diminish resonant oscillation, which contradicts common sense based on the conventional reasoning for a non-resonant harbor, where less wave energy is expected to be transmitted into the harbor through a smaller opening. Miles & Munk (1961) referred to this phenomenon as the harbor paradox. Additional important contributions were made by Le Méhauté (1961), Ippen & Goda (1963), Raichlen & Ippen (1965), and Raichlen (1966). These studies considered the effect of radiation through the entrance of the harbor and the resulting frequency responses of the harbor oscillations became fairly close to the experimentally observed ones. Other rigorous solutions for the problem were presented by Lee (1969, 1971), who considered rectangular and circular harbors with openings located on a straight coastline. He discovered that the trapping of energy by the harbor leads to an amplitude of oscillation that is far greater than the one of the incident wave. Similarly, Mei & Petroni (1973) dealt with a circular harbor protruding halfway into the open sea. Theories to deal with arbitrary harbor configurations were available after Hwang & Tuck (1970) and Lee (1969, 1971), who worked with boundary integral equation methods to calculate the oscillation in harbors of constant depth with arbitrary shape. Mei & Chen (1975) developed a hybrid-boundary-element technique to also study harbors of arbitrary geometry. Harbor resonances using the FEM are likewise computed in Walker & Brebbia (1978). A comprehensive list of references can be found in Yu & Chwang (1994).

The mild-slope equation, which describes the combined effects of refraction and diffraction of linear water waves, was first suggested by Eckart (1952) and later rederived by Berkhoff (1972*a*,*b*, 1976), Smith & Sprinks (1975), and others, and is now well-accepted as the method for estimating coastal wave conditions. The underlying assumption of this equation is that evanescent modes (locally emanated waves) are not important, and that the rate of change of depth and current within a wavelength is small. The mild-slope equation is a usually expressed in an elliptic form, and it turns into the Helmholtz equation for uniform water depths. Since then, different kinds of mild-slope equations have been derived (Liu & Shi 2008). A detailed survey of the literature on the mild-slope and its related equations is provided by Hsu, Lin, Wen & Ou (2006). Some examinations on the validity of the theory are performed by Booij (1983) and Ehrenmark & Williams (2001).

Along rigid, impermeable vertical walls a Neumann boundary condition is used, since there is no flow normal to the surface. However, in general an impedance boundary condition is used along coastlines or permeable structures, to account for a partial reflection of the flow on the boundary (Demirbilek & Panchang 1998). A study of harbor resonances using an approximated DtN operator and a model based on the Helmholtz equation with an impedance boundary condition on the coast was done by Quaas (2003).

An alternative parabolic equation method to solve the problem was developed by Radder (1979) and Kirby & Dalrymple (1983), which approximates the mild-slope equation. A sea-bottom friction and absorption boundary was considered by Chen (1986) for a hybrid BEM to analyze wave-induced oscillation in a harbor with arbitrary shape and depth. Berkhoff, Booy & Radder (1982) described and compared the computational results for the models of refraction, of parabolic refraction-diffraction, and of full refraction-diffraction. Tsay, Zhu & Liu (1989) considered the effects of topographical variation and energy dissipation, and developed a finite element numerical model to investigate wave refraction, diffraction, reflection, and dissipation. Chou & Han (1993) employed a boundary element method and under the consideration of the effect of partial reflection along boundaries to develop a numerical method for predicting wave height distribution in a harbor of arbitrary shape and variable water depth. Nardini & Brebbia (1982) proposed a DRBEM (dual reciprocity boundary element method), which was also studied by Hsiao, Lin & Fang (2001) and Hsiao, Lin & Hu (2002). The infinite element method was applied to the problem by Chen (1990). Interesting reviews of the theoretical advances on wave propagation modeling in coastal engineering can be found in Mei & Liu (1993) and Liu & Losada (2002). A review that brings together the large amount of literature on the analytical study of free-surface wave motion past porous structures is performed by Chwang & Chan (1998).

The study of harbor resonances becomes particularly important for countries with high seismicity and maritime harbors subject to tsunamis such as Chile. A tragical and recent example of the involved devastation was given by the 2010 Chilean earthquake, which occurred offshore from the Maule Region in south central Chile on February 27, 2010. Noteworthy, it had already been predicted by Ruegg et al. (2009). After the earthquake, the coast was afflicted by tsunami waves. At the port city of Talcahuano waves with amplitude up to 5 meters high were observed and the sea level rose above 2.4 meters. The tsunami caused serious damage to port facilities and lifted boats out of the water. A good harbor design should protect the waters of the harbor from such events as best as possible, and it is therefore of great interest to have a good knowledge of the appearing resonances.

b) Oblique-derivative half-plane Laplace problem

As a more theoretical application, we are interested in the study of elliptic partial differential operators, particularly the Laplace equation, with an oblique-derivative (impedance) boundary condition. This kind of operators is characterized by the inclusion of tangential derivatives in the boundary condition. We speak of a (purely) oblique-derivative boundary condition when it combines only tangential and normal derivatives, whereas a combination of tangential derivatives and an impedance boundary condition is referred to as an oblique-derivative impedance boundary condition.

The purely oblique-derivative problem for a second-order elliptic partial differential operator was first stated by the great French mathematician, theoretical physicist, and philosopher of science Jules Henri Poincaré (1854–1912) in his studies on the theory of tides (Poincaré 1910). Since then, the so-called Poincaré problem has been the subject of many publications (cf. Egorov & Kondrat'ev 1969, Paneah 2000), and it arises naturally when determining the gravitational fields of celestial bodies. Its main interest lies in the fact that it corresponds to a typical degenerate elliptic boundary-value problem where the vector field of its solution is tangent to the boundary of the domain on some subset. The Poincaré problem for harmonic functions, in particular, arises in semiconductor physics and considers constant coefficients for the oblique derivative in the boundary condition (Krutitskii & Chikilev 2000). It allows to describe the Hall effect, i.e., when the direction of an electric current and the direction of an electric field do not coincide in a semiconductor due the presence of a magnetic field (Krutitskii, Krutitskaya & Malysheva 1999). The two-dimensional Poincaré problem for the Laplace equation is treated in Lesnic (2007), Trefethen & Williams (1986), and further references can be also found in Lions (1956).

Of special interest is the oblique-derivative impedance Laplace problem stated in a half-space, and particularly the determination of its Green's function, which describes outgoing oblique surface waves that emanate from a point source and which increase or decrease exponentially along the boundary, depending on the obliqueness of the derivative in the boundary condition. An integral representation for this Green's function in half-spaces of three and higher dimensions was developed by Gilbarg & Trudinger (1983). Using an image method, it was later generalized by Keller (1981) to a wider class of equations, including the wave equation, the heat equation, and the Laplace equation. Its use for more general linear uniformly elliptic equations with discontinuous coefficients can be found in the articles of Di Fazio & Palagachev (1996) and Palagachev, Ragusa & Softova (2000). The generalization of this image method to wedges is performed by Gautesen (1988).

For the two-dimensional case and when dealing with the Laplace equation, there exists no representation of the Green's function, except the already mentioned cases when the oblique derivative becomes a normal one.

1.3 Objectives

The main objective of this thesis is to compute the Green's function for the Laplace and Helmholtz equations in two- and three-dimensional impedance half-spaces, and to use it for solving direct wave scattering problems in compactly perturbed half-spaces by developing appropriate integral equation techniques and a corresponding boundary element method. The goal is to give a numerically effective and efficient expression for the Green's function, and to determine its far field. The developed integral equations are to be supported only on a bounded portion of the boundary, and they have to work well for arbitrary compact perturbations towards the upper half-space, as long as the considered boundary is regular enough. It is also of interest to derive expressions for the far field of the solution of the scattering problem. The developed techniques are to be programmed in Fortran, implementing benchmark problems to test these calculations and the computational subroutines. Thus the idea in this thesis is to continue and extend the preliminary work performed in Hein (2006, 2007) and in Durán, Hein & Nédélec (2007*a*,*b*).

Another objective is to use the developed expressions and techniques to solve some interesting applications in science and engineering. One of the applications to consider deals with the computation of harbor resonances in coastal engineering, enhancing the model of Quaas (2003) by working with an impedance boundary condition and solving the problem by using integral equations instead of a DtN operator. The other application considers the calculation of the Green's function for the oblique-derivative impedance half-plane Laplace problem, which generalizes the techniques used in the computation of the other Green's functions from this thesis.

The interest behind this study is to comprehend better, from the mathematical point of view, the interaction between volume and surface waves caused by a point source in impedance half-spaces, and their application to some scattering problems in engineering. Only the linear, scalar, and time-harmonic cases are considered here, to simplify the analysis and to avoid additional complications. We include the study of the Laplace equation, where only surface waves appear, since the problem is somewhat simpler and permits a far better understanding of the treatment for the Helmholtz equation, particularly in the two-dimensional case.

To allow a better comprehension of the treated topics, this thesis is intended to be as self-contained as possible. Therefore a quick survey of the most important aspects of the mathematical and physical background and a detailed analysis of the relatively well-known full-space problems are also included. Additionally, a comprehensive list of references is given whenever possible, so as to ensure extensive further reading on the involved subjects if such an interest arises.

1.4 Contributions

Essentially, this thesis concentrates and recreates some of the most important elements of the widely dispersed knowledge on full- and half-space Green's functions for the Laplace and Helmholtz operators, and their associated integral equations, in a single document with a coherent and homogeneous notation. By doing so, new expressions are found and a better understanding of the involved techniques is achieved.

The main contribution of the thesis is the rigorous development of expressions for the Green's functions of the Helmholtz and Laplace operators in impedance half-spaces, in two and three dimensions, and their use to solve direct wave scattering problems by means of boundary integral equations. These expressions are characterized in terms of finite combinations of elementary functions, known special functions, and their primitives. In the case of the two-dimensional Laplace equation even a new explicit representation is found, based on exponential integrals and expressed in (2.94). A more general representation, based likewise on exponential integrals, is also developed for the Green's function of the oblique-derivative half-plane Laplace problem, which has not been computed before and is given explicitly in (7.41). For the other cases, effective numerical procedures are derived to evaluate the Green's functions everywhere and on all the values of interest. For the two-dimensional Helmholtz equation, we perform an improvement over our previous results in the numerical procedure (Durán et al. 2007*a*,*b*), which is now more efficient, uses a numerical quadrature formula instead of a fast Fourier transform, works better with complex impedances and wave numbers, and may be also evaluated in the complementary half-plane. The details are delineated in Section 3.5. The series-based representation for the Green's function of the three-dimensional Laplace equation (4.113), even if it is similar in a certain way to others found in the literature (cf., e.g., Noblesse 1982), it is derived in an rigorous and independent manner that sheds new light on its properties. The evaluation of the representation for the three-dimensional Helmholtz equation, specified in Section 5.5, corresponds to a direct numerical integration of the primitive-based expression of the Green's function, which can be adapted without difficulty to the other cases.

Another important contribution is the proper understanding of the limiting absorption principle and its interpretation, in the sense of distributions, as the appearance of additional Dirac masses for the spectral Green's function. This effect, which has not been particularly pointed out in the literature, allows us to treat all the involved Fourier integrals in the sense of Cauchy principal values and is expressed in (2.64), (3.59), (4.70), and (5.65). A different approach for the same topic is undertaken in Section 7.3 for the oblique-derivative case, where the additional appearing terms are interpreted as the solution of the homogeneous problem with a proper scaling, which is justified from the radiation condition, and their effect is expressed in (7.22).

The derived expressions for the Green's function yield better light on the interaction between the volume and the surface wave parts of the system's response to a point source, even in the presence of dissipation, and are coherent with results for the complex image method used to solve this problem (cf. Casciato & Sarabandi 2000, Taraldsen 2004, 2005). In particular, they retrieve the image source point on the complementary half-space and the continuous source distribution that stems from this point towards infinity along a line that is perpendicular to the half-space's boundary, increasing exponentially.

The herein treated wave scattering problems consider arbitrary compact perturbations towards the upper half-space and the associated integral representations and equations used to solve them are derived with great detail and have their support only on the perturbed portion of the boundary. In particular, a correct expression is given for the boundary integral representation on the unperturbed portion of the boundary (cf. Durán et al. 2007*a*,*b*). The integral equations are solved by using a boundary element method, and neither hybrid techniques nor domain truncation are required. Compact perturbations towards the lower half-space are not considered herein, but the thorough study of the singularities of the Green's functions (another contribution of this thesis) is the first step towards that direction to develop them in the near future.

A state of the art is developed for the full-space impedance Laplace and Helmholtz problems, since the theory for them is more or less well-known and they are closely related to the half-space problems. The main singularity of the associated Green's functions is the same, and several other aspects are analogous in both kinds of problems.

Another contribution is the development of computational subroutines to solve the considered problems, and the numerical results that are obtained by their execution. The programming is in general not easy and requires a careful treatment of the involved singular integrals (due the singularities of the Green's functions) to build the full matrixes that stem from the boundary element method. The subroutines are likewise programmed and tested for the full-space problems.

The application of the developed techniques to the computation of harbor resonances in coastal engineering is also a contribution of this thesis, which shows their use in the resolution of a practical problem in engineering.

1.5 Outline

To fulfill the objectives, this thesis is structured in eight chapters and five appendixes. Each chapter and each appendix is in his turn divided into sections and further into subsections in order to expose the contents in the hopefully most clear and accessible way for the reader. Each one starts with a short introduction that yields more light about its contents. A list of references is also included in each one of them.

Chapter I, the current chapter, presents a broad introduction to the thesis. The more general aspects are discussed and the framework that connects its different parts is described. It includes a short foreword, the motivation and overview, the objectives, the contributions, and the current outline.

In Chapters II, III, IV, and V we study the perturbed half-space impedance problems of the Laplace and Helmholtz equations in two and three dimensions respectively, using integral equation techniques and the boundary element method. These chapters include the main contributions of this thesis, particularly the computation of the Green's functions and their far-field expressions, and the development of the associated integral equations.

The following two chapters contain the applications of the developed techniques. Chapter VI deals with the computation of harbor resonances in coastal engineering, and in Chapter VII the Green's function for the oblique-derivative half-plane Laplace problem is derived and given explicitly.

Chapter VIII incorporates the conclusion of this thesis, including a short discussion on the results and some perspectives for future research. It is followed by the bibliographical references and afterwards by the appendixes.

In Appendix A we present a short survey of the mathematical and physical background of the thesis. The most important aspects are discussed and several references are given for each topic. It is intended as a quick reference guide to understand or refresh some deeper technical aspects mentioned throughout the thesis.

Appendixes B, C, D, and E, on the other hand, deal with the perturbed full-space impedance problems of the Laplace and Helmholtz equations in two and three dimensions respectively, using integral equation techniques and the boundary element method. These problems are relatively well-known (at least in theory) and the full extent of the mathematical techniques are illustrated on them.

For the not so experienced reader it is recommended to read first, after this introduction, Appendix A, and particularly the sections which contain lesser-known subjects. The references mentioned throughout should be consulted whenever some topic is not so well understood. Afterwards we recommend to read at least one of the appendixes that contain the full-space problems, i.e., Appendixes B, C, D, and E. The most detailed account of the theory is given in Appendix B, so that other chapters and appendixes may refer to it whenever necessary. Of course, if the reader is more interested in the Helmholtz equation or in the three-dimensional problems, then the corresponding appendixes should be consulted, since they contain all the important and related details. The experienced reader, on the other hand, may prefer eventually to pass straightforwardly to Chapter II. By following this itinerary, the reading experience of this thesis should be (hopefully) more delightful and instructive.