

FULL-PLANE IMPEDANCE HELMHOLTZ PROBLEM

C.1 Introduction

In this appendix we study the perturbed full-plane or free-plane impedance Helmholtz problem, also known as the exterior impedance Helmholtz problem in 2D, using integral equation techniques and the boundary element method.

We consider the problem of the Helmholtz equation in two dimensions on the exterior of a bounded obstacle with an impedance boundary condition. The perturbed full-plane impedance Helmholtz problem is a wave scattering problem around a bounded two-dimensional obstacle. In acoustic obstacle scattering the impedance boundary-value problem appears when we suppose that the normal velocity is proportional to the excess pressure on the boundary of the impenetrable obstacle. The special case of frequency zero for the volume waves has been treated already in Appendix B, since then we deal with the Laplace equation. The three-dimensional Helmholtz problem is treated thoroughly in Appendix E.

The main references for the problem treated herein are Kress (2002), Lenoir (2005), Nédélec (2001), and Terrasse & Abboud (2006). Additional related books and doctorate theses are the ones of Chen & Zhou (1992), Colton & Kress (1983), Ha-Duong (1987), Hsiao & Wendland (2008), Rjasanow & Steinbach (2007), and Steinbach (2008). Articles that take the Helmholtz equation with an impedance boundary condition into account are Angell & Kleinman (1982), Angell & Kress (1984), Angell, Kleinman & Hettlich (1990), Cakoni, Colton & Monk (2001), and Krutitskii (2002, 2003*a,b*). Interesting theoretical details on transmission problems can be found in Costabel & Stephan (1985). For more information on resonances of volume waves we refer to Poisson & Joly (1991). Eigenvalues for the far-field operator are computed in Colton & Kress (1995). The boundary element calculations are performed in the report of Bendali & Devys (1986) and in the article of Bendali & Souilah (1994). Hypersingular integral equations are considered by Feistauer, Hsiao & Kleinman (1996) and Kress (1995). The use of cracked domains is studied by Kress & Lee (2003), and the inverse problem in the articles of Cakoni et al. (2001) and Smith (1985). An optimal control problem is treated by Kirsch (1981). Applications for the Helmholtz problem can be found, among others, for acoustics (Morse & Ingard 1961) and for ultrasound imaging (Ammari 2008).

The Helmholtz equation allows the propagation of volume waves inside the considered domain, and when supplied with an impedance boundary condition it allows also the propagation of surface waves along the domain's boundary. The main difficulty in the numerical treatment and resolution of our problem is the fact that the exterior domain is unbounded. We solve it therefore with integral equation techniques and the boundary element method, which require the knowledge of the Green's function.

This appendix is structured in 14 sections, including this introduction. The direct scattering problem of the Helmholtz equation in a two-dimensional exterior domain with an impedance boundary condition is presented in Section C.2. The Green's function and its

far-field expression are computed respectively in Sections C.3 and C.4. Extending the direct scattering problem towards a transmission problem, as done in Section C.5, allows its resolution by using integral equation techniques, which is discussed in Section C.6. These techniques allow also to represent the far field of the solution, as shown in Section C.7. A particular problem that takes as domain the exterior of a circle is solved analytically in Section C.8. The appropriate function spaces and some existence and uniqueness results for the solution of the problem are presented in Section C.9. The dissipative problem is studied in Section C.10. By means of the variational formulation developed in Section C.11, the obtained integral equation is discretized using the boundary element method, which is described in Section C.12. The boundary element calculations required to build the matrix of the linear system resulting from the numerical discretization are explained in Section C.13. Finally, in Section C.14 a benchmark problem based on the exterior circle problem is solved numerically.

C.2 Direct scattering problem

We consider the direct scattering problem of linear time-harmonic acoustic waves on an exterior domain $\Omega_e \subset \mathbb{R}^2$, lying outside a bounded obstacle Ω_i and having a regular boundary $\Gamma = \partial\Omega_e = \partial\Omega_i$, as shown in Figure C.1. The time convention $e^{-i\omega t}$ is taken and the incident field u_I is known. The goal is to find the scattered field u as a solution to the Helmholtz equation in Ω_e , satisfying an outgoing radiation condition, and such that the total field u_T , decomposed as $u_T = u_I + u$, satisfies a homogeneous impedance boundary condition on the regular boundary Γ (e.g., of class C^2). The unit normal \mathbf{n} is taken outwards oriented of Ω_e . A given wave number $k > 0$ is considered, which depends on the pulsation ω and the speed of wave propagation c through the ratio $k = \omega/c$.

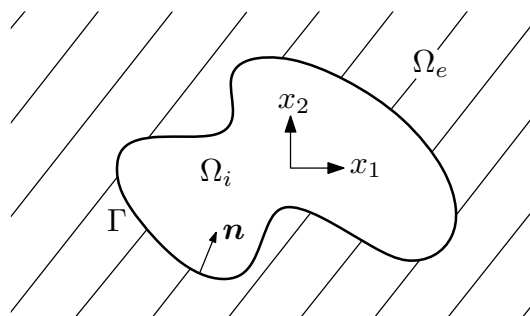


FIGURE C.1. Perturbed full-plane impedance Helmholtz problem domain.

The total field u_T satisfies thus the Helmholtz equation

$$\Delta u_T + k^2 u_T = 0 \quad \text{in } \Omega_e, \quad (\text{C.1})$$

which is also satisfied by the incident field u_I and the scattered field u , due linearity. For the total field u_T we take the homogeneous impedance boundary condition

$$-\frac{\partial u_T}{\partial n} + Zu_T = 0 \quad \text{on } \Gamma, \quad (\text{C.2})$$

where Z is the impedance on the boundary. If $Z = 0$ or $Z = \infty$, then we retrieve respectively the classical Neumann or Dirichlet boundary conditions. In general, we consider a complex-valued impedance $Z(\mathbf{x})$ that depends on the position \mathbf{x} and that may depend also on the pulsation ω . The scattered field u satisfies the non-homogeneous impedance boundary condition

$$-\frac{\partial u}{\partial n} + Zu = f_z \quad \text{on } \Gamma, \quad (\text{C.3})$$

where the impedance data function f_z is given by

$$f_z = \frac{\partial u_I}{\partial n} - Zu_I \quad \text{on } \Gamma. \quad (\text{C.4})$$

The solutions of the Helmholtz equation (C.1) in the full-plane \mathbb{R}^2 are the so-called plane waves, which we take as the known incident field u_I . Up to an arbitrary multiplicative factor, they are given by

$$u_I(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}}, \quad (\mathbf{k} \cdot \mathbf{k}) = k^2, \quad (\text{C.5})$$

where the wave propagation vector \mathbf{k} is taken such that $\mathbf{k} \in \mathbb{R}^2$ to obtain physically admissible waves which do not explode towards infinity. By considering a parametrization through the angle of incidence θ_I , for $0 \leq \theta_I < 2\pi$, we can express the wave propagation vector as $\mathbf{k} = (-k \cos \theta_I, -k \sin \theta_I)$. The plane waves can be thus also represented as

$$u_I(\mathbf{x}) = e^{-ik(x_1 \cos \theta_I + x_2 \sin \theta_I)}. \quad (\text{C.6})$$

An outgoing radiation condition is also imposed for the scattered field u , which specifies its decaying behavior at infinity and eliminates the non-physical solutions, e.g., plane waves and ingoing waves from infinity. It is known as the Sommerfeld radiation condition and receives its name from the German theoretical physicist Arnold Johannes Wilhelm Sommerfeld (1868–1951). This radiation condition allows only outgoing waves, i.e., waves moving away from the obstacle, and therefore characterizes an outward energy flux. It is also closely related with causality and fixes the positive sense of time (cf. Terrasse & Aboud 2006). The described outgoing waves have bounded energy and are thus physically admissible. The Sommerfeld radiation condition is stated either as

$$\frac{\partial u}{\partial r} - iku = \mathcal{O}\left(\frac{1}{r}\right) \quad (\text{C.7})$$

for $r = |\mathbf{x}|$, or, for some constant $C > 0$, by

$$\left| \frac{\partial u}{\partial r} - iku \right| \leq \frac{C}{r} \quad \text{as } r \rightarrow \infty. \quad (\text{C.8})$$

Alternatively it can be also expressed as

$$\lim_{r \rightarrow \infty} \sqrt{r} \left(\frac{\partial u}{\partial r} - iku \right) = 0, \quad (\text{C.9})$$

or even as

$$\frac{\partial u}{\partial r} - iku = \mathcal{O}\left(\frac{1}{r^\alpha}\right) \quad \text{for } \frac{1}{2} < \alpha < \frac{3}{2}. \quad (\text{C.10})$$

Likewise, a weaker and more general formulation of this radiation condition is

$$\lim_{R \rightarrow \infty} \int_{S_R} \left| \frac{\partial u}{\partial r} - iku \right|^2 d\gamma = 0, \quad (\text{C.11})$$

where $S_R = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| = R\}$ is the circle of radius R that is centered at the origin. If the opposite sign is taken, then we obtain a radiation condition for ingoing waves, namely

$$\lim_{r \rightarrow \infty} \sqrt{r} \left(\frac{\partial u}{\partial r} + iku \right) = 0. \quad (\text{C.12})$$

It describes ingoing waves of unbounded energy coming from infinity, which are not physically admissible and therefore not appropriate for our scattering problem. We remark that the correct sign for the ingoing and outgoing radiation conditions is determined exclusively by the chosen time convention. If we used the time convention $e^{i\omega t}$ instead of $e^{-i\omega t}$, then (C.12) would have been the outgoing radiation condition and (C.9) the ingoing one.

The perturbed full-plane impedance Helmholtz problem can be finally stated as

$$\left\{ \begin{array}{ll} \text{Find } u : \Omega_e \rightarrow \mathbb{C} \text{ such that} \\ \Delta u + k^2 u = 0 & \text{in } \Omega_e, \\ -\frac{\partial u}{\partial n} + Zu = f_z & \text{on } \Gamma, \\ \left| \frac{\partial u}{\partial r} - iku \right| \leq \frac{C}{r} & \text{as } r \rightarrow \infty. \end{array} \right. \quad (\text{C.13})$$

C.3 Green's function

The Green's function represents the response of the unperturbed system (without an obstacle) to a Dirac mass. It corresponds to a function G , which depends on a fixed source point $\mathbf{x} \in \mathbb{R}^2$ and an observation point $\mathbf{y} \in \mathbb{R}^2$. The Green's function is computed in the sense of distributions for the variable \mathbf{y} in the full-plane \mathbb{R}^2 by placing at the right-hand side of the Helmholtz equation a Dirac mass $\delta_{\mathbf{x}}$, centered at the point \mathbf{x} . It is therefore a solution $G(\mathbf{x}, \cdot) : \mathbb{R}^2 \rightarrow \mathbb{C}$ for the radiation problem of a point source, namely

$$\Delta_{\mathbf{y}} G(\mathbf{x}, \mathbf{y}) + k^2 G(\mathbf{x}, \mathbf{y}) = \delta_{\mathbf{x}}(\mathbf{y}) \quad \text{in } \mathcal{D}'(\mathbb{R}^2). \quad (\text{C.14})$$

The solution of this equation is not unique, and therefore its behavior at infinity has to be specified. For this purpose we impose on the Green's function also the outgoing radiation condition (C.8).

Due to the radial symmetry of the problem (C.14), it is natural to look for solutions in the form $G = G(r)$, where $r = |\mathbf{y} - \mathbf{x}|$. By considering only the radial component, the

Helmholtz equation in \mathbb{R}^2 becomes

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dG}{dr} \right) + k^2 G = 0, \quad r > 0. \quad (\text{C.15})$$

Replacing now $z = kr$ and considering $\psi(z) = G(r)$ yields $\frac{dG}{dr} = k \frac{d\psi}{dz}$ and consequently

$$\frac{k^2}{z} \left(z \frac{d^2\psi}{dz^2} + \frac{d\psi}{dz} \right) + k^2\psi = 0, \quad (\text{C.16})$$

which is equivalent to the zeroth order Bessel differential equation (vid. Subsection A.2.4)

$$z^2 \frac{d^2\psi}{dz^2} + z \frac{d\psi}{dz} + z^2\psi = 0. \quad (\text{C.17})$$

Independent solutions for this equation are the zeroth order Bessel functions of the first and second kinds, $J_0(z)$ and $Y_0(z)$, and equally the zeroth order Hankel functions of the first and second kinds, $H_0^{(1)}(z)$ and $H_0^{(2)}(z)$. The latter satisfy respectively the outgoing and ingoing radiation conditions and behave for small arguments, as $z \rightarrow 0$, like

$$H_0^{(1)}(z) \sim \frac{2i}{\pi} \ln(z), \quad H_0^{(2)}(z) \sim -\frac{2i}{\pi} \ln(z). \quad (\text{C.18})$$

For large arguments, as $|z| \rightarrow \infty$, they behave like

$$H_0^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}} e^{i(z - \frac{\pi}{4})}, \quad H_0^{(2)}(z) \sim \sqrt{\frac{2}{\pi z}} e^{-i(z - \frac{\pi}{4})}. \quad (\text{C.19})$$

Thus the solution of (C.17) is given by

$$\psi(z) = \alpha H_0^{(1)}(z) + \beta H_0^{(2)}(z), \quad \alpha, \beta \in \mathbb{C}, \quad (\text{C.20})$$

and consequently

$$G(r) = \alpha H_0^{(1)}(kr) + \beta H_0^{(2)}(kr), \quad \alpha, \beta \in \mathbb{C}. \quad (\text{C.21})$$

An outgoing wave behavior for the Green's function implies that $\beta = 0$, due (C.8). We know from (C.18) that the singularity of the Green's function is of logarithmic type. The multiplicative constant α can be thus determined in the same way as for the Green's function of the Laplace equation in (B.20) by means of a computation in the sense of distributions for (C.14). The unique radial outgoing fundamental solution of the Helmholtz equation turns out to be

$$G(r) = -\frac{i}{4} H_0^{(1)}(kr). \quad (\text{C.22})$$

The Green's function for outgoing waves is then finally given by

$$G(\mathbf{x}, \mathbf{y}) = -\frac{i}{4} H_0^{(1)}(k|\mathbf{y} - \mathbf{x}|). \quad (\text{C.23})$$

We remark that the Green's function for ingoing waves would have been

$$G(\mathbf{x}, \mathbf{y}) = \frac{i}{4} H_0^{(2)}(k|\mathbf{y} - \mathbf{x}|). \quad (\text{C.24})$$

To compute the derivatives of the Green's function we require some additional properties of Hankel functions. It holds that

$$\frac{d}{dz}H_0^{(1)}(z) = -H_1^{(1)}(z), \quad \frac{d}{dz}H_0^{(2)}(z) = -H_1^{(2)}(z), \quad (\text{C.25})$$

and

$$\frac{d}{dz}H_1^{(1)}(z) = H_0^{(1)}(z) - \frac{1}{z}H_1^{(1)}(z), \quad \frac{d}{dz}H_1^{(2)}(z) = H_0^{(2)}(z) - \frac{1}{z}H_1^{(2)}(z), \quad (\text{C.26})$$

where $H_1^{(1)}(z)$ and $H_1^{(2)}(z)$ denote the first order Hankel functions of the first and second kinds, respectively. For small arguments, as $z \rightarrow 0$, they behave like

$$H_1^{(1)}(z) \sim -\frac{2i}{\pi z}, \quad H_1^{(2)}(z) \sim \frac{2i}{\pi z}, \quad (\text{C.27})$$

and for large arguments, as $|z| \rightarrow \infty$, they behave like

$$H_1^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}}e^{i(z-\frac{3\pi}{4})}, \quad H_1^{(2)}(z) \sim \sqrt{\frac{2}{\pi z}}e^{-i(z-\frac{3\pi}{4})}. \quad (\text{C.28})$$

The gradient of the Green's function (C.23) is therefore given by

$$\nabla_{\mathbf{y}}G(\mathbf{x}, \mathbf{y}) = \frac{ik}{4}H_1^{(1)}(k|\mathbf{y} - \mathbf{x}|) \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|}, \quad (\text{C.29})$$

and the gradient with respect to the \mathbf{x} variable by

$$\nabla_{\mathbf{x}}G(\mathbf{x}, \mathbf{y}) = \frac{ik}{4}H_1^{(1)}(k|\mathbf{x} - \mathbf{y}|) \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|}. \quad (\text{C.30})$$

The double-gradient matrix is given by

$$\begin{aligned} \nabla_{\mathbf{x}}\nabla_{\mathbf{y}}G(\mathbf{x}, \mathbf{y}) &= \frac{ik}{4}H_1^{(1)}(k|\mathbf{x} - \mathbf{y}|) \left(-\frac{\mathbf{I}}{|\mathbf{x} - \mathbf{y}|} + 2\frac{(\mathbf{x} - \mathbf{y}) \otimes (\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^3} \right) \\ &\quad - \frac{ik^2}{4}H_0^{(1)}(k|\mathbf{x} - \mathbf{y}|) \frac{(\mathbf{x} - \mathbf{y}) \otimes (\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^2}, \end{aligned} \quad (\text{C.31})$$

where \mathbf{I} denotes a 2×2 identity matrix and where \otimes denotes the dyadic or outer product of two vectors, which results in a matrix and is defined in (A.573).

We note that the Green's function (C.23) is symmetric in the sense that

$$G(\mathbf{x}, \mathbf{y}) = G(\mathbf{y}, \mathbf{x}), \quad (\text{C.32})$$

and it fulfills similarly

$$\nabla_{\mathbf{y}}G(\mathbf{x}, \mathbf{y}) = \nabla_{\mathbf{y}}G(\mathbf{y}, \mathbf{x}) = -\nabla_{\mathbf{x}}G(\mathbf{x}, \mathbf{y}) = -\nabla_{\mathbf{x}}G(\mathbf{y}, \mathbf{x}), \quad (\text{C.33})$$

and

$$\nabla_{\mathbf{x}}\nabla_{\mathbf{y}}G(\mathbf{x}, \mathbf{y}) = \nabla_{\mathbf{y}}\nabla_{\mathbf{x}}G(\mathbf{x}, \mathbf{y}) = \nabla_{\mathbf{x}}\nabla_{\mathbf{y}}G(\mathbf{y}, \mathbf{x}) = \nabla_{\mathbf{y}}\nabla_{\mathbf{x}}G(\mathbf{y}, \mathbf{x}). \quad (\text{C.34})$$

Furthermore, due the exponential decrease of the Hankel functions at infinity, we observe that the expression (C.23) of the Green's function for outgoing waves is still valid if a complex wave number $k \in \mathbb{C}$ such that $\Im\{k\} > 0$ is used, which holds also for its derivatives (C.29), (C.30), and (C.31). In the case of ingoing waves, the expression (C.24)

and its derivatives are valid if a complex wave number $k \in \mathbb{C}$ now such that $\Im\{k\} < 0$ is taken into account.

On the account of performing the numerical evaluation of the Hankel functions, for real and complex arguments, we mention the polynomial approximations described in Abramowitz & Stegun (1972) and Newman (1984a), and the algorithms developed by Amos (1986, 1990c, 1995) and Morris (1993).

C.4 Far field of the Green's function

The far field of the Green's function describes its asymptotic behavior at infinity, i.e., when $|\mathbf{x}| \rightarrow \infty$ and assuming that \mathbf{y} is fixed. In this case and due (C.19), we have that

$$H_0^{(1)}(k|\mathbf{x} - \mathbf{y}|) \sim e^{-i\pi/4} \sqrt{\frac{2}{\pi k}} \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{\sqrt{|\mathbf{x} - \mathbf{y}|}}. \quad (\text{C.35})$$

By using a Taylor expansion we obtain that

$$|\mathbf{x} - \mathbf{y}| = |\mathbf{x}| \left(1 - 2 \frac{\mathbf{y} \cdot \mathbf{x}}{|\mathbf{x}|^2} + \frac{|\mathbf{y}|^2}{|\mathbf{x}|^2} \right)^{1/2} = |\mathbf{x}| - \frac{\mathbf{y} \cdot \mathbf{x}}{|\mathbf{x}|} + \mathcal{O}\left(\frac{1}{|\mathbf{x}|}\right). \quad (\text{C.36})$$

A similar expansion yields

$$\frac{1}{\sqrt{|\mathbf{x} - \mathbf{y}|}} = \frac{1}{\sqrt{|\mathbf{x}|}} + \mathcal{O}\left(\frac{1}{|\mathbf{x}|^{3/2}}\right), \quad (\text{C.37})$$

and we have also that

$$e^{ik|\mathbf{x}-\mathbf{y}|} = e^{ik|\mathbf{x}|} e^{-ik\mathbf{y}\cdot\mathbf{x}/|\mathbf{x}|} \left(1 + \mathcal{O}\left(\frac{1}{|\mathbf{x}|}\right) \right). \quad (\text{C.38})$$

We express the point \mathbf{x} as $\mathbf{x} = |\mathbf{x}| \hat{\mathbf{x}}$, being $\hat{\mathbf{x}}$ a unitary vector. The far field of the Green's function, as $|\mathbf{x}| \rightarrow \infty$, is thus given by

$$G^{ff}(\mathbf{x}, \mathbf{y}) = -\frac{e^{i\pi/4}}{\sqrt{8\pi k}} \frac{e^{ik|\mathbf{x}|}}{\sqrt{|\mathbf{x}|}} e^{-ik\hat{\mathbf{x}}\cdot\mathbf{y}}. \quad (\text{C.39})$$

Similarly, as $|\mathbf{x}| \rightarrow \infty$, we have for its gradient with respect to \mathbf{y} , that

$$\nabla_{\mathbf{y}} G^{ff}(\mathbf{x}, \mathbf{y}) = i e^{i\pi/4} \sqrt{\frac{k}{8\pi}} \frac{e^{ik|\mathbf{x}|}}{\sqrt{|\mathbf{x}|}} e^{-ik\hat{\mathbf{x}}\cdot\mathbf{y}} \hat{\mathbf{x}}, \quad (\text{C.40})$$

for its gradient with respect to \mathbf{x} , that

$$\nabla_{\mathbf{x}} G^{ff}(\mathbf{x}, \mathbf{y}) = -i e^{i\pi/4} \sqrt{\frac{k}{8\pi}} \frac{e^{ik|\mathbf{x}|}}{\sqrt{|\mathbf{x}|}} e^{-ik\hat{\mathbf{x}}\cdot\mathbf{y}} \hat{\mathbf{x}}, \quad (\text{C.41})$$

and for its double-gradient matrix, that

$$\nabla_{\mathbf{x}} \nabla_{\mathbf{y}} G^{ff}(\mathbf{x}, \mathbf{y}) = -e^{i\pi/4} \sqrt{\frac{k^3}{8\pi}} \frac{e^{ik|\mathbf{x}|}}{\sqrt{|\mathbf{x}|}} e^{-ik\hat{\mathbf{x}}\cdot\mathbf{y}} (\hat{\mathbf{x}} \otimes \hat{\mathbf{x}}). \quad (\text{C.42})$$

We remark that these far fields are still valid if a complex wave number $k \in \mathbb{C}$ such that $\Im\{k\} > 0$ is used, in which case the appearing complex square root is taken in such a way that its real part is nonnegative.

C.5 Transmission problem

We are interested in expressing the solution u of the direct scattering problem (C.13) by means of an integral representation formula over the boundary Γ . To study this kind of representations, the differential problem defined on Ω_e is extended as a transmission problem defined now on the whole plane \mathbb{R}^2 by combining (C.13) with a corresponding interior problem defined on Ω_i . For the transmission problem, which specifies jump conditions over the boundary Γ , a general integral representation can be developed, and the particular integral representations of interest are then established by the specific choice of the corresponding interior problem.

A transmission problem is then a differential problem for which the jump conditions of the solution field, rather than boundary conditions, are specified on the boundary Γ . As shown in Figure C.1, we consider the exterior domain Ω_e and the interior domain Ω_i , taking the unit normal \mathbf{n} pointing towards Ω_i . We search now a solution u defined in $\Omega_e \cup \Omega_i$, and use the notation $u_e = u|_{\Omega_e}$ and $u_i = u|_{\Omega_i}$. We define the jumps of the traces of u on both sides of the boundary Γ as

$$[u] = u_e - u_i \quad \text{and} \quad \left[\frac{\partial u}{\partial n} \right] = \frac{\partial u_e}{\partial n} - \frac{\partial u_i}{\partial n}. \quad (\text{C.43})$$

The transmission problem is now given by

$$\left\{ \begin{array}{l} \text{Find } u : \Omega_e \cup \Omega_i \rightarrow \mathbb{C} \text{ such that} \\ \Delta u + k^2 u = 0 \quad \text{in } \Omega_e \cup \Omega_i, \\ [u] = \mu \quad \text{on } \Gamma, \\ \left[\frac{\partial u}{\partial n} \right] = \nu \quad \text{on } \Gamma, \\ + \text{Outgoing radiation condition as } |\mathbf{x}| \rightarrow \infty, \end{array} \right. \quad (\text{C.44})$$

where $\mu, \nu : \Gamma \rightarrow \mathbb{C}$ are known functions. The outgoing radiation condition is still (C.8), and it is required to ensure uniqueness of the solution.

C.6 Integral representations and equations

C.6.1 Integral representation

To develop for the solution u an integral representation formula over the boundary Γ , we define by $\Omega_{R,\varepsilon}$ the domain $\Omega_e \cup \Omega_i$ without the ball B_ε of radius $\varepsilon > 0$ centered at the point $\mathbf{x} \in \Omega_e \cup \Omega_i$, and truncated at infinity by the ball B_R of radius $R > 0$ centered at the origin. We consider that the ball B_ε is entirely contained either in Ω_e or in Ω_i , depending

on the location of its center \mathbf{x} . Therefore, as shown in Figure C.2, we have that

$$\Omega_{R,\varepsilon} = ((\Omega_e \cup \Omega_i) \cap B_R) \setminus \overline{B_\varepsilon} \quad \text{and} \quad \Omega_R = (\Omega_e \cup \Omega_i) \cap B_R, \quad (\text{C.45})$$

where

$$B_R = \{\mathbf{y} \in \mathbb{R}^2 : |\mathbf{y}| < R\} \quad \text{and} \quad B_\varepsilon = \{\mathbf{y} \in \mathbb{R}^2 : |\mathbf{y} - \mathbf{x}| < \varepsilon\}. \quad (\text{C.46})$$

We consider similarly the boundaries of the balls

$$S_R = \{\mathbf{y} \in \mathbb{R}^2 : |\mathbf{y}| = R\} \quad \text{and} \quad S_\varepsilon = \{\mathbf{y} \in \mathbb{R}^2 : |\mathbf{y} - \mathbf{x}| = \varepsilon\}. \quad (\text{C.47})$$

The idea is to retrieve the domain $\Omega_e \cup \Omega_i$ at the end when the limits $R \rightarrow \infty$ and $\varepsilon \rightarrow 0$ are taken for the truncated domains $\Omega_{R,\varepsilon}$ and Ω_R .

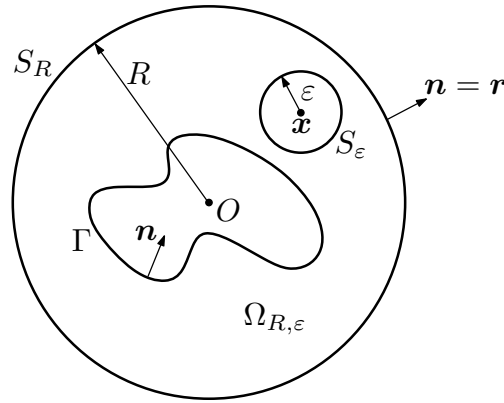


FIGURE C.2. Truncated domain $\Omega_{R,\varepsilon}$ for $\mathbf{x} \in \Omega_e \cup \Omega_i$.

Let us analyze first the asymptotic decaying behavior of the solution u , which satisfies the Helmholtz equation and the Sommerfeld radiation condition. For more generality, we assume here that the wave number k ($\neq 0$) is complex and such that $\Im\{k\} \geq 0$. We consider the weakest form of the radiation condition, namely (C.11), and develop

$$\int_{S_R} \left| \frac{\partial u}{\partial r} - iku \right|^2 d\gamma = \int_{S_R} \left[\left| \frac{\partial u}{\partial r} \right|^2 + |k|^2 |u|^2 + 2 \Im \left\{ ku \frac{\partial \bar{u}}{\partial r} \right\} \right] d\gamma. \quad (\text{C.48})$$

From the divergence theorem (A.614) applied on the truncated domain Ω_R and considering the complex conjugated Helmholtz equation we have

$$\begin{aligned} k \int_{S_R} u \frac{\partial \bar{u}}{\partial r} d\gamma + k \int_{\Gamma} u \frac{\partial \bar{u}}{\partial n} d\gamma &= k \int_{\Omega_R} \operatorname{div}(u \nabla \bar{u}) d\mathbf{x} \\ &= k \int_{\Omega_R} |\nabla u|^2 d\mathbf{x} - k \bar{k}^2 \int_{\Omega_R} |u|^2 d\mathbf{x}. \end{aligned} \quad (\text{C.49})$$

Replacing the imaginary part of (C.49) in (C.48) and taking the limit as $R \rightarrow \infty$, yields

$$\begin{aligned} \lim_{R \rightarrow \infty} \left[\int_{S_R} \left(\left| \frac{\partial u}{\partial r} \right|^2 + |k|^2 |u|^2 \right) d\gamma + 2 \Im\{k\} \int_{\Omega_R} (|\nabla u|^2 + |k|^2 |u|^2) d\mathbf{x} \right] \\ = 2 \Im \left\{ k \int_{\Gamma} u \frac{\partial \bar{u}}{\partial n} d\gamma \right\}. \end{aligned} \quad (\text{C.50})$$

Since the right-hand side is finite and since the left-hand side is nonnegative, we see that

$$\int_{S_R} |u|^2 d\gamma = \mathcal{O}(1) \quad \text{and} \quad \int_{S_R} \left| \frac{\partial u}{\partial r} \right|^2 d\gamma = \mathcal{O}(1) \quad \text{as } R \rightarrow \infty, \quad (\text{C.51})$$

and therefore it holds for a great value of $r = |\mathbf{x}|$ that

$$u = \mathcal{O}\left(\frac{1}{\sqrt{r}}\right) \quad \text{and} \quad |\nabla u| = \mathcal{O}\left(\frac{1}{\sqrt{r}}\right). \quad (\text{C.52})$$

We apply now Green's second integral theorem (A.613) to the functions u and $G(\mathbf{x}, \cdot)$ in the bounded domain $\Omega_{R,\varepsilon}$, by subtracting their respective Helmholtz equations, yielding

$$\begin{aligned} 0 &= \int_{\Omega_{R,\varepsilon}} (u(\mathbf{y}) \Delta_{\mathbf{y}} G(\mathbf{x}, \mathbf{y}) - G(\mathbf{x}, \mathbf{y}) \Delta u(\mathbf{y})) d\mathbf{y} \\ &= \int_{S_R} \left(u(\mathbf{y}) \frac{\partial G}{\partial r_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) - G(\mathbf{x}, \mathbf{y}) \frac{\partial u}{\partial r}(\mathbf{y}) \right) d\gamma(\mathbf{y}) \\ &\quad - \int_{S_{\varepsilon}} \left(u(\mathbf{y}) \frac{\partial G}{\partial r_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) - G(\mathbf{x}, \mathbf{y}) \frac{\partial u}{\partial r}(\mathbf{y}) \right) d\gamma(\mathbf{y}) \\ &\quad + \int_{\Gamma} \left([u](\mathbf{y}) \frac{\partial G}{\partial n_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) - G(\mathbf{x}, \mathbf{y}) \left[\frac{\partial u}{\partial n} \right](\mathbf{y}) \right) d\gamma(\mathbf{y}). \end{aligned} \quad (\text{C.53})$$

The integral on S_R can be rewritten as

$$\int_{S_R} \left[u(\mathbf{y}) \left(\frac{\partial G}{\partial r_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) - ikG(\mathbf{x}, \mathbf{y}) \right) - G(\mathbf{x}, \mathbf{y}) \left(\frac{\partial u}{\partial r}(\mathbf{y}) - ik u(\mathbf{y}) \right) \right] d\gamma(\mathbf{y}), \quad (\text{C.54})$$

which for R large enough and due the radiation condition (C.8) tends to zero, since

$$\left| \int_{S_R} u(\mathbf{y}) \left(\frac{\partial G}{\partial r_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) - ikG(\mathbf{x}, \mathbf{y}) \right) d\gamma(\mathbf{y}) \right| \leq \frac{C}{\sqrt{R}}, \quad (\text{C.55})$$

and

$$\left| \int_{S_R} G(\mathbf{x}, \mathbf{y}) \left(\frac{\partial u}{\partial r}(\mathbf{y}) - ik u(\mathbf{y}) \right) d\gamma(\mathbf{y}) \right| \leq \frac{C}{\sqrt{R}}, \quad (\text{C.56})$$

for some constants $C > 0$. If the function u is regular enough in the ball B_{ε} , then the second term of the integral on S_{ε} , when $\varepsilon \rightarrow 0$ and due (C.23), is bounded by

$$\left| \int_{S_{\varepsilon}} G(\mathbf{x}, \mathbf{y}) \frac{\partial u}{\partial r}(\mathbf{y}) d\gamma(\mathbf{y}) \right| \leq \frac{\pi\varepsilon}{2} \left| H_0^{(1)}(k\varepsilon) \right| \sup_{\mathbf{y} \in B_{\varepsilon}} \left| \frac{\partial u}{\partial r}(\mathbf{y}) \right|, \quad (\text{C.57})$$

and tends to zero due (C.18). The regularity of u can be specified afterwards once the integral representation has been determined and generalized by means of density arguments.

The first integral term on S_ε can be decomposed as

$$\begin{aligned} \int_{S_\varepsilon} u(\mathbf{y}) \frac{\partial G}{\partial r_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) d\gamma(\mathbf{y}) &= u(\mathbf{x}) \int_{S_\varepsilon} \frac{\partial G}{\partial r_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) d\gamma(\mathbf{y}) \\ &+ \int_{S_\varepsilon} \frac{\partial G}{\partial r_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) (u(\mathbf{y}) - u(\mathbf{x})) d\gamma(\mathbf{y}), \end{aligned} \quad (\text{C.58})$$

For the first term in the right-hand side of (C.58), by replacing (C.29), we have that

$$\int_{S_\varepsilon} \frac{\partial G}{\partial r_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) d\gamma(\mathbf{y}) = \frac{ik\pi\varepsilon}{2} H_1^{(1)}(k\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 1, \quad (\text{C.59})$$

which tends towards one due (C.27), while the second term is bounded by

$$\left| \int_{S_\varepsilon} (u(\mathbf{y}) - u(\mathbf{x})) \frac{\partial G}{\partial r_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) d\gamma(\mathbf{y}) \right| \leq \frac{k\pi\varepsilon}{2} \left| H_1^{(1)}(k\varepsilon) \right| \sup_{\mathbf{y} \in B_\varepsilon} |u(\mathbf{y}) - u(\mathbf{x})|, \quad (\text{C.60})$$

which tends towards zero when $\varepsilon \rightarrow 0$.

In conclusion, when the limits $R \rightarrow \infty$ and $\varepsilon \rightarrow 0$ are taken in (C.53), then the following integral representation formula holds for the solution u of the transmission problem:

$$u(\mathbf{x}) = \int_{\Gamma} \left([u](\mathbf{y}) \frac{\partial G}{\partial n_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) - G(\mathbf{x}, \mathbf{y}) \left[\frac{\partial u}{\partial n} \right](\mathbf{y}) \right) d\gamma(\mathbf{y}), \quad \mathbf{x} \in \Omega_e \cup \Omega_i. \quad (\text{C.61})$$

We observe thus that if the values of the jump of u and of its normal derivative are known on Γ , then the transmission problem (C.44) is readily solved and its solution given explicitly by (C.61), which, in terms of μ and ν , becomes

$$u(\mathbf{x}) = \int_{\Gamma} \left(\mu(\mathbf{y}) \frac{\partial G}{\partial n_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) - G(\mathbf{x}, \mathbf{y}) \nu(\mathbf{y}) \right) d\gamma(\mathbf{y}), \quad \mathbf{x} \in \Omega_e \cup \Omega_i. \quad (\text{C.62})$$

To determine the values of the jumps, an adequate integral equation has to be developed, i.e., an equation whose unknowns are the traces of the solution on Γ .

An alternative way to demonstrate the integral representation (C.61) is to proceed in the sense of distributions, in the same way as done in Section B.6. Again we obtain the single layer potential

$$\left\{ G * \left[\frac{\partial u}{\partial n} \right] \delta_{\Gamma} \right\}(\mathbf{x}) = \int_{\Gamma} G(\mathbf{x}, \mathbf{y}) \left[\frac{\partial u}{\partial n} \right](\mathbf{y}) d\gamma(\mathbf{y}) \quad (\text{C.63})$$

associated with the distribution of sources $[\partial u / \partial n] \delta_{\Gamma}$, and the double layer potential

$$\left\{ G * \frac{\partial}{\partial n} ([u] \delta_{\Gamma}) \right\}(\mathbf{x}) = - \int_{\Gamma} \frac{\partial G}{\partial n_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) [u](\mathbf{y}) d\gamma(\mathbf{y}) \quad (\text{C.64})$$

associated with the distribution of dipoles $\frac{\partial}{\partial n} ([u] \delta_{\Gamma})$. Combining properly (C.63) and (C.64) yields the desired integral representation (C.61).

We note that to obtain the gradient of the integral representation (C.61) we can pass directly the derivatives inside the integral, since there are no singularities if $\mathbf{x} \in \Omega_e \cup \Omega_i$. Therefore we have that

$$\nabla u(\mathbf{x}) = \int_{\Gamma} \left([u](\mathbf{y}) \nabla_{\mathbf{x}} \frac{\partial G}{\partial n_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) - \nabla_{\mathbf{x}} G(\mathbf{x}, \mathbf{y}) \left[\frac{\partial u}{\partial n} \right](\mathbf{y}) \right) d\gamma(\mathbf{y}). \quad (\text{C.65})$$

C.6.2 Integral equations

To determine the values of the traces that conform the jumps for the transmission problem (C.44), an integral equation has to be developed. For this purpose we place the source point \mathbf{x} on the boundary Γ and apply the same procedure as before for the integral representation (C.61), treating differently in (C.53) only the integrals on S_ε . The integrals on S_R still behave well and tend towards zero as $R \rightarrow \infty$. The Ball B_ε , though, is split in half into the two pieces $\Omega_e \cap B_\varepsilon$ and $\Omega_i \cap B_\varepsilon$, which are asymptotically separated by the tangent of the boundary if Γ is regular. Thus the associated integrals on S_ε give rise to a term $-(u_e(\mathbf{x}) + u_i(\mathbf{x}))/2$ instead of just $-u(\mathbf{x})$ as before. We must notice that in this case, the integrands associated with the boundary Γ admit an integrable singularity at the point \mathbf{x} . The desired integral equation related with (C.61) is then given by

$$\frac{u_e(\mathbf{x}) + u_i(\mathbf{x})}{2} = \int_{\Gamma} \left([u](\mathbf{y}) \frac{\partial G}{\partial n_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) - G(\mathbf{x}, \mathbf{y}) \left[\frac{\partial u}{\partial n} \right](\mathbf{y}) \right) d\gamma(\mathbf{y}), \quad \mathbf{x} \in \Gamma. \quad (\text{C.66})$$

By choosing adequately the boundary condition of the interior problem, and by considering also the boundary condition of the exterior problem and the jump definitions (C.43), this integral equation can be expressed in terms of only one unknown function on Γ . Thus, solving the problem (C.13) is equivalent to solve (C.66) and then replace the obtained solution in (C.61).

The integral equation holds only when the boundary Γ is regular (e.g., of class C^2). Otherwise, taking the limit $\varepsilon \rightarrow 0$ can no longer be well-defined and the result is false in general. In particular, if the boundary Γ has an angular point at $\mathbf{x} \in \Gamma$, then the left-hand side of the integral equation (C.66) is modified on that point according to the portion of the angle that remains inside Ω_e , in the same way as in (B.61).

Another integral equation can be also derived for the normal derivative of the solution u on the boundary Γ , by studying the jump properties of the single and double layer potentials. It is performed in the same manner as for the Laplace equation. If the boundary is regular at $\mathbf{x} \in \Gamma$, then it holds that

$$\frac{1}{2} \frac{\partial u_e}{\partial n}(\mathbf{x}) + \frac{1}{2} \frac{\partial u_i}{\partial n}(\mathbf{x}) = \int_{\Gamma} \left([u](\mathbf{y}) \frac{\partial^2 G}{\partial n_{\mathbf{x}} \partial n_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) - \frac{\partial G}{\partial n_{\mathbf{x}}}(\mathbf{x}, \mathbf{y}) \left[\frac{\partial u}{\partial n} \right](\mathbf{y}) \right) d\gamma(\mathbf{y}). \quad (\text{C.67})$$

This integral equation is modified correspondingly if \mathbf{x} is an angular point.

C.6.3 Integral kernels

In the same manner as for the Laplace equation, the integral kernels G , $\partial G/\partial n_{\mathbf{y}}$, and $\partial G/\partial n_{\mathbf{x}}$ are weakly singular, and thus integrable, whereas the kernel $\partial^2 G/\partial n_{\mathbf{x}} \partial n_{\mathbf{y}}$ is not integrable and therefore hypersingular.

The kernel G defined in (C.23) has the same logarithmic singularity as the Laplace equation, namely

$$G(\mathbf{x}, \mathbf{y}) \sim \frac{1}{2\pi} \ln |\mathbf{x} - \mathbf{y}| \quad \text{as } \mathbf{x} \rightarrow \mathbf{y}. \quad (\text{C.68})$$

It fulfills therefore (B.64) for any $\lambda > 0$. The kernels $\partial G/\partial n_{\mathbf{y}}$ and $\partial G/\partial n_{\mathbf{x}}$ are less singular along Γ than they appear at first sight, due the regularizing effect of the normal

derivatives. They are given respectively by

$$\frac{\partial G}{\partial n_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) = \frac{ik}{4} H_1^{(1)}(k|\mathbf{y} - \mathbf{x}|) \frac{(\mathbf{y} - \mathbf{x}) \cdot \mathbf{n}_{\mathbf{y}}}{|\mathbf{y} - \mathbf{x}|}, \quad (\text{C.69})$$

and

$$\frac{\partial G}{\partial n_{\mathbf{x}}}(\mathbf{x}, \mathbf{y}) = \frac{ik}{4} H_1^{(1)}(k|\mathbf{x} - \mathbf{y}|) \frac{(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}_{\mathbf{x}}}{|\mathbf{x} - \mathbf{y}|}, \quad (\text{C.70})$$

and their singularities, as $\mathbf{x} \rightarrow \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \Gamma$, adopt the form

$$\frac{\partial G}{\partial n_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) \sim \frac{(\mathbf{y} - \mathbf{x}) \cdot \mathbf{n}_{\mathbf{y}}}{2\pi|\mathbf{y} - \mathbf{x}|^2} \quad \text{and} \quad \frac{\partial G}{\partial n_{\mathbf{x}}}(\mathbf{x}, \mathbf{y}) \sim \frac{(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}_{\mathbf{x}}}{2\pi|\mathbf{x} - \mathbf{y}|^2}. \quad (\text{C.71})$$

Since the singularities are the same as for the Laplace equation, the estimates (B.70) and (B.71) continue to hold. Therefore we have that

$$\frac{\partial G}{\partial n_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) = \mathcal{O}(1) \quad \text{and} \quad \frac{\partial G}{\partial n_{\mathbf{x}}}(\mathbf{x}, \mathbf{y}) = \mathcal{O}(1). \quad (\text{C.72})$$

The singularities of the kernels $\partial G/\partial n_{\mathbf{y}}$ and $\partial G/\partial n_{\mathbf{x}}$ along Γ are thus only apparent and can be repaired by redefining the value of these kernels at $\mathbf{y} = \mathbf{x}$.

The kernel $\partial^2 G/\partial n_{\mathbf{x}}\partial n_{\mathbf{y}}$, on the other hand, adopts the form

$$\begin{aligned} \frac{\partial^2 G}{\partial n_{\mathbf{x}}\partial n_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) &= \frac{ik}{4} H_1^{(1)}(k|\mathbf{x} - \mathbf{y}|) \left(-\frac{\mathbf{n}_{\mathbf{x}} \cdot \mathbf{n}_{\mathbf{y}}}{|\mathbf{x} - \mathbf{y}|} - 2 \frac{((\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}_{\mathbf{x}})((\mathbf{y} - \mathbf{x}) \cdot \mathbf{n}_{\mathbf{y}})}{|\mathbf{x} - \mathbf{y}|^3} \right) \\ &+ \frac{ik^2}{4} H_0^{(1)}(k|\mathbf{x} - \mathbf{y}|) \frac{((\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}_{\mathbf{x}})((\mathbf{y} - \mathbf{x}) \cdot \mathbf{n}_{\mathbf{y}})}{|\mathbf{x} - \mathbf{y}|^2}. \end{aligned} \quad (\text{C.73})$$

Its singularity, when $\mathbf{x} \rightarrow \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \Gamma$, expresses itself as

$$\frac{\partial^2 G}{\partial n_{\mathbf{x}}\partial n_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) \sim -\frac{\mathbf{n}_{\mathbf{x}} \cdot \mathbf{n}_{\mathbf{y}}}{2\pi|\mathbf{y} - \mathbf{x}|^2} - \frac{((\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}_{\mathbf{x}})((\mathbf{y} - \mathbf{x}) \cdot \mathbf{n}_{\mathbf{y}})}{\pi|\mathbf{y} - \mathbf{x}|^4}. \quad (\text{C.74})$$

The regularizing effect of the normal derivatives applies only to its second term, but not to the first. Hence this kernel is hypersingular, with $\lambda = 2$, and it holds that

$$\frac{\partial^2 G}{\partial n_{\mathbf{x}}\partial n_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) = \mathcal{O}\left(\frac{1}{|\mathbf{y} - \mathbf{x}|^2}\right). \quad (\text{C.75})$$

The kernel is no longer integrable and the associated integral operator has to be thus interpreted in some appropriate sense as a divergent integral (cf., e.g., Hsiao & Wendland 2008, Lenoir 2005, Nédélec 2001).

C.6.4 Boundary layer potentials

We regard now the jump properties on the boundary Γ of the boundary layer potentials that have appeared in our calculations. For the development of the integral representation (C.62) we already made acquaintance with the single and double layer potentials, which we define now more precisely for $\mathbf{x} \in \Omega_e \cup \Omega_i$ as the integral operators

$$\mathcal{S}\nu(\mathbf{x}) = \int_{\Gamma} G(\mathbf{x}, \mathbf{y})\nu(\mathbf{y}) \, d\gamma(\mathbf{y}), \quad (\text{C.76})$$

$$\mathcal{D}\mu(\mathbf{x}) = \int_{\Gamma} \frac{\partial G}{\partial n_{\mathbf{y}}}(\mathbf{x}, \mathbf{y})\mu(\mathbf{y}) \, d\gamma(\mathbf{y}). \quad (\text{C.77})$$

The integral representation (C.62) can be now stated in terms of the layer potentials as

$$u = \mathcal{D}\mu - \mathcal{S}\nu. \quad (\text{C.78})$$

We remark that for any functions $\nu, \mu : \Gamma \rightarrow \mathbb{C}$ that are regular enough, the single and double layer potentials satisfy the Helmholtz equation, namely

$$(\Delta + k^2) \mathcal{S}\nu = 0 \quad \text{in } \Omega_e \cup \Omega_i, \quad (\text{C.79})$$

$$(\Delta + k^2) \mathcal{D}\mu = 0 \quad \text{in } \Omega_e \cup \Omega_i. \quad (\text{C.80})$$

For the integral equations (C.66) and (C.67), which are defined for $\mathbf{x} \in \Gamma$, we require the four boundary integral operators:

$$\mathcal{S}\nu(\mathbf{x}) = \int_{\Gamma} G(\mathbf{x}, \mathbf{y})\nu(\mathbf{y}) \, d\gamma(\mathbf{y}), \quad (\text{C.81})$$

$$\mathcal{D}\mu(\mathbf{x}) = \int_{\Gamma} \frac{\partial G}{\partial n_{\mathbf{y}}}(\mathbf{x}, \mathbf{y})\mu(\mathbf{y}) \, d\gamma(\mathbf{y}), \quad (\text{C.82})$$

$$\mathcal{D}^*\nu(\mathbf{x}) = \int_{\Gamma} \frac{\partial G}{\partial n_{\mathbf{x}}}(\mathbf{x}, \mathbf{y})\nu(\mathbf{y}) \, d\gamma(\mathbf{y}), \quad (\text{C.83})$$

$$\mathcal{N}\mu(\mathbf{x}) = \int_{\Gamma} \frac{\partial^2 G}{\partial n_{\mathbf{x}}\partial n_{\mathbf{y}}}(\mathbf{x}, \mathbf{y})\mu(\mathbf{y}) \, d\gamma(\mathbf{y}). \quad (\text{C.84})$$

The operator \mathcal{D}^* is in fact the adjoint of the operator \mathcal{D} . As we already mentioned, the kernel of the integral operator \mathcal{N} defined in (C.84) is not integrable, yet we write it formally as an improper integral. An appropriate sense for this integral will be given below. The integral equations (C.66) and (C.67) can be now stated in terms of the integral operators as

$$\frac{1}{2}(u_e + u_i) = \mathcal{D}\mu - \mathcal{S}\nu, \quad (\text{C.85})$$

$$\frac{1}{2} \left(\frac{\partial u_e}{\partial n} + \frac{\partial u_i}{\partial n} \right) = \mathcal{N}\mu - \mathcal{D}^*\nu. \quad (\text{C.86})$$

These integral equations can be easily derived from the jump properties of the single and double layer potentials. The single layer potential (C.76) is continuous and its normal derivative has a jump of size $-\nu$ across Γ , i.e.,

$$\mathcal{S}\nu|_{\Omega_e} = \mathcal{S}\nu = \mathcal{S}\nu|_{\Omega_i}, \quad (\text{C.87})$$

$$\frac{\partial}{\partial n} \mathcal{S}\nu|_{\Omega_e} = \left(-\frac{1}{2} + \mathcal{D}^* \right) \nu, \quad (\text{C.88})$$

$$\frac{\partial}{\partial n} \mathcal{S}\nu|_{\Omega_i} = \left(\frac{1}{2} + \mathcal{D}^* \right) \nu. \quad (\text{C.89})$$

The double layer potential (C.77), on the other hand, has a jump of size μ across Γ and its normal derivative is continuous, namely

$$\mathcal{D}\mu|_{\Omega_e} = \left(\frac{1}{2} + D\right) \mu, \quad (\text{C.90})$$

$$\mathcal{D}\mu|_{\Omega_i} = \left(-\frac{1}{2} + D\right) \mu, \quad (\text{C.91})$$

$$\frac{\partial}{\partial n} \mathcal{D}\mu|_{\Omega_e} = N\mu = \frac{\partial}{\partial n} \mathcal{D}\mu|_{\Omega_i}. \quad (\text{C.92})$$

The integral equation (C.85) is obtained directly either from (C.87) and (C.90), or from (C.87) and (C.91), by considering the appropriate trace of (C.78) and by defining the functions μ and ν as in (C.44). These three jump properties are easily proven by regarding the details of the proof for (C.66).

Similarly, the integral equation (C.86) for the normal derivative is obtained directly either from (C.88) and (C.92), or from (C.89) and (C.92), by considering the appropriate trace of the normal derivative of (C.78) and by defining again the functions μ and ν as in (C.44). The proof of the jump properties (C.88) and (C.89) is the same as for the Laplace equation, since the same singularities are involved, whereas the proof of (C.92) is similar, but with some differences, and is therefore replicated below.

a) Continuity of the normal derivative of the double layer potential

Differently as in the proof for the Laplace equation, in this case an additional term appears for the operator N , since it is the Helmholtz equation (C.80) that has to be considered in (B.104) and (B.105), yielding now for a test function $\varphi \in \mathcal{D}(\mathbb{R}^2)$ that

$$\left\langle \frac{\partial}{\partial n} \mathcal{D}\mu|_{\Omega_e}, \varphi \right\rangle = \int_{\Omega_e} \nabla \mathcal{D}\mu(\mathbf{x}) \cdot \nabla \varphi(\mathbf{x}) \, d\mathbf{x} - k^2 \int_{\Omega_e} \mathcal{D}\mu(\mathbf{x}) \varphi(\mathbf{x}) \, d\mathbf{x}, \quad (\text{C.93})$$

$$\left\langle \frac{\partial}{\partial n} \mathcal{D}\mu|_{\Omega_i}, \varphi \right\rangle = - \int_{\Omega_i} \nabla \mathcal{D}\mu(\mathbf{x}) \cdot \nabla \varphi(\mathbf{x}) \, d\mathbf{x} + k^2 \int_{\Omega_i} \mathcal{D}\mu(\mathbf{x}) \varphi(\mathbf{x}) \, d\mathbf{x}. \quad (\text{C.94})$$

From (A.588) and (C.33) we obtain the relation

$$\frac{\partial G}{\partial n_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) = \mathbf{n}_{\mathbf{y}} \cdot \nabla_{\mathbf{y}} G(\mathbf{x}, \mathbf{y}) = -\mathbf{n}_{\mathbf{y}} \cdot \nabla_{\mathbf{x}} G(\mathbf{x}, \mathbf{y}) = -\operatorname{div}_{\mathbf{x}}(G(\mathbf{x}, \mathbf{y}) \mathbf{n}_{\mathbf{y}}). \quad (\text{C.95})$$

Thus for the double layer potential (C.77) we have that

$$\mathcal{D}\mu(\mathbf{x}) = -\operatorname{div} \int_{\Gamma} G(\mathbf{x}, \mathbf{y}) \mu(\mathbf{y}) \mathbf{n}_{\mathbf{y}} \, d\gamma(\mathbf{y}) = -\operatorname{div} \mathcal{S}(\mu \mathbf{n}_{\mathbf{y}})(\mathbf{x}), \quad (\text{C.96})$$

being its gradient given by

$$\nabla \mathcal{D}\mu(\mathbf{x}) = -\nabla \operatorname{div} \int_{\Gamma} G(\mathbf{x}, \mathbf{y}) \mu(\mathbf{y}) \mathbf{n}_{\mathbf{y}} \, d\gamma(\mathbf{y}). \quad (\text{C.97})$$

From (A.589) we have that

$$\operatorname{curl}_{\mathbf{x}}(G(\mathbf{x}, \mathbf{y}) \mathbf{n}_{\mathbf{y}}) = \nabla_{\mathbf{x}} G(\mathbf{x}, \mathbf{y}) \times \mathbf{n}_{\mathbf{y}}. \quad (\text{C.98})$$

Hence, by considering (A.597), (C.80), and (C.98) in (C.97), we obtain that

$$\nabla \mathcal{D}\mu(\mathbf{x}) = \text{Curl} \int_{\Gamma} (\mathbf{n}_{\mathbf{y}} \times \nabla_{\mathbf{x}} G(\mathbf{x}, \mathbf{y})) \mu(\mathbf{y}) \, d\gamma(\mathbf{y}) + k^2 \int_{\Gamma} G(\mathbf{x}, \mathbf{y}) \mu(\mathbf{y}) \mathbf{n}_{\mathbf{y}} \, d\gamma(\mathbf{y}). \quad (\text{C.99})$$

From (C.33) and (A.659) we have that

$$\begin{aligned} \int_{\Gamma} (\mathbf{n}_{\mathbf{y}} \times \nabla_{\mathbf{x}} G(\mathbf{x}, \mathbf{y})) \mu(\mathbf{y}) \, d\gamma(\mathbf{y}) &= - \int_{\Gamma} \mathbf{n}_{\mathbf{y}} \times (\nabla_{\mathbf{y}} G(\mathbf{x}, \mathbf{y}) \mu(\mathbf{y})) \, d\gamma(\mathbf{y}) \\ &= \int_{\Gamma} \mathbf{n}_{\mathbf{y}} \times (G(\mathbf{x}, \mathbf{y}) \nabla \mu(\mathbf{y})) \, d\gamma(\mathbf{y}), \end{aligned} \quad (\text{C.100})$$

and consequently

$$\nabla \mathcal{D}\mu(\mathbf{x}) = \text{Curl} \int_{\Gamma} G(\mathbf{x}, \mathbf{y}) (\mathbf{n}_{\mathbf{y}} \times \nabla \mu(\mathbf{y})) \, d\gamma(\mathbf{y}) + k^2 \int_{\Gamma} G(\mathbf{x}, \mathbf{y}) \mu(\mathbf{y}) \mathbf{n}_{\mathbf{y}} \, d\gamma(\mathbf{y}). \quad (\text{C.101})$$

Now, the first expression in (C.93), due (A.608), (A.619), and (C.101), is given by

$$\begin{aligned} \int_{\Omega_e} \nabla \mathcal{D}\mu(\mathbf{x}) \cdot \nabla \varphi(\mathbf{x}) \, d\mathbf{x} &= - \int_{\Gamma} \int_{\Gamma} G(\mathbf{x}, \mathbf{y}) (\nabla \mu(\mathbf{y}) \times \mathbf{n}_{\mathbf{y}}) (\nabla \varphi(\mathbf{x}) \times \mathbf{n}_{\mathbf{x}}) \, d\gamma(\mathbf{y}) \, d\gamma(\mathbf{x}) \\ &\quad + k^2 \int_{\Omega_e} \left(\int_{\Gamma} G(\mathbf{x}, \mathbf{y}) \mu(\mathbf{y}) \mathbf{n}_{\mathbf{y}} \, d\gamma(\mathbf{y}) \right) \cdot \nabla \varphi(\mathbf{x}) \, d\mathbf{x}. \end{aligned} \quad (\text{C.102})$$

Applying (A.614) on the second term of (C.102) and considering (C.96), yields

$$\begin{aligned} \int_{\Omega_e} \nabla \mathcal{D}\mu(\mathbf{x}) \cdot \nabla \varphi(\mathbf{x}) \, d\mathbf{x} &= - \int_{\Gamma} \int_{\Gamma} G(\mathbf{x}, \mathbf{y}) (\nabla \mu(\mathbf{y}) \times \mathbf{n}_{\mathbf{y}}) (\nabla \varphi(\mathbf{x}) \times \mathbf{n}_{\mathbf{x}}) \, d\gamma(\mathbf{y}) \, d\gamma(\mathbf{x}) \\ &\quad + k^2 \int_{\Omega_e} \mathcal{D}\mu(\mathbf{x}) \varphi(\mathbf{x}) \, d\mathbf{x} + \int_{\Gamma} \int_{\Gamma} G(\mathbf{x}, \mathbf{y}) \mu(\mathbf{y}) \varphi(\mathbf{x}) (\mathbf{n}_{\mathbf{y}} \cdot \mathbf{n}_{\mathbf{x}}) \, d\gamma(\mathbf{y}) \, d\gamma(\mathbf{x}). \end{aligned} \quad (\text{C.103})$$

By replacing (C.103) in (C.93) we obtain finally that

$$\begin{aligned} \left\langle \frac{\partial}{\partial n} \mathcal{D}\mu|_{\Omega_e}, \varphi \right\rangle &= - \int_{\Gamma} \int_{\Gamma} G(\mathbf{x}, \mathbf{y}) (\nabla \mu(\mathbf{y}) \times \mathbf{n}_{\mathbf{y}}) (\nabla \varphi(\mathbf{x}) \times \mathbf{n}_{\mathbf{x}}) \, d\gamma(\mathbf{y}) \, d\gamma(\mathbf{x}) \\ &\quad + k^2 \int_{\Gamma} \int_{\Gamma} G(\mathbf{x}, \mathbf{y}) \mu(\mathbf{y}) \varphi(\mathbf{x}) (\mathbf{n}_{\mathbf{y}} \cdot \mathbf{n}_{\mathbf{x}}) \, d\gamma(\mathbf{y}) \, d\gamma(\mathbf{x}). \end{aligned} \quad (\text{C.104})$$

The analogous development for (C.94) yields

$$\begin{aligned} \left\langle \frac{\partial}{\partial n} \mathcal{D}\mu|_{\Omega_i}, \varphi \right\rangle &= - \int_{\Gamma} \int_{\Gamma} G(\mathbf{x}, \mathbf{y}) (\nabla \mu(\mathbf{y}) \times \mathbf{n}_{\mathbf{y}}) (\nabla \varphi(\mathbf{x}) \times \mathbf{n}_{\mathbf{x}}) \, d\gamma(\mathbf{y}) \, d\gamma(\mathbf{x}) \\ &\quad + k^2 \int_{\Gamma} \int_{\Gamma} G(\mathbf{x}, \mathbf{y}) \mu(\mathbf{y}) \varphi(\mathbf{x}) (\mathbf{n}_{\mathbf{y}} \cdot \mathbf{n}_{\mathbf{x}}) \, d\gamma(\mathbf{y}) \, d\gamma(\mathbf{x}). \end{aligned} \quad (\text{C.105})$$

This concludes the proof of (C.92), and shows that the integral operator (C.84) is properly defined in a weak sense for $\varphi \in \mathcal{D}(\mathbb{R}^2)$, instead of (B.115), by

$$\begin{aligned} \langle N\mu(\mathbf{x}), \varphi \rangle &= - \int_{\Gamma} \int_{\Gamma} G(\mathbf{x}, \mathbf{y}) (\nabla \mu(\mathbf{y}) \times \mathbf{n}_{\mathbf{y}}) (\nabla \varphi(\mathbf{x}) \times \mathbf{n}_{\mathbf{x}}) \, d\gamma(\mathbf{y}) \, d\gamma(\mathbf{x}) \\ &\quad + k^2 \int_{\Gamma} \int_{\Gamma} G(\mathbf{x}, \mathbf{y}) \mu(\mathbf{y}) \varphi(\mathbf{x}) (\mathbf{n}_{\mathbf{y}} \cdot \mathbf{n}_{\mathbf{x}}) \, d\gamma(\mathbf{y}) \, d\gamma(\mathbf{x}). \end{aligned} \quad (\text{C.106})$$

C.6.5 Alternatives for integral representations and equations

By taking into account the transmission problem (C.44), its integral representation formula (C.61), and its integral equations (C.66) and (C.67), several particular alternatives for integral representations and equations of the exterior problem (C.13) can be developed. The way to perform this is to extend properly the exterior problem towards the interior domain Ω_i , either by specifying explicitly this extension or by defining an associated interior problem, so as to become the desired jump properties across Γ . The extension has to satisfy the Helmholtz equation (C.1) in Ω_i and a boundary condition that corresponds adequately to the impedance boundary condition (C.3). The obtained system of integral representations and equations allows finally to solve the exterior problem (C.13), by using the solution of the integral equation in the integral representation formula.

a) Extension by zero

An extension by zero towards the interior domain Ω_i implies that

$$u_i = 0 \quad \text{in } \Omega_i. \quad (\text{C.107})$$

The jumps over Γ are characterized in this case by

$$[u] = u_e = \mu, \quad (\text{C.108})$$

$$\left[\frac{\partial u}{\partial n} \right] = \frac{\partial u_e}{\partial n} = Zu_e - f_z = Z\mu - f_z, \quad (\text{C.109})$$

where $\mu : \Gamma \rightarrow \mathbb{C}$ is a function to be determined.

An integral representation formula of the solution, for $\mathbf{x} \in \Omega_e \cup \Omega_i$, is given by

$$u(\mathbf{x}) = \int_{\Gamma} \left(\frac{\partial G}{\partial n_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) - Z(\mathbf{y})G(\mathbf{x}, \mathbf{y}) \right) \mu(\mathbf{y}) d\gamma(\mathbf{y}) + \int_{\Gamma} G(\mathbf{x}, \mathbf{y}) f_z(\mathbf{y}) d\gamma(\mathbf{y}). \quad (\text{C.110})$$

Since

$$\frac{1}{2}(u_e(\mathbf{x}) + u_i(\mathbf{x})) = \frac{\mu(\mathbf{x})}{2}, \quad \mathbf{x} \in \Gamma, \quad (\text{C.111})$$

we obtain, for $\mathbf{x} \in \Gamma$, the Fredholm integral equation of the second kind

$$\frac{\mu(\mathbf{x})}{2} + \int_{\Gamma} \left(Z(\mathbf{y})G(\mathbf{x}, \mathbf{y}) - \frac{\partial G}{\partial n_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) \right) \mu(\mathbf{y}) d\gamma(\mathbf{y}) = \int_{\Gamma} G(\mathbf{x}, \mathbf{y}) f_z(\mathbf{y}) d\gamma(\mathbf{y}), \quad (\text{C.112})$$

which has to be solved for the unknown μ . In terms of boundary layer potentials, the integral representation and the integral equation can be respectively expressed by

$$u = \mathcal{D}(\mu) - \mathcal{S}(Z\mu) + \mathcal{S}(f_z) \quad \text{in } \Omega_e \cup \Omega_i, \quad (\text{C.113})$$

$$\frac{\mu}{2} + \mathcal{S}(Z\mu) - \mathcal{D}(\mu) = \mathcal{S}(f_z) \quad \text{on } \Gamma. \quad (\text{C.114})$$

Alternatively, since

$$\frac{1}{2} \left(\frac{\partial u_e}{\partial n}(\mathbf{x}) + \frac{\partial u_i}{\partial n}(\mathbf{x}) \right) = \frac{Z(\mathbf{x})}{2} \mu(\mathbf{x}) - \frac{f_z(\mathbf{x})}{2}, \quad \mathbf{x} \in \Gamma, \quad (\text{C.115})$$

we obtain also, for $\mathbf{x} \in \Gamma$, the Fredholm integral equation of the second kind

$$\begin{aligned} \frac{Z(\mathbf{x})}{2}\mu(\mathbf{x}) + \int_{\Gamma} \left(-\frac{\partial^2 G}{\partial n_{\mathbf{x}} \partial n_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) + Z(\mathbf{y}) \frac{\partial G}{\partial n_{\mathbf{x}}}(\mathbf{x}, \mathbf{y}) \right) \mu(\mathbf{y}) \, d\gamma(\mathbf{y}) \\ = \frac{f_z(\mathbf{x})}{2} + \int_{\Gamma} \frac{\partial G}{\partial n_{\mathbf{x}}}(\mathbf{x}, \mathbf{y}) f_z(\mathbf{y}) \, d\gamma(\mathbf{y}), \end{aligned} \quad (\text{C.116})$$

which in terms of boundary layer potentials becomes

$$\frac{Z}{2}\mu - N(\mu) + D^*(Z\mu) = \frac{f_z}{2} + D^*(f_z) \quad \text{on } \Gamma. \quad (\text{C.117})$$

b) Continuous impedance

We associate to (C.13) the interior problem

$$\left\{ \begin{array}{ll} \text{Find } u_i : \Omega_i \rightarrow \mathbb{C} \text{ such that} \\ \Delta u_i + k^2 u_i = 0 & \text{in } \Omega_i, \\ -\frac{\partial u_i}{\partial n} + Z u_i = f_z & \text{on } \Gamma. \end{array} \right. \quad (\text{C.118})$$

The jumps over Γ are characterized in this case by

$$[u] = u_e - u_i = \mu, \quad (\text{C.119})$$

$$\left[\frac{\partial u}{\partial n} \right] = \frac{\partial u_e}{\partial n} - \frac{\partial u_i}{\partial n} = Z(u_e - u_i) = Z\mu, \quad (\text{C.120})$$

where $\mu : \Gamma \rightarrow \mathbb{C}$ is a function to be determined. In particular it holds that the jump of the impedance is zero, namely

$$\left[-\frac{\partial u}{\partial n} + Z u \right] = \left(-\frac{\partial u_e}{\partial n} + Z u_e \right) - \left(-\frac{\partial u_i}{\partial n} + Z u_i \right) = 0. \quad (\text{C.121})$$

An integral representation formula of the solution, for $\mathbf{x} \in \Omega_e \cup \Omega_i$, is given by

$$u(\mathbf{x}) = \int_{\Gamma} \left(\frac{\partial G}{\partial n_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) - Z(\mathbf{y}) G(\mathbf{x}, \mathbf{y}) \right) \mu(\mathbf{y}) \, d\gamma(\mathbf{y}). \quad (\text{C.122})$$

Since

$$-\frac{1}{2} \left(\frac{\partial u_e}{\partial n}(\mathbf{x}) + \frac{\partial u_i}{\partial n}(\mathbf{x}) \right) + \frac{Z(\mathbf{x})}{2} (u_e(\mathbf{x}) + u_i(\mathbf{x})) = f_z(\mathbf{x}), \quad \mathbf{x} \in \Gamma, \quad (\text{C.123})$$

we obtain, for $\mathbf{x} \in \Gamma$, the Fredholm integral equation of the first kind

$$\begin{aligned} \int_{\Gamma} \left(-\frac{\partial^2 G}{\partial n_{\mathbf{x}} \partial n_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) + Z(\mathbf{y}) \frac{\partial G}{\partial n_{\mathbf{x}}}(\mathbf{x}, \mathbf{y}) \right) \mu(\mathbf{y}) \, d\gamma(\mathbf{y}) \\ + Z(\mathbf{x}) \int_{\Gamma} \left(\frac{\partial G}{\partial n_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) - Z(\mathbf{y}) G(\mathbf{x}, \mathbf{y}) \right) \mu(\mathbf{y}) \, d\gamma(\mathbf{y}) = f_z(\mathbf{x}), \end{aligned} \quad (\text{C.124})$$

which has to be solved for the unknown μ . In terms of boundary layer potentials, the integral representation and the integral equation can be respectively expressed by

$$u = \mathcal{D}(\mu) - \mathcal{S}(Z\mu) \quad \text{in } \Omega_e \cup \Omega_i, \quad (\text{C.125})$$

$$-N(\mu) + D^*(Z\mu) + ZD(\mu) - ZS(Z\mu) = f_z \quad \text{on } \Gamma. \quad (\text{C.126})$$

We observe that the integral equation (C.126) is self-adjoint.

c) Continuous value

We associate to (C.13) the interior problem

$$\left\{ \begin{array}{ll} \text{Find } u_i : \Omega_i \rightarrow \mathbb{C} \text{ such that} \\ \Delta u_i + k^2 u_i = 0 & \text{in } \Omega_i, \\ -\frac{\partial u_e}{\partial n} + Z u_i = f_z & \text{on } \Gamma. \end{array} \right. \quad (\text{C.127})$$

The jumps over Γ are characterized in this case by

$$[u] = u_e - u_i = \frac{1}{Z} \left(\frac{\partial u_e}{\partial n} - f_z \right) - \frac{1}{Z} \left(\frac{\partial u_e}{\partial n} - f_z \right) = 0, \quad (\text{C.128})$$

$$\left[\frac{\partial u}{\partial n} \right] = \frac{\partial u_e}{\partial n} - \frac{\partial u_i}{\partial n} = \nu, \quad (\text{C.129})$$

where $\nu : \Gamma \rightarrow \mathbb{C}$ is a function to be determined.

An integral representation formula of the solution, for $\mathbf{x} \in \Omega_e \cup \Omega_i$, is given by the single layer potential

$$u(\mathbf{x}) = - \int_{\Gamma} G(\mathbf{x}, \mathbf{y}) \nu(\mathbf{y}) \, d\gamma(\mathbf{y}). \quad (\text{C.130})$$

Since

$$-\frac{1}{2} \left(\frac{\partial u_e}{\partial n}(\mathbf{x}) + \frac{\partial u_i}{\partial n}(\mathbf{x}) \right) + \frac{Z(\mathbf{x})}{2} (u_e(\mathbf{x}) + u_i(\mathbf{x})) = \frac{\nu(\mathbf{x})}{2} + f_z(\mathbf{x}), \quad \mathbf{x} \in \Gamma, \quad (\text{C.131})$$

we obtain, for $\mathbf{x} \in \Gamma$, the Fredholm integral equation of the second kind

$$-\frac{\nu(\mathbf{x})}{2} + \int_{\Gamma} \left(\frac{\partial G}{\partial n_{\mathbf{x}}}(\mathbf{x}, \mathbf{y}) - Z(\mathbf{x})G(\mathbf{x}, \mathbf{y}) \right) \nu(\mathbf{y}) \, d\gamma(\mathbf{y}) = f_z(\mathbf{x}), \quad (\text{C.132})$$

which has to be solved for the unknown ν . In terms of boundary layer potentials, the integral representation and the integral equation can be respectively expressed by

$$u = -\mathcal{S}(\nu) \quad \text{in } \Omega_e \cup \Omega_i, \quad (\text{C.133})$$

$$\frac{\nu}{2} + ZS(\nu) - D^*(\nu) = -f_z \quad \text{on } \Gamma. \quad (\text{C.134})$$

We observe that the integral equation (C.134) is mutually adjoint with (C.114).

d) Continuous normal derivative

We associate to (C.13) the interior problem

$$\left\{ \begin{array}{ll} \text{Find } u_i : \Omega_i \rightarrow \mathbb{C} \text{ such that} \\ \Delta u_i + k^2 u_i = 0 & \text{in } \Omega_i, \\ -\frac{\partial u_i}{\partial n} + Z u_e = f_z & \text{on } \Gamma. \end{array} \right. \quad (\text{C.135})$$

The jumps over Γ are characterized in this case by

$$[u] = u_e - u_i = \mu, \quad (\text{C.136})$$

$$\left[\frac{\partial u}{\partial n} \right] = \frac{\partial u_e}{\partial n} - \frac{\partial u_i}{\partial n} = (Zu_e - f_z) - (Zu_e - f_z) = 0, \quad (\text{C.137})$$

where $\mu : \Gamma \rightarrow \mathbb{C}$ is a function to be determined.

An integral representation formula of the solution, for $\mathbf{x} \in \Omega_e \cup \Omega_i$, is given by the double layer potential

$$u(\mathbf{x}) = \int_{\Gamma} \frac{\partial G}{\partial n_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) \mu(\mathbf{y}) \, d\gamma(\mathbf{y}). \quad (\text{C.138})$$

Since when $\mathbf{x} \in \Gamma$,

$$-\frac{1}{2} \left(\frac{\partial u_e}{\partial n}(\mathbf{x}) + \frac{\partial u_i}{\partial n}(\mathbf{x}) \right) + \frac{Z(\mathbf{x})}{2} (u_e(\mathbf{x}) + u_i(\mathbf{x})) = -\frac{Z(\mathbf{x})}{2} \mu(\mathbf{x}) + f_z(\mathbf{x}), \quad (\text{C.139})$$

we obtain, for $\mathbf{x} \in \Gamma$, the Fredholm integral equation of the second kind

$$\frac{Z(\mathbf{x})}{2} \mu(\mathbf{x}) + \int_{\Gamma} \left(-\frac{\partial^2 G}{\partial n_{\mathbf{x}} \partial n_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) + Z(\mathbf{x}) \frac{\partial G}{\partial n_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) \right) \mu(\mathbf{y}) \, d\gamma(\mathbf{y}) = f_z(\mathbf{x}), \quad (\text{C.140})$$

which has to be solved for the unknown μ . In terms of boundary layer potentials, the integral representation and the integral equation can be respectively expressed by

$$u = \mathcal{D}(\mu) \quad \text{in } \Omega_e \cup \Omega_i, \quad (\text{C.141})$$

$$\frac{Z}{2} \mu - N(\mu) + Z\mathcal{D}(\mu) = f_z \quad \text{on } \Gamma. \quad (\text{C.142})$$

We observe that the integral equation (C.142) is mutually adjoint with (C.117).

C.7 Far field of the solution

The asymptotic behavior at infinity of the solution u of (C.13) is described by the far field u^{ff} . Its expression can be deduced by replacing the far field of the Green's function G^{ff} and its derivatives in the integral representation formula (C.61), which yields

$$u^{ff}(\mathbf{x}) = \int_{\Gamma} \left([u](\mathbf{y}) \frac{\partial G^{ff}}{\partial n_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) - G^{ff}(\mathbf{x}, \mathbf{y}) \left[\frac{\partial u}{\partial n} \right](\mathbf{y}) \right) \, d\gamma(\mathbf{y}). \quad (\text{C.143})$$

By replacing now (C.39) and (C.40) in (C.143), we have that the far field of the solution is

$$u^{ff}(\mathbf{x}) = \frac{e^{ik|\mathbf{x}|}}{\sqrt{|\mathbf{x}|}} \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int_{\Gamma} e^{-ik\hat{\mathbf{x}} \cdot \mathbf{y}} \left(ik\hat{\mathbf{x}} \cdot \mathbf{n}_{\mathbf{y}} [u](\mathbf{y}) + \left[\frac{\partial u}{\partial n} \right](\mathbf{y}) \right) \, d\gamma(\mathbf{y}). \quad (\text{C.144})$$

The asymptotic behavior of the solution u at infinity is therefore given by

$$u(\mathbf{x}) = \frac{e^{ik|\mathbf{x}|}}{\sqrt{|\mathbf{x}|}} \left\{ u_{\infty}(\hat{\mathbf{x}}) + \mathcal{O}\left(\frac{1}{|\mathbf{x}|}\right) \right\}, \quad |\mathbf{x}| \rightarrow \infty, \quad (\text{C.145})$$

uniformly in all directions $\hat{\mathbf{x}}$ on the unit circle, where

$$u_{\infty}(\hat{\mathbf{x}}) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int_{\Gamma} e^{-ik\hat{\mathbf{x}} \cdot \mathbf{y}} \left(ik\hat{\mathbf{x}} \cdot \mathbf{n}_{\mathbf{y}} [u](\mathbf{y}) + \left[\frac{\partial u}{\partial n} \right](\mathbf{y}) \right) \, d\gamma(\mathbf{y}) \quad (\text{C.146})$$

is called the far-field pattern of u . It can be expressed in decibels (dB) by means of the scattering cross section

$$Q_s(\hat{\boldsymbol{x}}) \text{ [dB]} = 20 \log_{10} \left(\frac{|u_\infty(\hat{\boldsymbol{x}})|}{|u_0|} \right), \quad (\text{C.147})$$

where the reference level u_0 is typically taken as $u_0 = u_I$ when the incident field is given by a plane wave of the form (C.5), i.e., $|u_0| = 1$.

We remark that the far-field behavior (C.145) of the solution is in accordance with the Sommerfeld radiation condition (C.8), which justifies its choice.

C.8 Exterior circle problem

To understand better the resolution of the direct scattering problem (C.13), we study now the particular case when the domain $\Omega_e \subset \mathbb{R}^2$ is taken as the exterior of a circle of radius $R > 0$. The interior of the circle is then given by $\Omega_i = \{\boldsymbol{x} \in \mathbb{R}^2 : |\boldsymbol{x}| < R\}$ and its boundary by $\Gamma = \partial\Omega_e$, as shown in Figure C.3. We place the origin at the center of Ω_i and we consider that the unit normal \boldsymbol{n} is taken outwardly oriented of Ω_e , i.e., $\boldsymbol{n} = -\boldsymbol{r}$.

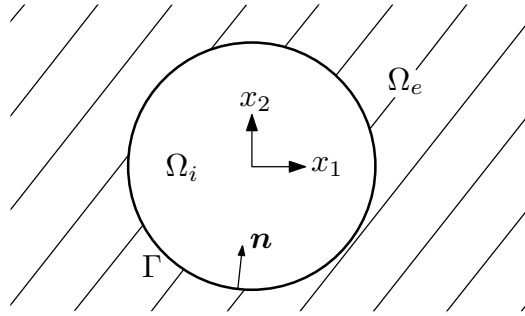


FIGURE C.3. Exterior of the circle.

The exterior circle problem is then stated as

$$\left\{ \begin{array}{l} \text{Find } u : \Omega_e \rightarrow \mathbb{C} \text{ such that} \\ \Delta u + k^2 u = 0 \quad \text{in } \Omega_e, \\ \frac{\partial u}{\partial r} + Zu = f_z \quad \text{on } \Gamma, \\ + \text{Outgoing radiation condition as } |\boldsymbol{x}| \rightarrow \infty, \end{array} \right. \quad (\text{C.148})$$

where we consider a constant impedance $Z \in \mathbb{C}$, a wave number $k > 0$, and where the radiation condition is as usual given by (C.8). As the incident field u_I we consider a plane wave in the form of (C.5), in which case the impedance data function f_z is given by

$$f_z = -\frac{\partial u_I}{\partial r} - Zu_I \quad \text{on } \Gamma. \quad (\text{C.149})$$

Due the particular chosen geometry, the solution u of (C.148) can be easily found analytically by using the method of variable separation, i.e., by supposing that

$$u(\mathbf{x}) = u(r, \theta) = h(r)g(\theta), \quad (\text{C.150})$$

where $r \geq 0$ and $-\pi < \theta \leq \pi$ are the polar coordinates in \mathbb{R}^2 . If the Helmholtz equation in (C.148) is expressed using polar coordinates, then

$$\Delta u + k^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + k^2 u = 0. \quad (\text{C.151})$$

By replacing now (C.150) in (C.151) we obtain

$$h''(r)g(\theta) + \frac{1}{r}h'(r)g(\theta) + \frac{1}{r^2}h(r)g''(\theta) + k^2h(r)g(\theta) = 0. \quad (\text{C.152})$$

Multiplying by r^2 , dividing by gh , and rearranging according to each variable yields

$$r^2 \frac{h''(r)}{h(r)} + r \frac{h'(r)}{h(r)} + k^2 r^2 = -\frac{g''(\theta)}{g(\theta)}. \quad (\text{C.153})$$

Since both sides in equation (C.153) involve different variables, therefore they are equal to a constant, denoted for convenience by n^2 , and we have that

$$r^2 \frac{h''(r)}{h(r)} + r \frac{h'(r)}{h(r)} + k^2 r^2 = -\frac{g''(\theta)}{g(\theta)} = n^2. \quad (\text{C.154})$$

From (C.154) we obtain the two ordinary differential equations

$$g''(\theta) + n^2 g(\theta) = 0, \quad (\text{C.155})$$

$$r^2 h''(r) + r h'(r) + (k^2 r^2 - n^2) h(r) = 0. \quad (\text{C.156})$$

The solutions for (C.155) have the general form

$$g(\theta) = a_n \cos(n\theta) + b_n \sin(n\theta), \quad n \in \mathbb{N}_0, \quad (\text{C.157})$$

where $a_n, b_n \in \mathbb{C}$ are arbitrary constants. The requirement that $n \in \mathbb{N}_0$ stems from the periodicity condition

$$g(\theta) = g(\theta + 2\pi n) \quad \forall n \in \mathbb{Z}, \quad (\text{C.158})$$

where we segregate positive and negative values for n . By considering for (C.156) the change of variables $z = kr$ and expressing $\psi(z) = h(r)$, we obtain the Bessel differential equation of order n , namely

$$z^2 \psi''(z) + z \psi'(z) + (z^2 - n^2) \psi(z) = 0. \quad (\text{C.159})$$

The independent solutions of (C.159) are $H_n^{(1)}(z)$ and $H_n^{(2)}(z)$, the Hankel functions of order n , and therefore the solutions of (C.156) have the general form

$$h(r) = c_n H_n^{(1)}(kr) + d_n H_n^{(2)}(kr), \quad n \geq 0, \quad (\text{C.160})$$

where $c_n, d_n \in \mathbb{C}$ are again arbitrary constants. The general solution for the Helmholtz equation considers the linear combination of all the solutions in the form of (C.150), namely

$$u(r, \theta) = \sum_{n=0}^{\infty} (c_n H_n^{(1)}(kr) + d_n H_n^{(2)}(kr)) (a_n \cos(n\theta) + b_n \sin(n\theta)). \quad (\text{C.161})$$

The radiation condition (C.8) implies that

$$d_n = 0, \quad n \in \mathbb{N}_0. \quad (\text{C.162})$$

Thus the general solution (C.161) turns into

$$u(r, \theta) = \sum_{n=0}^{\infty} H_n^{(1)}(kr) (a_n e^{in\theta} + b_n e^{-in\theta}), \quad (\text{C.163})$$

where all the undetermined constants have been merged into a_n and b_n , due their arbitrariness. Due the recurrence relation (A.121), the radial derivative of (C.163) is given by

$$\frac{\partial u}{\partial r}(r, \theta) = \sum_{n=0}^{\infty} \left(\frac{n}{r} H_n^{(1)}(kr) - k H_{n+1}^{(1)}(kr) \right) (a_n e^{in\theta} + b_n e^{-in\theta}). \quad (\text{C.164})$$

The constants a_n and b_n in (C.163) are determined through the impedance boundary condition on Γ . For this purpose, we expand the impedance data function f_z as a Fourier series:

$$f_z(\theta) = \sum_{n=-\infty}^{\infty} f_n e^{in\theta}, \quad -\pi < \theta \leq \pi, \quad (\text{C.165})$$

where

$$f_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_z(\theta) e^{-in\theta} d\theta, \quad n \in \mathbb{Z}. \quad (\text{C.166})$$

In particular, for a plane wave in the form of (C.5) we have the Jacobi-Anger expansion

$$u_I(\mathbf{x}) = e^{i\mathbf{k} \cdot \mathbf{x}} = e^{-ikr \cos(\theta - \theta_P)} = \sum_{n=-\infty}^{\infty} i^n J_n(kr) e^{in(\theta - \theta_P)}, \quad (\text{C.167})$$

where J_n is the Bessel function of order n , where $\theta_P = \theta_I + \pi$ is the propagation angle of the plane wave, and where

$$\mathbf{k} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = k \begin{pmatrix} \cos \theta_P \\ \sin \theta_P \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = r \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}. \quad (\text{C.168})$$

For a plane wave, the impedance data function (C.149) can be thus expressed as

$$f_z(\theta) = - \sum_{n=-\infty}^{\infty} i^n \left(\left(Z + \frac{n}{R} \right) J_n(kR) - k J_{n+1}(kR) \right) e^{in(\theta - \theta_P)}, \quad (\text{C.169})$$

which implies that

$$f_n = -i^n \left(\left(Z + \frac{n}{R} \right) J_n(kR) - k J_{n+1}(kR) \right) e^{-in\theta_P}, \quad n \in \mathbb{Z}. \quad (\text{C.170})$$

The impedance boundary condition takes therefore the form

$$\sum_{n=0}^{\infty} \left(\left(Z + \frac{n}{R} \right) H_n^{(1)}(kR) - k H_{n+1}^{(1)}(kR) \right) (a_n e^{in\theta} + b_n e^{-in\theta}) = \sum_{n=-\infty}^{\infty} f_n e^{in\theta}. \quad (\text{C.171})$$

We observe that the constants a_n and b_n can be uniquely determined only if

$$\left(Z + \frac{n}{R} \right) H_n^{(1)}(kR) - k H_{n+1}^{(1)}(kR) \neq 0 \quad \text{for } n \in \mathbb{N}_0. \quad (\text{C.172})$$

If this condition is not fulfilled, then the solution is no longer unique. The values $k, Z \in \mathbb{C}$ for which this occurs form a countable set. In particular, for a fixed k , the impedances Z which do not fulfill (C.172) can be explicitly characterized by

$$Z = k \frac{H_{n+1}^{(1)}(kR)}{H_n^{(1)}(kR)} - \frac{n}{R} \quad \text{for } n \in \mathbb{N}_0. \quad (\text{C.173})$$

The wave numbers k which do not fulfill (C.172), for a fixed Z , can only be characterized implicitly through the relation

$$\left(Z + \frac{n}{R}\right) H_n^{(1)}(kR) - k H_{n+1}^{(1)}(kR) = 0 \quad \text{for } n \in \mathbb{N}_0. \quad (\text{C.174})$$

If we suppose now that (C.172) takes place, then

$$a_0 = b_0 = \frac{f_0}{2ZH_0^{(1)}(kR) - 2kH_1^{(1)}(kR)}, \quad (\text{C.175})$$

$$a_n = \frac{Rf_n}{(ZR + n)H_n^{(1)}(kR) - kRH_{n+1}^{(1)}(kR)} \quad (n \geq 1), \quad (\text{C.176})$$

$$b_n = \frac{Rf_{-n}}{(ZR + n)H_n^{(1)}(kR) - kRH_{n+1}^{(1)}(kR)} \quad (n \geq 1). \quad (\text{C.177})$$

In the case of a plane wave we consider for f_n and f_{-n} the expression (C.170). The unique solution for the exterior circle problem (C.148) is then given by

$$u(r, \theta) = \frac{H_0^{(1)}(kr)f_0}{ZH_0^{(1)}(kR) - kH_1^{(1)}(kR)} + \sum_{n=1}^{\infty} \frac{RH_n^{(1)}(kr)(f_n e^{in\theta} + f_{-n} e^{-in\theta})}{(ZR + n)H_n^{(1)}(kR) - kRH_{n+1}^{(1)}(kR)}. \quad (\text{C.178})$$

We remark that there is no need here for an additional compatibility condition like (B.191).

If the condition (C.172) does not hold for some particular $m \in \mathbb{N}_0$, then the solution u is not unique. The constants a_m and b_m are then no longer defined by (C.176) and (C.176), and can be chosen in an arbitrary manner. For the existence of a solution in this case, however, we require also the orthogonality conditions $f_m = f_{-m} = 0$. Instead of (C.178), the solution of (C.148) is now given by the infinite family of functions

$$u(r, \theta) = \sum_{n=1}^{\infty} \frac{RH_n^{(1)}(kr)(f_n e^{in\theta} + f_{-n} e^{-in\theta})}{(ZR + n)H_n^{(1)}(kR) - kRH_{n+1}^{(1)}(kR)} + \alpha H_0^{(1)}(kr) \quad (m = 0), \quad (\text{C.179})$$

$$u(r, \theta) = \frac{H_0^{(1)}(kr)f_0}{ZH_0^{(1)}(kR) - kH_1^{(1)}(kR)} + \sum_{1 \leq n \neq m} \frac{RH_n^{(1)}(kr)(f_n e^{in\theta} + f_{-n} e^{-in\theta})}{(ZR + n)H_n^{(1)}(kR) - kRH_{n+1}^{(1)}(kR)} + H_m^{(1)}(kr)(\alpha e^{im\theta} + \beta e^{-im\theta}) \quad (m \geq 1), \quad (\text{C.180})$$

where $\alpha, \beta \in \mathbb{C}$ are arbitrary and where their associated terms have the form of volume waves, i.e., waves that propagate inside Ω_e . The exterior circle problem (C.148) admits thus a unique solution u , except on a countable set of values for k and Z which do not fulfill the condition (C.172). And even in this last case there exists a solution, although not unique, if two orthogonality conditions are additionally satisfied. This behavior for

the existence and uniqueness of the solution is typical of the Fredholm alternative, which applies when solving problems that involve compact perturbations of invertible operators.

C.9 Existence and uniqueness

C.9.1 Function spaces

To state a precise mathematical formulation of the herein treated problems, we have to define properly the involved function spaces. For the associated interior problems defined on the bounded set Ω_i we use the classical Sobolev space (vid. Section A.4)

$$H^1(\Omega_i) = \{v : v \in L^2(\Omega_i), \nabla v \in L^2(\Omega_i)^2\}, \quad (\text{C.181})$$

which is a Hilbert space and has the norm

$$\|v\|_{H^1(\Omega_i)} = \left(\|v\|_{L^2(\Omega_i)}^2 + \|\nabla v\|_{L^2(\Omega_i)^2}^2 \right)^{1/2}. \quad (\text{C.182})$$

For the exterior problem defined on the unbounded domain Ω_e , on the other hand, we introduce the weighted Sobolev space (cf., e.g., Nédélec 2001)

$$W^1(\Omega_e) = \left\{ v : \frac{v}{\sqrt{1+r^2} \ln(2+r^2)} \in L^2(\Omega_e), \frac{\nabla v}{\sqrt{1+r^2} \ln(2+r^2)} \in L^2(\Omega_e)^2, \frac{\partial v}{\partial r} - ikv \in L^2(\Omega_e) \right\}, \quad (\text{C.183})$$

where $r = |\mathbf{x}|$. If $W^1(\Omega_e)$ is provided with the norm

$$\|v\|_{W^1(\Omega_e)} = \left(\left\| \frac{v}{\sqrt{1+r^2} \ln(2+r^2)} \right\|_{L^2(\Omega_e)}^2 + \left\| \frac{\nabla v}{\sqrt{1+r^2} \ln(2+r^2)} \right\|_{L^2(\Omega_e)^2}^2 + \left\| \frac{\partial v}{\partial r} - ikv \right\|_{L^2(\Omega_e)}^2 \right)^{1/2}, \quad (\text{C.184})$$

then it becomes a Hilbert space. The restriction to any bounded open set $B \subset \Omega_e$ of the functions of $W^1(\Omega_e)$ belongs to $H^1(B)$, i.e., we have the inclusion $W^1(\Omega_e) \subset H_{\text{loc}}^1(\Omega_e)$, and the functions in these two spaces differ only by their behavior at infinity. We remark that the space $W^1(\Omega_e)$ contains the constant functions and all the functions of $H_{\text{loc}}^1(\Omega_e)$ that satisfy the radiation condition (C.8).

When dealing with Sobolev spaces, even a strong Lipschitz boundary $\Gamma \in C^{0,1}$ is admissible. In this case, and due the trace theorem (A.531), if $v \in H^1(\Omega_i)$ or $v \in W^1(\Omega_e)$, then the trace of v fulfills

$$\gamma_0 v = v|_{\Gamma} \in H^{1/2}(\Gamma). \quad (\text{C.185})$$

Moreover, the trace of the normal derivative can be also defined, and it holds that

$$\gamma_1 v = \frac{\partial v}{\partial n}|_{\Gamma} \in H^{-1/2}(\Gamma). \quad (\text{C.186})$$

C.9.2 Regularity of the integral operators

The boundary integral operators (C.81), (C.82), (C.83), and (C.84) can be characterized as linear and continuous applications such that

$$S : H^{-1/2+s}(\Gamma) \longrightarrow H^{1/2+s}(\Gamma), \quad D : H^{1/2+s}(\Gamma) \longrightarrow H^{3/2+s}(\Gamma), \quad (\text{C.187})$$

$$D^* : H^{-1/2+s}(\Gamma) \longrightarrow H^{1/2+s}(\Gamma), \quad N : H^{1/2+s}(\Gamma) \longrightarrow H^{-1/2+s}(\Gamma). \quad (\text{C.188})$$

This result holds for any $s \in \mathbb{R}$ if the boundary Γ is of class C^∞ , which can be derived from the theory of singular integral operators with pseudo-homogeneous kernels (cf., e.g., Nédélec 2001). Due the compact injection (A.554), it holds also that the operators

$$D : H^{1/2+s}(\Gamma) \longrightarrow H^{1/2+s}(\Gamma) \quad \text{and} \quad D^* : H^{-1/2+s}(\Gamma) \longrightarrow H^{-1/2+s}(\Gamma) \quad (\text{C.189})$$

are compact. For a strong Lipschitz boundary $\Gamma \in C^{0,1}$, on the other hand, these results hold only when $|s| < 1$ (cf. Costabel 1988). In the case of more regular boundaries, the range for s increases, but remains finite. For our purposes we use $s = 0$, namely

$$S : H^{-1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma), \quad D : H^{1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma), \quad (\text{C.190})$$

$$D^* : H^{-1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma), \quad N : H^{1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma), \quad (\text{C.191})$$

which are all linear and continuous operators, and where the operators D and D^* are compact. Similarly, we can characterize the single and double layer potentials defined respectively in (C.76) and (C.77) as linear and continuous integral operators such that

$$\mathcal{S} : H^{-1/2}(\Gamma) \longrightarrow W^1(\Omega_e \cup \Omega_i) \quad \text{and} \quad \mathcal{D} : H^{1/2}(\Gamma) \longrightarrow W^1(\Omega_e \cup \Omega_i). \quad (\text{C.192})$$

C.9.3 Application to the integral equations

It is not difficult to see that if $\mu \in H^{1/2}(\Gamma)$ and $\nu \in H^{-1/2}(\Gamma)$ are given, then the transmission problem (C.44) admits a unique solution $u \in W^1(\Omega_e \cup \Omega_i)$, as a consequence of the integral representation formula (C.62). For the direct scattering problem (C.13), though, this is not always the case, as was appreciated in the exterior circle problem (C.148). Nonetheless, if the Fredholm alternative applies, then we know that the existence and uniqueness of the problem can be ensured almost always, i.e., except on a countable set of values for the wave number and for the impedance.

We consider an impedance $Z \in L^\infty(\Gamma)$ and an impedance data function $f_z \in H^{-1/2}(\Gamma)$. In both cases all the continuous functions on Γ are included.

a) First extension by zero

Let us consider the first integral equation of the extension-by-zero alternative (C.112), which is given in terms of boundary layer potentials, for $\mu \in H^{1/2}(\Gamma)$, by

$$\frac{\mu}{2} + S(Z\mu) - D(\mu) = S(f_z) \quad \text{in } H^{1/2}(\Gamma). \quad (\text{C.193})$$

Due the imbedding properties of Sobolev spaces and in the same way as for the full-plane impedance Laplace problem, it holds that the left-hand side of the integral equation corresponds to an identity and two compact operators, and thus Fredholm's alternative applies.

b) Second extension by zero

The second integral equation of the extension-by-zero alternative (C.116) is given in terms of boundary layer potentials, for $\mu \in H^{1/2}(\Gamma)$, by

$$\frac{Z}{2}\mu - N(\mu) + D^*(Z\mu) = \frac{f_z}{2} + D^*(f_z) \quad \text{in } H^{-1/2}(\Gamma). \quad (\text{C.194})$$

The operator N plays the role of the identity and the other terms on the left-hand side are compact, thus Fredholm's alternative holds.

c) Continuous impedance

The integral equation of the continuous-impedance alternative (C.124) is given in terms of boundary layer potentials, for $\mu \in H^{1/2}(\Gamma)$, by

$$-N(\mu) + D^*(Z\mu) + ZD(\mu) - ZS(Z\mu) = f_z \quad \text{in } H^{-1/2}(\Gamma). \quad (\text{C.195})$$

Again, the operator N plays the role of the identity and the remaining terms on the left-hand side are compact, thus Fredholm's alternative applies.

d) Continuous value

The integral equation of the continuous-value alternative (C.132) is given in terms of boundary layer potentials, for $\nu \in H^{-1/2}(\Gamma)$, by

$$\frac{\nu}{2} + ZS(\nu) - D^*(\nu) = -f_z \quad \text{in } H^{-1/2}(\Gamma). \quad (\text{C.196})$$

On the left-hand side we have an identity operator and the remaining operators are compact, thus Fredholm's alternative holds.

e) Continuous normal derivative

The integral equation of the continuous-normal-derivative alternative (C.140) is given in terms of boundary layer potentials, for $\mu \in H^{1/2}(\Gamma)$, by

$$\frac{Z}{2}\mu - N(\mu) + ZD(\mu) = f_z \quad \text{in } H^{-1/2}(\Gamma). \quad (\text{C.197})$$

As before, Fredholm's alternative again applies, since on the left-hand side we have the operator N and two compact operators.

C.9.4 Consequences of Fredholm's alternative

Since the Fredholm alternative applies to each integral equation, therefore it applies also to the exterior differential problem (C.13) due the integral representation formula. The existence of the exterior problem's solution is thus determined by its uniqueness, and the wave numbers $k \in \mathbb{C}$ and impedances $Z \in \mathbb{C}$ for which the uniqueness is lost constitute a countable set, which we call respectively wave number spectrum and impedance spectrum of the exterior problem and denote them by σ_k and σ_Z . The spectrum σ_k considers a fixed Z and, conversely, the spectrum σ_Z considers a fixed k . The existence and uniqueness of the solution is therefore ensured almost everywhere. The same holds obviously for the solution of the integral equation, whose wave number spectrum and impedance spectrum we denote respectively by ς_k and ς_Z . Since each integral equation is derived from the exterior problem,

it holds that $\sigma_k \subset \varsigma_k$ and $\sigma_Z \subset \varsigma_Z$. The converse, though, is not necessarily true and depends on each particular integral equation. In any way, the sets $\varsigma_k \setminus \sigma_k$ and $\varsigma_Z \setminus \sigma_Z$ are at most countable.

Fredholm's alternative applies as much to the integral equation itself as to its adjoint counterpart, and equally to their homogeneous versions. Moreover, each integral equation solves at the same time an exterior and an interior differential problem. The loss of uniqueness of the integral equation's solution appears when the wave number k and the impedance Z are eigenvalues of some associated interior problem, either of the homogeneous integral equation or of its adjoint counterpart. Such a wave number k or impedance Z are contained respectively in ς_k or ς_Z .

The integral equation (C.114) is associated with the extension by zero (C.107), for which no eigenvalues appear. Nevertheless, its adjoint integral equation (C.134) of the continuous value is associated with the interior problem (C.127), which has a countable amount of eigenvalues k , but behaves otherwise well for all $Z \neq 0$.

The integral equation (C.117) is also associated with the extension by zero (C.107), for which no eigenvalues appear. Nonetheless, its adjoint integral equation (C.142) of the continuous normal derivative is associated with the interior problem (C.135), which has a countable amount of eigenvalues k , but behaves well for all Z , without restriction.

The integral equation (C.126) of the continuous impedance is self-adjoint and is associated with the interior problem (C.118), which has a countable quantity of eigenvalues k and Z .

Let us consider now the transmission problem generated by the homogeneous exterior problem

$$\left\{ \begin{array}{l} \text{Find } u_e : \Omega_e \rightarrow \mathbb{C} \text{ such that} \\ \Delta u_e + k^2 u_e = 0 \quad \text{in } \Omega_e, \\ -\frac{\partial u_e}{\partial n} + Z u_e = 0 \quad \text{on } \Gamma, \\ + \text{Outgoing radiation condition as } |\boldsymbol{x}| \rightarrow \infty, \end{array} \right. \quad (\text{C.198})$$

and the associated homogeneous interior problem

$$\left\{ \begin{array}{l} \text{Find } u_i : \Omega_i \rightarrow \mathbb{C} \text{ such that} \\ \Delta u_i + k^2 u_i = 0 \quad \text{in } \Omega_i, \\ \frac{\partial u_i}{\partial n} + Z u_i = 0 \quad \text{on } \Gamma, \end{array} \right. \quad (\text{C.199})$$

where the radiation condition is as usual given by (C.8), and where the unit normal \boldsymbol{n} always points outwards of Ω_e .

As for the Laplace equation, it holds again that the integral equations for this transmission problem have either the same left-hand side or are mutually adjoint to all other possible alternatives of integral equations that can be built for the exterior problem (C.13), and in particular to all the alternatives that were mentioned in the last subsection. The

eigenvalues k and Z of the homogeneous interior problem (C.199) are thus also contained respectively in ς_k and ς_Z .

We remark that additional alternatives for integral representations and equations based on non-homogeneous versions of the problem (C.199) can be also derived for the exterior impedance problem (cf. Ha-Duong 1987).

The determination of the wave number spectrum σ_k and the impedance spectrum σ_Z of the exterior problem (C.13) is not so easy, but can be achieved for simple geometries where an analytic solution is known.

In conclusion, the exterior problem (C.13) admits a unique solution u if $k \notin \sigma_k$, and $Z \notin \sigma_Z$, and each integral equation admits a unique solution, either μ or ν , if $k \notin \varsigma_k$ and $Z \notin \varsigma_Z$.

C.10 Dissipative problem

The dissipative problem considers waves that lose their amplitude as they travel through the medium. These waves dissipate their energy as they propagate and are modeled by a complex wave number $k \in \mathbb{C}$ whose imaginary part is strictly positive, i.e., $\Im\{k\} > 0$. This choice ensures that the Green's function (C.23) decreases exponentially at infinity. Due the dissipative nature of the medium, it is no longer suited to take plane waves in the form of (C.5) as the incident field u_I . Instead, we have to take a source of volume waves at a finite distance from the obstacle. For example, we can consider a point source located at $z \in \Omega_e$, in which case the incident field is given, up to a multiplicative constant, by

$$u_I(\mathbf{x}) = G(\mathbf{x}, z) = -\frac{i}{4} H_0^{(1)}(k|\mathbf{x} - z|). \quad (\text{C.200})$$

This incident field u_I satisfies the Helmholtz equation with a source term in the right-hand side, namely

$$\Delta u_I + k^2 u_I = \delta_z \quad \text{in } \mathcal{D}'(\Omega_e), \quad (\text{C.201})$$

which holds also for the total field u_T but not for the scattered field u , in which case the Helmholtz equation remains homogeneous. For a general source distribution g_s , whose support is contained in Ω_e , the incident field can be expressed by

$$u_I(\mathbf{x}) = G(\mathbf{x}, z) * g_s(z) = \int_{\Omega_e} G(\mathbf{x}, z) g_s(z) dz. \quad (\text{C.202})$$

This incident field u_I satisfies now

$$\Delta u_I + k^2 u_I = g_s \quad \text{in } \mathcal{D}'(\Omega_e), \quad (\text{C.203})$$

which holds again also for the total field u_T but not for the scattered field u .

The dissipative nature of the medium implies also that a radiation condition like (C.8) is no longer required. The ingoing waves are readily ruled out, since they verify $\Im\{k\} < 0$.

The dissipative scattering problem can be therefore stated as

$$\left\{ \begin{array}{ll} \text{Find } u : \Omega_e \rightarrow \mathbb{C} \text{ such that} \\ \Delta u + k^2 u = 0 & \text{in } \Omega_e, \\ -\frac{\partial u}{\partial n} + Zu = f_z & \text{on } \Gamma, \end{array} \right. \quad (\text{C.204})$$

where the impedance data function f_z is again given by

$$f_z = \frac{\partial u_I}{\partial n} - Zu_I \quad \text{on } \Gamma. \quad (\text{C.205})$$

The solution is now such that $u \in H^1(\Omega_e)$ (cf., e.g., Hazard & Lenoir 1998, Lenoir 2005), therefore, instead of (C.55) and (C.56), we obtain that

$$\left| \int_{S_R} \left(u(\mathbf{y}) \frac{\partial G}{\partial r_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) - G(\mathbf{x}, \mathbf{y}) \frac{\partial u}{\partial r}(\mathbf{y}) \right) d\gamma(\mathbf{y}) \right| \leq \frac{C}{\sqrt{R}} e^{-R\Im\{k\}}. \quad (\text{C.206})$$

It is not difficult to see that all the other developments performed for the non-dissipative case are also valid when considering dissipation. The only difference is that now a complex wave number k such that $\Im\{k\} > 0$ has to be taken everywhere into account and that the outgoing radiation condition is no longer needed.

C.11 Variational formulation

To solve a particular integral equation we convert it to its variational or weak formulation, i.e., we solve it with respect to certain test functions in a bilinear (or sesquilinear) form. Basically, the integral equation is multiplied by the (conjugated) test function and then the equation is integrated over the boundary of the domain. The test functions are taken in the same function space as the solution of the integral equation.

a) First extension by zero

The variational formulation for the first integral equation (C.193) of the extension-by-zero alternative searches $\mu \in H^{1/2}(\Gamma)$ such that $\forall \varphi \in H^{1/2}(\Gamma)$

$$\left\langle \frac{\mu}{2} + S(Z\mu) - D(\mu), \varphi \right\rangle = \langle S(f_z), \varphi \rangle. \quad (\text{C.207})$$

b) Second extension by zero

The variational formulation for the second integral equation (C.194) of the extension-by-zero alternative searches $\mu \in H^{1/2}(\Gamma)$ such that $\forall \varphi \in H^{1/2}(\Gamma)$

$$\left\langle \frac{Z}{2}\mu - N(\mu) + D^*(Z\mu), \varphi \right\rangle = \left\langle \frac{f_z}{2} + D^*(f_z), \varphi \right\rangle. \quad (\text{C.208})$$

c) Continuous impedance

The variational formulation for the integral equation (C.195) of the alternative of the continuous-impedance searches $\mu \in H^{1/2}(\Gamma)$ such that $\forall \varphi \in H^{1/2}(\Gamma)$

$$\left\langle -N(\mu) + D^*(Z\mu) + ZD(\mu) - ZS(Z\mu), \varphi \right\rangle = \langle f_z, \varphi \rangle. \quad (\text{C.209})$$

d) Continuous value

The variational formulation for the integral equation (C.196) of the continuous-value alternative searches $\nu \in H^{-1/2}(\Gamma)$ such that $\forall \psi \in H^{-1/2}(\Gamma)$

$$\left\langle \frac{\nu}{2} + ZS(\nu) - D^*(\nu), \psi \right\rangle = \langle -f_z, \psi \rangle. \quad (\text{C.210})$$

e) Continuous normal derivative

The variational formulation for the integral equation (C.197) of the continuous-normal-derivative alternative searches $\mu \in H^{1/2}(\Gamma)$ such that $\forall \varphi \in H^{1/2}(\Gamma)$

$$\left\langle \frac{Z}{2}\mu - N(\mu) + ZD(\mu), \varphi \right\rangle = \langle f_z, \varphi \rangle. \quad (\text{C.211})$$

C.12 Numerical discretization

C.12.1 Discretized function spaces

The exterior problem (C.13) is solved numerically with the boundary element method by employing a Galerkin scheme on the variational formulation of an integral equation. We use on the boundary curve Γ Lagrange finite elements of type either \mathbb{P}_1 or \mathbb{P}_0 . As shown in Figure C.4, the curve Γ is approximated by the discretized curve Γ^h , composed by I rectilinear segments T_j , sequentially ordered in clockwise direction for $1 \leq j \leq I$, such that their length $|T_j|$ is less or equal than h , and with their endpoints on top of Γ .

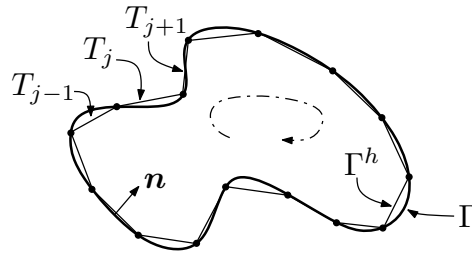


FIGURE C.4. Curve Γ^h , discretization of Γ .

The function space $H^{1/2}(\Gamma)$ is approximated using the conformal space of continuous piecewise linear polynomials with complex coefficients

$$Q_h = \{ \varphi_h \in C^0(\Gamma^h) : \varphi_h|_{T_j} \in \mathbb{P}_1(\mathbb{C}), \quad 1 \leq j \leq I \}. \quad (\text{C.212})$$

The space Q_h has a finite dimension I , and we describe it using the standard base functions for finite elements of type \mathbb{P}_1 , which we denote by $\{\chi_j\}_{j=1}^I$ as in (B.261) and where r_j and r_{j+1} represent the endpoints of segment T_j .

The function space $H^{-1/2}(\Gamma)$, on the other hand, is approximated using the conformal space of piecewise constant polynomials with complex coefficients

$$P_h = \{ \psi_h : \Gamma^h \rightarrow \mathbb{C} \mid \psi_h|_{T_j} \in \mathbb{P}_0(\mathbb{C}), \quad 1 \leq j \leq I \}. \quad (\text{C.213})$$

The space P_h has a finite dimension I , and is described using the standard base functions for finite elements of type \mathbb{P}_0 , which we denote by $\{\kappa_j\}_{j=1}^I$ as in (B.263).

In virtue of this discretization, any function $\varphi_h \in Q_h$ or $\psi_h \in P_h$ can be expressed as a linear combination of the elements of the base, namely

$$\varphi_h(\mathbf{x}) = \sum_{j=1}^I \varphi_j \chi_j(\mathbf{x}) \quad \text{and} \quad \psi_h(\mathbf{x}) = \sum_{j=1}^I \psi_j \kappa_j(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Gamma^h, \quad (\text{C.214})$$

where $\varphi_j, \psi_j \in \mathbb{C}$ for $1 \leq j \leq I$. The solutions $\mu \in H^{1/2}(\Gamma)$ and $\nu \in H^{-1/2}(\Gamma)$ of the variational formulations can be therefore approximated respectively by

$$\mu_h(\mathbf{x}) = \sum_{j=1}^I \mu_j \chi_j(\mathbf{x}) \quad \text{and} \quad \nu_h(\mathbf{x}) = \sum_{j=1}^I \nu_j \kappa_j(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Gamma^h, \quad (\text{C.215})$$

where $\mu_j, \nu_j \in \mathbb{C}$ for $1 \leq j \leq I$. The function f_z can be also approximated by

$$f_z^h(\mathbf{x}) = \sum_{j=1}^I f_j \chi_j(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Gamma^h, \quad \text{with } f_j = f_z(\mathbf{r}_j), \quad (\text{C.216})$$

or

$$f_z^h(\mathbf{x}) = \sum_{j=1}^I f_j \kappa_j(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Gamma^h, \quad \text{with } f_j = \frac{f_z(\mathbf{r}_j) + f_z(\mathbf{r}_{j+1})}{2}, \quad (\text{C.217})$$

depending on whether the original integral equation is stated in $H^{1/2}(\Gamma)$ or in $H^{-1/2}(\Gamma)$.

C.12.2 Discretized integral equations

a) First extension by zero

To see how the boundary element method operates, we apply it to the first integral equation of the extension-by-zero alternative, i.e., to the variational formulation (C.207). We characterize all the discrete approximations by the index h , including also the impedance and the boundary layer potentials. The numerical approximation of (C.207) leads to the discretized problem that searches $\mu_h \in Q_h$ such that $\forall \varphi_h \in Q_h$

$$\left\langle \frac{\mu_h}{2} + S_h(Z_h \mu_h) - D_h(\mu_h), \varphi_h \right\rangle = \langle S_h(f_z^h), \varphi_h \rangle. \quad (\text{C.218})$$

Considering the decomposition of μ_h in terms of the base $\{\chi_j\}$ and taking as test functions the same base functions, $\varphi_h = \chi_i$ for $1 \leq i \leq I$, yields the discrete linear system

$$\sum_{j=1}^I \mu_j \left(\frac{1}{2} \langle \chi_j, \chi_i \rangle + \langle S_h(Z_h \chi_j), \chi_i \rangle - \langle D_h(\chi_j), \chi_i \rangle \right) = \sum_{j=1}^I f_j \langle S_h(\chi_j), \chi_i \rangle. \quad (\text{C.219})$$

This constitutes a system of linear equations that can be expressed as a linear matrix system:

$$\begin{cases} \text{Find } \boldsymbol{\mu} \in \mathbb{C}^I \text{ such that} \\ \mathbf{M} \boldsymbol{\mu} = \mathbf{b}. \end{cases} \quad (\text{C.220})$$

The elements m_{ij} of the matrix \mathbf{M} are given by

$$m_{ij} = \frac{1}{2} \langle \chi_j, \chi_i \rangle + \langle S_h(Z_h \chi_j), \chi_i \rangle - \langle D_h(\chi_j), \chi_i \rangle \quad \text{for } 1 \leq i, j \leq I, \quad (\text{C.221})$$

and the elements b_i of the vector \mathbf{b} by

$$b_i = \langle S_h(f_z^h), \chi_i \rangle = \sum_{j=1}^I f_j \langle S_h(\chi_j), \chi_i \rangle \quad \text{for } 1 \leq i \leq I. \quad (\text{C.222})$$

The discretized solution u_h , which approximates u , is finally obtained by discretizing the integral representation formula (C.110) according to

$$u_h = \mathcal{D}_h(\mu_h) - \mathcal{S}_h(Z_h \mu_h) + \mathcal{S}_h(f_z^h), \quad (\text{C.223})$$

which, more specifically, can be expressed as

$$u_h = \sum_{j=1}^I \mu_j (\mathcal{D}_h(\chi_j) - \mathcal{S}_h(Z_h \chi_j)) + \sum_{j=1}^I f_j \mathcal{S}_h(\chi_j). \quad (\text{C.224})$$

By proceeding in the same way, the discretization of all the other alternatives of integral equations can be also expressed as a linear matrix system like (C.220). The resulting matrix \mathbf{M} is in general complex, full, non-symmetric, and with dimensions $I \times I$. The right-hand side vector \mathbf{b} is complex and of size I . The boundary element calculations required to compute numerically the elements of \mathbf{M} and \mathbf{b} have to be performed carefully, since the integrals that appear become singular when the involved segments are adjacent or coincident, due the singularity of the Green's function at its source point.

b) Second extension by zero

In the case of the second integral equation of the extension-by-zero alternative, i.e., of the variational formulation (C.208), the elements m_{ij} that constitute the matrix \mathbf{M} of the linear system (C.220) are given by

$$m_{ij} = \frac{1}{2} \langle Z_h \chi_j, \chi_i \rangle - \langle N_h(\chi_j), \chi_i \rangle + \langle D_h^*(Z_h \chi_j), \chi_i \rangle \quad \text{for } 1 \leq i, j \leq I, \quad (\text{C.225})$$

whereas the elements b_i of the vector \mathbf{b} are expressed as

$$b_i = \sum_{j=1}^I f_j \left(\frac{1}{2} \langle \chi_j, \chi_i \rangle + \langle D_h^*(Z_h \chi_j), \chi_i \rangle \right) \quad \text{for } 1 \leq i \leq I. \quad (\text{C.226})$$

The discretized solution u_h is again computed by (C.224).

c) Continuous impedance

In the case of the continuous-impedance alternative, i.e., of the variational formulation (C.209), the elements m_{ij} that constitute the matrix \mathbf{M} of the linear system (C.220) are given, for $1 \leq i, j \leq I$, by

$$m_{ij} = -\langle N_h(\chi_j), \chi_i \rangle + \langle D_h^*(Z_h \chi_j), \chi_i \rangle + \langle Z_h D_h(\chi_j), \chi_i \rangle - \langle Z_h S_h(Z_h \chi_j), \chi_i \rangle, \quad (\text{C.227})$$

whereas the elements b_i of the vector \mathbf{b} are expressed as

$$b_i = \sum_{j=1}^I f_j \langle \chi_j, \chi_i \rangle \quad \text{for } 1 \leq i \leq I. \quad (\text{C.228})$$

It can be observed that for this particular alternative the matrix \mathbf{M} turns out to be symmetric, since the integral equation is self-adjoint. The discretized solution u_h , due (C.125), is then computed by

$$u_h = \sum_{j=1}^I \mu_j (\mathcal{D}_h(\chi_j) - \mathcal{S}_h(Z_h \chi_j)). \quad (\text{C.229})$$

d) Continuous value

In the case of the continuous-value alternative, that is, of the variational formulation (C.210), the elements m_{ij} that constitute the matrix \mathbf{M} , now of the linear system

$$\begin{cases} \text{Find } \boldsymbol{\nu} \in \mathbb{C}^I \text{ such that} \\ \mathbf{M}\boldsymbol{\nu} = \mathbf{b}, \end{cases} \quad (\text{C.230})$$

are given by

$$m_{ij} = \frac{1}{2} \langle \kappa_j, \kappa_i \rangle + \langle Z_h \mathcal{S}_h(\kappa_j), \kappa_i \rangle - \langle D_h^*(\kappa_j), \kappa_i \rangle \quad \text{for } 1 \leq i, j \leq I, \quad (\text{C.231})$$

whereas the elements b_i of the vector \mathbf{b} are expressed as

$$b_i = - \sum_{j=1}^I f_j \langle \kappa_j, \kappa_i \rangle \quad \text{for } 1 \leq i \leq I. \quad (\text{C.232})$$

The discretized solution u_h , due (B.150), is then computed by

$$u_h = - \sum_{j=1}^I \nu_j \mathcal{S}_h(\kappa_j). \quad (\text{C.233})$$

e) Continuous normal derivative

In the case of the continuous-normal-derivative alternative, i.e., of the variational formulation (C.211), the elements m_{ij} that conform the matrix \mathbf{M} of the linear system (C.220) are given by

$$m_{ij} = \frac{1}{2} \langle Z_h \chi_j, \chi_i \rangle - \langle N_h(\chi_j), \chi_i \rangle + \langle Z_h D_h(\chi_j), \chi_i \rangle \quad \text{for } 1 \leq i, j \leq I, \quad (\text{C.234})$$

whereas the elements b_i of the vector \mathbf{b} are expressed as

$$b_i = \sum_{j=1}^I f_j \langle \chi_j, \chi_i \rangle \quad \text{for } 1 \leq i \leq I. \quad (\text{C.235})$$

The discretized solution u_h , due (C.141), is then computed by

$$u_h = \sum_{j=1}^I \mu_j \mathcal{D}_h(\chi_j). \quad (\text{C.236})$$