## FULL-PLANE IMPEDANCE LAPLACE PROBLEM

## B. 1 Introduction

In this appendix we study the perturbed full-plane or free-plane impedance Laplace problem, also known as the exterior impedance Laplace problem in 2D, using integral equation techniques and the boundary element method.

We consider the problem of the Laplace equation in two dimensions on the exterior of a bounded obstacle. The Laplace equation for an exterior domain, using typically either Dirichlet or Neumann boundary conditions, is a good example to illustrate the complexity of the integral equation techniques. For a more general treatment and in order to allow a better comparison with the development performed before for half-spaces, we consider in particular an impedance boundary condition. The perturbed full-plane impedance Laplace problem is not strictly speaking a wave scattering problem, but it can be regarded as a limit case of such a problem when the frequency tends towards zero (vid. Appendix C). It can be also regarded as a surface wave problem around a bounded two-dimensional obstacle. The three-dimensional case is treated thoroughly in Appendix D.

For the problem treated herein we follow mainly Nédélec (1977, 1979, 2001) and Raviart (1991). Further related books and doctorate theses are Chen \& Zhou (1992), Evans (1998), Giroire (1987), Hsiao \& Wendland (2008), Kellogg (1929), Kress (1989), Muskhelishvili (1953), Rjasanow \& Steinbach (2007), and Steinbach (2008). Some articles that consider the Laplace equation with an impedance boundary condition are Ahner \& Wiener (1991), Lanzani \& Shen (2004), and Medková (1998). Wendland, Stephan \& Hsiao (1979) treat the mixed boundary-value problem. Interesting theoretical details on transmission problems can be found in Costabel \& Stephan (1985). The boundary element calculations are performed in Bendali \& Devys (1986). The coupling of boundary integral equations and finite element methods is done in Johnson \& Nédélec (1980). The use of cracked domains is studied by Medková \& Krutitskii (2005), and the inverse problem by Fasino \& Inglese (1999) and Lin \& Fang (2005). Applications of the Laplace problem can be found, among others, for electrostatics (Jackson 1999), for conductivity in biomedical imaging (Ammari 2008), and for incompressible plane potential flows (Spurk 1997).

The Laplace equation does not allow the propagation of volume waves inside the considered domain, but the addition of an impedance boundary condition permits the propagation of surface waves along the boundary of the obstacle. The main difficulty in the numerical treatment and resolution of these problems is the fact that the exterior domain is unbounded. We treat this issue by using integral equation techniques and the boundary element method. The idea behind these techniques is to use Green's integral theorems to transform the problem and express it on the boundary of the obstacle, which is bounded. These methods require thus only the calculation of boundary values, rather than values throughout the unbounded exterior domain. They are in a significant manner more efficient in terms of computational resources for problems where the surface versus volume ratio is small. The drawback of these techniques is a more complex mathematical treatment and the requirement of knowing the Green's function of the system. It is the Green's function
which stores the information of the system's physics throughout the exterior domain and which allows to collapse the problem to hold only on the boundary. The dimension of a problem expressed in a volume is therefore reduced towards a surface, i.e., one dimension less, which is what makes these methods so interesting to consider.

This appendix is structured in 13 sections, including this introduction. The direct perturbation problem of the Laplace equation in a two-dimensional exterior domain with an impedance boundary condition is presented in Section B.2. The Green's function and its far-field expression are computed respectively in Sections B. 3 and B.4. Extending the direct perturbation problem towards a transmission problem, as done in Section B.5, allows its resolution by using integral equation techniques, which is discussed in Section B.6. These techniques allow also to represent the far field of the solution, as shown in Section B.7. A particular problem that takes as domain the exterior of a circle is solved analytically in Section B.8. The appropriate function spaces and some existence and uniqueness results for the solution of the problem are presented in Section B.9. By means of the variational formulation developed in Section B.10, the obtained integral equation is discretized using the boundary element method, which is described in Section B.11. The boundary element calculations required to build the matrix of the linear system resulting from the numerical discretization are explained in Section B.12. Finally, in Section B. 13 a benchmark problem based on the exterior circle problem is solved numerically.

## B. 2 Direct perturbation problem

We consider an exterior open and connected domain $\Omega_{e} \subset \mathbb{R}^{2}$ that lies outside a bounded obstacle $\Omega_{i}$ and whose boundary $\Gamma=\partial \Omega_{e}=\partial \Omega_{i}$ is regular (e.g., of class $C^{2}$ ), as shown in Figure B.1. As a perturbation problem, we decompose the total field $u_{T}$ as $u_{T}=u_{W}+u$, where $u_{W}$ represents the known field without obstacle, and where $u$ denotes the perturbed field due its presence, which has bounded energy. The direct perturbation problem of interest is to find the perturbed field $u$ that satisfies the Laplace equation in $\Omega_{e}$, an impedance boundary condition on $\Gamma$, and a decaying condition at infinity. We consider that the origin is located in $\Omega_{i}$ and that the unit normal $\boldsymbol{n}$ is taken always outwardly oriented of $\Omega_{e}$, i.e., pointing inwards of $\Omega_{i}$.


Figure B.1. Perturbed full-plane impedance Laplace problem domain.

The total field $u_{T}$ satisfies the Laplace equation

$$
\begin{equation*}
\Delta u_{T}=0 \quad \text { in } \Omega_{e}, \tag{B.1}
\end{equation*}
$$

which is also satisfied by the fields $u_{W}$ and $u$, due linearity. For the perturbed field $u$ we take also the inhomogeneous impedance boundary condition

$$
\begin{equation*}
-\frac{\partial u}{\partial n}+Z u=f_{z} \quad \text { on } \Gamma, \tag{B.2}
\end{equation*}
$$

where $Z$ is the impedance on the boundary, and where the impedance data function $f_{z}$ is assumed to be known. If $Z=0$ or $Z=\infty$, then we retrieve respectively the classical Neumann or Dirichlet boundary conditions. In general, we consider a complex-valued impedance $Z(\boldsymbol{x})$ depending on the position $\boldsymbol{x}$. The function $f_{z}(\boldsymbol{x})$ may depend on $Z$ and $u_{W}$, but is independent of $u$. If a homogeneous impedance boundary condition is desired for the total field $u_{T}$, then due linearity we can express the function $f_{z}$ as

$$
\begin{equation*}
f_{z}=\frac{\partial u_{W}}{\partial n}-Z u_{W} \quad \text { on } \Gamma . \tag{B.3}
\end{equation*}
$$

The Laplace equation (B.1) admits different kinds of non-trivial solutions $u_{W}$, when we consider the domain $\Omega_{e}$ as the unperturbed full-plane $\mathbb{R}^{2}$. One kind of solutions are the harmonic polynomials

$$
\begin{equation*}
u_{W}(\boldsymbol{x})=\mathfrak{R e}\{P(z)\}, \tag{B.4}
\end{equation*}
$$

where $P(z)$ denotes a polynomial in the complex variable $z=x_{1}+i x_{2}$. There exist in $\mathbb{R}^{2}$ likewise non-polynomial solutions of the form

$$
\begin{equation*}
u_{W}(\boldsymbol{x})=\mathfrak{R e}\{\phi(z)\}, \tag{B.5}
\end{equation*}
$$

where $\phi(z)$ is an entire function in the variable $z$, e.g., the exponential function $e^{z}$. From Liouville's theorem in complex variable theory (cf. Bak \& Newman 1997), we know that the growth at infinity of such a function $\phi$ is bigger than for any polynomial. Any such function can be taken as the known field without perturbation $u_{W}$, which holds in particular for all the constant and linear functions in $\mathbb{R}^{2}$.

For the perturbed field $u$ in the exterior domain $\Omega_{e}$, though, these functions represent undesired non-physical solutions, which have to be avoided in order to ensure uniqueness of the solution $u$. To eliminate them, it suffices to impose for $u$ an asymptotic decaying behavior at infinity that excludes the polynomials. This decaying condition involves finite energy throughout $\Omega_{e}$ and can be interpreted as an additional boundary condition at infinity. In our case it is given, for a great value of $|x|$, by

$$
\begin{equation*}
u(\boldsymbol{x})=\mathcal{O}\left(\frac{1}{|\boldsymbol{x}|}\right) \quad \text { and } \quad|\nabla u(\boldsymbol{x})|=\mathcal{O}\left(\frac{1}{|\boldsymbol{x}|^{2}}\right) \tag{B.6}
\end{equation*}
$$

where $\mathcal{O}(\cdot)$ describes the asymptotic upper bound in terms of simpler functions, known as the $\operatorname{big} \mathcal{O}$. The asymptotic decaying condition (B.6) can be expressed equivalently, for some constants $C>0$, by

$$
\begin{equation*}
|u(\boldsymbol{x})| \leq \frac{C}{|\boldsymbol{x}|} \quad \text { and } \quad|\nabla u(\boldsymbol{x})| \leq \frac{C}{|\boldsymbol{x}|^{2}} \quad \text { as } \quad|\boldsymbol{x}| \rightarrow \infty \tag{B.7}
\end{equation*}
$$

In fact, the decaying condition can be even stated as

$$
\begin{equation*}
u(\boldsymbol{x})=\mathcal{O}\left(\frac{1}{|\boldsymbol{x}|^{\alpha}}\right) \quad \text { and } \quad|\nabla u(\boldsymbol{x})|=\mathcal{O}\left(\frac{1}{|\boldsymbol{x}|^{1+\alpha}}\right) \quad \text { for } 0<\alpha \leq 1, \tag{B.8}
\end{equation*}
$$

or as the more weaker and general formulation

$$
\begin{equation*}
\lim _{R \rightarrow \infty} \int_{S_{R}} \frac{|u|^{2}}{R} \mathrm{~d} \gamma=0 \quad \text { and } \quad \lim _{R \rightarrow \infty} \int_{S_{R}} R|\nabla u|^{2} \mathrm{~d} \gamma=0 \tag{B.9}
\end{equation*}
$$

where $S_{R}=\left\{\boldsymbol{x} \in \mathbb{R}^{2}:|\boldsymbol{x}|=R\right\}$ is the circle of radius $R$ and where the boundary differential element in polar coordinates is given by $\mathrm{d} \gamma=R \mathrm{~d} \theta$. A different way to express the decaying condition, which is used, e.g., by Costabel \& Stephan (1985), is to specify some constants $a, b \in \mathbb{C}$ such that

$$
\begin{equation*}
|u(\boldsymbol{x})|=a+\frac{b}{2 \pi} \ln |\boldsymbol{x}|+\mathcal{O}\left(\frac{1}{|\boldsymbol{x}|}\right) \quad \text { and } \quad|\nabla u(\boldsymbol{x})|=\frac{b}{2 \pi|\boldsymbol{x}|}+\mathcal{O}\left(\frac{1}{|\boldsymbol{x}|^{2}}\right) . \tag{B.10}
\end{equation*}
$$

For simplicity, in our development we consider just $a=b=0$.
The perturbed full-plane impedance Laplace problem can be finally stated as

$$
\begin{cases}\text { Find } u: \Omega_{e} \rightarrow \mathbb{C} \text { such that }  \tag{B.11}\\ \Delta u=0 & \text { in } \Omega_{e}, \\ -\frac{\partial u}{\partial n}+Z u=f_{z} & \text { on } \Gamma, \\ |u(\boldsymbol{x})| \leq \frac{C}{|\boldsymbol{x}|} & \text { as }|\boldsymbol{x}| \rightarrow \infty \\ |\nabla u(\boldsymbol{x})| \leq \frac{C}{|\boldsymbol{x}|^{2}} & \text { as }|\boldsymbol{x}| \rightarrow \infty\end{cases}
$$

## B. 3 Green's function

The Green's function represents the response of the unperturbed system (without an obstacle) to a Dirac mass. It corresponds to a function $G$, which depends on a fixed source point $\boldsymbol{x} \in \mathbb{R}^{2}$ and an observation point $\boldsymbol{y} \in \mathbb{R}^{2}$. The Green's function is computed in the sense of distributions for the variable $\boldsymbol{y}$ in the full-plane $\mathbb{R}^{2}$ by placing at the right-hand side of the Laplace equation a Dirac mass $\delta_{\boldsymbol{x}}$, centered at the point $\boldsymbol{x}$. It is therefore a solution $G(\boldsymbol{x}, \cdot): \mathbb{R}^{2} \rightarrow \mathbb{C}$ for the radiation problem of a point source, namely

$$
\begin{equation*}
\Delta_{y} G(\boldsymbol{x}, \boldsymbol{y})=\delta_{\boldsymbol{x}}(\boldsymbol{y}) \quad \text { in } \mathcal{D}^{\prime}\left(\mathbb{R}^{2}\right) \tag{B.12}
\end{equation*}
$$

Due to the radial symmetry of the problem (B.12), it is natural to look for solutions in the form $G=G(r)$, where $r=|\boldsymbol{y}-\boldsymbol{x}|$. By considering only the radial component, the Laplace equation in $\mathbb{R}^{2}$ becomes

$$
\begin{equation*}
\frac{1}{r} \frac{\mathrm{~d}}{\mathrm{~d} r}\left(r \frac{\mathrm{~d} G}{\mathrm{~d} r}\right)=0, \quad r>0 \tag{B.13}
\end{equation*}
$$

The general solution of (B.13) is of the form

$$
\begin{equation*}
G(r)=C_{1} \ln r+C_{2}, \tag{B.14}
\end{equation*}
$$

for some constants $C_{1}$ and $C_{2}$. The choice of $C_{2}$ is arbitrary, while $C_{1}$ is fixed by the presence of the Dirac mass in (B.12). To determine $C_{1}$, we have to perform thus a computation in the sense of distributions (cf. Gel'fand \& Shilov 1964), using the fact that $G$ is harmonic for $r \neq 0$. For a test function $\varphi \in \mathcal{D}\left(\mathbb{R}^{2}\right)$, we have by definition that

$$
\begin{equation*}
\left\langle\Delta_{y} G, \varphi\right\rangle=\langle G, \Delta \varphi\rangle=\int_{\mathbb{R}^{2}} G \Delta \varphi \mathrm{~d} \boldsymbol{y}=\lim _{\varepsilon \rightarrow 0} \int_{r \geq \varepsilon} G \Delta \varphi \mathrm{~d} \boldsymbol{y} . \tag{B.15}
\end{equation*}
$$

We apply here Green's second integral theorem (A.613), choosing as bounded domain the circular shell $\varepsilon \leq r \leq a$, where $a$ is large enough so that the test function $\varphi(\boldsymbol{y})$, of bounded support, vanishes identically for $r \geq a$. Then

$$
\begin{equation*}
\int_{r \geq \varepsilon} G \Delta \varphi \mathrm{~d} \boldsymbol{y}=\int_{r \geq \varepsilon} \Delta_{y} G \varphi \mathrm{~d} \boldsymbol{y}-\int_{r=\varepsilon} G \frac{\partial \varphi}{\partial r} \mathrm{~d} \gamma+\int_{r=\varepsilon} \frac{\partial G}{\partial r_{\boldsymbol{y}}} \varphi \mathrm{d} \gamma, \tag{B.16}
\end{equation*}
$$

where $\mathrm{d} \gamma$ is the line element on the circle $r=\varepsilon$. Now

$$
\begin{equation*}
\int_{r \geq \varepsilon} \Delta_{y} G \varphi \mathrm{~d} \boldsymbol{y}=0 \tag{B.17}
\end{equation*}
$$

since outside the ball $r \leq \varepsilon$ the function $G$ is harmonic. As for the other terms, by replacing (B.14), we obtain that

$$
\begin{equation*}
\int_{r=\varepsilon} G \frac{\partial \varphi}{\partial r} \mathrm{~d} \gamma=\left(C_{1} \ln \varepsilon+C_{2}\right) \int_{r=\varepsilon} \frac{\partial \varphi}{\partial r} \mathrm{~d} \gamma=\mathcal{O}(\varepsilon \ln \varepsilon) \tag{B.18}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{r=\varepsilon} \frac{\partial G}{\partial r_{y}} \varphi \mathrm{~d} \gamma=\frac{C_{1}}{\varepsilon} \int_{r=\varepsilon} \varphi \mathrm{d} \gamma=2 \pi C_{1} S_{\varepsilon}(\varphi) \tag{B.19}
\end{equation*}
$$

where $S_{\varepsilon}(\varphi)$ is the mean value of $\varphi(\boldsymbol{y})$ on the circle of radius $\varepsilon$ and centered at $\boldsymbol{x}$. In the limit as $\varepsilon \rightarrow 0$, we obtain that $S_{\varepsilon}(\varphi) \rightarrow \varphi(\boldsymbol{x})$, so that

$$
\begin{equation*}
\left\langle\Delta_{y} G, \varphi\right\rangle=\lim _{\varepsilon \rightarrow 0} \int_{r \geq \varepsilon} G \Delta \varphi \mathrm{~d} \boldsymbol{y}=2 \pi C_{1} \varphi(\boldsymbol{x})=2 \pi C_{1}\left\langle\delta_{\boldsymbol{x}}, \varphi\right\rangle . \tag{B.20}
\end{equation*}
$$

Thus if $C_{1}=1 / 2 \pi$, then (B.12) is fulfilled. When we consider not only radial solutions, then the general solution of (B.12) is given by

$$
\begin{equation*}
G(\boldsymbol{x}, \boldsymbol{y})=\frac{1}{2 \pi} \ln |\boldsymbol{y}-\boldsymbol{x}|+\phi(\boldsymbol{x}, \boldsymbol{y}) \tag{B.21}
\end{equation*}
$$

where $\phi(\boldsymbol{x}, \boldsymbol{y})$ is any harmonic function in the variable $\boldsymbol{y}$, i.e., such that $\Delta_{\boldsymbol{y}} \phi=0$ in $\mathbb{R}^{2}$, which means that $\phi$ acquires the form of (B.4) or (B.5).

If we impose additionally, for a fixed $\boldsymbol{x}$, the asymptotic decaying condition

$$
\begin{equation*}
\left|\nabla_{\boldsymbol{y}} G(\boldsymbol{x}, \boldsymbol{y})\right|=\mathcal{O}\left(\frac{1}{|\boldsymbol{y}|}\right) \quad \text { as } \quad|\boldsymbol{y}| \longrightarrow \infty \tag{B.22}
\end{equation*}
$$

then we eliminate any polynomial (or bigger) growth at infinity, but we admit constant and logarithmic growth. By choosing arbitrarily that any constant has to be zero, we obtain finally that our Green's function satisfying (B.12) and (B.22) is given by

$$
\begin{equation*}
G(\boldsymbol{x}, \boldsymbol{y})=\frac{1}{2 \pi} \ln |\boldsymbol{y}-\boldsymbol{x}|, \tag{B.23}
\end{equation*}
$$

being its gradient

$$
\begin{equation*}
\nabla_{\boldsymbol{y}} G(\boldsymbol{x}, \boldsymbol{y})=\frac{\boldsymbol{y}-\boldsymbol{x}}{2 \pi|\boldsymbol{y}-\boldsymbol{x}|^{2}} . \tag{B.24}
\end{equation*}
$$

We can likewise define a gradient with respect to the $\boldsymbol{x}$ variable by

$$
\begin{equation*}
\nabla_{\boldsymbol{x}} G(\boldsymbol{x}, \boldsymbol{y})=\frac{\boldsymbol{x}-\boldsymbol{y}}{2 \pi|\boldsymbol{x}-\boldsymbol{y}|^{2}}, \tag{B.25}
\end{equation*}
$$

and a double-gradient matrix by

$$
\nabla_{\boldsymbol{x}} \nabla_{\boldsymbol{y}} G(\boldsymbol{x}, \boldsymbol{y})=\left[\begin{array}{cc}
\frac{\partial^{2} G}{\partial x_{1} \partial y_{1}} & \frac{\partial^{2} G}{\partial x_{1} \partial y_{2}}  \tag{B.26}\\
\frac{\partial^{2} G}{\partial x_{2} \partial y_{1}} & \frac{\partial^{2} G}{\partial x_{2} \partial y_{2}}
\end{array}\right]=-\frac{\boldsymbol{I}}{2 \pi|\boldsymbol{x}-\boldsymbol{y}|^{2}}+\frac{(\boldsymbol{x}-\boldsymbol{y}) \otimes(\boldsymbol{x}-\boldsymbol{y})}{\pi|\boldsymbol{x}-\boldsymbol{y}|^{4}},
$$

where $\boldsymbol{I}$ denotes a $2 \times 2$ identity matrix and where $\otimes$ denotes the dyadic or outer product of two vectors, which results in a matrix and is defined in (A.573).

We note that the Green's function (B.23) is symmetric in the sense that

$$
\begin{equation*}
G(\boldsymbol{x}, \boldsymbol{y})=G(\boldsymbol{y}, \boldsymbol{x}), \tag{B.27}
\end{equation*}
$$

and it fulfills similarly

$$
\begin{equation*}
\nabla_{y} G(\boldsymbol{x}, \boldsymbol{y})=\nabla_{\boldsymbol{y}} G(\boldsymbol{y}, \boldsymbol{x})=-\nabla_{\boldsymbol{x}} G(\boldsymbol{x}, \boldsymbol{y})=-\nabla_{\boldsymbol{x}} G(\boldsymbol{y}, \boldsymbol{x}), \tag{B.28}
\end{equation*}
$$

and

$$
\begin{equation*}
\nabla_{\boldsymbol{x}} \nabla_{\boldsymbol{y}} G(\boldsymbol{x}, \boldsymbol{y})=\nabla_{\boldsymbol{y}} \nabla_{\boldsymbol{x}} G(\boldsymbol{x}, \boldsymbol{y})=\nabla_{\boldsymbol{x}} \nabla_{\boldsymbol{y}} G(\boldsymbol{y}, \boldsymbol{x})=\nabla_{\boldsymbol{y}} \nabla_{\boldsymbol{x}} G(\boldsymbol{y}, \boldsymbol{x}) . \tag{B.29}
\end{equation*}
$$

## B. 4 Far field of the Green's function

The far field of the Green's function describes its asymptotic behavior at infinity, i.e., when $|\boldsymbol{x}| \rightarrow \infty$ and assuming that $\boldsymbol{y}$ is fixed. For this purpose, we search the terms of highest order at infinity by expanding the logarithm according to

$$
\begin{align*}
\ln |\boldsymbol{x}-\boldsymbol{y}| & =\frac{1}{2} \ln \left(|\boldsymbol{x}|^{2}\right)+\frac{1}{2} \ln \left(\frac{|\boldsymbol{x}-\boldsymbol{y}|^{2}}{|\boldsymbol{x}|^{2}}\right) \\
& =\ln |\boldsymbol{x}|+\frac{1}{2} \ln \left(1-2 \frac{\boldsymbol{y} \cdot \boldsymbol{x}}{|\boldsymbol{x}|^{2}}+\frac{|\boldsymbol{y}|^{2}}{|\boldsymbol{x}|^{2}}\right) . \tag{B.30}
\end{align*}
$$

Using a Taylor expansion of the logarithm around one yields

$$
\begin{equation*}
\ln |\boldsymbol{x}-\boldsymbol{y}|=\ln |\boldsymbol{x}|-\frac{\boldsymbol{y} \cdot \boldsymbol{x}}{|\boldsymbol{x}|^{2}}+\mathcal{O}\left(\frac{1}{|\boldsymbol{x}|^{2}}\right) . \tag{B.31}
\end{equation*}
$$

We express the point $\boldsymbol{x}$ as $\boldsymbol{x}=|\boldsymbol{x}| \hat{\boldsymbol{x}}$, being $\hat{\boldsymbol{x}}$ a unitary vector. The far field of the Green's function, as $|\boldsymbol{x}| \rightarrow \infty$, is thus given by

$$
\begin{equation*}
G^{f f}(\boldsymbol{x}, \boldsymbol{y})=\frac{1}{2 \pi} \ln |\boldsymbol{x}|-\frac{\boldsymbol{y} \cdot \hat{\boldsymbol{x}}}{2 \pi|\boldsymbol{x}|} . \tag{B.32}
\end{equation*}
$$

Similarly, as $|\boldsymbol{x}| \rightarrow \infty$, we have for its gradient with respect to $\boldsymbol{y}$, that

$$
\begin{equation*}
\nabla_{y} G^{f f}(\boldsymbol{x}, \boldsymbol{y})=-\frac{\hat{\boldsymbol{x}}}{2 \pi|\boldsymbol{x}|}, \tag{B.33}
\end{equation*}
$$

for its gradient with respect to $\boldsymbol{x}$, that

$$
\begin{equation*}
\nabla_{\boldsymbol{x}} G^{f f}(\boldsymbol{x}, \boldsymbol{y})=\frac{\hat{\boldsymbol{x}}}{2 \pi|\boldsymbol{x}|}, \tag{B.34}
\end{equation*}
$$

and for its double-gradient matrix, that

$$
\begin{equation*}
\nabla_{\boldsymbol{x}} \nabla_{\boldsymbol{y}} G^{f f}(\boldsymbol{x}, \boldsymbol{y})=-\frac{\boldsymbol{I}}{2 \pi|\boldsymbol{x}|^{2}}+\frac{\hat{\boldsymbol{x}} \otimes \hat{\boldsymbol{x}}}{\pi|\boldsymbol{x}|^{2}} . \tag{B.35}
\end{equation*}
$$

## B. 5 Transmission problem

We are interested in expressing the solution $u$ of the direct perturbation problem (B.11) by means of an integral representation formula over the boundary $\Gamma$. To study this kind of representations, the differential problem defined on $\Omega_{e}$ is extended as a transmission problem defined now on the whole plane $\mathbb{R}^{2}$ by combining (B.11) with a corresponding interior problem defined on $\Omega_{i}$. For the transmission problem, which specifies jump conditions over the boundary $\Gamma$, a general integral representation can be developed, and the particular integral representations of interest are then established by the specific choice of the corresponding interior problem.

A transmission problem is then a differential problem for which the jump conditions of the solution field, rather than boundary conditions, are specified on the boundary $\Gamma$. As shown in Figure B.1, we consider the exterior domain $\Omega_{e}$ and the interior domain $\Omega_{i}$, taking the unit normal $\boldsymbol{n}$ pointing towards $\Omega_{i}$. We search now a solution $u$ defined in $\Omega_{e} \cup \Omega_{i}$, and use the notation $u_{e}=\left.u\right|_{\Omega_{e}}$ and $u_{i}=\left.u\right|_{\Omega_{i}}$. We define the jumps of the traces of $u$ on both sides of the boundary $\Gamma$ as

$$
\begin{equation*}
[u]=u_{e}-u_{i} \quad \text { and } \quad\left[\frac{\partial u}{\partial n}\right]=\frac{\partial u_{e}}{\partial n}-\frac{\partial u_{i}}{\partial n} . \tag{B.36}
\end{equation*}
$$

The transmission problem is now given by

$$
\begin{cases}\text { Find } u: \Omega_{e} \cup \Omega_{i} & \rightarrow \mathbb{C} \text { such that }  \tag{B.37}\\ \Delta u=0 & \text { in } \Omega_{e} \cup \Omega_{i}, \\ {[u]=\mu} & \text { on } \Gamma \\ {\left[\frac{\partial u}{\partial n}\right]=\nu} & \text { on } \Gamma \\ + & \text { Decaying condition as }|\boldsymbol{x}| \rightarrow \infty\end{cases}
$$

where $\mu, \nu: \Gamma \rightarrow \mathbb{C}$ are known functions. The decaying condition is still (B.7), and it is required to ensure uniqueness of the solution.

## B. 6 Integral representations and equations

## B.6.1 Integral representation

To develop for the solution $u$ an integral representation formula over the boundary $\Gamma$, we define by $\Omega_{R, \varepsilon}$ the domain $\Omega_{e} \cup \Omega_{i}$ without the ball $B_{\varepsilon}$ of radius $\varepsilon>0$ centered at the
point $\boldsymbol{x} \in \Omega_{e} \cup \Omega_{i}$, and truncated at infinity by the ball $B_{R}$ of radius $R>0$ centered at the origin. We consider that the ball $B_{\varepsilon}$ is entirely contained either in $\Omega_{e}$ or in $\Omega_{i}$, depending on the location of its center $\boldsymbol{x}$. Therefore, as shown in Figure B.2, we have that

$$
\begin{equation*}
\Omega_{R, \varepsilon}=\left(\left(\Omega_{e} \cup \Omega_{i}\right) \cap B_{R}\right) \backslash \overline{B_{\varepsilon}}, \tag{B.38}
\end{equation*}
$$

where

$$
\begin{equation*}
B_{R}=\left\{\boldsymbol{y} \in \mathbb{R}^{2}:|\boldsymbol{y}|<R\right\} \quad \text { and } \quad B_{\varepsilon}=\left\{\boldsymbol{y} \in \mathbb{R}^{2}:|\boldsymbol{y}-\boldsymbol{x}|<\varepsilon\right\} . \tag{B.39}
\end{equation*}
$$

We consider similarly the boundaries of the balls

$$
\begin{equation*}
S_{R}=\left\{\boldsymbol{y} \in \mathbb{R}^{2}:|\boldsymbol{y}|=R\right\} \quad \text { and } \quad S_{\varepsilon}=\left\{\boldsymbol{y} \in \mathbb{R}^{2}:|\boldsymbol{y}-\boldsymbol{x}|=\varepsilon\right\} . \tag{B.40}
\end{equation*}
$$

The idea is to retrieve the domain $\Omega_{e} \cup \Omega_{i}$ at the end when the limits $R \rightarrow \infty$ and $\varepsilon \rightarrow 0$ are taken for the truncated domain $\Omega_{R, \varepsilon}$.


Figure B.2. Truncated domain $\Omega_{R, \varepsilon}$ for $\boldsymbol{x} \in \Omega_{e} \cup \Omega_{i}$.

We apply now Green's second integral theorem (A.613) to the functions $u$ and $G(\boldsymbol{x}, \cdot)$ in the bounded domain $\Omega_{R, \varepsilon}$, yielding

$$
\begin{align*}
0 & =\int_{\Omega_{R, \varepsilon}}\left(u(\boldsymbol{y}) \Delta_{y} G(\boldsymbol{x}, \boldsymbol{y})-G(\boldsymbol{x}, \boldsymbol{y}) \Delta u(\boldsymbol{y})\right) \mathrm{d} \boldsymbol{y} \\
& =\int_{S_{R}}\left(u(\boldsymbol{y}) \frac{\partial G}{\partial r_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y})-G(\boldsymbol{x}, \boldsymbol{y}) \frac{\partial u}{\partial r}(\boldsymbol{y})\right) \mathrm{d} \gamma(\boldsymbol{y}) \\
& -\int_{S_{\varepsilon}}\left(u(\boldsymbol{y}) \frac{\partial G}{\partial r_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y})-G(\boldsymbol{x}, \boldsymbol{y}) \frac{\partial u}{\partial r}(\boldsymbol{y})\right) \mathrm{d} \gamma(\boldsymbol{y}) \\
& +\int_{\Gamma}\left([u](\boldsymbol{y}) \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y})-G(\boldsymbol{x}, \boldsymbol{y})\left[\frac{\partial u}{\partial n}\right](\boldsymbol{y})\right) \mathrm{d} \gamma(\boldsymbol{y}) . \tag{B.41}
\end{align*}
$$

For $R$ large enough, the integral on $S_{R}$ tends to zero, since

$$
\begin{equation*}
\left|\int_{S_{R}} u(\boldsymbol{y}) \frac{\partial G}{\partial r_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y})\right| \leq \frac{C}{R} \tag{B.42}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|\int_{S_{R}} G(\boldsymbol{x}, \boldsymbol{y}) \frac{\partial u}{\partial r}(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y})\right| \leq \frac{C}{R} \ln R, \tag{B.43}
\end{equation*}
$$

for some constants $C>0$, due the asymptotic decaying behavior at infinity (B.7). If the function $u$ is regular enough in the ball $B_{\varepsilon}$, then the second term of the integral on $S_{\varepsilon}$, when $\varepsilon \rightarrow 0$ and due (B.23), is bounded by

$$
\begin{equation*}
\left|\int_{S_{\varepsilon}} G(\boldsymbol{x}, \boldsymbol{y}) \frac{\partial u}{\partial r}(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y})\right| \leq \varepsilon \ln \varepsilon \sup _{\boldsymbol{y} \in B_{\varepsilon}}\left|\frac{\partial u}{\partial r}(\boldsymbol{y})\right|, \tag{B.44}
\end{equation*}
$$

and tends to zero. The regularity of $u$ can be specified afterwards once the integral representation has been determined and generalized by means of density arguments. The first integral term on $S_{\varepsilon}$ can be decomposed as

$$
\begin{align*}
\int_{S_{\varepsilon}} u(\boldsymbol{y}) \frac{\partial G}{\partial r_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}) & =u(\boldsymbol{x}) \int_{S_{\varepsilon}} \frac{\partial G}{\partial r_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}) \\
& +\int_{S_{\varepsilon}} \frac{\partial G}{\partial r_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y})(u(\boldsymbol{y})-u(\boldsymbol{x})) \mathrm{d} \gamma(\boldsymbol{y}), \tag{B.45}
\end{align*}
$$

For the first term in the right-hand side of (B.45), by replacing (B.24), we have that

$$
\begin{equation*}
\int_{S_{\varepsilon}} \frac{\partial G}{\partial r_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y})=1 \tag{B.46}
\end{equation*}
$$

while the second term is bounded by

$$
\begin{equation*}
\left|\int_{S_{\varepsilon}}(u(\boldsymbol{y})-u(\boldsymbol{x})) \frac{\partial G}{\partial r_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y})\right| \leq \sup _{\boldsymbol{y} \in B_{\varepsilon}}|u(\boldsymbol{y})-u(\boldsymbol{x})|, \tag{B.47}
\end{equation*}
$$

which tends towards zero when $\varepsilon \rightarrow 0$.
In conclusion, when the limits $R \rightarrow \infty$ and $\varepsilon \rightarrow 0$ are taken in (B.41), then the following integral representation formula holds for the solution $u$ of the transmission problem:

$$
\begin{equation*}
u(\boldsymbol{x})=\int_{\Gamma}\left([u](\boldsymbol{y}) \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y})-G(\boldsymbol{x}, \boldsymbol{y})\left[\frac{\partial u}{\partial n}\right](\boldsymbol{y})\right) \mathrm{d} \gamma(\boldsymbol{y}), \quad \boldsymbol{x} \in \Omega_{e} \cup \Omega_{i} . \tag{B.48}
\end{equation*}
$$

We observe thus that if the values of the jump of $u$ and of its normal derivative are known on $\Gamma$, then the transmission problem (B.37) is readily solved and its solution given explicitly by (B.48), which, in terms of $\mu$ and $\nu$, becomes

$$
\begin{equation*}
u(\boldsymbol{x})=\int_{\Gamma}\left(\mu(\boldsymbol{y}) \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y})-G(\boldsymbol{x}, \boldsymbol{y}) \nu(\boldsymbol{y})\right) \mathrm{d} \gamma(\boldsymbol{y}), \quad \boldsymbol{x} \in \Omega_{e} \cup \Omega_{i} . \tag{B.49}
\end{equation*}
$$

To determine the values of the jumps, an adequate integral equation has to be developed, i.e., an equation whose unknowns are the traces of the solution on $\Gamma$.

An alternative way to demonstrate the integral representation (B.48) is to proceed in the sense of distributions. We consider in this case a test function $\varphi \in \mathcal{D}\left(\mathbb{R}^{2}\right)$ and use Green's second integral theorem (A.613) to obtain that

$$
\begin{equation*}
\langle\Delta u, \varphi\rangle=\langle u, \Delta \varphi\rangle=\int_{\Omega_{e}} u \Delta \varphi \mathrm{~d} \boldsymbol{x}=\int_{\Gamma}\left([u] \frac{\partial \varphi}{\partial n}-\left[\frac{\partial u}{\partial n}\right] \varphi\right) \mathrm{d} \gamma . \tag{B.50}
\end{equation*}
$$

For any function $f$, e.g., continuous over $\Gamma$, we define the distributions $f \delta_{\Gamma}$ and $\frac{\partial}{\partial n}\left(f \delta_{\Gamma}\right)$ of $\mathcal{D}^{\prime}\left(\mathbb{R}^{2}\right)$ respectively by

$$
\begin{equation*}
\left\langle f \delta_{\Gamma}, \varphi\right\rangle=\int_{\Gamma} f \varphi \mathrm{~d} \gamma \quad \text { and } \quad\left\langle\frac{\partial}{\partial n}\left(f \delta_{\Gamma}\right), \varphi\right\rangle=-\int_{\Gamma} f \frac{\partial \varphi}{\partial n} \mathrm{~d} \gamma . \tag{B.51}
\end{equation*}
$$

From a physical or mechanical point of view, the distribution $f \delta_{\Gamma}$ can be considered as a distribution of sources with density $f$ over $\Gamma$, while $\frac{\partial}{\partial n}\left(f \delta_{\Gamma}\right)$ is a distribution of dipoles oriented according to the unit normal $\boldsymbol{n}$ and of density $f$ over $\Gamma$. Using the notation (B.51) we have thus from (B.50) in the sense of distributions that

$$
\begin{equation*}
\Delta u=-\frac{\partial}{\partial n}\left([u] \delta_{\Gamma}\right)-\left[\frac{\partial u}{\partial n}\right] \delta_{\Gamma} \quad \text { in } \mathbb{R}^{2} . \tag{B.52}
\end{equation*}
$$

Hence $\Delta u$ can be interpreted as the sum of a distribution of sources and of a distribution of dipoles over $\Gamma$. Since the Green's function (B.23) is the fundamental solution of the Laplace operator $\Delta$, we have that a solution in $\mathcal{D}^{\prime}\left(\mathbb{R}^{2}\right)$ of the equation (B.52) is given by

$$
\begin{equation*}
u=G *\left(-\frac{\partial}{\partial n}\left([u] \delta_{\Gamma}\right)-\left[\frac{\partial u}{\partial n}\right] \delta_{\Gamma}\right) . \tag{B.53}
\end{equation*}
$$

This illustrates clearly how the solution $u$ is obtained as a convolution with the Green's function. Furthermore, the asymptotic decaying condition (B.7) implies that the solution (B.53) is unique. To obtain (B.48) it remains only to make (B.53) explicit. The term

$$
\begin{equation*}
\left\{G *\left[\frac{\partial u}{\partial n}\right] \delta_{\Gamma}\right\}(\boldsymbol{x})=\int_{\Gamma} G(\boldsymbol{x}, \boldsymbol{y})\left[\frac{\partial u}{\partial n}\right](\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}) \tag{B.54}
\end{equation*}
$$

is called single layer potential, associated with the distribution of sources $[\partial u / \partial n] \delta_{\Gamma}$, while

$$
\begin{equation*}
\left\{G * \frac{\partial}{\partial n}\left([u] \delta_{\Gamma}\right)\right\}(\boldsymbol{x})=-\int_{\Gamma} \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y})[u](\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}) \tag{B.55}
\end{equation*}
$$

represents a double layer potential, associated with the distribution of dipoles $\frac{\partial}{\partial n}\left([u] \delta_{\Gamma}\right)$. Combining (B.54) and (B.55) yields finally the desired integral representation (B.48).

We note that to obtain the gradient of the integral representation (B.48) we can pass directly the derivatives inside the integral, since there are no singularities if $\boldsymbol{x} \in \Omega_{e} \cup \Omega_{i}$. Therefore we have that

$$
\begin{equation*}
\nabla u(\boldsymbol{x})=\int_{\Gamma}\left([u](\boldsymbol{y}) \nabla_{\boldsymbol{x}} \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y})-\nabla_{\boldsymbol{x}} G(\boldsymbol{x}, \boldsymbol{y})\left[\frac{\partial u}{\partial n}\right](\boldsymbol{y})\right) \mathrm{d} \gamma(\boldsymbol{y}) . \tag{B.56}
\end{equation*}
$$

We remark also that the asymptotic decaying behavior (B.7) and Green's first integral theorem (A.612) imply that

$$
\begin{equation*}
\int_{\Gamma} \frac{\partial u_{e}}{\partial n} \mathrm{~d} \gamma=\int_{\Gamma} \frac{\partial u_{i}}{\partial n} \mathrm{~d} \gamma=0 \tag{B.57}
\end{equation*}
$$

since

$$
\begin{equation*}
\int_{\Gamma} \frac{\partial u_{e}}{\partial n} \mathrm{~d} \gamma=\int_{\Omega_{e} \cap B_{R}} \Delta u_{e} \mathrm{~d} \boldsymbol{x}-\int_{S_{R}} \frac{\partial u_{e}}{\partial r} \mathrm{~d} \gamma=-\int_{S_{R}} \frac{\partial u_{e}}{\partial r} \mathrm{~d} \gamma \xrightarrow[R \rightarrow \infty]{ } 0 \tag{B.58}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{\Gamma} \frac{\partial u_{i}}{\partial n} \mathrm{~d} \gamma=-\int_{\Omega_{i}} \Delta u_{i} \mathrm{~d} \boldsymbol{x}=0 \tag{B.59}
\end{equation*}
$$

Reciprocally, by using the integral representation formula (B.48) it can be verified that this hypothesis (B.57) implies the asymptotic decaying behavior (B.7).

## B.6.2 Integral equations

To determine the values of the traces that conform the jumps for the transmission problem (B.37), an integral equation has to be developed. For this purpose we place the source point $\boldsymbol{x}$ on the boundary $\Gamma$, as shown in Figure B.3, and apply the same procedure as before for the integral representation (B.48), treating differently in (B.41) only the integrals on $S_{\varepsilon}$. The integrals on $S_{R}$ still behave well and tend towards zero as $R \rightarrow \infty$. The Ball $B_{\varepsilon}$, though, is split in half into the two pieces $\Omega_{e} \cap B_{\varepsilon}$ and $\Omega_{i} \cap B_{\varepsilon}$, which are asymptotically separated by the tangent of the boundary if $\Gamma$ is regular. Thus the associated integrals on $S_{\varepsilon}$ give rise to a term $-\left(u_{e}(\boldsymbol{x})+u_{i}(\boldsymbol{x})\right) / 2$ instead of just $-u(\boldsymbol{x})$ as before. We must notice that in this case, the integrands associated with the boundary $\Gamma$ admit an integrable singularity at the point $\boldsymbol{x}$. The desired integral equation related with (B.48) is then given by

$$
\begin{equation*}
\frac{u_{e}(\boldsymbol{x})+u_{i}(\boldsymbol{x})}{2}=\int_{\Gamma}\left([u](\boldsymbol{y}) \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y})-G(\boldsymbol{x}, \boldsymbol{y})\left[\frac{\partial u}{\partial n}\right](\boldsymbol{y})\right) \mathrm{d} \gamma(\boldsymbol{y}), \quad \boldsymbol{x} \in \Gamma . \tag{B.60}
\end{equation*}
$$

By choosing adequately the boundary condition of the interior problem, and by considering also the boundary condition of the exterior problem and the jump definitions (B.36), this integral equation can be expressed in terms of only one unknown function on $\Gamma$. Thus, solving the problem (B.11) is equivalent to solve (B.60) and then replace the obtained solution in (B.48).


Figure B.3. Truncated domain $\Omega_{R, \varepsilon}$ for $\boldsymbol{x} \in \Gamma$.

We remark that the integral equation (B.60) has to be understood in the sense of a mean between the traces of the solution $u$ on both sides of $\Gamma$, as illustrated in Figure B.4. It gives information only for the jumps, but not for the solution of the problem. The true value of the solution on the boundary $\Gamma$ for the exterior and the interior problems is always given
by the limit case as $\boldsymbol{x}$ tends towards $\Gamma$ respectively from $\Omega_{e}$ and $\Omega_{i}$ of the representation formula (B.48).


Figure B.4. Jump over $\Gamma$ of the solution $u$.

The integral equation holds only when the boundary $\Gamma$ is regular (e.g., of class $C^{2}$ ). Otherwise, taking the limit $\varepsilon \rightarrow 0$ can no longer be well-defined and the result is false in general. In particular, if the boundary $\Gamma$ has an angular point at $\boldsymbol{x} \in \Gamma$, as shown in Figure B. 5 and where $\theta$ represents the angle in radians ( $0<\theta<2 \pi$ ) of the tangents of the boundary on that particular point $\boldsymbol{x}$ measured over $\Omega_{e}$, then the left-hand side of the integral equation (B.60) is modified on that point according to the portion of the ball $B_{\varepsilon}$ that remains inside $\Omega_{e}$, namely

$$
\begin{equation*}
\frac{\theta}{2 \pi} u_{e}(\boldsymbol{x})+\left(1-\frac{\theta}{2 \pi}\right) u_{i}(\boldsymbol{x})=\int_{\Gamma}\left([u](\boldsymbol{y}) \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y})-G(\boldsymbol{x}, \boldsymbol{y})\left[\frac{\partial u}{\partial n}\right](\boldsymbol{y})\right) \mathrm{d} \gamma(\boldsymbol{y}) \tag{B.61}
\end{equation*}
$$

The solution $u$ usually presents singularities on those points where $\Gamma$ fails to be regular.


Figure B.5. Angular point $\boldsymbol{x}$ of the boundary $\Gamma$.

Another integral equation can be also derived for the normal derivative of the solution $u$ on the boundary $\Gamma$, by studying the jump properties of the single and double layer potentials. Its derivation is more complicated than for (B.60), being the specific details explicited below in the subsection of boundary layer potentials. If the boundary is regular at $\boldsymbol{x} \in \Gamma$, then we obtain

$$
\begin{equation*}
\frac{1}{2} \frac{\partial u_{e}}{\partial n}(\boldsymbol{x})+\frac{1}{2} \frac{\partial u_{i}}{\partial n}(\boldsymbol{x})=\int_{\Gamma}\left([u](\boldsymbol{y}) \frac{\partial^{2} G}{\partial n_{\boldsymbol{x}} \partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y})-\frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x}, \boldsymbol{y})\left[\frac{\partial u}{\partial n}\right](\boldsymbol{y})\right) \mathrm{d} \gamma(\boldsymbol{y}) . \tag{B.62}
\end{equation*}
$$

This integral equation is modified in the same way as (B.61) if $\boldsymbol{x}$ is an angular point.

## B.6.3 Integral kernels

The integral kernels $G, \partial G / \partial n_{\boldsymbol{y}}$, and $\partial G / \partial n_{\boldsymbol{x}}$ are weakly singular, and thus integrable, whereas the kernel $\partial^{2} G / \partial n_{\boldsymbol{x}} \partial n_{\boldsymbol{y}}$ has a strong singularity at the point $\boldsymbol{x}$, which is not integrable and therefore referred to as a hypersingular kernel.

In general, a kernel $K(\boldsymbol{x}, \boldsymbol{y})$ of an integral operator of the form

$$
\begin{equation*}
T \varphi(\boldsymbol{x})=\int_{\Gamma} K(\boldsymbol{x}, \boldsymbol{y}) \varphi(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}), \quad \boldsymbol{x} \in \Gamma \subset \mathbb{R}^{N} \tag{B.63}
\end{equation*}
$$

is said to be weakly singular if it is defined and continuous for $\boldsymbol{x} \neq \boldsymbol{y}$, and if there exist some constants $C>0$ and $0<\lambda<N-1$ such that

$$
\begin{equation*}
|K(\boldsymbol{x}, \boldsymbol{y})| \leq \frac{C}{|\boldsymbol{x}-\boldsymbol{y}|^{\lambda}} \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \Gamma \tag{B.64}
\end{equation*}
$$

in which case the integral operator (B.63) is improper, but integrable, i.e., such that

$$
\begin{equation*}
\int_{\Gamma}|K(\boldsymbol{x}, \boldsymbol{y})| \mathrm{d} \gamma(\boldsymbol{y})<\infty \tag{B.65}
\end{equation*}
$$

If $K(\boldsymbol{x}, \boldsymbol{y})$ requires $\lambda \geq N-1$ in (B.64), then the kernel is said to be hypersingular.
The kernel $G$ defined in (B.23) is logarithmic and thus fulfills (B.64) for any $\lambda>0$. The kernels $\partial G / \partial n_{\boldsymbol{y}}$ and $\partial G / \partial n_{\boldsymbol{x}}$ are less singular along $\Gamma$ than they appear at first sight, due the regularizing effect of the normal derivatives. They are given respectively by

$$
\begin{equation*}
\frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y})=\frac{(\boldsymbol{y}-\boldsymbol{x}) \cdot \boldsymbol{n}_{\boldsymbol{y}}}{2 \pi|\boldsymbol{y}-\boldsymbol{x}|^{2}} \quad \text { and } \quad \frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x}, \boldsymbol{y})=\frac{(\boldsymbol{x}-\boldsymbol{y}) \cdot \boldsymbol{n}_{\boldsymbol{x}}}{2 \pi|\boldsymbol{x}-\boldsymbol{y}|^{2}} . \tag{B.66}
\end{equation*}
$$

Let us consider first the kernel $\partial G / \partial n_{\boldsymbol{y}}$. A regular boundary $\Gamma$ can be described in the neighborhood of a point $\boldsymbol{y}$ as the graph of a regular function $\varphi$ that takes variables on the tangent line at $\boldsymbol{y}$. We write $\eta_{2}=\varphi\left(\eta_{1}\right)$, being the origin of the coordinate system $\left(\eta_{1}, \eta_{2}\right)$ located at $\boldsymbol{y}$, where $\eta_{2}$ is aligned with $\boldsymbol{n}_{\boldsymbol{y}}$, and where $\eta_{1}$ lies on the tangent line at $\boldsymbol{y}$, as shown in Figure B.6. It holds thus that $\varphi(0)=0$ and $\varphi^{\prime}(0)=0$. A Taylor expansion around the origin yields

$$
\begin{equation*}
\eta_{2}=\varphi(0)+\varphi^{\prime}(0) \eta_{1}+\mathcal{O}\left(\left|\eta_{1}\right|^{2}\right)=\mathcal{O}\left(\left|\eta_{1}\right|^{2}\right), \tag{B.67}
\end{equation*}
$$

and therefore

$$
\begin{equation*}
(\boldsymbol{x}-\boldsymbol{y}) \cdot \boldsymbol{n}_{\boldsymbol{y}}=\eta_{2}=\varphi\left(\eta_{1}\right)=\mathcal{O}\left(\left|\eta_{1}\right|^{2}\right) . \tag{B.68}
\end{equation*}
$$

Since, on the other hand, we have

$$
\begin{equation*}
|\boldsymbol{y}-\boldsymbol{x}|^{2}=\left|\eta_{1}\right|^{2}+\left|\eta_{2}\right|^{2}=\mathcal{O}\left(\left|\eta_{1}\right|^{2}\right), \tag{B.69}
\end{equation*}
$$

consequently we obtain that

$$
\begin{equation*}
(\boldsymbol{y}-\boldsymbol{x}) \cdot \boldsymbol{n}_{\boldsymbol{y}}=\mathcal{O}\left(|\boldsymbol{y}-\boldsymbol{x}|^{2}\right) \tag{B.70}
\end{equation*}
$$

By inversing the roles, the same holds also when considering $\boldsymbol{n}_{\boldsymbol{x}}$ instead of $\boldsymbol{n}_{\boldsymbol{y}}$, i.e.,

$$
\begin{equation*}
(\boldsymbol{x}-\boldsymbol{y}) \cdot \boldsymbol{n}_{\boldsymbol{x}}=\mathcal{O}\left(|\boldsymbol{x}-\boldsymbol{y}|^{2}\right) \tag{B.71}
\end{equation*}
$$

This means that

$$
\begin{equation*}
\frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y})=\mathcal{O}(1) \quad \text { and } \quad \frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x}, \boldsymbol{y})=\mathcal{O}(1) \tag{B.72}
\end{equation*}
$$

The singularities of the kernels $\partial G / \partial n_{\boldsymbol{y}}$ and $\partial G / \partial n_{\boldsymbol{x}}$ along $\Gamma$ are thus only apparent and can be repaired by redefining the value of these kernels at $\boldsymbol{y}=\boldsymbol{x}$.


Figure B.6. Graph of the function $\varphi$ on the tangent line of $\Gamma$.

The kernel $\partial^{2} G / \partial n_{\boldsymbol{x}} \partial n_{\boldsymbol{y}}$, on the other hand, is strongly singular along $\Gamma$. It adopts the expression

$$
\begin{equation*}
\frac{\partial^{2} G}{\partial n_{\boldsymbol{x}} \partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y})=-\frac{\boldsymbol{n}_{\boldsymbol{x}} \cdot \boldsymbol{n}_{\boldsymbol{y}}}{2 \pi|\boldsymbol{y}-\boldsymbol{x}|^{2}}-\frac{\left((\boldsymbol{x}-\boldsymbol{y}) \cdot \boldsymbol{n}_{\boldsymbol{x}}\right)\left((\boldsymbol{y}-\boldsymbol{x}) \cdot \boldsymbol{n}_{\boldsymbol{y}}\right)}{\pi|\boldsymbol{y}-\boldsymbol{x}|^{4}} . \tag{B.73}
\end{equation*}
$$

The regularizing effect of the normal derivatives applies only to its second term, but not to the first, since

$$
\begin{equation*}
\boldsymbol{n}_{\boldsymbol{x}} \cdot \boldsymbol{n}_{\boldsymbol{y}}=\mathcal{O}(1) \tag{B.74}
\end{equation*}
$$

Hence the kernel (B.73) is clearly hypersingular, with $\lambda=2$, and it holds that

$$
\begin{equation*}
\frac{\partial^{2} G}{\partial n_{\boldsymbol{x}} \partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y})=\mathcal{O}\left(\frac{1}{|\boldsymbol{y}-\boldsymbol{x}|^{2}}\right) . \tag{B.75}
\end{equation*}
$$

This kernel is no longer integrable and the associated integral operator has to be thus interpreted in some appropriate sense as a divergent integral (cf., e.g., Hsiao \& Wendland 2008, Lenoir 2005, Nédélec 2001).

## B.6.4 Boundary layer potentials

We regard now the jump properties on the boundary $\Gamma$ of the boundary layer potentials that have appeared in our calculations. For the development of the integral representation (B.49) we already made acquaintance with the single and double layer potentials, which we define now more precisely for $\boldsymbol{x} \in \Omega_{e} \cup \Omega_{i}$ as the integral operators

$$
\begin{align*}
\mathcal{S} \nu(\boldsymbol{x}) & =\int_{\Gamma} G(\boldsymbol{x}, \boldsymbol{y}) \nu(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y})  \tag{B.76}\\
\mathcal{D} \mu(\boldsymbol{x}) & =\int_{\Gamma} \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) \mu(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}) \tag{B.77}
\end{align*}
$$

The integral representation (B.49) can be now stated in terms of the layer potentials as

$$
\begin{equation*}
u=\mathcal{D} \mu-\mathcal{S} \nu \tag{B.78}
\end{equation*}
$$

We remark that for any functions $\nu, \mu: \Gamma \rightarrow \mathbb{C}$ that are regular enough, the single and double layer potentials satisfy the Laplace equation, namely

$$
\begin{array}{ll}
\Delta \mathcal{S} \nu=0 & \text { in } \Omega_{e} \cup \Omega_{i}, \\
\Delta \mathcal{D} \mu=0 & \text { in } \Omega_{e} \cup \Omega_{i} \tag{B.80}
\end{array}
$$

For the integral equations (B.60) and (B.62), which are defined for $\boldsymbol{x} \in \Gamma$, we require the four boundary integral operators:

$$
\begin{align*}
S \nu(\boldsymbol{x}) & =\int_{\Gamma} G(\boldsymbol{x}, \boldsymbol{y}) \nu(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y})  \tag{B.81}\\
D \mu(\boldsymbol{x}) & =\int_{\Gamma} \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) \mu(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}),  \tag{B.82}\\
D^{*} \nu(\boldsymbol{x}) & =\int_{\Gamma} \frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x}, \boldsymbol{y}) \nu(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y})  \tag{B.83}\\
N \mu(\boldsymbol{x}) & =\int_{\Gamma} \frac{\partial^{2} G}{\partial n_{\boldsymbol{x}} \partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) \mu(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}) \tag{B.84}
\end{align*}
$$

The operator $D^{*}$ is in fact the adjoint of the operator $D$. As we already mentioned, the kernel of the integral operator $N$ defined in (B.84) is not integrable, yet we write it formally as an improper integral. An appropriate sense for this integral will be given below. The integral equations (B.60) and (B.62) can be now stated in terms of the integral operators as

$$
\begin{align*}
\frac{1}{2}\left(u_{e}+u_{i}\right) & =D \mu-S \nu  \tag{B.85}\\
\frac{1}{2}\left(\frac{\partial u_{e}}{\partial n}+\frac{\partial u_{i}}{\partial n}\right) & =N \mu-D^{*} \nu \tag{B.86}
\end{align*}
$$

These integral equations can be easily derived from the jump properties of the single and double layer potentials. The single layer potential (B.76) is continuous and its normal derivative has a jump of size $-\nu$ across $\Gamma$, i.e.,

$$
\begin{align*}
\left.\mathcal{S} \nu\right|_{\Omega_{e}} & =S \nu=\left.\mathcal{S} \nu\right|_{\Omega_{i}},  \tag{B.87}\\
\left.\frac{\partial}{\partial n} \mathcal{S} \nu\right|_{\Omega_{e}} & =\left(-\frac{1}{2}+D^{*}\right) \nu,  \tag{B.88}\\
\left.\frac{\partial}{\partial n} \mathcal{S} \nu\right|_{\Omega_{i}} & =\left(\frac{1}{2}+D^{*}\right) \nu . \tag{B.89}
\end{align*}
$$

The double layer potential (B.77), on the other hand, has a jump of size $\mu$ across $\Gamma$ and its normal derivative is continuous, namely

$$
\begin{align*}
\left.\mathcal{D} \mu\right|_{\Omega_{e}} & =\left(\frac{1}{2}+D\right) \mu,  \tag{B.90}\\
\left.\mathcal{D} \mu\right|_{\Omega_{i}} & =\left(-\frac{1}{2}+D\right) \mu, \tag{B.91}
\end{align*}
$$

$$
\begin{equation*}
\left.\frac{\partial}{\partial n} \mathcal{D} \mu\right|_{\Omega_{e}}=N \mu=\left.\frac{\partial}{\partial n} \mathcal{D} \mu\right|_{\Omega_{i}} . \tag{B.92}
\end{equation*}
$$

The integral equation (B.85) is obtained directly either from (B.87) and (B.90), or from (B.87) and (B.91), by considering the appropriate trace of (B.78) and by defining the functions $\mu$ and $\nu$ as in (B.37). These three jump properties are easily proven by regarding the details of the proof for (B.60).

Similarly, the integral equation (B.86) for the normal derivative is obtained directly either from (B.88) and (B.92), or from (B.89) and (B.92), by considering the appropriate trace of the normal derivative of (B.78) and by defining again the functions $\mu$ and $\nu$ as in (B.37). The proof of these other three jump properties is done below.
a) Jump of the normal derivative of the single layer potential

Let us then study first the proof of (B.88) and (B.89). The traces of the normal derivative of the single layer potential are given by

$$
\begin{align*}
\left.\frac{\partial}{\partial n} \mathcal{S} \nu(\boldsymbol{x})\right|_{\Omega_{e}} & =\lim _{\Omega_{e} \ni \boldsymbol{z} \rightarrow \boldsymbol{x}} \nabla \mathcal{S} \nu(\boldsymbol{z}) \cdot \boldsymbol{n}_{\boldsymbol{x}},  \tag{B.93}\\
\left.\frac{\partial}{\partial n} \mathcal{S} \nu(\boldsymbol{x})\right|_{\Omega_{i}} & =\lim _{\Omega_{i} \ni \boldsymbol{z} \rightarrow \boldsymbol{x}} \nabla \mathcal{S} \nu(\boldsymbol{z}) \cdot \boldsymbol{n}_{\boldsymbol{x}} . \tag{B.94}
\end{align*}
$$

Now we have that

$$
\begin{equation*}
\nabla \mathcal{S} \nu(\boldsymbol{z}) \cdot \boldsymbol{n}_{\boldsymbol{x}}=\int_{\Gamma} \boldsymbol{n}_{\boldsymbol{x}} \cdot \nabla_{\boldsymbol{z}} G(\boldsymbol{z}, \boldsymbol{y}) \nu(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}) \tag{B.95}
\end{equation*}
$$

For $\varepsilon>0$ we denote $\Gamma_{\varepsilon}=\Gamma \cap B_{\varepsilon}$, i.e., the portion of $\Gamma$ contained inside the ball $B_{\varepsilon}$ of radius $\varepsilon$ and centered at $\boldsymbol{x}$. By decomposing the integral we obtain that

$$
\begin{equation*}
\nabla \mathcal{S} \nu(\boldsymbol{z}) \cdot \boldsymbol{n}_{\boldsymbol{x}}=\int_{\Gamma \backslash \Gamma_{\varepsilon}} \boldsymbol{n}_{\boldsymbol{x}} \cdot \nabla_{\boldsymbol{z}} G(\boldsymbol{z}, \boldsymbol{y}) \nu(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y})+\int_{\Gamma_{\varepsilon}} \boldsymbol{n}_{\boldsymbol{x}} \cdot \nabla_{\boldsymbol{z}} G(\boldsymbol{z}, \boldsymbol{y}) \nu(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}) \tag{B.96}
\end{equation*}
$$

For the first integral in (B.96) we can take without problems the limit $\boldsymbol{z} \rightarrow \boldsymbol{x}$, since for a fixed $\varepsilon$ the integral is regular in $\boldsymbol{x}$. Since the singularity of the resulting kernel $\partial G / \partial n_{\boldsymbol{x}}$ is integrable, Lebesgue's dominated convergence theorem (cf. Royden 1988) implies that

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \int_{\Gamma \backslash \Gamma_{\varepsilon}} \frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x}, \boldsymbol{y}) \nu(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y})=\int_{\Gamma} \frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x}, \boldsymbol{y}) \nu(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y})=D^{*} \nu(\boldsymbol{x}) . \tag{B.97}
\end{equation*}
$$

Let us treat now the second integral in (B.96), which is again decomposed in different integrals in such a way that

$$
\begin{align*}
& \int_{\Gamma_{\varepsilon}} \boldsymbol{n}_{\boldsymbol{x}} \cdot \nabla_{\boldsymbol{z}} G(\boldsymbol{z}, \boldsymbol{y}) \nu(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y})=\int_{\Gamma_{\varepsilon}}\left(\boldsymbol{n}_{\boldsymbol{x}}-\boldsymbol{n}_{\boldsymbol{y}}\right) \cdot \nabla_{\boldsymbol{z}} G(\boldsymbol{z}, \boldsymbol{y}) \nu(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}) \\
& \quad+\int_{\Gamma_{\boldsymbol{\varepsilon}}} \boldsymbol{n}_{\boldsymbol{y}} \cdot \nabla_{\boldsymbol{z}} G(\boldsymbol{z}, \boldsymbol{y})(\nu(\boldsymbol{y})-\nu(\boldsymbol{x})) \mathrm{d} \gamma(\boldsymbol{y})+\nu(\boldsymbol{x}) \int_{\Gamma_{\varepsilon}} \boldsymbol{n}_{\boldsymbol{y}} \cdot \nabla_{\boldsymbol{z}} G(\boldsymbol{z}, \boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}) . \tag{B.98}
\end{align*}
$$

When $\varepsilon$ is small, and since $\Gamma$ is supposed to be regular, therefore $\Gamma_{\varepsilon}$ resembles a straight line segment of length $2 \varepsilon$. Thus we have that

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \int_{\Gamma_{\varepsilon}}\left(\boldsymbol{n}_{\boldsymbol{x}}-\boldsymbol{n}_{\boldsymbol{y}}\right) \cdot \nabla_{\boldsymbol{z}} G(\boldsymbol{z}, \boldsymbol{y}) \nu(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y})=0 . \tag{B.99}
\end{equation*}
$$

If $\nu$ is regular enough, then we have also that

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \int_{\Gamma_{\varepsilon}} \boldsymbol{n}_{\boldsymbol{y}} \cdot \nabla_{\boldsymbol{z}} G(\boldsymbol{z}, \boldsymbol{y})(\nu(\boldsymbol{y})-\nu(\boldsymbol{x})) \mathrm{d} \gamma(\boldsymbol{y})=0 \tag{B.100}
\end{equation*}
$$

For the remaining term in (B.98) we consider the angle $\theta$ under which the almost straight line segment $\Gamma_{\varepsilon}$ is seen from point $\boldsymbol{z}$ (cf. Figure B.7). If we denote $\boldsymbol{R}=\boldsymbol{y}-\boldsymbol{z}$ and $R=|\boldsymbol{R}|$, and consider an oriented boundary differential element $\mathrm{d} \boldsymbol{\gamma}=\boldsymbol{n}_{\boldsymbol{y}} \mathrm{d} \gamma(\boldsymbol{y})$ seen from point $\boldsymbol{z}$, then we can express the angle differential element by

$$
\begin{equation*}
\mathrm{d} \theta=\frac{\boldsymbol{R}}{R^{2}} \cdot \mathrm{~d} \boldsymbol{\gamma}=\frac{\boldsymbol{R} \cdot \boldsymbol{n}_{\boldsymbol{y}}}{R^{2}} \mathrm{~d} \gamma(\boldsymbol{y})=2 \pi \boldsymbol{n}_{\boldsymbol{y}} \cdot \nabla_{\boldsymbol{y}} G(\boldsymbol{z}, \boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}) \tag{B.101}
\end{equation*}
$$

Integrating over the segment $\Gamma_{\varepsilon}$ and considering (B.28) yields the angle $\theta$, namely

$$
\begin{equation*}
\theta=\int_{\Gamma_{\varepsilon}} \mathrm{d} \theta=2 \pi \int_{\Gamma_{\varepsilon}} \boldsymbol{n}_{\boldsymbol{y}} \cdot \nabla_{\boldsymbol{y}} G(\boldsymbol{z}, \boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y})=-2 \pi \int_{\Gamma_{\varepsilon}} \boldsymbol{n}_{\boldsymbol{y}} \cdot \nabla_{\boldsymbol{z}} G(\boldsymbol{z}, \boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}) \tag{B.102}
\end{equation*}
$$

where $-\pi \leq \theta \leq \pi$. The angle $\theta$ is positive when the vectors $\boldsymbol{R}$ and $\boldsymbol{n}_{\boldsymbol{y}}$ point towards the same side of $\Gamma_{\varepsilon}$, and negative when they oppose each other. Thus if $\boldsymbol{z}$ is very close to $\boldsymbol{x}$ and if $\varepsilon$ is small enough so that $\Gamma_{\varepsilon}$ behaves as a straight line segment, then

$$
\int_{\Gamma_{\varepsilon}} \boldsymbol{n}_{\boldsymbol{y}} \cdot \nabla_{\boldsymbol{z}} G(\boldsymbol{z}, \boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}) \approx \begin{cases}-1 / 2 & \text { if } \boldsymbol{z} \in \Omega_{e}  \tag{B.103}\\ 1 / 2 & \text { if } \boldsymbol{z} \in \Omega_{i} .\end{cases}
$$

Hence we obtain the desired jump formulae (B.88) and (B.89).


Figure B.7. Angle under which $\Gamma_{\varepsilon}$ is seen from point $\boldsymbol{z}$.
b) Continuity of the normal derivative of the double layer potential

We are now interested in proving the continuity of the normal derivative of the double layer potential across $\Gamma$, as expressed in (B.92). This will allow us at the same time to define an appropriate sense for the improper integral (B.84). This integral is divergent in a classical sense, but it can be nonetheless properly defined in a weak or distributional sense by considering it as a linear functional acting on a test function $\varphi \in \mathcal{D}\left(\mathbb{R}^{2}\right)$. By considering (B.80) and Green's first integral theorem (A.612), we can express our values of interest in a weak sense as

$$
\begin{align*}
\left\langle\left.\frac{\partial}{\partial n} \mathcal{D} \mu\right|_{\Omega_{e}}, \varphi\right\rangle & =\left.\int_{\Gamma} \frac{\partial}{\partial n} \mathcal{D} \mu(\boldsymbol{x})\right|_{\Omega_{e}} \varphi(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{x})=\int_{\Omega_{e}} \nabla \mathcal{D} \mu(\boldsymbol{x}) \cdot \nabla \varphi(\boldsymbol{x}) \mathrm{d} \boldsymbol{x}  \tag{B.104}\\
\left\langle\left.\frac{\partial}{\partial n} \mathcal{D} \mu\right|_{\Omega_{i}}, \varphi\right\rangle & =\left.\int_{\Gamma} \frac{\partial}{\partial n} \mathcal{D} \mu(\boldsymbol{x})\right|_{\Omega_{i}} \varphi(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{x})=-\int_{\Omega_{i}} \nabla \mathcal{D} \mu(\boldsymbol{x}) \cdot \nabla \varphi(\boldsymbol{x}) \mathrm{d} \boldsymbol{x} \tag{B.105}
\end{align*}
$$

From (A.588) and (B.28) we obtain the relation

$$
\begin{equation*}
\frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y})=\boldsymbol{n}_{\boldsymbol{y}} \cdot \nabla_{\boldsymbol{y}} G(\boldsymbol{x}, \boldsymbol{y})=-\boldsymbol{n}_{\boldsymbol{y}} \cdot \nabla_{\boldsymbol{x}} G(\boldsymbol{x}, \boldsymbol{y})=-\operatorname{div}_{\boldsymbol{x}}\left(G(\boldsymbol{x}, \boldsymbol{y}) \boldsymbol{n}_{\boldsymbol{y}}\right) \tag{B.106}
\end{equation*}
$$

Thus for the double layer potential (B.77) we have that

$$
\begin{equation*}
\mathcal{D} \mu(\boldsymbol{x})=-\operatorname{div} \int_{\Gamma} G(\boldsymbol{x}, \boldsymbol{y}) \mu(\boldsymbol{y}) \boldsymbol{n}_{\boldsymbol{y}} \mathrm{d} \gamma(\boldsymbol{y})=-\operatorname{div} \mathcal{S}\left(\mu \boldsymbol{n}_{\boldsymbol{y}}\right)(\boldsymbol{x}) \tag{B.107}
\end{equation*}
$$

being its gradient given by

$$
\begin{equation*}
\nabla \mathcal{D} \mu(\boldsymbol{x})=-\nabla \operatorname{div} \int_{\Gamma} G(\boldsymbol{x}, \boldsymbol{y}) \mu(\boldsymbol{y}) \boldsymbol{n}_{\boldsymbol{y}} \mathrm{d} \gamma(\boldsymbol{y}) . \tag{B.108}
\end{equation*}
$$

From (A.589) we have that

$$
\begin{equation*}
\operatorname{curl}_{\boldsymbol{x}}\left(G(\boldsymbol{x}, \boldsymbol{y}) \boldsymbol{n}_{\boldsymbol{y}}\right)=\nabla_{\boldsymbol{x}} G(\boldsymbol{x}, \boldsymbol{y}) \times \boldsymbol{n}_{\boldsymbol{y}} . \tag{B.109}
\end{equation*}
$$

Hence, by considering (A.597), (B.80), and (B.109) in (B.108), we obtain that

$$
\begin{equation*}
\nabla \mathcal{D} \mu(\boldsymbol{x})=\operatorname{Curl} \int_{\Gamma}\left(\boldsymbol{n}_{\boldsymbol{y}} \times \nabla_{\boldsymbol{x}} G(\boldsymbol{x}, \boldsymbol{y})\right) \mu(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}) . \tag{B.110}
\end{equation*}
$$

From (B.28) and (A.659) we have that

$$
\begin{align*}
\int_{\Gamma}\left(\boldsymbol{n}_{\boldsymbol{y}} \times \nabla_{\boldsymbol{x}} G(\boldsymbol{x}, \boldsymbol{y})\right) \mu(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}) & =-\int_{\Gamma} \boldsymbol{n}_{\boldsymbol{y}} \times\left(\nabla_{\boldsymbol{y}} G(\boldsymbol{x}, \boldsymbol{y}) \mu(\boldsymbol{y})\right) \mathrm{d} \gamma(\boldsymbol{y}) \\
& =\int_{\Gamma} \boldsymbol{n}_{\boldsymbol{y}} \times(G(\boldsymbol{x}, \boldsymbol{y}) \nabla \mu(\boldsymbol{y})) \mathrm{d} \gamma(\boldsymbol{y}), \tag{B.111}
\end{align*}
$$

and consequently

$$
\begin{equation*}
\nabla \mathcal{D} \mu(\boldsymbol{x})=\operatorname{Curl} \int_{\Gamma} G(\boldsymbol{x}, \boldsymbol{y})\left(\boldsymbol{n}_{\boldsymbol{y}} \times \nabla \mu(\boldsymbol{y})\right) \mathrm{d} \gamma(\boldsymbol{y}) \tag{B.112}
\end{equation*}
$$

Now, considering (A.608) and (A.619), and replacing (B.112) in (B.104), implies that

$$
\begin{equation*}
\int_{\Omega_{e}} \nabla \mathcal{D} \mu(\boldsymbol{x}) \cdot \nabla \varphi(\boldsymbol{x}) \mathrm{d} \boldsymbol{x}=-\int_{\Gamma} \int_{\Gamma} G(\boldsymbol{x}, \boldsymbol{y})\left(\nabla \mu(\boldsymbol{y}) \times \boldsymbol{n}_{\boldsymbol{y}}\right)\left(\nabla \varphi(\boldsymbol{x}) \times \boldsymbol{n}_{\boldsymbol{x}}\right) \mathrm{d} \gamma(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{x}) . \tag{B.113}
\end{equation*}
$$

Analogously, when replacing in (B.105) we have that

$$
\begin{equation*}
\int_{\Omega_{i}} \nabla \mathcal{D} \mu(\boldsymbol{x}) \cdot \nabla \varphi(\boldsymbol{x}) \mathrm{d} \boldsymbol{x}=\int_{\Gamma} \int_{\Gamma} G(\boldsymbol{x}, \boldsymbol{y})\left(\nabla \mu(\boldsymbol{y}) \times \boldsymbol{n}_{\boldsymbol{y}}\right)\left(\nabla \varphi(\boldsymbol{x}) \times \boldsymbol{n}_{\boldsymbol{x}}\right) \mathrm{d} \gamma(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{x}) \tag{B.114}
\end{equation*}
$$

Hence, from (B.104), (B.105), (B.113), and (B.114) we conclude the proof of (B.92). The integral operator (B.84) is thus properly defined in a weak sense for $\varphi \in \mathcal{D}\left(\mathbb{R}^{2}\right)$ by

$$
\begin{equation*}
\langle N \mu(\boldsymbol{x}), \varphi\rangle=-\int_{\Gamma} \int_{\Gamma} G(\boldsymbol{x}, \boldsymbol{y})\left(\nabla \mu(\boldsymbol{y}) \times \boldsymbol{n}_{\boldsymbol{y}}\right)\left(\nabla \varphi(\boldsymbol{x}) \times \boldsymbol{n}_{\boldsymbol{x}}\right) \mathrm{d} \gamma(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{x}) . \tag{B.115}
\end{equation*}
$$

## B.6.5 Calderón projectors

The surface layer potentials (B.81)-(B.84) are linked together by means of the socalled Calderón relations, which receive their name from the Argentine mathematician Alberto Pedro Calderón (1920-1998), who is best known for his work on the theory of partial differential equations and singular integral operators. The exterior and interior traces of a
function $u$ defined by (B.78) can be characterized, due (B.85) and (B.86), by

$$
\begin{align*}
& \binom{u_{e}}{\frac{\partial u_{e}}{\partial n}}=\left(\begin{array}{cc}
\frac{I}{2}+D & -S \\
N & \frac{I}{2}-D^{*}
\end{array}\right)\binom{\mu}{\nu}=\left(\frac{I}{2}+H\right)\binom{\mu}{\nu},  \tag{B.116}\\
& \binom{u_{i}}{\frac{\partial u_{i}}{\partial n}}=\left(\begin{array}{cc}
-\frac{I}{2}+D & -S \\
N & -\frac{I}{2}-D^{*}
\end{array}\right)\binom{\mu}{\nu}=\left(-\frac{I}{2}+H\right)\binom{\mu}{\nu}, \tag{B.117}
\end{align*}
$$

where

$$
H=\left(\begin{array}{cc}
D & -S  \tag{B.118}\\
N & -D^{*}
\end{array}\right)
$$

and where the vector $(\mu, \nu)^{T}$ is known as the Cauchy data on $\Gamma$. We define the exterior and interior Calderón projectors respectively by the operators

$$
\begin{equation*}
C_{e}=\frac{I}{2}+H \quad \text { and } \quad C_{i}=\frac{I}{2}-H, \tag{B.119}
\end{equation*}
$$

which satisfy

$$
\begin{equation*}
C_{e}^{2}=C_{e}, \quad C_{i}^{2}=C_{i}, \quad C_{e}+C_{i}=I . \tag{B.120}
\end{equation*}
$$

The identities (B.120) are equivalent to the set of relations

$$
\begin{equation*}
H^{2}=\frac{I}{4}, \tag{B.121}
\end{equation*}
$$

or more explicitly

$$
\begin{align*}
D S=S D^{*}, & D^{2}-S N=\frac{I}{4}  \tag{B.122}\\
N D=D^{*} N, & D^{* 2}-N S=\frac{I}{4} \tag{B.123}
\end{align*}
$$

Calderón projectors and relations synthesize in another way the structure of the integral equations, and are used more for theoretical purposes (e.g., matrix preconditioning).

## B.6.6 Alternatives for integral representations and equations

By taking into account the transmission problem (B.37), its integral representation formula (B.48), and its integral equations (B.60) and (B.62), several particular alternatives for integral representations and equations of the exterior problem (B.11) can be developed. The way to perform this is to extend properly the exterior problem towards the interior domain $\Omega_{i}$, either by specifying explicitly this extension or by defining an associated interior problem, so as to become the desired jump properties across $\Gamma$. The extension has to satisfy the Laplace equation (B.1) in $\Omega_{i}$ and a boundary condition that corresponds adequately to the impedance boundary condition (B.2). The obtained system of integral representations and equations allows finally to solve the exterior problem (B.11), by using the solution of the integral equation in the integral representation formula.

## a) Extension by zero

An extension by zero towards the interior domain $\Omega_{i}$ implies that

$$
\begin{equation*}
u_{i}=0 \quad \text { in } \Omega_{i} . \tag{B.124}
\end{equation*}
$$

The jumps over $\Gamma$ are characterized in this case by

$$
\begin{align*}
{[u] } & =u_{e}=\mu,  \tag{B.125}\\
{\left[\frac{\partial u}{\partial n}\right] } & =\frac{\partial u_{e}}{\partial n}=Z u_{e}-f_{z}=Z \mu-f_{z}, \tag{B.126}
\end{align*}
$$

where $\mu: \Gamma \rightarrow \mathbb{C}$ is a function to be determined.
An integral representation formula of the solution, for $\boldsymbol{x} \in \Omega_{e} \cup \Omega_{i}$, is given by

$$
\begin{equation*}
u(\boldsymbol{x})=\int_{\Gamma}\left(\frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y})-Z(\boldsymbol{y}) G(\boldsymbol{x}, \boldsymbol{y})\right) \mu(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y})+\int_{\Gamma} G(\boldsymbol{x}, \boldsymbol{y}) f_{z}(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}) \tag{B.127}
\end{equation*}
$$

Since

$$
\begin{equation*}
\frac{1}{2}\left(u_{e}(\boldsymbol{x})+u_{i}(\boldsymbol{x})\right)=\frac{\mu(\boldsymbol{x})}{2}, \quad \boldsymbol{x} \in \Gamma, \tag{B.128}
\end{equation*}
$$

we obtain, for $\boldsymbol{x} \in \Gamma$, the Fredholm integral equation of the second kind

$$
\begin{equation*}
\frac{\mu(\boldsymbol{x})}{2}+\int_{\Gamma}\left(Z(\boldsymbol{y}) G(\boldsymbol{x}, \boldsymbol{y})-\frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y})\right) \mu(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y})=\int_{\Gamma} G(\boldsymbol{x}, \boldsymbol{y}) f_{z}(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}) \tag{B.129}
\end{equation*}
$$

which has to be solved for the unknown $\mu$. In terms of boundary layer potentials, the integral representation and the integral equation can be respectively expressed by

$$
\begin{align*}
u=\mathcal{D}(\mu)-\mathcal{S}(Z \mu)+\mathcal{S}\left(f_{z}\right) & \text { in } \Omega_{e} \cup \Omega_{i},  \tag{B.130}\\
\frac{\mu}{2}+S(Z \mu)-D(\mu)=S\left(f_{z}\right) & \text { on } \Gamma . \tag{B.131}
\end{align*}
$$

Alternatively, since

$$
\begin{equation*}
\frac{1}{2}\left(\frac{\partial u_{e}}{\partial n}(\boldsymbol{x})+\frac{\partial u_{i}}{\partial n}(\boldsymbol{x})\right)=\frac{Z(\boldsymbol{x})}{2} \mu(\boldsymbol{x})-\frac{f_{z}(\boldsymbol{x})}{2}, \quad \boldsymbol{x} \in \Gamma \tag{B.132}
\end{equation*}
$$

we obtain also, for $\boldsymbol{x} \in \Gamma$, the Fredholm integral equation of the second kind

$$
\begin{array}{r}
\frac{Z(\boldsymbol{x})}{2} \mu(\boldsymbol{x})+\int_{\Gamma}\left(-\frac{\partial^{2} G}{\partial n_{\boldsymbol{x}} \partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y})+Z(\boldsymbol{y}) \frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x}, \boldsymbol{y})\right) \mu(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}) \\
=\frac{f_{z}(\boldsymbol{x})}{2}+\int_{\Gamma} \frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x}, \boldsymbol{y}) f_{z}(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}), \tag{B.133}
\end{array}
$$

which in terms of boundary layer potentials becomes

$$
\begin{equation*}
\frac{Z}{2} \mu-N(\mu)+D^{*}(Z \mu)=\frac{f_{z}}{2}+D^{*}\left(f_{z}\right) \quad \text { on } \Gamma . \tag{B.134}
\end{equation*}
$$

b) Continuous impedance

We associate to (B.11) the interior problem

$$
\begin{cases}\text { Find } u_{i}: \Omega_{i} \rightarrow \mathbb{C} \text { such that }  \tag{B.135}\\ \Delta u_{i}=0 & \text { in } \Omega_{i}, \\ -\frac{\partial u_{i}}{\partial n}+Z u_{i}=f_{z} & \text { on } \Gamma .\end{cases}
$$

The jumps over $\Gamma$ are characterized in this case by

$$
\begin{align*}
{[u] } & =u_{e}-u_{i}=\mu,  \tag{B.136}\\
{\left[\frac{\partial u}{\partial n}\right] } & =\frac{\partial u_{e}}{\partial n}-\frac{\partial u_{i}}{\partial n}=Z\left(u_{e}-u_{i}\right)=Z \mu \tag{B.137}
\end{align*}
$$

where $\mu: \Gamma \rightarrow \mathbb{C}$ is a function to be determined. In particular it holds that the jump of the impedance is zero, namely

$$
\begin{equation*}
\left[-\frac{\partial u}{\partial n}+Z u\right]=\left(-\frac{\partial u_{e}}{\partial n}+Z u_{e}\right)-\left(-\frac{\partial u_{i}}{\partial n}+Z u_{i}\right)=0 \tag{B.138}
\end{equation*}
$$

An integral representation formula of the solution, for $\boldsymbol{x} \in \Omega_{e} \cup \Omega_{i}$, is given by

$$
\begin{equation*}
u(\boldsymbol{x})=\int_{\Gamma}\left(\frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y})-Z(\boldsymbol{y}) G(\boldsymbol{x}, \boldsymbol{y})\right) \mu(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}) \tag{B.139}
\end{equation*}
$$

Since

$$
\begin{equation*}
-\frac{1}{2}\left(\frac{\partial u_{e}}{\partial n}(\boldsymbol{x})+\frac{\partial u_{i}}{\partial n}(\boldsymbol{x})\right)+\frac{Z(\boldsymbol{x})}{2}\left(u_{e}(\boldsymbol{x})+u_{i}(\boldsymbol{x})\right)=f_{z}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma \tag{B.140}
\end{equation*}
$$

we obtain, for $\boldsymbol{x} \in \Gamma$, the Fredholm integral equation of the first kind

$$
\begin{align*}
& \int_{\Gamma}\left(-\frac{\partial^{2} G}{\partial n_{\boldsymbol{x}} \partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y})+Z(\boldsymbol{y}) \frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x}, \boldsymbol{y})\right) \mu(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}) \\
&+Z(\boldsymbol{x}) \int_{\Gamma}\left(\frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y})-Z(\boldsymbol{y}) G(\boldsymbol{x}, \boldsymbol{y})\right) \mu(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y})=f_{z}(\boldsymbol{x}) \tag{B.141}
\end{align*}
$$

which has to be solved for the unknown $\mu$. In terms of boundary layer potentials, the integral representation and the integral equation can be respectively expressed by

$$
\begin{array}{cl}
u=\mathcal{D}(\mu)-\mathcal{S}(Z \mu) & \text { in } \Omega_{e} \cup \Omega_{i}, \\
-N(\mu)+D^{*}(Z \mu)+Z D(\mu)-Z S(Z \mu)=f_{z} & \text { on } \Gamma . \tag{B.143}
\end{array}
$$

c) Continuous value

We associate to (B.11) the interior problem

$$
\begin{cases}\text { Find } u_{i}: \Omega_{i} \rightarrow \mathbb{C} \text { such that }  \tag{B.144}\\ \Delta u_{i}=0 & \text { in } \Omega_{i} \\ -\frac{\partial u_{e}}{\partial n}+Z u_{i}=f_{z} & \text { on } \Gamma .\end{cases}
$$

The jumps over $\Gamma$ are characterized in this case by

$$
\begin{align*}
{[u] } & =u_{e}-u_{i}=\frac{1}{Z}\left(\frac{\partial u_{e}}{\partial n}-f_{z}\right)-\frac{1}{Z}\left(\frac{\partial u_{e}}{\partial n}-f_{z}\right)=0,  \tag{B.145}\\
{\left[\frac{\partial u}{\partial n}\right] } & =\frac{\partial u_{e}}{\partial n}-\frac{\partial u_{i}}{\partial n}=\nu \tag{B.146}
\end{align*}
$$

where $\nu: \Gamma \rightarrow \mathbb{C}$ is a function to be determined.
An integral representation formula of the solution, for $\boldsymbol{x} \in \Omega_{e} \cup \Omega_{i}$, is given by the single layer potential

$$
\begin{equation*}
u(\boldsymbol{x})=-\int_{\Gamma} G(\boldsymbol{x}, \boldsymbol{y}) \nu(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}) \tag{B.147}
\end{equation*}
$$

Since

$$
\begin{equation*}
-\frac{1}{2}\left(\frac{\partial u_{e}}{\partial n}(\boldsymbol{x})+\frac{\partial u_{i}}{\partial n}(\boldsymbol{x})\right)+\frac{Z(\boldsymbol{x})}{2}\left(u_{e}(\boldsymbol{x})+u_{i}(\boldsymbol{x})\right)=\frac{\nu(\boldsymbol{x})}{2}+f_{z}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma \tag{B.148}
\end{equation*}
$$

we obtain, for $\boldsymbol{x} \in \Gamma$, the Fredholm integral equation of the second kind

$$
\begin{equation*}
\frac{\nu(\boldsymbol{x})}{2}+\int_{\Gamma}\left(Z(\boldsymbol{x}) G(\boldsymbol{x}, \boldsymbol{y})-\frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x}, \boldsymbol{y})\right) \nu(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y})=-f_{z}(\boldsymbol{x}) \tag{B.149}
\end{equation*}
$$

which has to be solved for the unknown $\nu$. In terms of boundary layer potentials, the integral representation and the integral equation can be respectively expressed by

$$
\begin{array}{cl}
u=-\mathcal{S}(\nu) & \text { in } \Omega_{e} \cup \Omega_{i}, \\
\frac{\nu}{2}+Z S(\nu)-D^{*}(\nu)=-f_{z} & \text { on } \Gamma . \tag{B.151}
\end{array}
$$

d) Continuous normal derivative

We associate to (B.11) the interior problem

$$
\begin{cases}\text { Find } u_{i}: \Omega_{i} \rightarrow \mathbb{C} \text { such that }  \tag{B.152}\\ \Delta u_{i}=0 & \text { in } \Omega_{i} \\ -\frac{\partial u_{i}}{\partial n}+Z u_{e}=f_{z} & \text { on } \Gamma .\end{cases}
$$

The jumps over $\Gamma$ are characterized in this case by

$$
\begin{align*}
{[u] } & =u_{e}-u_{i}=\mu  \tag{B.153}\\
{\left[\frac{\partial u}{\partial n}\right] } & =\frac{\partial u_{e}}{\partial n}-\frac{\partial u_{i}}{\partial n}=\left(Z u_{e}-f_{z}\right)-\left(Z u_{e}-f_{z}\right)=0 \tag{B.154}
\end{align*}
$$

where $\mu: \Gamma \rightarrow \mathbb{C}$ is a function to be determined.
An integral representation formula of the solution, for $\boldsymbol{x} \in \Omega_{e} \cup \Omega_{i}$, is given by the double layer potential

$$
\begin{equation*}
u(\boldsymbol{x})=\int_{\Gamma} \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) \mu(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}) \tag{B.155}
\end{equation*}
$$

Since when $\boldsymbol{x} \in \Gamma$,

$$
\begin{equation*}
-\frac{1}{2}\left(\frac{\partial u_{e}}{\partial n}(\boldsymbol{x})+\frac{\partial u_{i}}{\partial n}(\boldsymbol{x})\right)+\frac{Z(\boldsymbol{x})}{2}\left(u_{e}(\boldsymbol{x})+u_{i}(\boldsymbol{x})\right)=-\frac{Z(\boldsymbol{x})}{2} \mu(\boldsymbol{x})+f_{z}(\boldsymbol{x}), \tag{B.156}
\end{equation*}
$$

we obtain, for $\boldsymbol{x} \in \Gamma$, the Fredholm integral equation of the second kind

$$
\begin{equation*}
\frac{Z(\boldsymbol{x})}{2} \mu(\boldsymbol{x})+\int_{\Gamma}\left(-\frac{\partial^{2} G}{\partial n_{\boldsymbol{x}} \partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y})+Z(\boldsymbol{x}) \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y})\right) \mu(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y})=f_{z}(\boldsymbol{x}) \tag{B.157}
\end{equation*}
$$

which has to be solved for the unknown $\mu$. In terms of boundary layer potentials, the integral representation and the integral equation can be respectively expressed by

$$
\begin{array}{cl}
u=\mathcal{D}(\mu) & \text { in } \Omega_{e} \cup \Omega_{i}, \\
\frac{Z}{2} \mu-N(\mu)+Z D(\mu)=f_{z} & \text { on } \Gamma . \tag{B.159}
\end{array}
$$

## B.6.7 Adjoint integral equations

Due Fredholm's alternative, there is a close relation between the solution of an integral equation and the one of its adjoint counterpart. The so-called adjoint integral equation is obtained by taking the adjoint of the integral operators that appear in the integral equation, disregarding the source terms at the right-hand side. For a function $\varphi: \Gamma \subset \mathbb{R}^{N} \rightarrow \mathbb{C}$, the linear adjoint of an integral operator of the form

$$
\begin{equation*}
T \varphi(\boldsymbol{x})=\int_{\Gamma} K(\boldsymbol{x}, \boldsymbol{y}) \varphi(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}), \quad \boldsymbol{x} \in \Gamma \tag{B.160}
\end{equation*}
$$

is given by the integral operator

$$
\begin{equation*}
T^{*} \varphi(\boldsymbol{x})=\int_{\Gamma} K(\boldsymbol{y}, \boldsymbol{x}) \varphi(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}), \quad \boldsymbol{x} \in \Gamma \tag{B.161}
\end{equation*}
$$

It is not difficult to see that the boundary layer potentials $S$ and $N$ are self-adjoint due their symmetric kernels, and that $D$ and $D^{*}$ are mutually adjoint, i.e.,

$$
\begin{equation*}
S^{*}=S, \quad N^{*}=N, \quad \text { and } \quad D^{*}=D . \tag{B.162}
\end{equation*}
$$

When we include also the impedance, then it holds that

$$
\begin{equation*}
(S(Z \varphi))^{*}=Z S(\varphi), \quad\left(D^{*}(Z \varphi)\right)^{*}=Z D(\varphi), \quad(Z S(Z \varphi))^{*}=Z S(Z \varphi) \tag{B.163}
\end{equation*}
$$

It can be seen now that the integral equations (B.131) of the first extension by zero and (B.151) of the continuous value are mutually adjoint. The same holds for the integral equations (B.134) of the second extension by zero and (B.159) of the continuous normal derivative, which are also mutually adjoint. The integral equation (B.143) of the continuous impedance, on the other hand, is self-adjoint.

## B. 7 Far field of the solution

The asymptotic behavior at infinity of the solution $u$ of (B.11) is described by the far field. It is denoted by $u^{f f}$ and is characterized by

$$
\begin{equation*}
u(\boldsymbol{x}) \sim u^{f f}(\boldsymbol{x}) \quad \text { as } \quad|\boldsymbol{x}| \rightarrow \infty . \tag{B.164}
\end{equation*}
$$

Its expression can be deduced by replacing the far field of the Green's function $G^{f f}$ and its derivatives in the integral representation formula (B.48), which yields

$$
\begin{equation*}
u^{f f}(\boldsymbol{x})=\int_{\Gamma}\left([u](\boldsymbol{y}) \frac{\partial G^{f f}}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y})-G^{f f}(\boldsymbol{x}, \boldsymbol{y})\left[\frac{\partial u}{\partial n}\right](\boldsymbol{y})\right) \mathrm{d} \gamma(\boldsymbol{y}) . \tag{B.165}
\end{equation*}
$$

By replacing now (B.32) and (B.33) in (B.165), we obtain that

$$
\begin{align*}
u^{f f}(\boldsymbol{x})= & -\frac{1}{2 \pi|\boldsymbol{x}|} \int_{\Gamma}\left(\hat{\boldsymbol{x}} \cdot \boldsymbol{n}_{\boldsymbol{y}}[u](\boldsymbol{y})-\hat{\boldsymbol{x}} \cdot \boldsymbol{y}\left[\frac{\partial u}{\partial n}\right](\boldsymbol{y})\right) \mathrm{d} \gamma(\boldsymbol{y}) \\
& -\frac{1}{2 \pi} \ln |\boldsymbol{x}| \int_{\Gamma}\left[\frac{\partial u}{\partial n}\right](\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}) . \tag{B.166}
\end{align*}
$$

Due (B.57) the second integral in (B.166) is zero. Thus the far field of the solution $u$ is

$$
\begin{equation*}
u^{f f}(\boldsymbol{x})=-\frac{1}{2 \pi|\boldsymbol{x}|} \int_{\Gamma}\left(\hat{\boldsymbol{x}} \cdot \boldsymbol{n}_{\boldsymbol{y}}[u](\boldsymbol{y})-\hat{\boldsymbol{x}} \cdot \boldsymbol{y}\left[\frac{\partial u}{\partial n}\right](\boldsymbol{y})\right) \mathrm{d} \gamma(\boldsymbol{y}) . \tag{B.167}
\end{equation*}
$$

The asymptotic behavior of the solution $u$ at infinity is therefore given by

$$
\begin{equation*}
u(\boldsymbol{x})=\frac{1}{|\boldsymbol{x}|}\left\{u_{\infty}(\hat{\boldsymbol{x}})+\mathcal{O}\left(\frac{1}{|\boldsymbol{x}|}\right)\right\}, \quad|\boldsymbol{x}| \rightarrow \infty \tag{B.168}
\end{equation*}
$$

uniformly in all directions $\hat{\boldsymbol{x}}$ on the unit circle, where

$$
\begin{equation*}
u_{\infty}(\hat{\boldsymbol{x}})=-\frac{1}{2 \pi} \int_{\Gamma}\left(\hat{\boldsymbol{x}} \cdot \boldsymbol{n}_{\boldsymbol{y}}[u](\boldsymbol{y})-\hat{\boldsymbol{x}} \cdot \boldsymbol{y}\left[\frac{\partial u}{\partial n}\right](\boldsymbol{y})\right) \mathrm{d} \gamma(\boldsymbol{y}) \tag{B.169}
\end{equation*}
$$

is called the far-field pattern of $u$. It can be expressed in decibels (dB) by means of the asymptotic cross section

$$
\begin{equation*}
Q_{s}(\hat{\boldsymbol{x}})[\mathrm{dB}]=20 \log _{10}\left(\frac{\left|u_{\infty}(\hat{\boldsymbol{x}})\right|}{\left|u_{0}\right|}\right), \tag{B.170}
\end{equation*}
$$

where the reference level $u_{0}$ may typically depend on $u_{W}$, but for simplicity we take $u_{0}=1$.
We remark that the far-field behavior (B.168) of the solution is in accordance with the decaying condition (B.7), which justifies its choice.

## B. 8 Exterior circle problem

To understand better the resolution of the direct perturbation problem (B.11), we study now the particular case when the domain $\Omega_{e} \subset \mathbb{R}^{2}$ is taken as the exterior of a circle of radius $R>0$. The interior of the circle is then given by $\Omega_{i}=\left\{\boldsymbol{x} \in \mathbb{R}^{2}:|\boldsymbol{x}|<R\right\}$ and its boundary by $\Gamma=\partial \Omega_{e}$, as shown in Figure B.8. We place the origin at the center of $\Omega_{i}$ and we consider that the unit normal $\boldsymbol{n}$ is taken outwardly oriented of $\Omega_{e}$, i.e., $\boldsymbol{n}=-\boldsymbol{r}$.


Figure B.8. Exterior of the circle.
The exterior circle problem is then stated as

$$
\left\{\begin{array}{l}
\text { Find } u: \Omega_{e} \rightarrow \mathbb{C} \text { such that }  \tag{B.171}\\
\Delta u=0 \quad \text { in } \Omega_{e} \\
\frac{\partial u}{\partial r}+Z u=f_{z} \quad \text { on } \Gamma \\
+ \text { Decaying condition as }|\boldsymbol{x}| \rightarrow \infty
\end{array}\right.
$$

where we consider a constant impedance $Z \in \mathbb{C}$ and where the asymptotic decaying condition is as usual given by (B.7).

Due the particular chosen geometry, the solution $u$ of (B.171) can be easily found analytically by using the method of variable separation, i.e., by supposing that

$$
\begin{equation*}
u(\boldsymbol{x})=u(r, \theta)=h(r) g(\theta) \tag{B.172}
\end{equation*}
$$

where $r \geq 0$ and $-\pi<\theta \leq \pi$ are the polar coordinates in $\mathbb{R}^{2}$, characterized by

$$
\begin{equation*}
r=\sqrt{x_{1}^{2}+x_{2}^{2}} \quad \text { and } \quad \theta=\arctan \left(\frac{x_{2}}{x_{1}}\right) \tag{B.173}
\end{equation*}
$$

If the Laplace equation in (B.171) is expressed using polar coordinates, then

$$
\begin{equation*}
\Delta u=\frac{\partial^{2} u}{\partial r^{2}}+\frac{1}{r} \frac{\partial u}{\partial r}+\frac{1}{r^{2}} \frac{\partial^{2} u}{\partial \theta^{2}}=0 \tag{B.174}
\end{equation*}
$$

By replacing now (B.172) in (B.174) we obtain

$$
\begin{equation*}
h^{\prime \prime}(r) g(\theta)+\frac{1}{r} h^{\prime}(r) g(\theta)+\frac{1}{r^{2}} h(r) g^{\prime \prime}(\theta)=0 . \tag{B.175}
\end{equation*}
$$

Multiplying by $r^{2}$, dividing by $g h$, and rearranging according to each variable yields

$$
\begin{equation*}
r^{2} \frac{h^{\prime \prime}(r)}{h(r)}+r \frac{h^{\prime}(r)}{h(r)}=-\frac{g^{\prime \prime}(\theta)}{g(\theta)} \tag{B.176}
\end{equation*}
$$

Since both sides in equation (B.176) involve different variables, therefore they are equal to a constant, denoted for convenience by $n^{2}$, and we have that

$$
\begin{equation*}
r^{2} \frac{h^{\prime \prime}(r)}{h(r)}+r \frac{h^{\prime}(r)}{h(r)}=-\frac{g^{\prime \prime}(\theta)}{g(\theta)}=n^{2} \tag{B.177}
\end{equation*}
$$

From (B.177) we obtain the two ordinary differential equations

$$
\begin{gather*}
g^{\prime \prime}(\theta)+n^{2} g(\theta)=0  \tag{B.178}\\
r^{2} h^{\prime \prime}(r)+r h^{\prime}(r)-n^{2} h(r)=0 \tag{B.179}
\end{gather*}
$$

The solutions for (B.178) have the general form

$$
\begin{equation*}
g(\theta)=a_{n} \cos (n \theta)+b_{n} \sin (n \theta), \quad n \in \mathbb{N}_{0}, \tag{B.180}
\end{equation*}
$$

where $a_{n}, b_{n} \in \mathbb{C}$ are arbitrary constants. The requirement that $n \in \mathbb{N}_{0}$ stems from the periodicity condition

$$
\begin{equation*}
g(\theta)=g(\theta+2 \pi n) \quad \forall n \in \mathbb{Z}, \tag{B.181}
\end{equation*}
$$

where we segregate positive and negative values for $n$. The solutions for (B.179), on the other hand, have the general form

$$
\begin{equation*}
h(r)=c_{n} r^{-n}+d_{n} r^{n}, \quad n>0, \tag{B.182}
\end{equation*}
$$

and for the particular case $n=0$, as already done in (B.14), it holds that

$$
\begin{equation*}
h(r)=c_{0}+d_{0} \ln r, \tag{B.183}
\end{equation*}
$$

where $c_{n}, d_{n} \in \mathbb{C}$ are again arbitrary constants. The general solution for the Laplace equation considers the linear combination of all the solutions in the form of (B.172), namely

$$
\begin{equation*}
u(r, \theta)=a_{0}\left(c_{0}+d_{0} \ln r\right)+\sum_{n=1}^{\infty}\left(c_{n} r^{-n}+d_{n} r^{n}\right)\left(a_{n} \cos (n \theta)+b_{n} \sin (n \theta)\right) \tag{B.184}
\end{equation*}
$$

The decaying condition (B.7) implies that

$$
\begin{equation*}
c_{0}=d_{0}=d_{n}=0, \quad n \in \mathbb{N} . \tag{B.185}
\end{equation*}
$$

Thus the general solution (B.184) turns into

$$
\begin{equation*}
u(r, \theta)=\sum_{n=1}^{\infty} r^{-n}\left(a_{n} e^{i n \theta}+b_{n} e^{-i n \theta}\right) \tag{B.186}
\end{equation*}
$$

where all the undetermined constants have been merged into $a_{n}$ and $b_{n}$, due their arbitrariness. The radial derivative of (B.186) is given by

$$
\begin{equation*}
\frac{\partial u}{\partial r}(r, \theta)=-\sum_{n=1}^{\infty} n r^{-(n+1)}\left(a_{n} e^{i n \theta}+b_{n} e^{-i n \theta}\right) \tag{B.187}
\end{equation*}
$$

The constants $a_{n}$ and $b_{n}$ in (B.186) are determined through the impedance boundary condition on $\Gamma$. For this purpose, we expand the impedance data function $f_{z}$ as a Fourier series:

$$
\begin{equation*}
f_{z}(\theta)=\sum_{n=-\infty}^{\infty} f_{n} e^{i n \theta}, \quad-\pi<\theta \leq \pi \tag{B.188}
\end{equation*}
$$

where

$$
\begin{equation*}
f_{n}=\frac{1}{2 \pi} \int_{-\pi}^{\pi} f_{z}(\theta) e^{-i n \theta} \mathrm{~d} \theta, \quad n \in \mathbb{Z} \tag{B.189}
\end{equation*}
$$

The impedance boundary condition considers $r=R$ and thus takes the form

$$
\begin{equation*}
\sum_{n=1}^{\infty}\left(\frac{Z R-n}{R^{n+1}}\right)\left(a_{n} e^{i n \theta}+b_{n} e^{-i n \theta}\right)=f_{z}(\theta)=\sum_{n=-\infty}^{\infty} f_{n} e^{i n \theta} \tag{B.190}
\end{equation*}
$$

We observe that the constants $a_{n}$ and $b_{n}$ can be uniquely determined only if $f_{0}=0$ and if $Z R \neq n$, for $n \in \mathbb{N}$ and $n \geq 1$. The first condition, which is usually referred to as a compatibility condition, is necessary to ensure the existence of the solution $u$, and can be restated as

$$
\begin{equation*}
\int_{\Gamma} f_{z} \mathrm{~d} \gamma=0 \tag{B.191}
\end{equation*}
$$

The second condition is more related with the loss of the solution's uniqueness. Therefore, if we suppose, for $n \in \mathbb{N}$ and $n \geq 1$, that $Z R \neq n$ and (B.191) hold, then

$$
\begin{equation*}
a_{n}=\frac{R^{n+1} f_{n}}{Z R-n} \quad \text { and } \quad b_{n}=\frac{R^{n+1} f_{-n}}{Z R-n} . \tag{B.192}
\end{equation*}
$$

The unique solution for the exterior circle problem (B.171) is then given by

$$
\begin{equation*}
u(r, \theta)=\sum_{n=1}^{\infty}\left(\frac{R^{n+1}}{Z R-n}\right) r^{-n}\left(f_{n} e^{i n \theta}+f_{-n} e^{-i n \theta}\right) \tag{B.193}
\end{equation*}
$$

If we consider now the case when $Z R=m$, for some particular integer $m \geq 1$, then the solution $u$ is not unique. The constants $a_{m}$ and $b_{m}$ are then no longer defined by (B.192), and can be chosen in an arbitrary manner. For the existence of a solution in this case, however, we require, together with the compatibility condition (B.191), also the orthogonality conditions $f_{m}=f_{-m}=0$, which are equivalent to

$$
\begin{equation*}
\int_{-\pi}^{\pi} f_{z}(\theta) e^{i m \theta} \mathrm{~d} \theta=\int_{-\pi}^{\pi} f_{z}(\theta) e^{-i m \theta} \mathrm{~d} \theta=0 \tag{B.194}
\end{equation*}
$$

Instead of (B.193), the solution of (B.171) is now given by the infinite family of functions

$$
\begin{equation*}
u(r, \theta)=\sum_{1 \leq n \neq m}\left(\frac{R^{n+1}}{Z R-n}\right) r^{-n}\left(f_{n} e^{i n \theta}+f_{-n} e^{-i n \theta}\right)+\alpha \frac{e^{i m \theta}}{r^{m}}+\beta \frac{e^{-i m \theta}}{r^{m}} \tag{B.195}
\end{equation*}
$$

where $\alpha, \beta \in \mathbb{C}$ are arbitrary and where their associated terms have the form of surface waves, i.e., waves that propagate along $\Gamma$ and decrease towards the interior of $\Omega_{e}$. Thus, if the compatibility condition (B.191) is satisfied, then the exterior circle problem (B.171) admits a unique solution $u$, except on a countable set of values for $Z R$. And even in this last case there exists a solution, although not unique, if two orthogonality conditions are additionally satisfied. This behavior for the existence and uniqueness of the solution is typical of the Fredholm alternative, which applies when solving problems that involve compact perturbations of invertible operators.

We remark that when a non-constant impedance $Z(\theta)$ is taken, then the compatibility condition (B.191) is no longer required for the existence of the solution $u$, a fact that can be inferred from (B.190) by considering the Fourier series terms of the impedance. An analytic formula for the solution is more difficult to obtain in this case, but it holds again that this solution will exist and be unique, except possibly for some at most countable set
of values where the uniqueness is lost and where additional orthogonality conditions have to be satisfied, which depend on $Z(\theta)$.

## B. 9 Existence and uniqueness

## B.9.1 Function spaces

To state a precise mathematical formulation of the herein treated problems, we have to define properly the involved function spaces. For the associated interior problems defined on the bounded set $\Omega_{i}$ we use the classical Sobolev space (vid. Section A.4)

$$
\begin{equation*}
H^{1}\left(\Omega_{i}\right)=\left\{v: v \in L^{2}\left(\Omega_{i}\right), \nabla v \in L^{2}\left(\Omega_{i}\right)^{2}\right\}, \tag{B.196}
\end{equation*}
$$

which is a Hilbert space and has the norm

$$
\begin{equation*}
\|v\|_{H^{1}\left(\Omega_{i}\right)}=\left(\|v\|_{L^{2}\left(\Omega_{i}\right)}^{2}+\|\nabla v\|_{L^{2}\left(\Omega_{i}\right)^{2}}^{2}\right)^{1 / 2} . \tag{B.197}
\end{equation*}
$$

For the exterior problem defined on the unbounded domain $\Omega_{e}$, on the other hand, we introduce the weighted Sobolev space (cf., e.g., Raviart 1991)

$$
\begin{equation*}
W^{1}\left(\Omega_{e}\right)=\left\{v: \frac{v}{\sqrt{1+r^{2}} \ln \left(2+r^{2}\right)} \in L^{2}\left(\Omega_{e}\right), \frac{\partial v}{\partial x_{i}} \in L^{2}\left(\Omega_{e}\right) \forall i \in\{1,2\}\right\} \tag{B.198}
\end{equation*}
$$

where $r=|\boldsymbol{x}|$. If $W^{1}\left(\Omega_{e}\right)$ is provided with the norm

$$
\begin{equation*}
\|v\|_{W^{1}\left(\Omega_{e}\right)}=\left(\left\|\frac{v}{\sqrt{1+r^{2}} \ln \left(2+r^{2}\right)}\right\|_{L^{2}\left(\Omega_{e}\right)}^{2}+\|\nabla v\|_{L^{2}\left(\Omega_{e}\right)^{2}}^{2}\right)^{1 / 2}, \tag{B.199}
\end{equation*}
$$

then it becomes a Hilbert space. The restriction to any bounded open set $B \subset \Omega_{e}$ of the functions of $W^{1}\left(\Omega_{e}\right)$ belongs to $H^{1}(B)$, i.e., we have the inclusion $W^{1}\left(\Omega_{e}\right) \subset H_{\mathrm{loc}}^{1}\left(\Omega_{e}\right)$, and the functions in these two spaces differ only by their behavior at infinity. We remark that the space $W^{1}\left(\Omega_{e}\right)$ contains the constant functions and all the functions of $H_{\mathrm{loc}}^{1}\left(\Omega_{e}\right)$ that satisfy the decaying condition (B.7). The justification for the use of these function spaces lies in the variational formulation of the differential problem, and they remain valid even when considering a source term with the same decaying behavior in the right-hand side of the Laplace equation, i.e., when working with the Poisson equation.

When dealing with Sobolev spaces, even a strong Lipschitz boundary $\Gamma \in C^{0,1}$ is admissible. In this case, and due the trace theorem (A.531), if $v \in H^{1}\left(\Omega_{i}\right)$ or $v \in W^{1}\left(\Omega_{e}\right)$, then the trace of $v$ fulfills

$$
\begin{equation*}
\gamma_{0} v=\left.v\right|_{\Gamma} \in H^{1 / 2}(\Gamma) \tag{B.200}
\end{equation*}
$$

Moreover, the trace of the normal derivative can be also defined, and it holds that

$$
\begin{equation*}
\gamma_{1} v=\left.\frac{\partial v}{\partial n}\right|_{\Gamma} \in H^{-1 / 2}(\Gamma) \tag{B.201}
\end{equation*}
$$

since $\Delta v=0 \in L^{2}\left(\Omega_{i} \cup \Omega_{e}\right)$. This way we do not need to work with the more cumbersome spaces $H^{1}\left(\Delta ; \Omega_{i}\right)$ and $W^{1}\left(\Delta ; \Omega_{e}\right)$, being the former defined in (A.535) and the latter in an analogous manner, but for (B.198).

## B.9.2 Regularity of the integral operators

The boundary integral operators (B.81), (B.82), (B.83), and (B.84) can be characterized as linear and continuous applications such that

$$
\begin{align*}
S: H^{-1 / 2+s}(\Gamma) \longrightarrow H^{1 / 2+s}(\Gamma), & D: H^{1 / 2+s}(\Gamma) \longrightarrow H^{3 / 2+s}(\Gamma)  \tag{B.202}\\
D^{*}: H^{-1 / 2+s}(\Gamma) \longrightarrow H^{1 / 2+s}(\Gamma), & N: H^{1 / 2+s}(\Gamma) \longrightarrow H^{-1 / 2+s}(\Gamma) . \tag{B.203}
\end{align*}
$$

This result holds for any $s \in \mathbb{R}$ if the boundary $\Gamma$ is of class $C^{\infty}$, which can be derived from the theory of singular integral operators with pseudo-homogeneous kernels (cf., e.g., Nédélec 2001). Due the compact injection (A.554), it holds also that the operators

$$
\begin{equation*}
D: H^{1 / 2+s}(\Gamma) \longrightarrow H^{1 / 2+s}(\Gamma) \quad \text { and } \quad D^{*}: H^{-1 / 2+s}(\Gamma) \longrightarrow H^{-1 / 2+s}(\Gamma) \tag{B.204}
\end{equation*}
$$

are compact. For a strong Lipschitz boundary $\Gamma \in C^{0,1}$, on the other hand, these results hold only when $|s|<1$ (cf. Costabel 1988). In the case of more regular boundaries, the range for $s$ increases, but remains finite. For our purposes we use $s=0$, namely

$$
\begin{align*}
S: H^{-1 / 2}(\Gamma) \longrightarrow H^{1 / 2}(\Gamma), & D: H^{1 / 2}(\Gamma) \longrightarrow H^{1 / 2}(\Gamma)  \tag{B.205}\\
D^{*}: H^{-1 / 2}(\Gamma) \longrightarrow H^{-1 / 2}(\Gamma), & N: H^{1 / 2}(\Gamma) \longrightarrow H^{-1 / 2}(\Gamma), \tag{B.206}
\end{align*}
$$

which are all linear and continuous operators, and where the operators $D$ and $D^{*}$ are compact. Similarly, we can characterize the single and double layer potentials defined respectively in (B.76) and (B.77) as linear and continuous integral operators such that

$$
\begin{equation*}
\mathcal{S}: H^{-1 / 2}(\Gamma) \longrightarrow W^{1}\left(\Omega_{e} \cup \Omega_{i}\right) \quad \text { and } \quad \mathcal{D}: H^{1 / 2}(\Gamma) \longrightarrow W^{1}\left(\Omega_{e} \cup \Omega_{i}\right) \tag{B.207}
\end{equation*}
$$

## B.9.3 Application to the integral equations

It is not difficult to see that if $\mu \in H^{1 / 2}(\Gamma)$ and $\nu \in H^{-1 / 2}(\Gamma)$ are given, then the transmission problem (B.37) admits a unique solution $u \in W^{1}\left(\Omega_{e} \cup \Omega_{i}\right)$, as a consequence of the integral representation formula (B.49). For the direct perturbation problem (B.11), though, this is not always the case, as was appreciated in the exterior circle problem (B.171). Nonetheless, if the Fredholm alternative applies, then we know that the existence and uniqueness of the problem can be ensured almost always, i.e., except on a countable set of values for the impedance.

We consider an impedance $Z \in L^{\infty}(\Gamma)$ and an impedance data function $f_{z} \in H^{-1 / 2}(\Gamma)$. In both cases all the continuous functions on $\Gamma$ are included. We remark that the product of a function $f \in L^{\infty}(\Gamma)$ by a function $g \in H^{1 / 2}(\Gamma)$ most likely does not appertain to $H^{1 / 2}(\Gamma)$, but is rather such that $f g \in H^{1 / 2-\epsilon}(\Gamma)$ for some $\epsilon>0$. What we can state for sure in this case is that $f g \in L^{2}(\Gamma)$, since $H^{1 / 2}(\Gamma) \subset L^{2}(\Gamma)$ and the product of a function in $L^{\infty}(\Gamma)$ by a function in $L^{2}(\Gamma)$ is in $L^{2}(\Gamma)$, as stated in (A.471). It holds similarly that if $f \in L^{\infty}(\Gamma)$ and $g \in H^{1}(\Gamma)$, then $f g \in H^{1}(\Gamma)$.

## a) First extension by zero

Let us study the first integral equation of the extension-by-zero alternative (B.129), which is given in terms of boundary layer potentials, for $\mu \in H^{1 / 2}(\Gamma)$, by

$$
\begin{equation*}
\frac{\mu}{2}+S(Z \mu)-D(\mu)=S\left(f_{z}\right) \quad \text { in } H^{1 / 2}(\Gamma) \tag{B.208}
\end{equation*}
$$

The following mapping properties hold:

$$
\begin{align*}
\mu \in H^{1 / 2}(\Gamma) & \longmapsto \frac{\mu}{2} \in H^{1 / 2}(\Gamma),  \tag{B.209}\\
Z \mu \in L^{2}(\Gamma) & \longmapsto S(Z \mu) \in H^{1}(\Gamma) \stackrel{c}{\hookrightarrow} H^{1 / 2}(\Gamma),  \tag{B.210}\\
\mu \in H^{1 / 2}(\Gamma) & \longmapsto D(\mu) \in H^{3 / 2}(\Gamma) \hookrightarrow H^{1 / 2}(\Gamma),  \tag{B.211}\\
f_{z} \in H^{-1 / 2}(\Gamma) & \longmapsto S\left(f_{z}\right) \in H^{1 / 2}(\Gamma) . \tag{B.212}
\end{align*}
$$

We observe that (B.209) is the identity operator (disregarding the multiplicative constant), and that (B.210) and (B.211) are compact, due the imbeddings of Sobolev spaces. Thus the integral equation (B.208) has the form of (A.441) and the Fredholm alternative holds.
b) Second extension by zero

The second integral equation of the extension-by-zero alternative (B.133) is given in terms of boundary layer potentials, for $\mu \in H^{1 / 2}(\Gamma)$, by

$$
\begin{equation*}
\frac{Z}{2} \mu-N(\mu)+D^{*}(Z \mu)=\frac{f_{z}}{2}+D^{*}\left(f_{z}\right) \quad \text { in } H^{-1 / 2}(\Gamma) \tag{B.213}
\end{equation*}
$$

In this case we have the mapping properties:

$$
\begin{align*}
\mu \in H^{1 / 2}(\Gamma) & \longmapsto \frac{Z}{2} \mu \in L^{2}(\Gamma) \stackrel{c}{\hookrightarrow} H^{-1 / 2}(\Gamma),  \tag{B.214}\\
\mu \in H^{1 / 2}(\Gamma) & \longmapsto N(\mu) \in H^{-1 / 2}(\Gamma),  \tag{B.215}\\
Z \mu \in L^{2}(\Gamma) & \longmapsto D^{*}(Z \mu) \in H^{1}(\Gamma) \stackrel{c}{\hookrightarrow} H^{-1 / 2}(\Gamma),  \tag{B.216}\\
f_{z} \in H^{-1 / 2}(\Gamma) & \longmapsto \frac{f_{z}}{2} \in H^{-1 / 2}(\Gamma),  \tag{B.217}\\
f_{z} \in H^{-1 / 2}(\Gamma) & \longmapsto D^{*}\left(f_{z}\right) \in H^{1 / 2}(\Gamma) \hookrightarrow H^{-1 / 2}(\Gamma) . \tag{B.218}
\end{align*}
$$

We see that the operators (B.214) and (B.216) are compact, whereas (B.215) represents the term of leading order and plays the role of the identity. In fact, by applying the operator $S$ on the integral equation (B.213) and due the second Calderón identity in (B.122), the resulting operator $S N$ can be decomposed as an identity and a compact operator. Thus again the Fredholm alternative holds.
c) Continuous impedance

The integral equation of the continuous-impedance alternative (B.141) is given in terms of boundary layer potentials, for $\mu \in H^{1 / 2}(\Gamma)$, by

$$
\begin{equation*}
-N(\mu)+D^{*}(Z \mu)+Z D(\mu)-Z S(Z \mu)=f_{z} \quad \text { in } H^{-1 / 2}(\Gamma) \tag{B.219}
\end{equation*}
$$

We have the mapping properties:

$$
\begin{align*}
\mu \in H^{1 / 2}(\Gamma) & \longmapsto N(\mu) \in H^{-1 / 2}(\Gamma),  \tag{B.220}\\
Z \mu \in L^{2}(\Gamma) & \longmapsto D^{*}(Z \mu) \in H^{1}(\Gamma) \stackrel{c}{\hookrightarrow} H^{-1 / 2}(\Gamma),  \tag{B.221}\\
\mu \in H^{1 / 2}(\Gamma) & \longmapsto Z D(\mu) \in H^{1}(\Gamma) \hookrightarrow H^{-1 / 2}(\Gamma),  \tag{B.222}\\
Z \mu \in L^{2}(\Gamma) & \longmapsto Z S(Z \mu) \in H^{1}(\Gamma) \hookrightarrow H^{-1 / 2}(\Gamma),  \tag{B.223}\\
f_{z} \in H^{-1 / 2}(\Gamma) & \longmapsto f_{z} \in H^{-1 / 2}(\Gamma) . \tag{B.224}
\end{align*}
$$

The operators (B.221), (B.222), and (B.223) are compact, whereas (B.220) plays the role of the identity. Thus the Fredholm alternative applies.
d) Continuous value

The integral equation of the continuous-value alternative (B.149) is given in terms of boundary layer potentials, for $\nu \in H^{-1 / 2}(\Gamma)$, by

$$
\begin{equation*}
\frac{\nu}{2}+Z S(\nu)-D^{*}(\nu)=-f_{z} \quad \text { in } H^{-1 / 2}(\Gamma) \tag{B.225}
\end{equation*}
$$

We have the mapping properties:

$$
\begin{align*}
\nu \in H^{-1 / 2}(\Gamma) & \longmapsto \frac{\nu}{2} \in H^{-1 / 2}(\Gamma)  \tag{B.226}\\
\nu \in H^{-1 / 2}(\Gamma) & \longmapsto Z S(\nu) \in L^{2}(\Gamma) \stackrel{c}{\hookrightarrow} H^{-1 / 2}(\Gamma)  \tag{B.227}\\
\nu \in H^{-1 / 2}(\Gamma) & \longmapsto D^{*}(\nu) \in H^{1 / 2}(\Gamma) \stackrel{c}{\hookrightarrow} H^{-1 / 2}(\Gamma),  \tag{B.228}\\
f_{z} \in H^{-1 / 2}(\Gamma) & \longmapsto-f_{z} \in H^{-1 / 2}(\Gamma) . \tag{B.229}
\end{align*}
$$

We observe that (B.226) is the identity operator, whereas (B.227) and (B.228) are compact. Thus the Fredholm alternative holds.
e) Continuous normal derivative

The integral equation of the continuous-normal-derivative alternative (B.157) is given in terms of boundary layer potentials, for $\mu \in H^{1 / 2}(\Gamma)$, by

$$
\begin{equation*}
\frac{Z}{2} \mu-N(\mu)+Z D(\mu)=f_{z} \quad \text { in } H^{-1 / 2}(\Gamma) \tag{B.230}
\end{equation*}
$$

We have the following mapping properties:

$$
\begin{align*}
\mu \in H^{1 / 2}(\Gamma) & \longmapsto \frac{Z}{2} \mu \in L^{2}(\Gamma) \stackrel{c}{\hookrightarrow} H^{-1 / 2}(\Gamma),  \tag{B.231}\\
\mu \in H^{1 / 2}(\Gamma) & \longmapsto N(\mu) \in H^{-1 / 2}(\Gamma),  \tag{B.232}\\
\mu \in H^{1 / 2}(\Gamma) & \longmapsto Z D(\mu) \in H^{1}(\Gamma) \stackrel{c}{\hookrightarrow} H^{-1 / 2}(\Gamma),  \tag{B.233}\\
f_{z} \in H^{-1 / 2}(\Gamma) & \longmapsto f_{z} \in H^{-1 / 2}(\Gamma) . \tag{B.234}
\end{align*}
$$

The operators (B.231) and (B.233) are compact, whereas (B.232) plays the role of the identity. Thus the Fredholm alternative again applies.

## B.9.4 Consequences of Fredholm's alternative

Since the Fredholm alternative applies to each integral equation, therefore it applies also to the exterior differential problem (B.11) due the integral representation formula. The existence of the exterior problem's solution is thus determined by its uniqueness, and the impedances $Z \in \mathbb{C}$ for which the uniqueness is lost constitute a countable set, which we call the impedance spectrum of the exterior problem and denote it by $\sigma_{Z}$. The existence and uniqueness of the solution is therefore ensured almost everywhere. The same holds obviously for the solution of the integral equation, whose impedance spectrum we denote by $\varsigma_{z}$. Since each integral equation is derived from the exterior problem, it holds that $\sigma_{Z} \subset \varsigma_{Z}$. The converse, though, is not necessarily true and depends on each particular integral equation. In any way, the set $\varsigma_{Z} \backslash \sigma_{Z}$ is at most countable.

Fredholm's alternative applies as much to the integral equation itself as to its adjoint counterpart, and equally to their homogeneous versions. Moreover, each integral equation solves at the same time an exterior and an interior differential problem. The loss of uniqueness of the integral equation's solution appears when the impedance $Z$ is an eigenvalue of some associated interior problem, either of the homogeneous integral equation or of its adjoint counterpart. Such an impedance $Z$ is contained in $\varsigma_{Z}$.

The integral equation (B.131) is associated with the extension by zero (B.124), for which no eigenvalues appear. Nevertheless, its adjoint integral equation (B.151) of the continuous value is associated with the interior problem (B.144), whose solution is unique for all $Z \neq 0$.

The integral equation (B.134) is also associated with the extension by zero (B.124), for which no eigenvalues appear. Nonetheless, its adjoint integral equation (B.159) of the continuous normal derivative is associated with the interior problem (B.152), whose solution is unique for all $Z$, without restriction.

The integral equation (B.143) of the continuous impedance is self-adjoint and is associated with the interior problem (B.135), which has a countable quantity of eigenvalues $Z$.

Let us consider now the transmission problem generated by the homogeneous exterior problem

$$
\left\{\begin{array}{l}
\text { Find } u_{e}: \Omega_{e} \rightarrow \mathbb{C} \text { such that }  \tag{B.235}\\
\Delta u_{e}=0 \quad \text { in } \Omega_{e}, \\
-\frac{\partial u_{e}}{\partial n}+Z u_{e}=0 \quad \text { on } \Gamma \\
+ \text { Decaying condition as }|\boldsymbol{x}| \rightarrow \infty
\end{array}\right.
$$

and the associated homogeneous interior problem

$$
\begin{cases}\text { Find } u_{i}: \Omega_{i} \rightarrow \mathbb{C} \text { such that }  \tag{B.236}\\ \Delta u_{i}=0 & \text { in } \Omega_{i}, \\ \frac{\partial u_{i}}{\partial n}+Z u_{i}=0 & \text { on } \Gamma\end{cases}
$$

where the asymptotic decaying condition is as usual given by (B.7), and where the unit normal $\boldsymbol{n}$ always points outwards of $\Omega_{e}$. Its jumps are characterized by

$$
\begin{align*}
{[u] } & =u_{e}-u_{i}=\frac{1}{Z}\left(\frac{\partial u_{e}}{\partial n}+\frac{\partial u_{i}}{\partial n}\right),  \tag{B.237}\\
{\left[\frac{\partial u}{\partial n}\right] } & =\frac{\partial u_{e}}{\partial n}-\frac{\partial u_{i}}{\partial n}=Z\left(u_{e}+u_{i}\right) . \tag{B.238}
\end{align*}
$$

It holds that the integral equations for this transmission problem composed by (B.235) and (B.236) have either the same left-hand side or are mutually adjoint to all other possible alternatives of integral equations that can be built for the exterior problem (B.11), and in particular to all the alternatives that were mentioned in the last subsection. The eigenvalues $Z$ of the homogeneous interior problem (B.236) are thus also contained in $\varsigma_{Z}$. To see this, let us construct the corresponding integral equations. By adding the Calderón relations (B.116) and (B.117) for the jumps (B.237) and (B.238), we obtain a system of integral equations that only relates these jumps, namely

$$
\frac{1}{2}\binom{u_{e}+u_{i}}{\frac{\partial u_{e}}{\partial n}+\frac{\partial u_{i}}{\partial n}}=\left(\begin{array}{cc}
D & -S  \tag{B.239}\\
N & -D^{*}
\end{array}\right)\binom{[u]}{\left[\frac{\partial u}{\partial n}\right]}=\binom{\frac{1}{2 Z}\left[\frac{\partial u}{\partial n}\right]}{\frac{Z}{2}[u]}
$$

We observe that even if the problems (B.235) and (B.236) are homogeneous, any possible jump condition can be assigned to them. The resulting system of integral equations can then be always combined in such a way that it has the same left-hand side or is mutually adjoint to any integral equation derived for the exterior problem (B.11).

In the case of the extension by zero we use the jumps (B.125) and (B.126). By replacing them in (B.239), we obtain the integral equations

$$
\begin{align*}
\frac{\mu}{2}+S(Z \mu)-D(\mu) & =S\left(f_{z}\right)+\frac{f_{z}}{2 Z} & & \text { in } H^{1 / 2}(\Gamma)  \tag{B.240}\\
\frac{Z}{2} \mu-N(\mu)+D^{*}(Z \mu) & =D^{*}\left(f_{z}\right) & & \text { in } H^{-1 / 2}(\Gamma) \tag{B.241}
\end{align*}
$$

It can be clearly observed that the equations (B.240) and (B.241) have the same left-hand side as (B.208) and (B.213), respectively.

For the continuous impedance we use the jumps (B.136) and (B.137). By replacing them in (B.239), multiplying the first row by $Z$, and subtracting it from the second row, we obtain the integral equation

$$
\begin{equation*}
-N(\mu)+D^{*}(Z \mu)+Z D(\mu)-Z S(Z \mu)=0 \quad \text { in } H^{-1 / 2}(\Gamma) \tag{B.242}
\end{equation*}
$$

This integral equation has the same left-hand side as (B.219).
In the case of the continuous value we consider the jumps (B.145) and (B.146). By replacing them in (B.239) and subtracting the second row from the first, we obtain the integral equation

$$
\begin{equation*}
\frac{\nu}{2}+Z S(\nu)-D^{*}(\nu)=0 \quad \text { in } H^{-1 / 2}(\Gamma) \tag{B.243}
\end{equation*}
$$

Again, this integral equation has the same left-hand side as (B.225).
For the continuous normal derivative we use the jumps (B.153) and (B.154). By replacing them in (B.239), multiplying the first row by $Z$ and adding the second row to the first, we obtain the integral equation

$$
\begin{equation*}
\frac{Z}{2} \mu-N(\mu)+Z D(\mu)=0 \quad \text { in } H^{-1 / 2}(\Gamma) \tag{B.244}
\end{equation*}
$$

This integral equation has the same left-hand side as (B.230).
We remark that additional alternatives for integral representations and equations based on non-homogeneous versions of the problem (B.236) can be also derived for the exterior impedance problem (cf. Ha-Duong 1987).

The determination of the impedance spectrum $\sigma_{Z}$ of the exterior problem (B.11) is not so easy, but can be achieved for simple geometries where an analytic solution is known.

In conclusion, the exterior problem (B.11) admits a unique solution $u$ if $Z \notin \sigma_{Z}$, and each integral equation admits a unique solution, either $\mu$ or $\nu$, if $Z \notin \varsigma_{Z}$.

## B.9.5 Compatibility condition

As we appreciated for the exterior circle problem, if a constant impedance $Z \in \mathbb{C}$ is considered, then the impedance data function $f_{z}$ has to satisfy some sort of compatibility condition like

$$
\begin{equation*}
\int_{\Gamma} f_{z} \mathrm{~d} \gamma=0, \tag{B.245}
\end{equation*}
$$

which is required for the existence of a solution $u$ of the exterior problem (B.11). To understand this better, we assume that $u$ is the solution of (B.11) and that $f_{z}$ satisfies (B.245). If we consider a constant $f_{0} \in \mathbb{C}$ and a constant impedance $Z \neq 0$, then

$$
\begin{equation*}
\widetilde{u}=u+\frac{f_{0}}{Z} \tag{B.246}
\end{equation*}
$$

satisfies the Laplace equation

$$
\begin{equation*}
\Delta \widetilde{u}=\Delta\left(u+\frac{f_{0}}{Z}\right)=0 \quad \text { in } \Omega_{e} \tag{B.247}
\end{equation*}
$$

and the impedance boundary condition

$$
\begin{equation*}
-\frac{\partial \widetilde{u}}{\partial n}+Z \widetilde{u}=-\frac{\partial u}{\partial n}+Z u+f_{0}=f_{z}+f_{0}=\widetilde{f}_{z} \quad \text { on } \Gamma, \tag{B.248}
\end{equation*}
$$

where

$$
\begin{equation*}
\int_{\Gamma} \widetilde{f}_{z} \mathrm{~d} \gamma=f_{0} \tag{B.249}
\end{equation*}
$$

Nonetheless, we observe that the function $\widetilde{u}$ does not fulfill the decaying condition (B.7) if $f_{0} \neq 0$ and is thus not admissible as a solution for the exterior problem with the impedance data function $\widetilde{f}_{z}$.

If we consider now a Neumann boundary condition ( $Z=0$ ), then the compatibility condition (B.245) is obtained by replacing the data function $f_{z}$ in (B.57).

In any case, it is the decaying condition (B.7) that generates the need of the compatibility condition (B.245). If we disregard the latter, then the exterior problem (B.11) still admits a solution that not necessarily satisfies the decaying condition.

## B. 10 Variational formulation

To solve a particular integral equation we convert it to its variational or weak formulation, i.e., we solve it with respect to certain test functions in a bilinear (or sesquilinear) form. Basically, the integral equation is multiplied by the (conjugated) test function and then the equation is integrated over the boundary of the domain. The test functions are taken in the same function space as the solution of the integral equation.
a) First extension by zero

The variational formulation for the first integral equation (B.208) of the extension-byzero alternative searches $\mu \in H^{1 / 2}(\Gamma)$ such that $\forall \varphi \in H^{1 / 2}(\Gamma)$

$$
\begin{equation*}
\left\langle\frac{\mu}{2}+S(Z \mu)-D(\mu), \varphi\right\rangle=\left\langle S\left(f_{z}\right), \varphi\right\rangle \tag{B.250}
\end{equation*}
$$

which in terms of integrals is expressed as

$$
\begin{align*}
\int_{\Gamma} \int_{\Gamma} & \left(Z(\boldsymbol{y}) G(\boldsymbol{x}, \boldsymbol{y})-\frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y})\right) \mu(\boldsymbol{y}) \varphi(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{x}) \\
& +\frac{1}{2} \int_{\Gamma} \mu(\boldsymbol{x}) \varphi(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{x})=\int_{\Gamma} \int_{\Gamma} G(\boldsymbol{x}, \boldsymbol{y}) f_{z}(\boldsymbol{y}) \varphi(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{x}) \tag{B.251}
\end{align*}
$$

b) Second extension by zero

The variational formulation for the second integral equation (B.213) of the extension-by-zero alternative searches $\mu \in H^{1 / 2}(\Gamma)$ such that $\forall \varphi \in H^{1 / 2}(\Gamma)$

$$
\begin{equation*}
\left\langle\frac{Z}{2} \mu-N(\mu)+D^{*}(Z \mu), \varphi\right\rangle=\left\langle\frac{f_{z}}{2}+D^{*}\left(f_{z}\right), \varphi\right\rangle \tag{B.252}
\end{equation*}
$$

which in terms of integrals is expressed as

$$
\begin{align*}
& \int_{\Gamma} \int_{\Gamma} G(\boldsymbol{x}, \boldsymbol{y})\left(\nabla \mu(\boldsymbol{y}) \times \boldsymbol{n}_{\boldsymbol{y}}\right)\left(\nabla \varphi(\boldsymbol{x}) \times \boldsymbol{n}_{\boldsymbol{x}}\right) \mathrm{d} \gamma(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{x}) \\
&+\int_{\Gamma} \int_{\Gamma} Z(\boldsymbol{y}) \frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x}, \boldsymbol{y}) \mu(\boldsymbol{y}) \varphi(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{x})+\frac{1}{2} \int_{\Gamma} Z(\boldsymbol{x}) \mu(\boldsymbol{x}) \varphi(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{x}) \\
&= \frac{1}{2} \int_{\Gamma} f_{z}(\boldsymbol{x}) \varphi(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{x})+\int_{\Gamma} \int_{\Gamma} \frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x}, \boldsymbol{y}) f_{z}(\boldsymbol{y}) \varphi(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{x}) . \tag{B.253}
\end{align*}
$$

c) Continuous impedance

The variational formulation for the integral equation (B.219) of the alternative of the continuous-impedance searches $\mu \in H^{1 / 2}(\Gamma)$ such that $\forall \varphi \in H^{1 / 2}(\Gamma)$

$$
\begin{equation*}
\left\langle-N(\mu)+D^{*}(Z \mu)+Z D(\mu)-Z S(Z \mu), \varphi\right\rangle=\left\langle f_{z}, \varphi\right\rangle \tag{B.254}
\end{equation*}
$$

which in terms of integrals is expressed as

$$
\begin{align*}
& \int_{\Gamma} \int_{\Gamma} G(\boldsymbol{x}, \boldsymbol{y})\left[\left(\nabla \mu(\boldsymbol{y}) \times \boldsymbol{n}_{\boldsymbol{y}}\right)\left(\nabla \varphi(\boldsymbol{x}) \times \boldsymbol{n}_{\boldsymbol{x}}\right)-Z(\boldsymbol{x}) Z(\boldsymbol{y}) \mu(\boldsymbol{y}) \varphi(\boldsymbol{x})\right] \mathrm{d} \gamma(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{x}) \\
&+\int_{\Gamma} \int_{\Gamma}\left(Z(\boldsymbol{y}) \frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x}, \boldsymbol{y})+Z(\boldsymbol{x}) \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y})\right) \mu(\boldsymbol{y}) \varphi(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{x}) \\
&=\int_{\Gamma} f_{z}(\boldsymbol{x}) \varphi(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{x}) . \tag{B.255}
\end{align*}
$$

d) Continuous value

The variational formulation for the integral equation (B.225) of the continuous-value alternative searches $\nu \in H^{-1 / 2}(\Gamma)$ such that $\forall \psi \in H^{-1 / 2}(\Gamma)$

$$
\begin{equation*}
\left\langle\frac{\nu}{2}+Z S(\nu)-D^{*}(\nu), \psi\right\rangle=\left\langle-f_{z}, \psi\right\rangle, \tag{B.256}
\end{equation*}
$$

which in terms of integrals is expressed as

$$
\begin{array}{r}
\int_{\Gamma} \int_{\Gamma}\left(Z(\boldsymbol{x}) G(\boldsymbol{x}, \boldsymbol{y})-\frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x}, \boldsymbol{y})\right) \nu(\boldsymbol{y}) \psi(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{x}) \\
+\frac{1}{2} \int_{\Gamma} \nu(\boldsymbol{x}) \psi(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{x})=-\int_{\Gamma} f_{z}(\boldsymbol{x}) \psi(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{x}) . \tag{B.257}
\end{array}
$$

e) Continuous normal derivative

The variational formulation for the integral equation (B.230) of the continuous-normalderivative alternative searches $\mu \in H^{1 / 2}(\Gamma)$ such that $\forall \varphi \in H^{1 / 2}(\Gamma)$

$$
\begin{equation*}
\left\langle\frac{Z}{2} \mu-N(\mu)+Z D(\mu), \varphi\right\rangle=\left\langle f_{z}, \varphi\right\rangle \tag{B.258}
\end{equation*}
$$

which in terms of integrals is expressed as

$$
\begin{align*}
& \int_{\Gamma} \int_{\Gamma} G(\boldsymbol{x}, \boldsymbol{y})\left(\nabla \mu(\boldsymbol{y}) \times \boldsymbol{n}_{\boldsymbol{y}}\right)\left(\nabla \varphi(\boldsymbol{x}) \times \boldsymbol{n}_{\boldsymbol{x}}\right) \mathrm{d} \gamma(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{x}) \\
&+\int_{\Gamma} \int_{\Gamma} Z(\boldsymbol{x}) \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) \mu(\boldsymbol{y}) \varphi(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{x})+\frac{1}{2} \int_{\Gamma} Z(\boldsymbol{x}) \mu(\boldsymbol{x}) \varphi(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{x}) \\
&=\int_{\Gamma} f_{z}(\boldsymbol{x}) \varphi(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{x}) . \tag{B.259}
\end{align*}
$$

## B. 11 Numerical discretization

## B.11.1 Discretized function spaces

The exterior problem (B.11) is solved numerically with the boundary element method by employing a Galerkin scheme on the variational formulation of an integral equation. We use on the boundary curve $\Gamma$ Lagrange finite elements of type either $\mathbb{P}_{1}$ or $\mathbb{P}_{0}$. As shown in Figure B.9, the curve $\Gamma$ is approximated by the discretized curve $\Gamma^{h}$, composed by $I$ rectilinear segments $T_{j}$, sequentially ordered in clockwise direction for $1 \leq j \leq I$, such that their length $\left|T_{j}\right|$ is less or equal than $h$, and with their endpoints on top of $\Gamma$.


Figure B.9. Curve $\Gamma^{h}$, discretization of $\Gamma$.

The function space $H^{1 / 2}(\Gamma)$ is approximated using the conformal space of continuous piecewise linear polynomials with complex coefficients

$$
\begin{equation*}
Q_{h}=\left\{\varphi_{h} \in C^{0}\left(\Gamma^{h}\right):\left.\varphi_{h}\right|_{T_{j}} \in \mathbb{P}_{1}(\mathbb{C}), \quad 1 \leq j \leq I\right\} \tag{B.260}
\end{equation*}
$$

The space $Q_{h}$ has a finite dimension $I$, and we describe it using the standard base functions for finite elements of type $\mathbb{P}_{1}$, denoted by $\left\{\chi_{j}\right\}_{j=1}^{I}$, shown in Figure B.10, and expressed as

$$
\chi_{j}(\boldsymbol{x})=\left\{\begin{array}{cl}
\frac{\left|\boldsymbol{x}-\boldsymbol{r}_{j-1}\right|}{\left|T_{j-1}\right|} & \text { if } \boldsymbol{x} \in T_{j-1}  \tag{B.261}\\
\frac{\left|\boldsymbol{r}_{j+1}-\boldsymbol{x}\right|}{\left|T_{j}\right|} & \text { if } \boldsymbol{x} \in T_{j} \\
0 & \text { if } \boldsymbol{x} \notin T_{j-1} \cup T_{j}
\end{array}\right.
$$

where segment $T_{j-1}$ has as endpoints $\boldsymbol{r}_{j-1}$ and $\boldsymbol{r}_{j}$, while the endpoints of segment $T_{j}$ are given by $\boldsymbol{r}_{j}$ and $\boldsymbol{r}_{j+1}$.


Figure B.10. Base function $\chi_{j}$ for finite elements of type $\mathbb{P}_{1}$.

The function space $H^{-1 / 2}(\Gamma)$, on the other hand, is approximated using the conformal space of piecewise constant polynomials with complex coefficients

$$
\begin{equation*}
P_{h}=\left\{\psi_{h}: \Gamma^{h} \rightarrow \mathbb{C}\left|\psi_{h}\right|_{T_{j}} \in \mathbb{P}_{0}(\mathbb{C}), \quad 1 \leq j \leq I\right\} \tag{B.262}
\end{equation*}
$$

The space $P_{h}$ has a finite dimension $I$, and is described using the standard base functions for finite elements of type $\mathbb{P}_{0}$, denoted by $\left\{\kappa_{j}\right\}_{j=1}^{I}$, shown in Figure B.11, and expressed as

$$
\kappa_{j}(\boldsymbol{x})= \begin{cases}1 & \text { if } \boldsymbol{x} \in T_{j}  \tag{B.263}\\ 0 & \text { if } \boldsymbol{x} \notin T_{j} .\end{cases}
$$

Again, we denote by $\boldsymbol{r}_{j}$ and $\boldsymbol{r}_{j+1}$ the endpoints of segment $T_{j}$.


Figure B.11. Base function $\kappa_{j}$ for finite elements of type $\mathbb{P}_{0}$.

In virtue of this discretization, any function $\varphi_{h} \in Q_{h}$ or $\psi_{h} \in P_{h}$ can be expressed as a linear combination of the elements of the base, namely

$$
\begin{equation*}
\varphi_{h}(\boldsymbol{x})=\sum_{j=1}^{I} \varphi_{j} \chi_{j}(\boldsymbol{x}) \quad \text { and } \quad \psi_{h}(\boldsymbol{x})=\sum_{j=1}^{I} \psi_{j} \kappa_{j}(\boldsymbol{x}) \quad \text { for } \boldsymbol{x} \in \Gamma^{h} \tag{B.264}
\end{equation*}
$$

where $\varphi_{j}, \psi_{j} \in \mathbb{C}$ for $1 \leq j \leq I$. The solutions $\mu \in H^{1 / 2}(\Gamma)$ and $\nu \in H^{-1 / 2}(\Gamma)$ of the variational formulations can be therefore approximated respectively by

$$
\begin{equation*}
\mu_{h}(\boldsymbol{x})=\sum_{j=1}^{I} \mu_{j} \chi_{j}(\boldsymbol{x}) \quad \text { and } \quad \nu_{h}(\boldsymbol{x})=\sum_{j=1}^{I} \nu_{j} \kappa_{j}(\boldsymbol{x}) \quad \text { for } \boldsymbol{x} \in \Gamma^{h} \tag{B.265}
\end{equation*}
$$

where $\mu_{j}, \nu_{j} \in \mathbb{C}$ for $1 \leq j \leq I$. The function $f_{z}$ can be also approximated by

$$
\begin{equation*}
f_{z}^{h}(\boldsymbol{x})=\sum_{j=1}^{I} f_{j} \chi_{j}(\boldsymbol{x}) \quad \text { for } \boldsymbol{x} \in \Gamma^{h}, \quad \text { with } f_{j}=f_{z}\left(\boldsymbol{r}_{j}\right), \tag{B.266}
\end{equation*}
$$

or

$$
\begin{equation*}
f_{z}^{h}(\boldsymbol{x})=\sum_{j=1}^{I} f_{j} \kappa_{j}(\boldsymbol{x}) \quad \text { for } \boldsymbol{x} \in \Gamma^{h}, \quad \text { with } f_{j}=\frac{f_{z}\left(\boldsymbol{r}_{j}\right)+f_{z}\left(\boldsymbol{r}_{j+1}\right)}{2} \tag{B.267}
\end{equation*}
$$

depending on whether the original integral equation is stated in $H^{1 / 2}(\Gamma)$ or in $H^{-1 / 2}(\Gamma)$.

## B.11.2 Discretized integral equations

a) First extension by zero

To see how the boundary element method operates, we apply it to the first integral equation of the extension-by-zero alternative, i.e., to the variational formulation (B.250). We characterize all the discrete approximations by the index $h$, including also the impedance and the boundary layer potentials. The numerical approximation of (B.250) leads to the
discretized problem that searches $\mu_{h} \in Q_{h}$ such that $\forall \varphi_{h} \in Q_{h}$

$$
\begin{align*}
& \int_{\Gamma^{h}} \int_{\Gamma^{h}}\left(Z_{h}(\boldsymbol{y}) G(\boldsymbol{x}, \boldsymbol{y})-\frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y})\right) \mu_{h}(\boldsymbol{y}) \varphi_{h}(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{x}) \\
& \quad+\frac{1}{2} \int_{\Gamma^{h}} \mu_{h}(\boldsymbol{x}) \varphi_{h}(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{x})=\int_{\Gamma^{h}} \int_{\Gamma^{h}} G(\boldsymbol{x}, \boldsymbol{y}) f_{z}^{h}(\boldsymbol{y}) \varphi_{h}(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{x}), \tag{B.268}
\end{align*}
$$

which in terms of boundary layer potentials becomes

$$
\begin{equation*}
\left\langle\frac{\mu_{h}}{2}+S_{h}\left(Z_{h} \mu_{h}\right)-D_{h}\left(\mu_{h}\right), \varphi_{h}\right\rangle=\left\langle S_{h}\left(f_{z}^{h}\right), \varphi_{h}\right\rangle . \tag{B.269}
\end{equation*}
$$

Considering the decomposition of $\mu_{h}$ in terms of the base $\left\{\chi_{j}\right\}$ and taking as test functions the same base functions, $\varphi_{h}=\chi_{i}$ for $1 \leq i \leq I$, yields the discrete linear system

$$
\begin{equation*}
\sum_{j=1}^{I} \mu_{j}\left(\frac{1}{2}\left\langle\chi_{j}, \chi_{i}\right\rangle+\left\langle S_{h}\left(Z_{h} \chi_{j}\right), \chi_{i}\right\rangle-\left\langle D_{h}\left(\chi_{j}\right), \chi_{i}\right\rangle\right)=\sum_{j=1}^{I} f_{j}\left\langle S_{h}\left(\chi_{j}\right), \chi_{i}\right\rangle \tag{B.270}
\end{equation*}
$$

This constitutes a system of linear equations that can be expressed as a linear matrix system:

$$
\left\{\begin{array}{l}
\text { Find } \boldsymbol{\mu} \in \mathbb{C}^{I} \text { such that }  \tag{B.271}\\
\boldsymbol{M \boldsymbol { \mu } = \boldsymbol { b }} .
\end{array}\right.
$$

The elements $m_{i j}$ of the matrix $\boldsymbol{M}$ are given by

$$
\begin{equation*}
m_{i j}=\frac{1}{2}\left\langle\chi_{j}, \chi_{i}\right\rangle+\left\langle S_{h}\left(Z_{h} \chi_{j}\right), \chi_{i}\right\rangle-\left\langle D_{h}\left(\chi_{j}\right), \chi_{i}\right\rangle \quad \text { for } 1 \leq i, j \leq I \tag{B.272}
\end{equation*}
$$

and the elements $b_{i}$ of the vector $\boldsymbol{b}$ by

$$
\begin{equation*}
b_{i}=\left\langle S_{h}\left(f_{z}^{h}\right), \chi_{i}\right\rangle=\sum_{j=1}^{I} f_{j}\left\langle S_{h}\left(\chi_{j}\right), \chi_{i}\right\rangle \quad \text { for } 1 \leq i \leq I \tag{B.273}
\end{equation*}
$$

The discretized solution $u_{h}$, which approximates $u$, is finally obtained by discretizing the integral representation formula (B.127) for $\boldsymbol{x} \in \Omega_{e} \cup \Omega_{i}$ according to

$$
\begin{equation*}
u_{h}(\boldsymbol{x})=\int_{\Gamma^{h}}\left(\frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y})-Z_{h}(\boldsymbol{y}) G(\boldsymbol{x}, \boldsymbol{y})\right) \mu_{h}(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y})+\int_{\Gamma^{h}} G(\boldsymbol{x}, \boldsymbol{y}) f_{z}^{h}(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}), \tag{B.274}
\end{equation*}
$$

or, in terms of boundary layer potentials, according to

$$
\begin{equation*}
u_{h}=\mathcal{D}_{h}\left(\mu_{h}\right)-\mathcal{S}_{h}\left(Z_{h} \mu_{h}\right)+\mathcal{S}_{h}\left(f_{z}^{h}\right) . \tag{B.275}
\end{equation*}
$$

More specifically, the solution is computed by

$$
\begin{equation*}
u_{h}=\sum_{j=1}^{I} \mu_{j}\left(\mathcal{D}_{h}\left(\chi_{j}\right)-\mathcal{S}_{h}\left(Z_{h} \chi_{j}\right)\right)+\sum_{j=1}^{I} f_{j} \mathcal{S}_{h}\left(\chi_{j}\right) \tag{B.276}
\end{equation*}
$$

By proceeding in the same way, the discretization of all the other alternatives of integral equations can be also expressed as a linear matrix system like (B.271). The resulting matrix $\boldsymbol{M}$ is in general complex, full, non-symmetric, and with dimensions $I \times I$. The right-hand side vector $\boldsymbol{b}$ is complex and of size $I$. The boundary element calculations required to compute numerically the elements of $\boldsymbol{M}$ and $\boldsymbol{b}$ have to be performed carefully,
since the integrals that appear become singular when the involved segments are adjacent or coincident, due the singularity of the Green's function at its source point.
b) Second extension by zero

In the case of the second integral equation of the extension-by-zero alternative, i.e., of the variational formulation (B.252), the elements $m_{i j}$ that constitute the matrix $M$ of the linear system (B.271) are given by

$$
\begin{equation*}
m_{i j}=\frac{1}{2}\left\langle Z_{h} \chi_{j}, \chi_{i}\right\rangle-\left\langle N_{h}\left(\chi_{j}\right), \chi_{i}\right\rangle+\left\langle D_{h}^{*}\left(Z_{h} \chi_{j}\right), \chi_{i}\right\rangle \quad \text { for } 1 \leq i, j \leq I \tag{B.277}
\end{equation*}
$$

whereas the elements $b_{i}$ of the vector $\boldsymbol{b}$ are expressed as

$$
\begin{equation*}
b_{i}=\sum_{j=1}^{I} f_{j}\left(\frac{1}{2}\left\langle\chi_{j}, \chi_{i}\right\rangle+\left\langle D_{h}^{*}\left(Z_{h} \chi_{j}\right), \chi_{i}\right\rangle\right) \quad \text { for } 1 \leq i \leq I \tag{B.278}
\end{equation*}
$$

The discretized solution $u_{h}$ is again computed by (B.276).
c) Continuous impedance

In the case of the continuous-impedance alternative, i.e., of the variational formulation (B.254), the elements $m_{i j}$ that constitute the matrix $M$ of the linear system (B.271) are given, for $1 \leq i, j \leq I$, by

$$
\begin{equation*}
m_{i j}=-\left\langle N_{h}\left(\chi_{j}\right), \chi_{i}\right\rangle+\left\langle D_{h}^{*}\left(Z_{h} \chi_{j}\right), \chi_{i}\right\rangle+\left\langle Z_{h} D_{h}\left(\chi_{j}\right), \chi_{i}\right\rangle-\left\langle Z_{h} S_{h}\left(Z_{h} \chi_{j}\right), \chi_{i}\right\rangle \tag{B.279}
\end{equation*}
$$

whereas the elements $b_{i}$ of the vector $\boldsymbol{b}$ are expressed as

$$
\begin{equation*}
b_{i}=\sum_{j=1}^{I} f_{j}\left\langle\chi_{j}, \chi_{i}\right\rangle \quad \text { for } 1 \leq i \leq I \tag{B.280}
\end{equation*}
$$

It can be observed that for this particular alternative the matrix $M$ turns out to be symmetric, since the integral equation is self-adjoint. The discretized solution $u_{h}$, due (B.142), is then computed by

$$
\begin{equation*}
u_{h}=\sum_{j=1}^{I} \mu_{j}\left(\mathcal{D}_{h}\left(\chi_{j}\right)-\mathcal{S}_{h}\left(Z_{h} \chi_{j}\right)\right) . \tag{B.281}
\end{equation*}
$$

d) Continuous value

In the case of the alternative of the continuous-value, i.e., of the variational formulation (B.256), the elements $m_{i j}$ that constitute the matrix $M$, now of the linear system

$$
\left\{\begin{array}{l}
\text { Find } \boldsymbol{\nu} \in \mathbb{C}^{I} \text { such that }  \tag{B.282}\\
\boldsymbol{M} \boldsymbol{\nu}=\boldsymbol{b},
\end{array}\right.
$$

are given by

$$
\begin{equation*}
m_{i j}=\frac{1}{2}\left\langle\kappa_{j}, \kappa_{i}\right\rangle+\left\langle Z_{h} S_{h}\left(\kappa_{j}\right), \kappa_{i}\right\rangle-\left\langle D_{h}^{*}\left(\kappa_{j}\right), \kappa_{i}\right\rangle \quad \text { for } 1 \leq i, j \leq I, \tag{B.283}
\end{equation*}
$$

whereas the elements $b_{i}$ of the vector $\boldsymbol{b}$ are expressed as

$$
\begin{equation*}
b_{i}=-\sum_{j=1}^{I} f_{j}\left\langle\kappa_{j}, \kappa_{i}\right\rangle \quad \text { for } 1 \leq i \leq I \tag{B.284}
\end{equation*}
$$

The discretized solution $u_{h}$, due (B.150), is then computed by

$$
\begin{equation*}
u_{h}=-\sum_{j=1}^{I} \nu_{j} \mathcal{S}_{h}\left(\kappa_{j}\right) \tag{B.285}
\end{equation*}
$$

e) Continuous normal derivative

In the case of the continuous-normal-derivative alternative, i.e., of the variational formulation (B.258), the elements $m_{i j}$ that conform the matrix $\boldsymbol{M}$ of the linear system (B.271) are given by

$$
\begin{equation*}
m_{i j}=\frac{1}{2}\left\langle Z_{h} \chi_{j}, \chi_{i}\right\rangle-\left\langle N_{h}\left(\chi_{j}\right), \chi_{i}\right\rangle+\left\langle Z_{h} D_{h}\left(\chi_{j}\right), \chi_{i}\right\rangle \quad \text { for } 1 \leq i, j \leq I \tag{B.286}
\end{equation*}
$$

whereas the elements $b_{i}$ of the vector $\boldsymbol{b}$ are expressed as

$$
\begin{equation*}
b_{i}=\sum_{j=1}^{I} f_{j}\left\langle\chi_{j}, \chi_{i}\right\rangle \quad \text { for } 1 \leq i \leq I . \tag{B.287}
\end{equation*}
$$

The discretized solution $u_{h}$, due (B.158), is then computed by

$$
\begin{equation*}
u_{h}=\sum_{j=1}^{I} \mu_{j} \mathcal{D}_{h}\left(\chi_{j}\right) \tag{B.288}
\end{equation*}
$$

## B. 12 Boundary element calculations

## B.12.1 Geometry

The boundary element calculations build the elements of the matrix $M$ resulting from the discretization of the integral equation, i.e., from (B.271) or (B.282). They permit thus to compute numerically expressions like (B.272). To evaluate the appearing singular integrals, we use the semi-numerical methods described in the report of Bendali \& Devys (1986).

Let us consider the elemental interactions between two straight segments $T_{K}$ and $T_{L}$ of a discrete closed curve $\Gamma^{h}$, which is composed by rectilinear segments and described in clockwise direction. The unit normal points always inwards of the domain encompassed by the curve $\Gamma^{h}$ (vid. Figure B.9).

We denote the segments more simply just as $K=T_{K}$ and $L=T_{L}$. As depicted in Figure B.12, the following notation is used:

- $|K|$ denotes the length of segment $K$.
- $|L|$ denotes the length of segment $L$.
- $\boldsymbol{\tau}_{K}, \boldsymbol{\tau}_{L}$ denote the unit tangents of segments $K$ and $L$.
- $\boldsymbol{n}_{K}, \boldsymbol{n}_{L}$ denote the unit normals of segments $K$ and $L$.
- $\boldsymbol{r}_{1}^{K}, \boldsymbol{r}_{2}^{K}$ denote the endpoints of segment $K$.
- $\boldsymbol{r}_{1}^{L}, \boldsymbol{r}_{2}^{L}$ denote the endpoints of segment $L$.
- $\boldsymbol{r}(\boldsymbol{x})$ denotes a variable location on segment $K$ (dependent on variable $\boldsymbol{x}$ ).
- $\boldsymbol{r}(\boldsymbol{y})$ denotes a variable location on segment $L$ (dependent on variable $\boldsymbol{y}$ ).


Figure B.12. Geometric characteristics of the segments $K$ and $L$.

Segment $K$ is parametrically described by

$$
\begin{equation*}
\boldsymbol{r}(\boldsymbol{x})=\boldsymbol{r}_{1}^{K}+s \boldsymbol{\tau}_{K}, \quad 0 \leq s \leq|K| \tag{B.289}
\end{equation*}
$$

In the same manner, segment $L$ is parametrically described by

$$
\begin{equation*}
\boldsymbol{r}(\boldsymbol{y})=\boldsymbol{r}_{1}^{L}+t \boldsymbol{\tau}_{L}, \quad 0 \leq t \leq|L| . \tag{B.290}
\end{equation*}
$$

Thus the parameters $s$ and $t$ can be expressed as

$$
\begin{align*}
& s=\left(\boldsymbol{r}(\boldsymbol{x})-\boldsymbol{r}_{1}^{K}\right) \cdot \boldsymbol{\tau}_{K},  \tag{B.291}\\
& t=\left(\boldsymbol{r}(\boldsymbol{y})-\boldsymbol{r}_{1}^{L}\right) \cdot \boldsymbol{\tau}_{L} . \tag{B.292}
\end{align*}
$$

The lengths of the segments are given by

$$
\begin{align*}
|K| & =\left|\boldsymbol{r}_{2}^{K}-\boldsymbol{r}_{1}^{K}\right|,  \tag{B.293}\\
|L| & =\left|\boldsymbol{r}_{2}^{L}-\boldsymbol{r}_{1}^{L}\right| . \tag{B.294}
\end{align*}
$$

The unit tangents of the segments, $\boldsymbol{\tau}_{K}=\left(\tau_{1}^{K}, \tau_{2}^{K}\right)$ and $\boldsymbol{\tau}_{L}=\left(\tau_{1}^{L}, \tau_{2}^{L}\right)$, are calculated as

$$
\begin{gather*}
\boldsymbol{\tau}_{K}=\frac{\boldsymbol{r}_{2}^{K}-\boldsymbol{r}_{1}^{K}}{|K|}  \tag{B.295}\\
\boldsymbol{\tau}_{L}=\frac{\boldsymbol{r}_{2}^{L}-\boldsymbol{r}_{1}^{L}}{|L|} \tag{B.296}
\end{gather*}
$$

The unit normals of the segments, $\boldsymbol{n}_{K}=\left(n_{1}^{K}, n_{2}^{K}\right)$ and $\boldsymbol{n}_{L}=\left(n_{1}^{L}, n_{2}^{L}\right)$, are perpendicular to the tangents and can be thus calculated as

$$
\begin{equation*}
\left(n_{1}^{K}, n_{2}^{K}\right)=\left(\tau_{2}^{K},-\tau_{1}^{K}\right), \tag{B.297}
\end{equation*}
$$

$$
\begin{equation*}
\left(n_{1}^{L}, n_{2}^{L}\right)=\left(\tau_{2}^{L},-\tau_{1}^{L}\right) \tag{B.298}
\end{equation*}
$$

For the elemental interactions between a point $\boldsymbol{x}$ on segment $K$ and a point $\boldsymbol{y}$ on segment $L$, the following notation is also used:

- $\boldsymbol{R}$ denotes the vector pointing from the point $\boldsymbol{x}$ towards the point $\boldsymbol{y}$.
- $R$ denotes the distance between the points $\boldsymbol{x}$ and $\boldsymbol{y}$.

These values are given by

$$
\begin{align*}
\boldsymbol{R} & =\boldsymbol{r}(\boldsymbol{y})-\boldsymbol{r}(\boldsymbol{x}),  \tag{B.299}\\
R & =|\boldsymbol{R}|=|\boldsymbol{y}-\boldsymbol{x}| \tag{B.300}
\end{align*}
$$

For the singular integral calculations, when considering the point $\boldsymbol{x}$ as a parameter, the following notation is also used (vid. Figure B.13):

- $\boldsymbol{R}_{1}^{L}, \boldsymbol{R}_{2}^{L}$ denote the vectors pointing from $\boldsymbol{x}$ towards the endpoints of segment $L$.
- $R_{1}^{L}, R_{2}^{L}$ denote the distances from $\boldsymbol{x}$ to the endpoints of segment $L$.
- $d_{L}$ denotes the signed distance from $\boldsymbol{x}$ to the line that contains segment $L$.
- $\theta_{L}$ denotes the angle formed by the vectors $\boldsymbol{R}_{1}^{L}$ and $\boldsymbol{R}_{2}^{L}\left(-\pi \leq \theta_{L} \leq \pi\right)$.

Thus on segment $L$ the following holds:

$$
\begin{array}{lll}
\boldsymbol{R}_{1}^{L}=\boldsymbol{r}_{1}^{L}-\boldsymbol{r}(\boldsymbol{x}), & & R_{1}^{L}=\left|\boldsymbol{R}_{1}^{L}\right|, \\
\boldsymbol{R}_{2}^{L}=\boldsymbol{r}_{2}^{L}-\boldsymbol{r}(\boldsymbol{x}), & R_{2}^{L}=\left|\boldsymbol{R}_{2}^{L}\right| . \tag{B.302}
\end{array}
$$

Likewise as before, we have that

$$
\begin{align*}
\boldsymbol{R} & =\boldsymbol{R}_{1}^{L}+t \boldsymbol{\tau}_{L}, \quad 0 \leq t \leq|L|,  \tag{B.303}\\
t & =\left(\boldsymbol{R}-\boldsymbol{R}_{1}^{L}\right) \cdot \boldsymbol{\tau}_{L} . \tag{B.304}
\end{align*}
$$

The signed distance $d_{L}$ is constant on $L$ and is characterized by

$$
\begin{equation*}
d_{L}=\boldsymbol{R} \cdot \boldsymbol{n}_{L}=\boldsymbol{R}_{1}^{L} \cdot \boldsymbol{n}_{L}=\boldsymbol{R}_{2}^{L} \cdot \boldsymbol{n}_{L} . \tag{B.305}
\end{equation*}
$$

Finally the signed angle $\theta_{L}$ is given by

$$
\begin{equation*}
\theta_{L}=\arccos \left(\frac{\boldsymbol{R}_{1}^{L} \cdot \boldsymbol{R}_{2}^{L}}{R_{1}^{L} R_{2}^{L}}\right) \operatorname{sign}\left(d_{L}\right), \quad-\pi \leq \theta_{L} \leq \pi \tag{B.306}
\end{equation*}
$$



Figure B.13. Geometric characteristics of the singular integral calculations.

## B.12.2 Boundary element integrals

The boundary element integrals are the basic integrals needed to perform the boundary element calculations. In our case, by considering $a, b \in\{0,1\}$, they can be expressed as

$$
\begin{align*}
Z A_{a, b} & =\int_{K} \int_{L}\left(\frac{s}{|K|}\right)^{a}\left(\frac{t}{|L|}\right)^{b} G(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d} L(\boldsymbol{y}) \mathrm{d} K(\boldsymbol{x})  \tag{B.307}\\
Z B_{a, b} & =\int_{K} \int_{L}\left(\frac{s}{|K|}\right)^{a}\left(\frac{t}{|L|}\right)^{b} \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d} L(\boldsymbol{y}) \mathrm{d} K(\boldsymbol{x})  \tag{B.308}\\
Z C_{a, b} & =\int_{K} \int_{L}\left(\frac{s}{|K|}\right)^{a}\left(\frac{t}{|L|}\right)^{b} \frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d} L(\boldsymbol{y}) \mathrm{d} K(\boldsymbol{x}) \tag{B.309}
\end{align*}
$$

where the parameters $s$ and $t$ depend respectively on the variables $\boldsymbol{x}$ and $\boldsymbol{y}$, as stated in (B.291) and (B.292). When the segments have to be specified, i.e., if $K=T_{i}$ and $L=T_{j}$, then we use respectively also the notation $Z A_{a, b}^{i, j}, Z B_{a, b}^{i, j}$, or $Z C_{a, b}^{i, j}$, e.g.,

$$
\begin{equation*}
Z A_{a, b}^{i, j}=\int_{T_{i}} \int_{T_{j}}\left(\frac{s}{|K|}\right)^{a}\left(\frac{t}{|L|}\right)^{b} G(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{x}) . \tag{B.310}
\end{equation*}
$$

It should be observed that (B.309) can be expressed in terms of (B.308):

$$
\begin{equation*}
Z C_{a, b}^{i, j}=Z B_{b, a}^{j, i}, \tag{B.311}
\end{equation*}
$$

since the involved operators are self-adjoint. It occurs therefore that all the integrals that stem from the numerical discretization can be expressed in terms of the two basic boundary element integrals (B.307) and (B.308).

For this to hold true, the impedance is discretized as a piecewise constant function $Z_{h}$, which on each segment $T_{j}$ adopts a constant value $Z_{j} \in \mathbb{C}$, e.g.,

$$
\begin{equation*}
\left.Z_{h}\right|_{T_{j}}=Z_{j}=\frac{1}{2}\left(Z\left(\boldsymbol{r}_{j}\right)+Z\left(\boldsymbol{r}_{j+1}\right)\right) . \tag{B.312}
\end{equation*}
$$

Now we can compute all the integrals of interest. We begin with the ones that are related with the finite elements of type $\mathbb{P}_{0}$, which are easier. It can be observed that

$$
\left\langle\kappa_{j}, \kappa_{i}\right\rangle=\int_{\Gamma^{h}} \kappa_{j}(\boldsymbol{x}) \kappa_{i}(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{x})= \begin{cases}\left|T_{i}\right| & \text { if } j=i,  \tag{B.313}\\ 0 & \text { if } j \neq i\end{cases}
$$

We have likewise that

$$
\begin{equation*}
\left\langle Z_{h} S_{h}\left(\kappa_{j}\right), \kappa_{i}\right\rangle=\int_{\Gamma^{h}} \int_{\Gamma^{h}} Z_{h}(\boldsymbol{x}) G(\boldsymbol{x}, \boldsymbol{y}) \kappa_{j}(\boldsymbol{y}) \kappa_{i}(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{x})=Z_{i} Z A_{0,0}^{i, j} \tag{B.314}
\end{equation*}
$$

It holds similarly that

$$
\begin{equation*}
\left\langle D_{h}^{*}\left(\kappa_{j}\right), \kappa_{i}\right\rangle=\int_{\Gamma^{h}} \int_{\Gamma^{h}} \frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x}, \boldsymbol{y}) \kappa_{j}(\boldsymbol{y}) \kappa_{i}(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{x})=Z B_{0,0}^{j, i} . \tag{B.315}
\end{equation*}
$$

We consider now the integrals for the finite elements of type $\mathbb{P}_{1}$. We have that

$$
\left\langle\chi_{j}, \chi_{i}\right\rangle=\int_{\Gamma^{h}} \chi_{j}(\boldsymbol{x}) \chi_{i}(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{x})=\left\{\begin{array}{ll}
\left|T_{i-1}\right| / 6 & \text { if } j=i-1,  \tag{B.316}\\
\left(\left|T_{i-1}\right|+\left|T_{i}\right|\right) / 3 & \text { if } j=i, \\
\left|T_{i}\right| / 6 & \text { if } j=i+1, \\
0 & \text { if } j \notin\{i-1, i, i+1\}
\end{array} \quad(\mathrm{B} .\right.
$$

In the same way, it occurs that

$$
\left\langle Z_{h} \chi_{j}, \chi_{i}\right\rangle= \begin{cases}Z_{i-1}\left|T_{i-1}\right| / 6 & \text { if } j=i-1,  \tag{B.317}\\ \left(Z_{i-1}\left|T_{i-1}\right|+Z_{i}\left|T_{i}\right|\right) / 3 & \text { if } j=i, \\ Z_{i}\left|T_{i}\right| / 6 & \text { if } j=i+1, \\ 0 & \text { if } j \notin\{i-1, i, i+1\} .\end{cases}
$$

We have also that

$$
\begin{align*}
\left\langle S_{h}\left(\chi_{j}\right), \chi_{i}\right\rangle= & \int_{\Gamma^{h}} \int_{\Gamma^{h}} G(\boldsymbol{x}, \boldsymbol{y}) \chi_{j}(\boldsymbol{y}) \chi_{i}(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{x}) \\
= & Z A_{1,1}^{i,-1,1}+Z A_{0,1}^{i, j-1}-Z A_{1,1}^{i, j-1}+Z A_{1,0}^{i-1, j}-Z A_{1,1}^{i-1, j} \\
& +Z A_{0,0}^{i, j}-Z A_{0,1}^{i, j}-Z A_{1,0}^{i, j}+Z A_{1,1}^{i, j} . \tag{B.318}
\end{align*}
$$

Additionally it holds that

$$
\begin{align*}
\left\langle S_{h}\left(Z_{h} \chi_{j}\right), \chi_{i}\right\rangle= & \int_{\Gamma^{h}} \int_{\Gamma^{h}} Z_{h}(\boldsymbol{y}) G(\boldsymbol{x}, \boldsymbol{y}) \chi_{j}(\boldsymbol{y}) \chi_{i}(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{x}) \\
= & Z_{j-1}\left(Z A_{1,1}^{i-1, j-1}+Z A_{0,1}^{i, j-1}-Z A_{1,1}^{i, j-1}\right) \\
& +Z_{j}\left(Z A_{1,0}^{i-1, j}-Z A_{1,1}^{i-1, j}+Z A_{0,0}^{i, j}-Z A_{0,1}^{i, j}-Z A_{1,0}^{i, j}+Z A_{1,1}^{i, j}\right) . \tag{B.319}
\end{align*}
$$

Furthermore we see that

$$
\begin{align*}
& \left\langle Z_{h} S_{h}\left(Z_{h} \chi_{j}\right), \chi_{i}\right\rangle=\int_{\Gamma^{h}} \int_{\Gamma^{h}} Z_{h}(\boldsymbol{x}) Z_{h}(\boldsymbol{y}) G(\boldsymbol{x}, \boldsymbol{y}) \chi_{j}(\boldsymbol{y}) \chi_{i}(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{x}) \\
& \quad=Z_{i-1} Z_{j-1} Z A_{1,1}^{i-1, j-1}+Z_{i} Z_{j-1}\left(Z A_{0,1}^{i, j-1}-Z A_{1,1}^{i, j-1}\right) \\
& \quad+Z_{i-1} Z_{j}\left(Z A_{1,0}^{i-1, j}-Z A_{1,1}^{i-1, j}\right)+Z_{i} Z_{j}\left(Z A_{0,0}^{i, j}-Z A_{0,1}^{i, j}-Z A_{1,0}^{i, j}+Z A_{1,1}^{i, j}\right) . \tag{B.320}
\end{align*}
$$

Likewise it occurs that

$$
\begin{align*}
\left\langle D_{h}\left(\chi_{j}\right), \chi_{i}\right\rangle= & \int_{\Gamma^{h}} \int_{\Gamma^{h}} \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) \chi_{j}(\boldsymbol{y}) \chi_{i}(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{x}) \\
= & Z B_{1,1}^{i-1, j-1}+Z B_{0,1}^{i, j-1}-Z B_{1,1}^{i, j-1}+Z B_{1,0}^{i-1, j}-Z B_{1,1}^{i-1, j} \\
& +Z B_{0,0}^{i, j}-Z B_{0,1}^{i, j}-Z B_{1,0}^{i, j}+Z B_{1,1}^{i, j} . \tag{B.321}
\end{align*}
$$

It holds moreover that

$$
\begin{align*}
\left\langle Z_{h} D_{h}\left(\chi_{j}\right), \chi_{i}\right\rangle= & \int_{\Gamma^{h}} \int_{\Gamma^{h}} Z_{h}(\boldsymbol{x}) \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) \chi_{j}(\boldsymbol{y}) \chi_{i}(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{x}) \\
= & Z_{i-1}\left(Z B_{1,1}^{i-1, j-1}+Z B_{1,0}^{i-1, j}-Z B_{1,1}^{i-1, j}\right) \\
& +Z_{i}\left(Z B_{0,1}^{i, j-1}-Z B_{1,1}^{i, j-1}+Z B_{0,0}^{i, j}-Z B_{0,1}^{i, j}-Z B_{1,0}^{i, j}+Z B_{1,1}^{i, j}\right) . \tag{B.322}
\end{align*}
$$

We have also that

$$
\begin{align*}
\left\langle D_{h}^{*}\left(\chi_{j}\right), \chi_{i}\right\rangle= & \int_{\Gamma^{h}} \int_{\Gamma^{h}} \frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x}, \boldsymbol{y}) \chi_{j}(\boldsymbol{y}) \chi_{i}(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{x}) \\
= & Z B_{1,1}^{j-1, i-1}+Z B_{1,0}^{j-1, i}-Z B_{1,1}^{j-1, i}+Z B_{0,1}^{j, i-1}-Z B_{1,1}^{j, i-1} \\
& +Z B_{0,0}^{j, i}-Z B_{1,0}^{j, i}-Z B_{0,1}^{j, i}+Z B_{1,1}^{j, i} . \tag{B.323}
\end{align*}
$$

Similarly it occurs that

$$
\begin{align*}
\left\langle D_{h}^{*}\left(Z_{h} \chi_{j}\right), \chi_{i}\right\rangle= & \int_{\Gamma^{h}} \int_{\Gamma^{h}} Z_{h}(\boldsymbol{y}) \frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x}, \boldsymbol{y}) \chi_{j}(\boldsymbol{y}) \chi_{i}(\boldsymbol{x}) \mathrm{d} \gamma(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{x}) \\
= & Z_{j-1}\left(Z B_{1,1}^{j-1, i-1}+Z B_{1,0}^{j-1, i}-Z B_{1,1}^{j-1, i}\right) \\
& +Z_{j}\left(Z B_{0,1}^{j, i-1}-Z B_{1,1}^{j, i-1}+Z B_{0,0}^{j, i}-Z B_{1,0}^{j, i}-Z B_{0,1}^{j, i}+Z B_{1,1}^{j, i}\right) \tag{B.324}
\end{align*}
$$

And finally, for the hypersingular term we have that

$$
\begin{align*}
& \left\langle N_{h}\left(\chi_{j}\right), \chi_{i}\right\rangle=-\int_{\Gamma^{h}} \int_{\Gamma^{h}} G(\boldsymbol{x}, \boldsymbol{y})\left(\nabla \chi_{j}(\boldsymbol{y}) \times \boldsymbol{n}_{\boldsymbol{y}}\right)\left(\nabla \chi_{i}(\boldsymbol{x}) \times \boldsymbol{n}_{\boldsymbol{x}}\right) \mathrm{d} \gamma(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{x}) \\
& =-Z A_{0,0}^{i-1, j-1} \frac{\left(\boldsymbol{\tau}_{j-1} \times \boldsymbol{n}_{j-1}\right)}{\left|T_{j-1}\right|} \frac{\left(\boldsymbol{\tau}_{i-1} \times \boldsymbol{n}_{i-1}\right)}{\left|T_{i-1}\right|}+Z A_{0,0}^{i, j-1} \frac{\left(\boldsymbol{\tau}_{j-1} \times \boldsymbol{n}_{j-1}\right)}{\left|T_{j-1}\right|} \frac{\left(\boldsymbol{\tau}_{i} \times \boldsymbol{n}_{i}\right)}{\left|T_{i}\right|} \\
&  \tag{B.325}\\
& \quad+Z A_{0,0}^{i-1, j} \frac{\left(\boldsymbol{\tau}_{j} \times \boldsymbol{n}_{j}\right)}{\left|T_{j}\right|} \frac{\left(\boldsymbol{\tau}_{i-1} \times \boldsymbol{n}_{i-1}\right)}{\left|T_{i-1}\right|}-Z A_{0,0}^{i, j} \frac{\left(\boldsymbol{\tau}_{j} \times \boldsymbol{n}_{j}\right)}{\left|T_{j}\right|} \frac{\left(\boldsymbol{\tau}_{i} \times \boldsymbol{n}_{i}\right)}{\left|T_{i}\right|} .
\end{align*}
$$

We remark that these formulae hold when the segments $T_{i-1}$ and $T_{i}$, as well as the segments $T_{j-1}$ and $T_{j}$, exist and are adjacent.

It remains now to compute the integrals (B.307) and (B.308), which are calculated in two steps with a semi-numerical integration, i.e., the singular parts are calculated analytically and the other parts numerically. First the internal integral for $\boldsymbol{y}$ is computed, then the external one for $\boldsymbol{x}$. This can be expressed as

$$
\begin{align*}
Z A_{a, b} & =\int_{K}\left(\frac{s}{|K|}\right)^{a} Z F_{b}(\boldsymbol{x}) \mathrm{d} K(\boldsymbol{x})  \tag{B.326}\\
Z F_{b}(\boldsymbol{x}) & =\int_{L}\left(\frac{t}{|L|}\right)^{b} G(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d} L(\boldsymbol{y}) \tag{B.327}
\end{align*}
$$

and

$$
\begin{align*}
Z B_{a, b} & =\int_{K}\left(\frac{s}{|K|}\right)^{a} Z G_{b}(\boldsymbol{x}) \mathrm{d} K(\boldsymbol{x})  \tag{B.328}\\
Z G_{b}(\boldsymbol{x}) & =\int_{L}\left(\frac{t}{|L|}\right)^{b} \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d} L(\boldsymbol{y}) \tag{B.329}
\end{align*}
$$

This kind of integrals can be also used to compute the terms associated with the discretized solution $u_{h}$. Using an analogous notation as in (B.310), we have that

$$
\begin{equation*}
\mathcal{S}_{h}\left(\kappa_{j}\right)=\int_{\Gamma^{h}} G(\boldsymbol{x}, \boldsymbol{y}) \kappa_{j}(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y})=Z F_{0}^{j}(\boldsymbol{x}) . \tag{B.330}
\end{equation*}
$$

Similarly it holds that

$$
\begin{equation*}
\mathcal{S}_{h}\left(\chi_{j}\right)=\int_{\Gamma^{h}} G(\boldsymbol{x}, \boldsymbol{y}) \chi_{j}(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y})=Z F_{1}^{j-1}(\boldsymbol{x})+Z F_{0}^{j}(\boldsymbol{x})-Z F_{1}^{j}(\boldsymbol{x}), \tag{B.331}
\end{equation*}
$$

and

$$
\begin{align*}
\mathcal{S}_{h}\left(Z_{h} \chi_{j}\right) & =\int_{\Gamma^{h}} Z_{h}(\boldsymbol{y}) G(\boldsymbol{x}, \boldsymbol{y}) \chi_{j}(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y}) \\
& =Z_{j-1} Z F_{1}^{j-1}(\boldsymbol{x})+Z_{j}\left(Z F_{0}^{j}(\boldsymbol{x})-Z F_{1}^{j}(\boldsymbol{x})\right) . \tag{B.332}
\end{align*}
$$

The remaining term is computed as

$$
\begin{equation*}
\mathcal{D}_{h}\left(\chi_{j}\right)=\int_{\Gamma^{h}} \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) \chi_{j}(\boldsymbol{y}) \mathrm{d} \gamma(\boldsymbol{y})=Z G_{1}^{j-1}(\boldsymbol{x})+Z G_{0}^{j}(\boldsymbol{x})-Z G_{1}^{j}(\boldsymbol{x}) . \tag{B.333}
\end{equation*}
$$

## B.12.3 Numerical integration for the non-singular integrals

The numerical integration of the non-singular integrals of the boundary element calculations is performed by a two-point Gauss quadrature formula (cf., e.g., Abramowitz \& Stegun 1972). The points considered on each segment are denoted as

$$
\begin{array}{ll}
\boldsymbol{x}_{1}=\alpha_{1} \boldsymbol{r}_{1}^{K}+\alpha_{2} \boldsymbol{r}_{2}^{K}, & \boldsymbol{x}_{2}=\alpha_{2} \boldsymbol{r}_{1}^{K}+\alpha_{1} \boldsymbol{r}_{2}^{K}, \\
\boldsymbol{y}_{1}=\alpha_{1} \boldsymbol{r}_{1}^{L}+\alpha_{2} \boldsymbol{r}_{2}^{L}, & \boldsymbol{y}_{2}=\alpha_{2} \boldsymbol{r}_{1}^{L}+\alpha_{1} \boldsymbol{r}_{2}^{L}, \tag{B.335}
\end{array}
$$

where

$$
\begin{equation*}
\alpha_{1}=\frac{1}{2}\left(1+\frac{1}{\sqrt{3}}\right) \quad \text { and } \quad \alpha_{2}=\frac{1}{2}\left(1-\frac{1}{\sqrt{3}}\right) . \tag{B.336}
\end{equation*}
$$

When considering a function $\varphi: L \rightarrow \mathbb{C}$, this formula is given by

$$
\begin{equation*}
\int_{\boldsymbol{r}_{1}^{L}}^{\boldsymbol{r}_{2}^{L}}\left(\frac{t}{|L|}\right)^{b} \varphi(\boldsymbol{y}) \mathrm{d} L(\boldsymbol{y}) \approx \frac{|L|}{2}\left(\alpha_{2}^{b} \varphi\left(\boldsymbol{y}_{1}\right)+\alpha_{1}^{b} \varphi\left(\boldsymbol{y}_{2}\right)\right) . \tag{B.337}
\end{equation*}
$$

An equivalent formula is used when considering a function $\phi: K \rightarrow \mathbb{C}$, given by

$$
\begin{equation*}
\int_{\boldsymbol{r}_{1}^{K}}^{\boldsymbol{r}_{2}^{K}}\left(\frac{s}{|K|}\right)^{a} \phi(\boldsymbol{x}) \mathrm{d} K(\boldsymbol{x}) \approx \frac{|K|}{2}\left(\alpha_{2}^{a} \phi\left(\boldsymbol{x}_{1}\right)+\alpha_{1}^{a} \phi\left(\boldsymbol{x}_{2}\right)\right) . \tag{B.338}
\end{equation*}
$$

The Gauss quadrature formula can be extended straightforwardly to a function of two variables, $\Phi: K \times L \rightarrow \mathbb{C}$, using both formulas shown above. Therefore

$$
\begin{gather*}
\int_{\boldsymbol{r}_{1}^{K}}^{\boldsymbol{r}_{2}^{K}} \int_{\boldsymbol{r}_{1}^{L}}^{\boldsymbol{r}_{2}^{L}}\left(\frac{s}{|K|}\right)^{a}\left(\frac{t}{|L|}\right)^{b} \Phi(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d} L(\boldsymbol{y}) \mathrm{d} K(\boldsymbol{x}) \approx \frac{|K||L|}{4}\left(\alpha_{2}^{a+b} \Phi\left(\boldsymbol{x}_{1}, \boldsymbol{y}_{1}\right)\right. \\
\left.+\alpha_{2}^{a} \alpha_{1}^{b} \Phi\left(\boldsymbol{x}_{1}, \boldsymbol{y}_{2}\right)+\alpha_{1}^{a} \alpha_{2}^{b} \Phi\left(\boldsymbol{x}_{2}, \boldsymbol{y}_{1}\right)+\alpha_{1}^{a+b} \Phi\left(\boldsymbol{x}_{2}, \boldsymbol{y}_{2}\right)\right) \tag{B.339}
\end{gather*}
$$

The points on which the non-singular integrals have to be evaluated to perform the numerical integration are depicted in Figure B.14.

We have that the integrals on $K$, (B.326) and (B.328), are non-singular and thus evaluated numerically with the two-point Gauss quadrature formula (B.338).

For the integrals on $L$, (B.327) and (B.329), two different cases have to be taken into account. If the segments $K$ and $L$ are not close together, e.g., neither adjacent nor equal,


Figure B.14. Evaluation points for the numerical integration.
then (B.327) and (B.329) can also be numerically integrated using the formula (B.337), i.e., in the whole, the integrals $Z A_{a, b}$ and $Z B_{a, b}$ are calculated employing (B.339).

For the computation of the discretized solution $u_{h}$, the quadrature formula (B.337) is taken into account if $\boldsymbol{x} \notin \Gamma^{h}$. Otherwise we use the analytical formulae for singular integrals that are below.

The quadrature formula (B.337) is likewise used in the computation of the far field, namely for the discretization of the far-field pattern (B.169).

## B.12.4 Analytical integration for the singular integrals

If the segments $K$ and $L$ are close together, then the integrals (B.327) and (B.329) are calculated analytically, treating $x$ as a given parameter. They are specifically given by

$$
\begin{align*}
& Z F_{0}(\boldsymbol{x})=\int_{L} \frac{\ln R}{2 \pi} \mathrm{~d} L(\boldsymbol{y}),  \tag{B.340}\\
& Z F_{1}(\boldsymbol{x})=\int_{L} t \frac{\ln R}{2 \pi|L|} \mathrm{d} L(\boldsymbol{y}), \tag{B.341}
\end{align*}
$$

and

$$
\begin{align*}
& Z G_{0}(\boldsymbol{x})=\int_{L} \frac{\boldsymbol{R} \cdot \boldsymbol{n}_{L}}{2 \pi R^{2}} \mathrm{~d} L(\boldsymbol{y}),  \tag{B.342}\\
& Z G_{1}(\boldsymbol{x})=\int_{L} t \frac{\boldsymbol{R} \cdot \boldsymbol{n}_{L}}{2 \pi R^{2}|L|} \mathrm{d} L(\boldsymbol{y}) \tag{B.343}
\end{align*}
$$

a) Computation of $Z G_{0}(\boldsymbol{x})$

The integral (B.342) is closely related with Gauss's divergence theorem. If we consider an oriented surface differential element $\mathrm{d} \boldsymbol{\gamma}=\boldsymbol{n}_{L} \mathrm{~d} L(\boldsymbol{y})$ seen from point $\boldsymbol{x}$, then we can express the angle differential element by

$$
\begin{equation*}
\mathrm{d} \theta=\frac{\boldsymbol{R}}{R^{2}} \cdot \mathrm{~d} \boldsymbol{\gamma}=\frac{\boldsymbol{R} \cdot \boldsymbol{n}_{L}}{R^{2}} \mathrm{~d} L(\boldsymbol{y})=2 \pi \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) \mathrm{d} L(\boldsymbol{y}) . \tag{B.344}
\end{equation*}
$$

Integrating over segment $L$ yields the angle $\theta_{L}$, as expressed in (B.306), namely

$$
\begin{equation*}
\theta_{L}=\int_{L} \mathrm{~d} \theta \quad\left(-\pi \leq \theta_{L} \leq \pi\right) \tag{B.345}
\end{equation*}
$$

The angle $\theta_{L}$ is positive when the vectors $\boldsymbol{R}$ and $\boldsymbol{n}_{L}$ point towards the same side of $L$. Thus integral (B.342) is obtained by integrating (B.344), which yields

$$
\begin{equation*}
Z G_{0}(\boldsymbol{x})=\int_{L} \frac{\boldsymbol{R} \cdot \boldsymbol{n}_{L}}{2 \pi R^{2}} \mathrm{~d} L(\boldsymbol{y})=\frac{\theta_{L}}{2 \pi} . \tag{B.346}
\end{equation*}
$$

b) Computation of $Z F_{1}(\boldsymbol{x})$

For the integral (B.341) we have that

$$
\begin{align*}
Z F_{1}(\boldsymbol{x}) & =\frac{1}{2 \pi|L|} \int_{L} \ln (R)\left(\boldsymbol{R}-\boldsymbol{R}_{1}^{L}\right) \cdot \boldsymbol{\tau}_{L} \mathrm{~d} L(\boldsymbol{y}) \\
& =\frac{1}{2 \pi|L|} \int_{L} R \ln (R) \frac{\boldsymbol{R}}{R} \cdot \boldsymbol{\tau}_{L} \mathrm{~d} L(\boldsymbol{y})-\frac{\boldsymbol{R}_{1}^{L} \cdot \boldsymbol{\tau}_{L}}{|L|} Z F_{0}(\boldsymbol{x}) . \tag{B.347}
\end{align*}
$$

If we denote the primitive of $R \ln R$ that vanishes for $R=0$ by

$$
\begin{equation*}
v(R)=\frac{R^{2}}{2}\left(\ln R-\frac{1}{2}\right) \tag{B.348}
\end{equation*}
$$

then (B.347) can be rewritten as

$$
\begin{equation*}
Z F_{1}(\boldsymbol{x})=\frac{1}{2 \pi|L|} \int_{L} \frac{\partial v}{\partial t} \mathrm{~d} L(\boldsymbol{y})-\frac{\boldsymbol{R}_{1}^{L} \cdot \boldsymbol{\tau}_{L}}{|L|} Z F_{0}(\boldsymbol{x}) \tag{B.349}
\end{equation*}
$$

and therefore $Z F_{1}(\boldsymbol{x})$ can be finally calculated as

$$
\begin{equation*}
Z F_{1}(\boldsymbol{x})=\frac{v\left(R_{2}^{L}\right)-v\left(R_{1}^{L}\right)}{2 \pi|L|}-\frac{\boldsymbol{R}_{1}^{L} \cdot \boldsymbol{\tau}_{L}}{|L|} Z F_{0}(\boldsymbol{x}) . \tag{B.350}
\end{equation*}
$$

c) Computation of $Z F_{0}(\boldsymbol{x})$

We consider now a function $w=w(R)$ that is bounded in the vicinity of zero and is such that

$$
\begin{equation*}
\Delta w=\frac{1}{R} \frac{\mathrm{~d}}{\mathrm{~d} R}\left(R \frac{\mathrm{~d} w}{\mathrm{~d} R}\right)=\ln R \tag{B.351}
\end{equation*}
$$

Hence, taking a primitive that vanishes at zero, it holds that

$$
\begin{equation*}
\frac{\mathrm{d} w}{\mathrm{~d} R}=\frac{R}{2}\left(\ln R-\frac{1}{2}\right) . \tag{B.352}
\end{equation*}
$$

We turn now to the local orthonormal variables $t$ and $n$, where

$$
\begin{equation*}
\boldsymbol{R}=\boldsymbol{R}_{1}^{L}+t \boldsymbol{\tau}_{L}+n \boldsymbol{n}_{L} . \tag{B.353}
\end{equation*}
$$

Since the Laplace operator $\Delta$ is invariant under orthonormal variable changes, we have from (B.351) that

$$
\begin{equation*}
Z F_{0}(\boldsymbol{x})=\frac{1}{2 \pi} \int_{L}\left(\frac{\partial^{2} w}{\partial t^{2}}+\frac{\partial^{2} w}{\partial n^{2}}\right) \mathrm{d} L(\boldsymbol{y}) \tag{B.354}
\end{equation*}
$$

By considering (B.352) we obtain that

$$
\begin{equation*}
\nabla w=\frac{\mathrm{d} w}{\mathrm{~d} R} \frac{\boldsymbol{R}}{R}=\frac{1}{2}\left(\ln R-\frac{1}{2}\right) \boldsymbol{R} \tag{B.355}
\end{equation*}
$$

$$
\begin{align*}
\frac{\partial w}{\partial t} & =\nabla w \cdot \boldsymbol{\tau}_{L}=\frac{1}{2}\left(\ln R-\frac{1}{2}\right) \boldsymbol{R} \cdot \boldsymbol{\tau}_{L}  \tag{B.356}\\
\frac{\partial w}{\partial n} & =\nabla w \cdot \boldsymbol{n}_{L}=\frac{1}{2}\left(\ln R-\frac{1}{2}\right) \boldsymbol{R} \cdot \boldsymbol{n}_{L}  \tag{B.357}\\
\frac{\partial^{2} w}{\partial n^{2}} & =\frac{1}{2} \boldsymbol{R} \cdot \boldsymbol{n}_{L} \frac{\partial}{\partial n} \ln R+\frac{1}{2}\left(\ln R-\frac{1}{2}\right) \tag{B.358}
\end{align*}
$$

The first integral in (B.354) is therefore given by

$$
\begin{equation*}
\frac{1}{2 \pi} \int_{L} \frac{\partial^{2} w}{\partial t^{2}} \mathrm{~d} L(\boldsymbol{y})=\frac{1}{4 \pi}\left(\ln R_{2}^{L}-\frac{1}{2}\right) \boldsymbol{R}_{2}^{L} \cdot \boldsymbol{\tau}_{L}-\frac{1}{4 \pi}\left(\ln R_{1}^{L}-\frac{1}{2}\right) \boldsymbol{R}_{1}^{L} \cdot \boldsymbol{\tau}_{L} \tag{B.359}
\end{equation*}
$$

while for the second one, due (B.305), it holds that

$$
\begin{equation*}
\frac{1}{2 \pi} \int_{L} \frac{\partial^{2} w}{\partial n^{2}} \mathrm{~d} L(\boldsymbol{y})=\frac{d_{L}}{2} Z G_{0}(\boldsymbol{x})+\frac{1}{2} Z F_{0}(\boldsymbol{x})-\frac{|L|}{8 \pi} \tag{B.360}
\end{equation*}
$$

From (B.346), (B.354), (B.359), and (B.360), we obtain the desired expression

$$
\begin{equation*}
Z F_{0}(\boldsymbol{x})=\frac{1}{2 \pi}\left(\boldsymbol{R}_{2}^{L} \cdot \boldsymbol{\tau}_{L} \ln R_{2}^{L}-\boldsymbol{R}_{1}^{L} \cdot \boldsymbol{\tau}_{L} \ln R_{1}^{L}-|L|+d_{L} \theta_{L}\right) . \tag{B.361}
\end{equation*}
$$

d) Computation of $Z G_{1}(\boldsymbol{x})$

The integral (B.343) is found straightforwardly by replacing (B.304), yielding

$$
\begin{align*}
Z G_{1}(\boldsymbol{x}) & =\int_{L} \frac{\boldsymbol{R} \cdot \boldsymbol{n}_{L}}{2 \pi R^{2}|L|}\left(\boldsymbol{R}-\boldsymbol{R}_{1}^{L}\right) \cdot \boldsymbol{\tau}_{L} \mathrm{~d} L(\boldsymbol{y}) \\
& =\int_{L} \frac{\boldsymbol{R} \cdot \boldsymbol{n}_{L}}{2 \pi R^{2}|L|} \boldsymbol{R} \cdot \boldsymbol{\tau}_{L} \mathrm{~d} L(\boldsymbol{y})-\frac{\boldsymbol{R}_{1}^{L} \cdot \boldsymbol{\tau}_{L}}{|L|} Z G_{0}(\boldsymbol{x}) \tag{B.362}
\end{align*}
$$

Due (B.305) we have then

$$
\begin{equation*}
Z G_{1}(\boldsymbol{x})=\frac{\ln \left(R_{2}^{L} / R_{1}^{L}\right)}{2 \pi|L|} \boldsymbol{R}_{1}^{L} \cdot \boldsymbol{n}_{L}-\frac{\boldsymbol{R}_{1}^{L} \cdot \boldsymbol{\tau}_{L}}{|L|} Z G_{0}(\boldsymbol{x}) \tag{B.363}
\end{equation*}
$$

e) Final computation of the singular integrals

In conclusion, the singular integrals (B.327) and (B.329) are computed using the formulae (B.346), (B.350), (B.361), and (B.363).

It should be observed that $Z B_{a, b}=0$ when the segments coincide, i.e., when $K=L$, since in this case $d_{L}=0$, and thus (B.346) and (B.363) become zero.

## B.13 Benchmark problem

As benchmark problem we consider the exterior circle problem (B.171), whose domain is shown in Figure B.8. The exact solution of this problem is stated in (B.193), and the idea is to retrieve it numerically with the integral equation techniques and the boundary element method described throughout this chapter.

For the computational implementation and the numerical resolution of the benchmark problem, we consider only the first integral equation of the extension-by-zero alternative (B.129), which is given in terms of boundary layer potentials by (B.208). The linear system (B.271) resulting from the discretization (B.269) of its variational formulation (B.250) is solved computationally with finite boundary elements of type $\mathbb{P}_{1}$ by using subroutines programmed in Fortran 90, by generating the mesh $\Gamma^{h}$ of the boundary with the free software Gmsh 2.4, and by representing graphically the results in Matlab 7.5 (R2007b).

We consider a radius $R=1$ and a constant impedance $Z=0.8$. The discretized boundary curve $\Gamma^{h}$ has $I=120$ segments and a discretization step $h=0.05235$, being

$$
\begin{equation*}
h=\max _{1 \leq j \leq I}\left|T_{j}\right| . \tag{B.364}
\end{equation*}
$$

We observe that $h \approx 2 \pi / I$. As the known field without obstacle we take

$$
\begin{equation*}
u_{W}(r, \theta)=\frac{e^{i \theta}}{r}=\frac{x_{1}+i x_{2}}{x_{1}^{2}+x_{2}^{2}}, \tag{B.365}
\end{equation*}
$$

which implies that the impedance data function is given by

$$
\begin{equation*}
f_{z}(\theta)=-\frac{\partial u_{W}}{\partial r}(R, \theta)-Z u_{W}(R, \theta)=-\frac{e^{i \theta}}{R^{2}}(Z R-1) \tag{B.366}
\end{equation*}
$$

The exact solution of the problem and its trace on the boundary are thus given by

$$
\begin{equation*}
u(\boldsymbol{x})=-u_{W}(r, \theta)=-\frac{e^{i \theta}}{r} \quad \text { and } \quad \mu(\theta)=-u_{W}(R, \theta)=-\frac{e^{i \theta}}{R} \tag{B.367}
\end{equation*}
$$

The numerically calculated trace of the solution $\mu_{h}$ of the benchmark problem, which was computed by using the boundary element method, is depicted in Figure B.15. In the same manner, the numerical solution $u_{h}$ is illustrated in Figures B. 16 and B.17. It can be observed that the numerical solution is quite close to the exact one.


Figure B.15. Numerically computed trace of the solution $\mu_{h}$.


Figure B.16. Contour plot of the numerically computed solution $u_{h}$.


Figure B.17. Oblique view of the numerically computed solution $u_{h}$.

We define the relative error of the trace of the solution as

$$
\begin{equation*}
E_{2}\left(h, \Gamma^{h}\right)=\frac{\left\|\Pi_{h} \mu-\mu_{h}\right\|_{L^{2}\left(\Gamma^{h}\right)}}{\left\|\Pi_{h} \mu\right\|_{L^{2}\left(\Gamma^{h}\right)}} \tag{B.368}
\end{equation*}
$$

where $\Pi_{h} \mu$ denotes the Lagrange interpolating function of the exact solution's trace $\mu$, i.e.,

$$
\begin{equation*}
\Pi_{h} \mu(\boldsymbol{x})=\sum_{j=1}^{I} \mu\left(\boldsymbol{r}_{j}\right) \chi_{j}(\boldsymbol{x}) \quad \text { and } \quad \mu_{h}(\boldsymbol{x})=\sum_{j=1}^{I} \mu_{j} \chi_{j}(\boldsymbol{x}) \quad \text { for } \boldsymbol{x} \in \Gamma^{h} . \tag{B.369}
\end{equation*}
$$

It holds therefore that

$$
\begin{equation*}
\left\|\Pi_{h} \mu-\mu_{h}\right\|_{L^{2}\left(\Gamma^{h}\right)}^{2}=(\tilde{\boldsymbol{\mu}}-\boldsymbol{\mu})^{*} \boldsymbol{A}(\tilde{\boldsymbol{\mu}}-\boldsymbol{\mu}) \quad \text { and } \quad\left\|\Pi_{h} \mu\right\|_{L^{2}\left(\Gamma^{h}\right)}^{2}=\tilde{\boldsymbol{\mu}}^{*} \boldsymbol{A} \tilde{\boldsymbol{\mu}} \tag{B.370}
\end{equation*}
$$

where $\mu\left(\boldsymbol{r}_{j}\right)$ and $\mu_{j}$ are respectively the elements of vectors $\tilde{\boldsymbol{\mu}}$ and $\boldsymbol{\mu}$, for $1 \leq j \leq I$, and where the elements $a_{i j}$ of the matrix $\boldsymbol{A}$ are specified in (B.316) and given by

$$
\begin{equation*}
a_{i j}=\left\langle\chi_{j}, \chi_{i}\right\rangle \quad \text { for } 1 \leq i, j \leq I . \tag{B.371}
\end{equation*}
$$

In our case, for a step $h=0.05235$, we obtained a relative error of $E_{2}\left(h, \Gamma^{h}\right)=0.004571$.

Similarly as for the trace, we define the relative error of the solution as

$$
\begin{equation*}
E_{\infty}\left(h, \Omega_{L}\right)=\frac{\left\|u-u_{h}\right\|_{L^{\infty}\left(\Omega_{L}\right)}}{\|u\|_{L^{\infty}\left(\Omega_{L}\right)}} \tag{B.372}
\end{equation*}
$$

being $\Omega_{L}=\left\{\boldsymbol{x} \in \Omega_{e}:\|\boldsymbol{x}\|_{\infty}<L\right\}$ for $L>0$, and where

$$
\begin{equation*}
\left\|u-u_{h}\right\|_{L^{\infty}\left(\Omega_{L}\right)}=\max _{\boldsymbol{x} \in \Omega_{L}}\left|u(\boldsymbol{x})-u_{h}(\boldsymbol{x})\right| \quad \text { and } \quad\|u\|_{L^{\infty}\left(\Omega_{L}\right)}=\max _{\boldsymbol{x} \in \Omega_{L}}|u(\boldsymbol{x})| . \tag{B.373}
\end{equation*}
$$

We consider $L=3$ and approximate $\Omega_{L}$ by a triangular finite element mesh of refinement $h$ near the boundary. For $h=0.05235$, the relative error that we obtained for the solution was $E_{\infty}\left(h, \Omega_{L}\right)=0.004870$.

The results for different mesh refinements, i.e., for different numbers of segments $I$ and discretization steps $h$ for $\Gamma^{h}$, are listed in Table B.1. These results are illustrated graphically in Figure B.18. It can be observed that the relative errors are approximately of order $h^{2}$.

TABLE B.1. Relative errors for different mesh refinements.

| $I$ | $h$ | $E_{2}\left(h, \Gamma^{h}\right)$ | $E_{\infty}\left(h, \Omega_{L}\right)$ |
| :---: | :---: | :---: | :---: |
| 12 | 0.5176 | $4.330 \cdot 10^{-1}$ | $4.330 \cdot 10^{-1}$ |
| 40 | 0.1569 | $4.100 \cdot 10^{-2}$ | $4.100 \cdot 10^{-2}$ |
| 80 | 0.07852 | $1.027 \cdot 10^{-2}$ | $1.082 \cdot 10^{-2}$ |
| 120 | 0.05235 | $4.571 \cdot 10^{-3}$ | $4.870 \cdot 10^{-3}$ |
| 240 | 0.02618 | $1.143 \cdot 10^{-3}$ | $1.239 \cdot 10^{-3}$ |
| 500 | 0.01257 | $2.633 \cdot 10^{-4}$ | $2.879 \cdot 10^{-4}$ |
| 1000 | 0.006283 | $6.581 \cdot 10^{-5}$ | $7.222 \cdot 10^{-5}$ |



Figure B.18. Logarithmic plots of the relative errors versus the discretization step.

