# FULL-SPACE IMPEDANCE HELMHOLTZ PROBLEM

## E.1 Introduction

In this appendix we study the perturbed full-space or free-space impedance Helmholtz problem, also known as the exterior impedance Helmholtz problem in 3D, using integral equation techniques and the boundary element method.

We consider the problem of the Helmholtz equation in three dimensions on the exterior of a bounded obstacle with an impedance boundary condition. The perturbed full-plane impedance Helmholtz problem is a wave scattering problem around a bounded three-dimensional obstacle. In acoustic obstacle scattering the impedance boundary-value problem appears when we suppose that the normal velocity is proportional to the excess pressure on the boundary of the impenetrable obstacle. The special case of frequency zero for the volume waves has been treated already in Appendix D, since then we deal with the Laplace equation. The two-dimensional Helmholtz problem was treated thoroughly in Appendix C.

The main references for the problem treated herein are Kress (2002), Lenoir (2005), Nédélec (2001), and Terrasse & Abboud (2006). Additional related books and doctorate theses are the ones of Chen & Zhou (1992), Colton & Kress (1983), Ha-Duong (1987), Hsiao & Wendland (2008), Kirsch & Grinberg (2008), Rjasanow & Steinbach (2007), and Steinbach (2008). Articles where the Helmholtz equation with an impedance boundary condition is taken into account are Ahner (1978), Angell & Kleinman (1982), Angell & Kress (1984), Angell, Kleinman & Hettlich (1990), Dassios & Kamvyssas (1997), Krutitskii (2003*a*,*b*), and Lin (1987). Theoretical details on transmission problems are given in Costabel & Stephan (1985). The inverse problem is studied in Colton & Kirsch (1981). The boundary element calculations can be found in the report of Bendali & Devys (1986) and in the article by Bendali & Souilah (1994). Applications for the impedance Helmholtz problem can be found, among others, for acoustics (Morse & Ingard 1961) and for ultrasound imaging (Ammari 2008).

The Helmholtz equation allows the propagation of volume waves inside the considered domain, and when supplied with an impedance boundary condition it allows also the propagation of surface waves along the domain's boundary. The main difficulty in the numerical treatment and resolution of our problem is the fact that the exterior domain is unbounded. We solve it therefore with integral equation techniques and the boundary element method, which require the knowledge of the Green's function.

This appendix is structured in 14 sections, including this introduction. The direct scattering problem of the Helmholtz equation in a three-dimensional exterior domain with an impedance boundary condition is presented in Section E.2. The Green's function and its far-field expression are computed respectively in Sections E.3 and E.4. Extending the direct scattering problem towards a transmission problem, as done in Section E.5, allows its resolution by using integral equation techniques, which is discussed in Section E.6. These techniques allow also to represent the far field of the solution, as shown in Section E.7. A particular problem that takes as domain the exterior of a sphere is solved analytically in

Section E.8. The appropriate function spaces and some existence and uniqueness results for the solution of the problem are presented in Section E.9. The dissipative problem is studied in Section E.10. By means of the variational formulation developed in Section E.11, the obtained integral equation is discretized using the boundary element method, which is described in Section E.12. The boundary element calculations required to build the matrix of the linear system resulting from the numerical discretization are explained in Section E.13. Finally, in Section E.14 a benchmark problem based on the exterior sphere problem is solved numerically.

## E.2 Direct scattering problem

We consider the direct scattering problem of linear time-harmonic acoustic waves on an exterior domain $\Omega_e \subset \mathbb{R}^3$, lying outside a bounded obstacle $\Omega_i$ and having a regular boundary $\Gamma = \partial\Omega_e = \partial\Omega_i$, as shown in Figure E.1. The time convention $e^{-i\omega t}$ is taken and the incident field $u_I$ is known. The goal is to find the scattered field $u$ as a solution to the Helmholtz equation in $\Omega_e$, satisfying an outgoing radiation condition, and such that the total field $u_T$, decomposed as $u_T = u_I + u$, satisfies a homogeneous impedance boundary condition on the regular boundary $\Gamma$ (e.g., of class $C^2$). The unit normal $\boldsymbol{n}$ is taken outwardly oriented of $\Omega_e$. A given wave number $k > 0$ is considered, which depends on the pulsation $\omega$ and the speed of wave propagation $c$ through the ratio $k = \omega/c$.

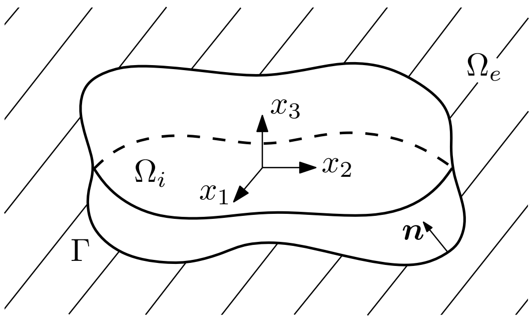



FIGURE E.1. Perturbed full-space impedance Helmholtz problem domain.

The total field $u_T$ satisfies thus the Helmholtz equation

$$\Delta u_T + k^2 u_T = 0 \qquad \text{in } \Omega_e, \tag{E.1}$$

which is also satisfied by the incident field $u_I$ and the scattered field $u$, due linearity. For the total field $u_T$ we take the homogeneous impedance boundary condition

$$-\frac{\partial u_T}{\partial n} + Z u_T = 0 \qquad \text{on } \Gamma, \tag{E.2}$$

where $Z$ is the impedance on the boundary. If $Z = 0$ or $Z = \infty$, then we retrieve respectively the classical Neumann or Dirichlet boundary conditions. In general, we consider a complex-valued impedance $Z(\boldsymbol{x})$ that depends on the position $\boldsymbol{x}$ and that may depend also on the pulsation $\omega$. The scattered field $u$ satisfies the non-homogeneous impedance boundary condition

$$-\frac{\partial u}{\partial n} + Zu = f_z \qquad \text{on } \Gamma, \tag{E.3}$$

where the impedance data function $f_z$ is given by

$$f_z = \frac{\partial u_I}{\partial n} - Zu_I \qquad \text{on } \Gamma. \tag{E.4}$$

The solutions of the Helmholtz equation (E.1) in the full-space $\mathbb{R}^3$ are the so-called plane waves, which we take as the known incident field $u_I$. Up to an arbitrary multiplicative factor, they are given by

$$u_I(\boldsymbol{x}) = e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \qquad (\boldsymbol{k}\cdot\boldsymbol{k}) = k^2, \tag{E.5}$$

where the wave propagation vector $\boldsymbol{k}$ is taken such that $\boldsymbol{k} \in \mathbb{R}^3$ to obtain physically admissible waves which do not explode towards infinity. By considering a parametrization through the angles of incidence $\theta_I$ and $\varphi_I$ for $0 \leq \theta_I \leq \pi$ and $-\pi < \varphi_I \leq \pi$, we can express the wave propagation vector as $\boldsymbol{k} = (-k\sin\theta_I\cos\varphi_I, -k\sin\theta_I\sin\varphi_I, -k\cos\theta_I)$. The plane waves can be thus also represented as

$$u_I(\boldsymbol{x}) = e^{-ik(x_1\sin\theta_I\cos\varphi_I + x_2\sin\theta_I\sin\varphi_I + x_3\cos\theta_I)}. \tag{E.6}$$

An outgoing radiation condition is also imposed for the scattered field $u$, which specifies its decaying behavior at infinity and eliminates the non-physical solutions. It is known as a Sommerfeld radiation condition and is stated either as

$$\frac{\partial u}{\partial r} - iku = \mathcal{O}\left(\frac{1}{r^2}\right) \tag{E.7}$$

for $r = |\boldsymbol{x}|$, or, for some constant $C > 0$, by

$$\left|\frac{\partial u}{\partial r} - iku\right| \leq \frac{C}{r^2} \qquad \text{as } r \to \infty. \tag{E.8}$$

Alternatively it can be also expressed as

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - iku\right) = 0, \tag{E.9}$$

or even as

$$\frac{\partial u}{\partial r} - iku = \mathcal{O}\left(\frac{1}{r^\alpha}\right) \qquad \text{for } 1 < \alpha < 3. \tag{E.10}$$

Likewise, a weaker and more general formulation of this radiation condition is

$$\lim_{R\to\infty} \int_{S_R} \left|\frac{\partial u}{\partial r} - iku\right|^2 \mathrm{d}\gamma = 0, \tag{E.11}$$

where $S_R = \{\boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x}| = R\}$ is the sphere of radius $R$ that is centered at the origin. We remark that an ingoing radiation condition would have the opposite sign, namely

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} + iku\right) = 0. \tag{E.12}$$

The perturbed full-space impedance Helmholtz problem can be finally stated as

$$\begin{cases} \text{Find } u : \Omega_e \to \mathbb{C} \text{ such that} \\ \Delta u + k^2 u = 0 & \text{in } \Omega_e, \\ -\dfrac{\partial u}{\partial n} + Zu = f_z & \text{on } \Gamma, \\ \left| \dfrac{\partial u}{\partial r} - iku \right| \leq \dfrac{C}{r^2} & \text{as } r \to \infty. \end{cases} \tag{E.13}$$

## E.3 Green's function

The Green's function represents the response of the unperturbed system (without an obstacle) to a Dirac mass. It corresponds to a function $G$, which depends on a fixed source point $\boldsymbol{x} \in \mathbb{R}^3$ and an observation point $\boldsymbol{y} \in \mathbb{R}^3$. The Green's function is computed in the sense of distributions for the variable $\boldsymbol{y}$ in the full-space $\mathbb{R}^3$ by placing at the right-hand side of the Helmholtz equation a Dirac mass $\delta_{\boldsymbol{x}}$, centered at the point $\boldsymbol{x}$. It is therefore a solution $G(\boldsymbol{x}, \cdot) : \mathbb{R}^3 \to \mathbb{C}$ for the radiation problem of a point source, namely

$$\Delta_{\boldsymbol{y}} G(\boldsymbol{x}, \boldsymbol{y}) + k^2 G(\boldsymbol{x}, \boldsymbol{y}) = \delta_{\boldsymbol{x}}(\boldsymbol{y}) \qquad \text{in } \mathcal{D}'(\mathbb{R}^3). \tag{E.14}$$

The solution of this equation is not unique, and therefore its behavior at infinity has to be specified. For this purpose we impose on the Green's function also the outgoing radiation condition (E.8).

Due to the radial symmetry of the problem (E.14), it is natural to look for solutions in the form $G = G(r)$, where $r = |\boldsymbol{y} - \boldsymbol{x}|$. By considering only the radial component, the Helmholtz equation in $\mathbb{R}^3$ becomes

$$\frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} \left( r^2 \frac{\mathrm{d}G}{\mathrm{d}r} \right) + k^2 G = 0, \qquad r > 0. \tag{E.15}$$

Replacing now $z = kr$ and considering $\psi(z) = G(r)$ yields $\frac{\mathrm{d}G}{\mathrm{d}r} = k \frac{\mathrm{d}\psi}{\mathrm{d}z}$ and consequently

$$k^2 \frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} + \frac{2k^2}{z} \frac{\mathrm{d}\psi}{\mathrm{d}z} + k^2 \psi = 0, \tag{E.16}$$

which is equivalent to the zeroth order spherical Bessel differential equation (vid. Subsection A.2.6)

$$z^2 \frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} + 2z \frac{\mathrm{d}\psi}{\mathrm{d}z} + z^2 \psi = 0. \tag{E.17}$$

Independent solutions for this equation are the zeroth order spherical Bessel functions of the first and second kinds, $j_0(z)$ and $y_0(z)$, and equally the zeroth order spherical Hankel functions of the first and second kinds, $h_0^{(1)}(z)$ and $h_0^{(2)}(z)$. The latter satisfy respectively the outgoing and ingoing radiation conditions and are expressed by

$$h_0^{(1)}(z) = -\frac{i}{z} e^{iz}, \qquad h_0^{(2)}(z) = \frac{i}{z} e^{-iz}. \tag{E.18}$$

Thus the solution of (E.17) is given by

$$\psi(z) = \alpha \frac{e^{iz}}{z} + \beta \frac{e^{-iz}}{z}, \qquad \alpha, \beta \in \mathbb{C}, \tag{E.19}$$

and consequently

$$G(r) = \alpha \frac{e^{ikr}}{r} + \beta \frac{e^{-ikr}}{r}, \qquad \alpha, \beta \in \mathbb{C}, \tag{E.20}$$

where $\alpha$ and $\beta$ are different than before, but still arbitrary. An outgoing wave behavior for the Green's function implies that $\beta = 0$, due (E.8). We observe from (E.18) that the singularity of the Green's function has the form $1/z$. The multiplicative constant $\alpha$ can be thus determined in the same way as for the Green's function of the Laplace equation in (D.17) by means of a computation in the sense of distributions for (E.14). The unique radial outgoing fundamental solution of the Helmholtz equation turns out to be

$$G(r) = -\frac{e^{ikr}}{4\pi r} = -\frac{ik}{4\pi} h_0^{(1)}(kr). \tag{E.21}$$

The Green's function for outgoing waves is then finally given by

$$G(\boldsymbol{x}, \boldsymbol{y}) = -\frac{e^{ik|\boldsymbol{y}-\boldsymbol{x}|}}{4\pi|\boldsymbol{y}-\boldsymbol{x}|} = -\frac{ik}{4\pi} h_0^{(1)}\big(k|\boldsymbol{y}-\boldsymbol{x}|\big). \tag{E.22}$$

We remark that the Green's function for ingoing waves would have been

$$G(\boldsymbol{x}, \boldsymbol{y}) = \frac{e^{-ik|\boldsymbol{y}-\boldsymbol{x}|}}{4\pi|\boldsymbol{y}-\boldsymbol{x}|} = -\frac{ik}{4\pi} h_0^{(2)}\big(k|\boldsymbol{y}-\boldsymbol{x}|\big). \tag{E.23}$$

To compute the derivatives of the Green's function we require some additional properties of spherical Hankel functions. It holds that

$$\frac{\mathrm{d}}{\mathrm{d}z} h_0^{(1)}(z) = -h_1^{(1)}(z), \qquad \frac{\mathrm{d}}{\mathrm{d}z} h_0^{(2)}(z) = -h_1^{(2)}(z), \tag{E.24}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}z} h_1^{(1)}(z) = h_0^{(1)}(z) - \frac{2}{z} h_1^{(1)}(z), \qquad \frac{\mathrm{d}}{\mathrm{d}z} h_1^{(2)}(z) = h_0^{(2)}(z) - \frac{2}{z} h_1^{(2)}(z), \tag{E.25}$$

where $h_1^{(1)}(z)$ and $h_1^{(2)}(z)$ denote the first order spherical Hankel functions of the first and second kinds, respectively, which are expressed as

$$h_1^{(1)}(z) = \left(-\frac{1}{z} - \frac{i}{z^2}\right) e^{iz}, \qquad h_1^{(2)}(z) = \left(-\frac{1}{z} + \frac{i}{z^2}\right) e^{-iz}. \tag{E.26}$$

The gradient of the Green's function (E.22) is therefore given by

$$\nabla_{\boldsymbol{y}} G(\boldsymbol{x}, \boldsymbol{y}) = \frac{e^{ik|\boldsymbol{y}-\boldsymbol{x}|}}{4\pi}\big(1 - ik|\boldsymbol{y}-\boldsymbol{x}|\big)\frac{\boldsymbol{y}-\boldsymbol{x}}{|\boldsymbol{y}-\boldsymbol{x}|^3} = \frac{ik^2}{4\pi} h_1^{(1)}\big(k|\boldsymbol{y}-\boldsymbol{x}|\big)\frac{\boldsymbol{y}-\boldsymbol{x}}{|\boldsymbol{y}-\boldsymbol{x}|}, \tag{E.27}$$

and the gradient with respect to the $\boldsymbol{x}$ variable by

$$\nabla_{\boldsymbol{x}} G(\boldsymbol{x}, \boldsymbol{y}) = \frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi}\big(1 - ik|\boldsymbol{x}-\boldsymbol{y}|\big)\frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^3} = \frac{ik^2}{4\pi} h_1^{(1)}\big(k|\boldsymbol{x}-\boldsymbol{y}|\big)\frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|}. \tag{E.28}$$

The double-gradient matrix is given by

$$\begin{aligned} \nabla_{\boldsymbol{x}} \nabla_{\boldsymbol{y}} G(\boldsymbol{x}, \boldsymbol{y}) = {} & \frac{ik^2}{4\pi} h_1^{(1)}\big(k|\boldsymbol{x}-\boldsymbol{y}|\big)\left(-\frac{\boldsymbol{I}}{|\boldsymbol{x}-\boldsymbol{y}|} + 3\frac{(\boldsymbol{x}-\boldsymbol{y})\otimes(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^3}\right) \\ & - \frac{ik^3}{4\pi} h_0^{(1)}\big(k|\boldsymbol{x}-\boldsymbol{y}|\big)\frac{(\boldsymbol{x}-\boldsymbol{y})\otimes(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2}, \end{aligned} \tag{E.29}$$

where $\boldsymbol{I}$ denotes a $3 \times 3$ identity matrix and where $\otimes$ denotes the dyadic or outer product of two vectors, which results in a matrix and is defined in (A.572).

We note that the Green's function (E.22) is symmetric in the sense that

$$G(\boldsymbol{x},\boldsymbol{y}) = G(\boldsymbol{y},\boldsymbol{x}), \tag{E.30}$$

and it fulfills similarly

$$\nabla_{\boldsymbol{y}} G(\boldsymbol{x},\boldsymbol{y}) = \nabla_{\boldsymbol{y}} G(\boldsymbol{y},\boldsymbol{x}) = -\nabla_{\boldsymbol{x}} G(\boldsymbol{x},\boldsymbol{y}) = -\nabla_{\boldsymbol{x}} G(\boldsymbol{y},\boldsymbol{x}), \tag{E.31}$$

and

$$\nabla_{\boldsymbol{x}} \nabla_{\boldsymbol{y}} G(\boldsymbol{x},\boldsymbol{y}) = \nabla_{\boldsymbol{y}} \nabla_{\boldsymbol{x}} G(\boldsymbol{x},\boldsymbol{y}) = \nabla_{\boldsymbol{x}} \nabla_{\boldsymbol{y}} G(\boldsymbol{y},\boldsymbol{x}) = \nabla_{\boldsymbol{y}} \nabla_{\boldsymbol{x}} G(\boldsymbol{y},\boldsymbol{x}). \tag{E.32}$$

Furthermore, due the exponential decrease of the spherical Hankel functions at infinity, we observe that the expression (E.22) of the Green's function for outgoing waves is still valid if a complex wave number $k \in \mathbb{C}$ such that $\mathfrak{Im}\{k\} > 0$ is used, which holds also for its derivatives (E.27), (E.28), and (E.29). In the case of ingoing waves, the expression (E.23) and its derivatives are valid if a complex wave number $k \in \mathbb{C}$ now such that $\mathfrak{Im}\{k\} < 0$ is taken into account.

## E.4 Far field of the Green's function

The far field of the Green's function describes its asymptotic behavior at infinity, i.e., when $|\boldsymbol{x}| \to \infty$ and assuming that $\boldsymbol{y}$ is fixed. By using a Taylor expansion we obtain that

$$|\boldsymbol{x}-\boldsymbol{y}| = |\boldsymbol{x}|\left(1 - 2\,\frac{\boldsymbol{y}\cdot\boldsymbol{x}}{|\boldsymbol{x}|^2} + \frac{|\boldsymbol{y}|^2}{|\boldsymbol{x}|^2}\right)^{1/2} = |\boldsymbol{x}| - \frac{\boldsymbol{y}\cdot\boldsymbol{x}}{|\boldsymbol{x}|} + \mathcal{O}\left(\frac{1}{|\boldsymbol{x}|}\right). \tag{E.33}$$

A similar expansion yields

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} = \frac{1}{|\boldsymbol{x}|} + \mathcal{O}\left(\frac{1}{|\boldsymbol{x}|^2}\right), \tag{E.34}$$

and we have also that

$$e^{ik|\boldsymbol{x}-\boldsymbol{y}|} = e^{ik|\boldsymbol{x}|} e^{-ik\boldsymbol{y}\cdot\boldsymbol{x}/|\boldsymbol{x}|}\left(1 + \mathcal{O}\left(\frac{1}{|\boldsymbol{x}|}\right)\right). \tag{E.35}$$

We express the point $\boldsymbol{x}$ as $\boldsymbol{x} = |\boldsymbol{x}|\,\hat{\boldsymbol{x}}$, being $\hat{\boldsymbol{x}}$ a unitary vector. The far field of the Green's function, as $|\boldsymbol{x}| \to \infty$, is thus given by

$$G^{ff}(\boldsymbol{x},\boldsymbol{y}) = -\frac{e^{ik|\boldsymbol{x}|}}{4\pi|\boldsymbol{x}|} e^{-ik\hat{\boldsymbol{x}}\cdot\boldsymbol{y}}. \tag{E.36}$$

Similarly, as $|\boldsymbol{x}| \to \infty$, we have for its gradient with respect to $\boldsymbol{y}$, that

$$\nabla_{\boldsymbol{y}} G^{ff}(\boldsymbol{x},\boldsymbol{y}) = \frac{ike^{ik|\boldsymbol{x}|}}{4\pi|\boldsymbol{x}|} e^{-ik\hat{\boldsymbol{x}}\cdot\boldsymbol{y}}\,\hat{\boldsymbol{x}}, \tag{E.37}$$

for its gradient with respect to $\boldsymbol{x}$, that

$$\nabla_{\boldsymbol{x}} G^{ff}(\boldsymbol{x},\boldsymbol{y}) = -\frac{ike^{ik|\boldsymbol{x}|}}{4\pi|\boldsymbol{x}|} e^{-ik\hat{\boldsymbol{x}}\cdot\boldsymbol{y}}\,\hat{\boldsymbol{x}}, \tag{E.38}$$

and for its double-gradient matrix, that

$$\nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{y}}G^{ff}(\boldsymbol{x},\boldsymbol{y}) = -\frac{k^2 e^{ik|\boldsymbol{x}|}}{4\pi|\boldsymbol{x}|}e^{-ik\hat{\boldsymbol{x}}\cdot\boldsymbol{y}}\,(\hat{\boldsymbol{x}}\otimes\hat{\boldsymbol{x}}). \tag{E.39}$$

We remark that these far fields are still valid if a complex wave number $k \in \mathbb{C}$ such that $\mathfrak{Im}\{k\} > 0$ is used.

## E.5 Transmission problem

We are interested in expressing the solution $u$ of the direct scattering problem (E.13) by means of an integral representation formula over the boundary $\Gamma$. To study this kind of representations, the differential problem defined on $\Omega_e$ is extended as a transmission problem defined now on the whole space $\mathbb{R}^3$ by combining (E.13) with a corresponding interior problem defined on $\Omega_i$. For the transmission problem, which specifies jump conditions over the boundary $\Gamma$, a general integral representation can be developed, and the particular integral representations of interest are then established by the specific choice of the corresponding interior problem.

A transmission problem is then a differential problem for which the jump conditions of the solution field, rather than boundary conditions, are specified on the boundary $\Gamma$. As shown in Figure E.1, we consider the exterior domain $\Omega_e$ and the interior domain $\Omega_i$, taking the unit normal $\boldsymbol{n}$ pointing towards $\Omega_i$. We search now a solution $u$ defined in $\Omega_e \cup \Omega_i$, and use the notation $u_e = u|_{\Omega_e}$ and $u_i = u|_{\Omega_i}$. We define the jumps of the traces of $u$ on both sides of the boundary $\Gamma$ as

$$[u] = u_e - u_i \qquad \text{and} \qquad \left[\frac{\partial u}{\partial n}\right] = \frac{\partial u_e}{\partial n} - \frac{\partial u_i}{\partial n}. \tag{E.40}$$

The transmission problem is now given by

$$\begin{cases} \text{Find } u : \Omega_e \cup \Omega_i \to \mathbb{C} \text{ such that} \\ \Delta u + k^2 u = 0 \qquad \text{in } \ \Omega_e \cup \Omega_i, \\ [u] = \mu \qquad \text{on } \ \Gamma, \\ \left[\dfrac{\partial u}{\partial n}\right] = \nu \qquad \text{on } \ \Gamma, \\ + \text{ Outgoing radiation condition as } |\boldsymbol{x}| \to \infty, \end{cases} \tag{E.41}$$

where $\mu, \nu : \Gamma \to \mathbb{C}$ are known functions. The outgoing radiation condition is still (E.8), and it is required to ensure uniqueness of the solution.

## E.6 Integral representations and equations

### E.6.1 Integral representation

To develop for the solution $u$ an integral representation formula over the boundary $\Gamma$, we define by $\Omega_{R,\varepsilon}$ the domain $\Omega_e \cup \Omega_i$ without the ball $B_\varepsilon$ of radius $\varepsilon > 0$ centered at the point $\boldsymbol{x} \in \Omega_e \cup \Omega_i$, and truncated at infinity by the ball $B_R$ of radius $R > 0$ centered at the origin. We consider that the ball $B_\varepsilon$ is entirely contained either in $\Omega_e$ or in $\Omega_i$, depending on the location of its center $\boldsymbol{x}$. Therefore, as shown in Figure E.2, we have that

$$\Omega_{R,\varepsilon} = \big((\Omega_e \cup \Omega_i) \cap B_R\big) \setminus \overline{B_\varepsilon} \qquad \text{and} \qquad \Omega_R = (\Omega_e \cup \Omega_i) \cap B_R, \tag{E.42}$$

where

$$B_R = \{\boldsymbol{y} \in \mathbb{R}^3 : |\boldsymbol{y}| < R\} \qquad \text{and} \qquad B_\varepsilon = \{\boldsymbol{y} \in \mathbb{R}^3 : |\boldsymbol{y} - \boldsymbol{x}| < \varepsilon\}. \tag{E.43}$$

We consider similarly the boundaries of the balls

$$S_R = \{\boldsymbol{y} \in \mathbb{R}^3 : |\boldsymbol{y}| = R\} \qquad \text{and} \qquad S_\varepsilon = \{\boldsymbol{y} \in \mathbb{R}^3 : |\boldsymbol{y} - \boldsymbol{x}| = \varepsilon\}. \tag{E.44}$$

The idea is to retrieve the domain $\Omega_e \cup \Omega_i$ at the end when the limits $R \to \infty$ and $\varepsilon \to 0$ are taken for the truncated domains $\Omega_{R,\varepsilon}$ and $\Omega_R$.

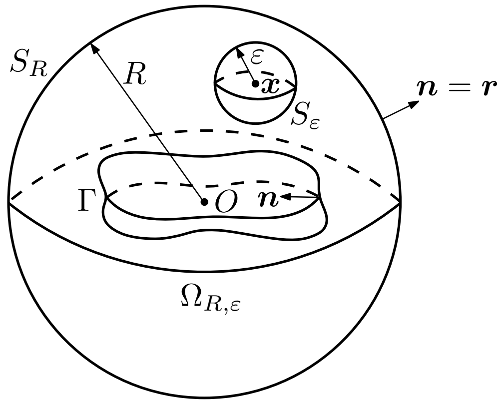



FIGURE E.2. Truncated domain $\Omega_{R,\varepsilon}$ for $\boldsymbol{x} \in \Omega_e \cup \Omega_i$.

Let us analyze first the asymptotic decaying behavior of the solution $u$, which satisfies the Helmholtz equation and the Sommerfeld radiation condition. For more generality, we assume here that the wave number $k$ $(\neq 0)$ is complex and such that $\mathfrak{Im}\{k\} \geq 0$. We consider the weakest form of the radiation condition, namely (E.11), and develop

$$\int_{S_R} \left| \frac{\partial u}{\partial r} - iku \right|^2 \mathrm{d}\gamma = \int_{S_R} \left[ \left| \frac{\partial u}{\partial r} \right|^2 + |k|^2 |u|^2 + 2\,\mathfrak{Im}\left\{ ku \frac{\partial \bar{u}}{\partial r} \right\} \right] \mathrm{d}\gamma. \tag{E.45}$$

From the divergence theorem (A.614) applied on the truncated domain $\Omega_R$ and considering the complex conjugated Helmholtz equation we have

$$\begin{aligned} k \int_{S_R} u \frac{\partial \bar{u}}{\partial r} \,\mathrm{d}\gamma + k \int_\Gamma u \frac{\partial \bar{u}}{\partial n} \,\mathrm{d}\gamma &= k \int_{\Omega_R} \operatorname{div}(u \nabla \bar{u}) \,\mathrm{d}\boldsymbol{x} \\ &= k \int_{\Omega_R} |\nabla u|^2 \,\mathrm{d}\boldsymbol{x} - k\bar{k}^2 \int_{\Omega_R} |u|^2 \,\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{E.46}$$

Replacing the imaginary part of (E.46) in (E.45) and taking the limit as $R \to \infty$, yields

$$\lim_{R\to\infty} \left[ \int_{S_R} \left( \left| \frac{\partial u}{\partial r} \right|^2 + |k|^2 |u|^2 \right) \mathrm{d}\gamma + 2\,\mathfrak{Im}\{k\} \int_{\Omega_R} \left( |\nabla u|^2 + |k|^2 |u|^2 \right) \mathrm{d}\boldsymbol{x} \right] \\ = 2\,\mathfrak{Im}\left\{ k \int_\Gamma u \frac{\partial \bar{u}}{\partial n} \,\mathrm{d}\gamma \right\}. \tag{E.47}$$

Since the right-hand side is finite and since the left-hand side is nonnegative, we see that

$$\int_{S_R} |u|^2 \,\mathrm{d}\gamma = \mathcal{O}(1) \qquad \text{and} \qquad \int_{S_R} \left| \frac{\partial u}{\partial r} \right|^2 \mathrm{d}\gamma = \mathcal{O}(1) \qquad \text{as } R \to \infty, \tag{E.48}$$

and therefore it holds for a great value of $r = |\boldsymbol{x}|$ that

$$u = \mathcal{O}\left(\frac{1}{r}\right) \qquad \text{and} \qquad |\nabla u| = \mathcal{O}\left(\frac{1}{r}\right). \tag{E.49}$$

We apply now Green's second integral theorem (A.613) to the functions $u$ and $G(\boldsymbol{x}, \cdot)$ in the bounded domain $\Omega_{R,\varepsilon}$, by subtracting their respective Helmholtz equations, yielding

$$\begin{aligned} 0 &= \int_{\Omega_{R,\varepsilon}} \big( u(\boldsymbol{y}) \Delta_{\boldsymbol{y}} G(\boldsymbol{x}, \boldsymbol{y}) - G(\boldsymbol{x}, \boldsymbol{y}) \Delta u(\boldsymbol{y}) \big) \mathrm{d}\boldsymbol{y} \\ &= \int_{S_R} \left( u(\boldsymbol{y}) \frac{\partial G}{\partial r_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) - G(\boldsymbol{x}, \boldsymbol{y}) \frac{\partial u}{\partial r}(\boldsymbol{y}) \right) \mathrm{d}\gamma(\boldsymbol{y}) \\ &\quad - \int_{S_\varepsilon} \left( u(\boldsymbol{y}) \frac{\partial G}{\partial r_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) - G(\boldsymbol{x}, \boldsymbol{y}) \frac{\partial u}{\partial r}(\boldsymbol{y}) \right) \mathrm{d}\gamma(\boldsymbol{y}) \\ &\quad + \int_\Gamma \left( [u](\boldsymbol{y}) \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) - G(\boldsymbol{x}, \boldsymbol{y}) \left[ \frac{\partial u}{\partial n} \right](\boldsymbol{y}) \right) \mathrm{d}\gamma(\boldsymbol{y}). \end{aligned} \tag{E.50}$$

The integral on $S_R$ can be rewritten as

$$\int_{S_R} \left[ u(\boldsymbol{y}) \left( \frac{\partial G}{\partial r_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) - ikG(\boldsymbol{x}, \boldsymbol{y}) \right) - G(\boldsymbol{x}, \boldsymbol{y}) \left( \frac{\partial u}{\partial r}(\boldsymbol{y}) - iku(\boldsymbol{y}) \right) \right] \mathrm{d}\gamma(\boldsymbol{y}), \tag{E.51}$$

which for $R$ large enough and due the radiation condition (E.8) tends to zero, since

$$\left| \int_{S_R} u(\boldsymbol{y}) \left( \frac{\partial G}{\partial r_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) - ikG(\boldsymbol{x}, \boldsymbol{y}) \right) \mathrm{d}\gamma(\boldsymbol{y}) \right| \leq \frac{C}{R}, \tag{E.52}$$

and

$$\left| \int_{S_R} G(\boldsymbol{x}, \boldsymbol{y}) \left( \frac{\partial u}{\partial r}(\boldsymbol{y}) - iku(\boldsymbol{y}) \right) \mathrm{d}\gamma(\boldsymbol{y}) \right| \leq \frac{C}{R}, \tag{E.53}$$

for some constants $C > 0$. If the function $u$ is regular enough in the ball $B_\varepsilon$, then the second term of the integral on $S_\varepsilon$, when $\varepsilon \to 0$ and due (E.22), is bounded by

$$\left| \int_{S_\varepsilon} G(\boldsymbol{x}, \boldsymbol{y}) \frac{\partial u}{\partial r}(\boldsymbol{y}) \,\mathrm{d}\gamma(\boldsymbol{y}) \right| \leq \varepsilon \, |e^{ik\varepsilon}| \sup_{\boldsymbol{y} \in B_\varepsilon} \left| \frac{\partial u}{\partial r}(\boldsymbol{y}) \right|, \tag{E.54}$$

and tends to zero. The regularity of $u$ can be specified afterwards once the integral representation has been determined and generalized by means of density arguments. The first integral term on $S_\varepsilon$ can be decomposed as

$$\int_{S_\varepsilon} u(\boldsymbol{y})\frac{\partial G}{\partial r_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}) = u(\boldsymbol{x})\int_{S_\varepsilon}\frac{\partial G}{\partial r_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}) + \int_{S_\varepsilon}\frac{\partial G}{\partial r_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y})\big(u(\boldsymbol{y})-u(\boldsymbol{x})\big)\,\mathrm{d}\gamma(\boldsymbol{y}), \tag{E.55}$$

For the first term in the right-hand side of (E.55), by replacing (E.27), we have that

$$\int_{S_\varepsilon}\frac{\partial G}{\partial r_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}) = (1-ik\varepsilon)\,e^{ik\varepsilon} \xrightarrow[\varepsilon\to 0]{} 1, \tag{E.56}$$

which tends towards one, while the second term is bounded by

$$\left|\int_{S_\varepsilon}\big(u(\boldsymbol{y})-u(\boldsymbol{x})\big)\frac{\partial G}{\partial r_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y})\right| \leq |1-ik\varepsilon|\,|e^{ik\varepsilon}|\sup_{\boldsymbol{y}\in B_\varepsilon}|u(\boldsymbol{y})-u(\boldsymbol{x})|, \tag{E.57}$$

which tends towards zero when $\varepsilon \to 0$.

In conclusion, when the limits $R\to\infty$ and $\varepsilon\to 0$ are taken in (E.50), then the following integral representation formula holds for the solution $u$ of the transmission problem:

$$u(\boldsymbol{x}) = \int_\Gamma\left([u](\boldsymbol{y})\frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) - G(\boldsymbol{x},\boldsymbol{y})\left[\frac{\partial u}{\partial n}\right](\boldsymbol{y})\right)\mathrm{d}\gamma(\boldsymbol{y}), \qquad \boldsymbol{x}\in\Omega_e\cup\Omega_i. \tag{E.58}$$

We observe thus that if the values of the jump of $u$ and of its normal derivative are known on $\Gamma$, then the transmission problem (E.41) is readily solved and its solution given explicitly by (E.58), which, in terms of $\mu$ and $\nu$, becomes

$$u(\boldsymbol{x}) = \int_\Gamma\left(\mu(\boldsymbol{y})\frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) - G(\boldsymbol{x},\boldsymbol{y})\nu(\boldsymbol{y})\right)\mathrm{d}\gamma(\boldsymbol{y}), \qquad \boldsymbol{x}\in\Omega_e\cup\Omega_i. \tag{E.59}$$

To determine the values of the jumps, an adequate integral equation has to be developed, i.e., an equation whose unknowns are the traces of the solution on $\Gamma$.

An alternative way to demonstrate the integral representation (E.58) is to proceed in the sense of distributions, in the same way as done in Section B.6. Again we obtain the single layer potential

$$\left\{G * \left[\frac{\partial u}{\partial n}\right]\delta_\Gamma\right\}(\boldsymbol{x}) = \int_\Gamma G(\boldsymbol{x},\boldsymbol{y})\left[\frac{\partial u}{\partial n}\right](\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}) \tag{E.60}$$

associated with the distribution of sources $[\partial u/\partial n]\delta_\Gamma$, and the double layer potential

$$\left\{G * \frac{\partial}{\partial n}\big([u]\delta_\Gamma\big)\right\}(\boldsymbol{x}) = -\int_\Gamma \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y})[u](\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}) \tag{E.61}$$

associated with the distribution of dipoles $\frac{\partial}{\partial n}([u]\delta_\Gamma)$. Combining properly (E.60) and (E.61) yields the desired integral representation (E.58).

We note that to obtain the gradient of the integral representation (E.58) we can pass directly the derivatives inside the integral, since there are no singularities if $\boldsymbol{x}\in\Omega_e\cup\Omega_i$. Therefore we have that

$$\nabla u(\boldsymbol{x}) = \int_\Gamma\left([u](\boldsymbol{y})\nabla_{\boldsymbol{x}}\frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) - \nabla_{\boldsymbol{x}}G(\boldsymbol{x},\boldsymbol{y})\left[\frac{\partial u}{\partial n}\right](\boldsymbol{y})\right)\mathrm{d}\gamma(\boldsymbol{y}). \tag{E.62}$$

### E.6.2 Integral equations

To determine the values of the traces that conform the jumps for the transmission problem (E.41), an integral equation has to be developed. For this purpose we place the source point $\boldsymbol{x}$ on the boundary $\Gamma$ and apply the same procedure as before for the integral representation (E.58), treating differently in (E.50) only the integrals on $S_\varepsilon$. The integrals on $S_R$ still behave well and tend towards zero as $R \to \infty$. The Ball $B_\varepsilon$, though, is split in half into the two pieces $\Omega_e \cap B_\varepsilon$ and $\Omega_i \cap B_\varepsilon$, which are asymptotically separated by the tangent of the boundary if $\Gamma$ is regular. Thus the associated integrals on $S_\varepsilon$ give rise to a term $-(u_e(\boldsymbol{x}) + u_i(\boldsymbol{x}))/2$ instead of just $-u(\boldsymbol{x})$ as before. We must notice that in this case, the integrands associated with the boundary $\Gamma$ admit an integrable singularity at the point $\boldsymbol{x}$. The desired integral equation related with (E.58) is then given by

$$\frac{u_e(\boldsymbol{x}) + u_i(\boldsymbol{x})}{2} = \int_\Gamma \left( [u](\boldsymbol{y}) \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) - G(\boldsymbol{x},\boldsymbol{y}) \left[\frac{\partial u}{\partial n}\right](\boldsymbol{y}) \right) \mathrm{d}\gamma(\boldsymbol{y}), \quad \boldsymbol{x} \in \Gamma. \tag{E.63}$$

By choosing adequately the boundary condition of the interior problem, and by considering also the boundary condition of the exterior problem and the jump definitions (E.40), this integral equation can be expressed in terms of only one unknown function on $\Gamma$. Thus, solving the problem (E.13) is equivalent to solve (E.63) and then replace the obtained solution in (E.58).

The integral equation holds only when the boundary $\Gamma$ is regular (e.g., of class $C^2$). Otherwise, taking the limit $\varepsilon \to 0$ can no longer be well-defined and the result is false in general. In particular, if the boundary $\Gamma$ has an angular point at $\boldsymbol{x} \in \Gamma$, then the left-hand side of the integral equation (E.63) is modified on that point according to the portion of the ball $B_\varepsilon$ that remains inside $\Omega_e$, analogously as was done for the two-dimensional case in (B.61), but now for solid angles.

Another integral equation can be also derived for the normal derivative of the solution $u$ on the boundary $\Gamma$, by studying the jump properties of the single and double layer potentials. It is performed in the same manner as for the Laplace equation. If the boundary is regular at $\boldsymbol{x} \in \Gamma$, then it holds that

$$\frac{1}{2}\frac{\partial u_e}{\partial n}(\boldsymbol{x}) + \frac{1}{2}\frac{\partial u_i}{\partial n}(\boldsymbol{x}) = \int_\Gamma \left( [u](\boldsymbol{y}) \frac{\partial^2 G}{\partial n_{\boldsymbol{x}} \partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) - \frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x},\boldsymbol{y}) \left[\frac{\partial u}{\partial n}\right](\boldsymbol{y}) \right) \mathrm{d}\gamma(\boldsymbol{y}). \tag{E.64}$$

This integral equation is modified correspondingly if $\boldsymbol{x}$ is an angular point.

### E.6.3 Integral kernels

In the same manner as for the Laplace equation, the integral kernels $G$, $\partial G/\partial n_{\boldsymbol{y}}$, and $\partial G/\partial n_{\boldsymbol{x}}$ are weakly singular, and thus integrable, whereas the kernel $\partial^2 G/\partial n_{\boldsymbol{x}} \partial n_{\boldsymbol{y}}$ is not integrable and therefore hypersingular.

The kernel $G$ defined in (E.22) has the same singularity as the Laplace equation, namely

$$G(\boldsymbol{x},\boldsymbol{y}) \sim -\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \qquad \text{as } \boldsymbol{x} \to \boldsymbol{y}. \tag{E.65}$$

It fulfills therefore (B.64) with $\lambda = 1$. The kernels $\partial G/\partial n_{\boldsymbol{y}}$ and $\partial G/\partial n_{\boldsymbol{x}}$ are less singular along $\Gamma$ than they appear at first sight, due the regularizing effect of the normal derivatives. They are given respectively by

$$\frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) = \frac{e^{ik|\boldsymbol{y}-\boldsymbol{x}|}}{4\pi}\big(1 - ik|\boldsymbol{y}-\boldsymbol{x}|\big)\frac{(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{n}_{\boldsymbol{y}}}{|\boldsymbol{y}-\boldsymbol{x}|^3}, \tag{E.66}$$

and

$$\frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x},\boldsymbol{y}) = \frac{e^{ik|\boldsymbol{y}-\boldsymbol{x}|}}{4\pi}\big(1 - ik|\boldsymbol{y}-\boldsymbol{x}|\big)\frac{(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{n}_{\boldsymbol{x}}}{|\boldsymbol{y}-\boldsymbol{x}|^3}, \tag{E.67}$$

and their singularities, as $\boldsymbol{x}\to\boldsymbol{y}$ for $\boldsymbol{x},\boldsymbol{y}\in\Gamma$, adopt the form

$$\frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) \sim \frac{(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{n}_{\boldsymbol{y}}}{4\pi|\boldsymbol{y}-\boldsymbol{x}|^3} \qquad \text{and} \qquad \frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x},\boldsymbol{y}) \sim \frac{(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{n}_{\boldsymbol{x}}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|^3}. \tag{E.68}$$

The appearing singularities are the same as for the Laplace equation and it can be shown that for the singularity the estimates (B.70) and (B.71) hold also in three dimensions, by using the same reasoning as in the two-dimensional case for the graph of a regular function $\varphi$ that takes variables now on the tangent plane. Therefore we have that

$$\frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) = \mathcal{O}\bigg(\frac{1}{|\boldsymbol{y}-\boldsymbol{x}|}\bigg) \qquad \text{and} \qquad \frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x},\boldsymbol{y}) = \mathcal{O}\bigg(\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\bigg), \tag{E.69}$$

and hence these kernels satisfy (B.64) with $\lambda = 1$.

The kernel $\partial^2 G/\partial n_{\boldsymbol{x}}\partial n_{\boldsymbol{y}}$, on the other hand, adopts the form

$$\begin{aligned}\frac{\partial^2 G}{\partial n_{\boldsymbol{x}}\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) &= \frac{ik^2}{4\pi}h_1^{(1)}\big(k|\boldsymbol{x}-\boldsymbol{y}|\big)\left(-\frac{\boldsymbol{n}_{\boldsymbol{x}}\cdot\boldsymbol{n}_{\boldsymbol{y}}}{|\boldsymbol{x}-\boldsymbol{y}|} - 3\frac{\big((\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{n}_{\boldsymbol{x}}\big)\big((\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{n}_{\boldsymbol{y}}\big)}{|\boldsymbol{x}-\boldsymbol{y}|^3}\right) \\ &\quad + \frac{ik^3}{4\pi}h_0^{(1)}\big(k|\boldsymbol{x}-\boldsymbol{y}|\big)\frac{\big((\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{n}_{\boldsymbol{x}}\big)\big((\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{n}_{\boldsymbol{y}}\big)}{|\boldsymbol{x}-\boldsymbol{y}|^2}.\end{aligned} \tag{E.70}$$

Its singularity, when $\boldsymbol{x}\to\boldsymbol{y}$ for $\boldsymbol{x},\boldsymbol{y}\in\Gamma$, expresses itself as

$$\frac{\partial^2 G}{\partial n_{\boldsymbol{x}}\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) \sim -\frac{\boldsymbol{n}_{\boldsymbol{x}}\cdot\boldsymbol{n}_{\boldsymbol{y}}}{4\pi|\boldsymbol{y}-\boldsymbol{x}|^3} - \frac{3\big((\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{n}_{\boldsymbol{x}}\big)\big((\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{n}_{\boldsymbol{y}}\big)}{4\pi|\boldsymbol{y}-\boldsymbol{x}|^5}. \tag{E.71}$$

The regularizing effect of the normal derivatives applies only to its second term, but not to the first. Hence this kernel is hypersingular, with $\lambda = 3$, and it holds that

$$\frac{\partial^2 G}{\partial n_{\boldsymbol{x}}\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) = \mathcal{O}\bigg(\frac{1}{|\boldsymbol{y}-\boldsymbol{x}|^3}\bigg). \tag{E.72}$$

The kernel is no longer integrable and the associated integral operator has to be thus interpreted in some appropriate sense as a divergent integral (cf., e.g., Hsiao & Wendland 2008, Lenoir 2005, Nédélec 2001).

### E.6.4 Boundary layer potentials

We regard now the jump properties on the boundary $\Gamma$ of the boundary layer potentials that have appeared in our calculations. For the development of the integral representation (E.59) we already made acquaintance with the single and double layer potentials, which we define now more precisely for $\boldsymbol{x} \in \Omega_e \cup \Omega_i$ as the integral operators

$$\mathcal{S}\nu(\boldsymbol{x}) = \int_\Gamma G(\boldsymbol{x},\boldsymbol{y})\nu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}), \tag{E.73}$$

$$\mathcal{D}\mu(\boldsymbol{x}) = \int_\Gamma \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y})\mu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}). \tag{E.74}$$

The integral representation (E.59) can be now stated in terms of the layer potentials as

$$u = \mathcal{D}\mu - \mathcal{S}\nu. \tag{E.75}$$

We remark that for any functions $\nu, \mu : \Gamma \to \mathbb{C}$ that are regular enough, the single and double layer potentials satisfy the Helmholtz equation, namely

$$(\Delta + k^2)\,\mathcal{S}\nu = 0 \qquad \text{in } \Omega_e \cup \Omega_i, \tag{E.76}$$

$$(\Delta + k^2)\,\mathcal{D}\mu = 0 \qquad \text{in } \Omega_e \cup \Omega_i. \tag{E.77}$$

For the integral equations (E.63) and (E.64), which are defined for $\boldsymbol{x} \in \Gamma$, we require the four boundary integral operators:

$$S\nu(\boldsymbol{x}) = \int_\Gamma G(\boldsymbol{x},\boldsymbol{y})\nu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}), \tag{E.78}$$

$$D\mu(\boldsymbol{x}) = \int_\Gamma \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y})\mu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}), \tag{E.79}$$

$$D^*\nu(\boldsymbol{x}) = \int_\Gamma \frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x},\boldsymbol{y})\nu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}), \tag{E.80}$$

$$N\mu(\boldsymbol{x}) = \int_\Gamma \frac{\partial^2 G}{\partial n_{\boldsymbol{x}}\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y})\mu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}). \tag{E.81}$$

The operator $D^*$ is in fact the adjoint of the operator $D$. As we already mentioned, the kernel of the integral operator $N$ defined in (E.81) is not integrable, yet we write it formally as an improper integral. An appropriate sense for this integral will be given below. The integral equations (E.63) and (E.64) can be now stated in terms of the integral operators as

$$\frac{1}{2}(u_e + u_i) = D\mu - S\nu, \tag{E.82}$$

$$\frac{1}{2}\left(\frac{\partial u_e}{\partial n} + \frac{\partial u_i}{\partial n}\right) = N\mu - D^*\nu. \tag{E.83}$$

These integral equations can be easily derived from the jump properties of the single and double layer potentials. The single layer potential (E.73) is continuous and its normal derivative has a jump of size $-\nu$ across $\Gamma$, i.e.,

$$\mathcal{S}\nu|_{\Omega_e} = S\nu = \mathcal{S}\nu|_{\Omega_i}, \tag{E.84}$$

$$\frac{\partial}{\partial n}\mathcal{S}\nu|_{\Omega_e} = \left(-\frac{1}{2} + D^*\right)\nu, \tag{E.85}$$

$$\frac{\partial}{\partial n}\mathcal{S}\nu|_{\Omega_i} = \left(\frac{1}{2} + D^*\right)\nu. \tag{E.86}$$

The double layer potential (E.74), on the other hand, has a jump of size $\mu$ across $\Gamma$ and its normal derivative is continuous, namely

$$\mathcal{D}\mu|_{\Omega_e} = \left(\frac{1}{2} + D\right)\mu, \tag{E.87}$$

$$\mathcal{D}\mu|_{\Omega_i} = \left(-\frac{1}{2} + D\right)\mu, \tag{E.88}$$

$$\frac{\partial}{\partial n}\mathcal{D}\mu|_{\Omega_e} = N\mu = \frac{\partial}{\partial n}\mathcal{D}\mu|_{\Omega_i}. \tag{E.89}$$

The integral equation (E.82) is obtained directly either from (E.84) and (E.87), or from (E.84) and (E.88), by considering the appropriate trace of (E.75) and by defining the functions $\mu$ and $\nu$ as in (E.41). These three jump properties are easily proven by regarding the details of the proof for (E.63).

Similarly, the integral equation (E.83) for the normal derivative is obtained directly either from (E.85) and (E.89), or from (E.86) and (E.89), by considering the appropriate trace of the normal derivative of (E.75) and by defining again the functions $\mu$ and $\nu$ as in (E.41). The proof of the jump properties (E.85) and (E.86) is the same as for the Laplace equation, since the same singularities are involved, whereas the proof of (E.89) is similar, but with some differences, and is therefore replicated below.

a) Continuity of the normal derivative of the double layer potential

Differently as in the proof for the Laplace equation, in this case an additional term appears for the operator $N$, since it is the Helmholtz equation (E.77) that has to be considered in (D.86) and (D.87), yielding now for a test function $\varphi \in \mathcal{D}(\mathbb{R}^3)$ that

$$\left\langle \frac{\partial}{\partial n}\mathcal{D}\mu|_{\Omega_e}, \varphi \right\rangle = \int_{\Omega_e} \nabla\mathcal{D}\mu(\boldsymbol{x}) \cdot \nabla\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - k^2 \int_{\Omega_e} \mathcal{D}\mu(\boldsymbol{x})\,\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \tag{E.90}$$

$$\left\langle \frac{\partial}{\partial n}\mathcal{D}\mu|_{\Omega_i}, \varphi \right\rangle = -\int_{\Omega_i} \nabla\mathcal{D}\mu(\boldsymbol{x}) \cdot \nabla\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} + k^2 \int_{\Omega_i} \mathcal{D}\mu(\boldsymbol{x})\,\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{E.91}$$

From (A.588) and (E.31) we obtain the relation

$$\frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{n_y} \cdot \nabla_{\boldsymbol{y}} G(\boldsymbol{x},\boldsymbol{y}) = -\boldsymbol{n_y} \cdot \nabla_{\boldsymbol{x}} G(\boldsymbol{x},\boldsymbol{y}) = -\operatorname{div}_{\boldsymbol{x}}\big(G(\boldsymbol{x},\boldsymbol{y})\boldsymbol{n_y}\big). \tag{E.92}$$

Thus for the double layer potential (E.74) we have that

$$\mathcal{D}\mu(\boldsymbol{x}) = -\operatorname{div}\int_\Gamma G(\boldsymbol{x},\boldsymbol{y})\mu(\boldsymbol{y})\boldsymbol{n_y}\,\mathrm{d}\gamma(\boldsymbol{y}) = -\operatorname{div}\mathcal{S}(\mu\,\boldsymbol{n_y})(\boldsymbol{x}), \tag{E.93}$$

being its gradient given by

$$\nabla\mathcal{D}\mu(\boldsymbol{x}) = -\nabla\operatorname{div}\int_\Gamma G(\boldsymbol{x},\boldsymbol{y})\mu(\boldsymbol{y})\boldsymbol{n_y}\,\mathrm{d}\gamma(\boldsymbol{y}). \tag{E.94}$$

From (A.589) we have that

$$\operatorname{curl}_{\boldsymbol{x}}\big(G(\boldsymbol{x},\boldsymbol{y})\boldsymbol{n}_{\boldsymbol{y}}\big) = \nabla_{\boldsymbol{x}} G(\boldsymbol{x},\boldsymbol{y}) \times \boldsymbol{n}_{\boldsymbol{y}}. \tag{E.95}$$

Hence, by considering (A.590), (E.77), and (E.95) in (E.94), we obtain that

$$\nabla \mathcal{D}\mu(\boldsymbol{x}) = \operatorname{curl} \int_\Gamma \big(\boldsymbol{n}_{\boldsymbol{y}} \times \nabla_{\boldsymbol{x}} G(\boldsymbol{x},\boldsymbol{y})\big)\mu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}) + k^2 \int_\Gamma G(\boldsymbol{x},\boldsymbol{y})\mu(\boldsymbol{y})\boldsymbol{n}_{\boldsymbol{y}}\,\mathrm{d}\gamma(\boldsymbol{y}). \tag{E.96}$$

From (E.31) and (A.658) we have that

$$\begin{aligned}\int_\Gamma \big(\boldsymbol{n}_{\boldsymbol{y}} \times \nabla_{\boldsymbol{x}} G(\boldsymbol{x},\boldsymbol{y})\big)\mu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}) &= -\int_\Gamma \boldsymbol{n}_{\boldsymbol{y}} \times \big(\nabla_{\boldsymbol{y}} G(\boldsymbol{x},\boldsymbol{y})\mu(\boldsymbol{y})\big)\,\mathrm{d}\gamma(\boldsymbol{y}) \\ &= \int_\Gamma \boldsymbol{n}_{\boldsymbol{y}} \times \big(G(\boldsymbol{x},\boldsymbol{y})\nabla\mu(\boldsymbol{y})\big)\,\mathrm{d}\gamma(\boldsymbol{y}),\end{aligned} \tag{E.97}$$

and consequently

$$\nabla \mathcal{D}\mu(\boldsymbol{x}) = \operatorname{curl} \int_\Gamma G(\boldsymbol{x},\boldsymbol{y})\big(\boldsymbol{n}_{\boldsymbol{y}} \times \nabla\mu(\boldsymbol{y})\big)\,\mathrm{d}\gamma(\boldsymbol{y}) + k^2 \int_\Gamma G(\boldsymbol{x},\boldsymbol{y})\mu(\boldsymbol{y})\boldsymbol{n}_{\boldsymbol{y}}\,\mathrm{d}\gamma(\boldsymbol{y}). \tag{E.98}$$

Now, the first expression in (E.90), due (A.596), (A.618), and (E.98), is given by

$$\begin{aligned}\int_{\Omega_e} \nabla \mathcal{D}\mu(\boldsymbol{x}) \cdot \nabla\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = &-\int_\Gamma\int_\Gamma G(\boldsymbol{x},\boldsymbol{y})\big(\nabla\mu(\boldsymbol{y}) \times \boldsymbol{n}_{\boldsymbol{y}}\big)\cdot\big(\nabla\varphi(\boldsymbol{x}) \times \boldsymbol{n}_{\boldsymbol{x}}\big)\,\mathrm{d}\gamma(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{x}) \\ &+ k^2 \int_{\Omega_e}\left(\int_\Gamma G(\boldsymbol{x},\boldsymbol{y})\mu(\boldsymbol{y})\boldsymbol{n}_{\boldsymbol{y}}\,\mathrm{d}\gamma(\boldsymbol{y})\right)\cdot\nabla\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{E.99}$$

Applying (A.614) on the second term of (E.99) and considering (E.93), yields

$$\begin{aligned}\int_{\Omega_e} \nabla \mathcal{D}\mu(\boldsymbol{x}) \cdot \nabla\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = &-\int_\Gamma\int_\Gamma G(\boldsymbol{x},\boldsymbol{y})\big(\nabla\mu(\boldsymbol{y}) \times \boldsymbol{n}_{\boldsymbol{y}}\big)\cdot\big(\nabla\varphi(\boldsymbol{x}) \times \boldsymbol{n}_{\boldsymbol{x}}\big)\,\mathrm{d}\gamma(\boldsymbol{y})\mathrm{d}\gamma(\boldsymbol{x}) \\ + k^2 \int_{\Omega_e} \mathcal{D}\mu(\boldsymbol{x})\,\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} &+ \int_\Gamma\int_\Gamma G(\boldsymbol{x},\boldsymbol{y})\mu(\boldsymbol{y})\varphi(\boldsymbol{x})(\boldsymbol{n}_{\boldsymbol{y}}\cdot\boldsymbol{n}_{\boldsymbol{x}})\,\mathrm{d}\gamma(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{x}).\end{aligned} \tag{E.100}$$

By replacing (E.100) in (E.90) we obtain finally that

$$\begin{aligned}\left\langle \frac{\partial}{\partial n}\mathcal{D}\mu|_{\Omega_e}, \varphi\right\rangle = &-\int_\Gamma\int_\Gamma G(\boldsymbol{x},\boldsymbol{y})\big(\nabla\mu(\boldsymbol{y}) \times \boldsymbol{n}_{\boldsymbol{y}}\big)\cdot\big(\nabla\varphi(\boldsymbol{x}) \times \boldsymbol{n}_{\boldsymbol{x}}\big)\,\mathrm{d}\gamma(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{x}) \\ &+ k^2 \int_\Gamma\int_\Gamma G(\boldsymbol{x},\boldsymbol{y})\mu(\boldsymbol{y})\varphi(\boldsymbol{x})(\boldsymbol{n}_{\boldsymbol{y}}\cdot\boldsymbol{n}_{\boldsymbol{x}})\,\mathrm{d}\gamma(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{x}).\end{aligned} \tag{E.101}$$

The analogous development for (E.91) yields

$$\begin{aligned}\left\langle \frac{\partial}{\partial n}\mathcal{D}\mu|_{\Omega_i}, \varphi\right\rangle = &-\int_\Gamma\int_\Gamma G(\boldsymbol{x},\boldsymbol{y})\big(\nabla\mu(\boldsymbol{y}) \times \boldsymbol{n}_{\boldsymbol{y}}\big)\cdot\big(\nabla\varphi(\boldsymbol{x}) \times \boldsymbol{n}_{\boldsymbol{x}}\big)\,\mathrm{d}\gamma(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{x}) \\ &+ k^2 \int_\Gamma\int_\Gamma G(\boldsymbol{x},\boldsymbol{y})\mu(\boldsymbol{y})\varphi(\boldsymbol{x})(\boldsymbol{n}_{\boldsymbol{y}}\cdot\boldsymbol{n}_{\boldsymbol{x}})\,\mathrm{d}\gamma(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{x}).\end{aligned} \tag{E.102}$$

This concludes the proof of (E.89), and shows that the integral operator (E.81) is properly defined in a weak sense for $\varphi \in \mathcal{D}(\mathbb{R}^3)$, instead of (D.97), by

$$\begin{aligned}\langle N\mu(\boldsymbol{x}), \varphi\rangle = &-\int_\Gamma\int_\Gamma G(\boldsymbol{x},\boldsymbol{y})\big(\nabla\mu(\boldsymbol{y})\times\boldsymbol{n}_{\boldsymbol{y}}\big)\cdot\big(\nabla\varphi(\boldsymbol{x})\times\boldsymbol{n}_{\boldsymbol{x}}\big)\,\mathrm{d}\gamma(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{x})\\ &+k^2\int_\Gamma\int_\Gamma G(\boldsymbol{x},\boldsymbol{y})\mu(\boldsymbol{y})\varphi(\boldsymbol{x})(\boldsymbol{n}_{\boldsymbol{y}}\cdot\boldsymbol{n}_{\boldsymbol{x}})\,\mathrm{d}\gamma(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{x}).\end{aligned} \tag{E.103}$$

### E.6.5 Alternatives for integral representations and equations

By taking into account the transmission problem (E.41), its integral representation formula (E.58), and its integral equations (E.63) and (E.64), several particular alternatives for integral representations and equations of the exterior problem (E.13) can be developed. The way to perform this is to extend properly the exterior problem towards the interior domain $\Omega_i$, either by specifying explicitly this extension or by defining an associated interior problem, so as to become the desired jump properties across $\Gamma$. The extension has to satisfy the Helmholtz equation (E.1) in $\Omega_i$ and a boundary condition that corresponds adequately to the impedance boundary condition (E.3). The obtained system of integral representations and equations allows finally to solve the exterior problem (E.13), by using the solution of the integral equation in the integral representation formula.

a) Extension by zero

An extension by zero towards the interior domain $\Omega_i$ implies that

$$u_i = 0 \qquad \text{in } \Omega_i. \tag{E.104}$$

The jumps over $\Gamma$ are characterized in this case by

$$[u] = u_e = \mu, \tag{E.105}$$

$$\left[\frac{\partial u}{\partial n}\right] = \frac{\partial u_e}{\partial n} = Zu_e - f_z = Z\mu - f_z, \tag{E.106}$$

where $\mu : \Gamma \to \mathbb{C}$ is a function to be determined.

An integral representation formula of the solution, for $\boldsymbol{x} \in \Omega_e \cup \Omega_i$, is given by

$$u(\boldsymbol{x}) = \int_\Gamma \left(\frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) - Z(\boldsymbol{y})G(\boldsymbol{x},\boldsymbol{y})\right)\mu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}) + \int_\Gamma G(\boldsymbol{x},\boldsymbol{y})f_z(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}). \tag{E.107}$$

Since

$$\frac{1}{2}\big(u_e(\boldsymbol{x}) + u_i(\boldsymbol{x})\big) = \frac{\mu(\boldsymbol{x})}{2}, \qquad \boldsymbol{x} \in \Gamma, \tag{E.108}$$

we obtain, for $\boldsymbol{x} \in \Gamma$, the Fredholm integral equation of the second kind

$$\frac{\mu(\boldsymbol{x})}{2} + \int_\Gamma \left(Z(\boldsymbol{y})G(\boldsymbol{x},\boldsymbol{y}) - \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y})\right)\mu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}) = \int_\Gamma G(\boldsymbol{x},\boldsymbol{y})f_z(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}), \tag{E.109}$$

which has to be solved for the unknown $\mu$. In terms of boundary layer potentials, the integral representation and the integral equation can be respectively expressed by

$$u = \mathcal{D}(\mu) - \mathcal{S}(Z\mu) + \mathcal{S}(f_z) \qquad \text{in } \Omega_e \cup \Omega_i, \tag{E.110}$$

$$\frac{\mu}{2} + S(Z\mu) - D(\mu) = S(f_z) \qquad \text{on } \Gamma. \tag{E.111}$$

Alternatively, since

$$\frac{1}{2}\left(\frac{\partial u_e}{\partial n}(\boldsymbol{x}) + \frac{\partial u_i}{\partial n}(\boldsymbol{x})\right) = \frac{Z(\boldsymbol{x})}{2}\mu(\boldsymbol{x}) - \frac{f_z(\boldsymbol{x})}{2}, \qquad \boldsymbol{x} \in \Gamma, \tag{E.112}$$

we obtain also, for $\boldsymbol{x} \in \Gamma$, the Fredholm integral equation of the second kind

$$\begin{aligned} \frac{Z(\boldsymbol{x})}{2}\mu(\boldsymbol{x}) + \int_\Gamma \left(-\frac{\partial^2 G}{\partial n_{\boldsymbol{x}} \partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) + Z(\boldsymbol{y})\frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x},\boldsymbol{y})\right)\mu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}) \\ = \frac{f_z(\boldsymbol{x})}{2} + \int_\Gamma \frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x},\boldsymbol{y}) f_z(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}), \end{aligned} \tag{E.113}$$

which in terms of boundary layer potentials becomes

$$\frac{Z}{2}\mu - N(\mu) + D^*(Z\mu) = \frac{f_z}{2} + D^*(f_z) \qquad \text{on } \Gamma. \tag{E.114}$$

b) Continuous impedance

We associate to (E.13) the interior problem

$$\begin{cases} \text{Find } u_i : \Omega_i \to \mathbb{C} \text{ such that} \\ \Delta u_i + k^2 u_i = 0 & \text{in } \Omega_i, \\ -\dfrac{\partial u_i}{\partial n} + Z u_i = f_z & \text{on } \Gamma. \end{cases} \tag{E.115}$$

The jumps over $\Gamma$ are characterized in this case by

$$[u] = u_e - u_i = \mu, \tag{E.116}$$

$$\left[\frac{\partial u}{\partial n}\right] = \frac{\partial u_e}{\partial n} - \frac{\partial u_i}{\partial n} = Z(u_e - u_i) = Z\mu, \tag{E.117}$$

where $\mu : \Gamma \to \mathbb{C}$ is a function to be determined. In particular it holds that the jump of the impedance is zero, namely

$$\left[-\frac{\partial u}{\partial n} + Zu\right] = \left(-\frac{\partial u_e}{\partial n} + Zu_e\right) - \left(-\frac{\partial u_i}{\partial n} + Zu_i\right) = 0. \tag{E.118}$$

An integral representation formula of the solution, for $\boldsymbol{x} \in \Omega_e \cup \Omega_i$, is given by

$$u(\boldsymbol{x}) = \int_\Gamma \left(\frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) - Z(\boldsymbol{y})G(\boldsymbol{x},\boldsymbol{y})\right)\mu(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{y}). \tag{E.119}$$

Since

$$-\frac{1}{2}\left(\frac{\partial u_e}{\partial n}(\boldsymbol{x}) + \frac{\partial u_i}{\partial n}(\boldsymbol{x})\right) + \frac{Z(\boldsymbol{x})}{2}\big(u_e(\boldsymbol{x}) + u_i(\boldsymbol{x})\big) = f_z(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Gamma, \tag{E.120}$$

we obtain, for $\boldsymbol{x} \in \Gamma$, the Fredholm integral equation of the first kind

$$\int_\Gamma \left( -\frac{\partial^2 G}{\partial n_{\boldsymbol{x}} \partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) + Z(\boldsymbol{y}) \frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x}, \boldsymbol{y}) \right) \mu(\boldsymbol{y}) \, \mathrm{d}\gamma(\boldsymbol{y}) + Z(\boldsymbol{x}) \int_\Gamma \left( \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) - Z(\boldsymbol{y}) G(\boldsymbol{x}, \boldsymbol{y}) \right) \mu(\boldsymbol{y}) \, \mathrm{d}\gamma(\boldsymbol{y}) = f_z(\boldsymbol{x}), \tag{E.121}$$

which has to be solved for the unknown $\mu$. In terms of boundary layer potentials, the integral representation and the integral equation can be respectively expressed by

$$u = \mathcal{D}(\mu) - \mathcal{S}(Z\mu) \qquad \text{in } \Omega_e \cup \Omega_i, \tag{E.122}$$

$$-N(\mu) + D^*(Z\mu) + ZD(\mu) - ZS(Z\mu) = f_z \qquad \text{on } \Gamma. \tag{E.123}$$

We observe that the integral equation (E.123) is self-adjoint.

c) Continuous value

We associate to (E.13) the interior problem

$$\begin{cases} \text{Find } u_i : \Omega_i \to \mathbb{C} \text{ such that} \\ \Delta u_i + k^2 u_i = 0 & \text{in } \Omega_i, \\ -\dfrac{\partial u_e}{\partial n} + Z u_i = f_z & \text{on } \Gamma. \end{cases} \tag{E.124}$$

The jumps over $\Gamma$ are characterized in this case by

$$[u] = u_e - u_i = \frac{1}{Z}\left( \frac{\partial u_e}{\partial n} - f_z \right) - \frac{1}{Z}\left( \frac{\partial u_e}{\partial n} - f_z \right) = 0, \tag{E.125}$$

$$\left[ \frac{\partial u}{\partial n} \right] = \frac{\partial u_e}{\partial n} - \frac{\partial u_i}{\partial n} = \nu, \tag{E.126}$$

where $\nu : \Gamma \to \mathbb{C}$ is a function to be determined.

An integral representation formula of the solution, for $\boldsymbol{x} \in \Omega_e \cup \Omega_i$, is given by the single layer potential

$$u(\boldsymbol{x}) = -\int_\Gamma G(\boldsymbol{x}, \boldsymbol{y}) \nu(\boldsymbol{y}) \, \mathrm{d}\gamma(\boldsymbol{y}). \tag{E.127}$$

Since

$$-\frac{1}{2}\left( \frac{\partial u_e}{\partial n}(\boldsymbol{x}) + \frac{\partial u_i}{\partial n}(\boldsymbol{x}) \right) + \frac{Z(\boldsymbol{x})}{2}\big( u_e(\boldsymbol{x}) + u_i(\boldsymbol{x}) \big) = \frac{\nu(\boldsymbol{x})}{2} + f_z(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma, \tag{E.128}$$

we obtain, for $\boldsymbol{x} \in \Gamma$, the Fredholm integral equation of the second kind

$$-\frac{\nu(\boldsymbol{x})}{2} + \int_\Gamma \left( \frac{\partial G}{\partial n_{\boldsymbol{x}}}(\boldsymbol{x}, \boldsymbol{y}) - Z(\boldsymbol{x}) G(\boldsymbol{x}, \boldsymbol{y}) \right) \nu(\boldsymbol{y}) \, \mathrm{d}\gamma(\boldsymbol{y}) = f_z(\boldsymbol{x}), \tag{E.129}$$

which has to be solved for the unknown $\nu$. In terms of boundary layer potentials, the integral representation and the integral equation can be respectively expressed by

$$u = -\mathcal{S}(\nu) \qquad \text{in } \Omega_e \cup \Omega_i, \tag{E.130}$$

$$\frac{\nu}{2} + ZS(\nu) - D^*(\nu) = -f_z \qquad \text{on } \Gamma. \tag{E.131}$$

We observe that the integral equation (E.131) is mutually adjoint with (E.111).

d) Continuous normal derivative

We associate to (E.13) the interior problem

$$\begin{cases} \text{Find } u_i : \Omega_i \to \mathbb{C} \text{ such that} \\ \Delta u_i + k^2 u_i = 0 & \text{in } \Omega_i, \\ -\dfrac{\partial u_i}{\partial n} + Z u_e = f_z & \text{on } \Gamma. \end{cases} \tag{E.132}$$

The jumps over $\Gamma$ are characterized in this case by

$$[u] = u_e - u_i = \mu, \tag{E.133}$$

$$\left[\frac{\partial u}{\partial n}\right] = \frac{\partial u_e}{\partial n} - \frac{\partial u_i}{\partial n} = \big(Z u_e - f_z\big) - \big(Z u_e - f_z\big) = 0, \tag{E.134}$$

where $\mu : \Gamma \to \mathbb{C}$ is a function to be determined.

An integral representation formula of the solution, for $\boldsymbol{x} \in \Omega_e \cup \Omega_i$, is given by the double layer potential

$$u(\boldsymbol{x}) = \int_\Gamma \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) \mu(\boldsymbol{y}) \, \mathrm{d}\gamma(\boldsymbol{y}). \tag{E.135}$$

Since when $\boldsymbol{x} \in \Gamma$,

$$-\frac{1}{2}\left(\frac{\partial u_e}{\partial n}(\boldsymbol{x}) + \frac{\partial u_i}{\partial n}(\boldsymbol{x})\right) + \frac{Z(\boldsymbol{x})}{2}\big(u_e(\boldsymbol{x}) + u_i(\boldsymbol{x})\big) = -\frac{Z(\boldsymbol{x})}{2}\mu(\boldsymbol{x}) + f_z(\boldsymbol{x}), \tag{E.136}$$

we obtain, for $\boldsymbol{x} \in \Gamma$, the Fredholm integral equation of the second kind

$$\frac{Z(\boldsymbol{x})}{2}\mu(\boldsymbol{x}) + \int_\Gamma \left(-\frac{\partial^2 G}{\partial n_{\boldsymbol{x}} \partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) + Z(\boldsymbol{x}) \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y})\right) \mu(\boldsymbol{y}) \, \mathrm{d}\gamma(\boldsymbol{y}) = f_z(\boldsymbol{x}), \tag{E.137}$$

which has to be solved for the unknown $\mu$. In terms of boundary layer potentials, the integral representation and the integral equation can be respectively expressed by

$$u = \mathcal{D}(\mu) \qquad \text{in } \Omega_e \cup \Omega_i, \tag{E.138}$$

$$\frac{Z}{2}\mu - N(\mu) + ZD(\mu) = f_z \qquad \text{on } \Gamma. \tag{E.139}$$

We observe that the integral equation (E.139) is mutually adjoint with (E.114).

## E.7 Far field of the solution

The asymptotic behavior at infinity of the solution $u$ of (E.13) is described by the far field $u^{ff}$. Its expression can be deduced by replacing the far field of the Green's function $G^{ff}$ and its derivatives in the integral representation formula (E.58), which yields

$$u^{ff}(\boldsymbol{x}) = \int_\Gamma \left([u](\boldsymbol{y}) \frac{\partial G^{ff}}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) - G^{ff}(\boldsymbol{x}, \boldsymbol{y}) \left[\frac{\partial u}{\partial n}\right](\boldsymbol{y})\right) \mathrm{d}\gamma(\boldsymbol{y}). \tag{E.140}$$

By replacing now (E.36) and (E.37) in (E.140), we have that the far field of the solution is

$$u^{ff}(\boldsymbol{x}) = \frac{e^{ik|\boldsymbol{x}|}}{4\pi|\boldsymbol{x}|} \int_\Gamma e^{-ik\hat{\boldsymbol{x}}\cdot\boldsymbol{y}} \left(ik\hat{\boldsymbol{x}} \cdot \boldsymbol{n}_{\boldsymbol{y}} \, [u](\boldsymbol{y}) + \left[\frac{\partial u}{\partial n}\right](\boldsymbol{y})\right) \mathrm{d}\gamma(\boldsymbol{y}). \tag{E.141}$$

The asymptotic behavior of the solution $u$ at infinity is therefore given by

$$u(\boldsymbol{x}) = \frac{e^{ik|\boldsymbol{x}|}}{|\boldsymbol{x}|}\left\{u_\infty(\hat{\boldsymbol{x}}) + \mathcal{O}\left(\frac{1}{|\boldsymbol{x}|}\right)\right\}, \qquad |\boldsymbol{x}| \to \infty, \tag{E.142}$$

uniformly in all directions $\hat{\boldsymbol{x}}$ on the unit sphere, where

$$u_\infty(\hat{\boldsymbol{x}}) = \frac{1}{4\pi}\int_\Gamma e^{-ik\hat{\boldsymbol{x}}\cdot\boldsymbol{y}}\left(ik\hat{\boldsymbol{x}}\cdot\boldsymbol{n}_{\boldsymbol{y}}\,[u](\boldsymbol{y}) + \left[\frac{\partial u}{\partial n}\right](\boldsymbol{y})\right)\mathrm{d}\gamma(\boldsymbol{y}) \tag{E.143}$$

is called the far-field pattern of $u$. It can be expressed in decibels (dB) by means of the scattering cross section

$$Q_s(\hat{\boldsymbol{x}}) \;\; [\text{dB}] = 20\log_{10}\left(\frac{|u_\infty(\hat{\boldsymbol{x}})|}{|u_0|}\right), \tag{E.144}$$

where the reference level $u_0$ is typically taken as $u_0 = u_I$ when the incident field is given by a plane wave of the form (E.5), i.e., $|u_0| = 1$.

We remark that the far-field behavior (E.142) of the solution is in accordance with the Sommerfeld radiation condition (E.8), which justifies its choice.

## E.8 Exterior sphere problem

To understand better the resolution of the direct scattering problem (E.13), we study now the particular case when the domain $\Omega_e \subset \mathbb{R}^3$ is taken as the exterior of a sphere of radius $R > 0$. The interior of the sphere is then given by $\Omega_i = \{\boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x}| < R\}$ and its boundary by $\Gamma = \partial\Omega_e$, as shown in Figure E.3. We place the origin at the center of $\Omega_i$ and we consider that the unit normal $\boldsymbol{n}$ is taken outwardly oriented of $\Omega_e$, i.e., $\boldsymbol{n} = -\boldsymbol{r}$.

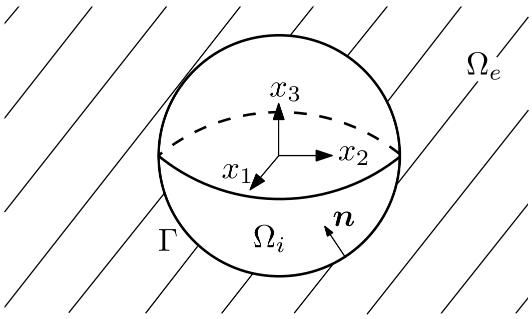



FIGURE E.3. Exterior of the sphere.

The exterior sphere problem is then stated as

$$\begin{cases} \text{Find } u : \Omega_e \to \mathbb{C} \text{ such that} \\ \Delta u + k^2 u = 0 \qquad \text{in } \Omega_e, \\ \dfrac{\partial u}{\partial r} + Zu = f_z \qquad \text{on } \Gamma, \\ \text{+ Outgoing Radiation condition as } |\boldsymbol{x}| \to \infty, \end{cases} \tag{E.145}$$

where we consider a constant impedance $Z \in \mathbb{C}$, a wave number $k > 0$, and where the radiation condition is as usual given by (E.8). As the incident field $u_I$ we consider a plane wave in the form of (E.5), in which case the impedance data function $f_z$ is given by

$$f_z = -\frac{\partial u_I}{\partial r} - Z u_I \qquad \text{on } \Gamma. \tag{E.146}$$

Due the particular chosen geometry, the solution $u$ of (E.145) can be easily found analytically by using the method of variable separation, i.e., by supposing that

$$u(\boldsymbol{x}) = u(r, \theta, \varphi) = h(r) g(\theta) f(\varphi), \tag{E.147}$$

where the radius $r \geq 0$, the polar angle $0 \leq \theta \leq \pi$, and the azimuthal angle $-\pi < \varphi \leq \pi$ denote the spherical coordinates in $\mathbb{R}^3$. If the Helmholtz equation in (E.145) is expressed using spherical coordinates, then

$$\Delta u + k^2 u = \frac{1}{r}\frac{\partial^2}{\partial r^2}(ru) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2} + k^2 u = 0. \tag{E.148}$$

By replacing now (E.147) in (E.148) we obtain

$$h''(r)g(\theta)f(\varphi) + \frac{2}{r}h'(r)g(\theta)f(\varphi) + \frac{h(r)f(\varphi)}{r^2 \sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}g}{\mathrm{d}\theta}(\theta)\right) \\ + \frac{h(r)g(\theta)f''(\varphi)}{r^2\sin^2\theta} + k^2 h(r)g(\theta)f(\varphi) = 0. \tag{E.149}$$

Multiplying by $r^2 \sin^2\theta$, dividing by $hgf$, and rearranging yields

$$r^2\sin^2\theta\left[\frac{h''(r)}{h(r)} + \frac{2}{r}\frac{h'(r)}{h(r)} + \frac{1}{g(\theta)r^2\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}g}{\mathrm{d}\theta}(\theta)\right) + k^2\right] + \frac{f''(\varphi)}{f(\varphi)} = 0. \tag{E.150}$$

The dependence on $\varphi$ has now been isolated in the last term. Consequently this term must be equal to a constant, which for convenience we denote by $-m^2$, i.e.,

$$\frac{f''(\varphi)}{f(\varphi)} = -m^2. \tag{E.151}$$

The solution of (E.151), up to a multiplicative constant, is of the form

$$f(\varphi) = e^{\pm i m \varphi}. \tag{E.152}$$

For $f(\varphi)$ to be single-valued, $m$ must be an integer if the full azimuthal range is allowed. By similar considerations we find the following separate equations for $g(\theta)$ and $h(r)$:

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}g}{\mathrm{d}\theta}(\theta)\right) + \left(l(l+1) - \frac{m^2}{\sin^2\theta}\right)g(\theta) = 0, \tag{E.153}$$

$$r^2 h''(r) + 2r h'(r) + \big(k^2 r^2 - l(l+1)\big)h(r) = 0, \tag{E.154}$$

where $l(l+1)$ is another conveniently denoted real constant. For the equation of the polar angle $\theta$ we consider the change of variables $x = \cos\theta$. In this case (E.153) turns into

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}g}{\mathrm{d}x}(x)\right) + \left(l(l+1) - \frac{m^2}{1-x^2}\right)g(x) = 0, \tag{E.155}$$

which corresponds to the generalized or associated Legendre differential equation (A.323), whose solutions on the interval $-1 \leq x \leq 1$ are the associated Legendre functions $P_l^m$ and $Q_l^m$, which are characterized respectively by (A.330) and (A.331). If the solution is to be single-valued, finite, and continuous in $-1 \leq x \leq 1$, then we have to exclude the solutions $Q_l^m$, take $l$ as a positive integer or zero, and admit for the integer $m$ only the values $-l, -(l-1), \ldots, 0, \ldots, (l-1), l$. The solution of (E.153), up to an arbitrary multiplicative constant, is therefore given by

$$g(\theta) = P_l^m(\cos\theta). \tag{E.156}$$

As for the Laplace equation, we combine the angular factors $g(\theta)$ and $f(\varphi)$ into the spherical harmonics $Y_l^m(\theta, \varphi)$, which are defined in (A.380). For the radial equation (E.154) we consider the change of variables $z = kr$ and express $\psi(z) = h(r)$, which yields the spherical Bessel differential equation of order $l$, namely

$$z^2\psi''(z) + 2z\psi'(z) + \big(z^2 - l(l+1)\big)\psi(z) = 0. \tag{E.157}$$

The independent solutions of (E.157) are $h_l^{(1)}(z)$ and $h_l^{(2)}(z)$, the spherical Hankel functions of order $l$, and therefore the solutions of (E.154) have the general form

$$h(r) = a_l\, h_l^{(1)}(kr) + b_l\, h_l^{(2)}(kr), \qquad l \geq 0, \tag{E.158}$$

where $a_l, b_l \in \mathbb{C}$ are arbitrary constants. The general solution for the Helmholtz equation considers the linear combination of all the solutions in the form (E.147), namely

$$u(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \big(A_{lm}\, h_l^{(1)}(kr) + B_{lm}\, h_l^{(2)}(kr)\big)\, Y_l^m(\theta, \varphi), \tag{E.159}$$

for some undetermined arbitrary constants $A_{lm}, B_{lm} \in \mathbb{C}$. The radiation condition (E.8) implies that

$$B_{lm} = 0, \quad -l \leq m \leq l, \quad l \geq 0. \tag{E.160}$$

Thus the general solution (E.159) turns into

$$u(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm}\, h_l^{(1)}(kr)\, Y_l^m(\theta, \varphi). \tag{E.161}$$

Due the recurrence relation (A.216), the radial derivative of (E.161) is given by

$$\frac{\partial u}{\partial r}(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} \left( \frac{l}{r} h_l^{(1)}(kr) - k h_{l+1}^{(1)}(kr) \right) Y_l^m(\theta, \varphi). \tag{E.162}$$

The constants $A_{lm}$ in (E.161) are determined through the impedance boundary condition on $\Gamma$. For this purpose, we expand the impedance data function $f_z$ into spherical harmonics:

$$f_z(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{lm}\, Y_l^m(\theta, \varphi), \qquad 0 \leq \theta \leq \pi, \;\; -\pi < \varphi \leq \pi, \tag{E.163}$$

where

$$f_{lm} = \int_{-\pi}^{\pi} \int_0^{\pi} f_z(\theta, \varphi)\, \overline{Y_l^m(\theta, \varphi)} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi, \quad m \in \mathbb{Z}, \;\; -l \leq m \leq l. \tag{E.164}$$

In particular, for a plane wave in the form of (E.5) we have the Jacobi-Anger expansion

$$u_I(\boldsymbol{x}) = e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = 4\pi \sum_{l=0}^{\infty} i^l j_l(kr) \sum_{m=-l}^{l} \overline{Y_l^m(\theta_P, \varphi_P)}\, Y_l^m(\theta, \varphi), \tag{E.165}$$

where $j_l$ is the spherical Bessel function of order $l$, and where $\theta_P = \pi - \theta_I$ and $\varphi_P = \varphi_I - \pi$ are the propagation angles of the plane wave, i.e., of the wave vector $\boldsymbol{k}$. We observe that the expression (E.165) can be also written in a more compact manner by using the addition theorem (A.389) and eventually also the relation (A.385). For a plane wave, the impedance data function (E.146) can be thus expressed as

$$f_z(\theta) = -4\pi \sum_{l=0}^{\infty} i^l \left( \left( Z + \frac{l}{R} \right) j_l(kR) - k j_{l+1}(kR) \right) \sum_{m=-l}^{l} \overline{Y_l^m(\theta_P, \varphi_P)}\, Y_l^m(\theta, \varphi), \tag{E.166}$$

which implies that

$$f_{lm} = -4\pi i^l \left( \left( Z + \frac{l}{R} \right) j_l(kR) - k j_{l+1}(kR) \right) \overline{Y_l^m(\theta_P, \varphi_P)}. \tag{E.167}$$

The impedance boundary condition takes therefore the form

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} \left( \left( Z + \frac{l}{R} \right) h_l^{(1)}(kR) - k h_{l+1}^{(1)}(kR) \right) Y_l^m(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{lm}\, Y_l^m(\theta, \varphi). \tag{E.168}$$

We observe that the constants $A_{lm}$ can be uniquely determined only if

$$\left( Z + \frac{l}{R} \right) h_l^{(1)}(kR) - k h_{l+1}^{(1)}(kR) \neq 0 \qquad \text{for } l \in \mathbb{N}_0. \tag{E.169}$$

If this condition is not fulfilled, then the solution is no longer unique. The values $k, Z \in \mathbb{C}$ for which this occurs form a countable set. In particular, for a fixed $k$, the impedances $Z$ which do not fulfill (E.169) can be explicitly characterized by

$$Z = k \frac{h_{l+1}^{(1)}(kR)}{h_l^{(1)}(kR)} - \frac{l}{R} \qquad \text{for } l \in \mathbb{N}_0. \tag{E.170}$$

The wave numbers $k$ which do not fulfill (E.169), for a fixed $Z$, can only be characterized implicitly through the relation

$$\left( Z + \frac{l}{R} \right) h_l^{(1)}(kR) - k h_{l+1}^{(1)}(kR) = 0 \qquad \text{for } l \in \mathbb{N}_0. \tag{E.171}$$

If we suppose now that (E.169) takes place, then

$$A_{lm} = \frac{R f_{lm}}{(ZR + l)\, h_l^{(1)}(kR) - kR\, h_{l+1}^{(1)}(kR)}. \tag{E.172}$$

In the case of a plane wave we consider for $f_{lm}$ the expression (E.167). The unique solution for the exterior sphere problem (E.145) is then given by

$$u(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{R f_{lm}\, h_l^{(1)}(kr)\, Y_l^m(\theta, \varphi)}{(ZR + l)\, h_l^{(1)}(kR) - kR\, h_{l+1}^{(1)}(kR)}. \tag{E.173}$$

We remark that there is no need here for an additional compatibility condition like (B.191).

If the condition (E.169) does not hold for some particular $n \in \mathbb{N}_0$, then the solution $u$ is not unique. The constants $A_{nm}$ are then no longer defined by (E.172), and can be chosen in an arbitrary manner. For the existence of a solution in this case, however, we require also the orthogonality conditions $f_{nm} = 0$ for $-n \leq m \leq n$. Instead of (E.173), the solution of (E.145) is now given by the infinite family of functions

$$u(r,\theta,\varphi) = \sum_{0\leq l\neq n}\sum_{m=-l}^{l} \frac{Rf_{lm}\, h_l^{(1)}(kr)\, Y_l^m(\theta,\varphi)}{(ZR+l)\, h_l^{(1)}(kR) - kR\, h_{l+1}^{(1)}(kR)} + \sum_{m=-n}^{n} \alpha_m\, h_n^{(1)}(kr)\, Y_n^m(\theta,\varphi), \tag{E.174}$$

where $\alpha_m \in \mathbb{C}$ for $-n \leq m \leq n$ are arbitrary and where their associated terms have the form of volume waves, i.e., waves that propagate inside $\Omega_e$. The exterior sphere problem (E.145) admits thus a unique solution $u$, except on a countable set of values for $k$ and $Z$ which do not fulfill the condition (E.169). And even in this last case there exists a solution, although not unique, if $2n+1$ orthogonality conditions are additionally satisfied. This behavior for the existence and uniqueness of the solution is typical of the Fredholm alternative, which applies when solving problems that involve compact perturbations of invertible operators.

## E.9 Existence and uniqueness

### E.9.1 Function spaces

To state a precise mathematical formulation of the herein treated problems, we have to define properly the involved function spaces. For the associated interior problems defined on the bounded set $\Omega_i$ we use the classical Sobolev space (vid. Section A.4)

$$H^1(\Omega_i) = \{v : \ v \in L^2(\Omega_i),\ \nabla v \in L^2(\Omega_i)^3\}, \tag{E.175}$$

which is a Hilbert space and has the norm

$$\|v\|_{H^1(\Omega_i)} = \Big(\|v\|^2_{L^2(\Omega_i)} + \|\nabla v\|^2_{L^2(\Omega_i)^3}\Big)^{1/2}. \tag{E.176}$$

For the exterior problem defined on the unbounded domain $\Omega_e$, on the other hand, we introduce the weighted Sobolev space (cf. Nédélec 2001)

$$W^1(\Omega_e) = \left\{v : \ \frac{v}{(1+r^2)^{1/2}} \in L^2(\Omega_e),\ \frac{\nabla v}{(1+r^2)^{1/2}} \in L^2(\Omega_e)^3,\ \frac{\partial v}{\partial r} - ikv \in L^2(\Omega_e)\right\}, \tag{E.177}$$

where $r = |\boldsymbol{x}|$. If $W^1(\Omega_e)$ is provided with the norm

$$\|v\|_{W^1(\Omega_e)} = \left( \left\|\frac{v}{(1+r^2)^{1/2}}\right\|^2_{L^2(\Omega_e)} + \left\|\frac{\nabla v}{(1+r^2)^{1/2}}\right\|^2_{L^2(\Omega_e)^3} + \left\|\frac{\partial v}{\partial r} - ikv\right\|^2_{L^2(\Omega_e)} \right)^{1/2}, \tag{E.178}$$

then it becomes a Hilbert space. The restriction to any bounded open set $B \subset \Omega_e$ of the functions of $W^1(\Omega_e)$ belongs to $H^1(B)$, i.e., we have the inclusion $W^1(\Omega_e) \subset H^1_{\text{loc}}(\Omega_e)$, and the functions in these two spaces differ only by their behavior at infinity. We remark that the space $W^1(\Omega_e)$ contains the constant functions and all the functions of $H^1_{\text{loc}}(\Omega_e)$ that satisfy the radiation condition (E.8).

When dealing with Sobolev spaces, even a strong Lipschitz boundary $\Gamma \in C^{0,1}$ is admissible. In this case, and due the trace theorem (A.531), if $v \in H^1(\Omega_i)$ or $v \in W^1(\Omega_e)$, then the trace of $v$ fulfills

$$\gamma_0 v = v|_\Gamma \in H^{1/2}(\Gamma). \tag{E.179}$$

Moreover, the trace of the normal derivative can be also defined, and it holds that

$$\gamma_1 v = \frac{\partial v}{\partial n}|_\Gamma \in H^{-1/2}(\Gamma). \tag{E.180}$$

### E.9.2 Regularity of the integral operators

The boundary integral operators (E.78), (E.79), (E.80), and (E.81) can be characterized as linear and continuous applications such that

$$S : H^{-1/2+s}(\Gamma) \longrightarrow H^{1/2+s}(\Gamma), \qquad D : H^{1/2+s}(\Gamma) \longrightarrow H^{3/2+s}(\Gamma), \tag{E.181}$$

$$D^* : H^{-1/2+s}(\Gamma) \longrightarrow H^{1/2+s}(\Gamma), \qquad N : H^{1/2+s}(\Gamma) \longrightarrow H^{-1/2+s}(\Gamma). \tag{E.182}$$

This result holds for any $s \in \mathbb{R}$ if the boundary $\Gamma$ is of class $C^\infty$, which can be derived from the theory of singular integral operators with pseudo-homogeneous kernels (cf., e.g., Nédélec 2001). Due the compact injection (A.554), it holds also that the operators

$$D : H^{1/2+s}(\Gamma) \longrightarrow H^{1/2+s}(\Gamma) \quad \text{and} \quad D^* : H^{-1/2+s}(\Gamma) \longrightarrow H^{-1/2+s}(\Gamma) \tag{E.183}$$

are compact. For a strong Lipschitz boundary $\Gamma \in C^{0,1}$, on the other hand, these results hold only when $|s| < 1$ (cf. Costabel 1988). In the case of more regular boundaries, the range for $s$ increases, but remains finite. For our purposes we use $s = 0$, namely

$$S : H^{-1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma), \qquad D : H^{1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma), \tag{E.184}$$

$$D^* : H^{-1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma), \qquad N : H^{1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma), \tag{E.185}$$

which are all linear and continuous operators, and where the operators $D$ and $D^*$ are compact. Similarly, we can characterize the single and double layer potentials defined respectively in (E.73) and (E.74) as linear and continuous integral operators such that

$$\mathcal{S} : H^{-1/2}(\Gamma) \longrightarrow W^1(\Omega_e \cup \Omega_i) \quad \text{and} \quad \mathcal{D} : H^{1/2}(\Gamma) \longrightarrow W^1(\Omega_e \cup \Omega_i). \tag{E.186}$$

### E.9.3 Application to the integral equations

It is not difficult to see that if $\mu \in H^{1/2}(\Gamma)$ and $\nu \in H^{-1/2}(\Gamma)$ are given, then the transmission problem (E.41) admits a unique solution $u \in W^1(\Omega_e \cup \Omega_i)$, as a consequence of the integral representation formula (E.59). For the direct scattering problem (E.13), though, this is not always the case, as was appreciated in the exterior sphere problem (E.145). Nonetheless, if the Fredholm alternative applies, then we know that the existence and uniqueness of the problem can be ensured almost always, i.e., except on a countable set of values for the wave number and for the impedance.

We consider an impedance $Z \in L^\infty(\Gamma)$ and an impedance data function $f_z \in H^{-1/2}(\Gamma)$. In both cases all the continuous functions on $\Gamma$ are included.

a) First extension by zero

Let us consider the first integral equation of the extension-by-zero alternative (E.109), which is given in terms of boundary layer potentials, for $\mu \in H^{1/2}(\Gamma)$, by

$$\frac{\mu}{2} + S(Z\mu) - D(\mu) = S(f_z) \qquad \text{in } H^{1/2}(\Gamma). \tag{E.187}$$

Due the imbedding properties of Sobolev spaces and in the same way as for the full-plane impedance Laplace problem, it holds that the left-hand side of the integral equation corresponds to an identity and two compact operators, and thus Fredholm's alternative applies.

b) Second extension by zero

The second integral equation of the extension-by-zero alternative (E.113) is given in terms of boundary layer potentials, for $\mu \in H^{1/2}(\Gamma)$, by

$$\frac{Z}{2}\mu - N(\mu) + D^*(Z\mu) = \frac{f_z}{2} + D^*(f_z) \qquad \text{in } H^{-1/2}(\Gamma). \tag{E.188}$$

The operator $N$ plays the role of the identity and the other terms on the left-hand side are compact, thus Fredholm's alternative holds.

c) Continuous impedance

The integral equation of the continuous-impedance alternative (E.121) is given in terms of boundary layer potentials, for $\mu \in H^{1/2}(\Gamma)$, by

$$-N(\mu) + D^*(Z\mu) + ZD(\mu) - ZS(Z\mu) = f_z \qquad \text{in } H^{-1/2}(\Gamma). \tag{E.189}$$

Again, the operator $N$ plays the role of the identity and the remaining terms on the left-hand side are compact, thus Fredholm's alternative applies.

d) Continuous value

The integral equation of the continuous-value alternative (E.129) is given in terms of boundary layer potentials, for $\nu \in H^{-1/2}(\Gamma)$, by

$$\frac{\nu}{2} + ZS(\nu) - D^*(\nu) = -f_z \qquad \text{in } H^{-1/2}(\Gamma). \tag{E.190}$$

On the left-hand side we have an identity operator and the remaining operators are compact, thus Fredholm's alternative holds.

e) Continuous normal derivative

The integral equation of the continuous-normal-derivative alternative (E.137) is given in terms of boundary layer potentials, for $\mu \in H^{1/2}(\Gamma)$, by

$$\frac{Z}{2}\mu - N(\mu) + ZD(\mu) = f_z \qquad \text{in } H^{-1/2}(\Gamma). \tag{E.191}$$

As before, Fredholm's alternative again applies, since on the left-hand side we have the operator $N$ and two compact operators.

### E.9.4 Consequences of Fredholm's alternative

Since the Fredholm alternative applies to each integral equation, therefore it applies also to the exterior differential problem (E.13) due the integral representation formula. The existence of the exterior problem's solution is thus determined by its uniqueness, and the wave numbers $k \in \mathbb{C}$ and impedances $Z \in \mathbb{C}$ for which the uniqueness is lost constitute a countable set, which we call respectively wave number spectrum and impedance spectrum of the exterior problem and denote them by $\sigma_k$ and $\sigma_Z$. The spectrum $\sigma_k$ considers a fixed $Z$ and, conversely, the spectrum $\sigma_Z$ considers a fixed $k$. The existence and uniqueness of the solution is therefore ensured almost everywhere. The same holds obviously for the solution of the integral equation, whose wave number spectrum and impedance spectrum we denote respectively by $\varsigma_k$ and $\varsigma_Z$. Since each integral equation is derived from the exterior problem, it holds that $\sigma_k \subset \varsigma_k$ and $\sigma_Z \subset \varsigma_Z$. The converse, though, is not necessarily true and depends on each particular integral equation. In any way, the sets $\varsigma_k \setminus \sigma_k$ and $\varsigma_Z \setminus \sigma_Z$ are at most countable.

Fredholm's alternative applies as much to the integral equation itself as to its adjoint counterpart, and equally to their homogeneous versions. Moreover, each integral equation solves at the same time an exterior and an interior differential problem. The loss of uniqueness of the integral equation's solution appears when the wave number $k$ and the impedance $Z$ are eigenvalues of some associated interior problem, either of the homogeneous integral equation or of its adjoint counterpart. Such a wave number $k$ or impedance $Z$ are contained respectively in $\varsigma_k$ or $\varsigma_Z$.

The integral equation (E.111) is associated with the extension by zero (E.104), for which no eigenvalues appear. Nevertheless, its adjoint integral equation (E.131) of the continuous value is associated with the interior problem (E.124), which has a countable amount of eigenvalues $k$, but behaves otherwise well for all $Z \neq 0$.

The integral equation (E.114) is also associated with the extension by zero (E.104), for which no eigenvalues appear. Nonetheless, its adjoint integral equation (E.139) of the continuous normal derivative is associated with the interior problem (E.132), which has a countable amount of eigenvalues $k$, but behaves well for all $Z$, without restriction.

The integral equation (E.123) of the continuous impedance is self-adjoint and is associated with the interior problem (E.115), which has a countable quantity of eigenvalues $k$ and $Z$.

Let us consider now the transmission problem generated by the homogeneous exterior problem

$$\begin{cases} \text{Find } u_e : \Omega_e \to \mathbb{C} \text{ such that} \\ \Delta u_e + k^2 u_e = 0 \qquad \text{in } \Omega_e, \\ -\dfrac{\partial u_e}{\partial n} + Z u_e = 0 \qquad \text{on } \Gamma, \\ + \text{ Outgoing radiation condition as } |\boldsymbol{x}| \to \infty, \end{cases} \tag{E.192}$$

and the associated homogeneous interior problem

$$\begin{cases} \text{Find } u_i : \Omega_i \to \mathbb{C} \text{ such that} \\ \Delta u_i + k^2 u_i = 0 \qquad \text{in } \Omega_i, \\ \dfrac{\partial u_i}{\partial n} + Z u_i = 0 \qquad \text{on } \Gamma, \end{cases} \tag{E.193}$$

where the radiation condition is as usual given by (E.8), and where the unit normal $\boldsymbol{n}$ always points outwards of $\Omega_e$.

As in the two-dimensional case, it holds again that the integral equations for this transmission problem have either the same left-hand side or are mutually adjoint to all other possible alternatives of integral equations that can be built for the exterior problem (E.13), and in particular to all the alternatives that were mentioned in the last subsection. The eigenvalues $k$ and $Z$ of the homogeneous interior problem (E.193) are thus also contained respectively in $\varsigma_k$ and $\varsigma_Z$.

We remark that additional alternatives for integral representations and equations based on non-homogeneous versions of the problem (E.193) can be also derived for the exterior impedance problem (cf. Ha-Duong 1987).

The determination of the wave number spectrum $\sigma_k$ and the impedance spectrum $\sigma_Z$ of the exterior problem (E.13) is not so easy, but can be achieved for simple geometries where an analytic solution is known.

In conclusion, the exterior problem (E.13) admits a unique solution $u$ if $k \notin \sigma_k$, and $Z \notin \sigma_Z$, and each integral equation admits a unique solution, either $\mu$ or $\nu$, if $k \notin \varsigma_k$ and $Z \notin \varsigma_Z$.

## E.10 Dissipative problem

The dissipative problem considers waves that lose their amplitude as they travel through the medium. These waves dissipate their energy as they propagate and are modeled by a complex wave number $k \in \mathbb{C}$ whose imaginary part is strictly positive, i.e., $\mathfrak{Im}\{k\} > 0$. This choice ensures that the Green's function (E.22) decreases exponentially at infinity. Due the dissipative nature of the medium, it is no longer suited to take plane waves in the form of (E.5) as the incident field $u_I$. Instead, we have to take a source of volume waves at a finite distance from the obstacle. For example, we can consider a point source located at $\boldsymbol{z} \in \Omega_e$, in which case the incident field is given, up to a multiplicative constant, by

$$u_I(\boldsymbol{x}) = G(\boldsymbol{x}, \boldsymbol{z}) = -\frac{e^{ik|\boldsymbol{x}-\boldsymbol{z}|}}{4\pi|\boldsymbol{x}-\boldsymbol{z}|} = -\frac{ik}{4\pi} h_0^{(1)}\big(k|\boldsymbol{x}-\boldsymbol{z}|\big). \tag{E.194}$$

This incident field $u_I$ satisfies the Helmholtz equation with a source term in the right-hand side, namely

$$\Delta u_I + k^2 u_I = \delta_{\boldsymbol{z}} \qquad \text{in } \mathcal{D}'(\Omega_e), \tag{E.195}$$

which holds also for the total field $u_T$ but not for the scattered field $u$, in which case the Helmholtz equation remains homogeneous. For a general source distribution $g_s$, whose support is contained in $\Omega_e$, the incident field can be expressed by

$$u_I(\boldsymbol{x}) = G(\boldsymbol{x},\boldsymbol{z}) * g_s(\boldsymbol{z}) = \int_{\Omega_e} G(\boldsymbol{x},\boldsymbol{z})\, g_s(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}. \tag{E.196}$$

This incident field $u_I$ satisfies now

$$\Delta u_I + k^2 u_I = g_s \qquad \text{in } \mathcal{D}'(\Omega_e), \tag{E.197}$$

which holds again also for the total field $u_T$ but not for the scattered field $u$.

The dissipative nature of the medium implies also that a radiation condition like (E.8) is no longer required. The ingoing waves are ruled out, since they verify $\mathfrak{Im}\{k\} < 0$. The dissipative scattering problem can be therefore stated as

$$\begin{cases} \text{Find } u : \Omega_e \to \mathbb{C} \text{ such that} & \\ \Delta u + k^2 u = 0 & \text{in } \Omega_e, \\ -\dfrac{\partial u}{\partial n} + Zu = f_z & \text{on } \Gamma, \end{cases} \tag{E.198}$$

where the impedance data function $f_z$ is again given by

$$f_z = \frac{\partial u_I}{\partial n} - Zu_I \qquad \text{on } \Gamma. \tag{E.199}$$

The solution is now such that $u \in H^1(\Omega_e)$ (cf., e.g., Hazard & Lenoir 1998, Lenoir 2005), therefore, instead of (E.52) and (E.53), we obtain that

$$\left| \int_{S_R} \left( u(\boldsymbol{y}) \frac{\partial G}{\partial r_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) - G(\boldsymbol{x},\boldsymbol{y}) \frac{\partial u}{\partial r}(\boldsymbol{y}) \right) \mathrm{d}\gamma(\boldsymbol{y}) \right| \leq \frac{C}{R}\, e^{-R\,\mathfrak{Im}\{k\}}. \tag{E.200}$$

It is not difficult to see that all the other developments performed for the non-dissipative case are also valid when considering dissipation. The only difference is that now a complex wave number $k$ such that $\mathfrak{Im}\{k\} > 0$ has to be taken everywhere into account and that the outgoing radiation condition is no longer needed.

## E.11 Variational formulation

To solve a particular integral equation we convert it to its variational or weak formulation, i.e., we solve it with respect to certain test functions in a bilinear (or sesquilinear) form. Basically, the integral equation is multiplied by the (conjugated) test function and then the equation is integrated over the boundary of the domain. The test functions are taken in the same function space as the solution of the integral equation.

a) First extension by zero

The variational formulation for the first integral equation (E.187) of the extension-by-zero alternative searches $\mu \in H^{1/2}(\Gamma)$ such that $\forall \varphi \in H^{1/2}(\Gamma)$

$$\left\langle \frac{\mu}{2} + S(Z\mu) - D(\mu), \varphi \right\rangle = \big\langle S(f_z), \varphi \big\rangle. \tag{E.201}$$

b) Second extension by zero

The variational formulation for the second integral equation (E.188) of the extension-by-zero alternative searches $\mu \in H^{1/2}(\Gamma)$ such that $\forall \varphi \in H^{1/2}(\Gamma)$

$$\left\langle \frac{Z}{2}\mu - N(\mu) + D^*(Z\mu), \varphi \right\rangle = \left\langle \frac{f_z}{2} + D^*(f_z), \varphi \right\rangle. \tag{E.202}$$

c) Continuous impedance

The variational formulation for the integral equation (E.189) of the alternative of the continuous-impedance searches $\mu \in H^{1/2}(\Gamma)$ such that $\forall \varphi \in H^{1/2}(\Gamma)$

$$\big\langle -N(\mu) + D^*(Z\mu) + ZD(\mu) - ZS(Z\mu), \varphi \big\rangle = \big\langle f_z, \varphi \big\rangle. \tag{E.203}$$

d) Continuous value

The variational formulation for the integral equation (E.190) of the continuous-value alternative searches $\nu \in H^{-1/2}(\Gamma)$ such that $\forall \psi \in H^{-1/2}(\Gamma)$

$$\left\langle \frac{\nu}{2} + ZS(\nu) - D^*(\nu), \psi \right\rangle = \big\langle -f_z, \psi \big\rangle. \tag{E.204}$$

e) Continuous normal derivative

The variational formulation for the integral equation (E.191) of the continuous-normal-derivative alternative searches $\mu \in H^{1/2}(\Gamma)$ such that $\forall \varphi \in H^{1/2}(\Gamma)$

$$\left\langle \frac{Z}{2}\mu - N(\mu) + ZD(\mu), \varphi \right\rangle = \big\langle f_z, \varphi \big\rangle. \tag{E.205}$$

## E.12 Numerical discretization

### E.12.1 Discretized function spaces

The exterior problem (E.13) is solved numerically with the boundary element method by employing a Galerkin scheme on the variational formulation of an integral equation. We use on the boundary surface $\Gamma$ Lagrange finite elements of type either $\mathbb{P}_1$ or $\mathbb{P}_0$. The surface $\Gamma$ is approximated by the triangular mesh $\Gamma^h$, composed by $T$ flat triangles $T_j$, $1 \leq j \leq T$, and $I$ nodes $\boldsymbol{r}_i \in \mathbb{R}^3$, $1 \leq i \leq I$. The triangles have a diameter less or equal than $h$, and their vertices or corners, i.e., the nodes $\boldsymbol{r}_i$, are on top of $\Gamma$, as shown in Figure E.4. The diameter of a triangle $K$ is given by

$$\operatorname{diam}(K) = \sup_{\boldsymbol{x},\boldsymbol{y} \in K} |\boldsymbol{y} - \boldsymbol{x}|. \tag{E.206}$$

The function space $H^{1/2}(\Gamma)$ is approximated using the conformal space of continuous piecewise linear polynomials with complex coefficients

$$Q_h = \big\{ \varphi_h \in C^0(\Gamma^h) \,:\, \varphi_h|_{T_j} \in \mathbb{P}_1(\mathbb{C}), \;\; 1 \leq j \leq T \big\}. \tag{E.207}$$

The space $Q_h$ has a finite dimension $I$, and we describe it using the standard base functions for finite elements of type $\mathbb{P}_1$, which we denote by $\{\chi_j\}_{j=1}^{I}$. The base function $\chi_j$ is

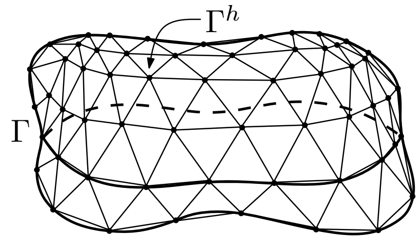



FIGURE E.4. Mesh $\Gamma^h$, discretization of $\Gamma$.

associated with the node $\boldsymbol{r}_j$ and has its support $\operatorname{supp}\chi_j$ on the triangles that have $\boldsymbol{r}_j$ as one of their vertices. On $\boldsymbol{r}_j$ it has a value of one and on the opposed edges of the triangles its value is zero, being linearly interpolated in between and zero otherwise.

The function space $H^{-1/2}(\Gamma)$, on the other hand, is approximated using the conformal space of piecewise constant polynomials with complex coefficients

$$P_h = \left\{\psi_h : \Gamma^h \to \mathbb{C} \mid \psi_h|_{T_j} \in \mathbb{P}_0(\mathbb{C}), \;\; 1 \leq j \leq T\right\}. \tag{E.208}$$

The space $P_h$ has a finite dimension $T$, and is described using the standard base functions for finite elements of type $\mathbb{P}_0$, which we denote by $\{\kappa_j\}_{j=1}^{T}$.

In virtue of this discretization, any function $\varphi_h \in Q_h$ or $\psi_h \in P_h$ can be expressed as a linear combination of the elements of the base, namely

$$\varphi_h(\boldsymbol{x}) = \sum_{j=1}^{I} \varphi_j\, \chi_j(\boldsymbol{x}) \qquad \text{and} \qquad \psi_h(\boldsymbol{x}) = \sum_{j=1}^{T} \psi_j\, \kappa_j(\boldsymbol{x}) \qquad \text{for } \boldsymbol{x} \in \Gamma^h, \tag{E.209}$$

where $\varphi_j, \psi_j \in \mathbb{C}$. The solutions $\mu \in H^{1/2}(\Gamma)$ and $\nu \in H^{-1/2}(\Gamma)$ of the variational formulations can be therefore approximated respectively by

$$\mu_h(\boldsymbol{x}) = \sum_{j=1}^{I} \mu_j\, \chi_j(\boldsymbol{x}) \qquad \text{and} \qquad \nu_h(\boldsymbol{x}) = \sum_{j=1}^{T} \nu_j\, \kappa_j(\boldsymbol{x}) \qquad \text{for } \boldsymbol{x} \in \Gamma^h, \tag{E.210}$$

where $\mu_j, \nu_j \in \mathbb{C}$. The function $f_z$ can be also approximated by

$$f_z^h(\boldsymbol{x}) = \sum_{j=1}^{I} f_j\, \chi_j(\boldsymbol{x}) \qquad \text{for } \boldsymbol{x} \in \Gamma^h, \quad \text{with } f_j = f_z(\boldsymbol{r}_j), \tag{E.211}$$

or

$$f_z^h(\boldsymbol{x}) = \sum_{j=1}^{T} f_j\, \kappa_j(\boldsymbol{x}) \qquad \text{for } \boldsymbol{x} \in \Gamma^h, \quad \text{with } f_j = \frac{f_z(\boldsymbol{r}_1^j) + f_z(\boldsymbol{r}_2^j) + f_z(\boldsymbol{r}_3^j)}{3}, \tag{E.212}$$

depending on whether the original integral equation is stated in $H^{1/2}(\Gamma)$ or in $H^{-1/2}(\Gamma)$. We denote by $\boldsymbol{r}_d^j$, for $d \in \{1, 2, 3\}$, the three vertices of triangle $T_j$.

### E.12.2 Discretized integral equations

a) First extension by zero

To see how the boundary element method operates, we apply it to the first integral equation of the extension-by-zero alternative, i.e., to the variational formulation (E.201). We characterize all the discrete approximations by the index $h$, including also the impedance and the boundary layer potentials. The numerical approximation of (E.201) leads to the discretized problem that searches $\mu_h \in Q_h$ such that $\forall \varphi_h \in Q_h$

$$\Big\langle \frac{\mu_h}{2} + S_h(Z_h\mu_h) - D_h(\mu_h), \varphi_h \Big\rangle = \big\langle S_h(f_z^h), \varphi_h \big\rangle. \qquad \text{(E.213)}$$

Considering the decomposition of $\mu_h$ in terms of the base $\{\chi_j\}$ and taking as test functions the same base functions, $\varphi_h = \chi_i$ for $1 \le i \le I$, yields the discrete linear system

$$\sum_{j=1}^{I} \mu_j \left( \frac{1}{2}\langle \chi_j, \chi_i\rangle + \langle S_h(Z_h\chi_j), \chi_i\rangle - \langle D_h(\chi_j), \chi_i\rangle \right) = \sum_{j=1}^{I} f_j \, \langle S_h(\chi_j), \chi_i\rangle. \qquad \text{(E.214)}$$

This constitutes a system of linear equations that can be expressed as a linear matrix system:

$$\begin{cases} \text{Find } \boldsymbol{\mu} \in \mathbb{C}^I \text{ such that} \\ \boldsymbol{M}\boldsymbol{\mu} = \boldsymbol{b}. \end{cases} \qquad \text{(E.215)}$$

The elements $m_{ij}$ of the matrix $\boldsymbol{M}$ are given by

$$m_{ij} = \frac{1}{2}\langle \chi_j, \chi_i\rangle + \langle S_h(Z_h\chi_j), \chi_i\rangle - \langle D_h(\chi_j), \chi_i\rangle \qquad \text{for } \; 1 \le i,j \le I, \qquad \text{(E.216)}$$

and the elements $b_i$ of the vector $\boldsymbol{b}$ by

$$b_i = \big\langle S_h(f_z^h), \chi_i \big\rangle = \sum_{j=1}^{I} f_j \, \langle S_h(\chi_j), \chi_i\rangle \qquad \text{for } \; 1 \le i \le I. \qquad \text{(E.217)}$$

The discretized solution $u_h$, which approximates $u$, is finally obtained by discretizing the integral representation formula (E.110) according to

$$u_h = \mathcal{D}_h(\mu_h) - \mathcal{S}_h(Z_h\mu_h) + \mathcal{S}_h(f_z^h), \qquad \text{(E.218)}$$

which, more specifically, can be expressed as

$$u_h = \sum_{j=1}^{I} \mu_j \big(\mathcal{D}_h(\chi_j) - \mathcal{S}_h(Z_h\chi_j)\big) + \sum_{j=1}^{I} f_j \, \mathcal{S}_h(\chi_j). \qquad \text{(E.219)}$$

By proceeding in the same way, the discretization of all the other alternatives of integral equations can be also expressed as a linear matrix system like (E.215). The resulting matrix $\boldsymbol{M}$ is in general complex, full, non-symmetric, and with dimensions $I \times I$ for elements of type $\mathbb{P}_1$ and $T \times T$ for elements of type $\mathbb{P}_0$. The right-hand side vector $\boldsymbol{b}$ is complex and of size either $I$ or $T$. The boundary element calculations required to compute numerically the elements of $\boldsymbol{M}$ and $\boldsymbol{b}$ have to be performed carefully, since the integrals that appear become singular when the involved triangles are coincident, or when they have a common vertex or edge, due the singularity of the Green's function at its source point.

b) Second extension by zero

In the case of the second integral equation of the extension-by-zero alternative, i.e., of the variational formulation (E.202), the elements $m_{ij}$ that constitute the matrix $\boldsymbol{M}$ of the linear system (E.215) are given by

$$m_{ij} = \frac{1}{2}\langle Z_h\chi_j, \chi_i\rangle - \langle N_h(\chi_j), \chi_i\rangle + \langle D_h^*(Z_h\chi_j), \chi_i\rangle \qquad \text{for } 1 \leq i, j \leq I, \tag{E.220}$$

whereas the elements $b_i$ of the vector $\boldsymbol{b}$ are expressed as

$$b_i = \sum_{j=1}^{I} f_j \left(\frac{1}{2}\langle \chi_j, \chi_i\rangle + \langle D_h^*(Z_h\chi_j), \chi_i\rangle\right) \qquad \text{for } 1 \leq i \leq I. \tag{E.221}$$

The discretized solution $u_h$ is again computed by (E.219).

c) Continuous impedance

In the case of the continuous-impedance alternative, i.e., of the variational formulation (E.203), the elements $m_{ij}$ that constitute the matrix $\boldsymbol{M}$ of the linear system (E.215) are given, for $1 \leq i, j \leq I$, by

$$m_{ij} = -\langle N_h(\chi_j), \chi_i\rangle + \langle D_h^*(Z_h\chi_j), \chi_i\rangle + \langle Z_h D_h(\chi_j), \chi_i\rangle - \langle Z_h S_h(Z_h\chi_j), \chi_i\rangle, \tag{E.222}$$

whereas the elements $b_i$ of the vector $\boldsymbol{b}$ are expressed as

$$b_i = \sum_{j=1}^{I} f_j \langle \chi_j, \chi_i\rangle \qquad \text{for } 1 \leq i \leq I. \tag{E.223}$$

It can be observed that for this particular alternative the matrix $\boldsymbol{M}$ turns out to be symmetric, since the integral equation is self-adjoint. The discretized solution $u_h$, due (E.122), is then computed by

$$u_h = \sum_{j=1}^{I} \mu_j \big(\mathcal{D}_h(\chi_j) - \mathcal{S}_h(Z_h\chi_j)\big). \tag{E.224}$$

d) Continuous value

In the case of the continuous-value alternative, that is, of the variational formulation (E.204), the elements $m_{ij}$ that constitute the matrix $\boldsymbol{M}$, now of the linear system

$$\begin{cases} \text{Find } \boldsymbol{\nu} \in \mathbb{C}^T \text{ such that} \\ \boldsymbol{M\nu} = \boldsymbol{b}, \end{cases} \tag{E.225}$$

are given by

$$m_{ij} = \frac{1}{2}\langle \kappa_j, \kappa_i\rangle + \langle Z_h S_h(\kappa_j), \kappa_i\rangle - \langle D_h^*(\kappa_j), \kappa_i\rangle \qquad \text{for } 1 \leq i, j \leq T, \tag{E.226}$$

whereas the elements $b_i$ of the vector $\boldsymbol{b}$ are expressed as

$$b_i = -\sum_{j=1}^{T} f_j \langle \kappa_j, \kappa_i\rangle \qquad \text{for } 1 \leq i \leq T. \tag{E.227}$$

The discretized solution $u_h$, due (E.130), is then computed by

$$u_h = -\sum_{j=1}^{T} \nu_j \, \mathcal{S}_h(\kappa_j). \tag{E.228}$$

e) Continuous normal derivative

In the case of the continuous-normal-derivative alternative, i.e., of the variational formulation (E.205), the elements $m_{ij}$ that conform the matrix $\boldsymbol{M}$ of the linear system (E.215) are given by

$$m_{ij} = \frac{1}{2}\langle Z_h\chi_j, \chi_i\rangle - \langle N_h(\chi_j), \chi_i\rangle + \langle Z_h D_h(\chi_j), \chi_i\rangle \qquad \text{for } 1 \leq i, j \leq I, \tag{E.229}$$

whereas the elements $b_i$ of the vector $\boldsymbol{b}$ are expressed as

$$b_i = \sum_{j=1}^{I} f_j \, \langle \chi_j, \chi_i\rangle \qquad \text{for } 1 \leq i \leq I. \tag{E.230}$$

The discretized solution $u_h$, due (E.138), is then computed by

$$u_h = \sum_{j=1}^{I} \mu_j \, \mathcal{D}_h(\chi_j). \tag{E.231}$$

## E.13 Boundary element calculations

The boundary element calculations build the elements of the matrix $\boldsymbol{M}$ resulting from the discretization of the integral equation, i.e., from (E.215) or (E.225). They permit thus to compute numerically expressions like (E.216). To evaluate the appearing singular integrals, we use the semi-numerical methods described in the report of Bendali & Devys (1986).

We use the same notation as in Section D.12, and the required boundary element integrals, for $a, b \in \{0, 1\}$ and $c, d \in \{1, 2, 3\}$, are again

$$ZA_{a,b}^{c,d} = \int_K \int_L \left(\frac{s_c}{h_c^K}\right)^a \left(\frac{t_d}{h_d^L}\right)^b G(\boldsymbol{x}, \boldsymbol{y}) \, \mathrm{d}L(\boldsymbol{y}) \, \mathrm{d}K(\boldsymbol{x}), \tag{E.232}$$

$$ZB_{a,b}^{c,d} = \int_K \int_L \left(\frac{s_c}{h_c^K}\right)^a \left(\frac{t_d}{h_d^L}\right)^b \frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x}, \boldsymbol{y}) \, \mathrm{d}L(\boldsymbol{y}) \, \mathrm{d}K(\boldsymbol{x}), \tag{E.233}$$

All the integrals that stem from the numerical discretization can be expressed in terms of these two basic boundary element integrals. The impedance is again discretized as a piecewise constant function $Z_h$, which on each triangle $T_j$ adopts a constant value $Z_j \in \mathbb{C}$. The integrals of interest are the same as for the Laplace equation, except for the hypersingular term, which is now given by

$$\begin{aligned}\langle N_h(\chi_j),\chi_i\rangle &= -\int_{\Gamma^h}\int_{\Gamma^h} G(\boldsymbol{x},\boldsymbol{y})\big(\nabla\chi_j(\boldsymbol{y})\times\boldsymbol{n}_{\boldsymbol{y}}\big)\cdot\big(\nabla\chi_i(\boldsymbol{x})\times\boldsymbol{n}_{\boldsymbol{x}}\big)\,\mathrm{d}\gamma(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{x})\\
&\quad + k^2\int_{\Gamma^h}\int_{\Gamma^h} G(\boldsymbol{x},\boldsymbol{y})\chi_j(\boldsymbol{y})\chi_i(\boldsymbol{x})(\boldsymbol{n}_{\boldsymbol{y}}\cdot\boldsymbol{n}_{\boldsymbol{x}})\,\mathrm{d}\gamma(\boldsymbol{y})\,\mathrm{d}\gamma(\boldsymbol{x})\\
&= -\sum_{K\ni\boldsymbol{r}_i}\sum_{L\ni\boldsymbol{r}_j}\frac{ZA_{0,0}^{c_i^K,d_j^L}}{h_{c_i^K}^K h_{d_j^L}^L}\big(\boldsymbol{\nu}_{c_i^K}^K\times\boldsymbol{n}_K\big)\cdot\big(\boldsymbol{\nu}_{d_j^L}^L\times\boldsymbol{n}_L\big)\\
&\quad + k^2\sum_{K\ni\boldsymbol{r}_i}\sum_{L\ni\boldsymbol{r}_j}\Big(ZA_{0,0}^{c_i^K,d_j^L} - ZA_{0,1}^{c_i^K,d_j^L} - ZA_{1,0}^{c_i^K,d_j^L} + ZA_{1,1}^{c_i^K,d_j^L}\Big)(\boldsymbol{n}_L\cdot\boldsymbol{n}_K).\end{aligned} \tag{E.234}$$

To compute the boundary element integrals (E.232) and (E.233), we isolate the singular part of the Green's function $G$ according to

$$G(R) = -\frac{1}{4\pi R} + \phi(R), \tag{E.235}$$

where $\phi(R)$ is a non-singular function, which is given by

$$\phi(R) = \frac{1-e^{ikR}}{4\pi R}. \tag{E.236}$$

For the derivative $G'(R)$ we have similarly that

$$G'(R) = \frac{1}{4\pi R^2} + \phi'(R), \tag{E.237}$$

where $\phi'(R)$ is also a non-singular function, which is given by

$$\phi'(R) = -\frac{1-(1-ikR)e^{ikR}}{4\pi R^2}. \tag{E.238}$$

We observe that

$$\frac{\partial G}{\partial n_{\boldsymbol{y}}}(\boldsymbol{x},\boldsymbol{y}) = G'(R)\,\frac{\boldsymbol{R}}{R}\cdot\boldsymbol{n}_{\boldsymbol{y}}. \tag{E.239}$$

It is not difficult to see that the singular part corresponds to the Green's function of the Laplace equation, and therefore the associated integrals are computed in the same way. For the integrals associated with $\phi(R)$ and $\phi'(R)$, which are non-singular, a three-point Gauss-Lobatto quadrature formula is used. All the other computations are performed in the same manner as in Section D.12 for the Laplace equation.

## E.14 Benchmark problem

As benchmark problem we consider the exterior sphere problem (E.145), whose domain is shown in Figure E.3. The exact solution of this problem is stated in (E.173), and the idea is to retrieve it numerically with the integral equation techniques and the boundary element method described throughout this chapter.

For the computational implementation and the numerical resolution of the benchmark problem, we consider only the first integral equation of the extension-by-zero alternative (E.109), which is given in terms of boundary layer potentials by (E.187). The linear system (E.215) resulting from the discretization (E.213) of its variational formulation (E.201) is solved computationally with finite boundary elements of type $\mathbb{P}_1$ by using subroutines programmed in Fortran 90, by generating the mesh $\Gamma^h$ of the boundary with the free software Gmsh 2.4, and by representing graphically the results in Matlab 7.5 (R2007b).

We consider a radius $R = 1$, a wave number $k = 3$, and a constant impedance $Z = 0.8$. The discretized boundary surface $\Gamma^h$ has $I = 702$ nodes, $T = 1400$ triangles, and a discretization step $h = 0.2136$, being

$$h = \max_{1\leq j\leq T} \operatorname{diam}(T_j). \tag{E.240}$$

As incident field $u_I$ we consider a plane wave in the form of (E.5) with a wave propagation vector $\boldsymbol{k} = (0, 1, 0)$, i.e., such that the angles of incidence in (E.6) are given by $\theta_I = \pi/2$ and $\varphi_I = -\pi/2$.

From (E.173) and (E.167), we can approximate the exact solution as the truncated series

$$u(r,\theta,\varphi) = -4\pi \sum_{l=0}^{40} i^l \frac{(ZR + l)\, j_l(kR) - kR\, j_{l+1}(kR)}{(ZR + l)\, h_l^{(1)}(kR) - kR\, h_{l+1}^{(1)}(kR)} h_l^{(1)}(kr)\Upsilon_l(\theta,\varphi), \tag{E.241}$$

where

$$\Upsilon_l(\theta,\varphi) = \sum_{m=-l}^{l} Y_l^m(\theta,\varphi)\overline{Y_l^m(\theta_P,\varphi_P)} = \frac{2l+1}{4\pi}\Bigg( P_l(\cos\theta)P_l(\cos\theta_P) + 2\sum_{m=1}^{l} \frac{(l-m)!}{(l+m)!} P_l^m(\cos\theta)P_l^m(\cos\theta_P)\cos\big(m(\varphi-\varphi_P)\big)\Bigg), \tag{E.242}$$

and where the trace on the boundary of the sphere is approximated by

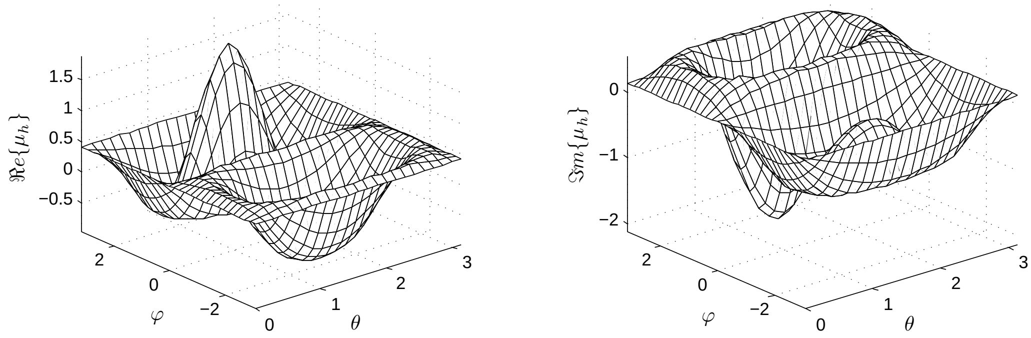
$$\mu(\theta,\varphi) = -4\pi \sum_{l=0}^{40} i^l \frac{(ZR + l)\, j_l(kR) - kR\, j_{l+1}(kR)}{(ZR + l)\, h_l^{(1)}(kR) - kR\, h_{l+1}^{(1)}(kR)} h_l^{(1)}(kR)\Upsilon_l(\theta,\varphi). \tag{E.243}$$

The numerically calculated trace of the solution $\mu_h$ of the benchmark problem, which was computed by using the boundary element method, is depicted in Figure E.5. In the same manner, the numerical solution $u_h$ is illustrated in Figures E.6 and E.7 for an angle $\theta = \pi/2$. It can be observed that the numerical solution is close to the exact one.

On behalf of the far field, two scattering cross sections are shown in Figure E.8. The bistatic radiation diagram represents the far-field pattern of the solution for a particular incident field in all observation directions. The monostatic radiation diagram, on the other hand, depicts the backscattering of incident fields from all directions, i.e., the far-field pattern in the same observation direction as for each incident field.

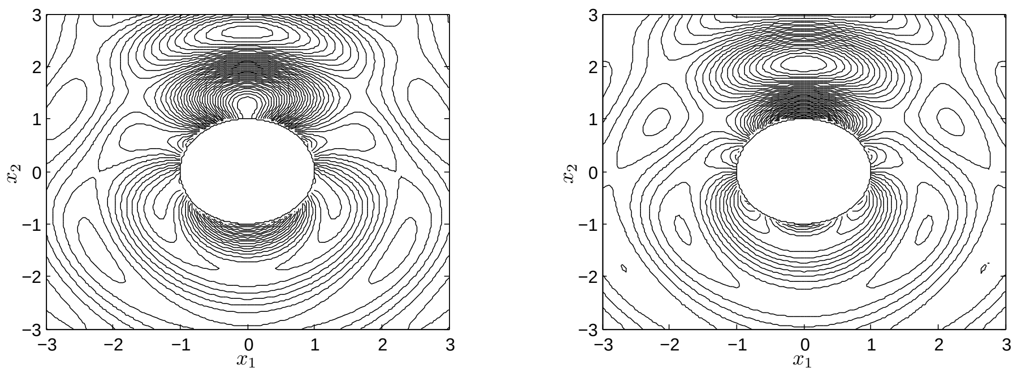
Likewise as in (D.346), we define the relative error of the trace of the solution as

$$E_2(h,\Gamma^h) = \frac{\|\Pi_h\mu - \mu_h\|_{L^2(\Gamma^h)}}{\|\Pi_h\mu\|_{L^2(\Gamma^h)}}, \tag{E.244}$$



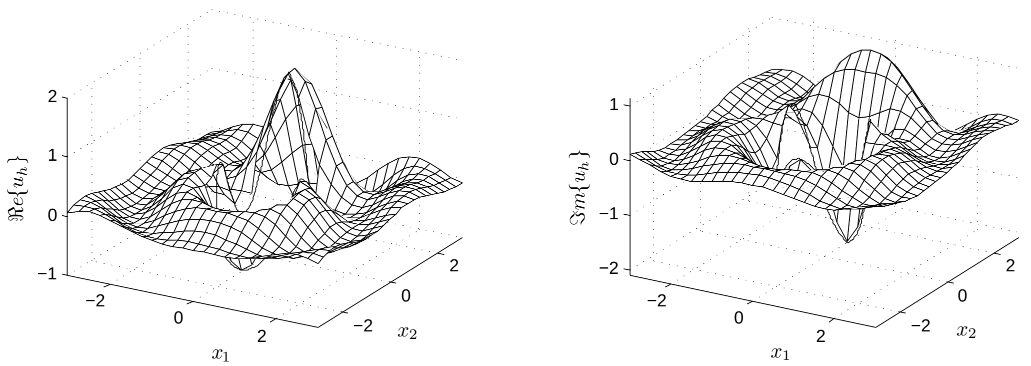

(a) Real part (b) Imaginary part

FIGURE E.5. Numerically computed trace of the solution $\mu_h$.



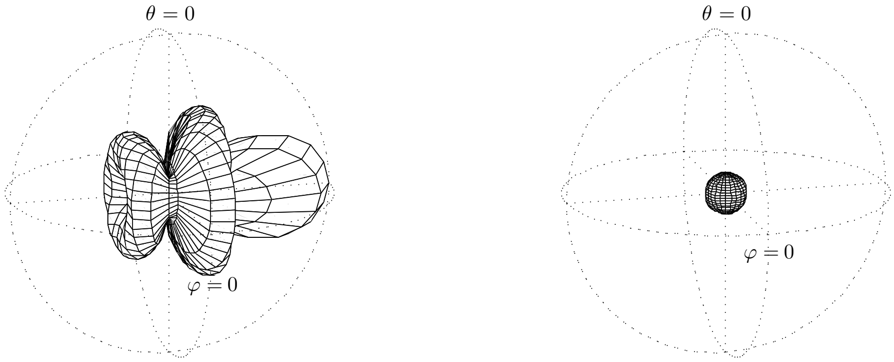

(a) Real part (b) Imaginary part

FIGURE E.6. Contour plot of the numerically computed solution $u_h$ for $\theta = \pi/2$.




(a) Real part (b) Imaginary part

FIGURE E.7. Oblique view of the numerically computed solution $u_h$ for $\theta = \pi/2$.




(a) Bistatic radiation diagram for $\theta_I = \frac{\pi}{2}$, $\varphi_I = -\frac{\pi}{2}$ (b) Monostatic radiation diagram

FIGURE E.8. Scattering cross sections ranging from -14 to 6 [dB].

where $\Pi_h \mu$ denotes the Lagrange interpolating function of the exact solution's trace $\mu$, i.e.,

$$\Pi_h \mu(\boldsymbol{x}) = \sum_{j=1}^{I} \mu(\boldsymbol{r}_j)\, \chi_j(\boldsymbol{x}) \quad \text{and} \quad \mu_h(\boldsymbol{x}) = \sum_{j=1}^{I} \mu_j\, \chi_j(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \Gamma^h. \tag{E.245}$$

In our case, for a step $h = 0.2136$, we obtained a relative error of $E_2(h, \Gamma^h) = 0.01400$.

As in (D.350), we define the relative error of the solution as

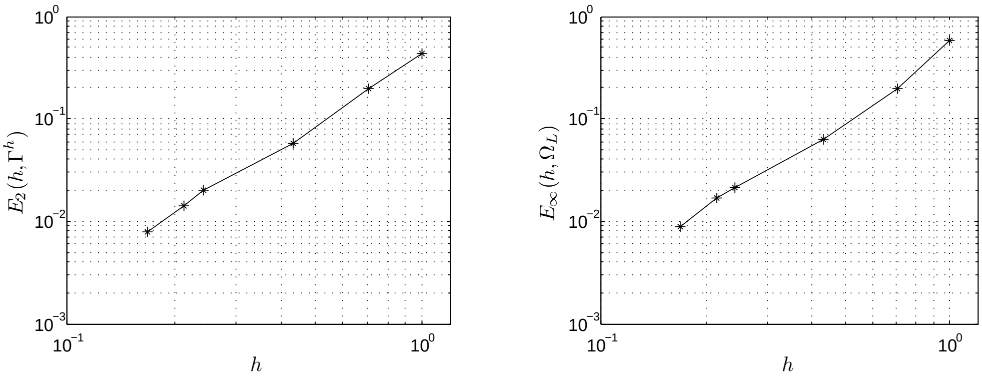
$$E_\infty(h, \Omega_L) = \frac{\|u - u_h\|_{L^\infty(\Omega_L)}}{\|u\|_{L^\infty(\Omega_L)}}, \tag{E.246}$$

being $\Omega_L = \{\boldsymbol{x} \in \Omega_e : \|\boldsymbol{x}\|_\infty < L\}$ for $L > 0$. We consider $L = 3$ and approximate $\Omega_L$ by a triangular finite element mesh of refinement $h$ near the boundary. For $h = 0.2136$, the relative error that we obtained for the solution was $E_\infty(h, \Omega_L) = 0.01667$.

The results for different mesh refinements, i.e., for different numbers of triangles $T$, nodes $I$, and discretization steps $h$ for $\Gamma^h$, are listed in Table E.1. These results are illustrated graphically in Figure E.9. It can be observed that the relative errors are approximately of order $h^2$.

TABLE E.1. Relative errors for different mesh refinements.

| $T$ | $I$ | $h$ | $E_2(h, \Gamma^h)$ | $E_\infty(h, \Omega_L)$ |
|---|---|---|---|---|
| 32 | 18 | 1.0000 | $4.286 \cdot 10^{-1}$ | $5.753 \cdot 10^{-1}$ |
| 90 | 47 | 0.7071 | $1.954 \cdot 10^{-1}$ | $1.986 \cdot 10^{-1}$ |
| 336 | 170 | 0.4334 | $5.821 \cdot 10^{-2}$ | $6.207 \cdot 10^{-2}$ |
| 930 | 467 | 0.2419 | $2.020 \cdot 10^{-2}$ | $2.148 \cdot 10^{-2}$ |
| 1400 | 702 | 0.2136 | $1.400 \cdot 10^{-2}$ | $1.667 \cdot 10^{-2}$ |
| 2448 | 1226 | 0.1676 | $7.892 \cdot 10^{-3}$ | $8.745 \cdot 10^{-3}$ |




(a) Relative error $E_2(h, \Gamma^h)$ (b) Relative error $E_\infty(h, \Omega_L)$

FIGURE E.9. Logarithmic plots of the relative errors versus the discretization step.