Functional analysis

Functional analysis is the branch of mathematics, and specifically of analysis, that is concerned with the study of infinite-dimensional vector spaces (mainly function spaces) and operators acting upon them. It is an essential tool in the proper understanding of all kind of problems in pure and applied mathematics, physics, biology, economics, etc. Functional analysis is particularly useful to state the adequate framework for the existence and uniqueness of the solution of these problems, and to characterize its dependence on different parameters of them. Some classical references are Brezis (1999) and Rudin (1973). Other references are Griffel (1985), Reed & Simon (1980), and Werner (1997).

Normed vector spaces

A vector space is a set E for which the operations of vector addition and scalar multiplication are well defined, i.e., such that the addition of any two elements of E (called vectors) belongs to E, and such that the multiplication of any element of E by a scalar of a field $\mathbb{K}$ (either $\mathbb{C}$ or $\mathbb{R}$) belongs also to E. A normed vector space corresponds to a vector space E that is supplied with a norm, i.e., with an application $\|\cdot\|_E : E \to \mathbb{R}_+$ that fulfills for all $u, v \in E$ and $\alpha \in \mathbb{K}$:

$$\|u\|_E = 0 \quad \Leftrightarrow \quad u = 0_E, \tag{A.402}$$

$$\|\alpha u\|_E = |\alpha| \, \|u\|_E, \tag{A.403}$$

$$\|u + v\|_E \leq \|u\|_E + \|v\|_E, \tag{A.404}$$

where 0_E denotes the null element or zero vector of E. A norm induces a distance on the set E that determines how far apart its elements are between each other. The distance $d(u, v)$ between any two elements $u, v \in E$ is then defined by

$$d(u, v) = \|u - v\|_E. \tag{A.405}$$

A norm characterizes the topology on E and thus the notion of convergence on this set.

a) Banach spaces

A Banach space is essentially a normed vector space that is complete with respect to the metric induced by the norm. It receives its name from the eminent Polish mathematician and university professor Stefan Banach (1892–1945), who was one of the founders of functional analysis. A normed vector space $(E, \|\cdot\|_E)$ is said to be complete if every Cauchy sequence in E has a limit in E. A sequence $\{u_n\} \subset E$ is of Cauchy if for all $\varepsilon > 0$ there exists an integer M such that $\|u_n - u_m\|_E \leq \varepsilon$ for all $n, m \geq M$. In other words, it holds in a Banach space that if the elements of a sequence become closer to each other as the sequence progresses, then the sequence is convergent.

b) Hilbert spaces

A Hilbert space H is a Banach space where the norm is defined by an inner product. It is named after the German mathematician David Hilbert (1862–1943), who is recognized as one of the most influential and universal mathematicians of the 19th and early 20th centuries. A Hilbert space is thus an abstract vector space that has geometric properties.

An inner or scalar product is a positive-definite sesquilinear form $(\cdot,\cdot)_H : H \times H \to \mathbb{K}$, which satisfies for all $u, v, w, x \in H$ and $\alpha, \beta \in \mathbb{K}$:

$$(u,u)_H > 0, \quad u \neq 0_H, \tag{A.406}$$

$$(u,v)_H = \overline{(v,u)_H}, \tag{A.407}$$

$$(u+v, w+x)_H = (u,w)_H + (u,x)_H + (v,w)_H + (v,x)_H, \tag{A.408}$$

$$(\alpha u, \beta v)_H = \alpha\overline{\beta}\,(u,v)_H, \tag{A.409}$$

where $\overline{\beta}$ denotes the complex conjugate of β. The property (A.406) implies the positive-definiteness, whereas the sesquilinearity is given by (A.408) and (A.409). In the case that the underlying field is real, i.e., $\mathbb{K} = \mathbb{R}$, the sesquilinearity turns into bilinearity and the inner product becomes symmetric due (A.407). The induced norm $\|\cdot\|_H$ is defined by

$$\|u\|_H = \sqrt{(u,u)_H} \qquad \forall u \in H, \tag{A.410}$$

and it satisfies the Cauchy-Schwartz inequality

$$|(u,v)_H| \leq \|u\|_H \|v\|_H \qquad \forall u, v \in H. \tag{A.411}$$

A.3.2 Linear operators and dual spaces

Let E and F be two Banach spaces with norms $\|\cdot\|_E$ and $\|\cdot\|_F$, respectively. We define a linear operator as an application $L : E \to F$ that satisfies for all $u, v \in E$ and $\alpha, \beta \in \mathbb{K}$:

$$L(\alpha u + \beta v) = \alpha L(u) + \beta L(v). \tag{A.412}$$

The linear operator L is continuous or bounded if there exists a constant C such that

$$\|L(v)\|_F \leq C\|v\|_E \qquad \forall v \in E. \tag{A.413}$$

We denote in particular by $\mathscr{L}(E,F)$ the space of all linear and continuous operators from E to F, which is also a Banach space when it is supplied with the norm

$$\|L\|_{\mathscr{L}(E,F)} = \sup_{v \neq 0_E} \frac{\|L(v)\|_F}{\|v\|_E} = \sup_{\|v\|_E \leq 1} \|L(v)\|_F = \sup_{\|v\|_E = 1} \|L(v)\|_F. \tag{A.414}$$

It holds therefore that

$$\|L(v)\|_F \leq \|L\|_{\mathscr{L}(E,F)} \|v\|_E \qquad \forall v \in E, \;\; \forall L \in \mathscr{L}(E,F). \tag{A.415}$$

The kernel, nucleus, or nullspace of a linear operator $L \in \mathscr{L}(E,F)$ is defined by

$$\mathcal{N}(L) = \{v \in E : \; L(v) = 0_F\}, \tag{A.416}$$

whereas its image or rang is given by

$$\mathcal{R}(L) = \{w \in F : \; w = L(v), \;\; v \in E\}. \tag{A.417}$$

When $F = E$, then we abbreviate $\mathscr{L}(E,E)$ simply by $\mathscr{L}(E)$.

a) Dual spaces

The dual space E' of a Banach space E corresponds to the space $\mathscr{L}(E,\mathbb{K})$ of all linear and continuous functionals from E to the field $\mathbb{K}$. The dual space E' is also a Banach space when it is supplied with the norm

$$\|L\|_{E'} = \sup_{v \neq 0_E} \frac{|L(v)|}{\|v\|_E} = \sup_{\|v\|_E \leq 1} |L(v)| = \sup_{\|v\|_E = 1} |L(v)|. \tag{A.418}$$

We denote by $\langle \cdot, \cdot \rangle_{E',E} : E' \times E \to \mathbb{K}$ the scalar duality product between both spaces, which is a bilinear form. If $L \in E'$ is given, then the application $\langle L, \cdot \rangle_{E',E} : E \to \mathbb{K}$ is linear and continuous. For $L \in E'$ and $v \in E$, the notation $\langle L, v \rangle_{E',E}$ is thus equivalent to $L(v)$, but can be also understood as $v(L)$. The duality product, analogously as in (A.415), fulfills

$$|\langle L, v \rangle_{E',E}| \leq \|L\|_{E'} \|v\|_E \qquad \forall v \in E, \ \forall L \in E'. \tag{A.419}$$

When the underlying field $\mathbb{K}$ is the set of complex numbers $\mathbb{C}$, then the dual space E' is frequently taken as the space $\mathscr{A}(E, \mathbb{K})$ of all antilinear and continuous functionals from E to the field $\mathbb{K}$. In this case the duality product becomes a sesquilinear form, i.e., a form that is linear in one argument and antilinear in the other. An operator $A \in \mathscr{A}(E, \mathbb{K})$ is said to be antilinear or conjugate linear if for all $u, v \in E$ and $\alpha, \beta \in \mathbb{K}$:

$$A(\alpha u + \beta v) = \overline{\alpha} A(u) + \overline{\beta} A(v). \tag{A.420}$$

The topological properties of linear and antilinear operators are the same, and they differ only on the issue of the complex conjugation. Clearly, if $\mathbb{K} = \mathbb{R}$, then the distinction between linearity and antilinearity disappears, and the sesquilinear forms become bilinear. We remark that the roles of linearity and antilinearity can be assigned at will in the duality product, when consistency is preserved. Duality can be thus understood either in a bilinear or in a sesquilinear sense (and even a biantilinear sense could be also used).

We can also define the bidual, double dual, or second dual space E'' of E, i.e., the dual space of E', which is the space $\mathscr{L}(E', \mathbb{K})$ of all linear and continuous functionals from E' to $\mathbb{K}$. In this case we consider the duality product $\langle \cdot, \cdot \rangle_{E',E''} : E' \times E'' \to \mathbb{K}$, which is again a bilinear (or sesquilinear) form. The space E can be then identified with a subspace of E'' if we use a linear mapping $J : E \to E''$ defined by

$$\langle L, J(v) \rangle_{E',E''} = \langle L, v \rangle_{E',E} \qquad \forall v \in E, \ \forall L \in E'. \tag{A.421}$$

The subspace $J(E)$ is closed in E'' and J is an isometry, i.e.,

$$\|J(v)\|_{E''} = \|v\|_E \qquad \forall v \in E. \tag{A.422}$$

Thus J is an isometric isomorphism of E onto a closed subspace of E''. Frequently E is identified with $J(E)$, in which case E is regarded as a subspace of E''. The spaces for which $J(E) = E''$ are called reflexive.

b) Orthogonal vector subspaces

Let E be a Banach space, E' its dual space, and $\langle \cdot, \cdot \rangle_{E',E}$ their duality product. We consider the vector subspaces $M \subset E$ and $N \subset E'$. We define the orthogonal vector space $M^\perp$ of M by

$$M^\perp = \{A \in E' : \langle A, v \rangle_{E',E} = 0 \quad \forall v \in M\}, \tag{A.423}$$

which is a closed vector subspace of E'. In the same way we define the orthogonal vector space $N^\perp$ of N by

$$N^\perp = \{v \in E : \langle A, v\rangle_{E',E} = 0 \quad \forall A \in N\}, \tag{A.424}$$

which is a closed vector subspace of E. If the duality product between $A \in E'$ and $v \in E$ becomes zero, then both elements can be considered as being in some way orthogonal, similarly as the orthogonality concept for the inner product in Hilbert spaces.

c) Riesz's representation theorem for Hilbert spaces

Every Hilbert space H is reflexive, i.e., it can be naturally identified with its double dual space H''. Furthermore, the Riesz representation theorem (cf., e.g. Brezis 1999), named after the Hungarian mathematician Frigyes Riesz (1880–1956), gives a complete and convenient description of the dual space H' of H, which is itself also a Hilbert space. It states that for each $L \in H'$ there exists a unique $u \in H$ such that

$$\langle L, v\rangle_{H',H} = (u, v)_H \qquad \forall v \in H, \tag{A.425}$$

where

$$\|u\|_H = \|L\|_{H'}. \tag{A.426}$$

This theorem implies that every linear and continuous functional L on H can be represented with the help of the inner product $(\cdot,\cdot)_H$. The application $L \mapsto u$ is an isometric isomorphism that identifies H and H'. We note that this identification is done often, but not always, since the simultaneous identification between a subspace of the Hilbert space and its dual does not work and yields absurd results (cf. Brezis 1999).

A.3.3 Adjoint and compact operators

Let E and F be two Banach spaces, whose dual spaces are given respectively by E' and F'. We define the adjoint operator of a linear operator $T \in \mathscr{L}(E, F)$ as the unique linear operator $T^* \in \mathscr{L}(F', E')$, or antilinear operator $T^* \in \mathscr{A}(F', E')$, that satisfies

$$\langle w, Tv\rangle_{F',F} = \langle T^*w, v\rangle_{E',E} \qquad \forall v \in E, \ \ \forall w \in F', \tag{A.427}$$

depending respectively on whether the duality product is bilinear or sesquilinear. Moreover, and depending again on the type of duality, the adjoint operator T^* is such that

$$\|T\|_{\mathscr{L}(E,F)} = \|T^*\|_{\mathscr{L}(F',E')} \qquad \text{or} \qquad \|T\|_{\mathscr{L}(E,F)} = \|T^*\|_{\mathscr{A}(F',E')}. \tag{A.428}$$

The adjoint operator T^* is thus either linear or antilinear. In finite-dimensional normed vector spaces, the linear operator T can be represented by a matrix and, in this case, its linear adjoint corresponds to its transposed matrix, whereas its antilinear adjoint corresponds to its hermitian matrix, i.e., its transposed and conjugated matrix.

In the case of a Hilbert space H, the adjoint of a linear operator $T \in \mathscr{L}(H)$ is the unique antilinear operator $T^* \in \mathscr{A}(H)$ that satisfies

$$(w, Tv)_H = (T^*w, v)_H \qquad \forall v, w \in H, \tag{A.429}$$

which is also such that

$$\|T\|_{\mathscr{L}(H)} = \|T^*\|_{\mathscr{A}(H)}. \tag{A.430}$$

The following properties hold for $S, T \in \mathscr{L}(H)$ and $\alpha \in \mathbb{K}$:

$$(S+T)^* = S^* + T^*, \qquad (\alpha T)^* = \overline{\alpha}\, T^*, \tag{A.431}$$

$$(ST)^* = T^* S^*, \qquad T^{**} = T. \tag{A.432}$$

A linear operator $T \in \mathscr{L}(E, F)$ is said to be compact if and only if for each bounded sequence $\{u_n\} \subset E$, the sequence $\{Tu_n\} \subset F$ admits a convergent subsequence. A compact operator thus maps bounded sets in E into a relatively compact sets in F, i.e., into sets whose closure is compact in F. It holds that any linear combination of compact operators is compact. Furthermore, the operator T is compact if and only if its adjoint operator $T^* \in \mathscr{L}(F', E')$ is also compact. If G denotes another Banach space, then the composition or product $ST \in \mathscr{L}(E, F)$ of two continuous linear operators $S \in \mathscr{L}(E, G)$ and $T \in \mathscr{L}(G, F)$ is compact if one of the two operators S or T is compact.

A.3.4 Imbeddings

Let E and F be two Banach spaces such that $E \subseteq F$. We say that E is continuously imbedded in F, written as $E \hookrightarrow F$, if E is a vector subspace of F and if the identity operator $I : E \to F$ defined by $I(v) = v$ for all $v \in E$ is continuous, i.e., if there exists a constant C such that

$$\|v\|_F \leq C \|v\|_E \qquad \forall v \in E. \tag{A.433}$$

Moreover, the space E is said to be compactly imbedded in F, written as $E \overset{c}{\hookrightarrow} F$, if E is continuously imbedded in F and if the identity operator $I : E \to F$ is a compact operator, i.e., if each bounded sequence in E admits a convergent subsequence in F.

A.3.5 Lax-Milgram's theorem

Lax-Milgram's theorem gives a sufficient condition to ensure the existence and uniqueness for the solution of a linear problem, which makes it a simple and powerful tool to solve partial differential equations of elliptic type. It was first established and proved by Lax & Milgram (1954) and constitutes a particular case of the projection theorem on convex closed sets in Hilbert spaces (cf., e.g., Brezis 1999).

The theorem is stated as follows. Let H be a Hilbert space and H' its dual space. Let $a : H \times H \to \mathbb{K}$ be a sesquilinear form on H, i.e., such that for all $u, v, w, x \in H$ and for all $\alpha, \beta \in \mathbb{K}$:

$$a(u+v, w+x) = a(u, w) + a(u, x) + a(v, w) + a(v, x), \tag{A.434}$$

$$a(\alpha u, \beta v) = \alpha \overline{\beta}\, a(u, v). \tag{A.435}$$

We suppose that the form $a(\cdot, \cdot)$ is continuous and coercive on $H \times H$, i.e., that there exist some constants $M > 0$ and $\alpha > 0$ such that for all $u, v \in H$:

$$|a(u, v)| \leq M \, \|u\|_H \|v\|_H, \tag{A.436}$$

$$\mathfrak{Re}\{a(u, u)\} \geq \alpha \, \|u\|_H^2. \tag{A.437}$$

Then, for any $f \in H'$ there exists a unique solution $u \in H$ such that

$$a(u, v) = \langle f, v \rangle_{H',H} \qquad \forall v \in H. \tag{A.438}$$

Moreover, the solution u depends continuously on f:

$$\|u\|_H \leq \frac{1}{\alpha} \|f\|_{H'}. \tag{A.439}$$

Lax-Milgram's theorem allows thus to state a sufficient condition to solve a linear problem of the form

$$Au = f, \tag{A.440}$$

where $A : H \to H'$ is a continuous linear operator and $f \in H'$. Typically (A.440) represents the differential problem, while (A.438) denotes its variational formulation.

A.3.6 Fredholm's alternative

The alternative of Fredholm is a theorem that characterizes the existence and uniqueness of the solution for a compactly perturbed linear problem. It is named after the Swedish mathematician Erik Ivar Fredholm (1866–1927), who established the modern theory of integral equations. The theorem generalizes the existence and uniqueness of the solution for a linear system in a finite-dimensional space. Some references are Brezis (1999), Colton & Kress (1983), Hsiao & Wendland (2008), and Ramm (2001, 2005).

Fredholm's alternative states that if E is a Banach space and if $T \in \mathscr{L}(E)$ is a compact operator, then

1. $\mathcal{N}(I - T)$ is of finite dimension,
2. $\mathcal{R}(I - T)$ is closed, i.e., $\mathcal{R}(I - T) = \mathcal{N}(I - T^*)^\perp$,
3. $\mathcal{N}(I - T) = \{0_E\} \;\; \Leftrightarrow \;\; \mathcal{R}(I - T) = E$,
4. $\dim \mathcal{N}(I - T) = \dim \mathcal{N}(I - T^*)$.

When solving an equation of the form $u - Tu = f$, the alternative is thus stated as follows. Either for any $f \in E$ the equation $u - Tu = f$ admits a unique solution $u \in E$ that depends continuously on f; or the homogeneous equation $u - Tu = 0_E$ admits n linearly independent solutions $u_1, u_2, \ldots, u_n \in \mathcal{N}(I - T) \subset E$ and, in this case, the inhomogeneous equation $u - Tu = f$ is solvable (not necessarily uniquely) if and only if f satisfies n orthogonality conditions, i.e., $f \in \mathcal{R}(I - T) = \mathcal{N}(I - T^*)^\perp$, which is of finite dimension.

The importance of Fredholm's alternative lies in the fact that it transforms the existence problem for the solution of the inhomogeneous equation $u - Tu = f$, which is quite difficult, into a uniqueness problem that removes the non-trivial solutions for the homogeneous equation $u - Tu = 0_E$, which is easier to accomplish. In other words, this theorem tells us that a compact perturbation of the identity operator is injective if and only if it is surjective. We remark that the alternative still remains valid when we replace $I - T$ by $S - T$, where $S \in \mathscr{L}(E)$ is a continuous and invertible linear operator whose inverse S^{-1} is also continuous. This stems from the fact that an equation of the form $Su - Tu = f$ can then be readily transformed into the equivalent form $u - S^{-1}Tu = S^{-1}f$, where $S^{-1}T$ is compact since T is compact.

Another way to express Fredholm's alternative is by considering the four operator equations

$$u - Tu = f \qquad \text{in } E, \tag{A.441}$$

$$u - Tu = 0_E \qquad \text{in } E, \tag{A.442}$$

$$w - T^*w = g \qquad \text{in } E', \tag{A.443}$$

$$w - T^*w = 0_{E'} \qquad \text{in } E'. \tag{A.444}$$

If $T \in \mathscr{L}(E)$ is a compact operator, then the following alternative holds. Either (A.442) has only the trivial solution $u = 0_E$, and then (A.444) has only the trivial solution $w = 0_{E'}$, and equations (A.441) and (A.443) are uniquely solvable for any right-hand sides $f \in E$ and $g \in E'$; or (A.442) has exactly n linearly independent solutions u_j, $1 \le j \le n$, and then (A.444) has also n linearly independent solutions w_j, $1 \le j \le n$, and equations (A.441) and (A.443) are solvable if and only if correspondingly

$$\langle w_j, f\rangle_{E',E} = 0 \quad \text{and} \quad \langle g, u_j\rangle_{E',E} = 0, \quad \text{for all } 1 \le j \le n. \tag{A.445}$$

If they are solvable, then their solutions are not unique and their general solutions are, respectively,

$$u = u_p + \sum_{j=1}^{n} \alpha_j u_j \qquad \text{and} \qquad w = w_p + \sum_{j=1}^{n} \beta_j w_j, \tag{A.446}$$

where α_j and β_j are arbitrary scalar constants in $\mathbb{K}$, and u_p and w_p are some particular solutions to (A.441) and (A.443), respectively.

Fredholm's alternative can be also interpreted from the point of view of eigenvalues and eigenvectors. It holds that the eigenvalues of a compact operator $T \in \mathscr{L}(E)$ form a discrete set in the complex plane, with zero as the only possible limit, and for each eigenvalue there are only a finite number of linearly independent eigenvectors. Roughly speaking, the eigenvalues $\lambda \in \mathbb{C}$ and eigenvectors $v \in E$, $v \neq 0_E$, of an operator $T \in \mathscr{L}(E)$ are such that $(T - \lambda I)v = 0_E$. The resolvent set is defined as

$$\rho(T) = \{\lambda \in \mathbb{C} \,:\, (T - \lambda I) \text{ is bijective from } E \text{ to } E\}. \tag{A.447}$$

We remark that if $\lambda \in \rho(T)$, then $(T - \lambda I)^{-1} \in \mathscr{L}(E)$. We define the spectrum $\sigma(T)$ of T as the complement of the resolvent set, i.e., $\sigma(T) = \mathbb{C} \setminus \rho(T)$. The spectrum $\sigma(T)$ is a compact set and such that

$$\lambda \in \sigma(T) \quad \Rightarrow \quad |\lambda| \le \|T\|_{\mathscr{L}(E)}. \tag{A.448}$$

We say that $\lambda \in \mathbb{C}$ is an eigenvalue, written as $\lambda \in \mathcal{EV}(T)$, if $\mathcal{N}(T - \lambda I) \neq \{0_E\}$, where $\mathcal{N}(T - \lambda I)$ is the eigenspace associated with λ. We have that $\mathcal{EV}(T) \subset \sigma(T)$. If $T \in \mathscr{L}(E)$ is a compact operator and E an infinite-dimensional Banach space, then

1. $0 \in \sigma(T)$,
2. $\sigma(T) \setminus \{0\} = \mathcal{EV}(T) \setminus \{0\}$,
3. one of the following holds:
    - $\sigma(T) = \{0\}$,
    - $\sigma(T) \setminus \{0\}$ is finite,

- $\sigma(T) \setminus \{0\}$ is a sequence that tends towards zero.

In other words, the elements of $\sigma(T)\setminus\{0\}$ are isolated points and at most countably infinite. Fredholm's alternative can be thus restated in the following form: a nonzero λ is either an eigenvalue of T, or it lies in the resolvent set $\rho(T)$.

Furthermore, a generalization to Lax-Milgram's theorem can be stated by setting Fredholm's alternative in the framework of variational forms. We consider in this case a Hilbert space H with an inner product $(\cdot,\cdot)_H$ and a dual space H', where the duality product is denoted by $\langle\cdot,\cdot\rangle_{H',H}$. Let $a : H \times H \to \mathbb{C}$ be a continuous sesquilinear form, and we suppose that it satisfies a Gårding inequality of the form

$$\mathfrak{Re}\big\{a(u,u) + (Cu,u)_H\big\} \geq \alpha\|u\|_H^2 \qquad \forall u \in H, \tag{A.449}$$

for some constant $\alpha > 0$ and for some compact linear operator $C : H \to H$. This inequality is named after the Swedish mathematician Lars Gårding, and it generalizes the coercitivity condition (A.437) that is required for the Lax-Milgram theorem. We consider the four variational problems

$$a(u,v) = \langle f, v\rangle_{H',H} \qquad \forall v \in H, \tag{A.450}$$

$$a(u,v) = 0 \qquad \forall v \in H, \tag{A.451}$$

$$\overline{a(v,w)} = \langle g, v\rangle_{H',H} \qquad \forall v \in H, \tag{A.452}$$

$$\overline{a(v,w)} = 0 \qquad \forall v \in H. \tag{A.453}$$

Then there holds the following alternative. Either (A.450) has exactly one solution $u \in H$ for every given $f \in H'$ and (A.452) has exactly one solution $w \in H$ for every given $g \in H'$; or the homogeneous problems (A.451) and (A.453) have finite-dimensional nullspaces of the same dimension $k > 0$, and the non-homogeneous problems (A.450) and (A.452) admit solutions if and only if respectively the orthogonality conditions

$$\langle f, w_j\rangle_{H',H} = 0 \quad \text{and} \quad \langle g, u_j\rangle_{H',H} = 0 \quad \text{for all } 1 \leq j \leq n \tag{A.454}$$

are satisfied, where $\{u_j\}_{j=1}^k$ spans the eigenspace of (A.451) and $\{w_j\}_{j=1}^k$ spans the eigenspace of (A.453), respectively.

A.4 Sobolev spaces

Sobolev spaces are function spaces which play a fundamental role in the modern theory of partial differential equations (PDE). A wider range of solutions of PDE, so-called weak solutions, are naturally found in Sobolev spaces rather than in the classical spaces of continuous functions and with the derivatives understood in the classical sense. Sobolev spaces allow an easy characterization of the regularity of these solutions. They are named after the Russian mathematician Sergei L'vovich Sobolev (1908–1989), who introduced these spaces together with the notion of generalized functions or distributions.

In particular, the solutions of the wave propagation problems treated in this thesis are searched in Sobolev spaces. Other boundary-value problems of PDE may require sometimes adaptations of Sobolev spaces, so-called weighted spaces, which are not discussed here. Complete surveys of Sobolev spaces can be found in Adams (1975), Brezis (1999), Grisvard (1985), Hsiao & Wendland (2008), Lions & Magenes (1972), and Ziemer (1989). For further applications and properties of Sobolev spaces we mention also the references Atkinson & Han (2005), Bony (2001), Chen & Zhou (1992), Nédélec (1977, 2001), Raviart & Thomas (1983), and Steinbach (2008).

We consider a domain Ω in $\mathbb{R}^N$ with a regular boundary $\Gamma = \partial\Omega$. By domain we understand an open nonempty and connected set. What is understood by the regularity of the boundary is specified later on. For the moment let us assume simply that the domain lies locally on only one side of Γ, and that Γ does not have cusps. Thus the situations in Figure A.14 are ruled out.

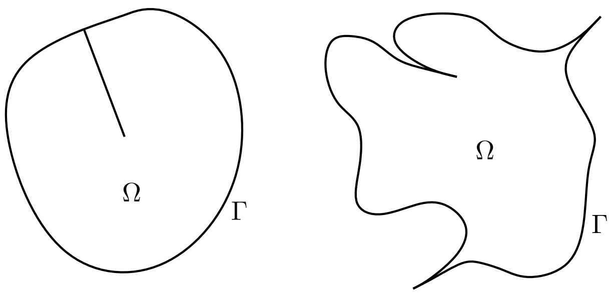

FIGURE A.14. Nonadmissible domains Ω.

Let f be a real-, or more generally, a complex-valued function defined on the domain Ω. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in \mathbb{N}_0^N$ be a multi-index of nonnegative integers. We write

$$D^{\boldsymbol{\alpha}} f = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_N}\right)^{\alpha_N} f \tag{A.455}$$

to denote a mixed partial derivative of f of order

$$|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_N. \tag{A.456}$$

A.4.1 Continuous function spaces

We denote by $C^m(\Omega)$ the space of all continuous functions whose derivatives up until order $m \in \mathbb{N}_0$ exist and are continuous in Ω. Thus, for $m = 0$, the space of all the continuous functions defined in Ω is denoted by $C^0(\Omega)$ or simply by $C(\Omega)$. Similarly, $C^\infty(\Omega)$ denotes the space of infinitely differentiable functions in Ω, which is such that

$$C^\infty(\Omega) = \bigcap_{m\in\mathbb{N}_0} C^m(\Omega). \tag{A.457}$$

It clearly holds that $C^\infty(\Omega) \subset C^{m+1}(\Omega) \subset C^m(\Omega)$ for all $m \in \mathbb{N}_0$. We remark that since Ω is open, the functions in $C^m(\Omega)$ need not to be bounded on Ω.

We represent by $C_0^m(\Omega)$ the space of functions in $C^m(\Omega)$ that have a compact support in Ω. By the support of a function we mean the closure of the set of points where the function is different from zero. A set in $\mathbb{R}^N$ is said to be compact if it is closed and bounded. In the same way as before, we denote by $C_0^\infty(\Omega)$ the set of all infinitely differentiable functions which, together with all of their derivatives, have compact support in Ω.

Similarly, one can define $C^m(\overline{\Omega})$ to be the space of functions in $C^m(\Omega)$ which, together with their derivatives of order $\leq m$, have continuous extensions to $\overline{\Omega} = \Omega \cup \Gamma$. If Ω is bounded and $m < \infty$, then $C^m(\overline{\Omega})$ is a Banach space (vid. Section A.3) with the norm

$$\|f\|_{C^m(\overline{\Omega})} = \sum_{|\boldsymbol{\alpha}|\leq m} \sup_{\boldsymbol{x}\in\overline{\Omega}} |D^{\boldsymbol{\alpha}} f(\boldsymbol{x})|. \tag{A.458}$$

If the domain Ω is unbounded, then we consider as $C^m(\overline{\Omega})$ the space of all functions of class C^m that are bounded in $\overline{\Omega}$. This space is a Banach space with the norm (A.458).

A function f that is defined in Ω is said to be Hölder continuous with exponent α, for $0 < \alpha < 1$, if there exists a constant $C > 0$ such that

$$|f(\boldsymbol{x}) - f(\boldsymbol{y})| \leq C\,|\boldsymbol{x} - \boldsymbol{y}|^\alpha \qquad \forall \boldsymbol{x}, \boldsymbol{y} \in \Omega. \tag{A.459}$$

If f fulfills (A.459) for $\alpha = 1$, then the function is said to be Lipschitz continuous. We say that f is locally Hölder or Lipschitz continuous with exponent α in Ω if it is Hölder or Lipschitz continuous with exponent α in every compact subset of Ω, respectively. These names were given after the German mathematicians Otto Ludwig Hölder (1859–1937) and Rudolf Otto Sigismund Lipschitz (1832–1903).

By $C^{m,\alpha}(\Omega)$, $m \in \mathbb{N}_0$, $0 < \alpha \leq 1$, we denote the space of functions in $C^m(\Omega)$ whose derivatives of order m are locally Hölder or Lipschitz continuous with exponent α in Ω. We remark that Hölder continuity may be viewed as a fractional differentiability. For $\alpha = 0$, we set $C^{m,0}(\Omega) = C^m(\Omega)$.

Further, by $C^{m,\alpha}(\overline{\Omega})$ we denote the subspace of $C^m(\overline{\Omega})$ consisting of functions which have m-th order Hölder or Lipschitz continuous derivatives of exponent α in Ω. If Ω is bounded, then we define the Hölder or Lipschitz norm by

$$\|f\|_{C^{m,\alpha}(\overline{\Omega})} = \|f\|_{C^m(\overline{\Omega})} + \sum_{|\boldsymbol{\beta}|=m} \sup_{\substack{\boldsymbol{x},\boldsymbol{y}\in\overline{\Omega}\\ \boldsymbol{x}\neq\boldsymbol{y}}} \frac{|D^{\boldsymbol{\beta}} f(\boldsymbol{x}) - D^{\boldsymbol{\beta}} f(\boldsymbol{y})|}{|\boldsymbol{x} - \boldsymbol{y}|^\alpha}. \tag{A.460}$$

The so-called Hölder space $C^{m,\alpha}(\overline{\Omega})$, equipped with the norm $\|\cdot\|_{C^{m,\alpha}(\overline{\Omega})}$, becomes a Banach space. Again, for an unbounded domain Ω we consider as $C^{m,\alpha}(\overline{\Omega})$ the Banach space of all bounded functions of class C^m. We have for $0<\beta<\alpha\leq 1$ the inclusions

$$C^{m,\alpha}(\overline{\Omega}) \subset C^{m,\beta}(\overline{\Omega}) \subset C^{m}(\overline{\Omega}). \tag{A.461}$$

It is also clear that $C^{m,1}(\overline{\Omega}) \not\subset C^{m+1}(\overline{\Omega})$. In general $C^{m+1}(\overline{\Omega}) \not\subset C^{m,1}(\overline{\Omega})$ either, but for some particular domains Ω the inclusion applies, e.g., for convex domains.

Let $m \in \mathbb{N}_0$ and let $0<\beta<\alpha\leq 1$, then we have the continuous imbeddings

$$C^{m+1}(\overline{\Omega}) \hookrightarrow C^{m}(\overline{\Omega}), \tag{A.462}$$

$$C^{m,\alpha}(\overline{\Omega}) \hookrightarrow C^{m}(\overline{\Omega}), \tag{A.463}$$

$$C^{m,\alpha}(\overline{\Omega}) \hookrightarrow C^{m,\beta}(\overline{\Omega}). \tag{A.464}$$

If Ω is bounded, then the imbeddings (A.463) and (A.464) are compact. Furthermore, if Ω is convex, then we have also the continuous imbeddings

$$C^{m+1}(\overline{\Omega}) \hookrightarrow C^{m,1}(\overline{\Omega}), \tag{A.465}$$

$$C^{m+1}(\overline{\Omega}) \hookrightarrow C^{m,\alpha}(\overline{\Omega}). \tag{A.466}$$

If Ω is convex and bounded, then the imbeddings (A.462) and (A.466) are compact.

A.4.2 Lebesgue spaces

The Lebesgue or L^p spaces correspond to classes of Lebesgue measurable functions defined on the domain $\Omega \subset \mathbb{R}^N$. They are defined, for $1 \leq p \leq \infty$, by

$$L^p(\Omega) = \{ f : \Omega \to \mathbb{C} \mid \|f\|_{L^p(\Omega)} < \infty \}, \tag{A.467}$$

where the L^p-norm is given by

$$\|f\|_{L^p(\Omega)} = \begin{cases} \left(\displaystyle\int_\Omega |f(\boldsymbol{x})|^p \,\mathrm{d}\boldsymbol{x} \right)^{1/p}, & 1 \leq p < \infty, \\ \operatorname*{ess\,sup}_{\boldsymbol{x}\in\Omega} |f(\boldsymbol{x})|, & p = \infty. \end{cases} \tag{A.468}$$

The appearing integrals have to be understood in the sense of Lebesgue (cf. Royden 1988), which is named after the French mathematician Henri Léon Lebesgue (1875–1941), who became famous for his theory of integration. We say that two functions are equal almost everywhere if they are equal except on a set of measure zero. Functions which are equal almost everywhere in the domain Ω are therefore identified together in $L^p(\Omega)$. The essential supremum is likewise defined in this sense by

$$\operatorname*{ess\,sup}_{\boldsymbol{x}\in\Omega} |f(\boldsymbol{x})| = \inf\{ C > 0 : \ |f(\boldsymbol{x})| \leq C \ \text{ almost everywhere in } \Omega \}. \tag{A.469}$$

We remark that L^p spaces, supplied with the L^p-norm, are Banach spaces. A normed vector space is said to be separable if it contains a countable dense subset. For $1<p<\infty$, we have that the space $L^p(\Omega)$ is separable, reflexive, and its dual space $L^p(\Omega)'$ is identified with $L^q(\Omega)$, where $\frac{1}{p}+\frac{1}{q}=1$. The space $L^1(\Omega)$ is separable, but not reflexive, and its dual space $L^1(\Omega)'$ is identified with $L^\infty(\Omega)$. The space $L^\infty(\Omega)$ is neither separable nor reflexive, and its dual space $L^\infty(\Omega)'$ is strictly contained in $L^1(\Omega)$. If

$$f_i \in L^{p_i}(\Omega) \quad (1 \leq i \leq n) \qquad \text{with} \qquad \frac{1}{p} = \sum_{i=1}^{n} \frac{1}{p_i} \leq 1, \quad 1 \leq p_i \leq \infty, \tag{A.470}$$

then the multiplication of these functions f_i is such that

$$f = f_1 f_2 \cdots f_n \in L^p(\Omega), \tag{A.471}$$

and furthermore

$$\|f\|_{L^p(\Omega)} \leq \|f\|_{L^{p_1}(\Omega)} \|f\|_{L^{p_2}(\Omega)} \cdots \|f\|_{L^{p_n}(\Omega)}. \tag{A.472}$$

If $f \in L^p(\Omega) \cap L^q(\Omega)$ with $1 \leq p \leq q \leq \infty$, then $f \in L^r(\Omega)$ for all $p \leq r \leq q$, and we have moreover the interpolation inequality

$$\|f\|_{L^r(\Omega)} \leq \|f\|^{\alpha}_{L^p(\Omega)} \|f\|^{1-\alpha}_{L^q(\Omega)}, \quad \text{where} \quad \frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q} \quad (0 \leq \alpha \leq 1). \tag{A.473}$$

In the particular case when $p = 2$, it holds that $L^2(\Omega)$ is also a Hilbert space with respect to the inner product

$$(f, g)_{L^2(\Omega)} = \int_\Omega f(\boldsymbol{x})\, \overline{g(\boldsymbol{x})}\, \mathrm{d}\boldsymbol{x}, \qquad \forall f, g \in L^2(\Omega). \tag{A.474}$$

Its dual space $L^2(\Omega)'$ is identified with the space $L^2(\Omega)$ itself.

We can likewise define the L^p_{loc} spaces by

$$L^p_{\text{loc}}(\Omega) = \{f : \Omega \to \mathbb{C} \mid f \in L^p(K) \quad \forall K \subset \Omega,\ K \text{ compact}\}, \tag{A.475}$$

which behave locally as L^p spaces, i.e., on each compact subset K of Ω. These locally defined functional spaces can not be supplied with reasonable norms, but nevertheless a Fréchet space structure may be defined for them (cf. Bony 2001). Fréchet spaces are certain topological vector spaces which are locally convex and complete with respect to a translation invariant metric. They receive their name from the French mathematician Maurice Fréchet (1878–1973), who is responsible for introducing the concept of metric spaces.

A.4.3 Sobolev spaces of integer order

We define now the Sobolev spaces $W^{m,p}$, for $1 \leq p \leq \infty$ and $m \in \mathbb{N}_0$, by

$$W^{m,p}(\Omega) = \{f : \Omega \to \mathbb{C} \mid D^{\boldsymbol{\alpha}} f \in L^p(\Omega) \ \forall \boldsymbol{\alpha} \in \mathbb{N}_0^N,\ |\boldsymbol{\alpha}| \leq m\}, \tag{A.476}$$

or alternatively, by

$$W^{m,p}(\Omega) = \{f : \Omega \to \mathbb{C} \mid \|f\|_{W^{m,p}(\Omega)} < \infty\}, \tag{A.477}$$

where the $W^{m,p}$-norm is given by

$$\|f\|_{W^{m,p}(\Omega)} = \begin{cases} \left(\displaystyle\sum_{|\boldsymbol{\alpha}| \leq m} \|D^{\boldsymbol{\alpha}} f\|^p_{L^p(\Omega)} \right)^{1/p}, & 1 \leq p < \infty, \\ \displaystyle\max_{|\boldsymbol{\alpha}| \leq m} \|D^{\boldsymbol{\alpha}} f\|_{L^\infty(\Omega)}, & p = \infty. \end{cases} \tag{A.478}$$

The Sobolev spaces $W^{m,p}$ are actually Banach spaces, provided that the derivatives are taken in the sense of distributions (vid. Section A.6). If $m = 0$, then we retrieve

$$W^{0,p}(\Omega) = L^p(\Omega), \qquad 1 \leq p \leq \infty. \tag{A.479}$$

For $p = 2$ the space $W^{m,2}(\Omega)$ becomes a Hilbert space, and is denoted in particular by

$$H^m(\Omega) = W^{m,2}(\Omega). \tag{A.480}$$

The space $H^m(\Omega)$ is supplied with the inner product

$$(f, g)_{H^m(\Omega)} = \sum_{|\boldsymbol{\alpha}|\leq m} \int_\Omega D^{\boldsymbol{\alpha}} f(\boldsymbol{x})\, \overline{D^{\boldsymbol{\alpha}} g(\boldsymbol{x})}\, \mathrm{d}\boldsymbol{x} \qquad \forall f, g \in H^m(\Omega), \tag{A.481}$$

and hence with the norm

$$\|f\|_{H^m(\Omega)} = \left(\sum_{|\boldsymbol{\alpha}|\leq m} \int_\Omega |D^{\boldsymbol{\alpha}} f(\boldsymbol{x})|^2 \,\mathrm{d}\boldsymbol{x} \right)^{1/2} \qquad \forall f \in H^m(\Omega). \tag{A.482}$$

We refer to $H^m(\Omega)$ as the Sobolev space of order m. Sobolev spaces of higher order contain elements with a higher degree of smoothness or regularity. We remark that if $f \in H^m(\Omega)$, then $\partial f/\partial x_i \in H^{m-1}(\Omega)$ for $1 \leq i \leq N$.

Due density, we can define now the space $H_0^m(\Omega)$ as the closure of $C_0^m(\Omega)$ under the H^m-norm (A.482), i.e.,

$$H_0^m(\Omega) = \overline{C_0^m(\Omega)}^{\|\cdot\|_{H^m(\Omega)}}. \tag{A.483}$$

We remark that if the domain Ω is regular enough, then the space $H^m(\Omega)$ can be defined alternatively as the completion of $C^\infty(\overline{\Omega})$ with respect to the norm $\|\cdot\|_{H^m(\Omega)}$, which means that for every $f \in H^m(\Omega)$ there exists a sequence $\{f_k\}_{k\in\mathbb{N}} \subset C^\infty(\overline{\Omega})$ such that

$$\lim_{k\to\infty} \|f - f_k\|_{H^m(\Omega)} = 0. \tag{A.484}$$

In the same manner as for the L^p spaces, we can also consider locally defined H^m_{loc} Sobolev spaces, given by

$$H^m_{\mathrm{loc}}(\Omega) = \{f : \Omega \to \mathbb{C} \mid f \in H^m(K)\ \ \forall K \subset \Omega,\ K \text{ compact}\}, \tag{A.485}$$

which behave as H^m spaces on each compact subset K of Ω, and can be treated in the framework of Fréchet spaces.

A.4.4 Sobolev spaces of fractional order

Sobolev spaces can be also defined for non-integer values of m, so-called fractional orders and denoted by s. For this we consider first the particular case when the domain Ω is the full space $\mathbb{R}^N$, in which case the Sobolev spaces of fractional order are defined by means of a Fourier transform (vid. Section A.7). For a real value s we use the norm

$$\|f\|_{H^s(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} (1 + |\boldsymbol{\xi}|^2)^s |\widehat{f}(\boldsymbol{\xi})|^2 \,\mathrm{d}\boldsymbol{\xi} \right)^{1/2}, \tag{A.486}$$

where $\widehat{f}$ denotes the Fourier transform of f. The weighting factor $(1+|\boldsymbol{\xi}|^2)^{s/2}$ is known as Bessel's potential of order s. The expression (A.486) defines an equivalent norm to (A.482) in $H^m(\mathbb{R}^N)$ if $s=m$, but holds also for non-integer and even negative values of s. If s is real and positive, then the Sobolev spaces of fractional order are defined by

$$H^s(\mathbb{R}^N) = \{f \in L^2(\mathbb{R}^N) : \|f\|_{H^s(\mathbb{R}^N)} < \infty\}, \tag{A.487}$$

which is equivalent to the definition given previously, when $s=m$. If we allow negative values for s, then the definition (A.487) has to be extended to admit as well tempered distributions in $\mathcal{S}'(\mathbb{R}^N)$ (vid. Sections A.6 & A.7). Thus in general, if $s \in \mathbb{R}$, then the Sobolev spaces of fractional order are defined by

$$H^s(\mathbb{R}^N) = \{f \in \mathcal{S}'(\mathbb{R}^N) : \|f\|_{H^s(\mathbb{R}^N)} < \infty\}. \tag{A.488}$$

We observe that the Sobolev space $H^{-s}(\mathbb{R}^N)$ is the dual space of $H^s(\mathbb{R}^N)$.

If we consider now a proper subdomain Ω of $\mathbb{R}^N$, then the Sobolev spaces of fractional order, for $s \geq 0$, are defined by

$$H^s(\Omega) = \{f : \Omega \to \mathbb{C} \mid \exists F \in H^s(\mathbb{R}^N) \text{ such that } F|_\Omega = f\}, \tag{A.489}$$

and have the norm

$$\|f\|_{H^s(\Omega)} = \inf\{\|F\|_{H^s(\mathbb{R}^N)} : F|_\Omega = f\}. \tag{A.490}$$

We remark that if Ω is a pathological domain such as those depicted in Figure A.14, then the new definition (A.489) is not equivalent to the old one for $H^m(\Omega)$ if $s=m$.

Since $C_0^\infty(\Omega) \subset C^\infty(\overline{\Omega})$, where for any $f \in C_0^\infty(\Omega)$ the trivial extension $\widetilde{f}$ by zero outside of Ω is in $C_0^\infty(\mathbb{R}^N)$, we define the space $\widetilde{H}^s(\Omega)$ for $s \geq 0$ to be the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|f\|_{\widetilde{H}^s(\Omega)} = \|\widetilde{f}\|_{H^s(\mathbb{R}^N)}. \tag{A.491}$$

This definition implies that

$$\widetilde{H}^s(\Omega) = \{f \in H^s(\mathbb{R}^N) : \operatorname{supp} f \subset \overline{\Omega}\}. \tag{A.492}$$

We remark that the space $\widetilde{H}^s(\Omega)$ is often also denoted as $H^s_{00}(\Omega)$ (cf., e.g., Lions & Magenes 1972). If $\Omega = \mathbb{R}^N$, then the H^s and $\widetilde{H}^s$ spaces coincide, i.e.,

$$\widetilde{H}^s(\mathbb{R}^N) = H^s(\mathbb{R}^N). \tag{A.493}$$

For negative orders we have that $H^{-s}(\Omega)$ is the dual space of $\widetilde{H}^s(\Omega)$, i.e.,

$$H^{-s}(\Omega) = \widetilde{H}^s(\Omega)', \tag{A.494}$$

where the norm is defined by means of the inner product in $L^2(\Omega)$, namely

$$\|f\|_{H^{-s}(\Omega)} = \sup_{0\neq\varphi\in\widetilde{H}^s(\Omega)} \frac{|(f,\varphi)_{L^2(\Omega)}|}{\|\varphi\|_{\widetilde{H}^s(\Omega)}}, \qquad s>0. \tag{A.495}$$

In the same way, the space $\widetilde{H}^{-s}(\Omega)$ is the dual space of $H^s(\Omega)$, i.e.,

$$\widetilde{H}^{-s}(\Omega) = H^s(\Omega)', \tag{A.496}$$

and is provided with the norm of the dual space

$$\|f\|_{\widetilde{H}^{-s}(\Omega)} = \sup_{0\neq\psi\in H^s(\Omega)} \frac{|(f,\psi)_{L^2(\Omega)}|}{\|\psi\|_{H^s(\Omega)}}, \qquad s>0. \tag{A.497}$$

It can be shown that the definition (A.492) applies also for $s<0$ if Ω is regular enough. For $s>0$ we obtain the inclusions

$$\widetilde{H}^s(\Omega) \subset H^s(\Omega) \subset L^2(\Omega) \subset \widetilde{H}^{-s}(\Omega) \subset H^{-s}(\Omega). \tag{A.498}$$

It holds in particular for $0 \leq s < \frac{1}{2}$ that $\widetilde{H}^s(\Omega) = H^s(\Omega)$ and $\widetilde{H}^{-s}(\Omega) = H^{-s}(\Omega)$, which is not true anymore for $s \geq \frac{1}{2}$. We have in this chain that $L^2(\Omega)$ is the only Sobolev space that is identified with its dual space, and is therefore called pivot space. It is a standard practice to represent the duality pairings among Sobolev spaces just as inner products in $L^2(\Omega)$, that is, the integral notation is maintained even if the elements are no longer L^2-integrable. In fact, the norm definitions (A.495) and (A.497) for the dual spaces $V' = H^{-s}(\Omega)$ and $\widetilde{H}^{-s}(\Omega)$ for $s>0$ are based on this representation. In this case, if $f \in V'$ but $f \notin L^2(\Omega)$, then we define

$$\langle f,\varphi\rangle_{V',V} = \lim_{n\to\infty} (f_n,\varphi)_{L^2(\Omega)} = \lim_{n\to\infty} \int_\Omega f_n(\boldsymbol{x})\,\overline{\varphi(\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x} \qquad \forall\varphi\in V, \tag{A.499}$$

where V is correspondingly either $\widetilde{H}^s(\Omega)$ or $H^s(\Omega)$, where $\langle\cdot,\cdot\rangle_{V',V}$ denotes the sesquilinear duality product between V' and V, and where $\{f_n\} \subset L^2(\Omega)$ is a sequence such that

$$\lim_{n\to\infty} \|f - f_n\|_{V'} = 0. \tag{A.500}$$

We know that the sequence $\{f_n\}$ exists and that (A.499) makes sense, since $H^{-s}(\Omega)$ is the completion of $L^2(\Omega)$ with respect to the norm of the dual space (A.495). We write thus

$$\langle f,\varphi\rangle_{V',V} = (f,\varphi)_{L^2(\Omega)} \tag{A.501}$$

for the duality pairing $(f,\varphi) \in V' \times V$, where the L^2-inner product on the right-hand side is understood in the sense of (A.499) for $f \notin L^2(\Omega)$.

For $s>t$ it holds also that $H^s(\Omega) \subset H^t(\Omega)$ and $\widetilde{H}^s(\Omega) \subset \widetilde{H}^t(\Omega)$, i.e., as the order of the Sobolev spaces increases, so does the smoothness of their elements. If $s = m+\sigma \geq 0$, for $m \in \mathbb{N}_0$ and $0<\sigma<1$, then the space $\widetilde{H}^s(\Omega)$ can be characterized as the completion of the space $C_0^{m+1}(\Omega)$ with respect to the norm (A.491), namely

$$\widetilde{H}^s(\Omega) = \overline{C_0^{m+1}(\Omega)}^{\|\cdot\|_{H^s(\mathbb{R}^N)}}. \tag{A.502}$$

A closely related space is

$$H_0^s(\Omega) = \overline{C_0^{m+1}(\Omega)}^{\|\cdot\|_{H^s(\Omega)}}, \tag{A.503}$$

which considers the closure of $C_0^{m+1}(\Omega)$, but now under the norm (A.490). It holds that

$$\widetilde{H}^s(\Omega) = H_0^s(\Omega) \qquad \forall s = m+\sigma, \quad m\in\mathbb{N}_0, \quad |\sigma| < \frac{1}{2}, \tag{A.504}$$

and when $s = m+1/2$, then the space $\widetilde{H}^s(\Omega)$ is strictly contained in $H_0^s(\Omega)$.

We observe that the Sobolev space $H^s(\Omega)$ of fractional order $s = m + \sigma$, for $m \in \mathbb{N}_0$ and $0 < \sigma < 1$, can be alternatively defined as

$$H^s(\Omega) = \{f \in L^2(\Omega) : \|f\|_{H^s(\Omega)} < \infty\}, \tag{A.505}$$

by means of the norm

$$\|f\|_{H^s(\Omega)} = \left(\|f\|^2_{H^m(\Omega)} + \sum_{|\boldsymbol{\alpha}|=m} \int_\Omega \int_\Omega \frac{|D^{\boldsymbol{\alpha}} f(\boldsymbol{x}) - D^{\boldsymbol{\alpha}} f(\boldsymbol{y})|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{N+2\sigma}} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{y} \right)^{1/2}, \tag{A.506}$$

where $\|\cdot\|_{H^m(\Omega)}$ is the norm for the Sobolev space of integer order m defined in (A.482). For further details we refer to Hsiao & Wendland (2008).

A.4.5 Trace spaces

Trace spaces are Sobolev spaces for functions defined on the boundary. If $f \in H^s(\Omega)$ is continuous up to the boundary Γ of Ω, then one can say that the value which f takes on Γ is the restriction to Γ (of the extension by continuity to $\overline{\Omega}$) of the function f, which is denoted by $f|_\Gamma$. In general, however, the elements of $H^s(\Omega)$ are defined except for a set of N-dimensional zero measure and it is meaningless therefore to speak of their restrictions to Γ (which has an N-dimensional zero measure). Therefore we use the concept of the trace of a function on Γ, which substitutes and generalizes that of the restriction $f|_\Gamma$ whenever the latter in the classical sense is inapplicable.

We follow the approach found in standard text books of identifying the boundary Γ with $\mathbb{R}^{N-1}$ by means of local parametric representations of Γ. Roughly speaking, we define the trace spaces to be isomorphic to the Sobolev spaces $H^s(\mathbb{R}^{N-1})$.

a) Regularity of the boundary

To characterize properly the regularity of the domain Ω, its boundary Γ is described locally by the graph of a function φ, and the properties of Γ are then specified through the properties of φ. We say that the boundary Γ is of class $C^{m,\alpha}$, for $m \in \mathbb{N}_0$ and $0 \le \alpha \le 1$, if for each $\boldsymbol{x} \in \Gamma$ there exists a neighborhood Θ of $\boldsymbol{x}$ in $\mathbb{R}^N$ and a new orthogonal coordinate system $\boldsymbol{y} = (\boldsymbol{y}_s, y_N) \in \mathbb{R}^N$, being $\boldsymbol{y}_s = (y_1, \ldots, y_{N-1}) \in \mathbb{R}^{N-1}$, such that

1. for some $\delta, \varepsilon > 0$ the neighborhood Θ is a hypercylinder in the new coordinates:

$$\Theta = \{\boldsymbol{y} \in \mathbb{R}^N : \, |\boldsymbol{y}_s| < \delta, \;\; |y_N| < \varepsilon\}; \tag{A.507}$$

2. there exists a function φ of class $C^{m,\alpha}$ defined on $Q = \{\boldsymbol{y}_s : |\boldsymbol{y}_s| < \delta\}$ such that

$$|\varphi(\boldsymbol{y}_s)| \le \frac{\varepsilon}{2} \qquad \forall \boldsymbol{y}_s \in Q, \tag{A.508}$$

$$\Omega \cap \Theta = \{\boldsymbol{y} \in \Theta : \, y_N < \varphi(\boldsymbol{y}_s)\}, \tag{A.509}$$

$$\Gamma \cap \Theta = \{\boldsymbol{y} \in \Theta : \, y_N = \varphi(\boldsymbol{y}_s)\}. \tag{A.510}$$

In other words, in a neighborhood Θ of $\boldsymbol{x}$, the domain Ω is below the graph of φ and consequently the boundary Γ is the graph of φ, as illustrated in Figure A.15. The pair (Θ, φ) is called a local chart of Γ. The relation between the new coordinates $\boldsymbol{y} \in \mathbb{R}^N$ and the old ones $\boldsymbol{x} \in \mathbb{R}^N$ is given by

$$\boldsymbol{x} = \boldsymbol{b} + \boldsymbol{T}(\boldsymbol{y}), \tag{A.511}$$

where $\boldsymbol{b} \in \mathbb{R}^N$ is a constant translation vector (eventually $\boldsymbol{b} \in \Gamma$), and where $\boldsymbol{T}$ is an orthogonal linear transformation, i.e., an orthogonal $N \times N$ matrix.

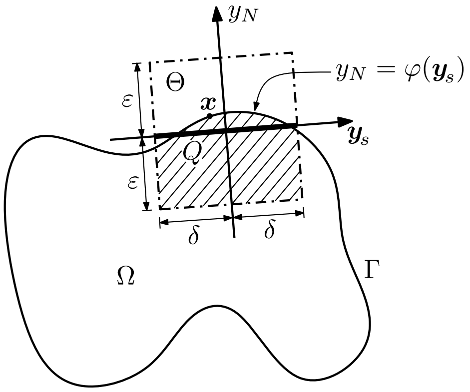

FIGURE A.15. Local chart of Γ.

For $\alpha = 0$, we say simply that Γ is of class C^m. By the regularity of the domain Ω we mean the regularity of its boundary Γ, and thus we may write indistinctly $\Omega \in C^m$ or $\Gamma \in C^m$. The boundary Γ is said to be of class C^∞ if $\Gamma \in \cap_{m=0}^{\infty} C^m$.

In the case when $\Gamma \in C^{0,1}$, the boundary is called a Lipschitz boundary (with a strong Lipschitz property) and Ω is called a (strong) Lipschitz domain, written as $\Omega \in C^{0,1}$. Such a boundary lies locally on only one side of Γ and does not have cusps, but can contain conical points or edges, which are not continuously differentiable. In particular, the domains shown in Figure A.14 are not strong Lipschitz domains. For strong Lipschitz domains a unique unit normal vector can be defined almost everywhere on Γ. These domains are useful for almost all practical purposes and they are regular enough so that the different definitions of Sobolev spaces on them usually coincide.

A boundary $\Gamma \in C^{1,\alpha}$ with $0 < \alpha < 1$ is called a Lyapunov boundary, and it has the property that a unique unit normal vector can be defined everywhere on Γ. It is named after the Russian mathematician and physicist Aleksandr Mikhailovich Lyapunov (1857–1918). In particular, we have the inclusions

$$C^{2,0} \subset C^{1,1} \subset C^{1,\alpha} \subset C^{1,0} \subset C^{0,1}, \tag{A.512}$$

and, more in general,

$$C^{m+1} \subset C^{m,1} \subset C^{m,\alpha} \subset C^m \qquad \forall m \in \mathbb{N}_0, \ \ 0 < \alpha < 1. \tag{A.513}$$

To prove them, let us consider a point $\boldsymbol{x} \in \Gamma$, which is contained in some local chart (Θ, φ) and described as $x_N = \varphi(\boldsymbol{x}_s)$, where $\boldsymbol{x} = (\boldsymbol{x}_s, x_N)$. Then there exists a neighborhood of $\boldsymbol{x}_s$ whose closure is convex and contained in the definition domain Q of the function φ. Hence, from (A.463), (A.464), and (A.465), we obtain the inclusions (A.512) and (A.513).

b) Definition of the trace spaces

Now let $L^2(\Gamma)$ be the completion of $C^0(\Gamma)$, the space of all continuous functions on Γ, with respect to the norm

$$\|f\|_{L^2(\Gamma)} = \left(\int_\Gamma |f(\boldsymbol{x})|^2 \, \mathrm{d}\gamma(\boldsymbol{x}) \right)^{1/2}, \tag{A.514}$$

which is a Hilbert space with the scalar product

$$(f,g)_{L^2(\Gamma)} = \int_\Gamma f(\boldsymbol{x}) \, \overline{g(\boldsymbol{x})} \, \mathrm{d}\gamma(\boldsymbol{x}) \qquad \forall f,g \in L^2(\Gamma). \tag{A.515}$$

For a strong Lipschitz domain $\Omega \in C^{0,1}$ it can be shown that there exists a unique linear mapping $\gamma_0 : H^1(\Omega) \to L^2(\Gamma)$ such that if $f \in C^0(\overline{\Omega})$ then $\gamma_0 f = f|_\Gamma$. For $f \in H^1(\Omega)$ we call $\gamma_0 f$ the trace of f on Γ and the mapping γ_0 the trace operator (of order 0). However, in order to characterize all those elements in $L^2(\Gamma)$ which can be the trace of elements of $H^1(\Omega)$, we introduce also the trace spaces $H^s(\Gamma)$. For $s = 0$ we set $H^0(\Gamma) = L^2(\Gamma)$.

Let the boundary Γ be bounded, in which case there exists a covering of Γ by a finite union of open neighborhoods $\Theta_j \subset \mathbb{R}^N$ in the form of (A.507), for $1 \leq j \leq p < \infty$, such that Γ is enclosed in the set $\bigcup_{j=1}^p \Theta_j$. Such an open covering of Γ and the collection of all the local parametric representations φ_j of Γ on each neighborhood Θ_j is called a finite atlas. Each function φ_j has a definition domain Q_j and is described by a different orthogonal coordinate system, which is obtained by means of a translation vector $\boldsymbol{b}_j$ and an orthogonal linear transformation $\boldsymbol{T}_j$, as described in (A.511). If the boundary Γ is unbounded, we still suppose that there exists a finite atlas of Γ, i.e., there is a finite amount of local charts that encompasses the unbounded portions of Γ, and therefore the same results apply also to this case. We consider now the parametric representation of Γ through the mappings $\Phi_j : Q_j \to \Gamma$ defined by

$$\boldsymbol{x} = \Phi_j(\boldsymbol{y}_s) = \boldsymbol{b}_j + \boldsymbol{T}_j\big(\boldsymbol{y}_s, \varphi_j(\boldsymbol{y}_s)\big), \qquad \boldsymbol{y}_s \in Q_j, \;\; \boldsymbol{x} \in \Gamma. \tag{A.516}$$

For $\Gamma \in C^{m,\alpha}$, this allows us to define in a first step the trace space $H^s(\Gamma)$, for all s with $0 \leq s < m + \alpha$ for non-integer $m + \alpha$ or $0 \leq s \leq m + \alpha$ for integer $m + \alpha$, by

$$H^s(\Gamma) = \{ f \in L^2(\Gamma) : \; f \circ \Phi_j \in H^s(Q_j), \;\; 1 \leq j \leq p \}, \tag{A.517}$$

where $\circ$ denotes the composition of two functions. This space is equipped with the norm

$$\|f\|_{H^s(\Gamma)} = \left(\sum_{j=1}^p \|f \circ \Phi_j\|^2_{H^s(Q_j)} \right)^{1/2}, \tag{A.518}$$

and it becomes a Hilbert space with the inner product

$$(f,g)_{H^s(\Gamma)} = \sum_{j=1}^p (f \circ \Phi_j, g \circ \Phi_j)_{H^s(Q_j)} \qquad \forall f,g \in H^s(\Gamma). \tag{A.519}$$

We note that the above restrictions for s are necessary since otherwise the differentiations with respect to $\boldsymbol{y}_s$ required in (A.518) and (A.519) may not be well defined. In an additional step, these definitions, (A.518) and (A.519), can be rewritten in terms of the Sobolev spaces $H^s(\mathbb{R}^{N-1})$ by using a partition of unity, i.e., a set of positive functions $\lambda_j \in C_0^\infty(\Theta_j)$ such that

$$\sum_{j=1}^{p} \lambda_j(\boldsymbol{x}) = 1 \tag{A.520}$$

in some neighborhood of Γ. For f given on Γ, we define the extended function on $\mathbb{R}^{N-1}$ by

$$\widetilde{(\lambda_j f)}(\boldsymbol{y}_s) = \begin{cases} (\lambda_j f)\big(\Phi_j(\boldsymbol{y}_s)\big) & \text{for } \boldsymbol{y}_s \in Q_j, \\ 0 & \text{otherwise.} \end{cases} \tag{A.521}$$

This allows us to redefine the trace space (A.517) as

$$H^s(\Gamma) = \{f \in L^2(\Gamma) : \widetilde{\lambda_j f} \in H^s(\mathbb{R}^{N-1}), \ \ 1 \leq j \leq p\}. \tag{A.522}$$

The corresponding norm now reads

$$\|f\|_{H^s(\Gamma)} = \left(\sum_{j=1}^{p} \|\widetilde{\lambda_j f}\|^2_{H^s(\mathbb{R}^{N-1})} \right)^{1/2}, \tag{A.523}$$

and is associated with the scalar product

$$(f, g)_{H^s(\Gamma)} = \sum_{j=1}^{p} (\widetilde{\lambda_j f}, \widetilde{\lambda_j g})_{H^s(\mathbb{R}^{N-1})} \qquad \forall f, g \in H^s(\Gamma). \tag{A.524}$$

Since the extended functions $\widetilde{\lambda_j f}$ are defined on $\mathbb{R}^{N-1}$ having compact supports in Q_j, and since in (A.523) and (A.524) we are using $H^s(\mathbb{R}^{N-1})$, we can introduce via L^2-duality the whole scale of Sobolev spaces $H^s(\Gamma)$, for all s with $-m-\alpha < s < m+\alpha$ for non-integer $m+\alpha$ or $-m-\alpha \leq s \leq m+\alpha$ for integer $m+\alpha$. We have that $H^{-s}(\Gamma)$ is the dual space of $H^s(\Gamma)$, and for $s > 0$ it can be defined as the completion of $L^2(\Gamma)$ with respect to the norm

$$\|f\|_{H^{-s}(\Gamma)} = \sup_{\|\varphi\|_{H^s(\Gamma)}=1} |(\varphi, f)_{L^2(\Gamma)}|. \tag{A.525}$$

The trace spaces can be alternatively defined in terms of boundary norms. We define the space $H^s(\Gamma)$, for $0 < s < 1$, as the completion of $C^0(\Gamma)$ with respect to the norm

$$\|f\|_{H^s(\Gamma)} = \left(\|f\|^2_{L^2(\Gamma)} + \int_\Gamma \int_\Gamma \frac{|f(\boldsymbol{x}) - f(\boldsymbol{y})|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{N-1+2s}} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \right)^{1/2}, \tag{A.526}$$

which means that we can define

$$H^s(\Gamma) = \{f \in L^2(\Gamma) : \|f\|_{H^s(\Gamma)} < \infty\}. \tag{A.527}$$

Again, $H^s(\Gamma)$ is a Hilbert space when equipped with the inner product

$$(f, g)_{H^s(\Gamma)} = (f, g)_{L^2(\Gamma)} + \int_\Gamma \int_\Gamma \frac{\big(f(\boldsymbol{x}) - f(\boldsymbol{y})\big)\overline{\big(g(\boldsymbol{x}) - g(\boldsymbol{y})\big)}}{|\boldsymbol{x}-\boldsymbol{y}|^{N-1+2s}} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}. \tag{A.528}$$

To use this definition for $s \geq 1$ is more complicated. Further details can be found in the book of Hsiao & Wendland (2008).

A third alternative to define the trace spaces on Γ is to use extensions of functions defined on Γ to Sobolev spaces defined in Ω. For $s > 0$ we define the Sobolev space

$$H^s(\Gamma) = \{f \in L^2(\Gamma) : \exists\, \widetilde{f} \in H^{s+\frac{1}{2}}(\Omega) \text{ such that } \gamma_0\widetilde{f} = \widetilde{f}|_\Gamma = f \text{ on } \Gamma\}, \tag{A.529}$$

which is supplied with the norm

$$\|f\|_{H^s(\Gamma)} = \inf_{\gamma_0\widetilde{f}=f} \|\widetilde{f}\|_{H^{s+1/2}(\Omega)}. \tag{A.530}$$

We observe that this definition for trace spaces can be used without problem for any $s > 0$, and it fulfills in a natural way the trace theorem.

As mentioned in Grisvard (1985), we remark that when a function f is a solution in Ω of an elliptic partial differential equation, then f has traces on the boundary provided it belongs to any Sobolev space, without any restriction to s.

c) Trace theorem

The trace theorem characterizes the conditions for the existence of the so-called trace operator. Let Ω be a domain with a boundary Γ of class $C^{m,1}$ with $m \in \mathbb{N}_0$ and where s is taken such that $\frac{1}{2} < s \leq m+1$. Under these conditions, the trace theorem states that there exists a linear continuous trace operator γ_0 with

$$\gamma_0 : H^s(\Omega) \longrightarrow H^{s-\frac{1}{2}}(\Gamma), \tag{A.531}$$

which is an extension of

$$\gamma_0 f = f|_\Gamma \quad \text{for } f \in C^0(\overline{\Omega}). \tag{A.532}$$

The theorem characterizes also traces of higher order. For a domain Ω with a boundary Γ of class $C^{m,1}$, we consider $j, m \in \mathbb{N}_0$ and we take s such that $\frac{1}{2} + j < s \leq m+1$. Then there exists a linear continuous trace operator γ_j with

$$\gamma_j : H^s(\Omega) \longrightarrow H^{s-j-\frac{1}{2}}(\Gamma), \tag{A.533}$$

which is an extension of the normal derivatives of order j

$$\gamma_j f = \frac{\partial^j f}{\partial n^j}\Big|_\Gamma = (\boldsymbol{n}\cdot\nabla)^j f|_\Gamma \quad \text{for } f \in C^\ell(\overline{\Omega}) \quad \text{with } s + j \leq \ell \in \mathbb{N}, \tag{A.534}$$

where $\boldsymbol{n}$ denotes the unit boundary normal vector that points outwards of the domain Ω. Moreover, the trace theorem states that under these conditions all the different definitions of trace spaces are equivalent.

d) The spaces $H^{1/2}(\Gamma)$, $H^{-1/2}(\Gamma)$, and $H^1(\Delta;\Omega)$

Of particular interest in our case are the trace spaces $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$. The trace space $H^{1/2}(\Gamma)$ can be defined either by (A.522), (A.527), or (A.529) for $s = \frac{1}{2}$, where the norm is given respectively by (A.523), (A.526), or (A.530). If $\Gamma \in C^{0,1}$, then the three presented alternative definitions for $H^{1/2}(\Gamma)$ coincide. Its dual space $H^{-1/2}(\Gamma)$ is given by the completion of $L^2(\Gamma)$ with respect to the norm of the dual space (A.525).

As mentioned in Raviart (1991), we have that a particularly interesting space to work with traces is

$$H^1(\Delta;\Omega) = \{f \in H^1(\Omega) : \Delta f \in L^2(\Omega)\}, \tag{A.535}$$

provided with the norm

$$\|f\|_{H^1(\Delta;\Omega)} = \left(\|f\|^2_{H^1(\Omega)} + \|\Delta f\|^2_{L^2(\Omega)}\right)^{1/2}, \tag{A.536}$$

since this space is adjusted enough so as to still allow to define the trace of the normal derivative. In fact, for $f \in H^1(\Delta;\Omega)$ and due the trace theorem, we have that

$$\gamma_0 f = f|_\Gamma \in H^{1/2}(\Gamma), \tag{A.537}$$

$$\gamma_1 f = \frac{\partial f}{\partial n}|_\Gamma \in H^{-1/2}(\Gamma). \tag{A.538}$$

e) Trace spaces on an open surface

In some applications we need trace spaces on an open connected part $\Gamma_0 \subset \Gamma$ of a closed boundary Γ. Let us assume that $\Gamma \in C^{m,1}$ with $m \in \mathbb{N}_0$. In the two-dimensional case $\Gamma_0 \subset \Gamma = \partial\Omega$ with $\Omega \in \mathbb{R}^2$, the boundary of Γ_0 is denoted by $\gamma = \partial\Gamma_0$ and consists just of two endpoints $\gamma = \{\boldsymbol{z}_1, \boldsymbol{z}_2\}$. In the three-dimensional case, the boundary $\partial\Gamma_0$ of Γ_0 is a closed curve γ. We assume that s satisfies $|s| \leq m+1$, and thus all the definitions for the trace space $H^s(\Gamma)$ coincide. As before, let us introduce the space of trivial extensions from $\overline{\Gamma}_0$ to Γ of functions f defined on $\overline{\Gamma}_0$ by zero outside of $\overline{\Gamma}_0$, which are denoted by $\widetilde{f}$. Thus we define

$$\widetilde{H}^s(\Gamma_0) = \{f \in H^s(\Gamma) : f|_{\Gamma\setminus\overline{\Gamma}_0} = 0\} = \{f \in H^s(\Gamma) : \operatorname{supp} f \subset \overline{\Gamma}_0\} \tag{A.539}$$

as a subspace of $H^s(\Gamma)$ with the corresponding norm

$$\|f\|_{\widetilde{H}^s(\Gamma_0)} = \|\widetilde{f}\|_{H^s(\Gamma)}. \tag{A.540}$$

By definition, $\widetilde{H}^s(\Gamma_0) \subset H^s(\Gamma)$. For $s \geq 0$ we also introduce the space

$$H^s(\Gamma_0) = \{f = F|_{\Gamma_0} : F \in H^s(\Gamma)\}, \tag{A.541}$$

equipped with the norm

$$\|f\|_{H^s(\Gamma_0)} = \inf_{\substack{F\in H^s(\Gamma)\\ F|_{\Gamma_0}=f}} \|F\|_{H^s(\Gamma)}. \tag{A.542}$$

Clearly $\widetilde{H}^s(\Gamma_0) \subset H^s(\Gamma_0)$. The dual space $H^{-s}(\Gamma_0)$ with respect to the inner product in $L^2(\Gamma_0)$ is well defined by the completion of $L^2(\Gamma_0)$ with respect to the norm

$$\|f\|_{H^{-s}(\Gamma_0)} = \sup_{0\neq\varphi\in\widetilde{H}^s(\Gamma_0)} \frac{|(f,\varphi)_{L^2(\Gamma_0)}|}{\|\varphi\|_{\widetilde{H}^s(\Gamma_0)}}, \qquad s > 0. \tag{A.543}$$

Correspondingly, we also have the dual space $\widetilde{H}^{-s}(\Gamma_0)$ with the norm

$$\|f\|_{\widetilde{H}^{-s}(\Gamma_0)} = \sup_{0\neq\psi\in H^s(\Gamma_0)} \frac{|(f,\psi)_{L^2(\Gamma_0)}|}{\|\psi\|_{H^s(\Gamma_0)}}, \qquad s > 0. \tag{A.544}$$

It holds therefore that

$$H^{-s}(\Gamma_0) = \widetilde{H}^s(\Gamma_0)', \tag{A.545}$$

$$\widetilde{H}^{-s}(\Gamma_0) = H^s(\Gamma_0)'. \tag{A.546}$$

We have for $s > 0$ also the inclusions

$$\widetilde{H}^s(\Gamma_0) \subset H^s(\Gamma_0) \subset L^2(\Gamma_0) \subset \widetilde{H}^{-s}(\Gamma_0) \subset H^{-s}(\Gamma_0). \tag{A.547}$$

Similar as before, if $s < \frac{1}{2}$, then $\widetilde{H}^s(\Gamma_0) = H^s(\Gamma_0)$. For $s > \frac{1}{2}$, we note that $f \in \widetilde{H}^s(\Gamma_0)$ satisfies $f|_\gamma = 0$. Hence, we can introduce the space $H_0^s(\Gamma_0)$ as the completion of $\widetilde{H}^s(\Gamma_0)$ with respect to the norm $\|\cdot\|_{H^s(\Gamma_0)}$. It holds then that $\widetilde{H}^s(\Gamma_0) = H_0^s(\Gamma_0)$ if $s \neq m + \frac{1}{2}$ for $m \in \mathbb{N}_0$, and that $\widetilde{H}^{m+1/2}(\Gamma_0)$ is strictly contained in $H_0^{m+1/2}(\Gamma_0)$.

A.4.6 Imbeddings of Sobolev spaces

It is primarily the imbedding characteristics (vid. Section A.3) of Sobolev spaces that render these spaces so useful in analysis, especially in the study of differential and integral operators. By knowing the mapping properties of such an operator in terms of Sobolev spaces, for example, it can be determined whether the operator is continuous or compact.

In $\mathbb{R}^N$ we have the continuous imbedding

$$H^s(\mathbb{R}^N) \hookrightarrow H^t(\mathbb{R}^N) \qquad \text{for } -\infty < t \leq s < \infty. \tag{A.548}$$

If $m \in \mathbb{N}_0$ and $0 \leq \alpha < 1$, then it holds that

$$H^s(\mathbb{R}^N) \hookrightarrow C^{m,\alpha}(\mathbb{R}^N) \qquad \text{for } s > m + \alpha + \frac{N}{2}, \tag{A.549}$$

which holds also if $s = m + \alpha + \frac{N}{2}$ and $0 < \alpha < 1$.

We consider now a bounded strong Lipschitz domain $\Omega \in C^{0,1}$. Then we have the compact and continuous imbeddings

$$H^s(\Omega) \overset{c}{\hookrightarrow} H^t(\Omega) \qquad \text{for } -\infty < t < s < \infty, \tag{A.550}$$

$$\widetilde{H}^s(\Omega) \overset{c}{\hookrightarrow} \widetilde{H}^t(\Omega) \qquad \text{for } -\infty < t < s < \infty, \tag{A.551}$$

$$H^s(\Omega) \overset{c}{\hookrightarrow} C^{m,\alpha}(\overline{\Omega}) \qquad \text{for } s > m + \alpha - \frac{N}{2},\ \ 0 \leq \alpha < 1,\ \ m \in \mathbb{N}_0. \tag{A.552}$$

We have also the continuous imbedding

$$H^s(\Omega) \hookrightarrow C^{m,\alpha}(\overline{\Omega}) \qquad \text{for } s = m + \alpha - \frac{N}{2},\ \ 0 < \alpha < 1,\ \ m \in \mathbb{N}_0. \tag{A.553}$$

Let Γ be a boundary of class $C^{k,1}$, $k \in \mathbb{N}_0$, and let $|t|, |s| \leq k + \frac{1}{2}$. Then we have the compact imbeddings

$$H^s(\Gamma) \overset{c}{\hookrightarrow} H^t(\Gamma) \qquad \text{for } t < s, \tag{A.554}$$

$$H^s(\Gamma) \overset{c}{\hookrightarrow} C^{m,\alpha}(\Gamma) \quad \text{for } s > m + \alpha + \frac{N}{2} - \frac{1}{2},\ \ 0 \leq \alpha < 1,\ \ m \in \mathbb{N}_0. \tag{A.555}$$