Green's functions and fundamental solutions

Green's functions are used to solve inhomogeneous boundary-value problems for differential equations subject to boundary conditions. They receive their name from the British mathematician and physicist George Green (1793–1841), who was the first to study a special case of this type of functions in his research on potential theory, which he developed in a famous essay (Green 1828).

The concept of a Green's function is essential throughout this thesis, so it becomes important to understand properly their significance. Our main references for these functions, treated in the sense of distributions, are Griffel (1985) and Terrasse & Abboud (2006). A more classical treatment of Green's functions in the context of mathematical physics can be found, e.g., in Bateman (1932), Courant & Hilbert (1966), and Morse & Feshbach (1953). There exist also several books that are almost entirely dedicated to Green's functions, like Barton (1989), DeSanto (1992), Duffy (2001), and Greenberg (1971). An exhaustive amount of them are likewise listed in Polyanin (2002).

The Green's function of a boundary-value problem for a linear differential equation is the fundamental solution of this equation satisfying homogeneous boundary conditions. It is thus the kernel of the integral operator that is the inverse of the differential operator generated by the given differential equation and the homogeneous boundary conditions. The Green's function yields therefore solutions for the inhomogeneous boundary-value problem. Finding the Green's function reduces the study of the properties for the differential operator to the study of similar properties for the corresponding integral operator.

A.8.1 Fundamental solutions

Technically, a fundamental solution for a partial differential operator $\mathcal{L}$, linear, with constant coefficients, and defined on the space of distributions $\mathcal{D}'(\mathbb{R}^N)$, is a distribution E that satisfies

$$\mathcal{L}E = \delta \qquad \text{in } \mathcal{D}'(\mathbb{R}^N), \tag{A.810}$$

where δ is the Dirac delta or impulse function, centered at the origin. The main interest of such a fundamental solution lies in the fact that if the convolution has a sense, then the solution of

$$\mathcal{L}u = f \qquad \text{in } \mathcal{D}'(\mathbb{R}^N), \tag{A.811}$$

for a known data function f, is given by

$$u = E * f. \tag{A.812}$$

In fact, due the linearity of $\mathcal{L}$, since E is a fundamental solution, and since δ is the neutral element of the convolution, we have

$$\mathcal{L}u = \mathcal{L}\{E * f\} = \mathcal{L}E * f = \delta * f = f. \tag{A.813}$$

By adding to the fundamental solution non-trivial solutions for the homogeneous problem, new fundamental solutions can be obtained. The fundamental solution for a well-posed problem is unique, if additional conditions are specified for the behavior of the solution, e.g., the decaying behavior at infinity, being these conditions often determined through physical considerations. In the construction of the fundamental solution it is permissible to use any methods to find the solutions of the equation, provided that the result is then justified by rigorous arguments.

We remark also that from the fundamental solution other solutions can be derived when, in the sense of distributions, derivatives of the Dirac delta function δ appear on the right-hand side. For example, the solution of

$$\mathcal{L}F = \frac{\partial \delta}{\partial x_i} \qquad \text{in } \mathcal{D}'(\mathbb{R}^N) \tag{A.814}$$

is given by

$$F = E * \frac{\partial \delta}{\partial x_i} = \frac{\partial E}{\partial x_i} * \delta = \frac{\partial E}{\partial x_i}. \tag{A.815}$$

A.8.2 Green's functions

In the case of the Green's function, the fundamental solution considers also homogeneous boundary conditions, and the Dirac delta function is no longer centered at the origin, but at a fixed source point. Thus, a Green's function of a partial differential operator $\mathcal{L}_{\boldsymbol{y}}$ with homogeneous boundary conditions, linear, with constant coefficients, acting on the variable $\boldsymbol{y}$, and defined on the space of distributions $\mathcal{D}'(\mathbb{R}^N)$, is a distribution G such that

$$\mathcal{L}_{\boldsymbol{y}}\{G(\boldsymbol{x},\boldsymbol{y})\} = \delta_{\boldsymbol{x}}(\boldsymbol{y}) \qquad \text{in } \mathcal{D}'(\mathbb{R}^N), \tag{A.816}$$

where $\delta_{\boldsymbol{x}}$ is the Dirac delta or impulse function with the Dirac mass centered at the source point $\boldsymbol{x}$, i.e., $\delta_{\boldsymbol{x}}(\boldsymbol{y}) = \delta(\boldsymbol{y}-\boldsymbol{x})$. The Green's function represents thus the impulse response of the operator $\mathcal{L}_{\boldsymbol{y}}$ with respect to the source point $\boldsymbol{x}$, being therefore the nucleus or kernel of the inverse operator of $\mathcal{L}_{\boldsymbol{y}}$, denoted by $\mathcal{L}_{\boldsymbol{y}}^{-1}$, which corresponds to an integral operator, and $G(\boldsymbol{x},\boldsymbol{y}) = \mathcal{L}_{\boldsymbol{y}}^{-1}\{\delta_{\boldsymbol{x}}(\boldsymbol{y})\}$. The Green's function, differently as the fundamental solution, is searched in some particular domain $\Omega \subset \mathbb{R}^N$ and satisfies some boundary conditions, but for simplicity we consider here just $\Omega = \mathbb{R}^N$.

The solution of the inhomogeneous partial differential boundary-value problem

$$\mathcal{L}_{\boldsymbol{x}}\{u(\boldsymbol{x})\} = f(\boldsymbol{x}) \qquad \text{in } \mathcal{D}'(\mathbb{R}^N), \tag{A.817}$$

is in this case given, if the convolution has a sense, by

$$u(\boldsymbol{x}) = G(\boldsymbol{x},\boldsymbol{y}) * f(\boldsymbol{y}), \tag{A.818}$$

where G is the Green's function of the operator $\mathcal{L}_{\boldsymbol{x}}$, which is symmetric, i.e.,

$$G(\boldsymbol{x},\boldsymbol{y}) = G(\boldsymbol{y},\boldsymbol{x}). \tag{A.819}$$

Again, as for the fundamental solution, we have

$$\begin{aligned}\mathcal{L}_{\boldsymbol{x}}\{u(\boldsymbol{x})\} &= \mathcal{L}_{\boldsymbol{x}}\{G(\boldsymbol{x},\boldsymbol{y}) * f(\boldsymbol{y})\} = \mathcal{L}_{\boldsymbol{x}}\{G(\boldsymbol{x},\boldsymbol{y})\} * f(\boldsymbol{y}) \\ &= \delta_{\boldsymbol{x}}(\boldsymbol{y}) * f(\boldsymbol{y}) = f(\boldsymbol{x}). \end{aligned} \tag{A.820}$$

We observe that the free- or full-space Green's function, i.e., without boundary conditions, is linked to the fundamental solution through the relation

$$G(\boldsymbol{x},\boldsymbol{y}) = E(\boldsymbol{x}-\boldsymbol{y}) = E(\boldsymbol{y}-\boldsymbol{x}). \tag{A.821}$$

A.8.3 Some free-space Green's functions

We consider now some examples of free-space Green's functions of our interest. The free-space Green's function for the Laplace equation satisfies in the sense of distributions

$$\Delta_{\boldsymbol{y}} G(\boldsymbol{x},\boldsymbol{y}) = \delta_{\boldsymbol{x}}(\boldsymbol{y}) \qquad \text{in } \mathcal{D}'(\mathbb{R}^N), \tag{A.822}$$

and is given by (Polyanin 2002)

$$G(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} \dfrac{|y-x|}{2} & \text{for } N=1, \\[2ex] \dfrac{1}{2\pi}\ln|\boldsymbol{y}-\boldsymbol{x}| & \text{for } N=2, \\[2ex] -\dfrac{1}{4\pi|\boldsymbol{y}-\boldsymbol{x}|} & \text{for } N=3, \\[2ex] -\dfrac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{N/2}(N-2)|\boldsymbol{y}-\boldsymbol{x}|^{N-2}} & \text{for } N\geq 4, \end{cases} \tag{A.823}$$

where Γ denotes the gamma function (vid. Subsection A.2.2).

The free-space Green's function of outgoing-wave behavior for the Helmholtz equation, on the other hand, satisfies in the sense of distributions

$$\Delta_{\boldsymbol{y}} G(\boldsymbol{x},\boldsymbol{y}) + k^2 G(\boldsymbol{x},\boldsymbol{y}) = \delta_{\boldsymbol{x}}(\boldsymbol{y}) \qquad \text{in } \mathcal{D}'(\mathbb{R}^N), \tag{A.824}$$

and has to be supplied with the Sommerfeld radiation condition

$$\lim_{|\boldsymbol{y}|\to\infty} |\boldsymbol{y}|^{\frac{N-1}{2}} \left(\frac{\partial G}{\partial|\boldsymbol{y}|}(\boldsymbol{x},\boldsymbol{y}) - ikG(\boldsymbol{x},\boldsymbol{y}) \right) = 0, \tag{A.825}$$

where $k \in \mathbb{C}$ corresponds to the wave number. By adapting the expressions listed in Polyanin (2002) we acquire in this case that

$$G(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} -\dfrac{i}{2k}\, e^{ik|y-x|} & \text{for } N=1, \\[2ex] -\dfrac{i}{4} H_0^{(1)}\big(k|\boldsymbol{y}-\boldsymbol{x}|\big) & \text{for } N=2, \\[2ex] -\dfrac{e^{ik|\boldsymbol{y}-\boldsymbol{x}|}}{4\pi|\boldsymbol{y}-\boldsymbol{x}|} & \text{for } N=3, \\[2ex] -\dfrac{i}{4}\left(\dfrac{k}{2\pi|\boldsymbol{y}-\boldsymbol{x}|}\right)^{\frac{N-2}{2}} H^{(1)}_{\frac{N-2}{2}}\big(k|\boldsymbol{y}-\boldsymbol{x}|\big) & \text{for } N\geq 4, \end{cases} \tag{A.826}$$

where $H_\nu^{(1)}$ denotes the Hankel function of the first kind of order ν (vid. Subsection A.2.4).

A.9 Wave propagation

Wave propagation is a complex physical phenomenon, whose mathematical description is in general not easy to accomplish. Some generalities concerning wave propagation and its mathematical modeling are presented below. Some references are Nédélec (2001), Jackson (1999), Kuttruff (2007), Wilcox (1975), Strauss (1992), and Evans (1998). An interesting survey of several research areas in wave propagation can be found in Keller (1979). A thorough discussion on the amount of samples per wavelength required in the discretization procedure and on some other related aspects is given in Marburg (2008).

A.9.1 Generalities on waves

A wave is a disturbance that propagates with time through a certain medium transferring energy progressively from point to point. The medium through which the wave travels may experience some local oscillations around fixed positions as the wave passes, but the particles in the medium do not travel with the wave, and are thus not displaced permanently. The medium could even be the vacuum as in the case of electromagnetic waves. The disturbance may take any of a number of shapes, from a finite width pulse to an infinitely long sine wave. Several kinds of waves exist, e.g., mechanical (sound, elastic, seismic, and ocean surface waves), electromagnetic (visible light, radio waves, X-rays), temperature, or gravitational waves.

Waves are characterized by crests and troughs, either perpendicular or parallel to the wave's motion. Waves in which the propagating disturbance is perpendicular to its motion are called transverse waves (waves on a string or electromagnetic waves), while waves in which it is parallel are called longitudinal waves (sound or pressure waves). Transverse waves can be polarized. Unpolarized waves can oscillate in any direction in the plane perpendicular to the direction of travel, while polarized waves oscillate in only one direction perpendicular to the line of travel.

All waves have a common behavior under a number of standard situations. They all can experience the phenomena of rectilinear propagation, interference, reflection, refraction, diffraction, and scattering. Rectilinear propagation states that waves in a homogeneous medium move or spread out in straight lines. Interference is the superposition of two or more waves resulting in a new wave pattern. The principle of linear superposition of waves states that the resultant displacement at a given point is equal to the sum of the displacements of different waves at that point. Reflection is an abrupt change in direction of a wave at an interface between two dissimilar media so that the wave returns into the medium from which it originated. Refraction is the change in direction of a wave due to a change in its velocity when entering a new medium with different refractive index. Diffraction is the bending of waves when they meet one (or more) partial obstacles, which deform the shape of the wavefronts as they pass. Scattering or dispersion is the process whereby waves are forced to deviate from a straight trajectory into many directions by one or more localized non-uniformities (called scatterers) in the medium through which they pass. Scattering is therefore a form of reflection in which a portion of the incident waves is redistributed into many directions by a scatterer.

A.9.2 Wave modeling

Waves are modeled physically and mathematically as solutions of a wave equation. Each kind of waves has its own wave equation and associated auxiliary conditions, e.g., boundary conditions, that can be applied. The most studied wave equation is probably the scalar wave equation of linear acoustics, which describes the propagation of sound in a homogeneous medium in the space $\mathbb{R}^N$ ($N = 1$, 2, or 3). It takes the form of the hyperbolic partial differential equation

$$\frac{\partial^2 p}{\partial t^2} - c^2 \Delta p = 0, \qquad \boldsymbol{x} \in \mathbb{R}^N, \;\; t \in \mathbb{R}_+, \tag{A.827}$$

where c is the speed of sound and $p = p(\boldsymbol{x}, t)$ is the induced pressure. By Δ we denote the Laplace operator

$$\Delta p = \sum_{j=1}^{N} \frac{\partial^2 p}{\partial x_j^2}, \tag{A.828}$$

named in honor of the French mathematician and astronomer Pierre-Simon, marquis de Laplace (1749–1827), whose work was pivotal to the development of mathematical astronomy. He formulated Laplace's equation and invented the Laplace transform, which appears in many branches of mathematical physics, a field that he took a leading role in forming.

After a mathematical trick attributed to the French mathematician, mechanician, physicist, and philosopher Jean le Rond d'Alembert (1717–1783), in a space of dimension $N = 1$ all regular solutions of (A.827) are of the form

$$p(x, t) = f(x - ct) + g(x + ct), \tag{A.829}$$

where f and g are arbitrary functions. This expression shows that if the functions f and g have compact support, then the solution propagates at a finite speed equal to c. Finite speed propagation is one of the essential characteristics of hyperbolic equations.

A time-harmonic solution of the wave equation (A.827) is a function of the form

$$p(\boldsymbol{x}, t) = \mathfrak{Re}\big\{u(\boldsymbol{x})e^{-i\omega t}\big\}, \tag{A.830}$$

where u is the amplitude of the pressure and i denotes the complex imaginary unit, which represents the square root of -1. The quantity ω is called the pulsation or angular frequency of the harmonic wave. Here the time convention $e^{-i\omega t}$ has been taken, which determines the sign of ingoing and outgoing waves, and thus also of the outgoing radiation condition when dealing with unbounded domains. After applying this separation of variables to (A.827), the function u becomes a solution of the Helmholtz equation

$$\Delta u + k^2 u = 0, \qquad k = \frac{\omega}{c}. \tag{A.831}$$

The number k is called wave number. The quantity $f = \omega/2\pi$ is called frequency and the length $\lambda = c/f = 2\pi/k$ is called wavelength. This equation carries the name of the German physician and physicist Hermann Ludwig Ferdinand von Helmholtz (1821–1894), for his contributions to mathematical acoustics and electromagnetism. When the frequency (or the wave number) is zero, then we obtain the Laplace equation

$$\Delta u = 0, \qquad \omega = 0. \tag{A.832}$$

The Helmholtz equation has a very special family of solutions called plane waves. Up to a multiplicative factor, they are the complex-valued functions of the form

$$u(\boldsymbol{x}) = e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \qquad (\boldsymbol{k}\cdot\boldsymbol{k}) = k^2. \tag{A.833}$$

They correspond to wavefronts that travel with velocity c in the direction given by the wave propagation vector $\boldsymbol{k}$. The vector $\boldsymbol{k}$ can be real, in which case $k = |\boldsymbol{k}|$ and these solutions are of modulus 1. When the vector $\boldsymbol{k}$ is complex, then the solutions are exponentially decreasing in a half-space determined by the imaginary part of the vector $\boldsymbol{k}$ and exponentially increasing in the other half-space, i.e., where they explode. They are called plane waves because $e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$ is constant on the planes $(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t) =$ constant.

A.9.3 Discretization requirements

Wave propagation problems dealing with geometries that are too complex to solve analytically are nowadays solved with the help of computers, by using appropriate numerical methods and discretization procedures. For this purpose, the considered geometry is discretized using a finite mesh to describe it. In computational linear time-harmonic wave propagation modeling, it is widely accepted that the appropriate refinement and configuration of this discretized mesh, i.e., the placement of its discretization nodes, should be related to the wavelength. The commonly applied rule of thumb is to use a fixed number of nodes per wavelength. In many cases, this number of nodes per wavelength varies typically between three and ten, although it is advised to use at least five or even six of them. Obviously, this number is closely related to a certain desired accuracy. Often the error is of an acceptable magnitude, which depends on the user and on certain technical requirements. A sine-wave discretization for different numbers of nodes per wavelength for an equidistant node distribution is depicted in Figure A.18.

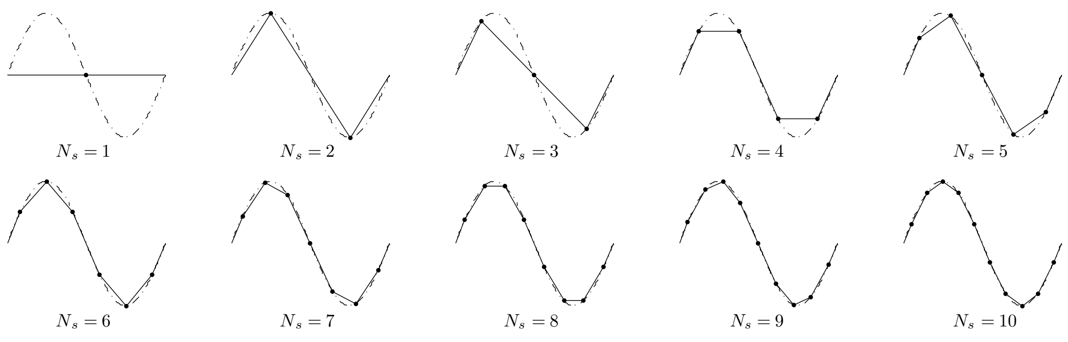

FIGURE A.18. Sine-wave discretization for different numbers of nodes per wavelength.

The idea of using a fixed number of nodes per wavelength is most likely a consequence of the Nyquist-Shannon sampling theorem, also known as Nyquist's sampling theorem, Shannon's sampling theorem, or simply as the sampling theorem. It is named after the Swedish electronic engineer Harry Nyquist (1889–1976) and the American electronic engineer and mathematician Claude Elwood Shannon (1916–2001), who laid the foundations that led to the development of information theory. Some references for this theorem are the extensive survey articles of Jerry (1977, 1979) and Unser (2000), and the books of Gasquet & Witomski (1999) and Irarrázaval (1999). The Nyquist-Shannon sampling theorem is of fundamental importance in wave propagation and in vibration analysis for experimental measurements and frequency detection. It states that at least two points per wavelength (or period of an oscillating function) are necessary to detect the corresponding frequency. However, a simple detection cannot be sufficient to approximate the function, as stated in Marburg (2008), who refers to several other authors and performs an extensive analysis on the discretization requirements for wave propagation problems, considering different types of finite elements. It is mentioned there that two points per wavelength are strictly sufficient, but would still not lead to an accurate reconstruction of the function, and it is therefore advised to take rather an amount of six to ten nodes per wavelength. In particular for boundary element methods, the common rule is to use six constant or linear boundary elements per wavelength. The concluding remarks recommend the use of discontinuous boundary elements with nodes located at the zeros of Legendre polynomials (vid. Subsection A.2.8), provided that the involved problem is essentially related to the inversion of the double layer potential operator. It is also mentioned that in the case of mixed problems and when the hypersingular operator is used, then probably other optimal locations for the nodes will be found.

A.10 Linear water-wave theory

The linear water-wave theory is concerned with the propagation of waves on the surface of the water, considered as small perturbations so that they can be linearly described. The study of these waves has many applications, including naval architecture, ocean engineering, and geophysical hydrodynamics. For example, it is required for predicting the behavior of floating structures (immersed totally or partially), such as ships, submarines, and tension-leg platforms, and for describing flows over bottom topography. Furthermore, the investigation of wave patterns of ships and other vehicles in forward motion is closely related to the calculation of the wave-making resistance and other hydrodynamic characteristics that are used in marine design. Another area of application is the mathematical modeling of unsteady waves resulting from such phenomena as underwater earthquakes, blasts, etc. We are herein interested in the derivation of the governing differential equations of these waves, obtained on the basis of general dynamics of an inviscid incompressible fluid (water is the standard example of such a fluid), and their linearization.

We are particularly devoted to waves arising in two closely related phenomena, which are radiation of waves by oscillating immersed bodies and scattering of incoming progressive waves by an obstacle (a floating body or variable bottom topography). Mathematically these phenomena give rise to a boundary-value problem that is usually referred to as the water-wave problem. The difficulty of this problem stems from several facts. First, it is essential that the water domain is infinite. Second, there is a spectral parameter (it is related to the radian frequency of waves) in a boundary condition on a semi-infinite part of the boundary (referred to as the free surface of water). Above all, the free surface may consist of more than one component as occurs for a surface-piercing toroidal body.

Good and complete references for the linear theory of water waves are Kuznetsov, Maz'ya & Vainberg (2002) and Wehausen & Laitone (1960), which are closely followed herein, in particular the former. Other references on this topic are Hazard & Lenoir (1998), Howe (2007), John (1949, 1950), Lamb (1916), Linton & McIver (2001), Mei (1983), Mei, Stiassnie & Yue (2005), Stoker (1957), and Wehausen (1971).

Water waves, also known as gravity waves, ocean surface waves, or simply surface waves, are created normally by a gravitational force in the presence of a free surface along which the pressure is constant. There are two ways to describe these waves mathematically. It is possible to trace the paths of individual particles (a Lagrangian description), but in this thesis an alternative form of equations (usually referred to as Eulerian) is adopted. The first description receives its name from the Italian-French mathematician and astronomer Joseph Louis Lagrange (1736–1813), who made important contributions to classical and celestial mechanics and to number theory. The second description is named after the already mentioned great Swissborn Russian mathematician and physicist Leonhard Euler (1707–1783). The motion is determined by the velocity field in the domain occupied by water at every moment of the time t.

Water is assumed to occupy a certain domain Ω bounded by one or more moving or fixed surfaces that separate water from some other medium. Actually we consider boundaries of two types: the above-mentioned free surface separating water from the atmosphere, and rigid surfaces including the bottom and surfaces of bodies floating in and/or beneath the free surface.

It is convenient to use rectangular coordinates $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ with the origin in the free surface at rest (which coincides with the mean free surface), and with the x_3 axis directed opposite to the acceleration caused by gravity. For the sake of brevity we will write $\boldsymbol{x}_s$ instead of (x_1, x_2). Two-dimensional problems can be treated simultaneously by considering the variables $(x_s, x_3) \in \mathbb{R}^2$, i.e., taking a scalar x_s instead of the vectorial $\boldsymbol{x}_s$, and renaming eventually x_3 by x_2. Two-dimensional problems form an important class of problems considering water motions that are the same in every plane orthogonal to a certain direction. As usual, $\nabla u = (\partial u/\partial x_1, \partial u/\partial x_2, \partial u/\partial x_3)$, and the horizontal component of ∇ will be denoted by ∇_s, that is, $\nabla_s u = (\partial u/\partial x_1, \partial u/\partial x_2, 0)$.

A.10.1 Equations of motion and boundary conditions

In the Eulerian formulation one seeks the velocity vector $\boldsymbol{v}$, the pressure p, and the fluid density ρ as functions of $\boldsymbol{x} \in \overline{\Omega}$ and $t \geq t_0$, where t_0 denotes a certain initial moment. Assuming the fluid to be inviscid without surface tension, one obtains the equations of motion from conservation laws. The conservation of mass implies the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \qquad \text{in } \Omega. \tag{A.834}$$

Under the assumption that the fluid is incompressible (which is usual in the water-wave theory), the last equation becomes

$$\nabla \cdot \boldsymbol{v} = 0 \qquad \text{in } \Omega. \tag{A.835}$$

The conservation of momentum in inviscid fluid leads to the so-called Euler equations. Taking into account the gravity force, one can write these three (or two) equations in the vector form

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\frac{1}{\rho} \nabla p + \boldsymbol{g} \qquad \text{in } \Omega. \tag{A.836}$$

Here $\boldsymbol{g}$ is the vector of the gravity force having zero horizontal components and the vertical one equal to $-g$, where g denotes the acceleration caused by gravity.

An irrotational character of motion is another usual assumption in the theory, i.e.,

$$\nabla \times \boldsymbol{v} = 0 \qquad \text{in } \Omega. \tag{A.837}$$

Note that one can prove that the motion is irrotational if it has this property at the initial moment. The last equation guarantees the existence of a velocity potential ϕ so that

$$\boldsymbol{v} = \nabla \phi \qquad \text{in } \overline{\Omega}. \tag{A.838}$$

This is obvious for simply connected domains, otherwise (for example, when one considers a two-dimensional problem for a totally immersed body), the so-called no-flow condition should be taken into account (vid. (A.843) below).

From (A.835) and (A.838) one obtains the Laplace equation

$$\Delta\phi = 0 \qquad \text{in } \Omega. \tag{A.839}$$

This greatly facilitates the theory but, in general, solutions of (A.839) do not manifest wave character. Waves are created by the boundary conditions on the free surface.

Let $x_3 = \eta(\boldsymbol{x}_s, t)$ be the equation of the free surface valid for $\boldsymbol{x}_s \in \Gamma$, where Γ is a union of some domains (generally depending on t) in $\mathbb{R}^{N-1}$, with $N = 2, 3$. The pressure is prescribed to be equal to the constant atmospheric pressure p_0 on $x_3 = \eta(\boldsymbol{x}_s, t)$, and the surface tension is neglected. From (A.837) and (A.838) one immediately obtains Bernoulli's equation

$$\frac{\partial\phi}{\partial t} + \frac{|\nabla\phi|^2}{2} = -\frac{p}{\rho} - gx_3 + C \qquad \text{in } \overline{\Omega}, \tag{A.840}$$

where C is a function of t alone. Indeed, applying ∇ to both sides in (A.840) and using (A.837) and (A.838), one obtains $\nabla C = 0$. Then, by changing ϕ by a suitable additive function of t, one can convert C into a constant having, for example, the value

$$C = \frac{p_0}{\rho}. \tag{A.841}$$

Now (A.840) gives the dynamic boundary condition on the free surface

$$g\eta + \frac{\partial\phi}{\partial t} + \frac{|\nabla\phi|^2}{2} = 0 \qquad \text{for } x_3 = \eta(\boldsymbol{x}_s, t), \;\; \boldsymbol{x}_s \in \Gamma. \tag{A.842}$$

Another boundary condition holds on every "physical" surface $\mathcal{S}$ bounding the fluid domain Ω and expressing the kinematic property that there is no transfer of matter across $\mathcal{S}$. Let $s(\boldsymbol{x}_s, x_3, t) = 0$ be the equation of $\mathcal{S}$, then

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \boldsymbol{v}\cdot\nabla s + \frac{\partial s}{\partial t} = 0 \qquad \text{on } \mathcal{S}. \tag{A.843}$$

Under assumption (A.838) this takes the form of

$$\frac{\partial\phi}{\partial n} = -\frac{1}{|\nabla s|}\frac{\partial s}{\partial t} = v_n \qquad \text{on } \mathcal{S}, \tag{A.844}$$

where v_n denotes the normal velocity of $\mathcal{S}$. Thus the kinematic boundary condition (A.844) means that the normal velocity of particles is continuous across a physical boundary.

On the fixed part of $\mathcal{S}$, (A.844) takes the form of

$$\frac{\partial\phi}{\partial n} = 0. \tag{A.845}$$

On the free surface, condition (A.843), written as follows,

$$\frac{\partial\eta}{\partial t} + \nabla_s\phi\cdot\nabla_s\eta - \frac{\partial\phi}{\partial x_3} = 0 \qquad \text{for } x_3 = \eta(\boldsymbol{x}_s, t), \;\; \boldsymbol{x}_s \in \Gamma, \tag{A.846}$$

complements the dynamic condition (A.842). Thus, in the present approach, two non-linear conditions (A.842) and (A.846) on the unknown boundary are responsible for waves, which constitutes the main characteristic feature of water-surface wave theory.

This brief account of governing equations can be summarized as follows. In the water-wave problem one seeks the velocity potential $\phi(\boldsymbol{x}_s, x_3, t)$ and the free surface elevation $\eta(\boldsymbol{x}_s, t)$ satisfying (A.839), (A.842), (A.844), and (A.846). The initial values of ϕ and η should also be prescribed, as well as the conditions at infinity (for unbounded Ω) to complete the problem, which is known as the Cauchy-Poisson problem.

A.10.2 Energy and its flow

Let Ω_0 be a subdomain of Ω, bounded by a "geometric" surface $\partial\Omega_0$ that may not be related to physical obstacles, and that is permitted to vary in time independently of moving water unlike the "physical" surfaces described below. Let $s_0(\boldsymbol{x}_s, x_3, t) = 0$ be the equation of $\partial\Omega_0$. The total energy contained in Ω_0 consists of kinetic and potential components, and is given by

$$E = \rho \int_{\Omega_0} \left(g x_3 + \frac{|\nabla\phi|^2}{2} \right) \mathrm{d}\boldsymbol{x}. \tag{A.847}$$

The first term related to the vertical displacement of a water particle corresponds to the potential energy, whereas the second one gives the kinetic energy that is proportional to the velocity squared. Using (A.840) and (A.841), one can write this in the form of

$$E = -\int_{\Omega_0} \left(\rho \frac{\partial\phi}{\partial t} + p - p_0 \right) \mathrm{d}\boldsymbol{x}. \tag{A.848}$$

Differentiating (A.848) with respect to t we get (John 1949, Lamb 1916)

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \rho \int_{\Omega_0} \nabla\phi \cdot \nabla \frac{\partial\phi}{\partial t} \,\mathrm{d}\boldsymbol{x} + \int_{\partial\Omega_0} \frac{1}{|\nabla s_0|} \frac{\partial s_0}{\partial t} \left(\rho \frac{\partial\phi}{\partial t} + p - p_0 \right) \mathrm{d}\gamma(\boldsymbol{x}). \tag{A.849}$$

Green's first integral theorem (A.612) applied to the first integral of (A.849) leads to

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_{\partial\Omega_0} \left\{ \rho \frac{\partial\phi}{\partial t} \left(\frac{\partial\phi}{\partial n} - v_n \right) - (p - p_0) v_n \right\} \mathrm{d}\gamma(\boldsymbol{x}), \tag{A.850}$$

where (A.839) is taken into account and v_n denotes the normal velocity of $\partial\Omega_0$. Hence the integrand in (A.850) is the rate of energy flow from Ω_0 through $\partial\Omega_0$ taken per units of time and area. The velocity of energy propagation is known as the group velocity. Further details can be found for this topic in Wehausen & Laitone (1960).

If a portion of $\partial\Omega_0$ is a fixed geometric surface, then $v_n = 0$ on this portion. The rate of energy flow is given by $-\rho(\partial\phi/\partial t)(\partial\phi/\partial n)$.

If a portion of $\partial\Omega_0$ is a "physical" boundary that is not penetrable by water particles, then (A.844) shows that the integrand in (A.850) is equal to $(p_0 - p)v_n$. Therefore, there is no energy flow through this portion of $\partial\Omega_0$ if either of two factors vanishes. In particular, this is true for the free surface ($p = p_0$) and for the bottom ($v_n = 0$).

A.10.3 Linearized unsteady problem

The presented problem is quite general, and it is very complicated to find an explicit solution for these equations. The difficulties arising from the fact that ϕ is a solution of the potential equation determined by non-linear boundary conditions on a variable boundary are considerable. A large number of papers has been published and great progress has been achieved in the mathematical treatment of non-linear water-wave problems. However, all rigorous results in this direction are concerned with water waves in the absence of floating bodies, although some numerical results treating different aspects of the non-linear problem have been achieved.

To be in a position to describe water waves in the presence of bodies, the equations should be approximated by more tractable ones. The usual and rather reasonable simplification consists in a linearization of the problem under certain assumptions concerning the motion of a floating body. An example of such assumptions (there are other ones leading to the same conclusions) suggests that a body's motion near the equilibrium position is so small that it produces only waves having a small amplitude and a small wavelength. There are three characteristic geometric parameters: a typical value of the wave height H, a typical wavelength L, and the water depth D. They give three characteristic quotients: H/L, H/D, and L/D. The relative importance of these quotients is different in different situations. Nevertheless, it was found that if

$$\frac{H}{D} \ll 1 \qquad \text{and} \qquad \frac{H}{L}\left(\frac{L}{D}\right)^3 \ll 1, \tag{A.851}$$

then the linearization can be justified by some heuristic considerations. The last parameter $(H/L)(L/D)^3 = (H/D)(L/D^2)$ is usually referred to as Ursell's number.

The linearized theory leads to results that are in a rather good agreement with experiments and observations. Furthermore, there is mathematical evidence that the linearized problem provides an approximation to the non-linear one. For the Cauchy-Poisson problem describing waves in a water layer caused by prescribed initial conditions, the linear approximation is justified rigorously. More precisely, under the assumption that the undisturbed water occupies a layer of constant depth, the following are proved. The non-linear problem is solvable for sufficiently small values of the linearization parameter. As this parameter tends to zero, solutions of the non-linear problem do converge to the solution of the linearized problem in the norm of some suitable function space.

A formal perturbation procedure leading to a sequence o linear problems can be developed as follows. Let us assume that the velocity potential ϕ and the free surface elevation η admit expansions with respect to a certain small parameter ϵ:

$$\phi(\boldsymbol{x}_s, x_3, t) = \epsilon\phi^{(1)}(\boldsymbol{x}_s, x_3, t) + \epsilon^2\phi^{(2)}(\boldsymbol{x}_s, x_3, t) + \epsilon^3\phi^{(3)}(\boldsymbol{x}_s, x_3, t) + \ldots, \tag{A.852}$$

$$\eta(\boldsymbol{x}_s, t) = \eta^{(0)}(\boldsymbol{x}_s, t) + \epsilon\eta^{(1)}(\boldsymbol{x}_s, t) + \epsilon^2\eta^{(2)}(\boldsymbol{x}_s, t) + \ldots, \tag{A.853}$$

where $\phi^{(1)}$, $\phi^{(2)}$, …, $\eta^{(0)}$, $\eta^{(1)}$, …, and all their derivatives are bounded. Consequently, the velocities of water particles are supposed to be small (proportional to ϵ), and $\epsilon = 0$ corresponds to water permanently at rest.

Substituting (A.852) into (A.839) gives

$$\Delta\phi^{(k)} = 0 \qquad \text{in } \Omega, \quad k = 1, 2, \ldots. \tag{A.854}$$

Furthermore, $\eta^{(0)}$ describing the free surface at rest cannot depend on t. When the expansions for ϕ and η are substituted into the Bernoulli boundary condition (A.842) and grouped according to powers of ϵ, one obtains

$$\eta^{(0)} = 0 \qquad \text{for } \boldsymbol{x}_s \in \Gamma. \tag{A.855}$$

This and Taylor's expansion of $\phi(\boldsymbol{x}_s, \eta(\boldsymbol{x}_s, t), t)$ in powers of ϵ yield the following for orders higher than zero:

$$\frac{\partial \phi^{(1)}}{\partial t} + g\eta^{(1)} = 0 \qquad \text{for } x_3 = 0,\ \ \boldsymbol{x}_s \in \Gamma, \tag{A.856}$$

$$\frac{\partial \phi^{(2)}}{\partial t} + g\eta^{(2)} = -\eta^{(1)} \frac{\partial^2 \phi^{(1)}}{\partial t \partial x_3} - \frac{|\nabla \phi^{(1)}|^2}{2} \qquad \text{for } x_3 = 0,\ \ \boldsymbol{x}_s \in \Gamma, \tag{A.857}$$

and so on, i.e., all these conditions hold on the mean position of the free surface at rest.

Similarly, the kinematic condition (A.846) leads to

$$\frac{\partial \phi^{(1)}}{\partial x_3} - \frac{\partial \eta^{(1)}}{\partial t} = 0 \qquad \text{for } x_3 = 0,\ \ \boldsymbol{x}_s \in \Gamma, \tag{A.858}$$

$$\frac{\partial \phi^{(2)}}{\partial x_3} - \frac{\partial \eta^{(2)}}{\partial t} = -\eta^{(1)} \frac{\partial^2 \phi^{(1)}}{\partial x_3^2} + \nabla_s \phi^{(1)} \cdot \nabla \eta^{(1)} \qquad \text{for } x_3 = 0,\ \ \boldsymbol{x}_s \in \Gamma, \tag{A.859}$$

and so on. Eliminating $\eta^{(1)}$ between (A.856) and (A.858), one finds the classical first-order linear free-surface condition

$$\frac{\partial^2 \phi^{(1)}}{\partial t^2} + g\frac{\partial \phi^{(1)}}{\partial x_3} = 0 \qquad \text{for } x_3 = 0,\ \ \boldsymbol{x}_s \in \Gamma. \tag{A.860}$$

In the same way, for $x_3 = 0$ and $\boldsymbol{x}_s \in \Gamma$, one obtains from (A.857) and (A.859) that

$$\frac{\partial^2 \phi^{(2)}}{\partial t^2} + g\frac{\partial \phi^{(2)}}{\partial x_3} = -\frac{\partial \phi^{(1)}}{\partial t} \nabla_s^2 \phi^{(1)} - \frac{1}{g^2}\frac{\partial}{\partial t}\left\{ \frac{\partial \phi^{(1)}}{\partial t}\frac{\partial^3 \phi^{(1)}}{\partial t^3} + |\nabla_s \phi^{(1)}|^2 \right\}. \tag{A.861}$$

Further free-surface conditions can be obtained for terms in (A.852) having higher orders in ϵ. All these conditions have the same operator in the left-hand side, and the right-hand term depends non-linearly on terms of smaller orders. It is worth mentioning that all of the high-order problems are formulated in the same domain Ω occupied by the water at rest. In particular, the free-surface boundary conditions are imposed at $\{x_3 = 0,\ \boldsymbol{x}_s \in \Gamma\}$.

A.10.4 Boundary condition on an immersed rigid surface

First, we note that the homogeneous Neumann condition (A.845) is linear on fixed surfaces. Hence, this condition is true for $\phi^{(k)}$, $k = 1, 2, \ldots$. The situation reverses for the inhomogeneous Neumann condition (A.844) on a moving surface $\mathcal{S}$, which can be subjected, for example, to a prescribed motion or freely floating. The problem of a body freely floating near its equilibrium position will not be treated here, and we restrict ourselves to the linearization of (A.844) for $\mathcal{S} = S(t, \epsilon)$ undergoing a given small amplitude motion near an equilibrium position S, i.e., when $S(t, \epsilon)$ tends to S as $\epsilon \to 0$.

It is convenient to carry out the linearization locally. Let us consider a neighborhood of $(\boldsymbol{x}_s^{(0)}, x_3^{(0)}) \in S$, where the surface is given explicitly in local cartesian coordinates $(\boldsymbol{\xi}_s, \xi_3)$, being in the three-dimensional case $\boldsymbol{\xi}_s = (\xi_1, \xi_2)$, and having an origin at $(\boldsymbol{x}_s^{(0)}, x_3^{(0)})$ and the ξ_3 axis directed into water normally to S. Let $\xi_3 = \zeta^{(0)}(\boldsymbol{\xi}_s)$ be the equation of S, and $S(t, \epsilon)$ be given by $\xi_3 = \zeta(\boldsymbol{\xi}_s, t, \epsilon)$, where

$$\zeta(\boldsymbol{\xi}_s, t, \epsilon) = \zeta^{(0)}(\boldsymbol{\xi}_s) + \epsilon\zeta^{(1)}(\boldsymbol{\xi}_s, t) + \epsilon^2\zeta^{(2)}(\boldsymbol{\xi}_s, t) + \dots. \tag{A.862}$$

After substituting (A.852) and $s = \xi_3 - \zeta(\boldsymbol{\xi}_s, t, \epsilon)$ into (A.843), we use (A.838), (A.862), and Taylor's expansion in the same way as, e.g., in (A.856). This gives the first-order equation

$$\frac{\partial \phi^{(1)}}{\partial \xi_3}(\boldsymbol{\xi}_s, \zeta^{(0)}, t) - \nabla_s \phi^{(1)}(\boldsymbol{\xi}_s, \zeta^{(0)}, t) \cdot \nabla_s \zeta^{(0)}(\boldsymbol{\xi}_s) = \frac{\partial \zeta^{(1)}}{\partial t}(\boldsymbol{\xi}_s, t), \tag{A.863}$$

which implies the linearized boundary condition

$$\frac{\partial \phi^{(1)}}{\partial n} = v_n^{(1)} \qquad \text{on } \mathcal{S}, \tag{A.864}$$

where

$$v_n^{(1)} = \frac{\partial \zeta^{(1)}/\partial t}{\sqrt{(1 + |\nabla_s \zeta^{(0)}|^2)}} \tag{A.865}$$

is the first-order approximation of the normal velocity of $S(t, \epsilon)$.

The second-order boundary condition on S has the form

$$\frac{\partial \phi^{(2)}}{\partial n} = \frac{\partial \zeta^{(2)}/\partial t}{\sqrt{(1 + |\nabla_s \zeta^{(0)}|^2)}} - \zeta^{(1)} \frac{\partial^2 \phi^{(1)}}{\partial n^2} - \sqrt{\frac{1 + |\nabla_s \zeta^{(1)}|^2}{1 + |\nabla_s \zeta^{(0)}|^2}} \frac{\partial \phi^{(1)}}{\partial n^{(1)}}, \tag{A.866}$$

where $\partial \phi^{(1)}/\partial n^{(1)}$ is the derivative in the direction of normal $\xi_3 = \zeta^{(1)}(\boldsymbol{\xi}_s, t)$ calculated on S. In addition, further conditions on S of the Neumann type can be obtained for terms of higher order in ϵ.

Thus, all $\phi^{(k)}$ satisfy the same linear boundary-value problem with different right-hand side terms in conditions on the free surface at rest and on the equilibrium surfaces of immersed bodies. These right-hand side terms depend on solutions obtained on previous steps. Solving these problems successively, beginning with problems (A.854), (A.860), and (A.864) complemented by some initial conditions, one can, generally speaking, find a solution to the non-linear problem in the form of (A.852) and (A.853). However, this procedure is not justified mathematically up to the present time. Therefore, we restrict ourselves to the first-order approximation, which on its own right gives rise to an extensive mathematical theory.

We summarize now the boundary-value problem for the first-order velocity potential $\phi^{(1)}(\boldsymbol{x}_s, x_3, t)$. It is defined in Ω occupied by water at rest with a boundary consisting of the free surface Γ, the bottom B, and the wetted surface of immersed bodies S, and it must satisfy

$$\Delta \phi^{(1)} = 0 \qquad \text{in } \Omega, \tag{A.867}$$

$$\frac{\partial^2 \phi^{(1)}}{\partial t^2} + g \frac{\partial \phi^{(1)}}{\partial x_3} = 0 \qquad \text{for } x_3 = 0, \ \boldsymbol{x}_s \in \Gamma, \tag{A.868}$$

$$\frac{\partial \phi^{(1)}}{\partial n} = v_n^{(1)} \qquad \text{on } S, \tag{A.869}$$

$$\frac{\partial \phi^{(1)}}{\partial n} = 0 \qquad \text{on } B, \tag{A.870}$$

$$\phi^{(1)}(\boldsymbol{x}_s, 0, 0) = \phi_0(\boldsymbol{x}_s) \quad \text{and} \quad \frac{\partial \phi^{(1)}}{\partial t}(\boldsymbol{x}_s, 0, 0) = -g\eta_0(\boldsymbol{x}_s), \tag{A.871}$$

where ϕ_0, $v_n^{(1)}$, and η_0 are given functions, and $\eta_0(\boldsymbol{x}_s) = \eta^{(1)}(\boldsymbol{x}_s, 0)$ (see (A.856)). Then

$$\eta^{(1)}(\boldsymbol{x}_s, t) = -\frac{1}{g}\frac{\partial \phi^{(1)}}{\partial t}(\boldsymbol{x}_s, 0, t) \tag{A.872}$$

gives the first-order approximation for the elevation of the free surface.

A.10.5 Linear time-harmonic waves

We are interested in the study of the steady-state problem of radiation and scattering of water waves by bodies floating in and/or beneath the free surface, assuming all motions to be simple harmonic in the time. The corresponding radian frequency is denoted by ω. Thus, the right-hand side term in (A.864) is

$$v_n^{(1)} = \mathfrak{Re}\{e^{-i\omega t} f\} \qquad \text{on } S, \tag{A.873}$$

where f is a complex function independent of t, and the first-order velocity potential $\phi^{(1)}$ can then be written in the form

$$\phi^{(1)}(\boldsymbol{x}_s, x_3, t) = \mathfrak{Re}\{e^{-i\omega t} u(\boldsymbol{x}_s, x_3)\}. \tag{A.874}$$

The latter assumption is justified by the so-called limiting amplitude principle, which is concerned with large-time behavior of a solution to the initial-boundary-value problem having (A.873) as the right-hand side term. According to this principle, such a solution tends to the potential (A.874) as $t \to \infty$, and u satisfies a steady-state problem. The limiting amplitude principle has general applicability in the theory of wave motions. Thus the problem of our interest describes waves developing at large time from time-periodic disturbances.

A complex function u in (A.874) is also referred to as velocity potential (but in this case with respect to time-harmonic dependence). We recall that u is defined in the fixed domain Ω occupied by water at rest outside any bodies present. The boundary $\partial\Omega$ consists of three disjoint sets: (i) S, which is the union of the wetted surfaces of bodies in equilibrium; (ii) Γ, denoting the free surface at rest that is the part of $x_3 = 0$ outside all the bodies; and (iii) B, which denotes the bottom positioned below $\Gamma \cup S$. Sometimes Ω is considered unbounded below and corresponding to infinitely deep water. This is the case in this thesis and it involves that $\partial\Omega = \Gamma \cup S$.

Substituting thus (A.873) and (A.874) into (A.867)–(A.870) gives the boundary-value problem for u:

$$\Delta u = 0 \qquad \text{in } \Omega, \tag{A.875}$$

$$\frac{\partial u}{\partial x_3} - \nu u = 0 \qquad \text{on } \Gamma, \tag{A.876}$$

$$\frac{\partial u}{\partial n} = f \qquad \text{on } S, \tag{A.877}$$

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on } B, \tag{A.878}$$

where $\nu = \omega^2/g$. We suppose that the normal $\boldsymbol{n}$ to a surface is always directed outwards of the water domain Ω.

For deep water ($B = \emptyset$), condition (A.878) should be replaced by something like

$$\sup_{(\boldsymbol{x}_s, x_3)\in\Omega} |u(\boldsymbol{x}_s, x_3)| < \infty. \tag{A.879}$$

This condition has no direct hydrodynamic meaning, apart from stating that the solution has to remain bounded in Ω. It implies the natural asymptotic behavior for the velocity field given by

$$|\nabla u| \longrightarrow 0 \qquad \text{as} \qquad x_3 \longrightarrow -\infty, \tag{A.880}$$

that is, the water motion decays with depth. Conditions at infinity that are similar to the last two conditions are usually imposed in the boundary-value problems for the Laplacian in domains exterior to a compact set in $\mathbb{R}^2$ and $\mathbb{R}^3$. A natural requirement that a solution to (A.875)–(A.879) should be unique also imposes a certain restriction on the behavior of u as $|\boldsymbol{x}_s| \to \infty$. We will return again to this topic below.

Let us consider now some simple examples of waves existing in the absence of bodies. The corresponding potentials can be easily obtained by separation of variables.

For a layer Ω of constant depth d, we consider the free surface $\Gamma = \{\boldsymbol{x}_s \in \mathbb{R}^2, x_3 = 0\}$ and the bottom $B = \{\boldsymbol{x}_s \in \mathbb{R}^2, x_3 = -d\}$. A plane progressive wave propagating in the direction of a wave vector $\boldsymbol{k}_s = (k_1, k_2)$ has the velocity potential

$$\mathfrak{Re}\big\{A \exp(i\boldsymbol{k}_s \cdot \boldsymbol{x}_s - i\omega t)\big\} \cosh\{k_s(x_3 + d)\}. \tag{A.881}$$

Here A is an arbitrary complex constant, $k_s = |\boldsymbol{k}_s|$, and the following relationship,

$$\nu = \frac{\omega^2}{g} = k_s \tanh(k_s d), \tag{A.882}$$

holds between ω and k_s. Tending d to infinity, we note that k_s becomes equal to ν and, instead of (A.881), we have

$$\mathfrak{Re}\big\{A \exp(i\boldsymbol{k}_s \cdot \boldsymbol{x}_s - i\omega t)\big\}\, e^{\nu x_3} \tag{A.883}$$

for the velocity potential of a plane progressive wave in deep water.

A sum of two potentials (A.881) corresponding to identical progressive waves propagating in opposite directions gives a standing wave. Putting the term $\exp(\nu x_3)$ instead of $\cosh\{k_s(x_3 + d)\}$ in (A.881) and omitting $\tanh(k_s d)$ in (A.882), one gets the potential of a progressive wave in deep water.

A standing cylindrical wave in a water layer of depth d has the potential

$$w_{\text{st}}(\boldsymbol{x}_s, x_3)\cos(\omega t), \tag{A.884}$$

where

$$w_{\mathrm{st}}(\boldsymbol{x}_s, x_3) = C_1 \cosh\{k_s(x_3 + d)\} J_0(k_s|\boldsymbol{x}_s|), \tag{A.885}$$

and where k_s is defined by (A.882), C_1 is a real constant, and J_0 denotes the Bessel function of order zero (vid. Subsection A.2.4). The same manipulation as above gives the standing wave in deep water.

A cylindrical wave having an arbitrary phase at infinity may be obtained as a combination of w_{st} and a similar potential with J_0 replaced by Y_0, the Neumann function of order zero. This allows one to construct a potential of outgoing waves as

$$\mathfrak{Re}\big\{e^{-i\omega t} w_{\mathrm{out}}(\boldsymbol{x}_s, x_3)\big\}, \tag{A.886}$$

where

$$w_{\mathrm{out}}(\boldsymbol{x}_s, x_3) = C_2 \cosh\{k_s(x_3 + d)\} H_0^{(1)}(k_s|\boldsymbol{x}_s|), \tag{A.887}$$

and where k_s is defined by (A.882), C_2 is a complex constant, and $H_0^{(1)}$ denotes the zeroth order Hankel function of the first kind. The outgoing behavior of this wave becomes clear from the asymptotic formula

$$H_0^{(1)}(k_s|\boldsymbol{x}_s|) = \sqrt{\frac{2}{\pi k_s |\boldsymbol{x}_s|}}\, e^{i(k_s|\boldsymbol{x}_s| - \pi/4)}\big(1 + \mathcal{O}(|\boldsymbol{x}_s|^{-1})\big) \qquad \text{as } \ |\boldsymbol{x}_s| \to \infty, \tag{A.888}$$

where $\mathcal{O}(\cdot)$ denotes the highest order of the remaining terms at infinity. Therefore, the wave w_{out} behaves at large distances like a radially outgoing progressive wave, but it is singular on the axis $|\boldsymbol{x}_s| = 0$. This is natural from a physical point of view, because outgoing waves should be radiated by a certain disturbance. In the case under consideration, the wave is produced by sources distributed with a suitable density over $\{|\boldsymbol{x}_s| = 0,\ -d < x_3 < 0\}$. Replacing $H_0^{(1)}$ in (A.887) by the zeroth order Hankel function of the second kind, $H_0^{(2)}$, one obtains an incoming wave.

A.10.6 Radiation conditions

The examples of waves existing in the absence of bodies, e.g., plane progressive and cylindrical waves, demonstrate that problem (A.875)–(A.878) should be complemented by an appropriate condition as $|\boldsymbol{x}_s| \to \infty$ to avoid non-uniqueness of the solution, which follows from the fact that there are infinitely many solutions in the form of (A.881). On the other hand, the energy dissipates when waves are radiated or scattered, i.e., there exists a flow of energy to infinity. On the contrary, there is no such flow for standing waves and no net flow for progressive waves. Since we are interested in describing radiation and scattering phenomena, a condition should be introduced to eliminate waves having no flow of energy to infinity. For this purpose a mathematical expression is used that is known as a radiation condition. To formulate this condition, we have to specify the geometry of the water domain at infinity.

Let Ω be an N-dimensional domain ($N = 2, 3$), which at infinity coincides with the layer $\{\boldsymbol{x}_s \in \mathbb{R}^{N-1},\ -d < x_3 < 0\}$, where $0 < d \leq \infty$. We say that u satisfies the radiation condition of the Sommerfeld type if

$$\frac{\partial u}{\partial |\boldsymbol{x}_s|} - ik_s u = \sigma(x_3)\mathcal{O}(|\boldsymbol{x}_s|^{(2-N)/2}) \qquad \text{as } \ |\boldsymbol{x}_s| \to \infty, \ \text{ uniformly in } x_3, \theta. \tag{A.889}$$

Here $\sigma(x_3) = (1 + |x_3|)^{-N+1}$ if $d = \infty$, $\sigma(x_3) = 1$ if $d < \infty$, k_s is defined by (A.882) for $d < \infty$, and $k_s = \nu$ for $d = \infty$, and $\theta \in [0, 2\pi)$ is the polar angle in the plane $\{x_3 = 0\}$. Uniformity in θ should be imposed only for the three-dimensional problem ($N = 3$).

Let us show that (A.889) guarantees dissipation of energy. For the sake of simplicity we assume that $d < \infty$. By $\mathcal{C}_r$ we denote a cylindrical surface $\Omega \cap \{|\boldsymbol{x}_s| = r\}$ contained inside Ω. By (A.850) the average energy flow to infinity through $\mathcal{C}_r$ over one period of oscillations is equal to

$$F_r = -\frac{\rho\omega}{2\pi} \int_0^{2\pi/\omega} \int_{\mathcal{C}_r} \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial |\boldsymbol{x}_s|} \,\mathrm{d}\gamma \,\mathrm{d}t. \tag{A.890}$$

Substituting (A.874) and taking into account that

$$\int_0^{2\pi/\omega} e^{\pm 2i\omega t} \,\mathrm{d}t = 0, \tag{A.891}$$

one finds that

$$\begin{aligned} F_r &= -\frac{\rho\omega^2}{8\pi} \int_0^{2\pi/\omega} \int_{\mathcal{C}_r} (ie^{i\omega t}\bar{u} - ie^{-i\omega t}u) \left(e^{-i\omega t} \frac{\partial u}{\partial |\boldsymbol{x}_s|} + e^{i\omega t} \frac{\partial \bar{u}}{\partial |\boldsymbol{x}_s|} \right) \mathrm{d}\gamma \,\mathrm{d}t \\ &= -\frac{\rho\omega}{4\pi} \int_{\mathcal{C}_r} \left(i\bar{u} \frac{\partial u}{\partial |\boldsymbol{x}_s|} - iu \frac{\partial \bar{u}}{\partial |\boldsymbol{x}_s|} \right) \mathrm{d}\gamma = \frac{\rho\omega}{2} \mathfrak{Im} \int_{\mathcal{C}_r} \bar{u} \frac{\partial u}{\partial |\boldsymbol{x}_s|} \,\mathrm{d}\gamma. \end{aligned} \tag{A.892}$$

This can be written as

$$F_r = \frac{\rho\omega}{4k_s} \left\{ \int_{\mathcal{C}_r} \left(\left| \frac{\partial u}{\partial |\boldsymbol{x}_s|} \right|^2 + k_s^2 |u|^2 \right) \mathrm{d}\gamma - \int_{\mathcal{C}_r} \left| \frac{\partial u}{\partial |\boldsymbol{x}_s|} - ik_s u \right|^2 \mathrm{d}\gamma \right\}. \tag{A.893}$$

Moreover, F_r does not depend on r when the obstacle surface S lies inside the cylinder $\{|\boldsymbol{x}_s| = r\}$, which can be proved as follows.

By Ω_r and Γ_r we denote $\Omega \cap \{|\boldsymbol{x}_s| < r\}$ and $\Gamma \cap \{|\boldsymbol{x}_s| < r\}$, respectively. Let us multiply (A.875) by $\bar{u}$ and integrate the result over Ω_r. By applying then Green's theorem we obtain

$$\int_{\Omega_r} |\nabla u|^2 \,\mathrm{d}\boldsymbol{x}_s \,\mathrm{d}x_3 = \int_{\partial\Omega_r} \bar{u} \frac{\partial u}{\partial n} \,\mathrm{d}\gamma, \tag{A.894}$$

where the normal $\boldsymbol{n}$ is directed outside of Ω_r. Using (A.876) and (A.878) we get

$$\int_{\Omega_r} |\nabla u|^2 \,\mathrm{d}\boldsymbol{x}_s \,\mathrm{d}x_3 = \nu \int_{\Gamma_r} |u|^2 \,\mathrm{d}\boldsymbol{x}_s + \int_{\mathcal{C}_r} \bar{u} \frac{\partial u}{\partial |\boldsymbol{x}_s|} \,\mathrm{d}\gamma - \int_S \bar{u} \frac{\partial u}{\partial n} \,\mathrm{d}\gamma. \tag{A.895}$$

Comparing this with (A.892) we find that

$$F_r = \frac{\rho\omega}{2} \mathfrak{Im} \int_S \bar{u} \frac{\partial u}{\partial |\boldsymbol{x}_s|} \,\mathrm{d}\gamma \tag{A.896}$$

is independent of r.

This fact yields that $F_r \geq 0$ because (A.889) implies that the last integral in (A.893) tends to zero as $r \to \infty$.

The crucial point in the proof that $F_r \geq 0$ is the equality (A.893), which suggests that (A.889) can be replaced by a weaker radiation condition of the Rellich type

$$\int_{\mathcal{C}_r} \left| \frac{\partial u}{\partial |\boldsymbol{x}_s|} - i k_s u \right|^2 \mathrm{d}\gamma = \mathcal{O}(1) \qquad \text{as } \ r \to \infty. \tag{A.897}$$

Actually, (A.889) and (A.897) are equivalent.

So, problem (A.875)–(A.878) has to be complemented by either (A.889) or (A.897). In various papers this problem appears under different names: the floating body problem, the sea-keeping problem, the wave-body interaction problem, the water-wave radiation (scattering) problem, and so on.