## Vector calculus and elementary differential geometry

Vector calculus, also known as vector analysis, is a field in mathematics that is concerned with multi-variable real or complex analysis of vectors. Vector calculus is concerned with scalar fields, which associate a scalar to every point in space, and vector fields, which associate a vector to every point in space. Differential geometry is a mathematical discipline that uses the methods of differential and integral calculus to study problems in geometry. It has grown into a field that is concerned more generally with geometric structures on differentiable manifolds, being closely related to differential topology and with the geometric aspects of the theory of differential equations.

Our goal is not to give a complete survey, but rather to define roughly operators that arise in these disciplines and use them to state some important integral theorems, which are used throughout this thesis. The main references for our approach on these subjects are Lenoir (2005), Nédélec (2001), and Terrasse & Abboud (2006).

### A.5.1 Differential operators on scalar and vector fields

We are herein interested in defining differential operators that act on complex scalar and vector fields in $\mathbb{R}^N$, in particular for $N = 2$ or 3. We define the scalar, inner, or dot product of two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^N$ by the scalar quantity

$$\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^{N} a_i \overline{b}_i, \tag{A.556}$$

where $\overline{z}$ stands for the complex conjugate of $z \in \mathbb{C}$. Some properties of the dot product, for $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{C}^N$, are

$$\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2, \tag{A.557}$$

$$\boldsymbol{a} \cdot \boldsymbol{b} = \overline{\boldsymbol{b} \cdot \boldsymbol{a}}, \tag{A.558}$$

$$\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}. \tag{A.559}$$

The vector or cross product of two vectors, on the other hand, is particular to three-dimensional space ($N = 3$). It is defined, for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^3$, by the vector

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{\boldsymbol{x}}_1 & \hat{\boldsymbol{x}}_2 & \hat{\boldsymbol{x}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2)\hat{\boldsymbol{x}}_1 + (a_3 b_1 - a_1 b_3)\hat{\boldsymbol{x}}_2 + (a_1 b_2 - a_2 b_1)\hat{\boldsymbol{x}}_3, \tag{A.560}$$

where $\hat{\boldsymbol{x}}_1$, $\hat{\boldsymbol{x}}_2$, and $\hat{\boldsymbol{x}}_3$ are the canonical cartesian unit vectors in $\mathbb{R}^3$. We can define also a cross product in two dimensions ($N = 2$), which yields for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^2$ the scalar value

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1. \tag{A.561}$$

The cross product satisfies, for $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{C}^N$ and $\alpha \in \mathbb{C}$, the identities

$$\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0}, \tag{A.562}$$

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}, \tag{A.563}$$

$$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}, \tag{A.564}$$

$$(\alpha \boldsymbol{a}) \times \boldsymbol{b} = \boldsymbol{a} \times (\alpha \boldsymbol{b}) = \alpha(\boldsymbol{a} \times \boldsymbol{b}). \tag{A.565}$$

In particular when $N = 3$, the dot and cross products satisfy, for $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \mathbb{C}^3$,

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}), \tag{A.566}$$

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}, \tag{A.567}$$

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c}). \tag{A.568}$$

For $N = 2$ and $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \mathbb{C}^2$, it holds that

$$\boldsymbol{a}(\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \times \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \times \boldsymbol{b}), \tag{A.569}$$

$$(\boldsymbol{a} \times \boldsymbol{b})(\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c}). \tag{A.570}$$

Another vector operation is given by the dyadic, tensor, or outer product of two vectors, which results in a matrix and is defined, for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^N$, by

$$\boldsymbol{a} \otimes \boldsymbol{b} = \boldsymbol{a}\,\boldsymbol{b}^* = \boldsymbol{a}\,\overline{\boldsymbol{b}}^T, \tag{A.571}$$

where $\boldsymbol{b}^*$ stands for the conjugated transpose of $\boldsymbol{b}$, being $\boldsymbol{b}^T$ the transposed vector. In three dimensions ($N = 3$) it is given by

$$\boldsymbol{a} \otimes \boldsymbol{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} \overline{b}_1 & \overline{b}_2 & \overline{b}_3 \end{bmatrix} = \begin{bmatrix} a_1\overline{b}_1 & a_1\overline{b}_2 & a_1\overline{b}_3 \\ a_2\overline{b}_1 & a_2\overline{b}_2 & a_2\overline{b}_3 \\ a_3\overline{b}_1 & a_3\overline{b}_2 & a_3\overline{b}_3 \end{bmatrix}, \tag{A.572}$$

whereas in two dimensions ($N = 2$) it takes the form of

$$\boldsymbol{a} \otimes \boldsymbol{b} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \begin{bmatrix} \overline{b}_1 & \overline{b}_2 \end{bmatrix} = \begin{bmatrix} a_1\overline{b}_1 & a_1\overline{b}_2 \\ a_2\overline{b}_1 & a_2\overline{b}_2 \end{bmatrix}. \tag{A.573}$$

The dyadic product satisfies, for $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{C}^N$ and $\alpha \in \mathbb{C}$, the properties

$$(\alpha \boldsymbol{a}) \otimes \boldsymbol{b} = \boldsymbol{a} \otimes (\alpha \boldsymbol{b}) = \alpha(\boldsymbol{a} \otimes \boldsymbol{b}), \tag{A.574}$$

$$\boldsymbol{a} \otimes (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{a} \otimes \boldsymbol{c}, \tag{A.575}$$

$$(\boldsymbol{a} + \boldsymbol{b}) \otimes \boldsymbol{c} = \boldsymbol{a} \otimes \boldsymbol{c} + \boldsymbol{b} \otimes \boldsymbol{c}. \tag{A.576}$$

It is interesting to observe that the $N \times N$ identity matrix $\boldsymbol{I}$ can be expressed as

$$\boldsymbol{I} = \sum_{i=1}^{N} \hat{\boldsymbol{x}}_i \otimes \hat{\boldsymbol{x}}_i, \tag{A.577}$$

being $\hat{\boldsymbol{x}}_i$, for $1 \leq i \leq N$, the canonical vectors in $\mathbb{R}^N$.

We define the gradient of a scalar field $f : \mathbb{R}^N \to \mathbb{C}$ as the vector field whose components are the partial derivatives of $f$, i.e.,

$$\operatorname{grad} f = \nabla f = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_N} \right). \tag{A.578}$$

The divergence of a vector field $\boldsymbol{v} : \mathbb{R}^N \to \mathbb{C}^N$ is defined as the scalar field

$$\operatorname{div} \boldsymbol{v} = \nabla \cdot \boldsymbol{v} = \sum_{i=1}^{N} \frac{\partial v_i}{\partial x_i}. \tag{A.579}$$

The common notation $\nabla \cdot \boldsymbol{v}$ for the divergence is a convenient mnemonic, although it constitutes a slight abuse of notation and therefore we rather denote it by $\operatorname{div} \boldsymbol{v}$.

The curl or rotor of a vector field has no general formula that holds for all dimensions. It is particular to three-dimensional space, although generalizations to other dimensions have been performed by using exterior or wedge products. In three dimensions and in cartesian coordinates, the curl of a vector field $\boldsymbol{v} : \mathbb{R}^3 \to \mathbb{C}^3$ is defined as the vector field

$$\operatorname{curl} \boldsymbol{v} = \nabla \times \boldsymbol{v} = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right)\hat{\boldsymbol{x}}_1 + \left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right)\hat{\boldsymbol{x}}_2 + \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)\hat{\boldsymbol{x}}_3. \tag{A.580}$$

The curl can be also rewritten as a determinant or a matrix operation, namely

$$\operatorname{curl} \boldsymbol{v} = \begin{vmatrix} \hat{\boldsymbol{x}}_1 & \hat{\boldsymbol{x}}_2 & \hat{\boldsymbol{x}}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{bmatrix} 0 & -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & -\frac{\partial}{\partial x_1} \\ -\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{bmatrix} \boldsymbol{v}. \tag{A.581}$$

In two dimensions we can define two different curls, a scalar and a vectorial one, which are respectively given, for $\boldsymbol{v} : \mathbb{R}^2 \to \mathbb{C}^2$ and $f : \mathbb{R}^2 \to \mathbb{C}$, by

$$\operatorname{curl} \boldsymbol{v} = \nabla \times \boldsymbol{v} = \begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} \\ v_1 & v_2 \end{vmatrix} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}, \tag{A.582}$$

$$\operatorname{Curl} f = \begin{vmatrix} \hat{\boldsymbol{x}}_1 & \hat{\boldsymbol{x}}_2 \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{vmatrix} = \frac{\partial f}{\partial x_2}\hat{\boldsymbol{x}}_1 - \frac{\partial f}{\partial x_1}\hat{\boldsymbol{x}}_2. \tag{A.583}$$

The Laplace operator for a scalar field $f : \mathbb{R}^N \to \mathbb{C}$ is defined by

$$\Delta f = \sum_{i=1}^{N} \frac{\partial^2 f}{\partial x_i^2}, \tag{A.584}$$

whereas the Laplace operator for a vectorial field $\boldsymbol{v} : \mathbb{R}^N \to \mathbb{C}^N$ is given by

$$\Delta \boldsymbol{v} = \sum_{i=1}^{N} \frac{\partial^2 \boldsymbol{v}}{\partial x_i^2}. \tag{A.585}$$

The double-gradient or Hessian matrix of a scalar field $f : \mathbb{R}^N \to \mathbb{C}$ is the square matrix of its second-order partial derivatives, which is defined by

$$\nabla\nabla f = \boldsymbol{H} f = \nabla \otimes \nabla f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_N \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_N^2} \end{bmatrix}. \tag{A.586}$$

The following vector identities hold for $\boldsymbol{v} : \mathbb{R}^N \to \mathbb{C}^N$ and $f, g : \mathbb{R}^N \to \mathbb{C}$:

$$\nabla(fg) = f\nabla g + g\nabla f, \tag{A.587}$$

$$\operatorname{div}(f\boldsymbol{v}) = f \operatorname{div} \boldsymbol{v} + \nabla f \cdot \boldsymbol{v}, \tag{A.588}$$

$$\operatorname{curl}(f\boldsymbol{v}) = f \operatorname{curl} \boldsymbol{v} + \nabla f \times \boldsymbol{v}, \tag{A.589}$$

In three dimensions, for $\boldsymbol{v}, \boldsymbol{u} : \mathbb{R}^3 \to \mathbb{C}^3$ and $f : \mathbb{R}^3 \to \mathbb{C}$, we have in particular that

$$\Delta \boldsymbol{v} = \nabla \operatorname{div} \boldsymbol{v} - \operatorname{curl} \operatorname{curl} \boldsymbol{v}, \tag{A.590}$$

$$\Delta f = \operatorname{div} \nabla f, \tag{A.591}$$

$$\operatorname{div}(\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot \operatorname{curl} \boldsymbol{u} - \boldsymbol{u} \cdot \operatorname{curl} \boldsymbol{v}, \tag{A.592}$$

$$\operatorname{curl}(\boldsymbol{u} \times \boldsymbol{v}) = (\boldsymbol{v} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{v} - \boldsymbol{v} \operatorname{div} \boldsymbol{u} + \boldsymbol{u} \operatorname{div} \boldsymbol{v}, \tag{A.593}$$

$$\nabla(\boldsymbol{u} \cdot \boldsymbol{v}) = (\boldsymbol{v} \cdot \nabla)\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{v} + \boldsymbol{v} \times \operatorname{curl} \boldsymbol{u} + \boldsymbol{u} \times \operatorname{curl} \boldsymbol{v}, \tag{A.594}$$

$$\operatorname{div} \operatorname{curl} \boldsymbol{v} = 0, \tag{A.595}$$

$$\operatorname{curl} \nabla f = \boldsymbol{0}, \tag{A.596}$$

whereas in two dimensions, for $\boldsymbol{v}, \boldsymbol{u} : \mathbb{R}^2 \to \mathbb{C}^2$ and $f, g : \mathbb{R}^2 \to \mathbb{C}$, it holds that

$$\Delta \boldsymbol{v} = \nabla \operatorname{div} \boldsymbol{v} - \operatorname{Curl} \operatorname{curl} \boldsymbol{v}, \tag{A.597}$$

$$\Delta f = \operatorname{div} \nabla f = - \operatorname{curl} \operatorname{Curl} f, \tag{A.598}$$

$$\operatorname{Curl}(fg) = f \operatorname{Curl} g + g \operatorname{Curl} f, \tag{A.599}$$

$$\operatorname{Curl}(\boldsymbol{u} \cdot \boldsymbol{v}) = \boldsymbol{u}^\perp \operatorname{div} \boldsymbol{v} + \boldsymbol{v}^\perp \operatorname{div} \boldsymbol{u} + (\boldsymbol{v} \times \nabla)\boldsymbol{u} + (\boldsymbol{u} \times \nabla)\boldsymbol{v}, \tag{A.600}$$

$$\operatorname{Curl}(\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{u} \operatorname{div} \boldsymbol{v} - \boldsymbol{v} \operatorname{div} \boldsymbol{u} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{u} - \boldsymbol{u} \cdot \nabla)\boldsymbol{v}, \tag{A.601}$$

$$\nabla(\boldsymbol{u} \cdot \boldsymbol{v}) = \boldsymbol{u} \operatorname{div} \boldsymbol{v} + \boldsymbol{v} \operatorname{div} \boldsymbol{u} - (\boldsymbol{v} \times \nabla)\boldsymbol{u}^\perp - (\boldsymbol{u} \times \nabla)\boldsymbol{v}^\perp, \tag{A.602}$$

$$\nabla(\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{u} \operatorname{curl} \boldsymbol{v} - \boldsymbol{v} \operatorname{curl} \boldsymbol{u} - (\boldsymbol{v} \times \nabla)\boldsymbol{u} + (\boldsymbol{u} \times \nabla)\boldsymbol{v}, \tag{A.603}$$

$$\operatorname{Curl} f \times \boldsymbol{v} = \nabla f \cdot \boldsymbol{v}, \tag{A.604}$$

$$\operatorname{Curl} f = \nabla f^\perp, \tag{A.605}$$

$$\operatorname{div} \operatorname{Curl} f = 0, \tag{A.606}$$

$$\operatorname{Curl} \operatorname{div} \boldsymbol{v} = \boldsymbol{0}, \tag{A.607}$$

$$\operatorname{curl} \nabla f = 0, \tag{A.608}$$

$$\nabla \operatorname{curl} \boldsymbol{v} = \boldsymbol{0}, \tag{A.609}$$

where $\boldsymbol{v}^\perp = (v_2, -v_1)$ denotes the orthogonal vector to $\boldsymbol{v}$, which fulfills $\boldsymbol{v} \cdot \boldsymbol{v}^\perp = 0$.

### A.5.2 Green's integral theorems

The Green's integral theorems constitute a generalization of the known integration-by-parts formula of integral calculus to functions with several variables. As is the case with the Green's function, these theorems are also named after the British mathematician and physicist George Green (1793–1841). They play a crucial role in the development of integral representations and equations for harmonic and scattering problems.

As shown in Figure A.16, we consider an open and bounded domain $\Omega \subset \mathbb{R}^N$, that has a regular (strong Lipschitz) boundary $\Gamma = \partial\Omega$, and where the unit surface normal $\boldsymbol{n}$ points outwards of $\Omega$.

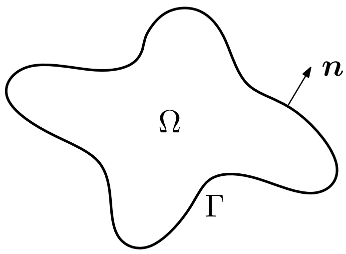


FIGURE A.16. Domain $\Omega$ for the Green's integral theorems.

The Gauss-Green theorem states that if $u \in H^1(\Omega)$, then

$$\int_\Omega \frac{\partial u}{\partial x_i}\,\mathrm{d}\boldsymbol{x} = \int_\Gamma u\, n_i\,\mathrm{d}\gamma \qquad (i = 1, \ldots, N), \tag{A.610}$$

which is directly related to the divergence theorem for a vector field (stated below).

The integration-by-parts formula in several variables is given, for $u, v \in H^1(\Omega)$, by

$$\int_\Omega \frac{\partial u}{\partial x_i} v\,\mathrm{d}\boldsymbol{x} = -\int_\Omega u \frac{\partial v}{\partial x_i}\,\mathrm{d}\boldsymbol{x} + \int_\Gamma u\, v\, n_i\,\mathrm{d}\gamma \qquad (i = 1, \ldots, N), \tag{A.611}$$

which is obtained by applying the Gauss-Green theorem (A.610) to $u\,v$.

Green's first integral theorem states, for $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, that

$$\int_\Omega \Delta u\, v\,\mathrm{d}\boldsymbol{x} = -\int_\Omega \nabla u \cdot \nabla v\,\mathrm{d}\boldsymbol{x} + \int_\Gamma \frac{\partial u}{\partial n} v\,\mathrm{d}\gamma, \tag{A.612}$$

obtained by employing (A.611) with $v = \partial u/\partial x_i$. The theorem still remains valid for somewhat less regular functions $u, v$ such that $u, v \in H^1(\Omega)$ and $\Delta u \in L^2(\Omega)$, that is, when $u \in H^1(\Delta; \Omega)$. In this case the integral on $\Gamma$ in (A.612) has to be understood in general in the sense of the duality product between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$.

Similarly, by combining adequately $u$ and $v$ in (A.612) we obtain Green's second integral theorem, given, for $u, v \in H^2(\Omega)$, by

$$\int_\Omega (u\Delta v - v\Delta u)\,\mathrm{d}\boldsymbol{x} = \int_\Gamma \left( u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) \mathrm{d}\gamma, \tag{A.613}$$

which holds also for $u, v \in H^1(\Omega)$ such that $\Delta u, \Delta v \in L^2(\Omega)$, i.e., for $u, v \in H^1(\Delta; \Omega)$. Again, in the latter case we have to consider in general the integrals on $\Gamma$ in the sense of the duality product between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$.

### A.5.3 Divergence integral theorem

The divergence theorem, also known as Gauss's theorem, is related to the divergence of a vector field. It states that if $\Omega \subset \mathbb{R}^N$ is an open and bounded domain with a regular (strong

Lipschitz) boundary $\Gamma$ and with a unit surface normal $\boldsymbol{n}$ pointing outwards of $\Omega$ as shown in Figure A.16, then we have for all $u \in H^1(\Omega)$ and $\boldsymbol{v} \in H^1(\Omega)^N$ that

$$\int_\Omega \operatorname{div}(u\,\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} = \int_\Omega (\nabla u \cdot \boldsymbol{v} + u \operatorname{div} \boldsymbol{v})\,\mathrm{d}\boldsymbol{x} = \int_\Gamma u\,\boldsymbol{v} \cdot \boldsymbol{n}\,\mathrm{d}\gamma. \tag{A.614}$$

By considering $u = 1$ we obtain the following simpler version of the divergence theorem:

$$\int_\Omega \operatorname{div} \boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = \int_\Gamma \boldsymbol{v} \cdot \boldsymbol{n}\,\mathrm{d}\gamma. \tag{A.615}$$

The divergence theorem can be proven from the integration-by-parts formula (A.611). In three-dimensional space, in particular, the divergence theorem relates a volume integral over $\Omega$ (on the left-hand side) with a surface integral on $\Gamma$ (on the right-hand side). More adjusted functional spaces for the divergence theorem that still allow to define traces on the boundary can be found in the book of Nédélec (2001).

### A.5.4 Curl integral theorem

The curl theorem, also known as Stokes' theorem after the Irish mathematician and physicist Sir George Gabriel Stokes (1819–1903), is related with the curl of a vector field and holds in three-dimensional space. There are, though, adaptations for other dimensions.

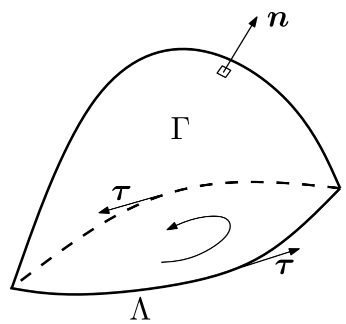


FIGURE A.17. Surface $\Gamma$ for Stokes' integral theorem.

This integral theorem considers an oriented smooth surface $\Gamma \subset \mathbb{R}^3$ that is bounded by a simple, closed, and smooth boundary curve $\Lambda = \partial\Gamma$. The curve $\Lambda$ has thus a positive orientation, i.e., it is described counterclockwise according to the direction of the unit tangent $\boldsymbol{\tau}$ when the unit normal $\boldsymbol{n}$ of the surface $\Gamma$ points towards the viewer, as shown in Figure A.17, following the right-hand rule. The curl theorem states then for $u \in H^1(\Gamma)$ and $\boldsymbol{v} \in H^1(\Gamma)^3$ that

$$\int_\Gamma (\nabla u \times \boldsymbol{v} + u \operatorname{curl} \boldsymbol{v}) \cdot \boldsymbol{n}\,\mathrm{d}\gamma = \int_\Lambda u\,\boldsymbol{v} \cdot \boldsymbol{\tau}\,\mathrm{d}\lambda. \tag{A.616}$$

By considering $u = 1$ we obtain the following simpler version of the curl theorem:

$$\int_\Gamma \operatorname{curl} \boldsymbol{v} \cdot \boldsymbol{n}\,\mathrm{d}\gamma = \int_\Lambda \boldsymbol{v} \cdot \boldsymbol{\tau}\,\mathrm{d}\lambda. \tag{A.617}$$

The curl theorem relates thus a surface integral over $\Gamma$ with a line integral on $\Lambda$. We remark that if the surface $\Gamma$ is closed, then the line integrals on $\Lambda$, located on the right-hand side of (A.616) and (A.617), become zero. As with Green's theorems, more adjusted functional spaces so as to still allow to define traces on the boundary can be also defined for the curl theorem. We refer to the book of Nédélec (2001) for further details.

### A.5.5 Other integral theorems

We can derive also other integral theorems from the previous ones, being particularly useful for this purpose the integration-by-parts formula (A.611). Let $\Omega$ be a domain in $\mathbb{R}^N$, for $N = 2$ or 3, whose boundary $\Gamma$ is regular and whose unit normal points outwards of the domain, as shown in Figure A.16.

In three-dimensional space ($N = 3$) and for $\boldsymbol{u}, \boldsymbol{v} \in H^1(\Omega)^3$ it holds that

$$\int_\Omega (\boldsymbol{v} \cdot \operatorname{curl} \boldsymbol{u} - \boldsymbol{u} \cdot \operatorname{curl} \boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} = \int_\Gamma \boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{n}) \,\mathrm{d}\gamma. \tag{A.618}$$

In two dimensions ($N = 2$), for $u \in H^1(\Omega)$ and $\boldsymbol{v} \in H^1(\Omega)^2$, we have that

$$\int_\Omega (\boldsymbol{v} \cdot \operatorname{Curl} u - u \operatorname{curl} \boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} = \int_\Gamma u\, (\boldsymbol{v} \times \boldsymbol{n}) \,\mathrm{d}\gamma. \tag{A.619}$$

By considering now the Gauss-Green theorem (A.610) and a function $u \in H^2(\Omega)$, we obtain the relation

$$\int_\Omega \frac{\partial^2 u}{\partial x_i \partial x_j} \,\mathrm{d}\boldsymbol{x} = \int_\Gamma \frac{\partial u}{\partial x_j} n_i \,\mathrm{d}\gamma = \int_\Gamma \frac{\partial u}{\partial x_i} n_j \,\mathrm{d}\gamma \qquad i, j = 1, \ldots, N. \tag{A.620}$$

### A.5.6 Elementary differential geometry

When dealing with trace spaces, we need to work sometimes with differential operators on a regular surface $\Gamma$ that is defined by a system of local charts, as the one shown in Figure A.15. We are interested herein in a short and elementary introduction to this kind of operators, and for simplicity we will avoid the language of differential forms that is usual in differential geometry, although all the operators which we will describe are of such nature.

Let $\Gamma$ be the regular boundary (e.g., of class $C^2$) of a domain $\Omega$ in $\mathbb{R}^N$, for $N = 2$ or 3, which has a unit normal $\boldsymbol{n}$ that points outwards of $\Omega$, as depicted in Figure A.16. For every point $\boldsymbol{x} \in \mathbb{R}^N$ we denote by $d(\boldsymbol{x}, \Gamma)$ the distance from $\boldsymbol{x}$ to the boundary $\Gamma$, given by

$$d(\boldsymbol{x}, \Gamma) = \inf_{\boldsymbol{y} \in \Gamma} |\boldsymbol{x} - \boldsymbol{y}|. \tag{A.621}$$

A collection of points whose distance to the boundary is less than $\varepsilon$ is called a tubular neighborhood of $\Gamma$. Such a neighborhood $\Omega_\varepsilon$ is thus defined by

$$\Omega_\varepsilon = \{\boldsymbol{x} \in \mathbb{R}^N : \ d(\boldsymbol{x}, \Gamma) < \varepsilon\} = \Omega_\varepsilon^+ \cup \Gamma \cup \Omega_\varepsilon^-, \tag{A.622}$$

where

$$\Omega_\varepsilon^+ = \{\boldsymbol{x} \in \overline{\Omega}^c : \ d(\boldsymbol{x}, \Gamma) < \varepsilon\} \qquad \text{and} \qquad \Omega_\varepsilon^- = \{\boldsymbol{x} \in \Omega : \ d(\boldsymbol{x}, \Gamma) < \varepsilon\}. \tag{A.623}$$

For $\varepsilon$ small enough and when the boundary is regular and oriented, any point $\boldsymbol{x}$ in such a neighborhood $\Omega_\varepsilon$ has a unique projection $\boldsymbol{x}_\Gamma = \mathcal{P}_\Gamma(\boldsymbol{x})$ on the boundary $\Gamma$ which satisfies

$$|\boldsymbol{x} - \boldsymbol{x}_\Gamma| = d(\boldsymbol{x}, \Gamma). \tag{A.624}$$

For a regular boundary $\Gamma$ that admits a tangent plane at the point $\boldsymbol{x}_\Gamma$, the line $\boldsymbol{x} - \boldsymbol{x}_\Gamma$ is directed along the normal of the boundary at this point. Inside $\Omega_\varepsilon$ the function $d(\boldsymbol{x}, \Gamma)$ is regular. We introduce the vector field

$$\boldsymbol{n}(\boldsymbol{x}) = \begin{cases} \nabla d(\boldsymbol{x}, \Gamma) & \text{if } \ \boldsymbol{x} \in \Omega_\varepsilon^+, \\ -\nabla d(\boldsymbol{x}, \Gamma) & \text{if } \ \boldsymbol{x} \in \Omega_\varepsilon^-, \end{cases} \tag{A.625}$$

which extends in a continuous manner the unit normal $\boldsymbol{n}$ on $\Gamma$, and is such that

$$\boldsymbol{n}(\boldsymbol{x}) = \boldsymbol{n}(\boldsymbol{x}_\Gamma) \quad \forall \boldsymbol{x} \in \Omega_\varepsilon, \quad \text{where } \ \boldsymbol{x}_\Gamma = \mathcal{P}_\Gamma(\boldsymbol{x}). \tag{A.626}$$

Any point $\boldsymbol{x}$ in the tubular neighborhood $\Omega_\varepsilon$ can be parametrically described by

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{x}_\Gamma, s) = \boldsymbol{x}_\Gamma + s\,\boldsymbol{n}(\boldsymbol{x}_\Gamma), \qquad -\varepsilon \le s \le \varepsilon, \tag{A.627}$$

where $\boldsymbol{x}_\Gamma \in \Gamma$ and

$$s = \begin{cases} d(\boldsymbol{x}, \Gamma), & \text{if } \ \boldsymbol{x} \in \Omega_\varepsilon^+, \\ -d(\boldsymbol{x}, \Gamma), & \text{if } \ \boldsymbol{x} \in \Omega_\varepsilon^-. \end{cases} \tag{A.628}$$

The tubular neighborhood can be parametrized as

$$\Omega_\varepsilon = \{\boldsymbol{x} = \boldsymbol{x}_\Gamma + s\,\boldsymbol{n}(\boldsymbol{x}_\Gamma) : \ \ \boldsymbol{x}_\Gamma \in \Gamma, \ \ -\varepsilon < s < \varepsilon\}, \tag{A.629}$$

and similarly

$$\Omega_\varepsilon^+ = \{\boldsymbol{x} = \boldsymbol{x}_\Gamma + s\,\boldsymbol{n}(\boldsymbol{x}_\Gamma) : \ \ \boldsymbol{x}_\Gamma \in \Gamma, \ \ 0 < s < \varepsilon\}, \tag{A.630}$$

$$\Omega_\varepsilon^- = \{\boldsymbol{x} = \boldsymbol{x}_\Gamma + s\,\boldsymbol{n}(\boldsymbol{x}_\Gamma) : \ \ \boldsymbol{x}_\Gamma \in \Gamma, \ \ -\varepsilon < s < 0\}. \tag{A.631}$$

For any fixed $s$ such that $-\varepsilon < s < \varepsilon$, we introduce the surface

$$\Gamma_s = \{\boldsymbol{x} = \boldsymbol{x}_\Gamma + s\,\boldsymbol{n}(\boldsymbol{x}_\Gamma) : \ \ \boldsymbol{x}_\Gamma \in \Gamma\}. \tag{A.632}$$

The field $\boldsymbol{n}(\boldsymbol{x})$ is always normal to $\Gamma_s$. We remark that

$$\boldsymbol{n}(\boldsymbol{x}) = \nabla s(\boldsymbol{x}) \qquad \forall \boldsymbol{x} \in \Omega_\varepsilon. \tag{A.633}$$

The derivative with respect to $s$ of a regular function defined on the tubular neighborhood $\Omega_\varepsilon$ is confounded with the normal derivative of the function on $\Gamma_s$. Let $u$ be a regular scalar function defined on $\Gamma$. We denote now by $\tilde{u}$ the lifting of $u$ defined on $\Omega_\varepsilon$ that is constant along the normal direction, and thus given by

$$\tilde{u}(\boldsymbol{x}) = \tilde{u}\big(\boldsymbol{x}_\Gamma + s\,\boldsymbol{n}(\boldsymbol{x}_\Gamma)\big) = u(\boldsymbol{x}_\Gamma). \tag{A.634}$$

We introduce now some differential operators, which act on functions defined on the surfaces $\Gamma$ and $\Gamma_s$. The tangential gradient $\nabla_\Gamma u$ is defined as

$$\nabla_\Gamma u = \operatorname{grad}_\Gamma u = \nabla \tilde{u}|_\Gamma, \tag{A.635}$$

which is the gradient of $\tilde{u}$ restricted to $\Gamma$. In the same way we can define the operator $\nabla_{\Gamma_s} u$. It can be proven that if $u$ is any regular function defined on the tubular neighborhood $\Omega_\varepsilon$, then for any point $\boldsymbol{x} = \boldsymbol{x}_\Gamma + s\,\boldsymbol{n}(\boldsymbol{x}_\Gamma)$, and in particular for $s = 0$, it holds that

$$\nabla u = \nabla_{\Gamma_s} u + \frac{\partial u}{\partial s}\boldsymbol{n}. \tag{A.636}$$

The tangential curl or rotational of the scalar function $u$ is defined as

$$\operatorname{Curl}_\Gamma u = \begin{cases} \operatorname{curl}(\tilde{u}\,\boldsymbol{n})|_\Gamma & \text{if } N = 3, \\ \operatorname{Curl} \tilde{u}|_\Gamma & \text{if } N = 2. \end{cases} \tag{A.637}$$

The field of normals is a gradient, which implies that when $N = 3$, then

$$\operatorname{curl} \boldsymbol{n} = \boldsymbol{0}. \tag{A.638}$$

By using (A.589) we obtain that the tangential curl in three dimensions is also given by

$$\operatorname{Curl}_\Gamma u = \nabla_\Gamma u \times \boldsymbol{n}. \tag{A.639}$$

The definition of a tangential vector field's lifting is not so straightforward as in (A.634) for a scalar field (cf. Nédélec 2001). In this case we have to consider also a curvature operator of the form

$$\mathcal{R}_s = \nabla \boldsymbol{n} = \nabla \otimes \boldsymbol{n}, \tag{A.640}$$

where the gradient of a vector is understood again in the sense of a dyadic or tensor product. The curvature operator $\mathcal{R}_s$ is a symmetric tensor acting on the tangent plane, and its normal derivative is given by

$$\frac{\partial}{\partial s}\mathcal{R}_s = -\mathcal{R}_s^2. \tag{A.641}$$

On the surface $\Gamma$ (when $s = 0$), we omit the index $s$. The diffeomorphism from $\Gamma$ onto $\Gamma_s$ defined by $\boldsymbol{x} = \boldsymbol{x}_\Gamma + s\,\boldsymbol{n}(\boldsymbol{x}_\Gamma)$ has now $\boldsymbol{x}_\Gamma = \boldsymbol{x} - s\,\boldsymbol{n}(\boldsymbol{x})$ as its inverse, and it satisfies

$$\mathcal{R}(\boldsymbol{x}_\Gamma) - \mathcal{R}_s(\boldsymbol{x}) = s\mathcal{R}_s(\boldsymbol{x})\mathcal{R}(\boldsymbol{x}_\Gamma) = s\mathcal{R}(\boldsymbol{x}_\Gamma)\mathcal{R}_s(\boldsymbol{x}), \tag{A.642}$$

$$\big(I + s\mathcal{R}(\boldsymbol{x}_\Gamma)\big)^{-1} = I - s\mathcal{R}_s(\boldsymbol{x}). \tag{A.643}$$

A regular tangential vector field $\boldsymbol{v}$ defined on $\Gamma$ has to be extended towards the tubular neighborhood $\Omega_\varepsilon$ as

$$\tilde{\boldsymbol{v}}(\boldsymbol{x}) = \boldsymbol{v}(\boldsymbol{x}_\Gamma) - s\mathcal{R}_s(\boldsymbol{x})\boldsymbol{v}(\boldsymbol{x}_\Gamma), \tag{A.644}$$

which corresponds to a constant extension along the normal direction, where the tangential components of the vector are rotated proportionally with the distance $s$. We note that in two dimensions the curvature operator has no effect, but it is important in three dimensions due the degrees of freedom of the tangent planes. The surface divergence of the vector field $\boldsymbol{v}$ is now defined as

$$\operatorname{div}_\Gamma \boldsymbol{v} = \operatorname{div} \tilde{\boldsymbol{v}}|_\Gamma, \tag{A.645}$$

while its surface curl is given by the scalar field

$$\operatorname{curl}_\Gamma \boldsymbol{v} = \begin{cases} (\operatorname{curl} \tilde{\boldsymbol{v}} \cdot \boldsymbol{n})|_\Gamma & \text{if } N = 3, \\ \operatorname{curl} \tilde{\boldsymbol{v}}|_\Gamma & \text{if } N = 2. \end{cases} \tag{A.646}$$

For $N = 3$ it holds that

$$\operatorname{curl}_\Gamma \boldsymbol{v} = \operatorname{div}_\Gamma(\boldsymbol{v} \times \boldsymbol{n}). \tag{A.647}$$

Similarly as in (A.639), we have in the two-dimensional case ($N = 2$) that

$$\operatorname{curl}_\Gamma(u\,\boldsymbol{n}) = \nabla_\Gamma u \times \boldsymbol{n}. \tag{A.648}$$

The Laplace-Beltrami operator or scalar surface Laplacian is defined by

$$\Delta_\Gamma u = \operatorname{div}_\Gamma \nabla_\Gamma u = -\operatorname{curl}_\Gamma \operatorname{Curl}_\Gamma u, \tag{A.649}$$

whereas the Hodge operator or vectorial Laplacian is given by

$$\Delta_\Gamma \boldsymbol{v} = \nabla_\Gamma \operatorname{div}_\Gamma \boldsymbol{v} - \operatorname{Curl}_\Gamma \operatorname{curl}_\Gamma \boldsymbol{v}. \tag{A.650}$$

It holds also that

$$\operatorname{div}_\Gamma \operatorname{Curl}_\Gamma u = 0, \qquad \operatorname{Curl}_\Gamma \operatorname{div}_\Gamma \boldsymbol{v} = \boldsymbol{0}, \tag{A.651}$$

$$\operatorname{curl}_\Gamma \nabla_\Gamma u = 0, \qquad \nabla_\Gamma \operatorname{curl}_\Gamma \boldsymbol{v} = \boldsymbol{0}. \tag{A.652}$$

If $\Gamma$ is a closed boundary surface, $u \in C^1(\Gamma)$ a scalar function, and $\boldsymbol{v} \in C^1(\Gamma)^{N-1}$ a tangential vector field, then the following Stokes' identities hold:

$$\int_\Gamma \nabla_\Gamma u \cdot \boldsymbol{v}\,\mathrm{d}\gamma = -\int_\Gamma u \operatorname{div}_\Gamma \boldsymbol{v}\,\mathrm{d}\gamma, \tag{A.653}$$

$$\int_\Gamma \operatorname{Curl}_\Gamma u \cdot \boldsymbol{v}\,\mathrm{d}\gamma = \int_\Gamma u \operatorname{curl}_\Gamma \boldsymbol{v}\,\mathrm{d}\gamma. \tag{A.654}$$

Similarly, if $u, v \in C^1(\Gamma)$, then we have also that

$$\int_\Gamma u \operatorname{Curl}_\Gamma v\,\mathrm{d}\gamma = -\int_\Gamma v \operatorname{Curl}_\Gamma u\,\mathrm{d}\gamma. \tag{A.655}$$

For $u \in C^2(\Gamma)$ and $v \in C^1(\Gamma)$ it holds that

$$-\int_\Gamma \Delta_\Gamma u\, v\,\mathrm{d}\gamma = \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma v\,\mathrm{d}\gamma = \int_\Gamma \operatorname{Curl}_\Gamma u \cdot \operatorname{Curl}_\Gamma v\,\mathrm{d}\gamma. \tag{A.656}$$

If $\boldsymbol{u} \in C^2(\Gamma)^{N-1}$ and $\boldsymbol{v} \in C^1(\Gamma)^{N-1}$ are tangential vector fields, then

$$-\int_\Gamma \Delta_\Gamma \boldsymbol{u} \cdot \boldsymbol{v}\,\mathrm{d}\gamma = \int_\Gamma \operatorname{div}_\Gamma \boldsymbol{u} \operatorname{div}_\Gamma \boldsymbol{v}\,\mathrm{d}\gamma + \int_\Gamma \operatorname{curl}_\Gamma \boldsymbol{u} \operatorname{curl}_\Gamma \boldsymbol{v}\,\mathrm{d}\gamma. \tag{A.657}$$

Finally, by considering (A.620) and $u \in C^2(\Gamma)$ we can derive the Stokes' type formulae

$$\int_\Gamma (\nabla_\Gamma u \times \boldsymbol{n})\,\mathrm{d}\gamma = \int_\Gamma \operatorname{Curl}_\Gamma u\,\mathrm{d}\gamma = \boldsymbol{0} \qquad (N = 3), \tag{A.658}$$

$$\int_\Gamma (\nabla_\Gamma u \times \boldsymbol{n})\,\mathrm{d}\gamma = \int_\Gamma \operatorname{curl}_\Gamma(u\,\boldsymbol{n})\,\mathrm{d}\gamma = 0 \qquad (N = 2). \tag{A.659}$$

## A.6 Theory of distributions

The theory of generalized functions or distributions was invented in order to give a solid theoretical foundation to the Dirac delta function. The solid foundation of the theory was developed in 1945 by the French mathematician Laurent Schwartz (1915–2002). Today, this theory is fundamental in the study of partial differential equations, and comes naturally into use in the treatment of boundary integral equations. Of special importance is the notion of weak or distributional derivative of an integrable function, which is used in the definition of Sobolev spaces (vid. Section A.4).

The computation of Green's functions is performed naturally in the framework of the theory of distributions, due the appearance of Dirac masses in its definition. It is therefore important to have some notions of its characteristics. A complete survey of the theory of distributions can be found in Gel'fand & Shilov (1964) and Schwartz (1978). Other references for this theory and its applications are Bony (2001), Bremermann (1965), Chen & Zhou (1992), Estrada & Kanwal (2002), Gasquet & Witomski (1999), Griffel (1985), Hsiao & Wendland (2008), and Rudin (1973).

### A.6.1 Definition of distribution

Let $\Omega$ be a domain in $\mathbb{R}^N$. We denote as test functions in $\Omega$ the elements of the space $C_0^\infty(\Omega)$ of indefinitely differentiable functions with compact support in $\Omega$. The support of a function is the closure of the set of points where the function does not vanish. The space $C_0^\infty(\Omega)$ is also denoted by $\mathcal{D}(\Omega)$ and has a Fréchet space structure. We say that a sequence $\{\varphi_n\}$ of test functions converges to $\varphi$ in $\mathcal{D}(\Omega)$ if there exists a compact set $K \subset \Omega$ such that $\operatorname{supp}(\varphi_n - \varphi) \subset K$ for every $n$, and if for each multi-index $\boldsymbol{\alpha} \in \mathbb{N}_0^N$,

$$\lim_{n\to\infty} D^{\boldsymbol{\alpha}}\varphi_n(\boldsymbol{x}) = D^{\boldsymbol{\alpha}}\varphi(\boldsymbol{x}), \quad \text{uniformly on } K. \tag{A.660}$$

We define a continuous linear functional $T$ on $\mathcal{D}(\Omega)$ as a mapping from $\mathcal{D}(\Omega)$ to the field $\mathbb{K}$ (either $\mathbb{C}$ or $\mathbb{R}$), denoted by $\langle T, \varphi\rangle$ for $\varphi \in \mathcal{D}(\Omega)$, that satisfies

$$\langle T, \alpha\varphi_1 + \beta\varphi_2\rangle = \alpha\,\langle T, \varphi_1\rangle + \beta\,\langle T, \varphi_2\rangle \qquad \forall\alpha,\beta \in \mathbb{K},\ \ \forall\varphi_1,\varphi_2 \in \mathcal{D}(\Omega), \tag{A.661}$$

and is such that

$$\varphi_n \to 0 \ \text{ in } \ \mathcal{D}(\Omega) \quad \Longrightarrow \quad \langle T, \varphi_n\rangle \to 0 \ \text{ in } \ \mathbb{K}. \tag{A.662}$$

Such a continuous linear functional is called a distribution or generalized function. The space of (Schwartz) distributions is denoted by $\mathcal{D}'(\Omega)$ and corresponds to the dual space of $\mathcal{D}(\Omega)$. Thus, the bilinear form $\langle\cdot,\cdot\rangle : \mathcal{D}'(\Omega)\times\mathcal{D}(\Omega) \to \mathbb{K}$ represents the duality product between both spaces. Strictly speaking, when the underlying field $\mathbb{K}$ is taken as $\mathbb{C}$, then the duality product should be considered as a sesquilinear form and the distributions as antilinear functionals. Nonetheless, for the sake of simplicity this is not usually done, since the results in $\mathcal{D}'(\Omega)$ are the same with the exception of a complex conjugation on the test functions in $\mathcal{D}(\Omega)$. We note that the space $\mathcal{D}'(\Omega)$ has the weak$^*$-topology of the dual space (cf. Rudin 1973). The vector space and convergence operations in $\mathcal{D}'(\Omega)$ can be summarized, if $T, S, T_n \in \mathcal{D}'(\Omega)$ and $\alpha, \beta \in \mathbb{K}$, by

$$\langle \alpha T + \beta S, \varphi \rangle = \alpha \langle T, \varphi \rangle + \beta \langle S, \varphi \rangle \qquad \forall \varphi \in \mathcal{D}(\Omega), \tag{A.663}$$

and

$$T_n \to T \text{ in } \mathcal{D}'(\Omega) \quad \Longleftrightarrow \quad \langle T_n, \varphi \rangle \to \langle T, \varphi \rangle \text{ in } \mathbb{K} \qquad \forall \varphi \in \mathcal{D}(\Omega). \tag{A.664}$$

Distributions may be also multiplied by indefinitely differentiable functions to form new distributions. If $T \in \mathcal{D}'(\Omega)$ and $\eta \in C^\infty(\Omega)$, then the product $\eta T \in \mathcal{D}'(\Omega)$ is defined by

$$\langle \eta T, \varphi \rangle = \langle T, \eta \varphi \rangle \qquad \forall \varphi \in \mathcal{D}(\Omega). \tag{A.665}$$

We remark, however, that the product of two distributions is not well-defined in general.

Every locally integrable function $f \in L^1_{\text{loc}}(\Omega)$ defines a distribution via

$$\langle f, \varphi \rangle = \int_\Omega f(\boldsymbol{x}) \varphi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \qquad \forall \varphi \in \mathcal{D}(\Omega). \tag{A.666}$$

The distribution $f$ is said to be generated by the function $f$. A distribution that is generated by a locally integrable function is called a regular distribution. All other distributions are called singular. This suggests the notation

$$\langle T, \varphi \rangle = \int_\Omega T(\boldsymbol{x}) \varphi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \tag{A.667}$$

for a continuous linear functional $T$ even when $T$ is not an $L^1_{\text{loc}}$ function.

### A.6.2 Differentiation of distributions

Let us now define the important operation of differentiation on distributions. For any $T \in \mathcal{D}'(\Omega)$, we define $D^{\boldsymbol{\alpha}} T$ to be a linear functional such that

$$\langle D^{\boldsymbol{\alpha}} T, \varphi \rangle = (-1)^{|\boldsymbol{\alpha}|} \langle T, D^{\boldsymbol{\alpha}} \varphi \rangle \qquad \forall \varphi \in \mathcal{D}(\Omega), \tag{A.668}$$

for a given multi-index $\boldsymbol{\alpha} \in \mathbb{N}_0^N$. It is not difficult to see that $D^{\boldsymbol{\alpha}} T$ itself is again a continuous linear functional, i.e., a distribution. When $T$ is a function such that $D^{\boldsymbol{\beta}} T \in L^1_{\text{loc}}(\Omega)$ for all $|\boldsymbol{\beta}| \leq |\boldsymbol{\alpha}|$, then the definition (A.668) amounts to no more than integration by parts. But when $T$ does not admit classical derivatives, then (A.668) still allows to differentiate in the sense of distributions, shifting the burden of differentiability from $T$ to $\varphi$. Thus every distribution in $\mathcal{D}'(\Omega)$ possesses derivatives of arbitrary orders. This is particularly useful when dealing with discontinuous functions, since even for them there exist well-defined derivatives in the distributional sense.

We now define the concept of a weak or distributional derivative of a locally integrable function $f \in L^1_{\text{loc}}(\Omega)$. There may or may not exist a function $g_{\boldsymbol{\alpha}} \in L^1_{\text{loc}}(\Omega)$ such that $D^{\boldsymbol{\alpha}} f = g_{\boldsymbol{\alpha}}$ in $\mathcal{D}'(\Omega)$. If such a $g_{\boldsymbol{\alpha}}$ exists, it is unique up to sets of measure zero, it is called the weak or distributional partial derivative of $f$, and it satisfies

$$\int_\Omega g_{\boldsymbol{\alpha}}(\boldsymbol{x}) \varphi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = (-1)^{|\boldsymbol{\alpha}|} \int_\Omega f(\boldsymbol{x}) D^{\boldsymbol{\alpha}} \varphi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \qquad \forall \varphi \in \mathcal{D}(\Omega). \tag{A.669}$$

If $f$ is sufficiently smooth to have a continuous partial derivative $D^{\alpha} f$ in the classical sense, then $D^{\alpha} f$ is also a distributional partial derivative of $f$. Of course $D^{\alpha} f$ may exist in the distributional sense without existing in the classical sense.

### A.6.3 Primitives of distributions

Taking a primitive from a distribution amounts to the same as when dealing with functions. Let us begin with the case $N = 1$ by supposing that $\Omega \subset \mathbb{R}$. In this case, if $T \in \mathcal{D}'(\Omega)$, then a distribution $S$ such that

$$\langle S', \varphi \rangle = \langle T, \varphi \rangle \qquad \forall \varphi \in \mathcal{D}(\Omega) \tag{A.670}$$

is called a primitive or antiderivative of $T$. Any distribution $T \in \mathcal{D}'(\Omega)$ has a primitive $S$ in $\mathcal{D}'(\Omega)$ which is unique up to an additive constant, i.e., all the primitives of $T$ are of the form $S + C$, where $C$ is some constant.

We have further that any distribution $T \in \mathcal{D}'(\Omega)$, for $N = 1$, has primitives of any order. A primitive of $m$-th order of $T$ is a distribution $R \in \mathcal{D}'(\Omega)$ such that

$$\langle R^{(m)}, \varphi \rangle = \langle T, \varphi \rangle \qquad \forall \varphi \in \mathcal{D}(\Omega). \tag{A.671}$$

The primitive of $m$-th order is unique up to an additive polynomial of order $m - 1$.

Furthermore, in the general case when $N \geq 1$, for any $T \in \mathcal{D}'(\Omega)$ there exists a distribution $S$ such that $\partial S / \partial x_j = T$ in $\mathcal{D}'(\Omega)$, being $j \in \{1, \ldots, N\}$. This primitive is unique up to an additive locally integrable function that does not depend upon $x_j$. Thus every distribution possesses primitives of arbitrary order.

### A.6.4 Dirac's delta function

The Dirac delta or impulse function $\delta$, which is not strictly speaking a function, was introduced by the British theoretical physicist Paul Adrien Maurice Dirac (1902–1984) as a technical device in the mathematical formulation of quantum mechanics. The Dirac delta vanishes everywhere except at the origin, where its value is infinite, and so that its integral has a value of one. It is therefore defined by

$$\delta(\boldsymbol{x}) = \begin{cases} \infty & \text{if } \boldsymbol{x} = \mathbf{0}, \\ 0 & \text{if } \boldsymbol{x} \neq \mathbf{0}, \end{cases} \tag{A.672}$$

and has the property

$$\int_{\Omega} \delta(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = 1 \qquad \text{if } \mathbf{0} \in \Omega. \tag{A.673}$$

There exists no function with these properties. However, the Dirac delta is well-defined as a distribution, in which case it associates to each test function $\varphi$ its value at the origin. Assuming that $\mathbf{0} \in \Omega$, the Dirac delta is defined as the distribution $\delta$ that satisfies

$$\int_{\Omega} \delta(\boldsymbol{x}) \varphi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = \varphi(\mathbf{0}) \qquad \forall \varphi \in \mathcal{D}(\Omega). \tag{A.674}$$

The linear functional $\delta$ defined on $\mathcal{D}(\Omega)$ by

$$\langle \delta, \varphi \rangle = \varphi(\mathbf{0}) \tag{A.675}$$

is continuous, and hence clearly a distribution on $\Omega$.

From (A.674) several other properties for the Dirac delta $\delta$ can be derived. It is a symmetric distribution, i.e., $\delta(\boldsymbol{x}) = \delta(-\boldsymbol{x})$, and its support is the point $\boldsymbol{x} = \boldsymbol{0}$. The shifted Dirac mass, $\delta_{\boldsymbol{a}}(\boldsymbol{x}) = \delta(\boldsymbol{x} - \boldsymbol{a})$, has its mass concentrated at the point $\boldsymbol{a} \in \Omega$. It thus picks out the conjugated value of a test function $\varphi$ at the point $\boldsymbol{a}$, namely

$$\langle \delta_{\boldsymbol{a}}, \varphi \rangle = \langle \delta(\boldsymbol{x} - \boldsymbol{a}), \varphi(\boldsymbol{x}) \rangle = \langle \delta(\boldsymbol{x}), \varphi(\boldsymbol{x} + \boldsymbol{a}) \rangle = \varphi(\boldsymbol{a}) \qquad \forall \varphi \in \mathcal{D}(\Omega). \tag{A.676}$$

A scaling of the Dirac mass by $\lambda \in \mathbb{K}$, $\lambda \neq 0$, yields

$$\langle \delta(\lambda \boldsymbol{x}), \varphi(\boldsymbol{x}) \rangle = |\lambda|^{-N} \langle \delta(\boldsymbol{x}), \varphi(\boldsymbol{x}/\lambda) \rangle = |\lambda|^{-N} \varphi(\boldsymbol{0}) \qquad \forall \varphi \in \mathcal{D}(\Omega), \tag{A.677}$$

and hence

$$\delta(\lambda \boldsymbol{x}) = |\lambda|^{-N} \delta(\boldsymbol{x}). \tag{A.678}$$

An arbitrary derivative of the dirac Delta function, $D^{\boldsymbol{\alpha}}\delta$, is given by

$$\langle D^{\boldsymbol{\alpha}}\delta, \varphi \rangle = (-1)^{|\boldsymbol{\alpha}|} D^{\boldsymbol{\alpha}} \varphi(\boldsymbol{0}) \qquad \forall \varphi \in \mathcal{D}(\Omega). \tag{A.679}$$

We remark that the multi-dimensional Dirac mass can be decomposed as a multiplication of one-dimensional Dirac deltas, namely

$$\delta(\boldsymbol{x}) = \prod_{j=1}^{N} \delta(x_j). \tag{A.680}$$

An important fact is that Dirac distributions appear when differentiating functions that have jumps. To see this, we consider, e.g., for $\Omega = \mathbb{R}$, the Heaviside step function

$$H(x) = \begin{cases} 1 & \text{if } \ x > 0, \\ 0 & \text{if } \ x < 0, \end{cases} \tag{A.681}$$

which is named after the self-taught English electrical engineer, mathematician, and physicist Oliver Heaviside (1850–1925), who developed this function among several other important contributions. The Heaviside function belongs to $L^1_{\text{loc}}(\mathbb{R})$, and defines thus a regular distribution, namely

$$\langle H, \varphi \rangle = \int_0^\infty \varphi(x) \, \mathrm{d}x \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}). \tag{A.682}$$

The function $H$ is differentiable everywhere with pointwise derivative zero, except at the origin, where it is non-differentiable in the classical sense. The distributional derivative $H'$ of $H$ satisfies

$$\langle H', \varphi \rangle = -\langle H, \varphi' \rangle = -\int_0^\infty \varphi'(x) \, \mathrm{d}x = \varphi(0). \tag{A.683}$$

Therefore we have

$$H'(x) = \delta(x) \qquad \text{in } \ \mathbb{R}. \tag{A.684}$$

The Dirac delta can be also generalized to consider line or surface mass distributions. For a line or a surface $\Gamma \subset \Omega$, we define the Dirac distribution $\delta_\Gamma$ as

$$\langle \delta_\Gamma, \varphi \rangle = \int_\Gamma \varphi(\boldsymbol{x}) \, \mathrm{d}\gamma(\boldsymbol{x}) \qquad \forall \varphi \in \mathcal{D}(\Omega). \tag{A.685}$$

This type of Dirac distributions appear, e.g., when differentiating over a jump that extends along a line or a surface. Further generalizations of the Dirac distribution that use the language of differential forms can be found in Gel'fand & Shilov (1964).

### A.6.5 Principal value and finite parts

Let us study some special singular distributions. For the sake of simplicity we consider $\Omega = \mathbb{R}$, i.e., $N = 1$. In this case, the function $f(x) = 1/x$, defined for $x \neq 0$, is not integrable around the origin. Thus we cannot associate a distribution with $f$, and we will have the same problem with any rational function having a real pole. The difficulty is that the integrand has a singularity so strong that it must be excised from the domain and the integral has to be defined by a limiting process, the result of which is called an improper integral. This inconvenient, though, can be solved. Although $f$ is not locally integrable, its primitive $F(x) = \ln|x|$ is locally integrable, being its indefinite integral $x \ln|x| - x$. The distribution that helps to solve our problem is simply the derivative of $F$ in the sense of distributions. This distribution is now well-defined and is called the principal value, being denoted by $\mathrm{pv}(1/x)$. We take symmetric limits ($\epsilon$ and $-\epsilon$) around the origin and obtain

$$\lim_{\epsilon\to 0^+} \left\{ \int_{-\infty}^{-\epsilon} \frac{\varphi(x)}{x}\,\mathrm{d}x + \int_{\epsilon}^{\infty} \frac{\varphi(x)}{x}\,\mathrm{d}x \right\} = -\int_{-\infty}^{\infty} \ln|x|\,\varphi'(x)\,\mathrm{d}x \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}). \quad \text{(A.686)}$$

The distribution $\mathrm{pv}(1/x)$, which is the natural choice for a distribution corresponding to $1/x$, is thus defined by

$$\langle \mathrm{pv}(1/x), \varphi \rangle = -\langle \ln|x|, \varphi' \rangle \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}). \quad \text{(A.687)}$$

We can interpret this equation as follows: to evaluate the improper integral $\int_{-\infty}^{\infty} \varphi(x)/x\,\mathrm{d}x$, integrate it by parts as if it were a convergent integral. The result is the convergent integral $\int_{-\infty}^{\infty} \ln|x|\varphi'(x)\,\mathrm{d}x$. The integration by parts is not justified, but this procedure gives the result (A.687) of our rigorous definitions, and can be therefore regarded as a formal procedure to obtain the results of the correct theory. The principal value of $1/x$ satisfies

$$x\ \mathrm{pv}\left(\frac{1}{x}\right) = 1, \quad \text{(A.688)}$$

and is characterized by

$$\mathrm{pv}\left(\frac{1}{x}\right) = (\ln|x|)'. \quad \text{(A.689)}$$

The converse of (A.688), though, does not apply. A distribution $T$ satisfies $x\,T = 1$ if and only if for some constant $C$

$$T(x) = \mathrm{pv}\left(\frac{1}{x}\right) + C\,\delta(x). \quad \text{(A.690)}$$

In general, if $f$ is a function defined for $x \neq 0$, then we define the (Cauchy) principal value of the integral $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ by

$$\mathrm{pv}\left(\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x\right) = \fint_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{\epsilon\to 0^+} \int_{|x|\geq\epsilon} f(x)\,\mathrm{d}x, \quad \text{(A.691)}$$

whenever the limit exists. As expressed in (A.691), the notation $⨍$ is also used to denote a Cauchy principal value for the integral.

We remark that the concept of principal value applies likewise and more in general to contour integrals in the complex plane. In this case we consider a complex-valued function $f(z)$, for $z \in \mathbb{C}$, with a pole on the integration contour $L$. The pole is enclosed with a circle of radius $\epsilon$ and the portion of the path lying outside this circle is denoted by $L(\epsilon)$. Provided that the function $f(z)$ is integrable over $L(\epsilon)$, the Cauchy principal value is defined now as the limit

$$\mathrm{pv}\left(\int_L f(z)\,\mathrm{d}z\right) = ⨍_L f(z)\,\mathrm{d}z = \lim_{\epsilon\longrightarrow 0}\int_{L(\epsilon)} f(z)\,\mathrm{d}z. \tag{A.692}$$

We can define distributions corresponding to other negative powers of $x$, but the principal value cannot be used to assign a definite value to $\int_{-\infty}^{\infty}\varphi(x)/x^n\,\mathrm{d}x$, because it does not exist if $n > 1$. In this case the integral is truly divergent. We therefore define negative powers directly as derivatives of $\ln|x|$. For any integer $n > 1$, we define the distribution $x^{-n}$ to be the $n$-th derivative of

$$F(x) = \frac{(-1)^{n-1}}{(n-1)!}\ln|x|. \tag{A.693}$$

This procedure is known as extracting the finite part of a divergent integral, and is denoted by $\mathrm{fp}(1/x^n)$. It is equivalent to

$$\langle \mathrm{fp}(1/x^n), \varphi\rangle = \frac{(-1)^{n-1}}{(n-1)!}\langle \ln|x|, \varphi^{(n)}\rangle \qquad \forall\varphi\in\mathcal{D}(\mathbb{R}), \tag{A.694}$$

and can be again interpreted as integrating $n$ times by parts until the integral becomes convergent. This formal procedure was invented in 1932 by the French mathematician Jacques Salomon Hadamard (1865–1963), long before the development of the theory of distributions, as a convenient device for dealing with divergent integrals appearing in the theory of wave propagation.

## A.7 Fourier transforms

The Fourier transform is a special integral transform that decomposes a function described in the spatial (or temporal) domain into a continuous spectrum of its frequency components. It is named in honor of the French mathematician and physicist Jean Baptiste Joseph Fourier (1768–1830), who initiated the investigation of Fourier series and their application to problems of heat flow. Fourier transforms have many applications, particularly because they allow to treat differential equations as algebraic equations in the spectral domain. Sobolev spaces of fractional order are also defined by means of Fourier transforms (vid. Section A.4).

Fourier transforms are frequently used in the computation of Green's functions in free-space or in half-spaces, since usually explicit expressions of them in the spectral domain can easily be found. It is, however, sometimes quite difficult to find the corresponding spatial counterpart. In this thesis, in particular, we deal widely with Fourier transforms to find Green's functions in the half-space problems. Some references are Bony (2001), Bremermann (1965), Gasquet & Witomski (1999), Griffel (1985), Reed & Simon (1980), and Weisstein (2002). Applications for Fourier transforms in signal analysis and complex variables may be found respectively in Irarrázaval (1999) and Weinberger (1995). For further studies on signals and wavelets we refer to Mallat (2000), and for applications in biomedical imaging, to Ammari (2008). Useful tables of integrals to compute Fourier transforms can be found in Bateman (1954) and Gradshteyn & Ryzhik (2007). Other Fourier transforms of special functions, particularly of Bessel functions and their spherical versions, are listed in Magnus & Oberhettinger (1954).

### A.7.1 Definition of Fourier transform

We define the direct or forward Fourier transform $\widehat{f} = \mathcal{F}\{f\}$ of an integrable function $f \in L^1(\mathbb{R}^N)$ as

$$\widehat{f}(\boldsymbol{\xi}) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} f(\boldsymbol{x})\, e^{-i\boldsymbol{\xi}\cdot\boldsymbol{x}}\, \mathrm{d}\boldsymbol{x}, \qquad \boldsymbol{\xi} \in \mathbb{R}^N, \tag{A.695}$$

and its inverse or backward Fourier transform $f = \mathcal{F}^{-1}\{\widehat{f}\}$ by

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \widehat{f}(\boldsymbol{\xi})\, e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}}\, \mathrm{d}\boldsymbol{\xi}, \qquad \boldsymbol{x} \in \mathbb{R}^N. \tag{A.696}$$

We remark that there exist several different definitions for the Fourier transform. Some authors do not distribute the $(2\pi)^N$ coefficient that lies before the integrals symmetrically between both transforms as we do, but assign it completely to the inverse Fourier transform. Other authors prefer to consider in the Fourier domain a frequency variable $\boldsymbol{\nu}$ instead of our pulsation variable $\boldsymbol{\xi}$, being their relation $\boldsymbol{\xi} = 2\pi\boldsymbol{\nu}$, and avoiding thus the need of the beforementioned coefficient $(2\pi)^N$. Thus, care has to be taken to identify the definition used by each author, since different Fourier transform pairs result from them.

The Fourier transforms (A.695) and (A.696) can be used also for a more general class of functions $f$, such as for functions in $L^2(\mathbb{R}^N)$ or even for tempered distributions in the space $\mathcal{S}'(\mathbb{R}^N)$, the dual of the Schwartz space of rapidly decreasing functions

$$\mathcal{S}(\mathbb{R}^N) = \left\{ f \in C^\infty(\mathbb{R}^N) \mid \boldsymbol{x}^{\boldsymbol{\beta}} D^{\boldsymbol{\alpha}} f \in L^\infty(\mathbb{R}^N) \quad \forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}_0^N \right\}, \tag{A.697}$$

where $\boldsymbol{x}^{\boldsymbol{\beta}} = x_1^{\beta_1} x_2^{\beta_2} \cdots x_N^{\beta_N}$ for a multi-index $\boldsymbol{\beta} \in \mathbb{N}_0^N$. The space $\mathcal{S}(\mathbb{R}^N)$ has the important property of being invariant under Fourier transforms, i.e., $\varphi \in \mathcal{S}(\mathbb{R}^N) \Leftrightarrow \widehat{\varphi} \in \mathcal{S}(\mathbb{R}^N)$. We have in particular the inclusion $\mathcal{D}(\mathbb{R}^N) \subset \mathcal{S}(\mathbb{R}^N)$, and thus $\mathcal{S}'(\mathbb{R}^N) \subset \mathcal{D}'(\mathbb{R}^N)$. The convergence in $\mathcal{S}'(\mathbb{R}^N)$ is the same as for distributions (vid. Section A.6), but with respect to test functions in $\mathcal{S}(\mathbb{R}^N)$. In effect, if $T_n, T \in \mathcal{S}'(\mathbb{R}^N)$, then

$$T_n \to T \;\text{ in }\; \mathcal{S}'(\mathbb{R}^N) \quad \Longleftrightarrow \quad \langle T_n, \varphi \rangle \to \langle T, \varphi \rangle \;\text{ in }\; \mathbb{K} \qquad \forall \varphi \in \mathcal{S}(\mathbb{R}^N). \tag{A.698}$$

A distribution $T \in \mathcal{D}'(\mathbb{R}^N)$ is at the same time a tempered distribution, i.e., $T \in \mathcal{S}'(\mathbb{R}^N)$, if and only if $T$ is a continuous linear functional on $\mathcal{D}(\mathbb{R}^N)$ in the topology of $\mathcal{S}(\mathbb{R}^N)$. In particular, every function in $L^p(\mathbb{R}^N)$, $p \geq 1$, is a tempered distribution. Every slowly increasing function $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ such that

$$|f(\boldsymbol{x})| \leq C\Big(1 + |\boldsymbol{x}|^M\Big) \qquad \forall \boldsymbol{x} \in \mathbb{R}^N, \tag{A.699}$$

for some constant $C > 0$ and some integer $M \in \mathbb{N}$, is also a tempered distribution. In general, for any tempered distribution $T \in \mathcal{S}'(\mathbb{R}^N)$, there are integers $n_1, n_2, \ldots, n_p$ and slowly increasing continuous functions $f_1, f_2, \ldots, f_p$ such that

$$T = \sum_{j=1}^{p} f_j^{(n_j)}. \tag{A.700}$$

The direct Fourier transform $\widehat{T} = \mathcal{F}\{T\}$ of a tempered distribution $T \in \mathcal{S}'(\mathbb{R}^N)$ is now defined by

$$\langle \widehat{T}, \varphi \rangle = \langle T, \widehat{\varphi} \rangle \qquad \forall \varphi \in \mathcal{S}(\mathbb{R}^N). \tag{A.701}$$

We have that $\widehat{T}$ is also a tempered distribution, because the Fourier transform is a continuous linear operator on $\mathcal{S}(\mathbb{R}^N)$. Formula (A.701) extends the Fourier transform from $L^1(\mathbb{R}^N)$ or $L^2(\mathbb{R}^N)$ to tempered distributions. The inverse Fourier transform $T = \mathcal{F}^{-1}\{\widehat{T}\}$ of a tempered distribution $\widehat{T} \in \mathcal{S}'(\mathbb{R}^N)$ is defined by

$$\langle T, \widehat{\varphi} \rangle = \langle \widehat{T}, \varphi \rangle \qquad \forall \widehat{\varphi} \in \mathcal{S}(\mathbb{R}^N). \tag{A.702}$$

The Fourier transform is thus a linear, 1-to-1, bicontinuous mapping from $\mathcal{S}'(\mathbb{R}^N)$ to $\mathcal{S}'(\mathbb{R}^N)$. For all $T \in \mathcal{S}'(\mathbb{R}^N)$ we have

$$\mathcal{F}^{-1}\left\{\mathcal{F}\{T\}\right\} = \mathcal{F}\left\{\mathcal{F}^{-1}\{T\}\right\} = T. \tag{A.703}$$

### A.7.2 Properties of Fourier transforms

In what follows, we consider arbitrary distributions $S, T \in \mathcal{S}'(\mathbb{R}^N)$, and arbitrary constants $\alpha, \beta \in \mathbb{K}$, $\boldsymbol{a} \in \mathbb{R}^N$, and $b \in \mathbb{R}$. We write

$$T(\boldsymbol{x}) \quad \xrightarrow{\;\;\mathcal{F}\;\;} \quad \widehat{T}(\boldsymbol{\xi}) \tag{A.704}$$

to denote that $\widehat{T}(\boldsymbol{\xi})$ is the Fourier transform of $T(\boldsymbol{x})$, i.e., $\widehat{T} = \mathcal{F}\{T\}$. The linearity of the Fourier transform implies that

$$\alpha\, S(\boldsymbol{x}) + \beta\, T(\boldsymbol{x}) \quad \xrightarrow{\mathcal{F}} \quad \alpha\, \widehat{S}(\boldsymbol{\xi}) + \beta\, \widehat{T}(\boldsymbol{\xi}). \tag{A.705}$$

The duality or symmetry property of the Fourier transform means that

$$\widehat{T}(\boldsymbol{x}) \quad \xrightarrow{\mathcal{F}} \quad T(-\boldsymbol{\xi}). \tag{A.706}$$

The reflection property yields

$$T(-\boldsymbol{x}) \quad \xrightarrow{\mathcal{F}} \quad \widehat{T}(-\boldsymbol{\xi}). \tag{A.707}$$

The translation or shifting property states that

$$T(\boldsymbol{x} - \boldsymbol{a}) \quad \xrightarrow{\mathcal{F}} \quad e^{-i\boldsymbol{a}\cdot\boldsymbol{\xi}}\, \widehat{T}(\boldsymbol{\xi}), \tag{A.708}$$

$$e^{i\boldsymbol{a}\cdot\boldsymbol{x}}\, T(\boldsymbol{x}) \quad \xrightarrow{\mathcal{F}} \quad \widehat{T}(\boldsymbol{\xi} - \boldsymbol{a}). \tag{A.709}$$

The scaling property, for $a_1, a_2, \ldots, a_N \neq 0$, yields

$$T\left(\frac{x_1}{a_1}, \frac{x_2}{a_2}, \ldots, \frac{x_N}{a_N}\right) \quad \xrightarrow{\mathcal{F}} \quad |a_1 a_2 \cdots a_N|\, \widehat{T}(a_1\xi_1, a_2\xi_2, \ldots, a_N\xi_N), \tag{A.710}$$

$$T(a_1x_1, a_2x_2, \ldots, a_Nx_N) \quad \xrightarrow{\mathcal{F}} \quad \frac{1}{|a_1 a_2 \cdots a_N|}\, \widehat{T}\left(\frac{\xi_1}{a_1}, \frac{\xi_2}{a_2}, \ldots, \frac{\xi_N}{a_N}\right), \tag{A.711}$$

and, in particular, for $b \neq 0$,

$$T\left(\frac{\boldsymbol{x}}{b}\right) \quad \xrightarrow{\mathcal{F}} \quad |b|^N\, \widehat{T}(b\,\boldsymbol{\xi}), \tag{A.712}$$

$$T(b\,\boldsymbol{x}) \quad \xrightarrow{\mathcal{F}} \quad \frac{1}{|b|^N}\, \widehat{T}\left(\frac{\boldsymbol{\xi}}{b}\right). \tag{A.713}$$

The modulation property implies that

$$T(\boldsymbol{x}) \cos(\boldsymbol{a}\cdot\boldsymbol{x}) \quad \xrightarrow{\mathcal{F}} \quad \frac{1}{2}\Big(\widehat{T}(\boldsymbol{\xi} - \boldsymbol{a}) + \widehat{T}(\boldsymbol{\xi} + \boldsymbol{a})\Big), \tag{A.714}$$

$$\frac{1}{2}\Big(T(\boldsymbol{x} - \boldsymbol{a}) + T(\boldsymbol{x} + \boldsymbol{a})\Big) \quad \xrightarrow{\mathcal{F}} \quad \widehat{T}(\boldsymbol{\xi}) \cos(\boldsymbol{a}\cdot\boldsymbol{\xi}). \tag{A.715}$$

The parity property of the Fourier transform involves that

$$T \text{ even} \quad \xrightarrow{\mathcal{F}} \quad \widehat{T} \text{ even}, \tag{A.716}$$

$$T \text{ odd} \quad \xrightarrow{\mathcal{F}} \quad \widehat{T} \text{ odd}, \tag{A.717}$$

$$T \text{ real and even} \quad \xrightarrow{\mathcal{F}} \quad \widehat{T} \text{ real and even}, \tag{A.718}$$

$$T \text{ real and odd} \quad \xrightarrow{\mathcal{F}} \quad \widehat{T} \text{ imaginary and odd}, \tag{A.719}$$

$$T \text{ imaginary and even} \quad \xrightarrow{\mathcal{F}} \quad \widehat{T} \text{ imaginary and even}, \tag{A.720}$$

$$T \text{ imaginary and odd} \quad \xrightarrow{\mathcal{F}} \quad \widehat{T} \text{ real and odd}. \tag{A.721}$$

For the complex conjugation we have that

$$\overline{T(\boldsymbol{x})} \quad \xrightarrow{\mathcal{F}} \quad \overline{\widehat{T}(-\boldsymbol{\xi})}. \tag{A.722}$$

The important derivation property of the Fourier transform, that transforms derivatives into multiplications by monomials, is given by

$$\frac{\partial T}{\partial x_j}(\boldsymbol{x}) \quad \xrightarrow{\mathcal{F}} \quad i\xi_j\,\widehat{T}(\boldsymbol{\xi}), \qquad j \in \{1,2,\ldots,N\}, \tag{A.723}$$

$$D^{\boldsymbol{\alpha}}T(\boldsymbol{x}) \quad \xrightarrow{\mathcal{F}} \quad (i\boldsymbol{\xi})^{\boldsymbol{\alpha}}\,\widehat{T}(\boldsymbol{\xi}), \qquad \boldsymbol{\alpha} \in \mathbb{N}_0^N, \tag{A.724}$$

which holds also for the inverses

$$-ix_j\,T(\boldsymbol{x}) \quad \xrightarrow{\mathcal{F}} \quad \frac{\partial \widehat{T}}{\partial \xi_j}(\boldsymbol{\xi}), \qquad j \in \{1,2,\ldots,N\}, \tag{A.725}$$

$$(-i\boldsymbol{x})^{\boldsymbol{\alpha}}\,T(\boldsymbol{x}) \quad \xrightarrow{\mathcal{F}} \quad D^{\boldsymbol{\alpha}}\widehat{T}(\boldsymbol{\xi}), \qquad \boldsymbol{\alpha} \in \mathbb{N}_0^N. \tag{A.726}$$

The integration property, for $j \in \{1,2,\ldots,N\}$, states that

$$\int_{-\infty}^{x_j} T|_{x_j=y_j}(\boldsymbol{x})\,\mathrm{d}y_j \quad \xrightarrow{\mathcal{F}} \quad \frac{\widehat{T}(\boldsymbol{\xi})}{i\xi_j} + \pi\delta(\xi_j)\widehat{T}|_{\xi_j=0}(\boldsymbol{\xi}), \tag{A.727}$$

and similarly

$$-\frac{T(\boldsymbol{x})}{ix_j} + \pi\delta(x_j)T|_{x_j=0}(\boldsymbol{x}) \quad \xrightarrow{\mathcal{F}} \quad \int_{-\infty}^{\xi_j} \widehat{T}|_{\xi_j=\eta_j}(\boldsymbol{\xi})\,\mathrm{d}\eta_j. \tag{A.728}$$

We say that a distribution $T \in \mathcal{S}'(\mathbb{R}^N)$, is separable if there exist some distributions $T_j \in \mathcal{S}'(\mathbb{R})$, for $j \in \{1,2,\ldots,N\}$, such that

$$T(\boldsymbol{x}) = T_1(x_1)T_2(x_2)\cdots T_N(x_N). \tag{A.729}$$

The separability property of the Fourier transform states that if $T$ is a separable distribution, then so is $\widehat{T}$, i.e.,

$$T_1(x_1)T_2(x_2)\cdots T_N(x_N) \quad \xrightarrow{\mathcal{F}} \quad \widehat{T}_1(\xi_1)\widehat{T}_2(\xi_2)\cdots\widehat{T}_N(\xi_N). \tag{A.730}$$

This means that for separable distributions we can compute independently the partial Fourier transform of each factor, and multiply the results at the end. This property holds also if a distribution is partially separable, i.e., separable for only some of its variables.

We have that if $f \in L^2(\mathbb{R}^N)$, then its Fourier transform $\widehat{f}$ is in $L^2(\mathbb{R}^N)$ too. We have also, for $f, g \in L^1(\mathbb{R}^N)$ or $f, g \in L^2(\mathbb{R}^N)$, that

$$\int_{\mathbb{R}^N} \widehat{f}(\boldsymbol{x})g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^N} f(\boldsymbol{x})\widehat{g}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{A.731}$$

Furthermore, if $f, g \in L^2(\mathbb{R}^N)$, then we have Parseval's formula

$$\int_{\mathbb{R}^N} f(\boldsymbol{x})\overline{g(\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^N} \widehat{f}(\boldsymbol{\xi})\overline{\widehat{g}(\boldsymbol{\xi})}\,\mathrm{d}\boldsymbol{\xi}, \tag{A.732}$$

named after the French mathematician Marc-Antoine Parseval des Chênes (1755–1836). In particular, when $f = g$, then (A.732) turns into Plancherel's formula

$$\int_{\mathbb{R}^N} |f(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^N} |\widehat{f}(\boldsymbol{\xi})|^2\,\mathrm{d}\boldsymbol{\xi}, \tag{A.733}$$

which is named after the Swiss mathematician Michel Plancherel (1885–1967).

### A.7.3 Convolution

We define the convolution or faltung $f * g$ of two functions $f$ and $g$ from $\mathbb{R}^N$ to $\mathbb{K}$, if it exists, as

$$f(\boldsymbol{x}) * g(\boldsymbol{x}) = \int_{\mathbb{R}^N} f(\boldsymbol{y})g(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = \int_{\mathbb{R}^N} f(\boldsymbol{x}-\boldsymbol{y})g(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}. \tag{A.734}$$

The convolution has the property of regularizing a function by averaging, and is a commutative operation, i.e.,

$$f(\boldsymbol{x}) * g(\boldsymbol{x}) = g(\boldsymbol{x}) * f(\boldsymbol{x}). \tag{A.735}$$

The convolution is well-defined if $f, g \in L^2(\mathbb{R}^N)$. It can be further shown that the convolution $L^p(\mathbb{R}^N) * L^q(\mathbb{R}^N)$ is well-defined for $p, q, r \geq 1$ and such that $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$. In this case, if $f \in L^p(\mathbb{R}^N)$ and $g \in L^q(\mathbb{R}^N)$, then $f * g$ is in $L^r(\mathbb{R}^N)$. Moreover, the notion of convolution can be extended to the framework of distributions, in which case the convolutions $\mathcal{D}(\mathbb{R}^N) * \mathcal{D}'(\mathbb{R}^N)$, $\mathcal{S}(\mathbb{R}^N) * \mathcal{S}'(\mathbb{R}^N)$, $\mathcal{E}(\mathbb{R}^N) * \mathcal{E}'(\mathbb{R}^N)$, and even $\mathcal{E}'(\mathbb{R}^N) * \mathcal{S}'(\mathbb{R}^N)$ and $\mathcal{E}'(\mathbb{R}^N) * \mathcal{D}'(\mathbb{R}^N)$ are well-defined. By $\mathcal{E}'(\mathbb{R}^N)$ we denote the subspace of $\mathcal{D}'(\mathbb{R}^N)$ of those distributions that have compact support, which is the dual of $\mathcal{E}(\mathbb{R}^N) = C^\infty(\mathbb{R}^N)$. It can be shown that $\mathcal{E}'(\mathbb{R}^N)$ is also a linear subspace of $\mathcal{S}'(\mathbb{R}^N)$. The inclusions are such that

$$\mathcal{D} \subset \mathcal{E}', \quad \mathcal{S} \subset \mathcal{S}', \quad \mathcal{E} \subset \mathcal{D}', \quad \mathcal{D} \subset \mathcal{S} \subset \mathcal{E}, \quad \text{and} \quad \mathcal{E}' \subset \mathcal{S}' \subset \mathcal{D}'. \tag{A.736}$$

If $T \in \mathcal{D}'(\mathbb{R}^N)$ and $\varphi \in C^\infty(\mathbb{R}^N)$, then the convolution $T * \varphi$ is defined by

$$T(\boldsymbol{x}) * \varphi(\boldsymbol{x}) = \langle T(\boldsymbol{y}), \varphi(\boldsymbol{x}-\boldsymbol{y})\rangle = \langle T(\boldsymbol{x}-\boldsymbol{y}), \varphi(\boldsymbol{y})\rangle. \tag{A.737}$$

If $S \in \mathcal{E}'(\mathbb{R}^N)$ and $T \in \mathcal{D}'(\mathbb{R}^N)$, then

$$\psi_T(\boldsymbol{y}) = \langle T(\boldsymbol{x}), \varphi(\boldsymbol{x}+\boldsymbol{y})\rangle \in C^\infty(\mathbb{R}^N), \tag{A.738}$$

$$\psi_S(\boldsymbol{y}) = \langle S(\boldsymbol{x}), \varphi(\boldsymbol{x}+\boldsymbol{y})\rangle \in \mathcal{D}(\mathbb{R}^N), \tag{A.739}$$

and therefore the convolution $S * T$ is defined by

$$\langle S(\boldsymbol{x}) * T(\boldsymbol{x}), \varphi(\boldsymbol{x})\rangle = \big\langle S(\boldsymbol{y}), \langle T(\boldsymbol{x}), \varphi(\boldsymbol{x}+\boldsymbol{y})\rangle\big\rangle = \big\langle T(\boldsymbol{y}), \langle S(\boldsymbol{x}), \varphi(\boldsymbol{x}+\boldsymbol{y})\rangle\big\rangle \tag{A.740}$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^N)$.

Let $T \in \mathcal{D}'(\mathbb{R}^N)$ be a distribution. Then the Dirac delta function $\delta$ acts like a unit element for the convolution, namely

$$D^{\boldsymbol{\alpha}}\delta * T = T * D^{\boldsymbol{\alpha}}\delta = D^{\boldsymbol{\alpha}}T, \qquad \boldsymbol{\alpha} \in \mathbb{N}_0^N, \tag{A.741}$$

and is, in particular, its neuter element, i.e.,

$$\delta * T = T * \delta = T. \tag{A.742}$$

The $\delta$-function allows also to shift arguments by means of

$$\delta_{\boldsymbol{a}}(\boldsymbol{x}) * T(\boldsymbol{x}) = T(\boldsymbol{x}) * \delta_{\boldsymbol{a}}(\boldsymbol{x}) = T(\boldsymbol{x}-\boldsymbol{a}). \tag{A.743}$$

The convolution has the property of distributing the derivatives among its members. Thus, if $S \in \mathcal{E}'(\mathbb{R}^N)$ and $T \in \mathcal{D}'(\mathbb{R}^N)$, then

$$\frac{\partial}{\partial x_j}\{S * T\} = \frac{\partial S}{\partial x_j} * T = S * \frac{\partial T}{\partial x_j}, \qquad j \in \{1, 2, \dots, N\}, \tag{A.744}$$

and, more generally,

$$D^{\boldsymbol{\alpha}}\{S * T\} = D^{\boldsymbol{\alpha}} S * T = S * D^{\boldsymbol{\alpha}} T, \qquad \boldsymbol{\alpha} \in \mathbb{N}_0^N. \tag{A.745}$$

An important property of the Fourier transform is that it turns convolutions into multiplications and viceversa. Thus, if $S \in \mathcal{E}'(\mathbb{R}^N)$ and $T \in \mathcal{S}'(\mathbb{R}^N)$, then we have that

$$T(\boldsymbol{x}) * S(\boldsymbol{x}) \quad \xrightarrow{\mathcal{F}} \quad (2\pi)^{N/2}\, \widehat{T}(\boldsymbol{\xi})\widehat{S}(\boldsymbol{\xi}), \tag{A.746}$$

$$(2\pi)^{N/2}\, T(\boldsymbol{x})S(\boldsymbol{x}) \quad \xrightarrow{\mathcal{F}} \quad \widehat{T}(\boldsymbol{\xi}) * \widehat{S}(\boldsymbol{\xi}). \tag{A.747}$$

### A.7.4 Some Fourier transform pairs

We consider now some Fourier transform pairs, defined on $\mathbb{R}^N$, that use the definitions (A.695) and (A.696). For the Dirac delta $\delta$ holds that

$$\delta(\boldsymbol{x}) \quad \xrightarrow{\mathcal{F}} \quad \frac{1}{(2\pi)^{N/2}}, \tag{A.748}$$

$$\frac{1}{(2\pi)^{N/2}} \quad \xrightarrow{\mathcal{F}} \quad \delta(\boldsymbol{\xi}). \tag{A.749}$$

The complex exponential function, for $\boldsymbol{a} \in \mathbb{R}^N$, satisfies

$$e^{i\boldsymbol{a}\cdot\boldsymbol{x}} \quad \xrightarrow{\mathcal{F}} \quad (2\pi)^{N/2}\delta(\boldsymbol{\xi} - \boldsymbol{a}), \tag{A.750}$$

$$(2\pi)^{N/2}\delta(\boldsymbol{x} + \boldsymbol{a}) \quad \xrightarrow{\mathcal{F}} \quad e^{i\boldsymbol{a}\cdot\boldsymbol{\xi}}. \tag{A.751}$$

For the cosine function we have

$$\cos(\boldsymbol{a}\cdot\boldsymbol{x}) \quad \xrightarrow{\mathcal{F}} \quad \frac{(2\pi)^{N/2}}{2}\big(\delta(\boldsymbol{\xi} - \boldsymbol{a}) + \delta(\boldsymbol{\xi} + \boldsymbol{a})\big), \tag{A.752}$$

$$\frac{(2\pi)^{N/2}}{2}\big(\delta(\boldsymbol{x} + \boldsymbol{a}) + \delta(\boldsymbol{x} - \boldsymbol{a})\big) \quad \xrightarrow{\mathcal{F}} \quad \cos(\boldsymbol{a}\cdot\boldsymbol{\xi}), \tag{A.753}$$

and for the sine function we have

$$\sin(\boldsymbol{a}\cdot\boldsymbol{x}) \quad \xrightarrow{\mathcal{F}} \quad \frac{(2\pi)^{N/2}}{2\,i}\big(\delta(\boldsymbol{\xi} - \boldsymbol{a}) - \delta(\boldsymbol{\xi} + \boldsymbol{a})\big), \tag{A.754}$$

$$\frac{(2\pi)^{N/2}}{2\,i}\big(\delta(\boldsymbol{x} + \boldsymbol{a}) - \delta(\boldsymbol{x} - \boldsymbol{a})\big) \quad \xrightarrow{\mathcal{F}} \quad \sin(\boldsymbol{a}\cdot\boldsymbol{\xi}). \tag{A.755}$$

Powers of monomials, for $n \in \mathbb{N}_0$ and $j \in \{1, 2, \dots, N\}$, yield

$$x_j^n \quad \xrightarrow{\mathcal{F}} \quad i^n (2\pi)^{N/2} \frac{\partial^n \delta}{\partial \xi_j^n}(\boldsymbol{\xi}), \tag{A.756}$$

$$(-i)^n (2\pi)^{N/2} \frac{\partial^n \delta}{\partial x_j^n}(\boldsymbol{x}) \quad \xrightarrow{\mathcal{F}} \quad \xi_j^n, \tag{A.757}$$

and, for the general case when $\boldsymbol{\alpha} \in \mathbb{N}_0^N$ is a multi-index, yield

$$\boldsymbol{x}^{\boldsymbol{\alpha}} \quad \xrightarrow{\mathcal{F}} \quad i^{|\boldsymbol{\alpha}|}(2\pi)^{N/2} D^{\boldsymbol{\alpha}}\delta(\boldsymbol{\xi}), \tag{A.758}$$

$$(-i)^{|\boldsymbol{\alpha}|}(2\pi)^{N/2} D^{\boldsymbol{\alpha}}\delta(\boldsymbol{x}) \quad \xrightarrow{\mathcal{F}} \quad \boldsymbol{\xi}^{\boldsymbol{\alpha}}. \tag{A.759}$$

### A.7.5 Fourier transforms in 1D

The direct Fourier transform $\widehat{f}$ of an integrable function or tempered distribution $f$ in the one-dimensional case, i.e., when $N = 1$, is defined by

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\xi x}\, \mathrm{d}x, \qquad \xi \in \mathbb{R}, \tag{A.760}$$

and its inverse Fourier transform by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(\xi)\, e^{i\xi x}\, \mathrm{d}\xi, \qquad x \in \mathbb{R}. \tag{A.761}$$

Several signals, either functions or distributions, are commonly used for the 1D case. Among them we have the Heaviside step function $H(x)$, which is defined in (A.681). We have further the sign function

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases} \tag{A.762}$$

The rect function $\sqcap(x)$ is defined by

$$\sqcap(x) = \begin{cases} 1 & \text{if } |x| < \frac{1}{2}, \\ 0 & \text{if } |x| > \frac{1}{2}. \end{cases} \tag{A.763}$$

The triangle function $\wedge(x)$ is given by

$$\wedge(x) = \begin{cases} 1 - |x| & \text{if } |x| \le 1, \\ 0 & \text{if } |x| > 1. \end{cases} \tag{A.764}$$

We have now the 1D Fourier transform pairs

$$\delta(x) \quad \xrightarrow{\mathcal{F}} \quad \frac{1}{\sqrt{2\pi}}, \tag{A.765}$$

$$\frac{1}{\sqrt{2\pi}} \quad \xrightarrow{\mathcal{F}} \quad \delta(\xi), \tag{A.766}$$

$$\operatorname{sign}(x) \quad \xrightarrow{\mathcal{F}} \quad -i\sqrt{\frac{2}{\pi}}\, \operatorname{pv}\!\left(\frac{1}{\xi}\right), \tag{A.767}$$

$$H(x) \quad \xrightarrow{\mathcal{F}} \quad \frac{1}{i\sqrt{2\pi}}\, \operatorname{pv}\!\left(\frac{1}{\xi}\right) + \sqrt{\frac{\pi}{2}}\, \delta(\xi), \tag{A.768}$$

$$x^n \quad \xrightarrow{\mathcal{F}} \quad i^n \sqrt{2\pi}\, \delta^{(n)}(\xi) \qquad (n \ge 1), \tag{A.769}$$

$$\operatorname{pv}\!\left(\frac{1}{x}\right) \quad \xrightarrow{\mathcal{F}} \quad -i\sqrt{\frac{\pi}{2}}\, \operatorname{sign}(\xi), \tag{A.770}$$

$$\mathrm{fp}\left(\frac{1}{x^n}\right) \quad \xrightarrow{\mathcal{F}} \quad -i\sqrt{\frac{\pi}{2}}\,\frac{(-i\xi)^{n-1}}{(n-1)!}\,\operatorname{sign}(\xi) \qquad (n \geq 1), \tag{A.771}$$

$$\sqcap(x) \quad \xrightarrow{\mathcal{F}} \quad \frac{1}{\sqrt{2\pi}}\,\frac{\sin(\xi/2)}{\xi/2}, \tag{A.772}$$

$$\wedge(x) \quad \xrightarrow{\mathcal{F}} \quad \frac{1}{\sqrt{2\pi}}\left(\frac{\sin(\xi/2)}{\xi/2}\right)^2, \tag{A.773}$$

$$\frac{\sin(\pi x)}{\pi x} \quad \xrightarrow{\mathcal{F}} \quad \frac{1}{\sqrt{2\pi}}\,\sqcap\!\left(\frac{\xi}{2\pi}\right), \tag{A.774}$$

$$e^{-a|x|} \quad \xrightarrow{\mathcal{F}} \quad \sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+\xi^2} \qquad (\mathfrak{Re}\,a > 0), \tag{A.775}$$

$$e^{-ax^2} \quad \xrightarrow{\mathcal{F}} \quad \frac{1}{\sqrt{2a}}\,e^{-\xi^2/4a} \qquad (a > 0), \tag{A.776}$$

$$e^{-ax}H(x) \quad \xrightarrow{\mathcal{F}} \quad \frac{1}{\sqrt{2\pi}\,(a+i\xi)} \qquad (\mathfrak{Re}\,a > 0), \tag{A.777}$$

$$\cos(ax) \quad \xrightarrow{\mathcal{F}} \quad \sqrt{\frac{\pi}{2}}\,\big(\delta(\xi+a)+\delta(\xi-a)\big) \qquad (a \in \mathbb{R}), \tag{A.778}$$

$$\sin(ax) \quad \xrightarrow{\mathcal{F}} \quad i\sqrt{\frac{\pi}{2}}\,\big(\delta(\xi+a)-\delta(\xi-a)\big) \qquad (a \in \mathbb{R}), \tag{A.779}$$

$$\frac{1}{\sqrt{|x|}} \quad \xrightarrow{\mathcal{F}} \quad \frac{1}{\sqrt{|\xi|}}. \tag{A.780}$$

In the sense of homogeneous distributions (cf. Gel'fand & Shilov 1964), we have that

$$\ln\left(\sqrt{x^2+a^2}\right) \quad \xrightarrow{\mathcal{F}} \quad -\sqrt{\frac{\pi}{2}}\,\frac{e^{-|a||\xi|}}{|\xi|} \qquad (a \in \mathbb{R}). \tag{A.781}$$

Some Fourier transforms involving Bessel and Hankel functions (vid. Subsection A.2.4), for $a \in \mathbb{R}$ and $b > 0$, are

$$J_0(x) \quad \xrightarrow{\mathcal{F}} \quad \sqrt{\frac{2}{\pi}}\,\frac{\sqcap(\xi/2)}{\sqrt{1-\xi^2}}, \tag{A.782}$$

$$J_0\left(b\sqrt{x^2+a^2}\right) \quad \xrightarrow{\mathcal{F}} \quad \sqrt{\frac{2}{\pi}}\,\frac{\sqcap(\xi/2b)}{\sqrt{b^2-\xi^2}}\cos\!\Big(\sqrt{b^2-\xi^2}\,|a|\Big), \tag{A.783}$$

$$Y_0\left(b\sqrt{x^2+a^2}\right) \quad \xrightarrow{\mathcal{F}} \quad \sqrt{\frac{2}{\pi}}\,\frac{\sqcap(\xi/2b)}{\sqrt{b^2-\xi^2}}\sin\!\Big(\sqrt{b^2-\xi^2}\,|a|\Big) - \sqrt{\frac{2}{\pi}}\,\frac{e^{-\sqrt{\xi^2-b^2}\,|a|}}{\sqrt{\xi^2-b^2}}\big(1-\sqcap(\xi/2b)\big), \tag{A.784}$$

$$H_0^{(1)}\left(b\sqrt{x^2+a^2}\right) \quad \xrightarrow{\mathcal{F}} \quad -i\sqrt{\frac{2}{\pi}}\,\frac{e^{-\sqrt{\xi^2-b^2}\,|a|}}{\sqrt{\xi^2-b^2}}, \tag{A.785}$$

where the complex square root in (A.785) is defined in such a way that

$$\sqrt{\xi^2-b^2} = -i\sqrt{b^2-\xi^2}. \tag{A.786}$$

### A.7.6 Fourier transforms in 2D

The direct Fourier transform $\widehat{f}$ of an integrable function or tempered distribution $f$ in the two-dimensional case, i.e., when $N = 2$, is defined by

$$\widehat{f}(\xi_1, \xi_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2)\, e^{-i(\xi_1 x_1 + \xi_2 x_2)}\, \mathrm{d}x_1\, \mathrm{d}x_2, \qquad \xi_1, \xi_2 \in \mathbb{R}, \tag{A.787}$$

and its inverse Fourier transform by

$$f(x_1, x_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widehat{f}(\xi_1, \xi_2)\, e^{i(\xi_1 x_1 + \xi_2 x_2)}\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2, \qquad x_1, x_2 \in \mathbb{R}. \tag{A.788}$$

To express the radial components we use the notation

$$r = |\boldsymbol{x}| = \sqrt{x_1^2 + x_2^2} \qquad \text{and} \qquad \rho = |\boldsymbol{\xi}| = \sqrt{\xi_1^2 + \xi_2^2}. \tag{A.789}$$

It holds that the two-dimensional Fourier transform of a circularly symmetric function is also circularly symmetric and the same is true for the converse. The 2D Fourier transform turns in this case into the Hankel transform of order zero, which is given by

$$\widehat{f}(\rho) = \int_0^{\infty} f(r) J_0(\rho r)\, r\, \mathrm{d}r, \qquad \rho \geq 0, \tag{A.790}$$

and its inverse by

$$f(r) = \int_0^{\infty} \widehat{f}(\rho) J_0(\rho r)\, \rho\, \mathrm{d}\rho, \qquad r \geq 0. \tag{A.791}$$

This relation between both integral transforms stems from the integral representation of the zeroth-order Bessel function (A.112), which implies that

$$J_0(\rho r) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\rho r \cos\psi}\, \mathrm{d}\psi = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\rho r \cos\psi}\, \mathrm{d}\psi. \tag{A.792}$$

If we denote the polar angles by

$$\theta = \arctan\left(\frac{x_2}{x_1}\right) \qquad \text{and} \qquad \psi = \arctan\left(\frac{\xi_2}{\xi_1}\right), \tag{A.793}$$

then we can relate (A.788) and (A.791), due (A.792), by means of

$$\begin{aligned} f(x_1, x_2) = f(r) &= \frac{1}{2\pi} \int_0^{\infty} \int_0^{2\pi} \widehat{f}(\rho)\, \rho\, e^{i\rho r(\cos\theta\cos\psi + \sin\theta\sin\psi)}\, \mathrm{d}\psi\, \mathrm{d}\rho \\ &= \frac{1}{2\pi} \int_0^{\infty} \widehat{f}(\rho)\, \rho \int_0^{2\pi} e^{i\rho r\cos(\psi-\theta)}\, \mathrm{d}\psi\, \mathrm{d}\rho \\ &= \int_0^{\infty} \widehat{f}(\rho) J_0(\rho r)\, \rho\, \mathrm{d}\rho. \end{aligned} \tag{A.794}$$

The relation between (A.787) and (A.790) can be proved using a similar development.

For the 2D case there are also several signals that are commonly used. Among them we have the two-dimensional rect function $\sqcap(x_1, x_2)$, defined by

$$\sqcap(x_1, x_2) = \sqcap(x_1)\, \sqcap(x_2) = \begin{cases} 1 & \text{if } |x_1| < \frac{1}{2} \text{ and } |x_2| < \frac{1}{2}, \\ 0 & \text{elsewhere}, \end{cases} \tag{A.795}$$

and the circ function, defined by

$$\sqcap(r) = \begin{cases} 1 & \text{if } r < \frac{1}{2}, \\ 0 & \text{elsewhere.} \end{cases} \tag{A.796}$$

We have now the 2D Fourier transform pairs

$$\delta(x_1, x_2) \xrightarrow{\mathcal{F}} \frac{1}{2\pi}, \tag{A.797}$$

$$\frac{1}{2\pi} \xrightarrow{\mathcal{F}} \delta(\xi_1, \xi_2), \tag{A.798}$$

$$\delta(x_1) \xrightarrow{\mathcal{F}} \delta(\xi_2), \tag{A.799}$$

$$\delta(x_2) \xrightarrow{\mathcal{F}} \delta(\xi_1), \tag{A.800}$$

$$\sqcap(x_1, x_2) \xrightarrow{\mathcal{F}} \frac{1}{2\pi} \frac{\sin(\xi_1/2)}{\xi_1/2} \frac{\sin(\xi_2/2)}{\xi_2/2}, \tag{A.801}$$

$$\sqcap(r) \xrightarrow{\mathcal{F}} \frac{J_1(\rho/2)}{2\rho}, \tag{A.802}$$

$$e^{-ar^2} \xrightarrow{\mathcal{F}} \frac{1}{2a} e^{-\rho^2/4a} \qquad (a > 0), \tag{A.803}$$

$$\frac{1}{r} \xrightarrow{\mathcal{F}} \frac{1}{\rho}. \tag{A.804}$$

Other interesting 2D Fourier transforms, for $a \in \mathbb{R}$ and $b > 0$, are

$$\frac{1}{\sqrt{r^2 + a^2}} \xrightarrow{\mathcal{F}} \frac{e^{-\rho|a|}}{\rho}, \tag{A.805}$$

$$\frac{\sin\left(b\sqrt{r^2 + a^2}\right)}{\sqrt{r^2 + a^2}} \xrightarrow{\mathcal{F}} \frac{\cos\left(\sqrt{b^2 - \rho^2}\,|a|\right)}{\sqrt{b^2 - \rho^2}} \sqcap\left(\frac{\xi_1}{2b}, \frac{\xi_2}{2b}\right), \tag{A.806}$$

$$\begin{aligned} \frac{\cos\left(b\sqrt{r^2 + a^2}\right)}{\sqrt{r^2 + a^2}} \xrightarrow{\mathcal{F}} & -\frac{\sin\left(\sqrt{b^2 - \rho^2}\,|a|\right)}{\sqrt{b^2 - \rho^2}} \sqcap\left(\frac{\xi_1}{2b}, \frac{\xi_2}{2b}\right) \\ & + \frac{e^{-\sqrt{\rho^2 - b^2}\,|a|}}{\sqrt{\rho^2 - b^2}} \left(1 - \sqcap\left(\frac{\xi_1}{2b}, \frac{\xi_2}{2b}\right)\right), \end{aligned} \tag{A.807}$$

$$\frac{e^{ib\sqrt{r^2 + a^2}}}{\sqrt{r^2 + a^2}} \xrightarrow{\mathcal{F}} \frac{e^{-\sqrt{\rho^2 - b^2}\,|a|}}{\sqrt{\rho^2 - b^2}}, \tag{A.808}$$

where the complex square root in (A.808) is defined in such a way that

$$\sqrt{\rho^2 - b^2} = -i\sqrt{b^2 - \rho^2}. \tag{A.809}$$

We observe that the left-hand side of the expressions (A.806), (A.807), and (A.808) is closely related with the spherical Bessel and Hankel functions $j_0$, $y_0$, and $h_0^{(1)}$, respectively. For further details, see Subsection A.2.6.