Towards design stability margins estimations using Chetaev functions

Résumé long du chapitre 5

Dans le chapitre précédent, un modèle de l'ensemble pilote-véhicule a été développé à l'aide d'une approche par bond graphs afin d'étudier les RPCs aéroélastiques sur les axes latéral et de roulis. Cet outil a permis d'avoir une approche système de la modélisation qui est cruciale pour l'étude de systèmes dynamiques complexes. Avoir la possibilité de modéliser et de simuler le comportement d'un système physique est important lorsqu'il s'agit de valider un modèle vis-à-vis d'essais expérimentaux. Néanmoins, plus que le modèle par lui-même, c'est son analyse qui permettra d'améliorer la conception des hélicoptères. Presque par nature, un comportement aéroélastique émerge des aéronefs à voilure tournante, et celui-ci doit être pris en compte par les concepteurs qui doivent définir un domaine de vol stable dans lequel est incluse, avec des marges, l'enveloppe de vol de l'aéronef. Le développement de méthodes de calcul efficaces afin de s'assurer de la stabilité des aéronefs permettra d'explorer plus en profondeur l'espace de conception. Ceci permettra également de réduire le nombre d'essais en vol et les solutions curatives développées, par définition, tardivement dans le processus de conception. Ce chapitre pose les premières pierres de ce qui pourrait devenir une méthode d'analyse de la stabilité directement à partir d'un modèle non linéaire à l'aide de fonctions de Chetaev, à un coût de calcul potentiellement intéressant.
Formellement, la stabilité est un concept qui caractérise les positions d'équilibres d'un système dynamique. La Figure 5-1 montre des exemples d'équilibres d'une particule de masse m en présence uniquement de la gravité. Afin de qualifier la nature de l'équilibre, stable ou non, il faut imaginer le mouvement de la particule si on lui appliquait une force aussi faible que l'on peut le souhaiter. Si après perturbation, la particule revient à sa position d'équilibre, alors on peut dire que cet équilibre est stable. Mathématiquement, la définition proposée par (Perelmuter & Slivker, 2013) *est rappelée, voir développement sous la Figure 5-1.*
Par abus de langage et afin d'alléger l'écriture, on parlera souvent de « la stabilité du système » pour parler de la stabilité d'un équilibre donnée du système en question.
La théorie la plus générale pour étudier la stabilité de l'équilibre d'un système dynamique, décrit par des équations différentielles ordinaires, est la théorie d'Aleksandr Mikhailovich Lyapunov datant de la fin du XIX^ème^ siècle. Les méthodes de Lyapunov peuvent être appliquées à des systèmes d'équations non linéaires. Ces méthodes sont souvent classées dans deux catégories : les méthodes dites directe et indirecte. La méthode indirecte est appelée ainsi car elle demande une première étape de linéarisation autour d'un équilibre. Ensuite, le calcul des parties réelles des valeurs propres donnent l'information de la stabilité ou non de l'équilibre : si

toutes ces parties réelles sont strictement négatives alors l'équilibre est asymptotiquement stable. Par contre, si au moins une de ces parties réelles est strictement positive alors l'équilibre est instable. Le dernier cas possible est celui où toutes les parties réelles sont strictement négatives, sauf une qui est nulle. Dans ce cas, il n'est pas possible de conclure sur la nature de l'équilibre, des termes de plus haut degré sont nécessaires pour conclure.

Dans le cas où le système est non-autonome, c'est-à-dire que le temps apparaît explicitement dans les équations, la méthode précédente n'est pas directement applicable puisque le temps n'est a priori pas une variable d'état en mécanique classique.

Le cas particulier des équations linéaires à coefficients périodiques, voir équation (66), peut néanmoins être traité à l'aide de la théorie de Floquet (Bielawa, 2006). *Le traitement de ce type d'équations est intéressant dans le monde de l'hélicoptère, puisque même le modèle le plus simple de système rotor-fuselage mène à des équations différentielles à coefficients périodiques. De manière très synthétique, il peut être retenu que la méthode de Floquet demande de calculer la norme des valeurs propres d'une matrice intermédiaire, appelée matrice de monodromie. Les vecteurs colonnes de celle-ci sont des solutions indépendantes obtenues numériquement par un nombre d'intégration égal à la dimension du vecteur d'état et sur un temps égal à la période des équations. Si la plus grande norme des valeurs propres est strictement inférieure à 1 alors l'équilibre est asymptotiquement stable. Si la plus grande norme des valeurs propres est strictement supérieure à 1 alors l'équilibre est instable, voir équation (67).*

Afin de traiter le cas le plus général des systèmes non-autonomes, une méthode existe : il s'agit du calcul des exposants de Lyapunov (LCEs). (Bielawa, 2006) *pensait, que les outils de la théorie du chaos pourraient être utiles dans l'analyse de la dynamique des hélicoptères. Les exposants de Lyapunov sont des grandeurs utilisées dans littérature afin de mesurer la sensibilité aux conditions initiales, sensibilité qui caractérise les systèmes chaotiques. Très récemment* (Tamer & Masarati, 2015) *propose d'appliquer cette méthode à des modèles utiles dans le monde de l'hélicoptère. L'exemple de l'équation d'une pale qui bat et qui en même temps possède une vitesse d'avancement est reproduit, voir l'équation (68). Celle-ci est linéaire à coefficients périodiques et donc peut-être étudiée à l'aide de la théorie de Floquet, comme à l'aide des exposants caractéristiques de Lyapunov, voir Figure 5-2. Il faut noter que cette méthode peut aussi bien s'appliquer à des trajectoires obtenues par simulation numérique qu'à des résultats expérimentaux, voir* (Wolf, Swift, Swinney, & Vastano, 1985).

Cette méthode, bien que générale, ne possède pas de lien explicite avec l'énergie. D'autres méthodes de la littérature applicables aux systèmes mécaniques ont, elles, un lien bien plus fort avec l'énergie, voir les théorèmes 1 à 3 sous la Figure 5-2. Le théorème le plus connu parmi ceux-ci est celui de Lagrange-Dirichlet : ***si l'énergie potentielle d'un système conservatif est strictement minimale au voisinage d'un équilibre alors cet équilibre est stable****. Un rotor dont la vitesse angulaire est maintenue constante par une source extérieure est par définition non conservatif. Ce théorème trouvera donc peu d'applications à l'étude de la dynamique de l'hélicoptère. Mais comme dans toute histoire moderne liée à la stabilité, Lyapunov doit apparaître quelque part. Une généralisation du théorème de Lagrange-Dirichlet a été proposée dans ce que l'on appelle la méthode directe* de Lyapunov (Marquez, 2003). Cette méthode consiste à trouver une fonction appelée fonction candidate de Lyapunov qui vérifierait les 3 propriétés du théorème 5. Si l'on est capable de trouver une fonction qui vérifie ces propriétés alors ce théorème donne des conditions suffisantes de stabilité de l'équilibre autour duquel la fonction vérifie les propriétés. En conséquence, si l'on n'est pas capable de trouver une fonction de Lyapunov, on ne peut tout simplement pas conclure sur la stabilité ou non d'un équilibre. En 1961, Nikolai Gur'evich Cheatev (Marquez, 2003) proposa des conditions suffisantes d'instabilité que doivent vérifier une fonction candidate, voir théorème 6.

Afin d'illustrer l'intérêt que peut représenter l'application d'une telle approche à des problèmes plus complexes, il est proposé de commencer par traiter l'exemple classique du pendule simple. Ce système possède deux équilibres, l'un est stable, l'autre instable. Mais comment aurait-on pu le savoir sans faire aucune expérience ? Eh bien ceci aurait pu être conclu en prenant la fonction énergie totale ou Hamiltonien comme fonction candidate, voir équation (69) et Figure 5-3.

Plaçons-nous, mentalement, au voisinage de l'équilibre inférieur du pendule simple, $\alpha=0$ et à l'équilibre $\dot{\alpha}=0$, donc $H(0)=0$. En choisissant le domaine D, tel que $D=\{\{-\pi/2;\ \pi/2\},R\}$, dans $D - \{0\}$, $1-\cos\alpha>0$. De plus, $\dot{\alpha}^2 \geq 0$. Donc, dans $D - \{0\}$, $H(x)>0$. Finalement, $\dot{H}(x)$ est la puissance dissipée par le système. Dans ce cas, elle est nulle donc $\dot{H}(x) \leq 0$ dans $D - \{0\}$. Le théorème 5 permet donc de conclure que H est une fonction de Lyapunov sur D et que l'équilibre inférieur du pendule simple est une position d'équilibre stable.

Plaçons-nous maintenant au voisinage de la position d'équilibre supérieure, $\alpha=\pi$. Par définition, à l'équilibre, $\dot{\alpha}=0$, donc $H(\pi)-2mgl=0$. En posant comme fonction V candidate, non plus exactement H, mais H décalé d'une valeur statique, $V=H-2mgl$, on a bien $V=0$ à l'équilibre. Afin de pouvoir appliquer le théorème de Chetaev théorème 6, l'existence de deux conditions doit être prouvée. C'est ici que nous postulons que cette méthode à un intérêt en termes de puissance de calcul. En effet, les deux conditions d'existence qui doivent être vérifiée n'ont pas besoin d'être vérifiée tout autour de l'équilibre, comme pour une fonction de Lyapunov, mais sur n'importe quelle portion, de l'espace des phases, aussi petite soit-elle. Afin de montrer l'existence d'une solution x_0 et d'un domaine U comme l'exige le théorème, il est proposé de faire cela visuellement, voir Figure 5-4, bien que cela puisse aussi être démontré analytiquement. Afin de construire la Figure 5-4 (a), il faut imaginer la projection des lignes de niveaux (iso-énergie) de la Figure 5-3 (b). La partie (b) de la Figure 5-4, permet de distinguer un domaine gris, où V est strictement positive ; et un domaine blanc, où V est strictement négative. On ajoute alors une hypothèse : du frottement visqueux autour du pivot du pendule. La dérivée totale par rapport au temps de V est la puissance dissipée, $\dot{V}=\dot{H}=P_{dissipated} < 0$ dans R^2-{R, {0}} (ce domaine correspond à l'espace des phases auquel on retire la droite bleue en pointillés). Le signe de $\dot{V}$ est alors strictement négatif sur le domaine défini précédemment. Dès lors, en choisissant un des domaines U_i, de couleur orange, V et $\dot{V}$ sont de même signe et non nulles. De plus les domaines U_i ont un sommet qui part de l'équilibre, il existe donc un point x_0 dans le domaine U_i aussi proche que l'on peut le souhaiter de l'équilibre. Par conséquent, V est une fonction de Chetaev, ce qui nous permet de conclure que la position d'équilibre supérieure est instable.

***Une synthèse** des points évoqués jusqu'à maintenant s'impose. Les méthodes actuelles utilisées dans la communauté consistent à analyser la stabilité de systèmes uniquement à partir de modèles linéaires. Cette approche, limite de fait la prise en compte de non linéarités. Cela supprime aussi la possibilité de conclure sur la stabilité de l'équilibre d'un système, directement à partir d'un modèle non linéaire. Des méthodes permettant de s'affranchir de ces limites existent, comme le calcul des exposants de Lyapunov (LCEs). Mais cette méthode, bien que très générale n'a aucun lien explicite avec l'énergie, contrairement à la méthode directe de Lyapunov pour laquelle les fonctions de Lyapunov et de Chetaev sont souvent associés à des fonctions basées sur l'énergie. De plus, cette méthode n'a pas encore été appliquée à des problèmes associés aux hélicoptères, et permet, comme les LCEs, de conclure sur la stabilité d'un équilibre directement à partir d'un modèle non linéaire. Mais cette méthode possède deux désavantages non négligeables. Le premier est que cette méthode ne donne que des conditions suffisantes de stabilité ou d'instabilité. Donc si l'on n'est pas capable de trouver une fonction de Lyapunov ou Chetaev, rien ne peut être conclu en termes de stabilité. Le deuxième désavantage est qu'il n'existe pas de méthode systématique pour trouver une fonction de Lyapunov ou Chetaev. Une personne censée se serait probablement arrêtée là et aurait passé son chemin. Mais l'application du théorème de Chetaev à l'exemple du pendule simple montre qu'une fonction candidate de Chetaev ne doit vérifier des propriétés que sur une portion de l'espace des phases autour d'un équilibre. Si l'on devait reconstruire la surface (b) de la* Figure 5-3 *numériquement, la reconstruction d'une portion de cette surface aurait déjà suffi à conclure sur l'instabilité de l'équilibre, voir (b) de la* Figure 5-6.

*La synthèse du paragraphe précédent nous permet donc de formuler plus précisément notre **proposition de méthode**. Dans le cas du pendule simple, l'utilisation d'une simulation temporelle afin de reconstruire l'allure de l'énergie totale au voisinage d'un équilibre n'est pas la méthode la plus efficace pour arriver à conclure sur la stabilité. Mais déterminer le signe d'une fonction comme l'énergie totale dans le cas d'un système non linéaire de grande dimension peut se révéler impossible de manière analytique. Dès lors, une simulation temporelle peut s'avérer indispensable pour apporter une réponse aux questions de stabilité.*

1. *La première proposition consiste à vérifier le signe d'une éventuelle fonction candidate à l'aide d'une simulation temporelle.*

Une fois qu'une fonction de Chetaev est trouvée, le temps de calcul peut être réduit au temps minimal d'apparition du domaine U au sens du théorème 6. Etant donné que les systèmes physiques que nous étudions présentent des instabilités paramétriques (voir chapitre précédent),

2. *La deuxième proposition consiste à réaliser des balayages paramétriques du modèle du système physique, pour des paramètres qui sont en général des paramètres de conception de l'hélicoptère, et de vérifier à chaque fois le signe de la fonction candidate et de sa dérivée à l'aide d'une simulation numérique.*

a. *Si le domaine U peut encore être trouvé, la fonction candidate est une fonction de Chetaev.*
b. *Sinon, rien ne peut être conclu quant à l'équilibre du système dynamique.*

Le problème de trouver un moyen systématique d'obtenir l'expression d'une fonction de Chetaev n'est pas résolu. Néanmoins, des fonctions candidates peuvent être proposées selon le problème à traiter. Par exemple, comme illustré dans la suite du chapitre, une fonction construite à partir de l'énergie totale, ou Hamiltonien, semble une bonne fonction candidate dans le cas du phénomène de « résonance sol ».
L'approche proposée ici peut bien évidemment être appliquée à d'autres représentations du système que le bond graphs. Néanmoins, l'idée proposée ici est née en construisant des bond graphs, avec lesquels finalement, des flux d'énergies sont tracés. De plus, cette idée a été inspirée par les travaux de (Junco, 1993) *qui propose d'extraire des fonctions de Lyapunov directement à partir des bond graphs. Il est donc revendiqué ici que, même si l'approche peut être appliquée à un autre type de représentation du système, les bond graphs ont constitué une étape intellectuelle nécessaire à l'émergence de cette idée.*
Le reste du chapitre est consacré à l'illustration de la proposition en application au cas de la résonance sol, un phénomène dynamique important dans le monde de l'hélicoptère. Le point de départ de cette dernière partie est l'approche classique que l'on utilise, depuis 40 ans maintenant, afin de prédire le phénomène. Le modèle possède 3 degrés de libertés après application du changement de variable de Coleman, voir Figure 5-7 *et les 3 équations du mouvement (77). La résonance sol, est une instabilité paramétrique, bien qu'elle soit appelé « résonance », voir le diagramme de Campbell en Appendix 6 (la perte d'amortissement du mode régressif de traînée autour de la vitesse nominale du rotor ; c'est-à-dire une vitesse réduite égale à 1 sur le diagramme). Pour le jeu de données présenté en* Table 5, *le mode régressif de traînée est instable, voir* Figure 5-8.
La méthode développée ici, propose de retrouver ces zones d'instabilités paramétriques à partir de la représentation bond graph du système et de résultats de simulations dans le domaine temporel au voisinage de l'équilibre de ce modèle non linéaire. Dans un premier temps, il est justifié pourquoi le Hamiltonien, somme de l'énergie cinétique et potentielle du système en question, est une fonction candidate intéressante dans le cas de la résonance sol. En effet, dans (Oberinger & Hajek, 2013)*, il est affirmé que « la condition de la présence d'une instabilité dynamique dans un système mécanique, c'est-à-dire l'apparition d'oscillations divergentes, est une augmentation de l'énergie interne du système ». L'énergie interne est dans ce cas l'énergie totale ou Hamiltonien. Afin de mieux comprendre cette affirmation, deux résultats de simulation sont présentés* Figure 5-9*. En bleu, ce sont les résultats lorsque le système est paramétré selon les valeurs nominales ; on constate, comme on peut s'y attendre, que la réponse libre du système est divergente après une petite perturbation appliquée sur la masse représentant le fuselage. En effet pour ces paramètres, l'équilibre est prédit instable par le modèle linéaire. En orange, voir* Figure 5-9*, l'amortissement des amortisseurs de torsions, voir* Figure 5-7 *est multiplié par 3 afin d'éviter le phénomène de résonance sol : après perturbation le système revient à sa position d'équilibre. La dernière figure*

de Figure 5-9*, montre l'évolution du Hamiltonien. Dans le cas stable, le Hamiltonien est décroissant alors qu'il est croissant dans le cas instable. Pour* (Oberinger & Hajek, 2013) *cette croissance est une condition suffisante d'instabilité. Ceci est très intéressant puisque si le Hamiltonien était strictement croissant, ce serait automatiquement une fonction de Chetaev. Mais le Hamiltonien n'est pas strictement croissant, voir* Figure 5-10*, ce n'est donc pas une fonction de Chetaev et on ne peut pas l'utiliser en l'état pour conclure sur l'instabilité de l'équilibre. Afin de rendre cette fonction monotone, il est proposé de calculer sa régression linéaire sur 4 périodes de rotor, après avoir attendu 6 périodes de rotor de régime transitoire, voir* Figure 5-10 *et équation (80). De cette manière, et sur la portion longue de 4 périodes de rotor, ce qui semblerait être une fonction de Chetaev est obtenue. Il est proposé, comme pour l'exemple du pendule simple en (b)* Figure 5-6*, d'observer H et sa dérivée totale par rapport au temps, dans le plan* ($\dot{V}$, V)*, voir* Figure 5-11*. Dans le cas 2, en orange, il doit être noté, que la fonction ainsi construite n'est ni Chetaev, ni Lyapunov et que rien ne peut être conclu quant à la stabilité de cet équilibre. Pour être rigoureux il faut noter que la construction de la fonction de Chetaev proposée ici est incomplète puisqu'elle devrait être continue jusqu'à l'équilibre. Ceci reste donc en suspens et devra être abordé dans le futur. Néanmoins afin de vérifier le potentiel de l'approche proposée, la méthode a été appliquée sur un balayage plus important de paramètres, voir* Figure 5-12*. Afin d'estimer le domaine d'instabilité, il faut ne conserver que les points pour lesquels β est positif, ce qui permet d'obtenir le domaine d'instabilité en blanc de la* Figure 5-13*. Sur la même figure, ce domaine est comparé au domaine que l'on aurait obtenu à l'aide du modèle linéaire et d'un calcul de valeurs propres. On peut constater que les résultats sont très proches. Les points qui ne coïncident pas sont uniquement à la frontière du domaine. Sur cette frontière la méthode linéaire n'est plus valable. Mais la méthode proposée ici utilise une méthode numérique (BDF) qui elle-même possède des limites de stabilité et de précision dont il faudrait, dans le futur, quantifier l'impact.*

Finalement, les équations (83) à (86) donnent l'expression analytique du Hamiltonien à partir des variables d'état qui sont utilisées en bond graph. Le modèle bond graph du système rotor-fuselage est complétement développé Figure 5-14*. On peut notamment observer que des capteurs d'énergie ont été placés au pied des éléments de stockage I et C et des capteurs de puissance au pied des éléments de dissipation R et de la source d'énergie MSf, qui maintient le rotor à vitesse constante. La somme des valeurs indiquées par les capteurs d'énergie donne la valeur du Hamiltonien, et la somme des valeurs indiquées par les capteurs de puissance donne la dérivée totale par rapport au temps du Hamiltonien, qui n'est autre que la différence entre la puissance fournie par le moteur et la puissance dissipée par les amortisseurs de traînée et du train d'atterrissage. De ce point de vue, les résultats numériques montrent que, lorsque l'instabilité apparaît, le système se réorganise de telles manières qu'il tente de tirer de plus en plus d'énergie de la source, c'est-à-dire du moteur, afin de stocker celle-ci sous forme d'énergies cinétique et potentielle. Sur un système réel, les pièces mécaniques sont capables de stocker une quantité finie d'énergie après quoi leur rupture est inévitable, d'où l'intérêt des concepteurs d'éviter ce type de phénomènes.*

In the previous chapter, in order to investigate lateral-roll aeroelastic RPCs, a pilot-vehicle model has been proposed using bond graphs. This approach has allowed to have a system approach of modeling which is crucial when investigating complex dynamic systems. Having the possibility to model and simulate a model is important especially to validate models against experiments. However, more important than the model itself it is its analysis that will help designers improving rotorcraft designs. More than fifty years ago, (Chetaev, 1960)[21] already stated "*In modern engineering there arise new and increasingly more complex problems concerning the stability of motion. Looking at the past and anticipating the future, one can see that in order to keep up with technological progress it will be necessary to develop more and more precise methods for the investigation of these stability problems*". By nature rotorcraft exhibit an aeroelastic behavior for which designers need to define its stability domain that contains the rotorcraft flight envelop with some margins. The development of computationally efficient methods to investigate the stability of rotorcrafts will allow a deeper exploration of the design space in order to reduce flight tests costs and curative solutions costs that are by definition developed too late in the design process.

5.1. Definition

Formally, stability is a concept that characterizes the equilibrium positions of a system[22]. Imagine what would be the motion of the particles of mass *m* if they were slightly deviated from their equilibrium,

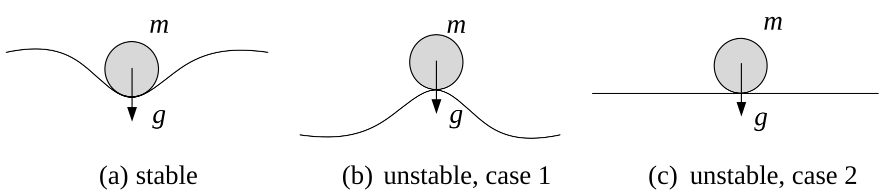

Figure 5-1. Simple examples of stable and unstable equilibriums

A definition of the equilibrium stability concept "*is connected with motions which a system is capable of making after it is moved from its equilibrium by having its points receive very small initial deviations from the equilibrium position and very small initial velocities. If, after this violation of equilibrium, the system will deviate very little in its subsequent motion from the equilibrium position of interest, then this position of equilibrium is said to be stable*", (Perelmuter & Slivker, 2013).

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[21] Nikolai Gur'evich Chetaev was a professor of Mechanics between 1930 and 1940, and gave among others, lectures on aircraft stability (Chetaev, 1960) at the University of Kazan; major city of the Russian helicopter industry.

[22] For simplicity we will very often speak about *the system's stability*, but one should keep in mind we refer to *the stability of a specific equilibrium of the system*.

Mathematically speaking, following (Perelmuter & Slivker, 2013): let $\mathbf{q}_0$ be the vector of a system's generalized coordinates for a given equilibrium. This state is said to be stable if for any small number ε there exists a respectively small number δ such that for any perturbations of the generalized coordinates, $\delta\mathbf{q}_0$, and initial velocities $\delta\dot{\mathbf{q}}_0$ which satisfy the conditions,

$$\left\|\delta\mathbf{q}_0\right\| < \delta, \quad \left\|\delta\dot{\mathbf{q}}_0\right\| < \delta$$

The following inequalities will hold at any time t,

$$\left\|\mathbf{q}(t) - \mathbf{q}_0\right\| < \varepsilon, \quad \left\|\dot{\mathbf{q}}(t)\right\| < \varepsilon$$

In other words, for an equilibrium to be stable, the generalized coordinates should not go beyond the ε neighborhood of the initial equilibrium state in the course of the system's subsequent motion. The same neighborhood should contain also the generalized velocities of the system which were zero in the initial state of equilibrium.

An equilibrium is said to be unstable if it is not stable.

It is critical to be able to determine as soon as possible in the design process whether a rotorcraft design is prone to any dynamic instability. Ideally, this should be assessed based on a model that represents the rotorcraft with the highest physical fidelity in terms of subsystems behaviors (structural mechanics, aerodynamics, hydraulics, pilot biodynamics, etc.) and independently from the linearity or nonlinearity of their associated mathematical models (due to geometry, large displacements or material behavior) at the lowest computational cost. In the rotorcraft community, stability studies usually limit to the investigation of linear models.

5.2. State-of-the art

The most general theory to investigate the stability of equilibriums of ordinary differential equations has been introduced by Aleksandr Mikhailovich Lyapunov at the end of the 19th century. Lyapunov's methods have the power of being applicable to nonlinear systems. His methods are usually classified in the direct and the indirect categories.

5.2.1. Lyapunov's indirect method

The indirect method allows to conclude on the local stability of an equilibrium of a nonlinear system $\dot{x}=f(x)$ by considering its linearization $\dot{x}=A.x$ around the equilibrium (Perelmuter & Slivker, 2013), A being the classic state-space matrix.

i. *If the real parts in all roots of the characteristic equation of the linearized system are negative, then the equilibrium of the nonlinear system is asymptotically stable*

ii. *If the real part in at least one root of the characteristic equation of the linearized system is positive, then the equilibrium of the nonlinear system is unstable*

However, one should keep in mind, that in the case the roots of the characteristic equation of the linearized system contain some purely imaginary ones, while all the others have negative real parts, the first approximation equations are not enough to decide whether the equilibrium is stable or not. In this case, higher order terms are necessary to conclude.

5.2.2. Periodic systems

Historically (Peters, Lieb, & Ahaus, 2011), back at the end of the 19th century, scientists investigated the stability of natural satellites' orbits around a planet under periodic variation in gravitational force. The problem can be casted in its simplest form in the following periodic differential equation, known as Mathieu equation,

$$\ddot{x}(t)+\left[\omega^2+\varepsilon\sin(t)\right]x=0 \tag{66}$$

It has a period of 2π and for $\varepsilon=0$ becomes aperiodic or constant. This kind of time periodic equations arise in rotating systems and especially in helicopter systems. The behavior of time periodic equations is particular in the sense it can lead to *parametric resonance* or *parametric instabilities*; this will be illustrated later with ground resonance.

To study the stability of periodic systems, many approaches exist (Peters, Lieb, & Ahaus, 2011). In the case of a rotor-airframe system, if the rotor is not isotropic and the equilibrium of the aircraft is far from hover flight or cannot be considered at low advancing speeds, the rotorcraft mathematical model takes after linearization the shape of a state-space system $\dot{x}=A(t).x$, with a time-periodic state space matrix, with $A(t+T)=A(t)$ where $T=2\pi/\Omega$ and Ω is the rotor angular velocity (Bielawa, 2006). In the rotorcraft and wind turbines community, the most popular method to investigate the stability of such systems was proposed by Gaston Floquet in 1883.

Floquet theory and linear time periodic systems

In this section, the Floquet method is presented (Bielawa, 2006), (Coisnon, 2014).
Let be a linear time periodic (LTP) system $\dot{x}=A(t).x$, with $A(t+T)=A(t)$ and $x \in R^n$.

When the system is linear-time invariant (LTI), time does not appear explicitly in *A* and a stability analysis can be resumed to an eigenvalue resolution. In the case of an LTP this is not possible anymore. Floquet solves the problem by introducing a matrix **B**, called monodromy matrix that can be constructed by concatenating *n* independent solutions of the system at time T. The monodromy matrix characterizes the system state over 1 period. The *n* independent solutions are *n* vectors that can be obtained numerically by integrating *n* times for *n* different initial conditions, over a time interval between *0* and *T* with the help of a numerical method. The determination of the stability of an equilibrium can be assessed by computing the absolute value of the eigenvalues λ_i of matrix **B**,

$$\begin{cases} \max(|\lambda_i|) > 1 \Rightarrow \text{unstable} \\ \max(|\lambda_i|) < 1 \Rightarrow \text{asymptotically stable} \end{cases} \tag{67}$$

Computationally speaking one should note this method demands to compute *n* numerical solutions of the system over one period *T* of the system, and then run an eigenvalue analysis of the monodromy matrix. When the number of degrees of freedom of the system rises, the computational cost might become excessive.

5.2.3. Lyapunov Characteristic Exponents

As imagined by (Bielawa, 2006) chaos theory tools are progressively emerging in the field of rotorcraft dynamics field: "*the engineering application of chaos theory are [...] in their infancy*", "*For most applications the existence of chaotic motion in a system would be, like vibrations and aeroelastic instability, a response condition to be avoided and/or designed out of the system*". It is the case, very recently in (Tamer & Masarati, 2015), in which Lyapunov Characteristic Exponents (LCEs) are proposed and applied to the investigation of stability of helicopter problems such as ground resonance with the inclusion of nonlinearities. LCEs are also used in literature to measure the sensitivity of solutions of dynamical systems to small perturbations which characterize chaotic systems. This method can be applied directly to: dynamic systems which are non-autonomous[23], time simulation results or time series from experiments (Wolf, Swift, Swinney, & Vastano, 1985). Stability estimation using LCEs can be seen as a generalization of conventional stability analysis of linear time invariant (LTI) or linear time periodic systems (LTP).
This is illustrated in particular in (Tamer & Masarati, 2015), in which a rigid blade flapping motion stability is investigated for varying values of advancing ratio μ,

$$\ddot{\beta} + \frac{\gamma}{8}\left(1 + \frac{4}{3}\mu\sin(t)\right)\dot{\beta} + \left(\nu_\beta^2 + \frac{\gamma}{8}\left(\frac{4}{3}\mu\cos(t) + \mu^2\sin(2t)\right)\right)\beta = 0 \tag{68}$$

---

[23] A non-autonomous dynamic system is system where the time *t* appears explicitly in the equations of motion, $\dot{x}=f(x,t)$.

The Figure 5-2 shows that the Floquet theory method and the estimation of LCEs give the same quantitative results.

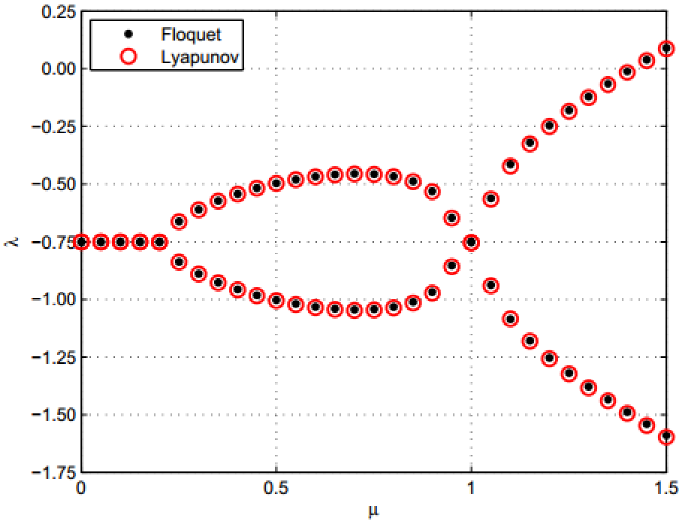


Figure 5-2. Lyapunov Characteristic Exponents and Floquet multipliers logarithm estimations for varying advancing ratios μ from (Tamer & Masarati, 2015)

5.2.4. Potential energy theorems

There exists in literature potential energy based theorems that give sufficient conditions of stability and instability (Perelmuter & Slivker, 2013). The most famous one is the Lagrange-Dirichlet[24] theorem that can be applied to conservative systems.

Theorem 1 - Lagrange-Dirichlet

If the potential energy of a system takes a strictly minimal value in the vicinity of an equilibrium state of interest, then this state of equilibrium of the conservative mechanical system is stable.

Remark

The potential energy has to be strictly minimal and not just minimal. An example of minimal potential energy that leads to an unstable equilibrium is the case 2 of Figure 5-1.

---

[24] The theorem was first proposed and demonstrated by Lagrange (1788) by neglecting higher order terms and later demonstrated for any order by Lejeune-Dirichlet (1846).

Theorem 2 - Lyapunov

The equilibrium is unstable if the absence of a potential energy minimum can be recognized by second-order terms in the expression of the potential energy, without the need to consider higher-order terms, (Perelmuter & Slivker, 2013).

Theorem 3 - Lyapunov

The equilibrium is unstable if the potential energy takes a maximum value, and the presence of this maximum can be found out from the lowest-order terms in the expansion of the potential energy into a power series, (Perelmuter & Slivker, 2013).

Theorem 4 –Chetaev's instability criterion generalization

If the potential energy of a non-conservative system does not have its minimum in a state of equilibrium, and this follows from considering lowest-order terms of the energy expansion, then this state of equilibrium is unstable, (Perelmuter & Slivker, 2013).

5.2.5. Lyapunov's direct method

Lyapunov's direct method consists in finding a candidate function that verifies three simple properties, see Theorem 5 (Marquez, 2003). If one is able to find such a function, then it is a sufficient condition for the stability of the equilibrium. Since this method gives only sufficient conditions of stability, as it is, not finding a Lyapunov function does not mean the equilibrium is unstable. In 1961, Chetaev proposed converse properties for candidate functions that give sufficient conditions of instability (Marquez, 2003).
A major barrier for the use in practical problems is there is no systematic method to find Lyapunov or Chetaev functions. Computationally speaking a major advantage of the method is there is no need to linearize the equations or compute eigenvalues and no time simulations need to be performed.

Theorem 5- Lyapunov functions

Let $x=0$ be an equilibrium point of the dynamic system described by $\dot{x}=f(x)$, and let be a function $V: D \rightarrow R$, where D is a region of R^n around 0 such that,

i. $V(0) = 0$
ii. $V(x) > 0$ in $D - \{0\}$
iii. $\dot{V}(x) \leq 0$ in $D - \{0\}$

Thus $x = 0$ is stable.
Furthermore if $\dot{V}(x) < 0$ in $D - \{0\}$, then the equilibrium is asymptotically stable.

Theorem 6 – Chetaev functions

Let $x=0$ be an equilibrium point of the dynamic system described by $\dot{x}=f(x)$, and let be a function $V: D\rightarrow R$, where D is a region of R^n around 0 such that,

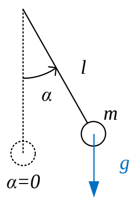
i. $V(0) = 0$

ii. $\exists\, x_0 \in R^n$, *arbitrarily close to* $x=0$ *such that* $V(x_0)>0$

iii. $\dot{V} > 0$ *for all* $x \in U$, *where* $U=\{x \in D: \|x\| \leq \varepsilon$, *and* $V(x) > 0\}$

Under these conditions, $x = 0$ is unstable.

Example of application, the simple pendulum equilibriums



Let us imagine a simple pendulum of mass m parameterized by an angle α in the gravity field in the absence of friction. The total energy of the system, sum of kinetic and potential energy is its Hamiltonian H,

$$H = \frac{1}{2} m \left(l \dot{\alpha} \right)^2 + mgl \left(1 - \cos \alpha \right) \tag{69}$$

The nonlinear equation of motion of the system can be put in the form,

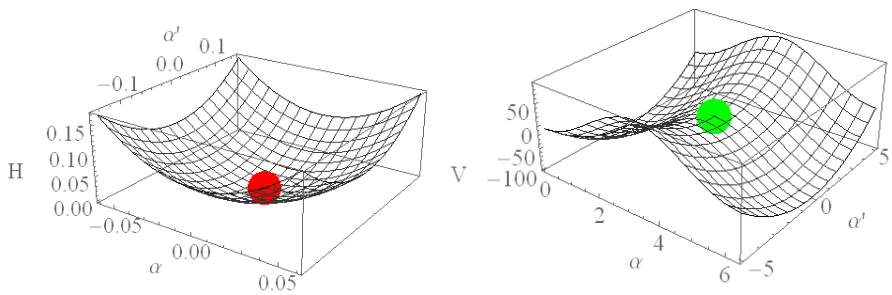
$$ml^2 \ddot{\alpha} + mgl \sin \alpha = 0 \tag{70}$$

And in the form of an autonomous system,

$$\dot{\mathbf{x}} = f(\mathbf{x}) \text{ with, } \mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} \alpha \\ \dot{\alpha} \end{Bmatrix} \text{ and } f \text{ such that} \begin{cases} \dot{x}_1 = \alpha \\ \dot{x}_2 = -\dfrac{g}{l} \sin x_1 \end{cases} \tag{71}$$

Lower equilibrium, $\alpha=0$

By definition at equilibrium $\dot{\alpha}=0$, therefore $H(0)=0$. By choosing $D=\{\{-\pi/2;\ \pi/2\},R\}$, in $D - \{0\}$, $1\text{-}\cos\alpha>0$. In addition $\dot{\alpha}^2 \geq 0$. Therefore, in $D - \{0\}$, $H(x)>0$. Finally, $\dot{H}(x)$ is the power dissipated by the system which in this case is null, therefore $\dot{H}(x) \leq 0$ *in* $D - \{0\}$. Theorem 5 allows to conclude that H is a Lyapunov function on D and that the lower equilibrium position of the system is stable.

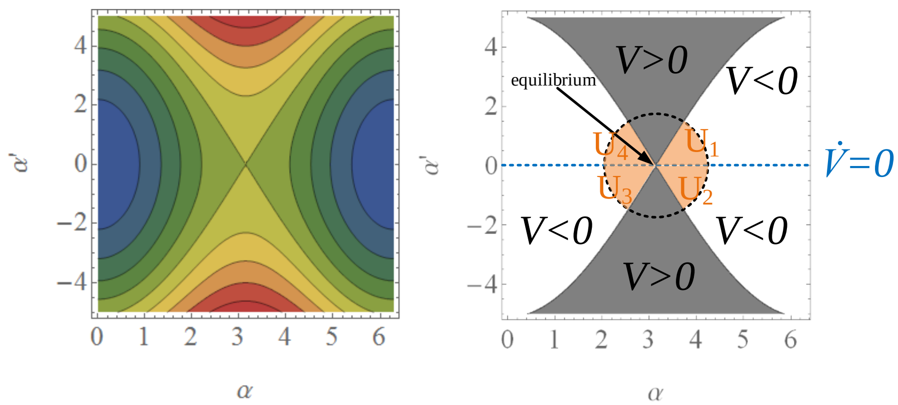


(a) Lower equilibrium, $\alpha=0$ (b) Upper equilibrium, $\alpha=\pi$

Figure 5-3. Hamiltonian in phase space around equilibriums of simple pendulum

Upper equilibrium, $\alpha=\pi$

By definition at equilibrium $\dot{\alpha}=0$, therefore $H(\pi)\text{-}2mgl=0$. By choosing $V=H\text{-}2mgl$, $V=0$ at equilibrium. In order to be able to apply Chetaev's theorem, the two existence conditions on x_0 and U need to be proved. This could be done analytically, however an explanation through figures is proposes, see Figure 5-4.



(a) Iso-V around $\alpha=\pi$ in phase space (b) Existence of at least one cone U in phase space

Figure 5-4. Geometric interpretation of Chetaev's theorem

To construct Figure 5-4 (a), one needs to imagine the isocurves of Figure 5-3 (b) projected on the phase space: in blue the energy is minimal and corresponds to the lower equilibrium points. On Figure 5-4 (b), colors distinguish two important domains: when Chetaev's candidate function is positive and negative definite. On the frontiers of Figure 5-4 (b), $V=0$. Since the orange domain takes its origin at the equilibrium, one can choose a point in the orange domain where $V<0$ as close as one can imagine close to the equilibrium: x_0 therefore exists.

Let us add some friction proportional to angular velocity to the system; the total time derivative of V is the power dissipated by friction: $\dot{V}=\dot{H}=P_{dissipated} < 0$ for x *in* R^2-{R, {0}}[25]. The sign of $\dot{V}$ is therefore fixed. By choosing U equal to one of the orange domains U_i is sufficient to find a domain where V and $\dot{V}$ have the same sign and are not null. V is therefore a Chetaev function and the equilibrium is unstable.

5.3. Synthesis

5.3.1. Challenges

Actual methods in the rotorcraft community consist in investigating the stability of linear systems. This approach limits the nonlinearities that can be taken into account. It also demands to actually linearize a model suppressing the possibility to conclude on the stability without manipulating extensively its mathematical model when possible. To unlock these limits as imagined by (Bielawa, 2006) chaos theory tools are progressively emerging in rotorcraft dynamics field: "*the engineering application of chaos theory are […] in their infancy*", "*For most applications the existence of chaotic motion in a system would be, like vibrations and aeroelastic instability, a response condition to be avoided and/or designed out of the system*". It is the case, very recently in (Tamer & Masarati, 2015), in which Lyapunov Characteristic Exponents (LCEs) are proposed and applied to the investigation of stability of helicopter problems such as ground resonance with the inclusion of nonlinearities. This method can be applied directly to time simulation results or time series from experiments. However, this very general method is not explicitly related to the energy relations of a given physical system.

Another method that has not been applied yet to the investigation of rotorcraft dynamics and that we propose to apply is the use of Lyapunov's direct method for which usually the candidate Lyapunov and Chetaev functions are energy-based functions. This method allows to conclude on the stability of equilibriums from a nonlinear set of ordinary differential equations. However, it gives sufficient conditions of stability and can therefore only generate conservative results. Furthermore, there is no systematic method for finding Lyapunov or Chetaev functions, the method is therefore not systematic.

[25] This domain corresponds to the plane without the blue dashed line on Figure 5-4 where the power dissipated is null.

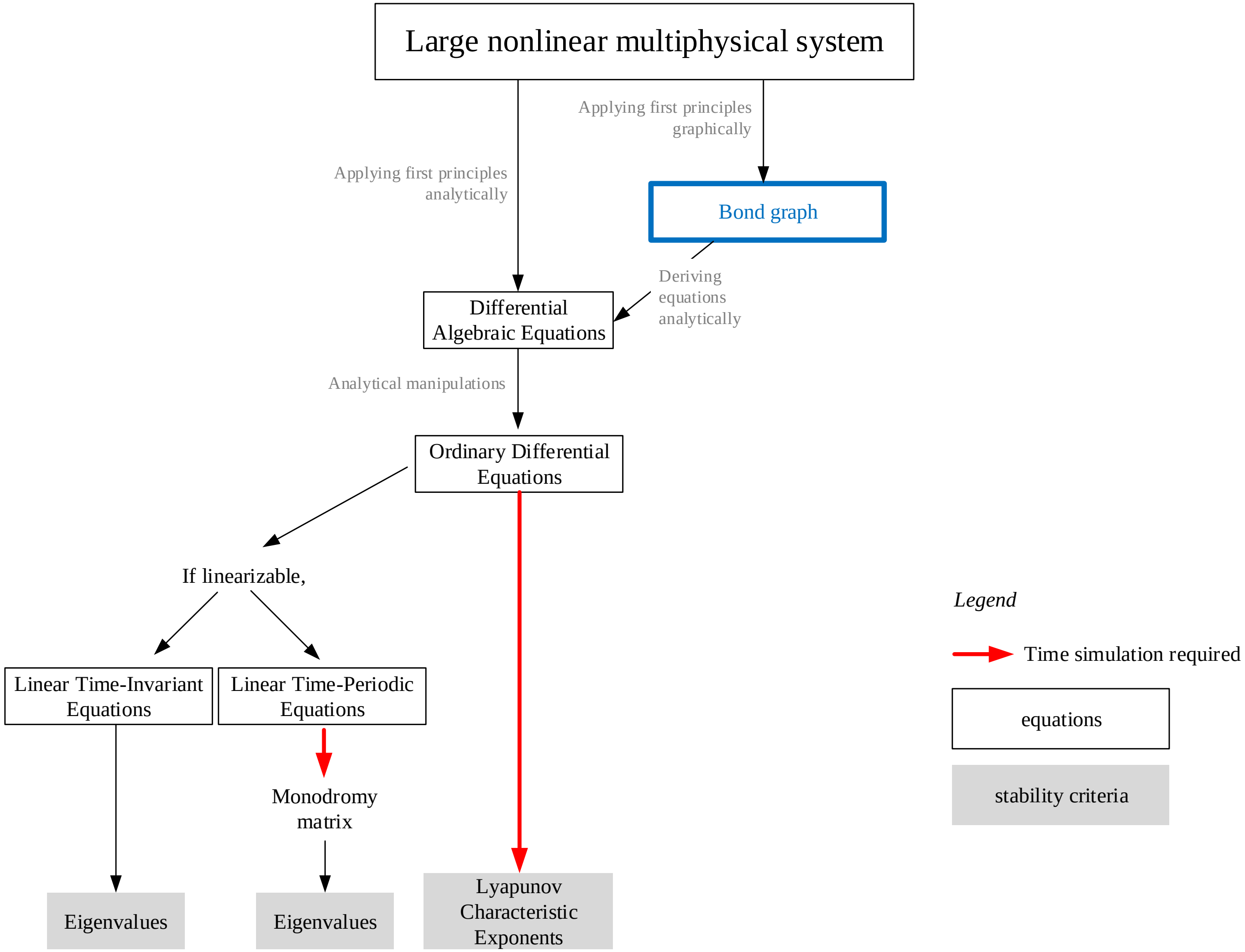


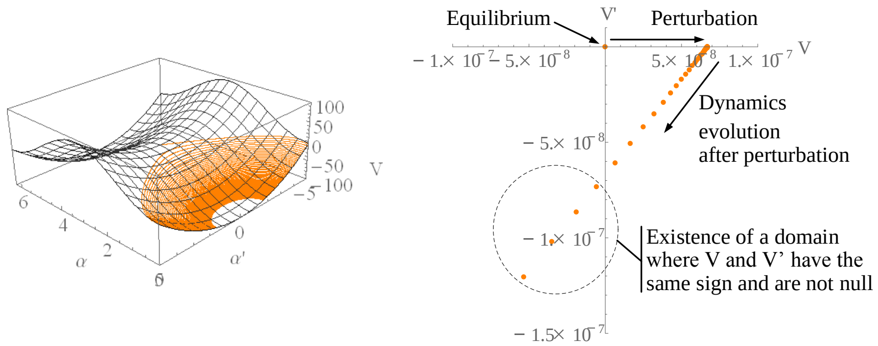
Figure 5-5. Synthesis of methods to evaluate the stability/instability of equilibriums

5.3.2. Proposal

An analysis of Chetaev's theorem proposed in the next lines reveals that proving the instability of an equilibrium using his theorem can be done with the help of a numerical simulation at a potentially interesting computational cost compared to other methods.

The justification of stability using Lyapunov candidate functions is more demanding in the sense the conditions need to be proved in every point of a domain *D-{0}* (which is the domain of stability). Whereas justifying the instability of an equilibrium demands to prove the existence of a small "cone" of vertex the equilibrium (Shnol, 2007) where V and $\dot{V}$ have the same sign and are not null.

The potential computational usefulness of this remark is proposed to be used and can be understood geometrically from Figure 5-3. Let us suppose the total time derivative of the candidate function is fixed. If we were to reconstruct the surfaces around equilibrium point by point to prove stability we would need to obtain points everywhere around the equilibrium to make sure its shape is a paraboloid of revolution (Figure 5-3 (a)) and not a hyperbolic paraboloid (Figure 5-3 (b)). While finding a domain U where V and $\dot{V}$ have the same sign and are not null is sufficient to prove instability.



(a) Candidate function in phase space after a small perturbation

(b) Existence of a domain U around equilibrium, 1.5s after a small perturbation, 0.011s of computation time

Figure 5-6. Sign of Chetaev candidate function determination using a numerical simulation

In the case of the pendulum, the use of a numerical simulation to prove the existence of a domain U is not necessary because it could have been done more efficiently analytically. However when it comes to large nonlinear systems, trying to find analytically the sign of a candidate function and its total time derivative, V and $\dot{V}$, around equilibrium might become impossible.

1. The first proposal consists in checking the sign of eventual candidate functions with the help of a numerical simulation around equilibrium

Once a suitable Chetaev candidate function is found, the computation time[26] can be reduced to a minimal time where the existence of the domain U appears. Since the physical systems that we are interested in present parametric instabilities,

2. The second proposal consists in performing parametric sweeps of the physical system model, which are usually helicopter design parameters, and verifying each time the behavior of the candidate function using numerical simulations.
    a. If a domain U can be found, the candidate function is a Chetaev function. Quantitative information about the instability of the equilibrium can be obtained by memorizing the value of $\dot{V}$. If one chooses V as H, the Hamiltonian, or total energy, sum of kinetic and potential energies, $\dot{V}$ is the difference between the power input and power output of the system through its boundaries.

    b. If a domain U cannot be found, one should keep in mind one cannot, theoretically speaking, conclude whereas the equilibrium is stable or not.

The weakness of Lyapunov's direct method for which there is no systematic method to find such a function is not solved. However, problem dependent candidate functions can be proposed. A candidate function is proposed for the special problem of 'helicopter ground resonance' in this chapter.

The approach proposed here can obviously be applied to other representations than bond graphs. However, this idea emerged while mapping energy flows in bond graphs and reading (Junco, 1993) in which Lyapunov functions are proposed to be extracted from BGs. It is therefore claimed that even if the approach can be applied to other representations, bond graphs were a necessary intellectual step to arrive to this idea.

---

[26] The computation time of this method still needs to be compared to the computation time needed by other methods such as Lyapunov Characteristic Exponents, in order to better position it

5.4. Illustration, 'helicopter ground resonance' instability

The rest of this chapter is devoted to an application of the method proposed in the previous section. The application concerns helicopter ground resonance, which is a major dynamic phenomena of interest in rotorcraft dynamics. The starting point of this section dedicated to the ones that do not work on helicopters is the most classic ground resonance 3 degrees of freedom analytical model and its parametric instability analysis using a Campbell diagram known for more than 40 years now. Then the choice of a Hamiltonian based function of the system is discussed to be an interesting Chetaev candidate function. Time simulations of H for different parameter sets are computed around equilibrium and the soundness of the approach is illustrated.

5.4.1. A mechanical parametric instability

The simplest example consists in deriving the equations of a rotor on moving base in the absence of aerodynamic forces, see (Krysinski & Malburet, 2011).

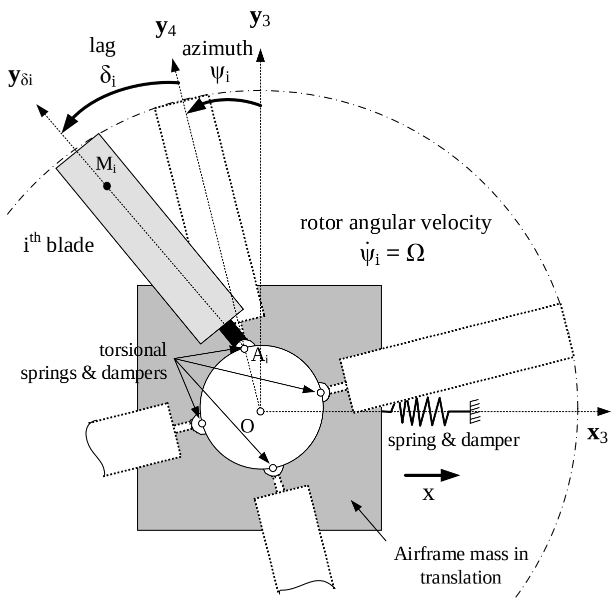


Figure 5-7. Simplified rotor-airframe model for ground resonance, 1 lag dof per blade, 1 translation dof for the airframe

Supposing the rotor has b blades, a massless hub-mast and a constant rotor angular velocity. By expressing the potential energy of the system, which is the sum of the contributions of the energy stored in the airframe landing gear represented by the spring & damper of characteristics k_f and c_x and the so-called rotor lag dampers represented by equivalent torsional springs & dampers of characteristics k_δ and c_δ,

$$V=\frac{1}{2}k_f x^2+\sum_i^b \frac{1}{2}k_\delta \delta_i^2 \tag{72}$$

By expressing the dissipation energy of the system,

$$D=\frac{1}{2}c_x \dot{x}^2+\sum_i^b \frac{1}{2}c_\delta \dot{\delta}_i^2 \tag{73}$$

Its kinetic energy,

$$T=\frac{1}{2}M_f \mathbf{V}^2{}_{G\in airframe/Rg}+\sum_i \frac{1}{2}\int_{blade_i} \mathbf{V}^2{}_{M\in blade_i/Rg}\,dm \tag{74}$$

The derivation of Lagrange equations and a linearization around small lag angles gives, for the airframe lateral motion,

$$\begin{aligned} &m_f\ddot{x}+bm_{bl}\ddot{x}+c_x\dot{x}+k_f x+m_s\Omega^2\sum_{i=1}^b \cos(\Omega\,\mathrm{t}+\frac{2\pi(\mathrm{i}-1)}{b})\delta_i \\ &+2m_s\Omega\sum_{i=1}^b \sin(\Omega\,\mathrm{t}+\frac{2\pi(\mathrm{i}-1)}{b})\dot{\delta}_i-m_s\sum_{i=1}^b \cos(\Omega\,\mathrm{t}+\frac{2\pi(\mathrm{i}-1)}{b})\ddot{\delta}_i=0 \end{aligned} \tag{75}$$

And for each i^{th} blade lag motion,

$$I_{bl}\ddot{\delta}_i+c_\delta\dot{\delta}_i+k_\delta\delta_i+em_s\Omega^2\delta_i-m_s\cos(\Omega\,\mathrm{t}+\frac{2\pi(\mathrm{i}-1)}{b})\ddot{x}=0 \tag{76}$$

In which one can see the time periodic coefficients in front of airframe generalized coordinates in blade equations and vice versa, the system of equations is a Linear Time Periodic one. Under the conditions of an isotropic rotor, hover flight or low advancing speeds, rotorcraft mathematical systems can be in addition put in the form of a linear time invariant (LTI) systems by using the multiblade coordinate transformation Appendix 2, and keep only the coupled equations,

$$\begin{cases} \left(bm_{bl}+m_f x\right)\ddot{x}+\frac{1}{2}bm_s\ddot{\delta}_{1s}+c_x\dot{x}+k_f x=0 \\ \frac{1}{2}bI_{bl}\ddot{\delta}_{1c}+\frac{1}{2}bc_\delta\dot{\delta}_{1c}+bI_{bl}\Omega\dot{\delta}_{1s}+\frac{1}{2}bI_{bl}\left(-1+\nu^2\right)\Omega^2\delta_{1c}+\frac{1}{2}bc_\delta\Omega\delta_{1s}=0 \\ \frac{1}{2}bm_s\ddot{x}+\frac{1}{2}bI_{bl}\ddot{\delta}_{1s}-bI_{bl}\Omega\dot{\delta}_{1c}+\frac{1}{2}bc_\delta\dot{\delta}_{1s}-\frac{1}{2}bc_\delta\Omega\delta_{1c}+\frac{1}{2}bI_{bl}\left(-1+\nu^2\right)\Omega^2\delta_{1s}=0 \end{cases} \tag{77}$$

With,

$$\nu=\sqrt{\frac{k_\delta}{\Omega^2 I_{bl}}+\frac{em_s}{I_{bl}}} \text{ and } m_s=\int_0^L \rho.r.dr \tag{78}$$

The set of equations can be then casted into **M**, **C** and **K** matrices and set of equations is set into state space form with a state vector being $\mathbf{x}= [\mathbf{q},\dot{\mathbf{q}}]^T$ and $\mathbf{q}= [x,\delta_{1c}, \delta_{1s}]^T$,

$$\dot{\mathbf{x}} = \mathbf{Ax} \text{ with } \mathbf{A} = \begin{bmatrix} -\mathbf{M}^{-1}\mathbf{C} & -\mathbf{M}^{-1}\mathbf{K} \\ \mathbf{I} & [\mathbf{0}] \end{bmatrix} \tag{79}$$

A classic eigenvalue analysis of matrix **A** allows to conclude on the stability of the equilibrium. The equilibrium position of the system presented above is known to be potentially unstable due to a parametric instability when the undamped frequencies of each blade lag motion in the fixed frame equals the airframe undamped frequency lateral motion,

Table 5. Helicopter data

Main rotor		
Number of blades	b	4
Blade root eccentricity	e (m)	0.198
Blade length	L (m)	5.50
Angular velocity	Ω (rad/s)	38.7
Individual blade		
Static moment	m_s (m.kg)	102
Inertia	I_{bl} (m².kg)	373
Mass	m_{bl} (kg)	37
Equivalent angular lag damper stiffness	k_δ (N.m/rad)	100 000
Equivalent angular lag damper damping	c_δ (N.m.s/rad)	2500
Airframe		
Mass	m_f (kg)	2000
Landing gear		
Equivalent stiffness	k_f (N.m/rad)	808 520
Equivalent damping	c_x (N.m/rad)	1300

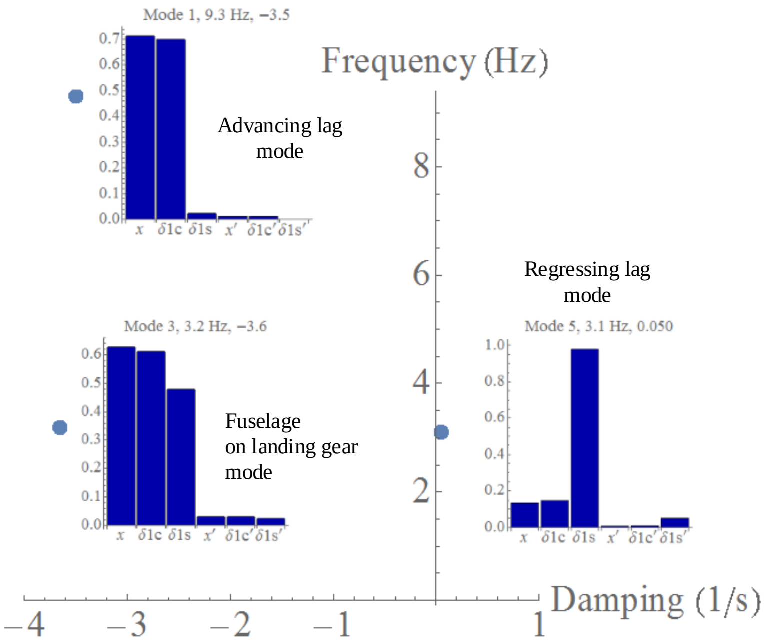


Figure 5-8. Modes and shapes in the complex plane at rotor nominal angular velocity

5.4.2. Average of the Hamiltonian function as a Chetaev candidate function

In (Oberinger & Hajek, 2013), the authors state that "*the condition for the presence of a dynamical instability of a mechanical system, i.e. divergent oscillations, is an increase of the system's internal energy*. Internal energy is defined in this case as the sum of kinetic and potential energies of the system and it is therefore the sum of total energy of the system, let us also call it the Hamiltonian. In that same paper the authors use this condition to trace the energy contributions of helicopter physical system's degrees of freedom to the increase of internal energy. They apply this to helicopter ground resonance and to more complex rotor-airframe couplings.

A numerical simulation of the response to a small force perturbation on the airframe of the idealized rotor-airframe physical system presented on Figure 5-7, is plotted below and illustrates what states Oberinger: in the blue case, nominal values of Table 5, according to Campbell diagram the system is unstable. Numerically the states appear to diverge and the Hamiltonian appears to grow. The system enters in a regime in which it demands power to the source. The model from which the results are obtained is a bond graph model that is presented in the next section.

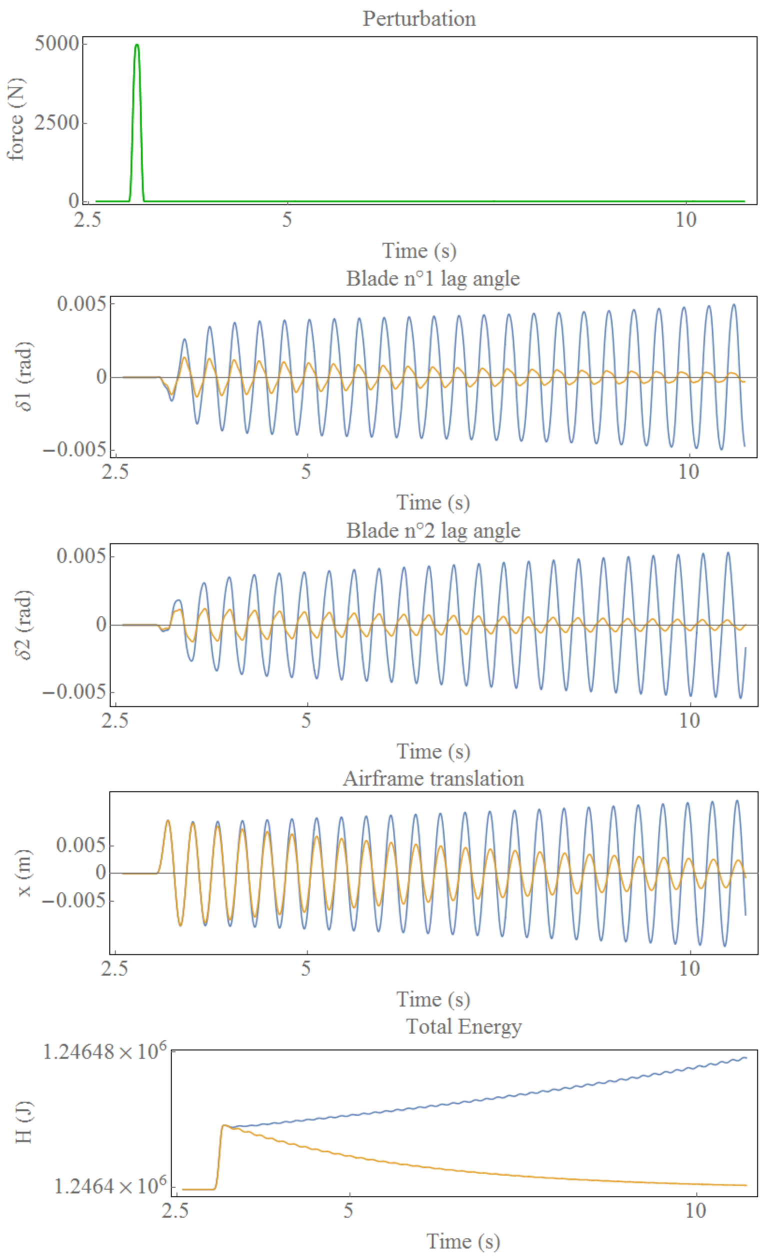


Figure 5-9. Free response of rotor-airframe for nominal parameters (blue) and three times higher damping in lag dampers (orange)

Stating that the internal energy increases is mathematically vague. As a matter of fact if the Hamiltonian would strictly increase in time around equilibrium, its total time derivative would have the same sign as its Hamiltonian, and the function would immediately be a Chetaev function. However, a closer look to the Hamiltonian plotted before[27] shows H tends to increase with time but *not strictly*: it oscillates while increasing, therefore its total time derivative changes sign around the equilibrium and it is not a Chetaev function.

However, in the particular case of ground resonance parametric instability, it appears numerically, that a linear regression of the Hamiltonian around the equilibrium over a few rotor periods, defined as $\hat{H}$, could be a Chetaev function, see Figure 5-10.

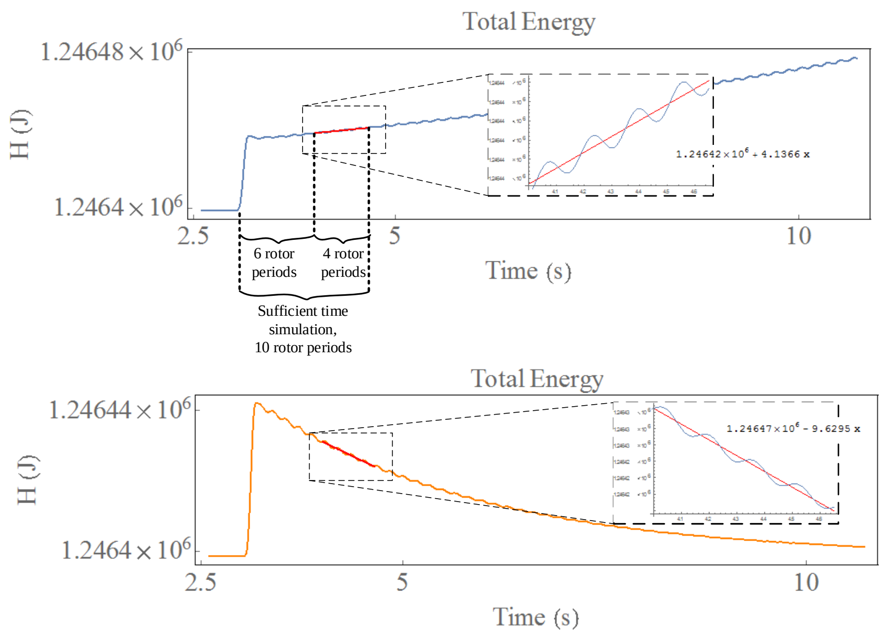


Figure 5-10. Computation of $\hat{H}$, a linear regression of total energy over a few rotor periods

Based on the numerical simulation results a least-squares fit is employed to find β and γ such that,

$$\hat{H}(t) = \beta.t + \gamma \tag{80}$$

This can be represented in the $(\dot{V}, V)$ plane, see Figure 5-11; the reader should also come back to (b) Figure 5-6, to see the analogy between the simple pendulum and this more complex system by relating these two figures. A physical meaning can be given to the Chetaev function,

[27] (Oberinger & Hajek, 2013) numerical simulation results of ground resonance exhibit similar oscillations.

$$\frac{d\hat{H}}{dt} = \beta = \hat{P}_{engine} - \hat{P}_{dissipated} > 0 \tag{81}$$

In other words, when the system becomes unstable, it reorganizes itself in such a way it demands more energy to sources outside its boundaries than it is able to dissipate. This surplus of energy is in return stored in potential and kinetic energies.

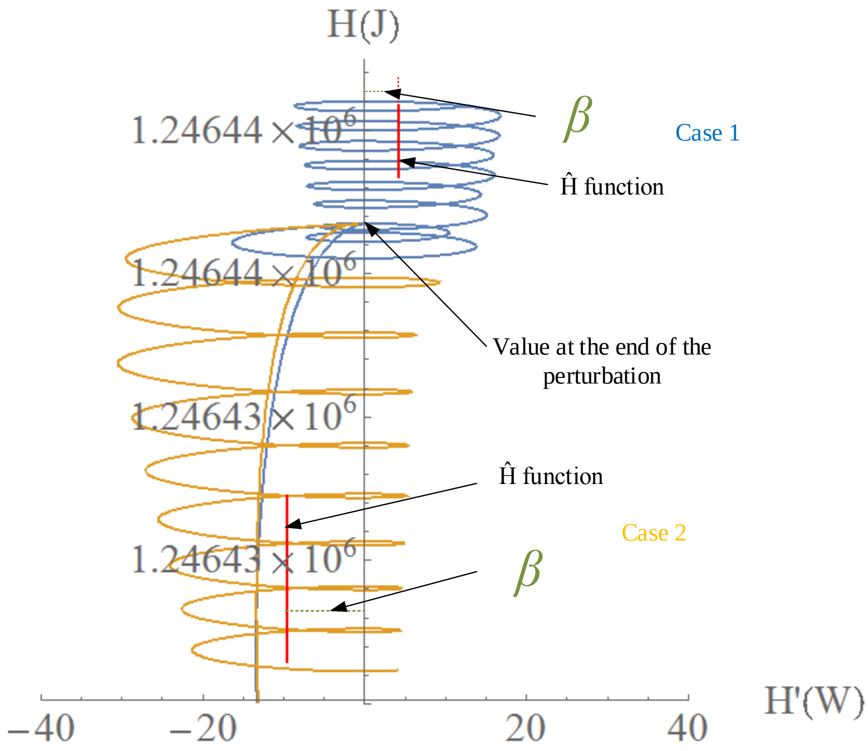


Figure 5-11. Projection in the ($\dot{V}$, V) plane after perturbation of equilibrium

5.4.3. Hamiltonian expression from a bond graph model of the physical system

The objective of this section is twofold, first to give an explicit graph that represents the physical system described before and secondly to explain how energy-based Chetaev candidate functions can be expressed.

Bong graph model of the physical system

Based on the rotor-airframe model using multibody dynamics developed in chapter 2, one can simplify it to obtain the bond graph presented on Figure 5-14. The first simplification consists in modifying the equations of motion of the airframe: here it is considered as a rigid body in translation, as a result the superior part of the graph that accounts for Euler equations has been removed. The rest of the simplifications consist in modifying the parameters of the R and C elements to enforce constraints in joints between rigid bodies. The airframe is constrained in such a way it can only translate in the x direction and each one of the blades can lag but not flap or pitch. It can also be remarked that the hub inertia and mass have not been taken into account by just removing the inertial elements on its graph; the resulting equations behind it, represent therefore pure kinematic relations. In order, to show all the bonds and causal strokes, the full graph is presented on Figure 5-14.

In addition, to implement the constraints on the degrees of freedom that cannot move, parasitic elements are introduced. As already discussed in the first chapters of this work, this approach has advantages and disadvantages. This choice is motivated by the need to keep as many elements as possible in integral causality. However, the inertial elements that represent the mass matrices of each blade are still in derivative causality. Therefore, the system of equations is a set of DAEs. Rigorously speaking Chetaev functions apply only to ODEs. It is conjectured here, they can still be applied directly to the underlying proposed set of DAEs. A transformation to a set of ODEs would consist in adding R and C to all the revolute joints.
Furthermore, the set of ODEs one would obtain would be non-autonomous; the time *t* would appear explicitly in the equations. A supplementary step would be needed to make the system autonomous and apply the proposed Chetaev theorem by for example invoking the periodicity of the equations and adding the time *t* as a state variable as proposed in (Masarati, Quaranta, Lanz, & Mantegazza, 2003). Another option would be to see to what extent the non-autonomous version of Chetaev theorem can be applied. These two options need in any case further investigations.

Hamiltonian function expression from the bond graph

Interestingly, a bond graph such as the one presented next page can be seen as structured around the total energy of the system or Hamiltonian in our case. The physical system on Figure 5-7 is modeled as containing:

1. **5 linear springs** for which their potential energies are proportional to the square of the associated generalized position by their stiffness characteristic.

2. **5 rigid bodies** that are characterized by 5 kinetic energy terms: one associated to the airframe and 4 for each blade rotor. Each one of this blade kinetic energies can be decomposed into two terms: one associated with the rotation of the rigid body and one associated to its translation. In our approach these terms are therefore decomposed into 2 terms, as a remainder of the equation (16) given in Chapter 1,

$$T_i = \frac{1}{2} m_i \mathbf{V}_{G_i,i/0}^{0}{}^{T} . \mathbf{V}_{G_i,i/0}^{0} + \frac{1}{2} \mathbf{\Omega}_{i/0}^{i}{}^{T} . \mathbf{I}_{G,i} . \mathbf{\Omega}_{i/0}^{i} \tag{82}$$

Finally, the bond graph of our system contains 5 potential energy terms and 10 kinetic energy terms. The expression of each of these terms is then decomposed in the bond graph into the three directions of the Cartesian space and their sum give the expressions of kinetic and potential energy, in other words total energy or the Hamiltonian of the system,

$$\begin{aligned} H(\mathbf{p},\mathbf{q}) &= T(\mathbf{p}) + V(\mathbf{q}) \\ &= \sum \frac{p_i^2}{2I_i} + \sum \frac{q_j^2}{2C_j} \end{aligned} \tag{83}$$

Where, p_i are the generalized momenta and q_i are the generalized coordinates,

$$\mathbf{p} = {}^T\left[m\dot{x}, \left\{ I(\dot{\Psi} + \dot{\delta}_k), m\mathbf{V}^0_{G_k,k/0}.\mathbf{x}_3, m\mathbf{V}^0_{G_k,k/0}.\mathbf{y}_3, \text{ for each blade } k \right\} \right] \tag{84}$$

$$\mathbf{q} = {}^T\left[x, \delta_1, \delta_2, \delta_3, \delta_4 \right] \tag{85}$$

And the bond graph storage elements contain linear constitutive laws,

$$I_i = m_i \text{ and } C_j = \frac{1}{k_j} \tag{86}$$

As described in vector **p**, the rotor angular velocity is a state variable using this approach. To keep the hypothesis of constant rotor angular velocity, a perfect source of flow, an MSf bond graph element, is introduced in the graph: it represents the engine and keeps this state variable at the desired value at any time *t*. Compared to the classic formulation of the problem using Lagrange equations, at this stage, the kinetic energy of the system has a very compact form and does not contain itself the nonlinearities due to kinematic transformations from rigid body frames to the inertial reference frame. These nonlinearities are outside *H* and but are of course still present at the level of transformation elements.
Under these hypotheses the first principle of thermodynamics gives,

$$H(\mathbf{p},\mathbf{q}) = W_{engine}(\mathbf{p},\mathbf{q}) - W_{dissipations}(\mathbf{p}) \tag{87}$$

Each of the energy terms $p_i^2/2I_i$ and $q_j^2/2C_j$ of $H(\mathbf{p},\mathbf{q})$ can be computed from the graph by the energy sensors that have been placed at the root of the bond graph I and C elements see Figure 5-14. Additional power sensors have also been placed on R and the MSf elements to capture the total time derivative of *H*. Interestingly, under this form, see equation (83), *H* is a quadratic form of the states of the system with positive coefficients I_i and C_j; which means it is a positive definite function. At equilibrium the rotor angular velocity is constant and therefore the rotor kinetic energy is not null. On the other side, the potential energy of the system is null at equilibrium and since it is a quadratic form of states with positive coefficients C_j it is strictly minimal at the equilibrium.

5.4.4. Estimating the instability domain by parametric sweep

In the previous sections, it has been explained how the total energy of the system as well as its time derivative could be expressed from the bond graph. In this section, the computation of β, for system parameter variations, gives the sign of the proposed Chetaev function. The computation is done directly from the set of nonlinear DAEs and with the help of a Backward Differentiation Formula (BDF) method available in the bond graph preprocessor 20-sim®.
Relevant parameters of interest for the ground resonance phenomenon are the rotor angular velocity and the damping of the regressing lag mode, see Appendix 6. The damping of this mode is modified by varying the characteristic of the lag dampers damping. The results are presented on Figure 5-12, in which an increase of β becomes obvious around the nominal rotor angular velocity and decreases with an increase of lag damping. The determination of the necessary damping on a rotor is a major concern for rotorcraft designers. In fact over sizing the dampers will mean a heavier damper. In addition to the mass of the damper, which is a penalty by itself, the dampers rotate and will therefore generate higher centrifugal forces on other rotor parts. In return these parts will probably become heavier in other to substantiate static and fatigue strength criteria. Downsizing the lag damping degrades the stability of the rotor-airframe system and leads to aircrafts prone to *ground resonance*, *air resonance* or *lateral-roll aeroelastic RPCs*. However it should be noticed that other solutions exist to improve the damping of the regressing lag mode, such as the implementation of active controls (Takahashi & Friedmann, 1991), (Krysinski & Malburet, 2011).

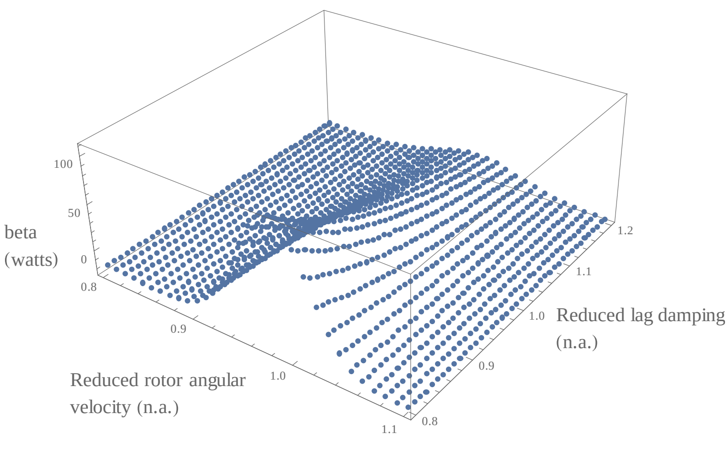


Figure 5-12. β for varying reduced rotor angular velocity and lag damping

To illustrate the soundness of the proposed approach, the results are compared to the linear stability results of the same physical system, see Figure 5-13. In orange, if the real part of all eigenvalues is strictly negative, an orange point is plotted. In blue, the criteria is on the sign of β: if beta is strictly negative, a blue point is plotted. A good agreement is found for the estimated instability domain between the two methods, see Figure 5-13. The minimal time computation to get this results have been of 10 rotor periods, as with Floquet theory. A higher number of periods converges to these results, but of course at an increased cost.

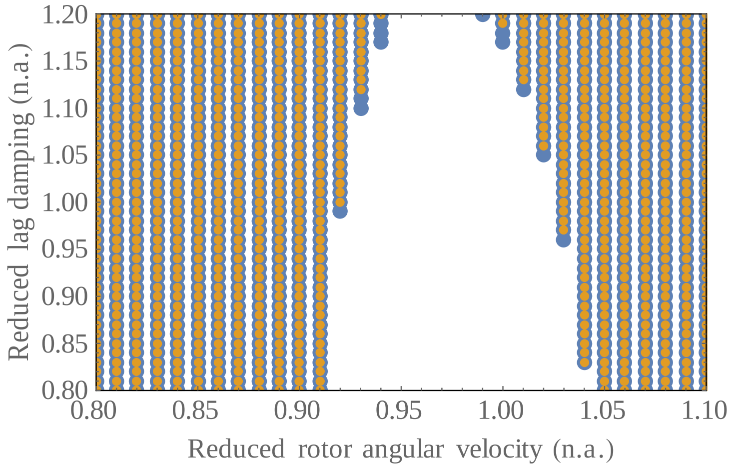


Figure 5-13. Instability domain estimations, negative real parts of the eigenvalues (orange) and negative beta (blue)

5.5. Conclusion

This chapter has presented and proposed an approach using Chetaev functions in order to determine if the equilibrium of a dynamic system is unstable directly from a nonlinear model. The use of this approach has a major limitations: there is no systematic method to find a Chetaev function and finding a Chetaev function gives only a sufficient condition of instability. As a result, if a Chetaev function is not found one cannot say the equilibrium is stable.

However, in the case where:

- There is a need to perform a parametric sweeps on a set of equations. It is in particular the case when investigating *parametric instabilities*.
- For one set of parameters, one has at disposal a Chetaev function. This would need to be proved analytically for that set.
- Then the method might have an interest in terms of computation time: indeed it is proposed to verify whether the Chetaev function candidate of the previous point is still Chetaev which demands a very short computation time.

This is illustrated on the ground resonance case for which total energy average seems to be an interesting Chetaev function candidate; an analytical proof still needs to be done for at least one case. It is shown that the computation time is in this case is around the same as what would be needed by applying the Floquet theory to the equivalent linear time periodic (LTP) system, which is 10 rotor periods for a system with 5 degrees of freedom. This computation time should be benchmarked to the time needed to compute the largest Lyapunov Characteristic Exponent. The approach proposed here can obviously be applied to other representations than bond graphs. However, this idea emerged while mapping energy flows in bond graphs and reading (Junco, 1993) in which Lyapunov functions are proposed to be extracted from BGs. It is therefore claimed that even if the approach can be applied to other representations, bond graphs were a necessary intellectual step to arrive to this idea.

Furthermore, extensive analytical work is still to be done. In particular the total energy average function proposed is not continuous till the equilibrium; to be rigorous it has to be continuous, so the connecting function expression needs to be found. The Chetaev theorem used in this chapter is the one that can be applied to autonomous systems; the transition from the nonlinear periodic equations to the nonlinear time invariant one needs to be justified more rigorously.

Finally, the algorithmic implications have not been discussed. As a matter of fact, *algebraic* instead of *ordinary* differential equations are numerically solved; but the Backward Differentiation Formula (BDF) method used here has its own stability and accuracy limitations that need to be taken into account to distinguish *physical instabilities* from *numerical* ones.

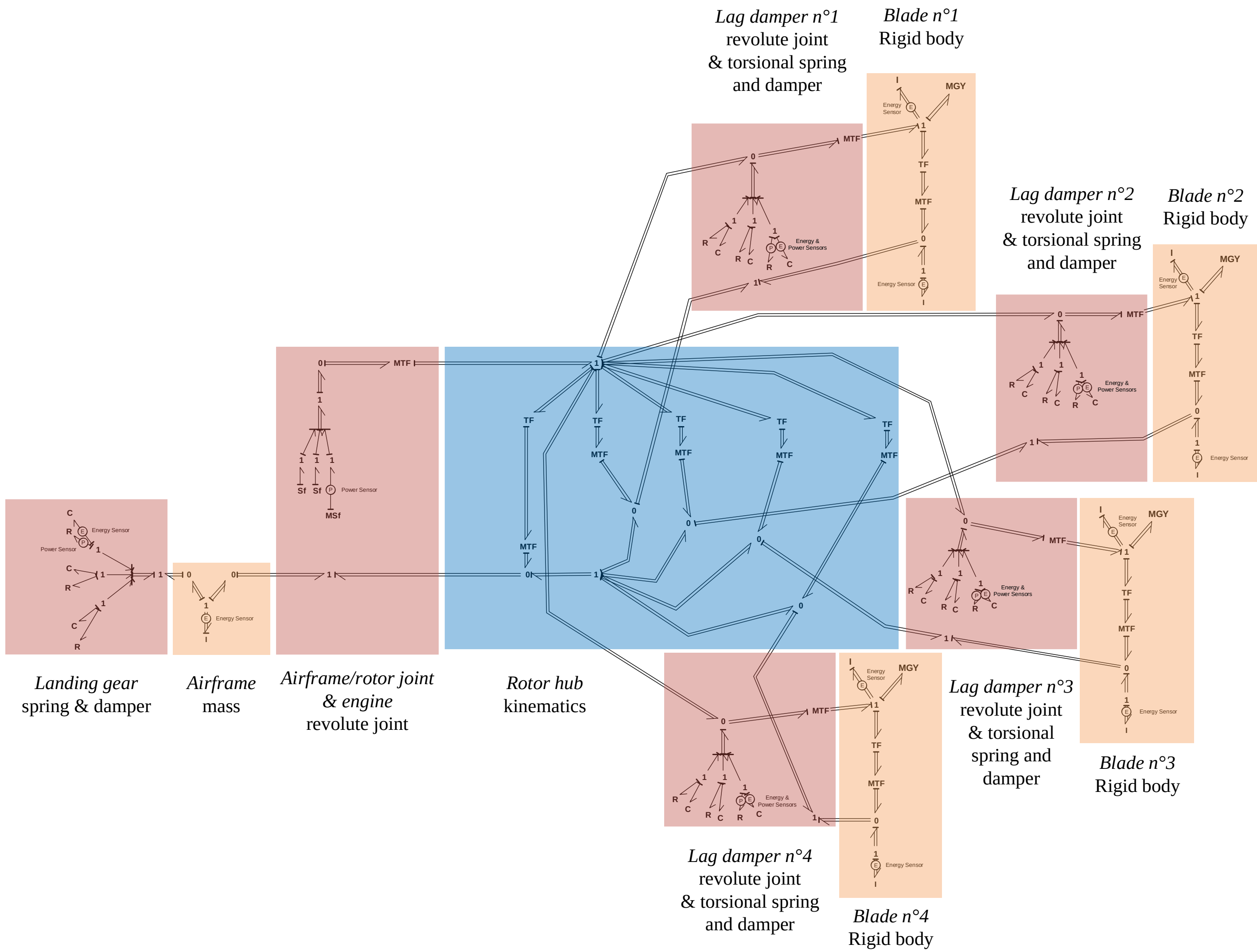


Figure 5-14. Rotor-airframe bond graph for ground resonance study using a multibody approach

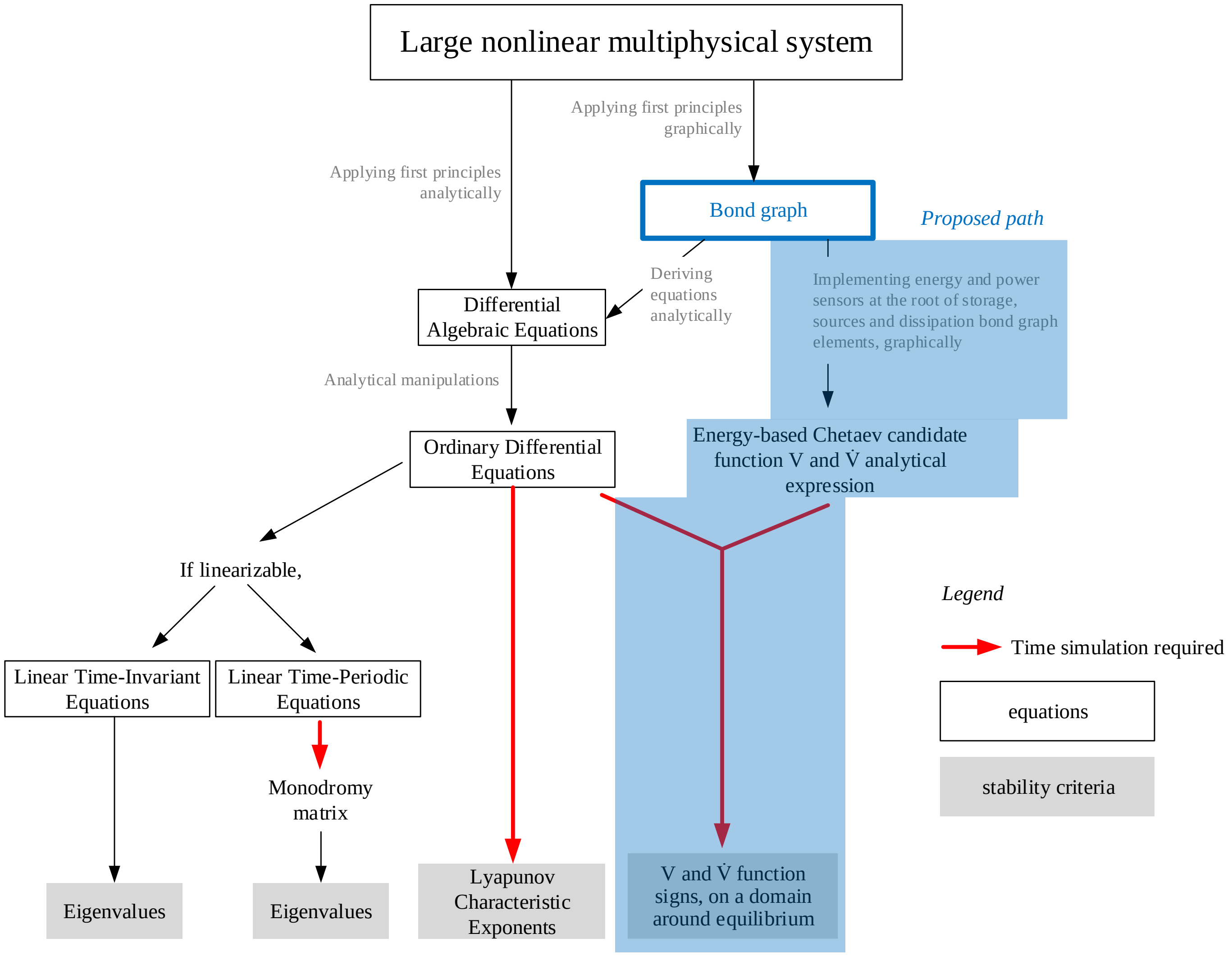


Figure 5-15. Path proposal, in the context of methods to evaluate the stability/instability of equilibriums

Conclusion

This thesis focuses on the modeling & analysis of complex dynamic systems using bond graphs. The modeling approach is in particular applied to the investigation of a dynamic phenomenon that appears on helicopters known as lateral-roll axes aeroelastic Rotorcraft-Pilot Couplings (RPCs). In this synthesis, three topics are put into perspective namely, the modeling method, the analysis of the resulting models and finally the application to RPCs themselves.

Concerning the modeling method

On the road towards a global energetic approach of helicopters in which as many subsystems and as physically detailed as necessary could be modeled, this investigation has illustrated two main advantages of the bond graph approach.
Modularity has been first illustrated when modeling multibody systems on chapters 2 & 3 using the bond graph pattern of a rigid body without manipulating equations. In fact, when modeled with such procedure, the equations of motion can be obtained relatively easily. However, as often in mechanics, the easier the equations of motion are obtained, the more complex the mathematical models are: not only Ordinary Differential Equations (ODEs) appear but also Differential Algebraic Equations (DAEs). The approach has of course the advantage of being applicable to a large class of problems but the drawback of probably not being the most computationally efficient for a particular problem. In addition in chapter 4, modularity was illustrated by showing how the frontiers between subsystems of a physical system are materialized at the graphical level of a BG.
A second advantage of BGs is that the form of their underlying mathematical models can be manipulated from the graphical level: "*An essential consequence of augmenting the bond graph with causal strokes is that the form of the mathematical model can be determined without formulating and manipulating any equation*", (Borutzky, 2009). This appears particularly useful to alleviate the level of complexity of the mathematical model prior to its resolution by a numerical method. This has been illustrated on chapter 2 when implementing the constraints between rigid bodies in a multibody system and in chapter 3 for systems that contain a closed kinematic chain (CKC). Furthermore an original bond graph has been proposed to model the concatenation of three revolute joints as needed when modeling the joint between articulated rotor blades and the rotor hub. The proposed bond graph allows to locally remove the algebraic constraints between blade and rotor hub joints at the graphical level.
It should be kept in mind that bond graphs represent naturally differential equations and not partial differential equations. If the need is such, for example to model complex aerodynamics, a more adapted energetic method could be used such as *Port-Hamiltonian Systems*, an evolution of bond graphs (Schaft, 2006).

Concerning the stability analysis of the resulting models

The last chapter presents and proposes the first blocks of an approach using Chetaev functions in order to determine if the equilibrium of a dynamic system is unstable directly from a nonlinear model. The approach proposed here can obviously be applied to other representations than bond graphs. However, this idea emerged while mapping energy flows in bond graphs and reading (Junco, 1993) in which energy based Lyapunov functions are proposed to be extracted from BGs. It is therefore claimed that even if the approach can be applied to other representations, bond graphs were a necessary intellectual step to arrive to this idea. The use of this approach has major limitations: there is no systematic method to find a Chetaev function and finding a Chetaev function gives only a sufficient condition of instability. As a result, if a Chetaev function is not found one cannot say the equilibrium is stable. However, the method might have an interest in terms of computation time in particular cases such as the investigation of *parametric instabilities*.

Concerning the application to lateral-roll aeroelastic RPCs

The development of an aeromechanical rotor-airframe model is proposed in chapter 2 to be representative of helicopter dynamics around hover at low frequencies. The aerodynamic model is a quasi-steady one for which the bond graph is considered to be an original contribution. It both allows to take into account for variable aerodynamic properties along blade spans and can be used without any modification to represent hover or forward flight configurations.
A pilot model has been developed in chapter 3. It consists in a neuromusculoskeletal model of pilot’s left upper limb. The individual subsystems that compose this model have been translated into bond graphs from literature. However, multibond graph representations are proposed for the first time to model individual neuromuscular forces between bones that have spatial motion. This model is applied for the first time for the prediction of BDFT on the lateral axis of a helicopter. The BDFT of a given human being depends on the settings of his neuromuscular system; for example whether he/she is stressed or not. It is known that the computation of the motion of human movement or posture leads to a mathematical indeterminacy because the human body possesses less degrees of freedom than muscles that act as actuators. As proposed in literature, the addition of an energetic principle can solve the indeterminacy problem. Postulating that the human body minimizes metabolic cost during motion allows to compute muscle activation unknowns. From a computational point of view an optimization algorithm is used to minimize muscle forces work, an energetic quantity that can be computed naturally from the bond graph representation. Numerical simulations are performed to compute pilot’s biodynamic feedthrough (BDFT). The results are quite encouraging: the task dependency of helicopter pilots BDFT can be predicted to a certain extent. An iterative procedure has allowed to find a set of parameters that have a physical meaning to reproduce what has been seen in experiments.
In chapter 4 the human-machine model obtained represents a bioaeroelastic behavior that is then analyzed more into detail; more precisely lateral-roll aeroelastic RPCs are investigated. A linear stability analysis confirms what has been conjectured in literature concerning the role played by the regressing lag mode in the phenomenon.

Furthermore, the results show that for higher neuromuscular pilot stiffness's, the higher frequencies of the advancing lag mode could also excite the airframe. This last result needs to be confirmed by some experiments. It should kept in mind that some of the assumptions taken in the full bioaeroelastic system are reductive; especially concerning the flight configuration and friction assumptions of the cyclic lever. However, even if the predicted instability domains will move with more detailed physics, the linear models used in chapter 4 will help understand the physical mechanisms behind the phenomena that could help explaining damping drops in real flights.

Future research

In the short term, the vehicle aeromechanical model needs to be further compared to flight tests to be considered as fully valid. One of the first improvements that will need to be done on the modeling hypothesis concerns the necessity to take into account rotor inflow velocity in the aerodynamic model. The implementation of unsteady aerodynamic models should also be investigated. The model prediction in terms of the pilot model kinematics should be compared to other literature models and supplementary experiments. More flight configurations should also be investigated and in particular forward flight.

In the long term, the next step would be to find an explicit mathematical relation between task and neuromuscular system parameters. This would allow obtaining a useful quantity for rotorcraft designers which would be BDFT maximal envelops rather than precise BDFT pilot behaviors.

Concerning the instability analysis using Chetaev functions an extensive analytical work is still to be done. In particular the total energy average function proposed is not continuous till the equilibrium; to be rigorous it has to be continuous, so the connecting expression needs to be found. The Chetaev theorem used in this chapter is the one that can be applied to autonomous systems; the transition from the nonlinear time periodic equations to the nonlinear time invariant ones needs to be justified more rigorously. In terms of computation time, the method should be benchmarked to the time needed to compute the largest Lyapunov Characteristic Exponent. Finally, the algorithmic implications have not been discussed. As a matter of fact, algebraic instead of ordinary differential equations are numerically solved; but the Backward Differentiation Formula (BDF) method used here has its own stability and accuracy limitations that need to be taken into account to distinguish physical instability estimation from numerical instabilities.

The models and the method employed to develop these models could be used in a wider range of applications, for example in the automotive industry, to investigate pilot-vehicle interactions. The neuromusculoskeletal model could be used in clinical applications to develop for example prosthesis based on muscle force estimations using non-invasive methods.

Finally, a characteristic of bond graphs that is probably underexploited is that not only the *equations of motion* can be obtained from the graphs, but at the same time, all the quantities necessary to compute the *conservation of energy* are represented.